Chapter 3

Problems

12.

$$P(A < C \mid A < B) = \frac{P(A < C, A < B)}{P(A < B)}$$
$$= \frac{1/3}{1/2} = 2/3$$

13. (a) (.9)(.8)(.7) = .504

(b) Let F_i denote the event that she failed the ith exam.

$$P(F_2 \mid (F_1^c F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1-.504} = \frac{(.9)(.2)}{.496} = .3629$$

14. $P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13}$, $P(E_2 \mid E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13}$

$P(E_3 \mid E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}$, $P(E_4 \mid E_1E_2E_3) = 1$.

Hence,

$$p = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} \cdot \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} \cdot \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}$$

17. With S being survival and C being C section of a randomly chosen delivery, we have that

$$.98 = P(S) = P(S \mid C).15 + P(S \mid C^2)\ .85$$
$$= .96(.15) + P(S \mid C^2)\ .85$$

Hence

$P(S \mid C^c) \approx .9835.$

23. a. $\dfrac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6}=\dfrac{5}{9}$

b. $\dfrac{1}{3!}=\dfrac{1}{6}$

c. $\dfrac{5}{9}\dfrac{1}{6}=\dfrac{5}{54}$

26. (a) $P\{g-g \mid \text{at least one } g\} = \dfrac{1/4}{3/4}$ = 1/3.

(b) Since we have no information about the ball in the urn, the answer is 1/2.

33. Let C be the event that the tumor is cancerous, and let N be the event that the doctor does not call. Then

$$\beta = P(C \mid N) = \frac{P(NC)}{P(N)}$$

$$= \frac{P(N \mid C)P(C)}{P(N \mid C)P(C) + P(N \mid C^c)P(C^c)}$$

$$= \frac{\alpha}{\alpha + \frac{1}{2}(1-\alpha)}$$

$$= \frac{2\alpha}{1+\alpha} \geq \alpha$$

with strict inequality unless $\alpha = 1$.

35. (a) Let G denote the event “Gloria used the key last,” D denote “Dominic used the key last,” and M denote “key was misplaced”. Then,

$$P(M) = P(M|G)P(G) + P(M|D)P(D) = 0.3 \times \frac{2}{3} + 0.45 \times \frac{1}{3} = .35$$

Thus, the probability of the key being put in its proper place is .65.

(b) $P(G|M) = \dfrac{P(M|G)P(G)}{P(M)} = \dfrac{0.3\times\frac{2}{3}}{0.35} = .5714$

40.

40.

$$P\{\text{tails} \mid w\} = \frac{\frac{1}{2} \times \frac{3}{15}}{\frac{1}{2} \times \frac{5}{12} + \frac{1}{2} \times \frac{3}{15}} = \frac{12}{25 + 12} = \frac{12}{37}$$

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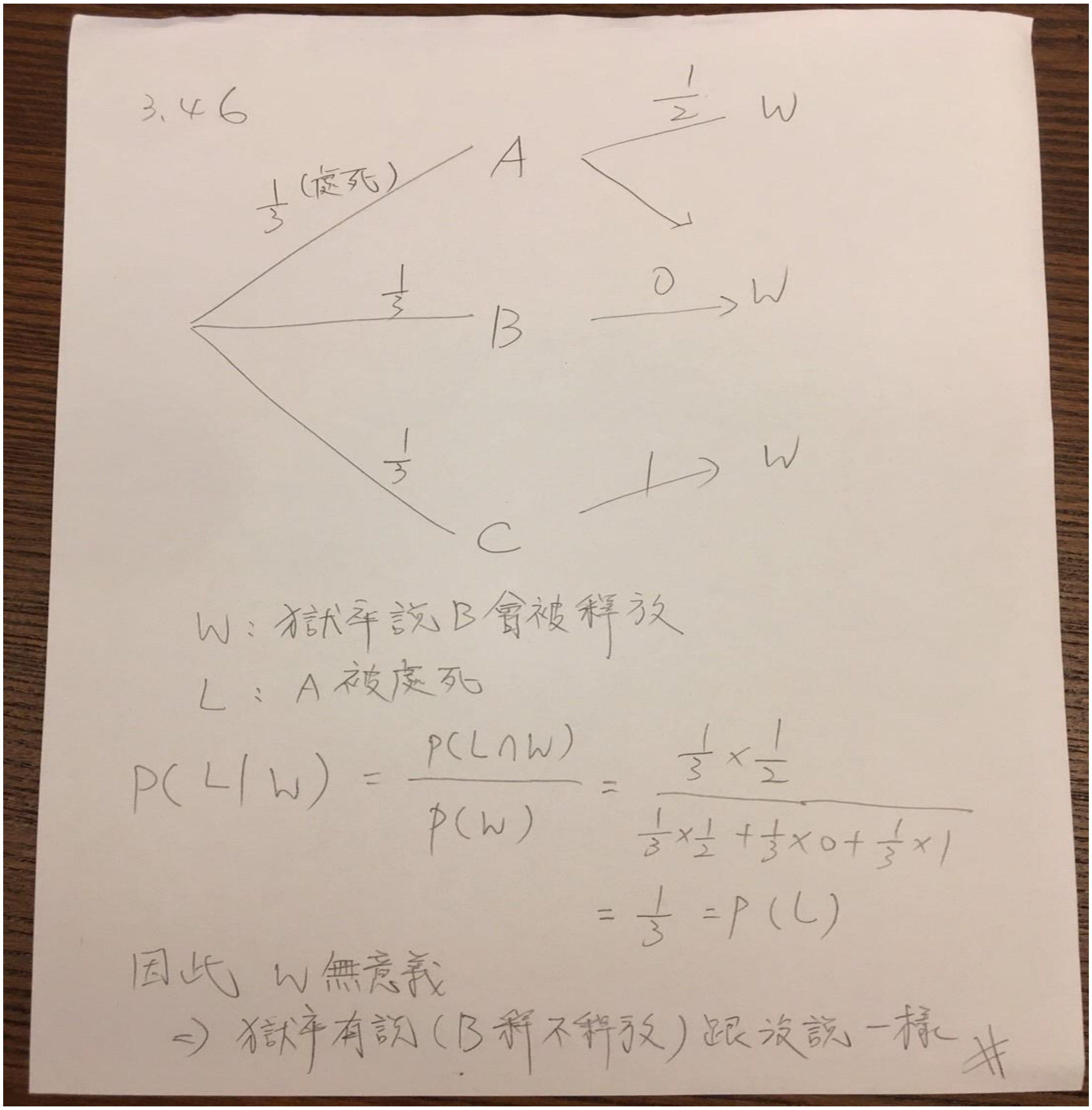
45. $P\{2 \text{ headed} \mid \text{heads}\} = \dfrac{\frac{1}{3}(1)}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{3}{4}} = \dfrac{4}{4+2+3} = \dfrac{4}{9}$.

46.



W：獄卒說B會被釋放

L：A被處死

$$P(L|W) = \frac{P(L\cap W)}{P(W)} = \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 0+\frac{1}{3}\times 1}$$

$$= \frac{1}{3} = P(L)$$

因此 W無意義

⇒ 獄卒有說(B釋不釋放)跟沒說一樣 #

47. Letting A and B, respectively, be the events that the computer can be fixed by A and that it can be fixed by B, then

$$\begin{aligned} P(A\cup B) &= P(A\cup A^c B) \\ &= P(A)+P(A^c B) \\ &= P(A)+P(A^c)P(B|A^c) = .3+.7(.4) = .58. \end{aligned}$$

Also,

$$P(A^c B|A\cup B) = \frac{P(A^c B)}{P(A\cup B)} = 28/58.$$

49. $P\{\text{all white}\} = \frac{1}{6}\left[\frac{5}{15} + \frac{5}{15}\frac{4}{14} + \frac{5}{15}\frac{4}{14}\frac{3}{13} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right]$

$P\{3 \mid \text{all white}\} = \dfrac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P\{\text{all white}\}}$

54. Let L_i be the event that A is in position i, i = 1, 2, 3, 4, and let A be the event that A wins the tournament. Then

$$P(A) = \frac{1}{4}(P(A|L_1) + P(A|L_2) + P(A|L_3) + P(A|L_4)) = \frac{1}{4}(p^3 + p^3 + p^2 + p).$$

57. (a) *Let T* be the event that one of the teams wins the first 3 games. Then, with A being the event that team A wins the first 3 games, we have

$$P(A|T) = \frac{P(AT)}{P(T)} = \frac{P(A)}{P(T)} = \frac{p^3}{p^3 + (1-p)^3}$$

(b) Letting W be the event that a team wins the first 3 games and also wins the series, we have

$$P(W|T) = \frac{P(WT)}{P(T)}$$
$$= \frac{P(W)}{P(T)}$$
$$= \frac{p^3(1-(1-p)^4) + (1-p)^3(1-p^4)}{p^3 + (1-p)^3}$$

where the preceding used that W occurs if either A wins the first 3 games and then does not lose 4 games in a row or if player B wins the first 3 games and then does not lose 4 games in a row.

60. $P\{\text{Boy}, F\} = \dfrac{4}{16+x}$ $\quad P\{\text{Boy}) = \dfrac{10}{16+x}$ $\quad P\{F\} = \dfrac{10}{16+x}$

so independence $\Rightarrow 4 = \dfrac{10 \cdot 10}{16+x} \Rightarrow 4x = 36$ or $x = 9$.

A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

62. (a) $2p(1 - p)$

(b) $\binom{3}{2} p^2 (1 - p)$

(c) $P\{\text{up on first} \mid \text{up 1 after 3}\}$

$= P\{\text{up first, up 1 after 3}\}/[3p^2(1 - p)]$

$= p2p(1 - p)/[3p^2(1 - p)] = 2/3.$

67. (a) $P\{\text{both hit} \mid \text{at least one hit}\} = \dfrac{P\{\text{both hit}\}}{P\{\text{at least one hit}\}}$

$= p_1p_2/(1 - q_1q_2)$

(b) $P\{\text{Barb hit} \mid \text{at least one hit}\} = p_1/(1 - q_1q_2)$

$Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

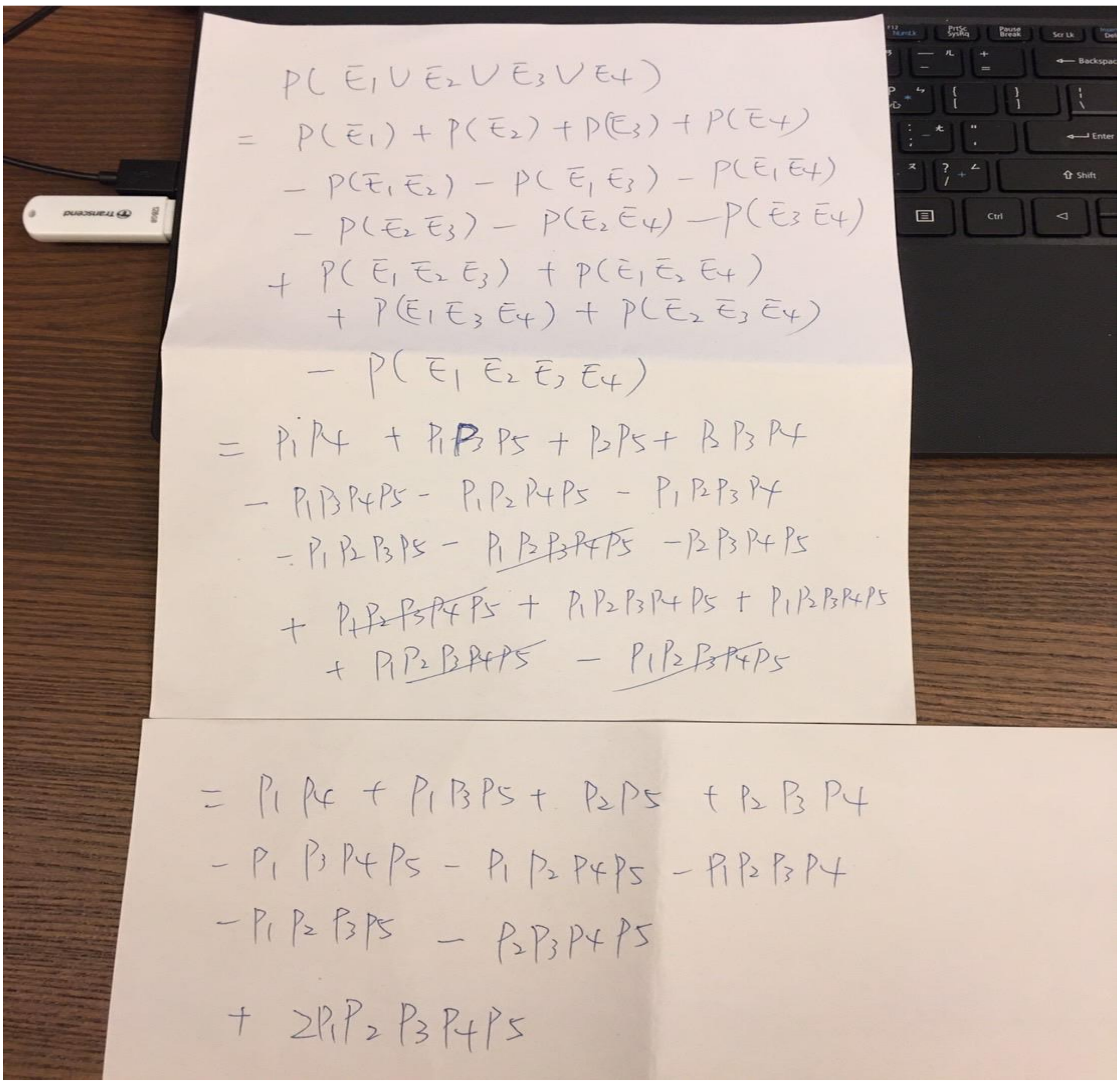
70. (a) $[1-(1-P_1P_2)(1-P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1, 3, 5 \text{ all close}\}$

$E_3 = \{2, 5 \text{ close}\}$, $E_4 = \{2, 3, 4 \text{ close}\}$. The desired probability is

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) = &P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) \\ &- P(E_2E_3) - P(E_2E_4) + P(E_3E_4) + P(E_1E_2E_3) + P(E_1E_2E_4) \\ &+ P(E_1E_3E_4) + P(E_2E_3E_4) + P(E_1E_2E_3E_4) \end{aligned}$$

$$\begin{aligned} = &P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 \\ &- P_1P_2P_3P_5 - P_2P_3P_4P_5 + 2P_1P_2P_3P_4P_5 + 3P_1P_2P_3P_4P_5. \end{aligned}$$



71. (a) $P_1P_2Q_3Q_4 + P_1P_3Q_2Q_4 + P_1P_4Q_2Q_3 + P_2P_3Q_1Q_4 + P_2P_4Q_1Q_3 + P_3P_4Q_1Q_2 + P_1P_2P_3Q_4$

$+ P_1P_2P_4Q_3 + P_1P_3P_4Q_2 + P_2P_3P_4Q_1 + P_1P_2P_3P_4$, where $Q_i = 1 - P_i$.

(c) $\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

77. Let P_A be the probability that A wins when A rolls first, and let P_B be the probability that B wins when B rolls first. Using that the sum of the dice is 9 with probability 1/9, we obtain upon conditioning on whether A rolls a 9 that

$$P_A = \frac{1}{9} + \frac{8}{9}(1 - P_B)$$

Similarly,

$$P_B = \frac{5}{36} + \frac{31}{36}(1 - P_A)$$

Solving these equations gives that $P_A = 9/19$ (and that $P_B = 45/76$.)

81. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1 - p)[p^2 + (1 - p)^2]$.

(b) Let A be the event that A wins. Conditioning on the outcome of the first two games gives

$$P(A) = P(A \mid a, a)p^2 + P(A \mid a, b)p(1 - p) + P(A \mid b, a)(1 - p)p + P(A \mid b, b)(1 - p)^2$$

$$= p^2 + P(A)2p(1 - p)$$

where the notation a, b means, for instance, that A wins the first and B wins the second game. The final equation used that $P(A \mid a, b) = P(A \mid b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1 - 2p(1 - p)}$$

Self-Test Problems and Exercises

9.

9. 令 A 表植物仍活著之事件令 W 為植物被澆水之事件。

(a)

$$P(A) = P(A \mid W)P(W) + P(A \mid W^c)P(W^c)$$
$$= (.85)(.9) + (.2)(.1) = .785$$

(b)

$$P(W^c \mid A^c) = \frac{P(A^c \mid W^c)P(W^c)}{P(A^c)}$$
$$= \frac{(.8)(.1)}{.215} = \frac{16}{43}$$

12.

12. 令 L_i 為 Maria 喜歡第 i 本書之事件，$i = 1, 2$。則

$$P(L_2 \mid L_1^c) = \frac{P(L_1^c L_2)}{P(L_1^c)} = \frac{P(L_1^c L_2)}{4}$$

利用 L_2 為互斥事件 L_1 和 L_2 和 $L_1^c L_2$ 之聯集，我們可得

$$.5 = P(L_2) = P(L_1 L_2) + P(L_1^c L_2) = .4 + P(L_1^c L_2)$$

所以

$$P(L_2 \mid L_1^c) = \frac{.1}{.4} = .25$$

25.

25. 令 D_i, $i = 1, 2$，表收音機 i 為不良品之事件。又令 A 和 B 分別表收音機是由工廠 A 和工廠 B 製造之事件。則

$$P(D_2 \mid D_1) = \frac{P(D_1 D_2)}{P(D_1)}$$
$$= \frac{P(D_1 D_2 \mid A)P(A) + P(D_1 D_2 \mid B)[(B)}{P(D_1 \mid A)P(A) + P(D_1 \mid B)P(B)}$$
$$= \frac{(.05)^2(1/2) + (.01)^2(1/2)}{(.05)(1/2) + (.01)(1/2)}$$
$$= 13/300$$

34.

3.34. Let W_1 be the event that player 1 wins the contest. Letting O be the event that player 1 does not play in round 1, we obtain by conditioning on whether or not O occurs, that

$$P(W_1) = P(W_1|O)P(O) + P(W_1|O^c)P(O^c)$$
$$= P(W_1|O)\frac{1}{3} + \frac{1}{3}\frac{1}{4}\frac{2}{3}$$

where the preceding used that if O^c occurs then 1 would have to beat both 2 and 3 to win the tournament. To compute $P(W_1|O)$, condition on which of 2 or 3 wins the first game. Letting B_i be the event that i wins the first game

$$P(W_1|O) = P(W_1|0, B_2)P(B_2|O) + P(W_1|0, B_3)P(B_3|O)$$
$$= \frac{1}{3}\frac{2}{5} + \frac{1}{4}\frac{3}{5} = 17/60$$

Hence, $P(W_1) = 3/20$. Also,

$$P(O|W_1) = \frac{P(W_1|O)P(O)}{P(W_1)} = \frac{(17/60)(1/3)}{3/20} = 17/27$$

36.

3.36. Let B_3 be the probability that 3 beats 4. Because 1 beats 2 with probability 1/3,

$$P(1) = P(1|B_3)P(B_3) + P(1|B_3^c)P(B_3^c) = (1/3)(1/4)(3/7)$$
$$+ (1/3)(1/5)(4/7) = 31/420$$

37.

3.37. (a) Condition on who wins the first game to obtain:

$$P(W_3) = P(W_3|1\text{ wins})(1/3) + P(W_3|2\text{ wins})(2/3)$$
$$= (1/3)(3/4)\prod_{i=4}^{n}\frac{3}{i+3} + (2/3)(3/5)\prod_{i=4}^{n}\frac{3}{i+3}$$
$$= \frac{13}{20}\prod_{i=4}^{n}\frac{3}{i+3}$$

(b) Condition on the opponent of player 4. If O_i is the event that i is the opponent, $i = 1, 2, 3$, then

$$P(O_1) = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$$
$$P(O_2) = \frac{2}{3}\frac{2}{5} = \frac{4}{15}$$
$$P(O_3) = 1 - \frac{1}{12} - \frac{4}{15} = \frac{13}{20}$$

Hence,

$$P(W_4) = \sum_{i=1}^{3} P(W_4|O_i)P(O_i) = \frac{4}{5}\frac{1}{12} + \frac{4}{6}\frac{4}{15} + \frac{4}{7}\frac{13}{20} = \frac{194}{315}$$