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Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same name by Garg, Kulkarni and Kulkarni [13].

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Abstract

Distributing goods fairly and efficiently between several recipients is of great interest in computer science, economics, politics, and many other areas. One common problem is the maximum Nash social welfare problem (MNP). The MNP asks to distribute unsharable and indivisible goods amongst recipients such that the geometric mean of the recipients' valuations for their assigned goods is maximised. In the asymmetric variant, the valuations are additionally multiplied by individual weights.

Garg, Kulkarni and Kulkarni [13] provide two new polynomial-time approximation algorithms for the MNP. Contrary to previously known algorithms, the approximation guarantees depend only on the number of recipients but not on the number of goods even in the asymmetric variant. More specifically, the algorithms have approximation factors of $\mathcal{O}(n)$ and $\mathcal{O}(n \log n)$ for the asymmetric MNP under additive and submodular valuation functions, respectively, where n is the number of recipients.

In this seminar report on the paper by Garg, Kulkarni and Kulkarni, we present both algorithms and analyse their efficiency. Furthermore, we prove that the MNP under submodular valuation functions cannot be approximated by a factor better than $\frac{e}{e-1}$, even when not asymmetric.

Contents

1 Introduction

1.1 Motivation

The study of distributing goods amongst several recipients is an interdisciplinary field, pursued scientifically as early as the 1940s [18]. It is interesting both computationally (how to distribute fast) and qualitatively (how to distribute well). Its areas of application are manifold:

- In industrial procurement, the preferences of buyers and sellers need be appropriately captured and real-world constraints on goods and services be taken into account [7].
- Mobile Edge Computing denotes a technique where mobile devices compete for computing and storage capabilities of physically close servers. However, the participation of users and the provision of the serves has to be incentivised and monetised [2, 10].
- Many production sites, machines, suppliers, etc. are required for manufacturing, and the tasks between them must be scheduled efficiently. Disturbances must be quickly paid heed to [7].
- Water is a crucial resource, and even hostile countries must come to mutual agreements on the withdrawal from contested rivers [9].

1.2 Preliminaries

In this seminar paper, we focus on unsharable and indivisible goods, which we term *items*. The recipients of those items are termed *agents*. The distributions of items amongst agents are modelled through allocations.

Definition 1. Let $\mathcal{G}$ be a set of m items and $\mathcal{A}$ be a set of n agents. An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_i)_{i \in \mathcal{A}}$ of *bundles* $\boldsymbol{x}_i \subset \mathcal{G}$ such that each item is element of precisely one bundle, that is, $\bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i = \mathcal{G}$ and $\boldsymbol{x}_i \cap \boldsymbol{x}_{i'} = \emptyset$ for all $i \neq i'$. An item $j \in \mathcal{G}$ is *assigned* to agent $i \in \mathcal{A}$ if $j \in \boldsymbol{x}_i$ holds.

The satisfaction of an agent i with her bundle $\boldsymbol{x}_i$ is measured by her *valuation function* v_i, which assigns each item set a real value. We always assume that valuation functions are monotonically non-decreasing, i. e., $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$, $\forall \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{G}$, and normalised, i. e., $v_i(\emptyset) = 0$. Note that this implies non-negativity, i. e., $v_i(\mathcal{S}) \geq 0$, $\forall \mathcal{S} \subset \mathcal{G}$. Besides fulfilling these properties, the valuation functions can come from a plethora of function families. We discuss additive and submodular functions in greater detail.

Additive The valuation $v_i(\mathcal{S})$ of an agent i for any item set $\mathcal{S} \subset \mathcal{G}$ is the sum of the valuations of the individual items $j \in \mathcal{S}$, that is, $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(\{j\})$.
Additive functions are fairly simple but also useful, and many expansions exist [11, 13].

Submodular Let $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \coloneqq v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$ denote the marginal valuation of agent i for an item set $\mathcal{S}_1 \subset \mathcal{G}$ over a disjoint set $\mathcal{S}_2 \subset \mathcal{G}$. This valuation function satisfies the submodularity constraint $v_i(\{j\} \mid \mathcal{S}_1 \cup \mathcal{S}_2) \leq v_i(\{j\} \mid \mathcal{S}_1)$ for all $\{j\}, \mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$.
Submodular valuation functions, which encompass the additive ones, have the property that the gain from assigning new items decreases with increasing bundle size. Diminishing returns are a common phenomenon in economics, making submodular functions worthwhile to study [16]. Their relations to matroids [12, 19, 20] make them interesting from a theoretical point of view, too.

In a slight abuse of notation, we sometimes omit curly braces delimiting a set, so we write $v_i(j_1, j_2, \ldots)$ instead of $v_i(\{j_1, j_2, \ldots\})$ for example.

In order to measure and maximise the overall satisfaction of all agents, one needs to combine their valuations. Several options arise here; common choices are the utilitarian social welfare, that is the sum of all valuations [2, 7, 9, 13, 16], and the egalitarian social welfare, that is the minimum of all valuations [7, 13]. We consider a third one, the Nash social welfare.

Problem 2. Given a set $\mathcal{G}$ of items and a set $\mathcal{A}$ of agents with valuation functions $v_i : 2^{\mathcal{G}} \to \mathbb{R}_0^+$ and weights $\eta_i \in \mathbb{R}^+$ for all agents $i \in \mathcal{A}$, the *maximum Nash social welfare problem (MNP)* is to find an allocation x^* which maximises the weighted geometric mean of valuations, that is,

$$x^* \overset{!}{=} \arg\max_{x \in X_{\mathcal{A}}(\mathcal{G})} \{\mathrm{NSW}(x)\} \quad \text{with } \mathrm{NSW}(x) := \Big(\prod_{i \in \mathcal{A}} v_i(x_i)^{\eta_i}\Big)^{1/\sum_{i \in \mathcal{A}} \eta_i}$$

where $X_{\mathcal{A}}(\mathcal{G})$ is the set of all possible allocations. The problem is called *symmetric* if all agent weights η_i are equal, and *asymmetric* otherwise.

For the techniques employed in later sections, it is beneficial to consider the logarithmic Nash social welfare, that is,

$$\log \mathrm{NSW}(x) = \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log v_i(x_i)\,, \tag{1}$$

which is a sum instead of a product. The Nash social welfare strikes a middle ground between the utilitarian and egalitarian social welfare, which focus on efficiency (height of overall satisfaction) and fairness (how agents value other agents' bundles), respectively. In addition, it exhibits scale-freeness, that is invariance to the scales in which the valuations are expressed. Even though the MNP is $\mathcal{APX}$-hard, approximate solutions largely keep the properties of optimal allocations [3, 6, 8, 17, see also Remark 8].

1.3 Related Work and Contribution by Garg, Kulkarni and Kulkarni

The research on the MNP is rather young and less advanced than the research on other allocation problems. As reference[1], for the submodular utilitarian social welfare problem, a hardness of $\frac{e}{e-1}$ was proven in 2007 [16], and an $\frac{e}{e-1}$-approximation algorithm was shown in 2008 [20] — the additive case is trivially solvable through repeated maximum matchings anyway. For the egalitarian social welfare problem, a randomised $(320\sqrt{n}\log^3 n)$-approximation algorithm for additive valuations [1] and a $(2n-1)$-approximation algorithm for submodular ones [15] have been devised in 2010 and 2007, respectively. These may not be the best algorithms though, as the best known lower bound on the approximation factor is 2 [5].

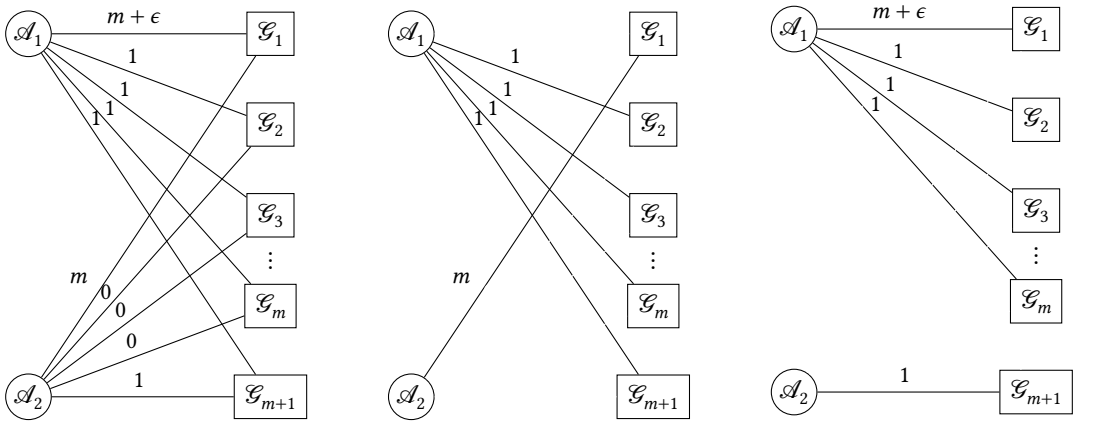
In contrast, for the symmetric additive MNP, an approximation hardness of 1.069 was shown in 2019 [11], but only a 1.45-approximation algorithm has been found in 2018 [3]. For the symmetric submodular MNP, an $(m-n+1)$-approximation algorithm has been devised in 2014 [17]. However, an approximation factor dependant on the number of items is not desirable as the number of items vastly exceeds the number of agents in many applications. Moreover, both approaches exploit the symmetry of the studied problem and fail to extend to the asymmetric case.

Garg, Kulkarni and Kulkarni [13] further the research by contributing two polynomial-time algorithms for the asymmetric MNP. The first one, *SMatch*, computes a $2n$-approximative allocation under additive valuation functions. It does so by *s*martly *match*ing agents and items which make up the parts of a bipartite graph. The second one, *RepReMatch*, computes a $2n(\log_2 n + 3)$-approximative allocation under submodular valuation functions. It does so by *rep*eatedly computing *match*ings, which then get partly annulled, so that items can be *re*matched.

1.4 Structure of the Report

We present and analyse SMatch in Section 2 and RepReMatch in Section 3. In Section 4, we give an analysis on the hardness of the submodular MNP. Section 5 contains the summary, an overview of newly published work since 2020, and an outlook on open questions.

1. The overview is given on the state of research as it was roughly at the end of the year 2019, when Garg, Kulkarni and Kulkarni published the version of their paper [13] on which this seminar report is based.



(a) problem instance **(b)** $\mathrm{NSW}(\boldsymbol{x}^*) = \sqrt{m \cdot m}$ **(c)** $\mathrm{NSW}(\boldsymbol{x}) \leq \sqrt{(2m+\epsilon-1)\cdot 1}$

Figure 1: Illustration of Example 3 with $m+1$ items and two equally weighted agents, their valuations shown as edge labels. It demonstrates that naïvely repeated matching without consideration of the future leads to an allocation $\boldsymbol{x}$ which is at best a $\sqrt{m/2}$-approximation of the optimum $\boldsymbol{x}^*$, meaning the approximation factor depends on the number of items.

2 SMatch

2.1 Presentation of the Algorithm

In the case of an equal number of agents and items, i. e., $n = m$, the *additive* MNP can be solved exactly by finding a maximum matching on a bipartite graph with the sets of agents and of items as its parts; the weight of an edge between agent i and item j would be the weighted logarithmic valuation of item j by agent i, i. e. $\eta_i \log v_i(j)$. Should there be more items than agents, then it would seem natural to just repeat this process until all items are assigned. The flaw of this idea is that such a greedy approach considers only the valuations of items in the current graph and perhaps the valuations of items already assigned. As the following examples demonstrates, this leads to an algorithm with an approximation factor dependent on the number m of items.

Example 3. There are $m+1$ items $\mathscr{G}_1, \ldots, \mathscr{G}_{m+1}$ and two equally weighted agents $\mathscr{A}_1$ and $\mathscr{A}_2$. Agent $\mathscr{A}_1$ values item $\mathscr{G}_1$ at $m+\epsilon$ and all other items at 1. Agent $\mathscr{A}_2$ values item $\mathscr{G}_1$ at m, item $\mathscr{G}_{m+1}$ at 1 and all other items at 0 (Fig. 1a). In an optimal allocation $\boldsymbol{x}^*$, item $\mathscr{G}_1$ would be assigned to agent $\mathscr{A}_2$ and all other items to agent $\mathscr{A}_1$, resulting in a Nash social welfare of $\sqrt{m \cdot m} = m$ (Fig. 1b). A repeated maximum matching algorithm would greedily assign item $\mathscr{G}_1$ to agent $\mathscr{A}_1$ and item $\mathscr{G}_{m+1}$ to agent $\mathscr{A}_2$ first. Even if all remaining items were going to be assigned to agent $\mathscr{A}_1$, the Nash social welfare would never surpass $\sqrt{(2m+\epsilon-1)\cdot 1} < \sqrt{2m}$ (Fig. 1c). The approximation factor of the output $\boldsymbol{x}$ therefore is at least $\sqrt{m/2}$ and, thus, depends on the number of items.

The geometric mean is high when bundles are valuated similarly, wherefore it may be beneficial to give an item to agents who cannot expect many more valuable ones in the future instead of to agents who value the item a bit more but do so for others as well. This is known as the Pigou-Dalton transfer principle [4].

The algorithm SMatch (Algorithm 1) eliminates the flaw by first gaining foresight of the valuations of items assigned after the first matching, thereby achieving an approximation factor of $2n$ (Theorem 7). For a fixed agent i, order the items in descending order of valuations and denote the jth most liked item by $\mathscr{G}_i^j$. We assume a well-defined order, so items of equal valuation shall be ordered further by ID or something similar. SMatch does in fact repeatedly match items. During the first matching, however, the edge weights are defined as $\eta_i \log\big(v_i(j) + u_i/n\big)$ for an edge

Algorithm 1: SMatch for the asymmetric additive MNP

Input : a set $\mathcal{G}$ of m items, a set $\mathcal{A}$ of n agents, additive valuation functions $v_i : 2^{\mathcal{G}} \to \mathbb{R}_0^+$ and weights $\eta_i \in \mathbb{R}^+$ for all agents $i \in \mathcal{A}$

Output: a $2n$-approximation $\boldsymbol{x} = (\boldsymbol{x}_i)_{i\in\mathcal{A}}$ of an optimal allocation

$u_i \leftarrow v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m) \quad \forall i \in \mathcal{A}$ ▷ *estimation of future valuations*

$E \leftarrow \left\{ \left(i, j, \eta_i \log(v_i(j) + u_i/n)\right) \mid i \in \mathcal{A}, j \in \mathcal{G} \right\}$ ▷ *weighted edges using estimation*

$G \leftarrow (\mathcal{A}, \mathcal{G}, E)$ ▷ *bipartite graph*

$\mathcal{M} \leftarrow$ max_weight_matching(G)

$\boldsymbol{x}_i \leftarrow \{ j \mid (i,j) \in \mathcal{M} \} \quad \forall i \in \mathcal{A}$ ▷ *assign according to matching*

$\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$ ▷ *set of unassigned items*

while $\mathcal{G}^{\text{rem}} \neq \emptyset$ **do**

  $E \leftarrow \left\{ \left(i, j, \eta_i \log(v_i(j) + v_i(\boldsymbol{x}_i))\right) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}} \right\}$ ▷ *weighted edges using only valuations*

  $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, E)$

  $\mathcal{M} \leftarrow$ max_weight_matching(G)

  $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i \cup \{ j \mid (i,j) \in \mathcal{M} \} \quad \forall i \in \mathcal{A}$

  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$

end while

return $\boldsymbol{x}$

between agent i and item j. The value u_i serves as estimation of the valuation of items assigned after the first matching (Lemma 5) and is defined as

$$u_i := \min_{\substack{\mathcal{S}\subset\mathcal{G} \\ |\mathcal{S}|\leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m). \tag{2}$$

Note that the set $\mathcal{S}$ realizing the minimum of $v_i(\mathcal{G} \setminus \mathcal{S})$ may have less than $2n$ elements if there are less than $2n$ items in total or if some items are valuated at 0. From the second matching onwards, the edge weights are defined as $\eta_i \log\big(v_i(j) + v_i(\boldsymbol{x}_i)\big)$, where $\boldsymbol{x}_i$ is the continuously updated bundle of agent i. The addend $v_i(\boldsymbol{x}_i)$ could lead to better allocations in applications but does not improve the approximation factor asymptotically and so is of no further interest in the analysis.

2.2 Analysis of the Algorithm

To calculate the approximation factor of SMatch, we first need to establish a lower bound on the valuation of single items. For convenience, we order the items of any set $\mathcal{S} = \{j^1, j^2, \dots\}$ in descreasing order of valuation, so that it holds $v_i(j^k) \geq v_i(j^{k'})$ for all $k < k'$. Note that if set $\mathcal{S}$ happens to be a bundle, then item j^k was assigned according to the kth matching.

Lemma 4. *For each agent $i \in \mathcal{A}$ and her final bundle $\boldsymbol{x}_i = \{a_i^1, \dots, a_i^{\tau_i}\}$, it holds $v_i(a_i^t) \geq v_i(\mathcal{G}_i^{tn})$ for all $t = 1, \dots, \tau_i$.*

Proof. Before the tth matching, no more than $(t-1)n$ items out of the tn most highly valued items $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ have been assigned in previous matchings since at most n many out of those items are assigned each time. Because of the tth matching, at most $n-1$ more items could be assigned to other agents, leaving at least one item of $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ attainable for agent i. Since $v_i(\mathcal{G}_i^k) \geq v_i(\mathcal{G}_i^{tn})$ for all $k \leq tn$ by definition of $\mathcal{G}_i^k$, the lemma follows. □

We can now use $u_i/n = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m)/n$ to establish a lower bound on the valuations of items assigned after the first matching.

Lemma 5. *For each agent $i \in \mathcal{A}$ and her final bundle $\boldsymbol{x}_i = \{a_i^1, \dots, a_i^{\tau_i}\}$, it holds $v_i(a_i^2, \dots, a_i^{\tau_i}) \geq u_i/n$.*

Proof. By Lemma 4 and definition of $\mathcal{G}_i^{tn}$, every item a_i^t is worth at least as much as each item $\mathcal{G}_i^{tn+k}$ with $k \in \{1, \dots, n\}$ and, consequently, its valuation $v_i(a_i^t)$ is at least as high as the mean valuation $\frac{1}{n} v_i(\mathcal{G}_i^{tn+1}, \dots, \mathcal{G}_i^{tn+n})$. Also, it holds $\tau_i n + n \geq m$ since agent i gets assigned $\tau_i \geq \left\lfloor \frac{m}{n} \right\rfloor \geq \frac{m}{n} - 1$ many items. This yields

$$v_i(a_i^2, \dots, a_i^{\tau_i}) = \sum_{t=2}^{\tau_i} v_i(a_i^t) \geq \sum_{t=2}^{\tau_i} \frac{1}{n} v_i(\mathcal{G}_i^{tn+1}, \dots, \mathcal{G}_i^{tn+n}) \geq \frac{1}{n} v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^{m}) = \frac{u_i}{n}\,. \tag{3}$$

□

Remark 6. In Lemma 5, we assumed non-zero valuations for all items, hence the bundle lengths of $\tau_i \geq \left\lfloor \frac{m}{n} \right\rfloor$. Of course, one would not assign items to agents who value them at zero in an actual program. Inasmuch as additional zeroes do not change the sum in Eq. (3), Lemma 5 still holds.

This allows us to calculate an approximation factor for SMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

Theorem 7. *SMatch has an approximation factor of* $2n$.

Proof. Lemma 5 can be plugged into the logarithmic Nash social welfare:

$$\log \mathrm{NSW}(\boldsymbol{x}) = \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log v_i(a_i^1, \dots, a_i^{\tau_i}) \tag{4}$$

$$= \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\big(v_i(a_i^1) + v_i(a_i^2, \dots, a_i^{\tau_i})\big) \tag{5}$$

$$\geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\big(v_i(a_i^1) + u_i/n\big) \tag{6}$$

Notice that the first matching of SMatch maximises the sum in Eq. (6). Thus, assigning all agents i their respective most highly valued item o_i^1 from an optimal bundle $\boldsymbol{x}_i^* = \{o_i^1, \dots, o_i^{\tau_i^*}\}$ yields the even lower bound

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\big(v_i(o_i^1) + u_i/n\big)\,. \tag{7}$$

Recall the definition of u_i from Eq. (2). Consider a slightly modified variant:

$$u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \text{ with } \mathcal{S}_i \coloneqq \operatorname*{arg\,min}_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \tag{8}$$

Now consider the set $\mathcal{S}_i^*$ of the (at most) $2n$ most highly valued items in the optimal bundle $\boldsymbol{x}_i^*$, that is,

$$\mathcal{S}_i^* \coloneqq \operatorname*{arg\,min}_{\substack{\mathcal{S} \subset \boldsymbol{x}_i^* \\ |\mathcal{S}| \leq 2n}} \{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S})\}\,. \tag{9}$$

It holds $v_i(o_i^1) \geq \frac{1}{2n} v_i(\mathcal{S}_i^*)$ because of $v_i(o_i^1) \geq v_i(j)$ for all items $j \in \mathcal{S}_i^*$. Furthermore, it holds $u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)$. We can insert these two inequalities into Eq. (7) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\mathcal{S}_i^*)}{2n} + \frac{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)}{n}\right) \tag{10}$$

$$\geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n}\right) \tag{11}$$

$$= \log\left(\frac{\mathrm{NSW}(\boldsymbol{x}^*)}{2n}\right) \tag{12}$$

□

The analysis is asymptotically tight [13, Section 6.3]. It is possible to design an instance for the asymmetric additive MNP such that SMatch achieves an approximation factor approaching $n/2$. It remains to be shown whether the symmetric additive MNP leads to an asymptotically equally bad approximation factor.

Remark 8. SMatch produces *fair* allocations which are *envy-free up to one item (EF1)* [13, Section 5.2]. An allocation $\boldsymbol{x}$ is EF1 if, for every pair $(i, i') \in \mathscr{A}^2$ of agents, one needs to remove at most one item from the bundle $\boldsymbol{x}_{i'}$ of agent i' such that agent i does not want to swap bundles. In other words, it holds $v_i(\boldsymbol{x}_i) \geq v_i(\boldsymbol{x}_{i'})$ or, if not, there is an item $j \in \boldsymbol{x}_{i'}$ such that $v_i(\boldsymbol{x}_i) \geq v_i(\boldsymbol{x}_{i'} \setminus \{j\})$.

However, SMatch does not produce allocations which are *Pareto-optimal (PO)*, another popular fairness property [13, Remark 5.2]. An allocation $\boldsymbol{x}$ is PO if there is no other allocation $\boldsymbol{x}'$ where every agent is at least as well off and one agent is strictly better off, i. e., it does not hold $v_i(\boldsymbol{x}'_i) \geq v_i(\boldsymbol{x}_i)$ for all agents $i \in \mathscr{A}$ and $v_{i'}(\boldsymbol{x}'_{i'}) > v_{i'}(\boldsymbol{x}_{i'})$ for some agent $i' \in \mathscr{A}$.

3 RepReMatch

3.1 Presentation of the Algorithm

The algorithm SMatch estimates the worst-case valuation of future items by determining the set of highest-value items and then valuing the remaining items. Unfortunately, this approach does not work for general *submodular* valuation functions because taking a set of highest valuation away does not necessarily leave a set of lowest valuation. In fact, it can be shown [19] that determining the set of lowest valuation is approximable only within a factor of $\Omega\left(\sqrt{m/\ln m}\right)$.

For this reason, the algorithm RepReMatch (Algorithm 2) relies on an approach with three phases, thereby achieving an approximation factor of $2n(\log_2 n + 3)$ (Theorem 16). In phase I, a sufficiently big set of high-value items is determined through repeated matchings. This phase serves merley to determine this set, so items are assigned temporarily only. The edge weights reflect this by taking the valuations of just single items into account.

Algorithm 2: RepReMatch for the asymmetric submodular MNP

Input : a set $\mathcal{G}$ of m items, a set $\mathcal{A}$ of n agents, submodular valuation functions $v_i : 2^{\mathcal{G}} \to \mathbb{R}_0^+$ and weights $\eta_i \in \mathbb{R}^+$ for all agents $i \in \mathcal{A}$

Output: a $2n(\log_2 n + 3)$-approximation $\boldsymbol{x}^{\mathrm{III}} = (\boldsymbol{x}_i^{\mathrm{III}})_{i\in\mathcal{A}}$ of an optimal allocation

Phase I:

$\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ ▷ *initialise temporary allocation*

$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}$ ▷ *set of unassigned items*

for $t \leftarrow 1, \dots, \lceil \log_2 n \rceil + 1$ **do**

  if $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **then**

    $E \leftarrow \left\{ \left(i, j, \eta_i \log v_i(j)\right) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \right\}$ ▷ *weighted edges using val. of single item*

    $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, E)$ ▷ *bipartite graph*

    $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$

    $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \boldsymbol{x}_i^{\mathrm{I}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$ ▷ *assign (temporarily) according to matching*

    $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$ ▷ *remove assigned items*

  end if

end for

Phase II:

$\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ ▷ *put allocation $\boldsymbol{x}^{\mathrm{I}}$ away & initialise new, definite one*

while $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **do**

  $E \leftarrow \left\{ \left(i, j, \eta_i \log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})\right) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \right\}$ ▷ *valuation of item & current bundle*

  $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, E)$

  $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$

  $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$ ▷ *assign (definitely) according to matching*

  $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$

end while

Phase III:

$\mathcal{G}^{\mathrm{rem}} \leftarrow \bigcup_{i\in\mathcal{A}} \boldsymbol{x}_i^{\mathrm{I}}$ ▷ *release items assigned in phase I*

$E \leftarrow \left\{ \left(i, j, \eta_i \log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})\right) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \right\}$ ▷ *valuation of item & current bundle*

$G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, E)$

$\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$

$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$ ▷ *initialise final allocation & assign according to matching*

$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$

$\boldsymbol{x}^{\mathrm{III}} \leftarrow \mathrm{arbitrary\_allocation}(\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \boldsymbol{x}^{\mathrm{III}}, (v_i)_{i\in\mathcal{A}})$ ▷ *assign remainder of items arbitrarily*

return $\boldsymbol{x}^{\mathrm{III}}$

In phase II, the remaining items are assigned definitely through repeated matchings. Consequently, each edge weight is updated in each round to reflect the valuation of both the respective item and the items assigned so far.

In phase III, the high-value items assigned in phase I are released. One maximum weight matching is calculated, and the matched items are assigned accordingly. Similarly to the previous phase, each edge weight uses the valuation of both the respective item and the items assigned so far, i. e., the bundle from phase II. The remainder of released items is assigned arbitrarily.

3.2 Analysis of the Algorithm

We use the term *round* to refer to the iterations of the loops in the phases I and II. For ease of notation, we refer to the moment right before the first iteration in phase II as round 0.

We start by analysing phase II as it is the first phase with definitive assignments. To this end, we introduce two types of item sets.

Definition 9. Let x_i^* be an optimal bundle of an agent $i \in \mathscr{A}$. For any round $r \geq 1$ in phase II, the set $\mathscr{L}_{i,r} \subset x_i^*$ of *lost items* is the set of all items $j \in x_i^*$ assigned to other agents $i' \neq i$ in that round.

Definition 10. Let x_i^* be an optimal bundle of an agent $i \in \mathscr{A}$ and $x_i^{\mathrm{II}} = \{a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}\}$ be her bundle at the end of phase II. The set of *optimal and attainable items* is defined as $\overline{x}_{i,0}^* := x_i^* \setminus \bigcup_{i' \in \mathscr{A}} x_{i'}^{\mathrm{I}}$ for round 0, as $\overline{x}_{i,1}^* := \overline{x}_{i,0}^* \setminus \mathscr{L}_{i,1}$ for round 1 and as $\overline{x}_{i,r}^* := \overline{x}_{i,r-1}^* \setminus (\mathscr{L}_{i,r} \cup \{a_i^{r-1}\})$ for round $r = 2, \dots, \tau_i^{\mathrm{II}}$ of phase II.

We denote their sizes by $\ell_{i,r} := |\mathscr{L}_{i,r}|$ and $\overline{\tau}_{i,r}^* := |\overline{x}_{i,r}^*|$, respectively. Firstly, we give a lower bound on the valuations of optimal and attainable items.

Lemma 11. *For each agent $i \in \mathscr{A}$ and her bundle $x_i^{\mathrm{II}} = \{a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}\}$ at the end of phase II, it holds in all rounds $r = 2, \dots, \tau_i^{\mathrm{II}}$ that*

$$v_i(\overline{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq v_i(\overline{x}_{i,1}^*) - \sum_{r'=1}^{r-1} \ell_{i,r'+1} \cdot v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1}) - v_i(a_i^1, \dots, a_i^{r-1}) .$$

Proof. The original proof [13, pp. 13 sq.] consists of a lengthy induction rich in formulae and case differentiations. We opt to give a different but more intuitive approach.

Writing the left-hand side of the lemma out in full gives

$$v_i(\overline{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = v_i(\overline{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) - v_i(a_i^1, \dots, a_i^{r-1}) , \tag{13}$$

which explains the last subtrahend on the right-hand side.

Next, we show a lower bound on $v_i(\overline{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\})$. The items which are optimal and attainable in round r were so in round $r-1$, too. Additionally, the optimal but lost items of round r were attainable in round $r-1$ as well. The item a_i^r assigned to agent i in round r was still attainable in round $r-1$ and may also be optimal. Therefore, it holds $(\overline{x}_{i,r}^* \cup \{a_i^{r-1}\}) \supset (\overline{x}_{i,r-1}^* \setminus \mathscr{L}_{i,r})$ and

$$v_i(\overline{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) \geq v_i(\overline{x}_{i,r-1}^* \setminus \mathscr{L}_{i,r} \cup \{a_i^1, \dots, a_i^{r-2}\}) \geq v_i(\overline{x}_{i,r-1}^* \cup \{a_i^1, \dots, a_i^{r-2}\}) - v_i(\mathscr{L}_{i,r}) . \tag{14}$$

The second inequality makes intuitively sense when one thinks of diminishing returns: The gain from adding the set $\mathscr{L}_{i,r}$ to the items in $\overline{x}_{i,r-1}^* \cup \{a_i^1, \dots, a_i^{r-2}\}$ cannot be higher than its subtracted valuation over the empty set. If we now recursively apply Eq. (14), we eventually arrive at

$$v_i(\overline{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) \geq v_i(\overline{x}_{i,1}^*) - \sum_{r'=1}^{r-1} v_i(\mathscr{L}_{i,r'+1}) . \tag{15}$$

In the last step, we give an upper bound on $v_i(\mathscr{L}_{i,r'+1})$. The valuation $v_i(j \mid a_i^1, \dots, a_i^{r'-1})$ of any item $j \in \mathscr{L}_{i,r'+1}$ in round r' could not have been higher than $v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1})$ as otherwise item $a_i^{r'}$ would not have been assigned before item j. From this follows

$$v_i(\mathscr{L}_{i,r'+1}) \leq \ell_{i,r'+1} \cdot v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1}). \tag{16}$$

Combining Eqs. (13), (15) and (16) proves the lemma. □

The lemma can be used to find a lower bound on the marginal valuation of the items actually assigned in each round r.

Corollary 12. *From Lemma 11 follows*

$$v_i(a_i^r \mid a_i^1, \dots, a_i^{r-1}) \geq \left(v_i(\overline{x}_{i,1}^*) - \sum_{r'=1}^{r-1} \ell_{i,r'+1} \cdot v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1}) - v_i(a_i^1, \dots, a_i^{r-1})\right) \Big/ \overline{\tau}_{i,r}^*.$$

Proof. Valuation functions are monotonic, so $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ holds for all sets $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{G}$. Induction shows that there must be an item $j \in \mathcal{S}_2$ with valuation $v_i(j) \geq v_i(\mathcal{S}_2)/|\mathcal{S}_2|$, otherwise it would hold $v_i(\emptyset) > 0$. Applied to Lemma 11, this means that there must be an item $j \in \overline{x}_{i,r}^*$ with a marginal valuation of at least $v_i(\overline{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})/\overline{\tau}_{i,r}^*$. As item a_i^r was the one to be assigned in round r, its marginal valuation cannot be smaller. □

This, finally, enables us to give a lower bound on the valuation of the bundles $\boldsymbol{x}_i^{\text{II}}$.

Lemma 13. *For each agent $i \in \mathscr{A}$ and her bundle $\boldsymbol{x}_i^{\text{II}} = \{a_i^1, \dots, a_i^{\tau_i^{\text{II}}}\}$, it holds*

$$v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}}) \geq v_i(\overline{x}_{i,1}^*)/n.$$

Proof. In each round $r = 1, \dots, \tau_i^{\text{II}}$, there are $\ell_{i,r}$ optimal and attainable items of agent i gettin assigned to other agents. As n is the number of agents, $n-1$ is an upper bound on $\ell_{i,r}$. Furthermore, after τ_i^{II} rounds, the number of unassigned items is at most $n-1$ elsewise agent i would have been assigned yet another item in a round $\tau_i^{\text{II}} + 1$. This implies $\overline{\tau}_{i,\tau_i^{\text{II}}}^* \leq n$, as those $n-1$ items could be optimal and item $a_i^{\tau_i^{\text{II}}}$ may be, too. Together with Corollary 12, the lemma is proven:

$$\begin{aligned}
v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}}) &= v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1}) + v_i(a_i^{\tau_i^{\text{II}}} \mid a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1}) && (17)\\
&\geq v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1}) + \Big(v_i(\overline{x}_{i,1}^*) - \sum_{r'=1}^{\tau_i^{\text{II}}-1} \ell_{i,r'+1} \cdot v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1}) && (18)\\
&\qquad - v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1})\Big) \Big/ \overline{\tau}_{i,\tau_i^{\text{II}}}^*\\
&\geq v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1}) + \Big(v_i(\overline{x}_{i,1}^*) - \sum_{r'=1}^{\tau_i^{\text{II}}-1} (n-1) v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1}) && (19)\\
&\qquad - v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1})\Big) \Big/ n\\
&\geq v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1}) + \big(v_i(\overline{x}_{i,1}^*) - (n-1) v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1}) - v_i(a_i^1, \dots, a_i^{\tau_i^{\text{II}}-1})\big)/n && (20)\\
&= v_i(\overline{x}_{i,1}^*)/n && (21)
\end{aligned}$$

□

After having obtained a lower bound on the valuation of items assigned in phase II, we need a lower bound for phase III as well. Therefor we introduce a third type of item set.

Definition 14. Let $\boldsymbol{x}_i^* = \{o_i^1, \dots, o_i^{\tau_i^*}\}$ be an optimal bundle of an agent $i \in \mathscr{A}$. The set of *outstanding items* is defined as $\mathcal{G}_i^+ := \{ j \in \mathcal{G} \mid v_i(j) \geq v_i(o_i^1) \}$.

Lemma 15. *In phase III, there exists a matching such that each agent $i \in \mathscr{A}$ is matched to one of her outstanding items in the set $\bigcup_{i' \in \mathscr{A}} \boldsymbol{x}_{i'}^{\mathrm{I}}$ of released items.*

Proof. If all items were matched in phase I, i. e., $\bigcup_{i' \in \mathscr{A}} \boldsymbol{x}_{i'}^{\mathrm{I}} = \mathscr{G}$, then all optimal items are released in phase III and each agent can be matched to one; the lemma is proven immediately. If not, imagine for some t that only the items assigned in the first t rounds of phase I were released. Now choose some matching $\mathscr{M}_t$ with the following properties:

1. If all outstanding items of an agent i were amongst the released items, she gets matched with an outstanding item $j \in \mathscr{G}_i^+$.
2. The number of agents matched with one of their outstanding items is maximal amongst all matchings fulfilling property 1.

Property 1 is always satisfiable as the union of k many sets $\mathscr{G}_i^+$ contains k different outstanding items o_i^1, which can be matched with agents i. Property 2 leads to all agents being matched with an outstanding item when $t = \lceil \log_2 n \rceil + 1$, i. e. the number of rounds in phase I, whence the lemma follows. To prove this, we denote by $\mathscr{A}_t^-$ the set of agents who are *not* matched with one of their outstanding items, and show by induction on t that it holds $|\mathscr{A}_t^-| \leq n/2^t$.

In the base case $t = 1$, none of the items are assigned initially. Denote by α the number of agents who were not assigned an outstanding item in the first round of phase I. If $\alpha \leq n/2$, then a matching $\mathscr{M}_1$ as described above obviously exists and the base case is immediately proven. Otherwise, all items from at least α many sets $\mathscr{G}_i^+$ got assigned to someone. Again: Each set $\mathscr{G}_i^+$ contains the outstanding item o_i^1, so the union of these sets contains at least α items which can be matched with at least α agents upon release. This, then, leaves at most $n - \alpha < n/2$ agents not matched with an outstanding item.

For the induction hypothesis, we assume that the statement holds true for all rounds up to some t. In the induction step $t \to t+1$, by property 1, there is at least one unassigned, outstanding item in each set $\mathscr{G}_{i'}^+$ of all agents $i' \in \mathscr{A}_t^-$ at the start of round $t+1$. Analogously to the base case, for at least half of those agents i', these unassigned items will be assigned to either them or someone else, and it can be argued accordingly. By the induction hypothesis, it holds $|\mathscr{A}_{t+1}^-| \leq |\mathscr{A}_t^-|/2 \leq (n/2^t)/2 = n/2^{t+1}$. □

This allows us to calculate an approximation factor for RepReMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

Theorem 16. *RepReMatch has an approximation factor of $2n(\log_2 n + 3)$.*

Proof. By Lemma 15, we can assign each agent i an outstanding item $j_i^+ \in \mathscr{G}_i^+$ in the beginning of phase III. RepReMatch maximises the logarithmic Nash social welfare, so

$$\log \mathrm{NSW}(\boldsymbol{x}^{\mathrm{III}}) \geq \frac{1}{\sum_{i \in \mathscr{A}} \eta_i} \cdot \sum_{i \in \mathscr{A}} \eta_i \log v_i(j_i^+, a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}) \tag{22}$$

is a lower bound on the logarithmic Nash social welfare after the first matching in phase III, with $\boldsymbol{x}_i^{\mathrm{II}} = \{a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}\}$ being the bundle of agent i at the end of phase II.

Lemma 13 delivers a straightforward lower bound on the valuations of bundles, namely

$$v_i(j_i^+, a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}) \geq v_i(a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}) \geq \frac{v_i(\overline{\boldsymbol{x}}_{i,1}^*)}{n} = \frac{v_i(\overline{\boldsymbol{x}}_{i,0}^* \setminus \mathscr{L}_{i,1})}{n}. \tag{23}$$

Next, consider the complement of $\overline{\boldsymbol{x}}_{i,0}^* \setminus \mathscr{L}_{i,1}$, i. e. $(\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*) \cup \mathscr{L}_{i,1}$. Phase I runs for at most $\lceil \log_2 n \rceil + 1$ rounds, and at most n items are assigned in each iteration. Therefore, at most $n(\log_2 n + 2)$ optimal items are assigned in that phase, i. e., $|\bigcup_{i' \in \mathscr{A}} \boldsymbol{x}_{i'}^{\mathrm{I}}| = |\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*| \leq n(\log_2 n + 2)$. As in Lemma 13, it also holds $n \geq \ell_{i,1} = |\mathscr{L}_{i,1}|$. Since $(\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*) \cup \mathscr{L}_{i,1} \subset \boldsymbol{x}_i^*$ and $v_i(o_i^1) \geq v_i(o_i^r)$ for all $o_i^r \in \boldsymbol{x}_i^*$, it holds

$$v_i(j_i^+, a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}) \geq v_i(j_i^+) \geq v_i(o_i^1) \geq \frac{v_i\big((\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*) \cup \mathscr{L}_{i,1}\big)}{n(\log_2 n + 3)}. \tag{24}$$

Equations (23) and (24), combined with the submodularity of the valuation functions, give the concise lower bound

$$v_i(j_i^+, a_i^1, \dots, a_i^{\tau_i^{\mathrm{II}}}) \geq \frac{1}{2}\left(\frac{v_i\big((\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*) \cup \mathscr{L}_{i,1}\big)}{n(\log_2 n + 3)} + \frac{v_i\big(\overline{\boldsymbol{x}}_{i,0}^* \setminus \mathscr{L}_{i,1}\big)}{n}\right) \tag{25}$$

$$\geq \frac{1}{2} \cdot \frac{v_i\big((\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*) \cup \mathscr{L}_{i,1}\big) + v_i\big(\overline{\boldsymbol{x}}_{i,0}^* \setminus \mathscr{L}_{i,1}\big)}{n(\log_2 n + 3)} \tag{26}$$

$$\geq \frac{1}{2} \cdot \frac{v_i\big(((\boldsymbol{x}_i^* \setminus \overline{\boldsymbol{x}}_{i,0}^*) \cup \mathscr{L}_{i,1}) \cup (\overline{\boldsymbol{x}}_{i,0}^* \setminus \mathscr{L}_{i,1})\big)}{n(\log_2 n + 3)} \tag{27}$$

$$= \frac{v_i(\boldsymbol{x}_i^*)}{2n(\log_2 n + 3)}\,. \tag{28}$$

We can insert this lower bound into Eq. (22) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}^{\mathrm{III}}) \geq \frac{1}{\sum_{i \in \mathscr{A}} \eta_i} \cdot \sum_{i \in \mathscr{A}} \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n(\log_2 n + 3)}\right) = \log\left(\frac{\mathrm{NSW}(\boldsymbol{x}^*)}{2n(\log_2 n + 3)}\right) \tag{29}$$

□

4 Hardness of Approximation

Garg, Kulkarni and Kulkarni [13, Sction 4] provide the following hardness result.

Theorem 17. *The submodular MNP is not approximable by a factor better than* $\frac{e}{e-1}$ *in polynomial time unless* $\mathcal{P} = \mathcal{NP}$*, even when the agents have equal weights and valuation functions.*

Proof. Consider the related maximum utilitarian social welfare problem[1].

Problem 18. Given a set $\mathcal{G}$ of items and a set $\mathcal{A}$ of agents with submodular valuation functions $v_i : 2^{\mathcal{G}} \to \mathbb{R}_0^+$ for all agents $i \in \mathcal{A}$, the symmetric submodular *maximum utilitarian social welfare problem (MUP)* is to find an allocation $\boldsymbol{x}^*$ which maximises the sum of valuations, that is,

$$\boldsymbol{x}^* \stackrel{!}{=} \arg\max_{\boldsymbol{x} \in \boldsymbol{X}_{\mathcal{A}}(\mathcal{G})} \{\mathrm{USW}(\boldsymbol{x})\} \quad \text{with } \mathrm{USW}(\boldsymbol{x}) \coloneqq \sum_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i)$$

where $\boldsymbol{X}_{\mathcal{A}}(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst n agents.

Note that this problem is identical to the symmetric submodular MNP except for the sum in the objective function instead of a product. We are going to exploit this fact to calculate the Nash social welfare of instances for MUP. Khot et al. [16] supply a polynomial-time reduction of MUP from the following problem:

Problem 19. Given a graph $G = (V, E)$ and a constant $c < 1$, the *c-Gap-Max-3-Colouring problem* is to decide whether, for any 3-colouring of graph G which maximises the number of edges with differently coloured endpoints, the number of such edges is $|E|$ (*Yes instance*) or $c|E|$ and below (*No instance*).

Proposition 20. *There exists a constant* $c < 1$ *such that the c-Gap-Max-3-Colouring problem is* $\mathcal{NP}$*-hard.*

Reducing an instance of the c-Gap-Max-3-Colouring problem yields an instance of the symmetric submodular MUP with identical valuation functions. Its properties are as follows:

Yes instance The utilitarian social welfare is nC because every agent values her bundle at n, whereby C is a constant depending on the input graph. The Nash social welfare of the instance is C.

No instance The utilitarian social welfare is no more than $\frac{e-1+\varepsilon}{e} \cdot nC$ with some $\varepsilon > 0$. Applying the inequality of arithmetic and geometric means, i. e., $(x_1 + \cdots + x_n)/n \geq \sqrt[n]{x_1 \cdots x_n}$ for all non-negative numbers $x_1, \ldots, x_n \in \mathbb{R}_0^+$, reveals that the Nash social welfare of the instance is at most $\frac{e-1+\varepsilon}{e} \cdot C$.

Thereout follows that the submodular MNP, even when symmetric and under identical valuation functions, cannot be approximated by a factor of $\frac{e}{e-1+\varepsilon}$, so $\frac{e}{e-1}$ constitutes a lower bound on the approximation factor. □

For a constant number of agents, Garg, Kulkarni and Kulkarni [13, Section 5.1] describe a family $(A_\epsilon)_{\epsilon>0}$ of algorithms for the asymmetric submodular MNP where each algorithm A_ϵ achieves an approximation factor of $\frac{e}{e-1} + \epsilon$.

1. Garg, Kulkarni and Kulkarni (and many others) call Problem 18 the 'Allocation problem'. We changed the name to match the naming scheme of Problem 2 and to avoid confusion, as both problems are about finding allocations.

5 Conclusion

5.1 Summary

The asymmetric maximum Nash social welfare problem (MNP) asks to find an allocation of unsharable and indivisible items amongst agents such that the weighted geometric mean of their valuations is maximised. Based on a paper by Garg, Kulkarni and Kulkarni [13], we presented two polynomial-time algorithms for the asymmetric MNP. The novelty of both algorithms lies in their approximation factors depending on the number n of agents but not on the number of items.

The first algorithm, SMatch, finds a $2n$-approximative allocation under additive valuation functions. It does so by repeatedly matching agents with items within a weighted bipartite graph. For the very first matching, the edge weights incorporate an estimation of the valuation of future items. The output allocation is envy-free up to one item.

The second algorithm, RepReMatch, finds a $2n(\log_2 n + 3)$-approximative allocation under submodular valuation functions. In phase I, a set of high-value items is determined through repeated matchings and then put away. In phase II, agents are repeatedly matched with the remaining items. In phase III, the high-value items are finally assigned to the agents.

Additionally, it was shown that any polynomial-time algorithm for the submodular MNP must have an approximation factor of at least $\frac{e}{e-1}$ unless $\mathcal{P} = \mathcal{NP}$. This holds even when the problem is symmetric and the valuation functions equal.

5.2 Open Questions and Recent Work

Garg, Kulkarni and Kulkarni [13] conjecture that SMatch has a better approximation factor for the symmetric additive MNP, though neither were they able to prove a better factor nor could they prove the tightness of their analysis. Additionally, they forbore to prove the tightness of the analysis of RepReMatch, and we suspect there to be quite some constants to be saved — admittedly, the benefits of a more thorough analysis are questionable.

The research on the MNP gained pace in the last years, and some publications are of especial interest in the context of this report:

- XOS functions encompass the submodular functions. Unfortunately, RepReMatch does not guarantee an approximation factor independent of the number of items even for the symmetric XOS MNP [13, Section 6.2]. Barman et al. [4] used both a modified RepReMatch and the discrete moving-knife method to get an $\mathcal{O}(n^{53/54})$-approximation algorithm. However, the algorithm uses demand and XOS queries, whereas RepReMatch needs only the weaker value queries.
- For the submodular MNP, Garg et al. [14] devised a family $(A_\epsilon)_{\epsilon>0}$ of algorithms which are $(4+\epsilon)$-approximative in the symmetric case and $e(n \cdot \max_{i \in \mathcal{A}}\{\eta_i\} + 2 + \epsilon)$-approximative in the asymmetric one.
- Rado valuations are a special class of submodular functions and stem from a generalisation of OXS valuations, in regard to which Garg, Kulkarni and Kulkarni explicitly mentioned a lack of research. Garg, Husic and Vegh [12] developed a five-phase algorithm, which shares ideas with RepReMatch. It achieves an approximation factor of $256e^{3/e} \approx 772$ in case of the symmetric Rado MNP and of $256\gamma^3$ with $\gamma := \max_{i \in \mathcal{A}}\{\eta_i\}/\min_{i \in \mathcal{A}}\{\eta_i\}$ in the asymmetric case. Interestingly, the factor is 16γ in case of the asymmetric additive MNP, so the algorithm outperforms both SMatch and RepReMatch in many instances.

Besides efficiency, fairness is also a property towards which algorithms can be designed, although SMatch and RepReMatch advance nothing in that regard. Apart from devising more algorithms for more valuation functions or with features inspired by reality, it could also be rewarding to look at basic cases. For example, the gap between the hardness of 1.069 [11] and the best currently achievable approximation factor of 1.45 [3] of the symmetric additive MNP has yet to be closed.

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