

Chapter 8

Problems

1. $P\{0 \leq X \leq 40\} = 1 - P\{|X - 20| > 20\} \geq 1 - 20/400 = 19/20$

Handwritten solution for problem 1:

$$\begin{aligned} & 1. \\ & P\{0 \leq X \leq 40\} \\ &= P\{0 - 20 \leq X - 20 \leq 40 - 20\} \\ &= P\{-20 \leq X - 20 \leq 20\} \\ &= P\{|X - 20| \leq 20\} \\ &= 1 - P\{|X - 20| > 20\} \\ &\geq 1 - \frac{20}{400} = \frac{19}{20} \quad \# \end{aligned}$$

$$4. \quad (a) \quad P\left\{\sum_{i=1}^{20} X_i > 15\right\} \leq 20/15$$

$$(b) \quad P\left\{\sum_{i=1}^{20} X_i > 15\right\} = P\left\{\sum_{i=1}^{20} X_i > 15.5\right\}$$

$$\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\}$$

$$= P\{Z > -1.118\}$$

$$\approx 0.8682165$$

6. If X_i is the outcome of the i^{th} roll then $E[X_i] = 7/2$ $\text{Var}(X_i) = 35/12$ and so

$$P\left\{\sum_{i=1}^{79} X_i \leq 300\right\} = P\left\{\sum_{i=1}^{79} X_i \leq 300.5\right\}$$

$$\approx P\left\{N(0,1) \leq \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \leq 1.58\} = .9429$$

$$7. \quad P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525 - 500}{\sqrt{(100 \times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$$

where the above uses that an exponential with mean 5 has variance 25.

8. If we let X_i denote the life of bulb i and let R_i be the time to replace bulb i then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \leq 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$

524.75 and variance $2500 + 99/48 = 2502$ it follows that the desired probability is approximately equal to $P\{N(0, 1) \leq [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \leq .505\} = .693$

It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2} \text{Unif}[0,1]\right) = 1/48$$

Self-Test Problems and Exercises

1. 與 2.

1. 某家經銷商每週賣出的汽車數量是一個隨機變數具有期望值 16。求下列各事件之機率的一個上界。

(a) 下週的銷售量大於 18；

(b) 下週的銷售量大於 25。

2. 假設問題中的每週汽車銷售量之變異數為 9。

(a) 求下週的銷售量在 10 到 22 之間

(b) (含) 的機率之一個下界。

(b) 求下週的銷售量大於 18 之機率的一個上界。

1. 令 X 表下週之銷售數量，注意 X 是整數的。由馬可夫不等式，我們得到下列：

$$(a) P\{X > 18\} = P\{X \geq 19\} \leq \frac{E[X]}{19} = 16/19$$

$$(b) P\{X > 25\} = P\{X \geq 26\} \leq \frac{E[X]}{26} = 16/26$$

2. (a) $P\{10 \leq X \leq 22\} = P\{|X - 16| \leq 6\}$

$$= P\{|X - \mu| \leq 6\}$$

$$= 1 - P\{|X - \mu| > 6\}$$

$$\geq 1 - 9/36 = 3/4$$

$$(b) P\{X \geq 19\} = P\{X - 16 \geq 3\} \leq \frac{9}{9-9} = 1/2$$

在 (a) 部分，我們使用謝比雪夫不等式；在 (b) 部分，我們使用其單邊型式。(見定理 5.1)

4.

8.4. If X is the number produced at factory A and Y the number produced at factory B , then

$$\begin{aligned} E[Y - X] &= -2, \quad \text{Var}(Y - X) = 36 + 9 = 45 \\ P\{Y - X > 0\} &= P\{Y - X \geq 1\} \\ &= P\{Y - X + 2 \geq 3\} \leq \frac{45}{45 + 9} = 45/54 \end{aligned}$$

4. 若 X 為工廠 A 所生產之數量且 Y 為工廠 B 所生產之數量，則

$$\begin{aligned} E[Y - X] &= -2, \quad \text{Var}(Y - X) = 36 + 9 = 45 \\ P\{Y - X > 0\} &= P\{Y - X \geq 1\} = P\{Y - X + 2 \geq 3\} \leq \frac{45}{45 + 9} = 45/54 \end{aligned}$$

7.

8.7. If X is the time required to service a machine, then

$$E[X] = .2 + .3 = .5$$

Also, since the variance of an exponential random variable is equal to the square of its mean, we have

$$\text{Var}(X) = (.2)^2 + (.3)^2 = .13$$

Therefore, with X_i being the time required to service job $i, i = 1, \dots, 20$, and Z being a standard normal random variable, it follows that

$$\begin{aligned} P\{X_1 + \dots + X_{20} < 8\} &= P\left\{ \frac{X_1 + \dots + X_{20} - 10}{\sqrt{2.6}} < \frac{8 - 10}{\sqrt{2.6}} \right\} \\ &\approx P\{Z < -1.24035\} \\ &\approx .1074 \end{aligned}$$

7. 若 X 表維修一部機器所需時間，則

$$E[X] = 0.2 + 0.3 = 0.5$$

而且，指數隨機變數的變異數等於其平均數的平方，所以

$$\text{Var}(X) = (0.2)^2 + (0.3)^2 = 0.13$$

因此，令 X_i 表服務工作 i 所花時間 $i = 1, \dots, 20$ ，且 Z 為一個標準常態隨機變數，則

$$\begin{aligned} P\{X_1 + \dots + X_{20} < 8\} &= P\left\{ \frac{X_1 + \dots + X_{20} - 10}{\sqrt{2.6}} < \frac{8 - 10}{\sqrt{2.6}} \right\} \\ &\approx P\{Z < -1.24035\} \\ &\approx .1074 \end{aligned}$$