Chapter 4

Problems

17.

17.

(a) $P(X<1) = \lim_{n\to\infty} P\{X \le 1-\frac{1}{n}\}$

$= \lim_{n\to\infty} F(1-\frac{1}{n})$

$= \frac{1}{2}$ #

(b) $P(X>2) = 1 - P\{X \le 2\}$

$= 1 - F(2)$

$= 1 - \frac{2+1}{4} = \frac{1}{4}$ #

(c) $P\{\frac{1}{3} < X < \frac{5}{3}\}$

$= \lim_{n\to\infty} F(\frac{5}{3}-\frac{1}{n}) - F(\frac{1}{3})$

$= \frac{\frac{5}{3}+1}{4} - \frac{\frac{1}{3}}{2} = \frac{1}{2}$ #

19.

19.

$P(X=1) = P(X\le 1) - P(X<1)$

$= F(1) - \lim_{n\to\infty} F(1-\frac{1}{n})$

$= \frac{1}{4} - 0 = \frac{1}{4}$ #

$P(X=3) = P(X\le 3) - P(X<3)$

$= F(3) - \lim_{n\to\infty} F(3-\frac{1}{n})$

$= \frac{5}{8} - \frac{1}{4} = \frac{3}{8}$ #

$P(X=4) = P(X\le 4) - P(X<4)$

$= F(4) - \lim_{n\to\infty} F(4-\frac{1}{n})$

$= \frac{3}{4} - \frac{5}{8} = \frac{1}{8}$ #

$P(X=6) = P(X\le 6) - P(X<6)$

$= F(6) - \lim_{n\to\infty} F(6-\frac{1}{n})$

$= \frac{7}{8} - \frac{3}{4} = \frac{1}{8}$ #

$P(X=7) = P(X\le 7) - P(X<7)$

$= F(7) - \lim_{n\to\infty} F(7-\frac{1}{n}) = 1 - \frac{7}{8} = \frac{1}{8}$ #

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X = i\} = i/148$, $i = 40, 33, 25, 50$

$$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$$

$$E[Y] = (40 + 33 + 25 + 50)/4 = 37$$

31. $E[\text{score}] = p^*[1 - (1 - P)^2] + (1 - p^*)(1 - p^2)$

$$\frac{d}{dp} = 2(1 - p)p^* - 2p(1 - p^*)$$

$$= 0 \Rightarrow p = p^*$$

31.

$$E[\text{score}] = p^*(1-(1-p)^2) + (1-p^*)(1-p^2)$$

$$= p^*(1-(1-2p+p^2)) + (1-p^*)(1-p^2)$$

$$= p^*(2p-p^2) + (1-p^*)(1-p^2)$$

$$= 2pp^* - \cancel{p^2p^*} + 1 - p^* - p^2 + \cancel{p^2p^*}$$

$$\Rightarrow \frac{dE[\text{score}]}{dp} = 2p^* - 2p = 0$$

$$\Rightarrow p^* = p$$

#

32. If T is the number of tests needed for a group of 10 people, then

$$E[T] = (.9)^{10} + 11[1 - (.9)^{10}] = 11 - 10(.9)^{10}$$

35. Let X denote the amount you win.

(a) $P(X=2)=\binom{4}{1}\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)=\frac{4}{17}$

$$P(X=1)=\binom{2}{1}\left(\frac{26}{52}\right)\left(\frac{25}{51}\right)=\frac{25}{51}$$
$$P(X=-0.50)=1-\frac{4}{17}-\frac{25}{51}=\frac{14}{51}$$
$$E(X)=2\left(\frac{4}{17}\right)+1\left(\frac{25}{51}\right)-0.50\left(\frac{14}{51}\right)\approx 0.8235.$$

(b) $E(X^2)=2^2\left(\frac{4}{17}\right)+1^2\left(\frac{25}{51}\right)+(-0.50)^2\left(\frac{14}{51}\right)=1.5.$

Therefore, $Var(X)=1.5-0.8235^2\approx 0.8218.$

39. $E[(4X-1)^2]=Var(4X-1)+(E[4X-1])^2=16Var(X)+(4E(X)-1)^2=16+11^2=137.$

(i)$Var(5-2X)=4Var(X)=4.$

43. (a) Because each question will, independently, be answered correctly by both A and B with probability .28, the mean number is 2.8.

(b) Because each question will, independently, be answered correctly either by A or by B with probability $1-.18=.82$, the number of questions so answered is binomial with parameters n = 10, p = .82, yielding that its variance is $np(1-p)=1.476$.

45. $$\alpha\sum_{i=k}^{n}\binom{n}{i}p_1^i(1-p_1)^{n-i}+(1-\alpha)\sum_{i=k}^{n}\binom{n}{i}p_2^i(1-p_2)^{n-i}$$

50. (a) $\frac{1}{2}\binom{10}{7}.4^7.6^3+\frac{1}{2}\binom{10}{7}.7^7.3^3$

(b) $$\frac{\frac{1}{2}\binom{9}{6}(.4)^7(.6)^3+\frac{1}{2}\binom{9}{6}(.7)^7(.3)^3}{.55}$$

53.$P(Win, Win, Lose|7\ Wins) = \frac{P(Win,Win,Lose\ and\ 7\ Wins)}{P(7\ Wins)} =$
$\frac{P(7\ Wins|Win, Win, Lose)P(Win,Win,Lose)}{P(7\ Wins)} = \frac{P(5\ Wins\ in\ 7\ Rounds)P(Win,Win,Lose)}{P(7\ Wins)} =$
$\frac{\left(\binom{7}{5}p^5(1-p)^2\right)(p^2(1-p)^1)}{\binom{10}{3}p^7(1-p)^3} = \frac{7}{40}$

(a)$P(Lose, Win, Lose|7\ Wins) = \frac{P(Lose,Win,Lose\ and\ 7\ Wins)}{P(7\ Wins)} =$
$\frac{P(7\ Wins|Lose, Win, Lose)P(Lose,Win,Lose)}{P(7\ Wins)} = \frac{P(6\ Wins\ in\ 7\ Rounds)P(Lose,Win,Lose)}{P(7\ Wins)} =$
$\frac{\left(\binom{7}{6}p^6(1-p)^1\right)(p^1(1-p)^2)}{\binom{10}{3}p^7(1-p)^3} = \frac{7}{120}$

54. The probability of a fall is small; thus, the distribution of the number of falls can be approximated by the Poisson distribution.

(a) $P(no\ falls) = e^{-0.3} \approx 0.7408.$
(b) $P(\geq 3\ falls) = 1 - P(\leq 2\ falls) = 1 - \left(e^{-0.3} + 0.3e^{-0.3} + \frac{0.3^2 e^{-0.3}}{2!}\right) \approx 0.0036.$

60. (a) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - \sum_{i=0}^{3} \frac{(7)^i e^{-7}}{i!} \approx 0.9182.$

(b) $P(X \leq 5|X \geq 1) = \frac{P(1 \leq X \leq 5)}{P(X \geq 1)} = \frac{\sum_{i=1}^{5} \frac{(7)^i e^{-7}}{i!}}{1-e^{-7}} \approx 0.3001.$

66. (a) $e^{-4} \approx 0.0183$

(b) $1 - \left(e^{-4} + 4e^{-4} + \frac{4^2 e^{-4}}{2} + \frac{4^3 e^{-4}}{3!} + \frac{4^4 e^{-4}}{4!}\right) \approx 0.3712$

74. (a) $\left(\frac{26}{38}\right)^5$

(b) $\left(\frac{26}{38}\right)^3 \frac{12}{38}$

77. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3+\binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2+\binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5\frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

77.

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)$

$= \underbrace{\binom{5}{4}\cdot\left(\frac{2}{3}\right)^4\cdot\left(\frac{1}{3}\right)}_{\text{前5次}}\cdot\underset{\substack{\uparrow\\ \text{第6次}}}{\frac{2}{3}}$

(d) $\binom{6}{4}\cdot\left(\frac{2}{3}\right)^5\cdot\left(\frac{1}{3}\right)^2$

$= \underbrace{\binom{6}{4}\cdot\left(\frac{2}{3}\right)^4\cdot\left(\frac{1}{3}\right)^2}_{\text{前6次}}\cdot\underset{\substack{\uparrow\\ \text{第7次}}}{\frac{2}{3}}$

82. Let F be the number of students in the group who have failed the test.

(a) $P(F=0)=\frac{\binom{41}{10}}{\binom{50}{10}}\approx 0.1091$

(b) $P(F\geq 3)=1-P(F\leq 2)=1-\frac{\binom{41}{10}+\binom{41}{9}\binom{9}{1}+\binom{41}{8}\binom{9}{2}}{\binom{50}{10}}\approx 0.2491$

86. Let X_i be the number of traffic accidents occurring on road i ($i=1,2,3,4,5$). Then $E(X_1+X_2+X_3+X_4+X_5)=0.45+0.2+0.4+0.5+0.35=1.9$.

88. Let B_i be equal to 1 if the necklace contains at least one bead of color i; otherwise, it is 0. Then,

$$E\left(\sum_{i=1}^{k} B_i\right) = \sum_{i=1}^{k} E(B_i) = \sum_{i=1}^{k} P(B_i = 1) = \sum_{i=1}^{k}(1-(1-p_i)^n) = k - \sum_{i=1}^{k}(1-p_i)^n.$$

89. (a) $P(X=0) = P(X=0|R)P(R) + P(X=0|B)P(B) + P(X=0|G)P(G)$

$$= \frac{1}{3}\left(\frac{\binom{15}{4}}{\binom{25}{4}} + \frac{\binom{17}{4}}{\binom{25}{4}} + \frac{\binom{18}{4}}{\binom{25}{4}}\right)$$

(b) $P(X_i = 1) = P(X_i = 1|R)P(R) + P(X_i = 1|B)P(B) + P(X_i = 1|G)$

$$P(G) = \frac{1}{3}(10/25 + 8/25 + 7/25) = 1/3$$

which can also be seen by noting that no matter what the color of the i^{th} ball selected, there is probability 1/3 that was the chosen color.

(c) $E[X] = E[X_1 + X_2 + X_3 + X_4] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = 4/3$

89.

(c)

$$E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4]$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$$

$$= 1 \cdot \frac{1}{3} = \frac{1}{3}, \quad i = 1, 2, 3, 4$$

(b) 小題答案

Self-Test Problems and Exercises

2.

2. 關係式保證 $p_i = c^i p_0,\ i = 1, 2$，其中 $p_i = P\{X = i\}$。因為這些機率總和為 1，所以

$$p_0(1+c+c^2)=1 \Rightarrow p_0 = \frac{1}{1+c+c^2}$$

因此，

$$E[X] = p_1 + 2p_2 = \frac{c+2c^2}{1+c+c^2}$$

3.

4.3. Let X be the number of flips. Then the probability mass function of X is

$$p_2 = p^2 + (1-p)^2, \quad p_3 = 1 - p_2 = 2p(1-p)$$

Hence,

$$E[X] = 2p_2 + 3p_3 = 2p_2 + 3(1-p_2) = 3 - p^2 - (1-p)^2$$

5.

5. 令 $p = P\{X = 1\}$。則 $E[X] = p$ 且 $\mathrm{Var}(X) = p(1-p)$，所以

$$p = 3p(1-p)$$

保證 $p = 2/3$。因此，$P\{X = 0\} = 1/3$。

9.

9. 因為 $E[X] = np$, $\mathrm{Var}(X) = np(1-p)$，且已知 $np = 6$, $np(1-p) = 2.4$。所以 $1 - p = 0.4$ 或 $p = 0.6$, $n = 10$。因此，

$$P\{X = 5\} = \binom{10}{5}(.6)^5(.4)^5$$

22.

22. 令 X 表你玩的局數且 Y 為你輸掉的局數。

(a) 在你玩了 4 局之後，你將持續玩下去直到輸為止。因此，$X-4$ 為一個幾何隨機變數具參數 $1-p$，所以

$$E[X]=E[4+(X-4)]=4+E[X-4]=4+\frac{1}{1-p}$$

(b) 若我們令 Z 表你在前 4 局輸掉的局數，則 Z 是一個二項隨機變數具參數 4 和 $1-p$。因為 $Y=Z+1$，我們有

$$E[Y]=E[Z+1]=E[Z]+1=4(1-p)+1$$