Chapter 8

Problems

1.

$$P(0 < X < 40)$$
$$= P(0-20 < X-20 < 40-20)$$
$$= P(-20 < X-20 < 20)$$
$$= P(|X-20| < 20)$$
$$= 1 - P(|X-20| \geq 20)$$
$$\because P(|X-20| \geq 20) \leq \frac{20}{20^2} = \frac{1}{20}$$
$$\Rightarrow -P(|X-20| \geq 20) \geq -\frac{1}{20}$$
$$\Rightarrow 1 - P(|X-20| \geq 20) \geq 1 - \frac{1}{20} = \frac{19}{20}$$

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4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

(b) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} = P\left\{\sum_{i=1}^{20} X_i > 15.5\right\}$

$$\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\}$$

$$= P\{Z > -1.118\}$$

$$\approx 0.8682165$$

6. If X_i is the outcome of the i^{th} roll then $E[X_i] = 7/2$ $\text{Var}(X_i) = 35/12$ and so

$$P\left\{\sum_{i=1}^{79} X_i \le 300\right\} = P\left\{\sum_{i=1}^{79} X_i \le 300.5\right\}$$

$$\approx P\left\{N(0,1) \le \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \le 1.58\} = .9429$$

7. $P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525 - 500}{\sqrt{(100 \times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$

where the above uses that an exponential with mean 5 has variance 25.

8. If we let X_i denote the life of bulb i and let R_i be the time to replace bulb i then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \le 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$ 524.75 and variance 2500 + 99/48 = 2502 it follows that the desired probability is approximately equal to $P\{N(0, 1) \le [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \le .505\} = .693$
It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48$$

Self-Test Problems and Exercises

1.與 2.

1. 某家經銷商每週賣出的汽車數量是一個隨機變數具有期望值 16。求下列各事件之機率的一個上界。

   (a) 下週的銷售量大於 18；

   (b) 下週的銷售量大於 25。

2. 假設問題中的每週汽車銷售量之變異數爲 9。

   (a) 求下週的銷售量在 10 到 22 之間（含）的機率之一個下界。

   (b) 求下週的銷售量大於 18 之機率的一個上界。

1. 令 X 表下週之銷售數量，注意 X 是整數的。由馬可夫不等式，我們得到下列：

   (a) $P\{X > 18\} = P\{X \geq 19\} \leq \dfrac{E[X]}{19} = 16/19$

   (b) $P\{X > 25\} = P\{X \geq 26\} \leq \dfrac{E[X]}{26} = 16/26$

2. (a)
$$\begin{aligned} P\{10 \leq X \leq 22\} &= P\{|X - 16| \leq 6\} \\ &= P\{|X - \mu| \leq 6\} \\ &= 1 - P\{|X - \mu| > 6\} \\ &\geq 1 - 9/36 = 3/4 \end{aligned}$$

   (b) $P\{X \geq 19\} = P\{X - 16 \geq 3\} \leq \dfrac{9}{9-9} = 1/2$

   在 (a) 部分，我們使用謝比雪夫不等式；在 (b) 部分，我們使用其單邊型式。（見定理 5.1）

4.

8.4. If X is the number produced at factory A and Y the number produced at factory B, then

$$E[Y - X] = -2, \quad \text{Var}(Y - X) = 36 + 9 = 45$$

$$\begin{aligned} P\{Y - X > 0\} &= P\{Y - X \geq 1\} \\ &= P\{Y - X + 2 \geq 3\} \leq \frac{45}{45 + 9} = 45/54 \end{aligned}$$

4. 若 X 為工廠 A 所生產之數量且 Y 為工廠 B 所生產之數量，則

$$E[Y-X]=-2, \quad \text{Var}(Y-X)=36+9=45$$

$$P\{Y-X>0\}=P\{Y-X\geq 1\}=P\{Y-X+2\geq 3\}\leq \frac{45}{45+9}=45/54$$

7.

8.7. If X is the time required to service a machine, then

$$E[X] = .2 + .3 = .5$$

Also, since the variance of an exponential random variable is equal to the square of its mean, we have

$$\text{Var}(X) = (.2)^2 + (.3)^2 = .13$$

Therefore, with X_i being the time required to service job $i, i = 1, \ldots, 20$, and Z being a standard normal random variable, it follows that

$$\begin{aligned} P\{X_1 + \cdots + X_{20} < 8\} &= P\left\{\frac{X_1 + \cdots + X_{20} - 10}{\sqrt{2.6}} < \frac{8-10}{\sqrt{2.6}}\right\} \\ &\approx P\{Z < -1.24035\} \\ &\approx .1074 \end{aligned}$$

7. 若 X 表維修一部機器所需時間，則

$$E[X] = 0.2 + 0.3 = 0.5$$

而且，指數隨機變數的變異數等於其平均數的平方，所以

$$\text{Var}(X) = (0.2)^2 + (0.3)^2 = 0.13$$

因此，令 X_i 表服務工作 i 所花時間 $i = 1, \cdots, 20$，且 Z 為一個標準常態隨機變數，則

$$\begin{aligned} P\{X_1 + \cdots + X_{20} < 8\} &= P\left\{\frac{X_1 + \cdots + X_{20} - 10}{\sqrt{2.6}} < \frac{8-10}{\sqrt{2.6}}\right\} \\ &\approx P\{Z < -1.24035\} \\ &\approx .1074 \end{aligned}$$