

Chapter 4

Problems

17.

17.

$$\begin{aligned} (a) \quad P(X < 1) &= \lim_{n \rightarrow \infty} P\left\{X \leq 1 - \frac{1}{n}\right\} \\ &= \lim_{n \rightarrow \infty} F\left(1 - \frac{1}{n}\right) \\ &= \frac{1}{2} \quad \# \end{aligned}$$
$$\begin{aligned} (b) \quad P(X > 2) &= 1 - P\{X \leq 2\} \\ &= 1 - F(2) \\ &= 1 - \frac{2+1}{4} = \frac{1}{4} \quad \# \end{aligned}$$
$$\begin{aligned} (c) \quad P\left\{\frac{1}{3} < X < \frac{5}{3}\right\} \\ &= \lim_{n \rightarrow \infty} F\left(\frac{5}{3} - \frac{1}{n}\right) - F\left(\frac{1}{3}\right) \\ &= \frac{\frac{5}{3} + 1}{4} - \frac{\frac{1}{3}}{2} = \frac{1}{2} \quad \# \end{aligned}$$

19.

19,

$$\begin{aligned}
 P(X=1) &= P(X \leq 1) - P(X < 1) \\
 &= F(1) - \lim_{n \rightarrow \infty} F(1 - \frac{1}{n}) \\
 &= \frac{1}{4} - 0 = \frac{1}{4} \#
 \end{aligned}$$

$$\begin{aligned}
 P(X=3) &= P(X \leq 3) - P(X < 3) \\
 &= F(3) - \lim_{n \rightarrow \infty} F(3 - \frac{1}{n}) \\
 &= \frac{5}{8} - \frac{1}{4} = \frac{3}{8} \#
 \end{aligned}$$

$$\begin{aligned}
 P(X=4) &= P(X \leq 4) - P(X < 4) \\
 &= F(4) - \lim_{n \rightarrow \infty} F(4 - \frac{1}{n}) \\
 &= \frac{3}{4} - \frac{5}{8} = \frac{1}{8} \#
 \end{aligned}$$

$$\begin{aligned}
 P(X=6) &= P(X \leq 6) - P(X < 6) \\
 &= F(6) - \lim_{n \rightarrow \infty} F(6 - \frac{1}{n}) \\
 &= \frac{7}{8} - \frac{3}{4} = \frac{1}{8} \#
 \end{aligned}$$

$$\begin{aligned}
 P(X=7) &= P(X \leq 7) - P(X < 7) \\
 &= F(7) - \lim_{n \rightarrow \infty} F(7 - \frac{1}{n}) = 1 - \frac{7}{8} = \frac{1}{8} \#
 \end{aligned}$$

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X = i\} = i/148, i = 40, 33, 25, 50$

$$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$$

$$E[Y] = (40 + 33 + 25 + 50)/4 = 37$$

31. $E[\text{score}] = p^*[1 - (1 - p)^2 + (1 - p^*)(1 - p^2)]$

$$\frac{d}{dp} = 2(1 - p)p^* - 2p(1 - p^*)$$

$$= 0 \Rightarrow p = p^*$$

32. If T is the number of tests needed for a group of 10 people, then

$$E[T] = (.9)^{10} + 11[1 - (.9)^{10}] = 11 - 10(.9)^{10}$$

39. $E[(4X - 1)^2] = \text{Var}(4X - 1) + (E[4X - 1])^2 = 16\text{Var}(X) + (4E(X) - 1)^2 = 16 + 11^2 = 137.$

(i) $\text{Var}(5 - 2X) = 4\text{Var}(X) = 4.$

42. $\sum_{i=0}^6 \binom{10}{i} \left(\frac{3}{6}\right)^{10} \approx 0.8281.$

43. (a) Because each question will, independently, be answered correctly by both A and B with probability .28, the mean number is 2.8.
- (b) Because each question will, independently, be answered correctly either by A or by B with probability $1 - .18 = .82$, the number of questions so answered is binomial with parameters $n = 10$, $p = .82$, yielding that its variance is $np(1 - p) = 1.476.$

45. $\alpha \sum_{i=k}^n \binom{n}{i} p_1^i (1 - p_1)^{n-i} + (1 - \alpha) \sum_{i=k}^n \binom{n}{i} p_2^i (1 - p_2)^{n-i}$

50. (a) $\frac{1}{2} \binom{10}{7} .4^7 .6^3 + \frac{1}{2} \binom{10}{7} .7^7 .3^3$

(b) $\frac{\frac{1}{2} \binom{9}{6} (.4)^7 (.6)^3 + \frac{1}{2} \binom{9}{6} (.7)^7 (.3)^3}{.55}$

60. (a) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - \sum_{i=0}^3 \frac{(7)^i e^{-7}}{i!} \approx 0.9182.$

(b) $P(X \leq 5 | X \geq 1) = \frac{P(1 \leq X \leq 5)}{P(X \geq 1)} = \frac{\sum_{i=1}^5 \frac{(7)^i e^{-7}}{i!}}{1 - e^{-7}} \approx 0.3001.$

66. (a) $e^{-4} \approx 0.0183$

(b) $1 - \left(e^{-4} + 4e^{-4} + \frac{4^2 e^{-4}}{2} + \frac{4^3 e^{-4}}{3!} + \frac{4^4 e^{-4}}{4!} \right) \approx 0.3712$

74. (a) $\left(\frac{26}{38} \right)^5$

(b) $\left(\frac{26}{38} \right)^3 \frac{12}{38}$

77. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 + \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2 + \binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5 \frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

82. Let F be the number of students in the group who have failed the test.

(a) $P(F = 0) = \frac{\binom{41}{10}}{\binom{50}{10}} \approx 0.1091$

(b) $P(F \geq 3) = 1 - P(F \leq 2) = 1 - \frac{\binom{41}{10} + \binom{41}{9}\binom{9}{1} + \binom{41}{8}\binom{9}{2}}{\binom{50}{10}} \approx 0.2491$

86. Let X_i be the number of traffic accidents occurring on road i ($i = 1, 2, 3, 4, 5$). Then $E(X_1 + X_2 + X_3 + X_4 + X_5) = 0.45 + 0.2 + 0.4 + 0.5 + 0.35 = 1.9$.

88. Let B_i be equal to 1 if the necklace contains at least one bead of color i ; otherwise, it is 0. Then,

$$E\left(\sum_{i=1}^k B_i\right) = \sum_{i=1}^k E(B_i) = \sum_{i=1}^k P(B_i = 1) = \sum_{i=1}^k (1 - (1 - p_i)^n) = k - \sum_{i=1}^k (1 - p_i)^n.$$

89. (a) $P(X = 0) \rightarrow P(X = 0|R)P(R) + P(X = 0|B)P(B) + P(X = 0|G)P(G)$

$$= \frac{1}{3} \left(\frac{\binom{15}{4}}{\binom{25}{4}} + \frac{\binom{17}{4}}{\binom{25}{4}} + \frac{\binom{18}{4}}{\binom{25}{4}} \right)$$

(b) $P(X_i = 1) = P(X_i = 1|R)P(R) + P(X_i = 1|B)P(B) + P(X_i = 1|G)P(G)$

$$P(G) = \frac{1}{3}(10/25 + 8/25 + 7/25) = 1/3$$

which can also be seen by noting that no matter what the color of the i^{th} ball selected, there is probability $1/3$ that was the chosen color.

(c) $E[X] = E[X_1 + X_2 + X_3 + X_4] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = 4/3$

Self-Test Problems and Exercises

2.

2. 關係式保證 $p_i = c^i p_0, i = 1, 2$ ，其中 $p_i = P\{X = i\}$ 。因為這些機率總和為 1，所以

$$p_0(1 + c + c^2) = 1 \Rightarrow p_0 = \frac{1}{1 + c + c^2}$$

因此，

$$E[X] = p_1 + 2p_2 = \frac{c + 2c^2}{1 + c + c^2}$$

3.

4.3. Let X be the number of flips. Then the probability mass function of X is

$$p_2 = p^2 + (1 - p)^2, \quad p_3 = 1 - p_2 = 2p(1 - p)$$

Hence,

$$E[X] = 2p_2 + 3p_3 = 2p_2 + 3(1 - p_2) = 3 - p^2 - (1 - p)^2$$

5.

5. 令 $p = P\{X = 1\}$ 。則 $E[X] = p$ 且 $\text{Var}(X) = p(1 - p)$ ，所以

$$p = 3p(1 - p)$$

保證 $p = 2/3$ 。因此， $P\{X = 0\} = 1/3$ 。

9.

9. 因為 $E[X] = np$, $\text{Var}(X) = np(1 - p)$ ，且已知 $np = 6$, $np(1 - p) = 2.4$ 。所以 $1 - p = 0.4$ 或 $p = 0.6$, $n = 10$ 。因此，

$$P\{X = 5\} = \binom{10}{5} (.6)^5 (.4)^5$$

22.

22. 令 X 表你玩的局數且 Y 為你輸掉的局數。

(a) 在你玩了 4 局之後，你將持續玩下去直到輸為止。因此， $X - 4$ 為一個幾何隨機變數具參數 $1 - p$ ，所以

$$E[X] = E[4 + (X - 4)] = 4 + E[X - 4] = 4 + \frac{1}{1 - p}$$

(b) 若我們令 Z 表你在前 4 局輸掉的局數，則 Z 是一個二項隨機變數具參數 4 和 $1 - p$ 。因為 $Y = Z + 1$ ，我們有

$$E[Y] = E[Z + 1] = E[Z] + 1 = 4(1 - p) + 1$$