Chapter 5

Problems

1. (a) $c\int_2^4 \left(x - \frac{3}{x^2}\right)dx = 1 \Rightarrow c = \frac{4}{21}$

   (b) $F(x) = \frac{4}{21}\int_2^x \left(x - \frac{3}{x^2}\right)dx = \frac{4}{21}\left(\frac{3}{x} + \frac{x^2}{2} - \frac{7}{2}\right),\ 2 < x < 4$

4. (a) $\int_{20}^{\infty} \frac{10}{x^2}dx = \frac{-10}{x}\Big|_{20}^{\infty} = 1/2$.

   (b) $F(y) = \int_{10}^{y} \frac{10}{x^2}dx = 1 - \frac{10}{y}$, $y > 10$. $F(y) = 0$ for $y < 10$.

   (c) $\sum_{i=3}^{6}\binom{6}{i}\left(\frac{2}{3}\right)^i\left(\frac{1}{3}\right)^{6-i}$ since $\bar{F}(15) = \frac{10}{15}$. Assuming independence of the events that the devices exceed 15 hours.

7. $\int_1^3 (a + bx^3)dx = 1 \Rightarrow 2a + 20b = 1$

$$\int_1^3 x(a + bx^3)dx = 5 \Rightarrow 4a + \frac{242b}{5} = 5 \Rightarrow 20a + 242b = 25$$

Solving the two simultaneous equations gives us the following: $a = -\frac{43}{14}, b = \frac{5}{14}$.

10. (a) $P\{\text{goes to } A\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$.

    $= 2/3$ since X is uniform $(0, 60)$.

    (b) same answer as in (a).

13. (a) Let X be the random variable represent the waiting time of the passenger.

    Then, $P(X > 6) = \int_6^{15} \frac{1}{15}dx = \frac{3}{5}$.

    (b) $P(X > 10\,|X > 8) = \frac{P(X>10)}{P(X>8)} = \frac{\int_{10}^{15}\frac{1}{15}dx}{\int_8^{15}\frac{1}{15}dx} = \frac{15}{21}$.

16. $P\{X > 50\} = P\left\{\frac{X-40}{4} > \frac{10}{4}\right\} = 1 - \Phi(2.5) = 1 - .9938$

    Hence, $(P\{X < 50\})^{10} = (.9938)^{10}$

22. Let S_n denote the number of times Tom hits the bullseye in n throws and Z a standard normal random variable. Then,

$$P(S_{50} < 1) = P(S_{50} < 0.5) = P\left(\frac{S_{50} - (50)(0.05)}{\sqrt{(50)(0.05)(0.95)}} < \frac{0.5 - (50)(0.05)}{\sqrt{(50)(0.05)(0.95)}}\right)$$

$$\approx P\left(Z < \frac{0.5 - (50)(0.05)}{\sqrt{(50)(0.05)(0.95)}}\right) = \Phi(1.6222) = 0.9476.$$

23. (a) Let R_n denote the number of times a red card appears in n picks and Z be a standard normal random variable. Then,

$P(249.5 < R_{100} < 300.5) = P\left(\frac{249.5-500(0.5)}{\sqrt{500(0.5)(0.5)}} < \frac{R_{100}-500(0.5)}{\sqrt{500(0.5)(0.5)}} < \frac{300.5-500(0.5)}{\sqrt{500(0.5)(0.5)}}\right) \approx$ $P\left(\frac{249.5-500(0.5)}{\sqrt{500(0.5)(0.5)}} < Z < \frac{300.5-500(0.5)}{\sqrt{500(0.5)(0.5)}}\right) = P(-0.0447 < Z < 4.5169) = \Phi(4.5169) -$ $\Phi(-0.0447) = 0.5178.$

(b) Let E_n denote the number of times an even card appears in n picks and Z be a standard normal random variable. Then,

$P(E_{100} > 200) = P\left(\frac{E_{100}-500\left(\frac{20}{52}\right)}{\sqrt{500\left(\frac{20}{52}\right)\left(\frac{32}{52}\right)}} > \frac{200-500\left(\frac{20}{52}\right)}{\sqrt{500\left(\frac{20}{52}\right)\left(\frac{32}{52}\right)}}\right) \approx P\left(Z > \frac{200-500\left(\frac{20}{52}\right)}{\sqrt{500\left(\frac{20}{52}\right)\left(\frac{32}{52}\right)}}\right) = P(Z >$ $0.7071) = 1 - \Phi(0.7071) = 0.2398.$

32.

$\frac{1}{\lambda} = 1.5 \Rightarrow \lambda = \frac{2}{3}$; therefore, $P(X > x) = 1 - F(x) = e^{-\frac{2}{3}x}$.

(a) $P(X > 2) = e^{-\frac{4}{3}} = 0.2636.$

(b) By memoryless property, $P(X > 2 \mid X > 1) = P(X > 1) = e^{-\frac{2}{3}} = 0.5134.$

38. (a) $F_Y(y) = P(Y \leq y) = P(log(X) \leq y) = P(X \leq e^y) = \frac{e^y-1}{4}$ for $log(1) \leq y \leq log(5)$. Therefore, the probability density function is given by the following:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{4}e^y, & log(1) \leq y \leq log(5) \\ 0, & otherwise \end{cases}$$

(b) $P\left(\frac{1}{3} < Y < \frac{2}{3}\right) = F_Y\left(\frac{2}{3}\right) - F_Y\left(\frac{1}{3}\right) = 0.1380.$

38. (a)

$$F_Y(y) = P(Y \le y)$$

$$= P(\log X \le y)$$

$$= P(e^{\log X} \le e^y)$$

$$= P(X \le e^y)$$

$$= F(e^y) = \frac{e^y - 1}{4}$$

$1 \le X \le 5$

$\Rightarrow \log 1 \le \log X \le \log 5$

$\Rightarrow \log 1 \le y \le \log 5$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= \begin{cases} \frac{1}{4}e^y, & \log 1 \le y \le \log 5 \\ 0, & \text{o.w} \end{cases}$$

(b)

$$P\left(\frac{1}{3} < Y < \frac{2}{3}\right)$$

$$= F_Y\left(\frac{2}{3}\right) - F_Y\left(\frac{1}{3}\right)$$

$$= \frac{e^{\frac{2}{3}} - 1}{4} - \frac{e^{\frac{1}{3}} - 1}{4}$$

$$= \frac{1}{4}\left(e^{\frac{2}{3}} - e^{\frac{1}{3}}\right)$$

#

40. $F_Y(y) = P(Y \le y) = P\left(e^{-\lambda X} \le y\right) = P\left(log\left(e^{-\lambda X}\right) \le log(y)\right) = P\left(X \ge -\frac{1}{\lambda} log(y)\right) = e^{-\lambda\left(-\frac{1}{\lambda} log\,(y)\right)} = y$ for $0 < y \le 1$.

Therefore, the probability density function is given by the following equation:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 1, & y \in (0,1] \\ 0, & otherwise, \end{cases}$$

, i.e., $Y$is a uniform random variable on (0,1].

4. $F_Y(y) = P(Y \le y)$

$= P(e^{-\lambda X} \le y)$

$= P(\log(e^{-\lambda X}) \le \log(y))$

$= P(-\lambda X \le \log(y))$

$= P(X \ge -\frac{1}{\lambda}\log(y))$

$X \ge 0$

$\Rightarrow -\lambda X \le 0$

$\Rightarrow e^{-\lambda X} \le e^0$

$\Rightarrow y \le 1$

$= e^{-\lambda\cdot(-\frac{1}{\lambda}\log(y))}$

$= e^{\log(y)} = y \Rightarrow 0 < y \le 1$

$\therefore f_Y(y) = \frac{dF_Y(y)}{dy}$

$= \begin{cases} 1, & 0 < y \le 1 \\ 0, & o.w. \end{cases}$ #

Self-Test Problems and Exercises

3.

5.3. First, let us find c by using

$$1 = \int_0^2 cx^4 dx = 32c/5 \Rightarrow c = 5/32$$

(a) $E[X] = \frac{5}{32}\int_0^2 x^5 dx = \frac{5}{32}\frac{64}{6} = 5/3$
(b) $E[X^2] = \frac{5}{32}\int_0^2 x^6 dx = \frac{5}{32}\frac{128}{7} = 20/7 \Rightarrow \text{Var}(X) = 20/7 - (5/3)^2 = 5/63$

3. 首先讓我們藉由下式求 c

$$1 = \int_0^2 cx^4 dx = 32c/5 \Rightarrow c = 5/32$$

(a) $E[X] = \frac{5}{32}\int_0^2 x^5 dx = \frac{5}{32}\frac{64}{6} = 5/3$

(b) $E[X^2] = \frac{5}{32}\int_0^2 x^6 dx = \frac{5}{32}\frac{128}{7} = 20/7 \Rightarrow \text{Var}(X) = 20/7 - (5/3)^2 = 5/63$

10.

5.10. Let X be the tire life in units of one thousand, and let $Z = (X - 34)/4$. Note that Z is a standard normal random variable.
(a) $P\{X > 40\} = P\{Z > 1.5\} \approx .0668$
(b) $P\{30 < X < 35\} = P\{-1 < Z < .25\} = P\{Z < .25\} - P\{Z > 1\} \approx .44$
(c) $P\{X > 40 | X > 30\} = P\{X > 40\}/P\{X > 30\}$
$= P\{Z > 1.5\}/P\{Z > -1\} \approx .079$

10. 令 X 為輪胎壽命以千里為單位，且令 $Z = (X - 34) / 4$。注意到 Z 為一個標準常態隨機變數。

(a) $P\{X > 40\} = P\{Z > 1.5\} \approx .0668$

(b) $P\{30 < X < 35\} = P\{-1 < Z < .25\} = P\{Z < .25\} - P\{Z > 1\} \approx .44$

(c) $P\{X > 40 \mid X > 30\} = P\{X > 40\} / P\{X > 30\}$
$= P\{Z > 1.5\} / P\{Z > -1\} \approx .079$

13.

5.13. The lack of memory property of the exponential gives the result $e^{-4/5}$.

13. 由指數分配的無記憶性可得 $e^{-4/5}$。