Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

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Supervised by Dr Giovanna Varricchio

1st August 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

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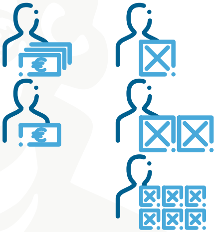

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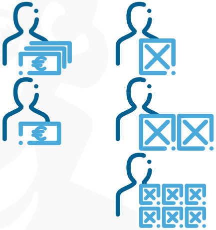

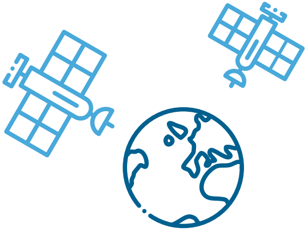

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- procurement
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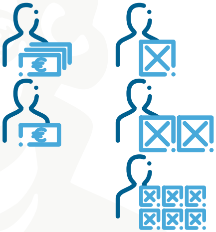

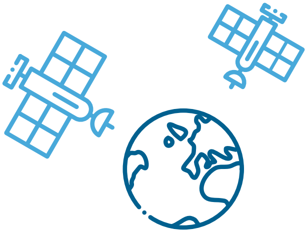

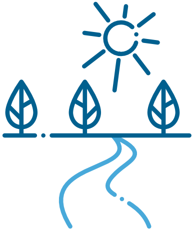

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Allocations

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But how to measure its efficiency and fairness?

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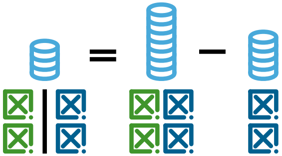

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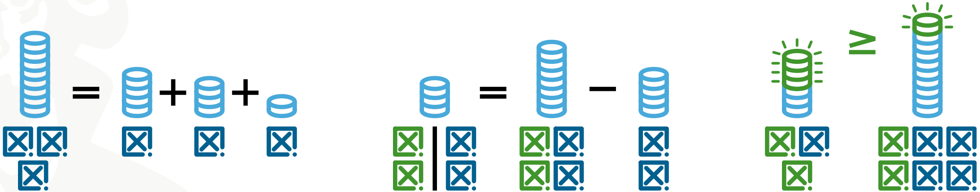
  - diminishing returns



Asymmetric Maximum Nash Social Welfare Problem

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$$\boldsymbol{x}^* \stackrel{!}{=} \underset{\boldsymbol{x} \in \boldsymbol{X}_{\mathscr{A}}(\mathscr{G})}{\arg\max} \{\mathrm{NSW}(\boldsymbol{x})\}$$

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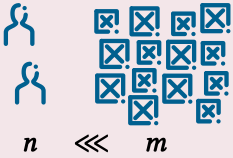

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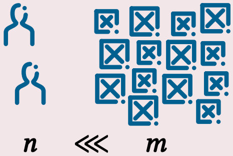

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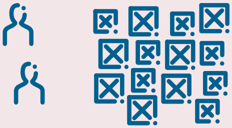

$n \lll m$

2 RepReMatch

Naïve Approach

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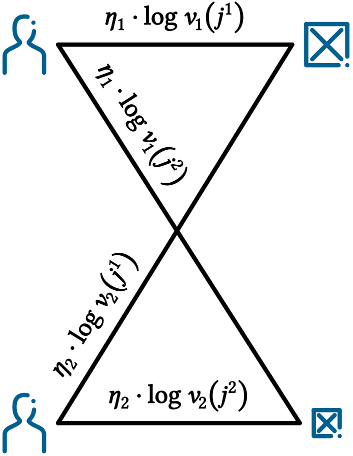

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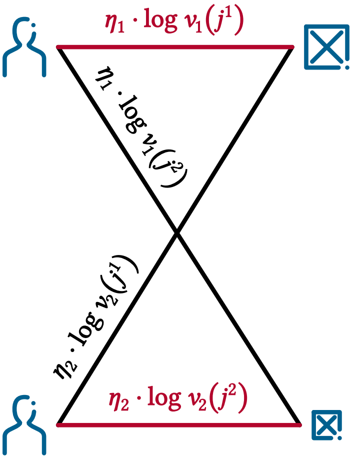

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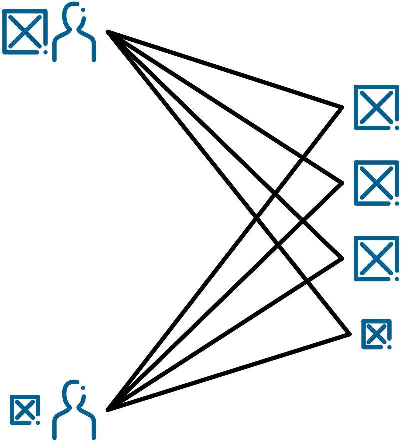
Greedy algorithm:

- repeatedly use maximum weight matchings



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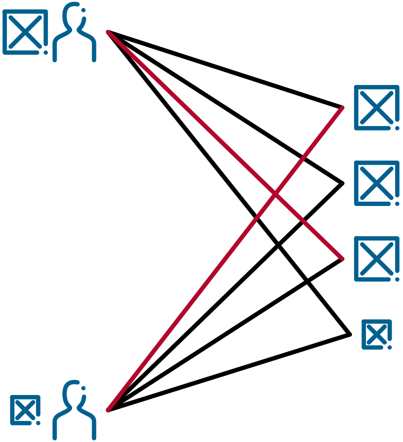

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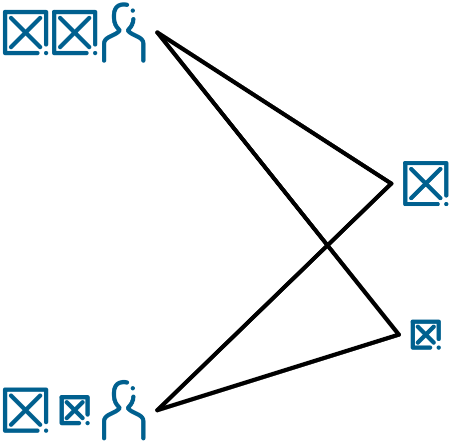

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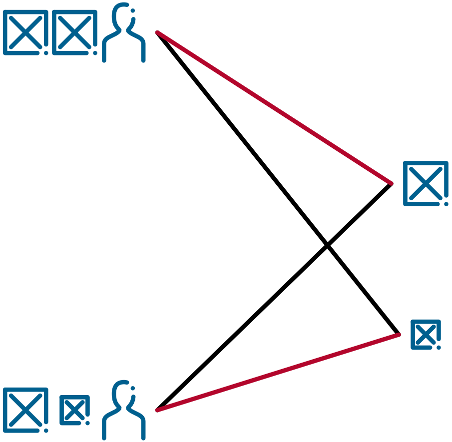

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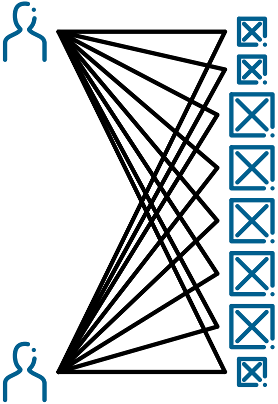
We need change the past in three phases:



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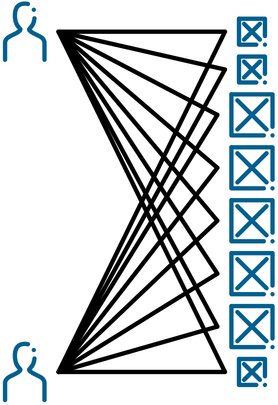
Phase 1 Assign enough high-value items **temporarily**.



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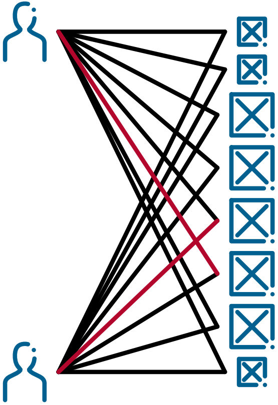

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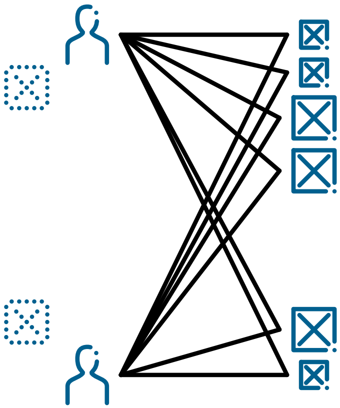

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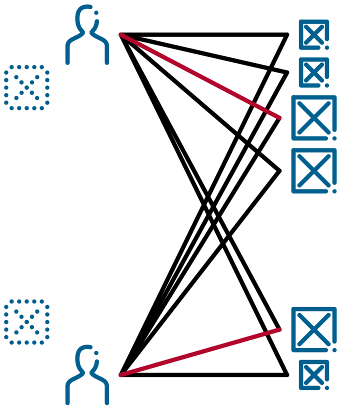

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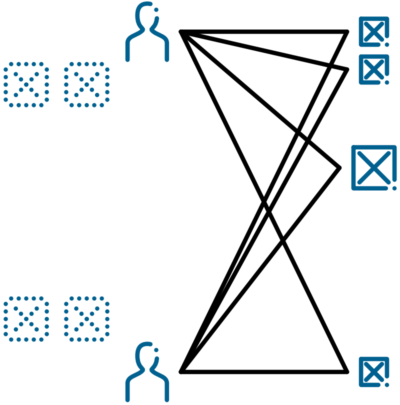

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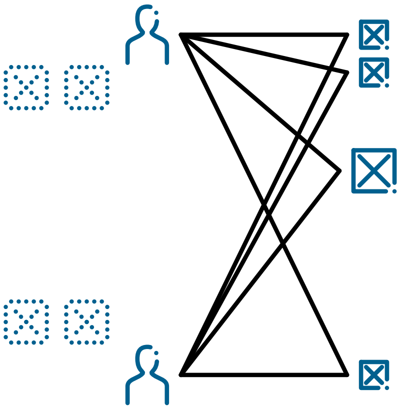

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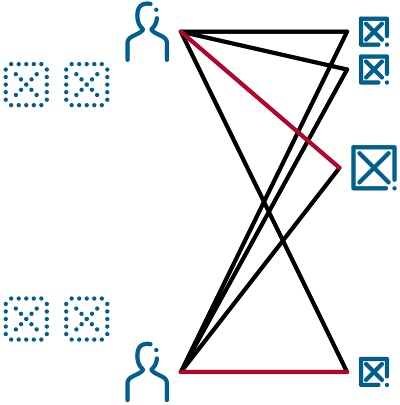

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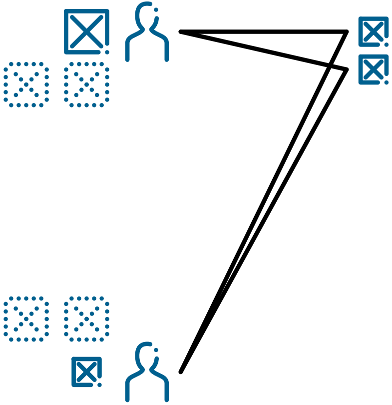

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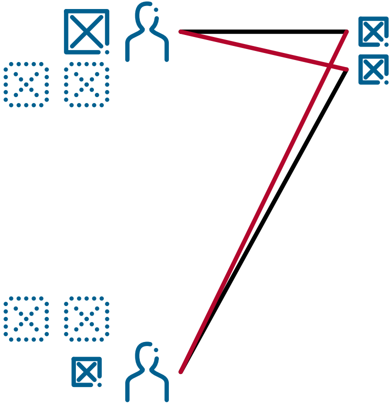

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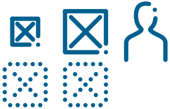

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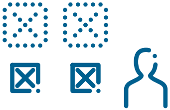

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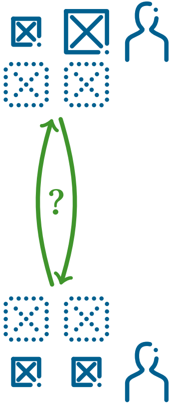

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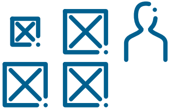

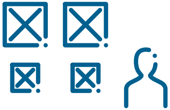

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Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$-approximation under submodular valuations.

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Phase I:

1. repeat $\lceil \log_2 n \rceil + 1$ times
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\eta_i \cdot \log v_i(j)$
   2. compute maximum weight matching
   3. update bundles $\boldsymbol{x}_i^{\mathrm{I}}$ & remove assigned items

Phase II:

2. repeat until $\mathscr{G} = \emptyset$
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\eta_i \cdot \log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})$
   2. compute maximum weight matching
   3. update bundles $\boldsymbol{x}_i^{\mathrm{II}}$ & remove assigned items

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Phase III:

The Algorithm

Phase I:

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   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\eta_i \cdot \log v_i(j)$
   2. compute maximum weight matching
   3. update bundles $\boldsymbol{x}_i^{\mathrm{I}}$ & remove assigned items

Phase II:

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Phase III:

3. create bipartite graph $\left(\mathscr{A}, \bigcup_{i \in \mathscr{A}} \boldsymbol{x}_i^{\mathrm{I}}, E\right)$ with edge weights $\eta_i \cdot \log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})$

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Analysing Phases I & III (1/2)

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Phase I reserves ‘high-value’ items.

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$\Rightarrow$ Are enough outstanding items reserved?

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Each agent can be matched with an outstanding item in phase III.

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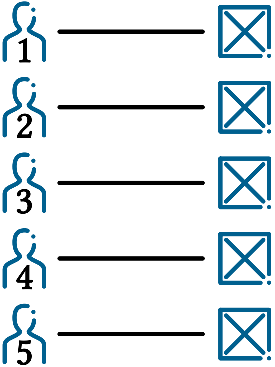

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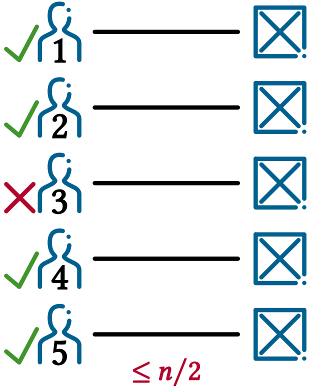

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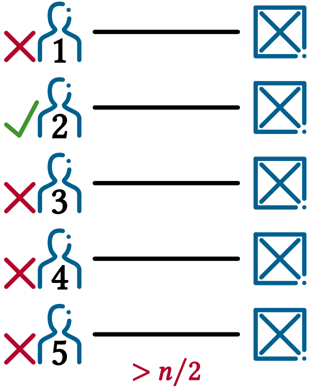

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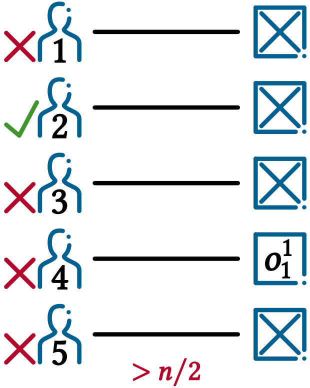

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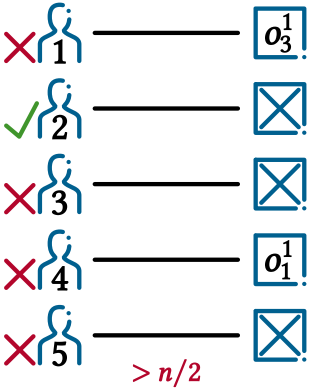

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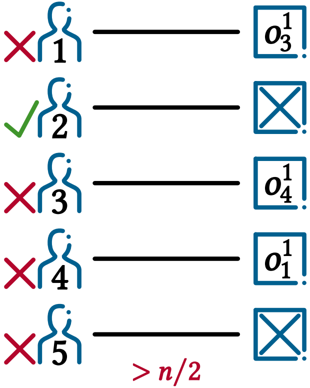

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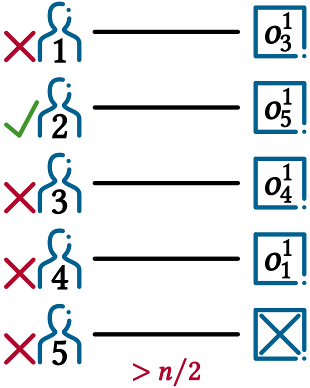

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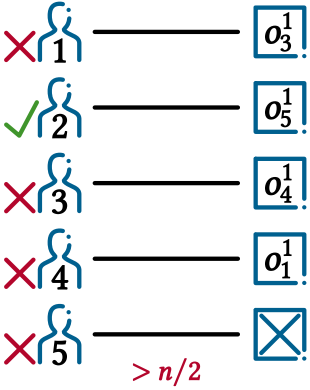

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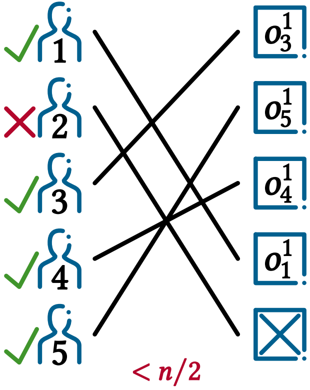

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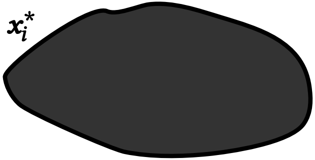

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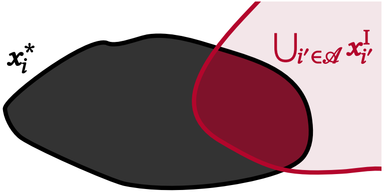

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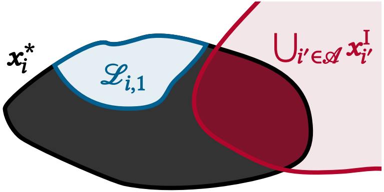

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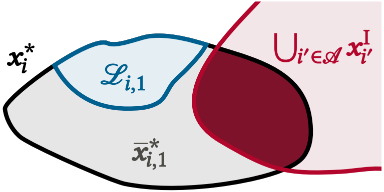

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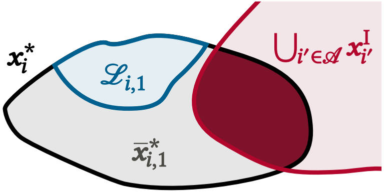

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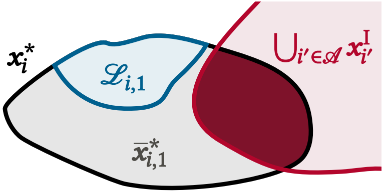

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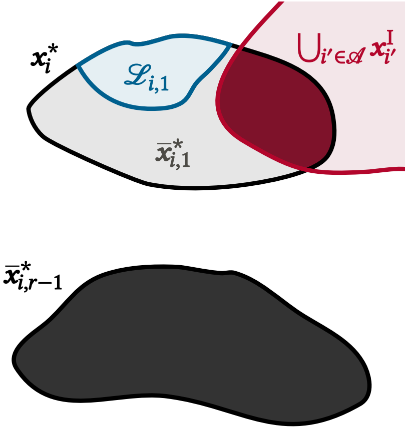

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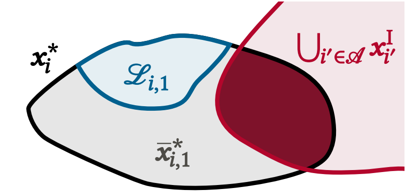

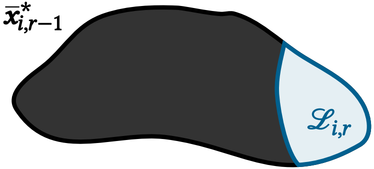

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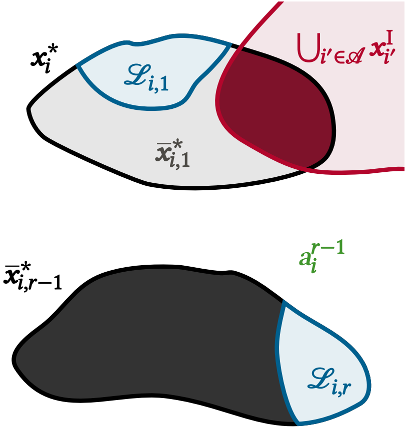

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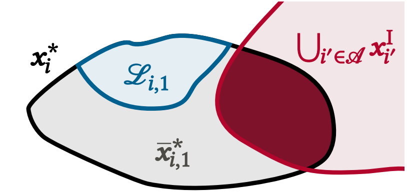

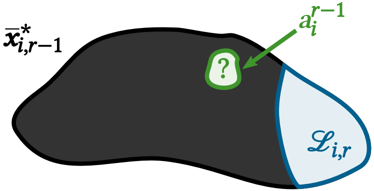

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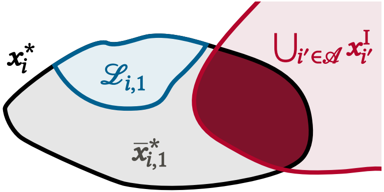

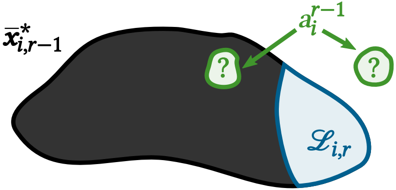

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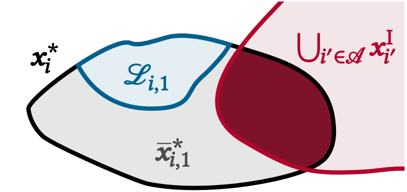
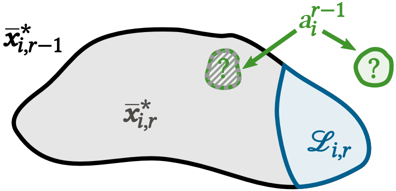
The set $\mathscr{L}_{i,r}$ of ***lost items*** is the set of all optimal items $j \in \boldsymbol{x}_i^*$ matched with other agents $i' \neq i$ in round r.

Definition

Let $\boldsymbol{x}_i^{\mathrm{II}} = \{a_i^1, a_i^2, \ldots\}$ be the bundle of agent i. The set of ***optimal and attainable items*** is defined as

$$\overline{\boldsymbol{x}}_{i,r}^* := \begin{cases} \boldsymbol{x}_i^* \setminus \left(\bigcup_{i' \in \mathscr{A}} \boldsymbol{x}_{i'}^{\mathrm{I}} \cup \mathscr{L}_{i,1}\right) & \text{in round } r = 1, \\ & \text{in round } r \geq 2. \end{cases}$$





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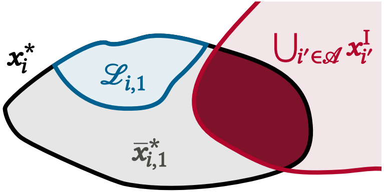

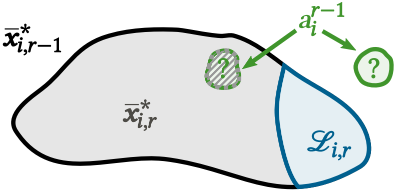

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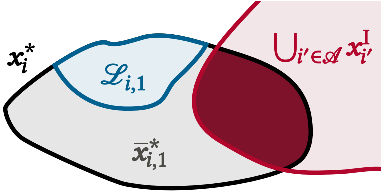
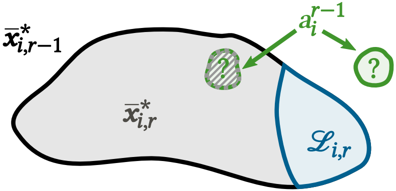
$\Rightarrow$ What is the valuation of the remaining items?





Analysing Phase II (2/2)

Lemma

$$v_i\left(\bar{x}^*_{i,r} \mid a_i^1, \ldots, a_i^{r-1}\right) \qquad \text{if } r \geq 2$$

Analysing Phase II (2/2)

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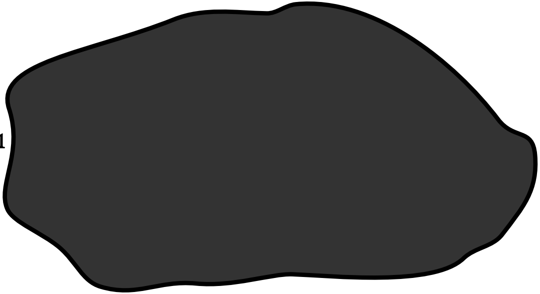
$\overline{\boldsymbol{x}}^*_{i,1}$



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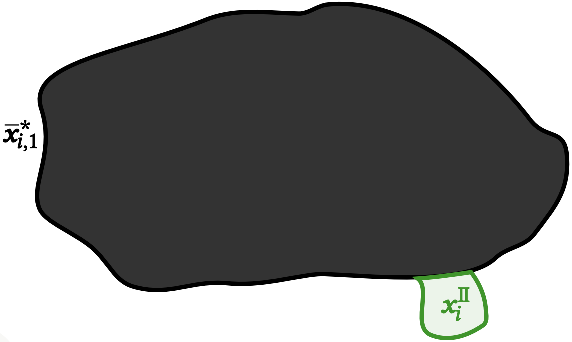

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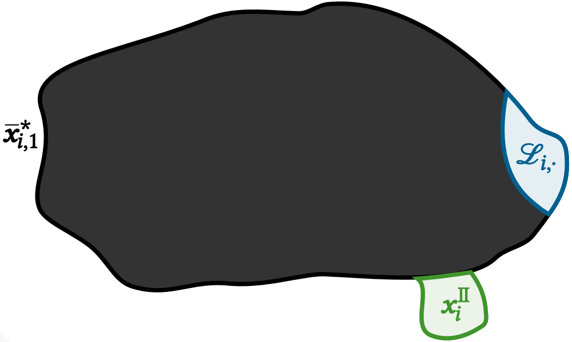

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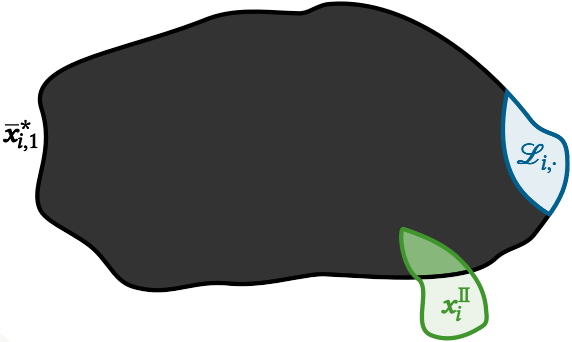

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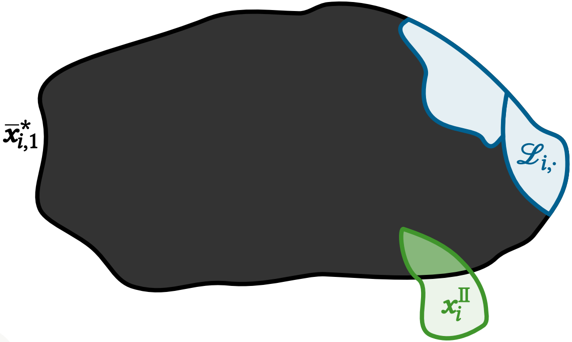

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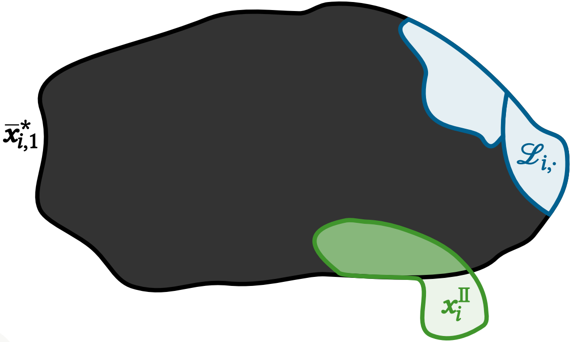

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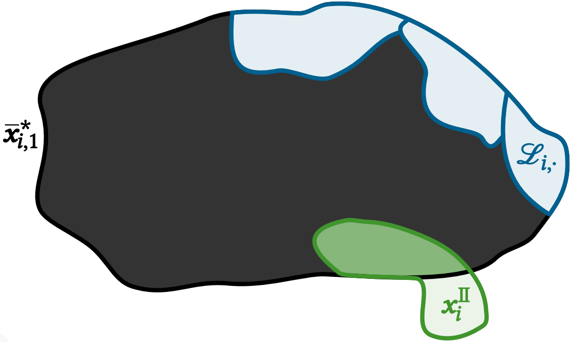

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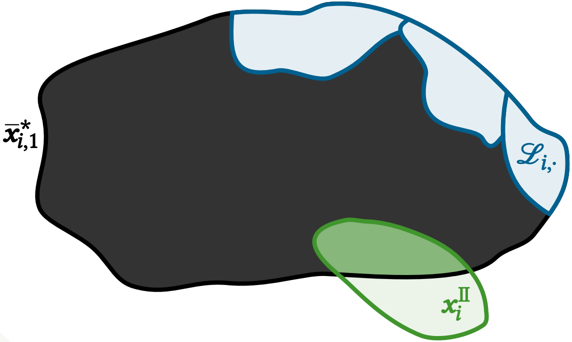

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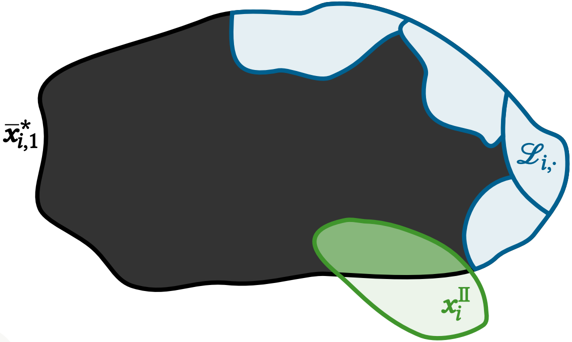

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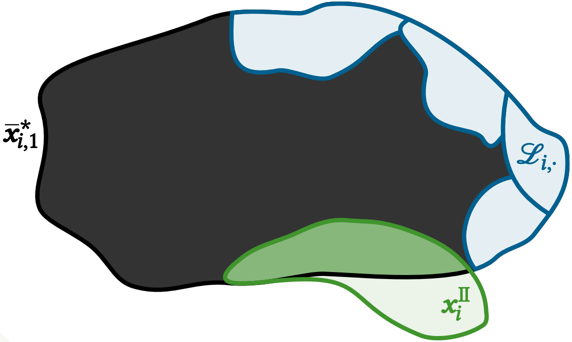

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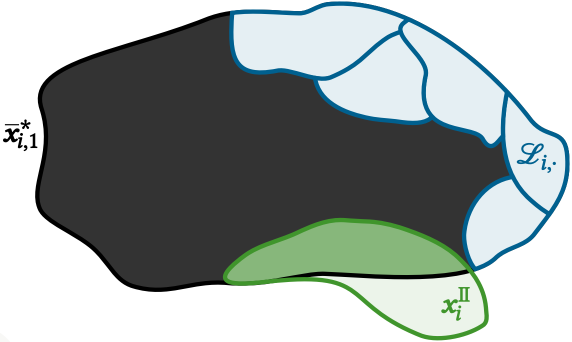

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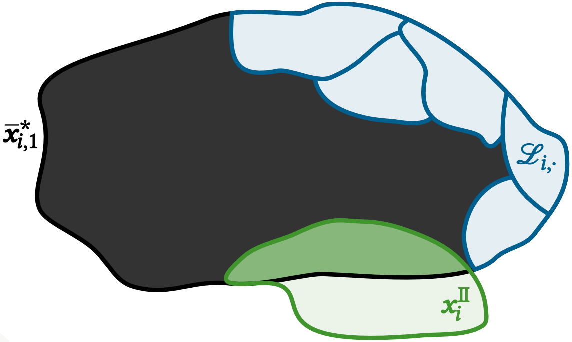

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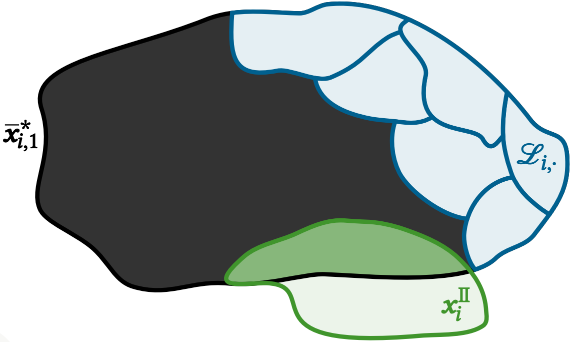

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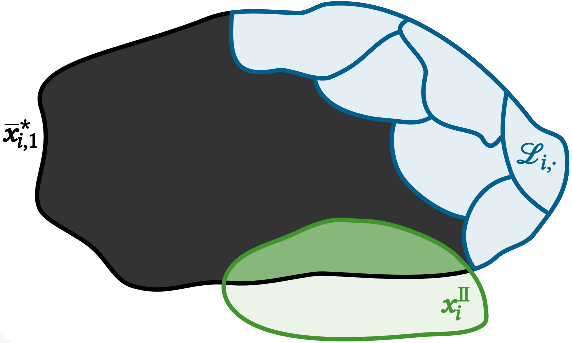

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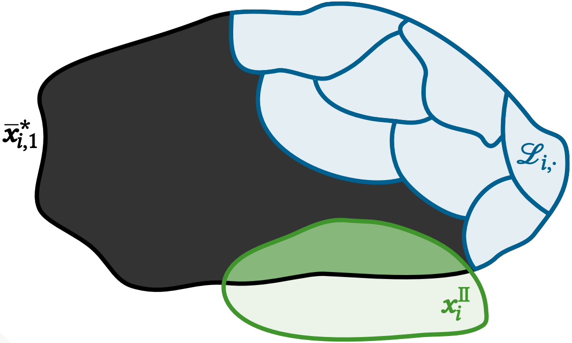

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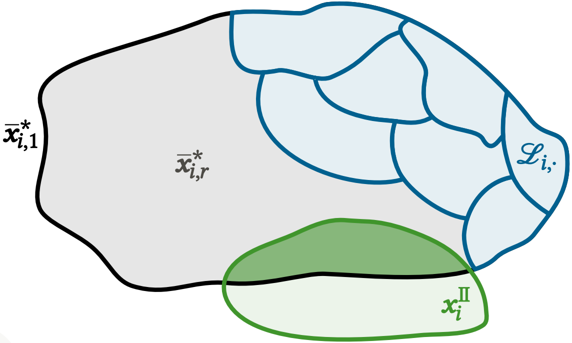

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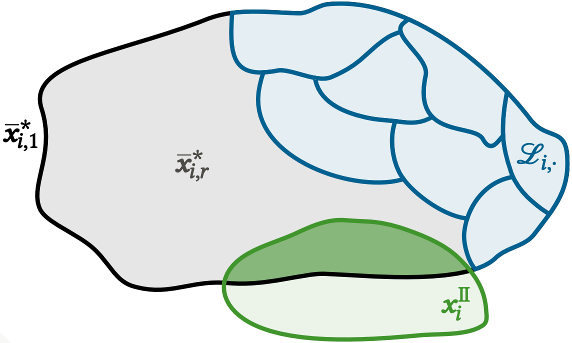

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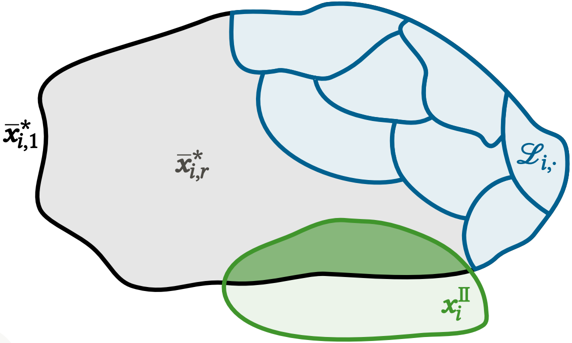

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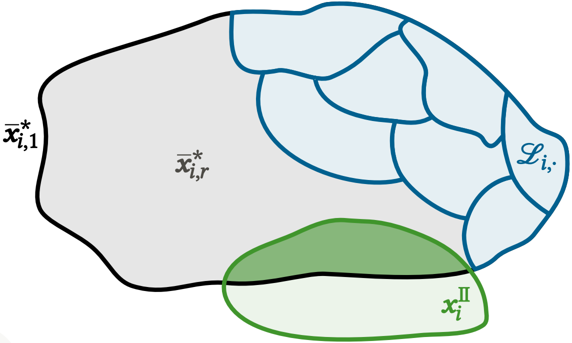

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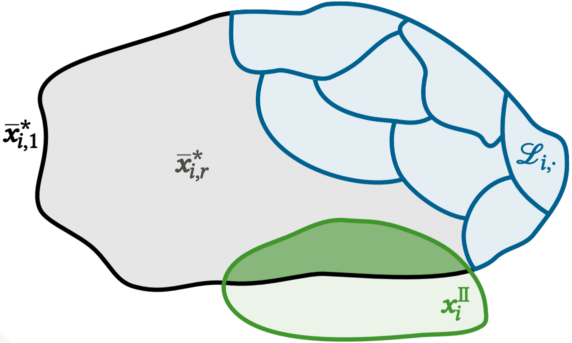

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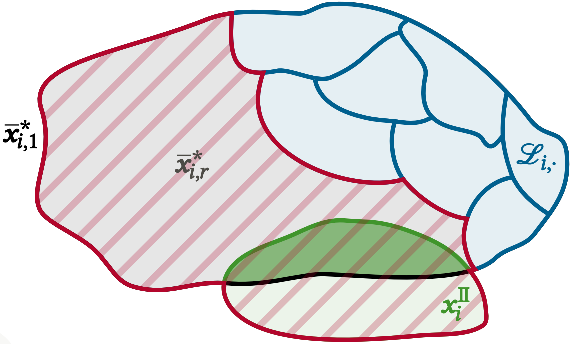

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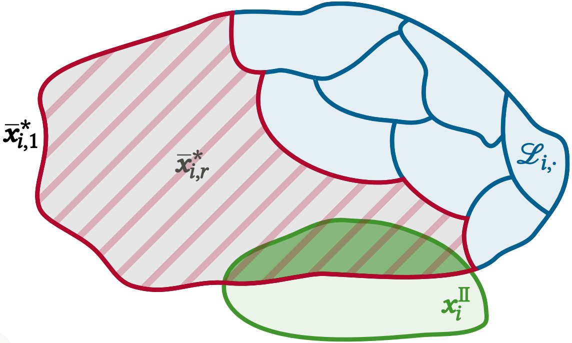

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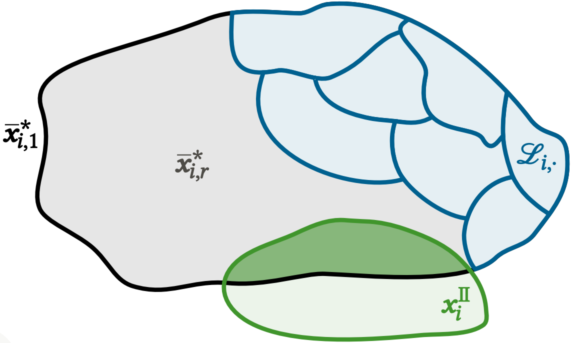

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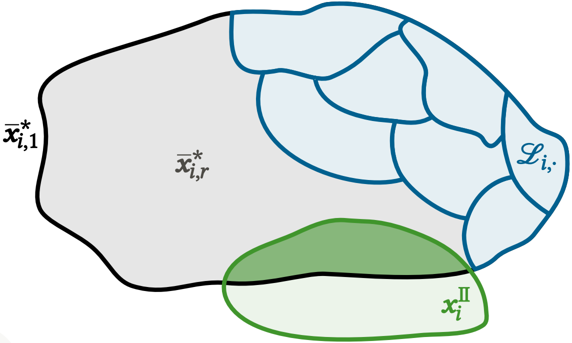

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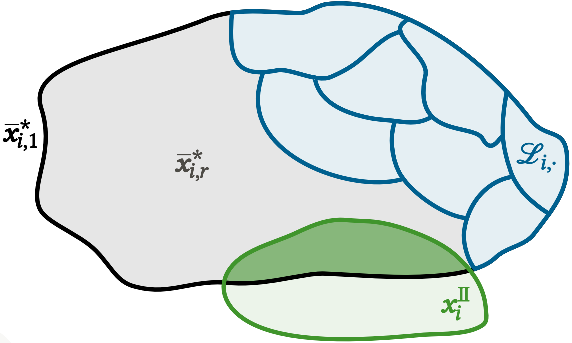

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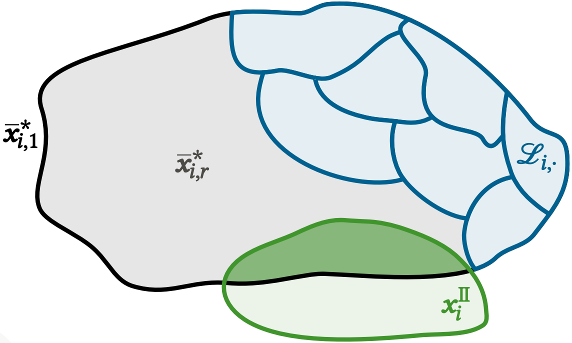

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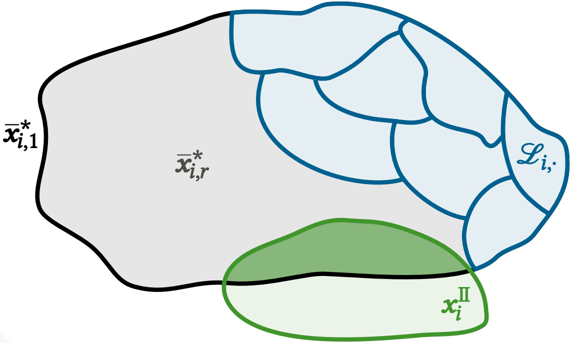

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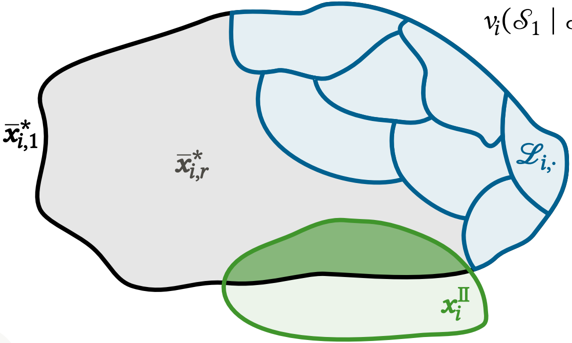
$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) =$



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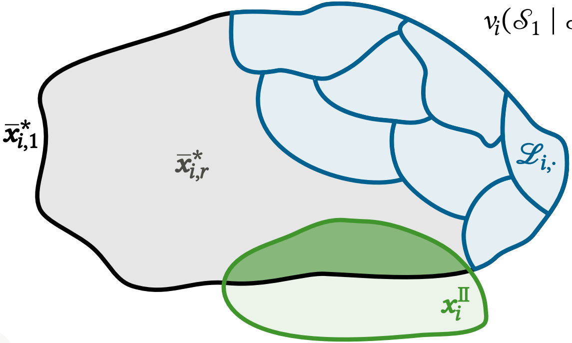
$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i\left(j^1 \mid \mathcal{S}_2\right)$$



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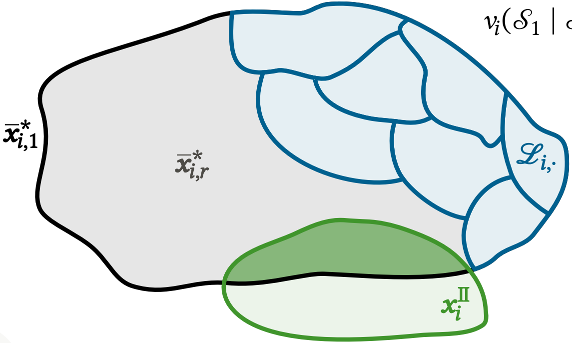

$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i\left(j^1 \mid \mathcal{S}_2\right) + v_i\left(j^2 \mid \mathcal{S}_2 \cup \{j^1\}\right)$$

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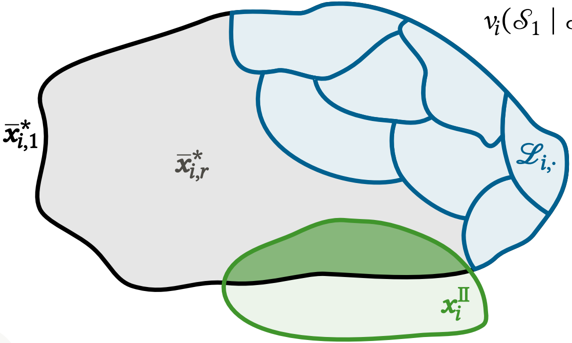

$$\begin{aligned} v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = {} & v_i\left(j^1 \mid \mathcal{S}_2\right) + \\ & v_i\left(j^2 \mid \mathcal{S}_2 \cup \{j^1\}\right) + \\ & v_i\left(j^3 \mid \mathcal{S}_2 \cup \{j^1, j^2\}\right) \end{aligned}$$

Analysing Phase II (2/2)

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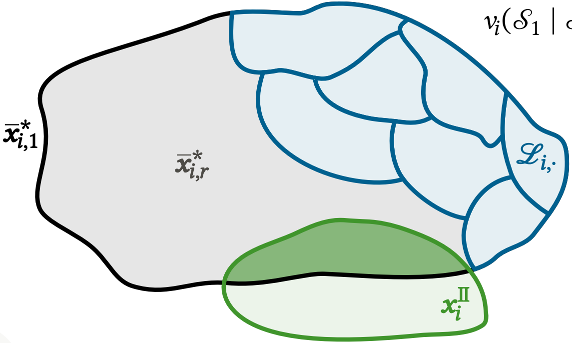

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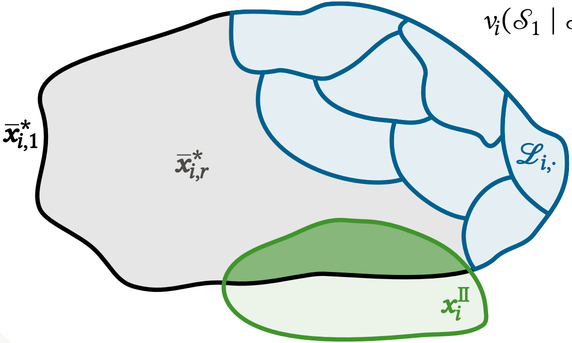

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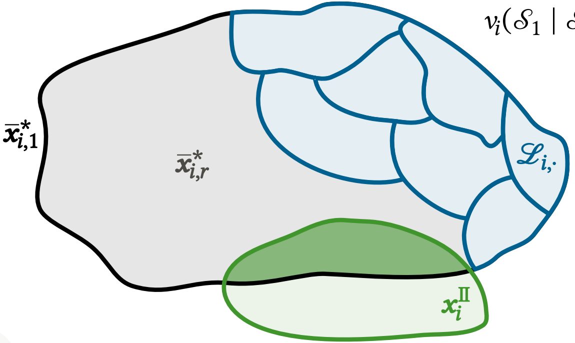

$$\begin{aligned} v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq v_i\left(j^1 \mid \mathcal{S}_2\right) &+ \\ v_i\left(j^2 \mid \mathcal{S}_2\right) &+ \\ v_i\left(j^3 \mid \mathcal{S}_2\right) &+ \\ \vdots \quad & \end{aligned}$$

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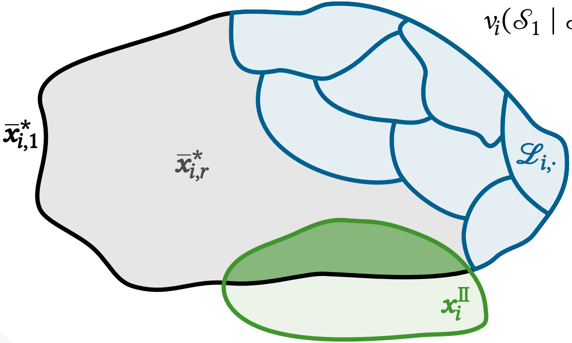

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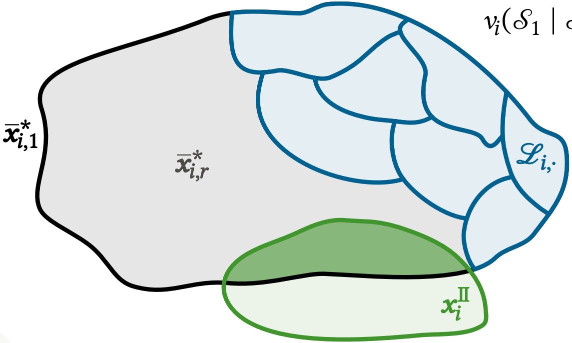

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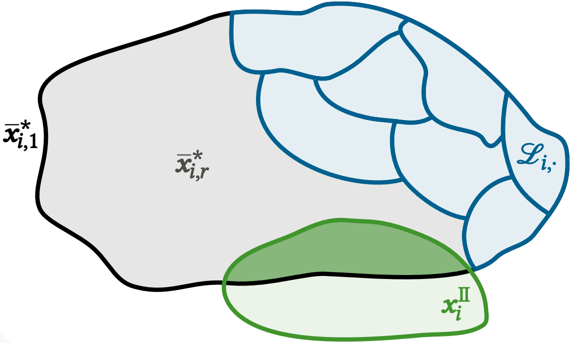

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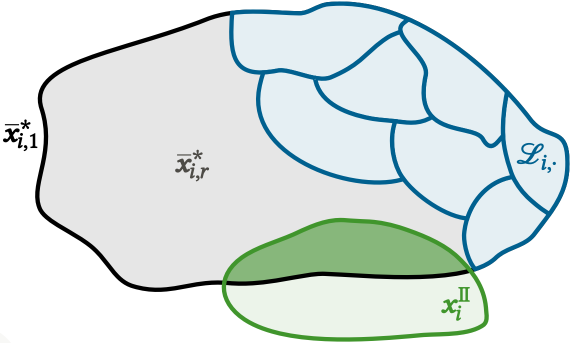

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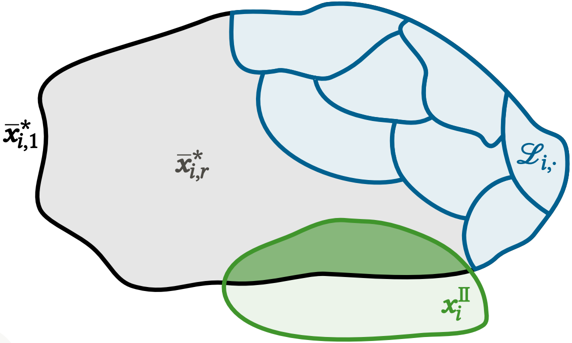

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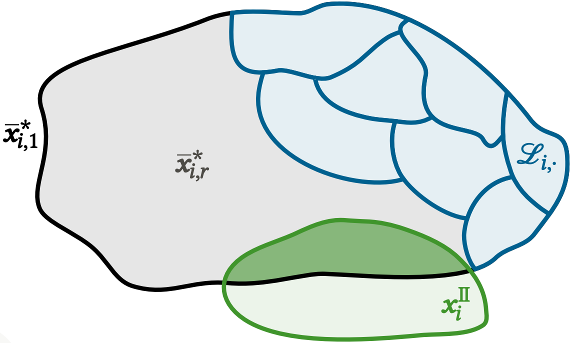

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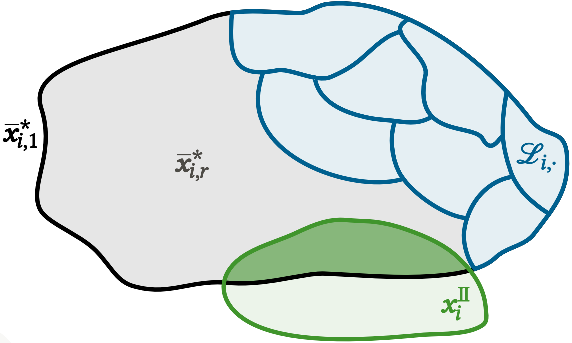

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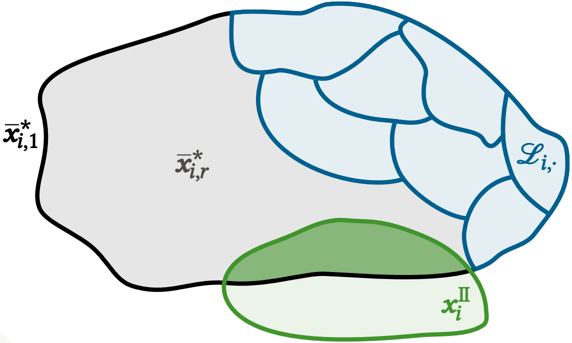

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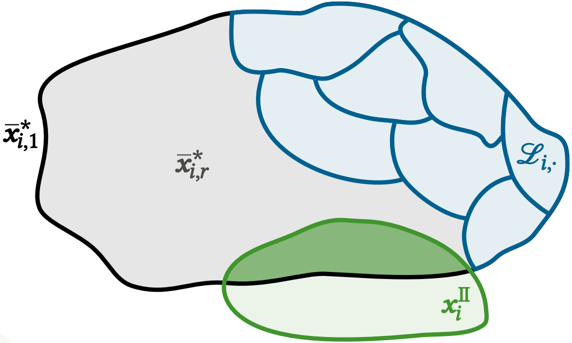

3

Conclusion

Summary & Outlook

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- allocation $\hat{=}$ tuple of bundles

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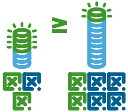
- allocation ≙ tuple of bundles
- valuation through submodular functions



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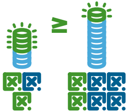

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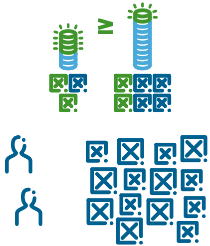

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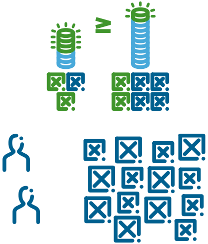

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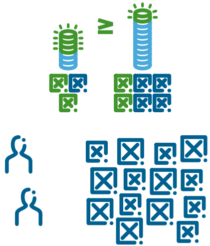

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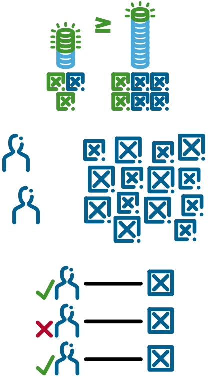

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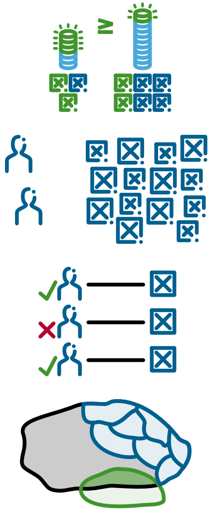

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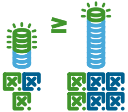
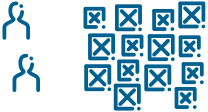
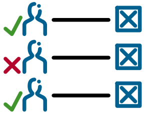
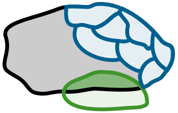
  Phase I finding enough outstanding items
  Phase II assigning remaining items
  Phase III re-assigning outstanding items

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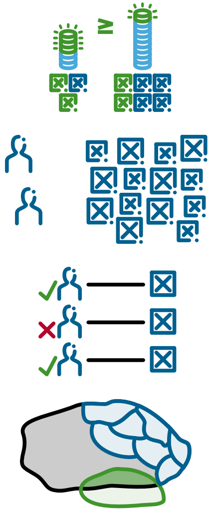

Any room for improvement? Lower bound of $\frac{e}{e-1} \approx 1.58$!

End of Talk