111-1 機率導論 期中考　　2022.11.22

班級：________________　學號：________________　姓名：_______________

1. (10%) An urn contains 3 red and 7 black balls. Players *A* and *B* withdraw balls from the urn consecutively until a red ball is selected. Find the probability that *A* selects the red ball. (*A* draws the first ball, then *B*, and so on. There is no replacement of the balls drawn.)
   (一個袋子中有 3 紅球和 7 黑球。*A*、*B* 兩人輪流自袋中取球，直到取到紅球才停止。則 *A* 取到紅球之機率為何？(*A* 先取球，*B* 再取，…等等，取出的球不再放回袋內。))

<解答>

27. Imagine that all 10 balls are withdrawn

$$P(A)=\frac{3\cdot 9!+7\cdot 6\cdot 3\cdot 7!+7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 5!+7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 3\cdot 3!}{10!}$$

2. (16%) You ask your neighbor to water a sickly plant while you are on vacation. Without water, it will die with probability 0.8; with water, it will die with probability 0.15. You are 90 percent certain that your neighbor will remember to water the plant. (你要求鄰居在你出門渡假時幫你澆水。若缺水時植物會死亡的機率為 0.8，不缺水時植物會死亡的機率為 0.15。你有 90% 的信心鄰居會記得澆水。)
   (a) (8%) What is the probability that the plant will be alive when you return? (則當你回家時，植物仍活著的機率為多少？)
   (b) (8%) If the plant is dead upon your return, what is the probability that your neighbor forgot to water it? (若植物已死，鄰居忘記澆水的機率為多少？)

<解答>

9. 令 *A* 表植物仍活著之事件令 *W* 為植物被澆水之事件。

(a)
$$P(A)=P(A\mid W)P(W)+P(A\mid W^c)P(W^c)$$
$$=(.85)(.9)+(.2)(.1)=.785$$

(b)
$$P(W^c\mid A^c)=\frac{P(A^c\mid W^c)P(W^c)}{P(A^c)}$$
$$=\frac{(.8)(.1)}{.215}=\frac{16}{43}$$

3. (10%) If 8 people, consisting of 4 couples, are randomly arranged in a row, find the probability that no person is next to their partner. (4 對夫妻坐成一排，求沒有夫妻坐在一起之機率。)

<解答>

53. Let A_i be the event that couple i sit next to each other. Then

$$P(\cup_{i=1}^{4} A_i) = 4\frac{2\cdot 7!}{8!} - 6\frac{2^2\cdot 6!}{8!} + 4\frac{2^3\cdot 5!}{8!} - \frac{2^4\cdot 4!}{8!}$$

and the desired probability is 1 minus the preceding.

$$P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c)$$
$$= P(A_1 \cup A_2 \cup A_3 \cup A_4)^c$$
$$= 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$$

4. (8%) In a class, there are 4 first-year boys, 6 first-year girls, and 6 sophomore boys. How many sophomore girls must be present if **sex** and **class** are to be **independent** when a student is selected at random?
(某班有 4 位大一男生，6 位大一女生和 6 位大二男生。隨機選出一位同學的情況下，需要有多少位大二女生出席才能使得**性別**和**年級**為**獨立事件**？)

<解答>

60. $P\{\text{Boy}, F\} = \frac{4}{16+x}$ $\quad P\{\text{Boy}) = \frac{10}{16+x}$ $\quad P\{F\} = \frac{10}{16+x}$

so independence $\Rightarrow 4 = \frac{10\cdot 10}{16+x} \Rightarrow 4x = 36$ or $x = 9$.

A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

5. (20%) A and B play a series of games. Each game is independently won by A with probability p and by B with probability $1-p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the winner of the series. (A 和 B 玩一序列的遊戲。每次遊戲 A 贏的機率為 p 且 B 贏的機率為 $1-p$，而遊戲的結果是互相獨立的。當其中一位玩家贏的總次數比另一位玩家多 2 次則停止遊戲。而贏的次數較多者即為該序列遊戲的優勝者。)
   (a) (10%) Find the probability that a total of 4 games are played. (求總共只玩 4 次遊戲之機率。)
   (b) (10%) Find the probability that A is the winner of the series. (求 A 是該序列遊戲之優勝者的機率。)

<解答>

81. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1-p)[p^2 + (1-p)^2]$.

(b) Let A be the event that A wins. Conditioning on the outcome of the first two games gives

$$P(A) = P(A|a, a)p^2 + P(A|a, b)p(1-p) + P(A|b, a)(1-p)p + P(A|b, b)(1-p)^2$$
$$= p^2 + P(A)2p(1-p)$$

where the notation a, b means, for instance, that A wins the first and B wins the second game. The final equation used that $P(A|a, b) = P(A|b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

6. (16%) A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be (一鞋櫃有 10 雙鞋子。若隨機取出 8 隻鞋子，則？)
(a) (8%) no complete pair? (沒有完整一雙鞋之機率為何？)
(b) (8%) exactly 1 complete pair? (恰有完整一雙鞋之機率為何？)

<解答>

52 (a) $\dfrac{\binom{10}{8}\left(\binom{2}{1}\right)^8}{\binom{20}{8}} = \dfrac{20\times18\times16\times14\times12\times10\times8\times6}{20\times19\times18\times17\times16\times15\times14\times13}$ #

(b) (法一) $\dfrac{\binom{10}{1}\binom{2}{2}\binom{9}{6}\left(\binom{2}{1}\right)^6}{\binom{20}{8}}$ #

(法二) $\dfrac{\binom{10}{1}\binom{2}{2}\binom{18}{1}\binom{16}{1}\binom{14}{1}\binom{12}{1}\binom{10}{1}\binom{8}{1}\times\frac{1}{6!}}{\binom{20}{8}}$ #

7. (20%) Suppose that 10 fish are caught at a lake that contains 5 distinct types of fish. (假設在包含 5 種不同類型魚的湖中捕獲了 10 條魚。)
   (a) (6%) How many different outcomes are possible, where an outcome specifies the numbers of caught fish of each of the 5 types? (有多少種不同的可能結果，其中每一個結果分別包含了 5 種分別不同類型中每一種魚的捕獲數量？)
   (b) (6%) How many outcomes are possible when 3 of the 10 fish caught are trout? (當捕獲的 10 條魚中有 3 條是鱒魚時，有多少種可能的結果？)
   (c) (8%) How many when at least 2 of the 10 are trout? (當 10 條中至少有 2 條是鱒魚時，有多少種可能的結果？)

<解答>

(a) $\binom{14}{4} = 1001$

(b) $\binom{10}{3} = 120$

(c) There are $\binom{13}{3} = 286$ possible outcomes having 0 trout caught and $\binom{12}{3} = 220$ possible outcomes having 1 trout caught. Hence, using (a), there are $1001 - 286 - 220 = 495$ possible outcomes in which at least 2 of the 10 are trout.