

Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

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26th July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

What is the issue?

We need to distribute goods amongst recipients *efficiently* and *fairly*.

Where is this encountered?

- industrial procurement
- satellites
- water withdrawal

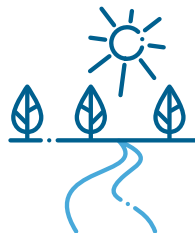
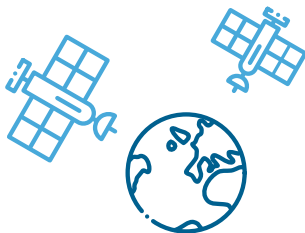
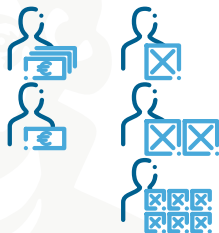


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1

Preliminaries



Setting:

- recipients: set \mathcal{A} of n agents
- goods: set \mathcal{G} of m items

Definition

An *allocation* is a tuple $\mathbf{x} = (\mathbf{x}_i)_{i \in \mathcal{A}}$ of bundles $\mathbf{x}_i \subset \mathcal{G}$ such that each item is element of precisely one bundle.

Item j is *assigned* to agent i if $j \in \mathbf{x}_i$.

But how to measure its efficiency and fairness?

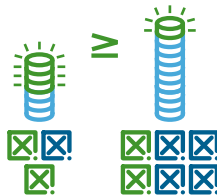
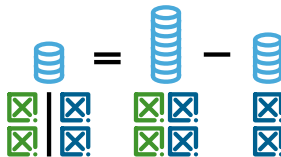
Valuation Functions

Requirements:

- monotonically non-decreasing: $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ if $\mathcal{S}_1 \subset \mathcal{S}_2$
- normalised: $v_i(\emptyset) = 0$

Types:

- additive: $v_i(\mathcal{S}) := \sum_{j \in \mathcal{S}} v_i(j)$
- submodular: $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$
 - diminishing returns



Asymmetric Maximum Nash Social Welfare Problem

Problem

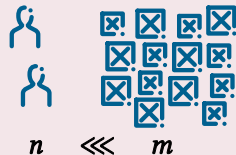
$$x^* \stackrel{!}{=} \arg \max_{x \in X_{\mathcal{A}}(\mathcal{G})} \{\text{NSW}(x)\} \quad \text{with } \text{NSW}(x) := \left(\prod_{i \in \mathcal{A}} v_i(x_i)^{\eta_i} \right)^{1 / \sum_{i \in \mathcal{A}} \eta_i}$$

- $X_{\mathcal{A}}(\mathcal{G})$: all possible allocations
- η_i : agent weight

The NSW strikes a middle ground between efficiency and fairness!

Is there a polynomial-time algorithm with an approximation factor ...

- ... dependent on n ?
- ... independent from m ?





2

RepReMatch



Naïve Approach

Naïve approach:

- repeatedly use maximum matchings
- fails because of missing foresight
 - additive valuations: sort items by valuation
 $\Rightarrow 2n$ -approximation (SMatch)
 - submodular valuations: lowest valuation
approximable only by $\Omega(\sqrt{m/\ln m})$ ⚡



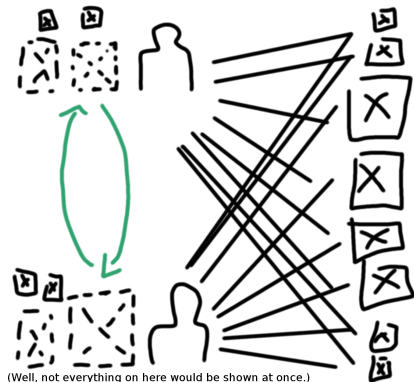
Key Ideas of the Algorithm

We need change the past in three phases:

- Phase I** Assign enough high-value items temporarily.
- Phase II** Assign the remaining items definitely.
- Phase III** Re-assign the items of phase I definitely.

Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$ -approximation under submodular valuations.



The Algorithm

Phase I:

- 1 repeat $\lceil \log_2 n \rceil + 1$ times
 - 1 create bipartite graph $(\mathcal{A}, \mathcal{G}, E)$ with edge weights $\log v_i(j)^{\eta_i}$
 - 2 compute maximum weight matching
 - 3 update bundles \mathbf{x}_i^I & remove assigned items

Phase II:

- 2 repeat until $\mathcal{G} = \emptyset$
 - 1 create bipartite graph $(\mathcal{A}, \mathcal{G}, E)$ with edge weights $\log v_i(\mathbf{x}_i^I \cup \{j\})^{\eta_i}$
 - 2 compute maximum weight matching
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Phase III:

- 3 create bipartite graph $(\mathcal{A}, \bigcup_{i \in \mathcal{A}} \mathbf{x}_i^I, E)$ with edge weights $\log v_i(\mathbf{x}_i^I \cup \{j\})^{\eta_i}$
- 4 compute maximum weight matching
- 5 create bundles \mathbf{x}_i^I

Analysing Phases I & III (1/2)



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Definition

Let $\mathbf{x}_i^* = \{o_i^1, o_i^2, \dots\}$ be an optimal bundle. An item $j \in \mathcal{G}$ is *outstanding* if $v_i(j) \geq v_i(o_i^1)$.



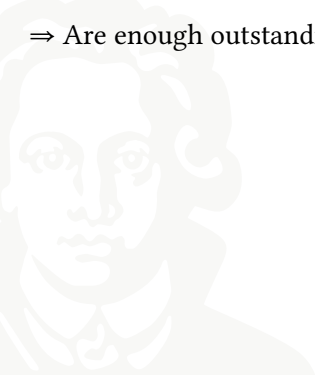
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⇒ Are enough outstanding items reserved?



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Lemma

Each agent can be matched with an outstanding item in phase III.

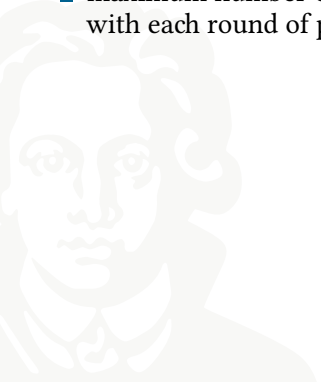


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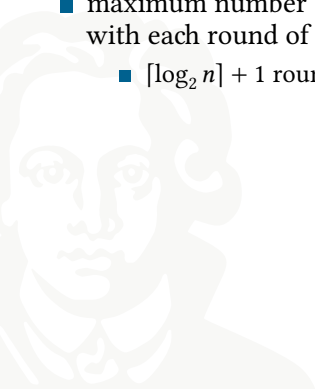


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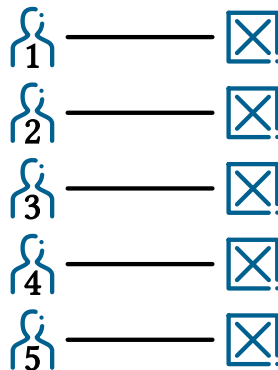
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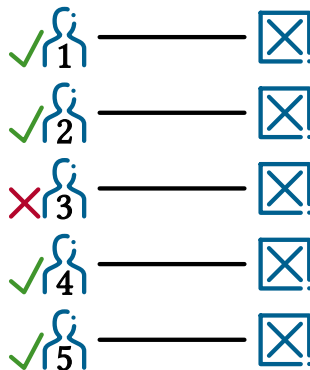
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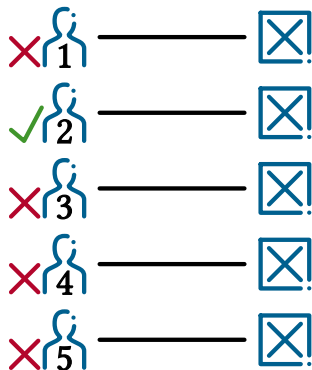
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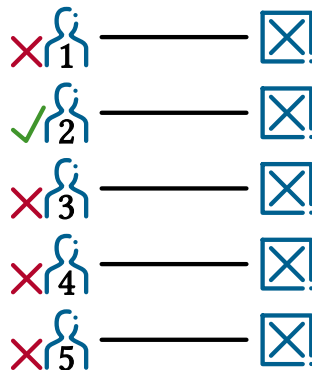
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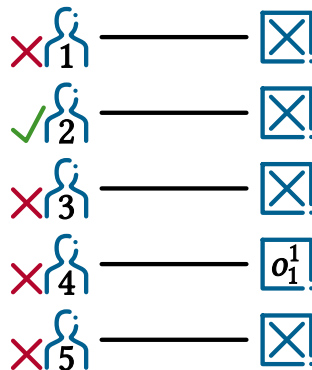
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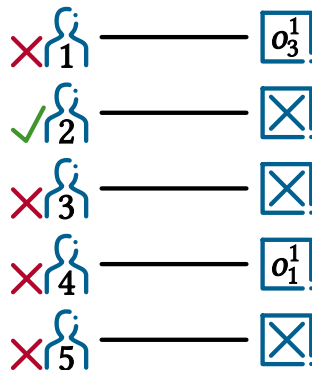
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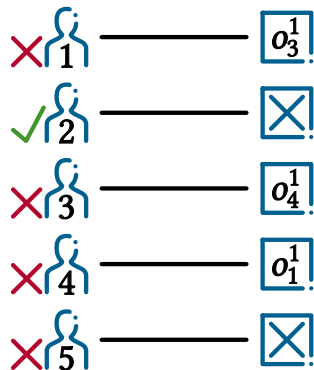
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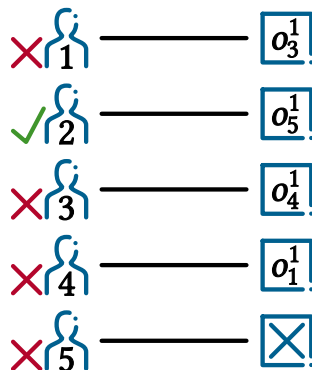
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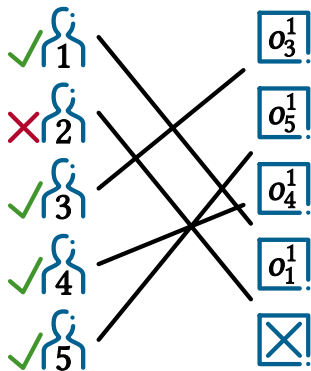
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 - $> n/2$ many agents matched upon release in phase III



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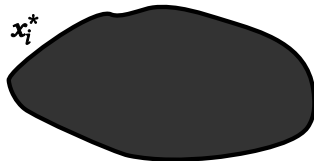
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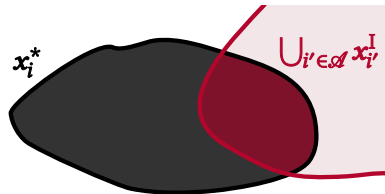
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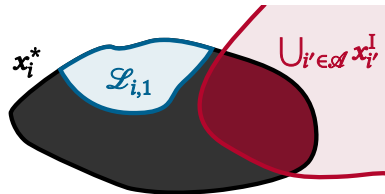
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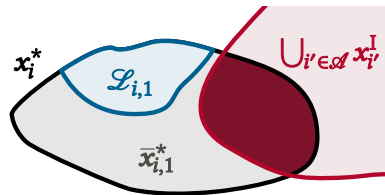
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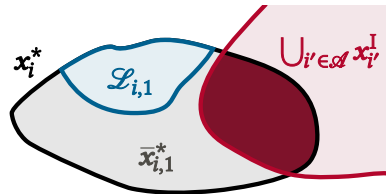
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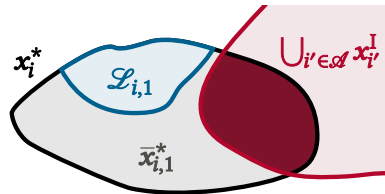
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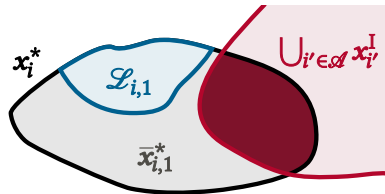
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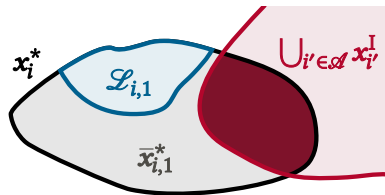
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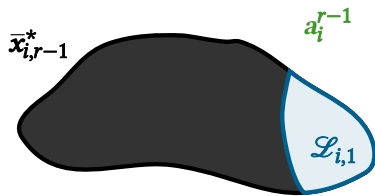
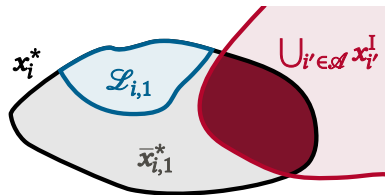
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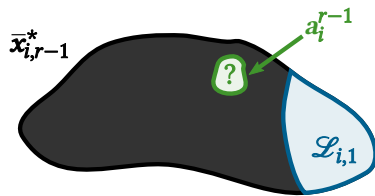
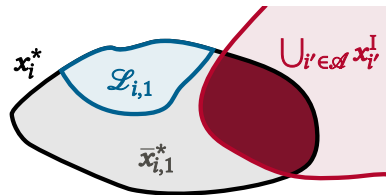
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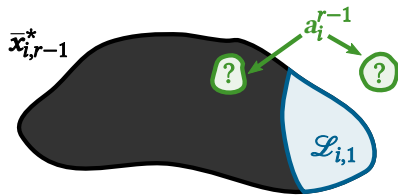
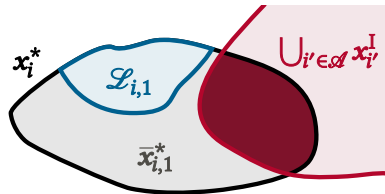
Definition

The set $\mathcal{L}_{i,r}$ of *lost items* is the set of all optimal items $j \in \mathbf{x}_i^*$ assigned to other agents $i' \neq i$ in round r .

Definition

Let $\mathbf{x}_i^{\text{II}} = \{a_i^1, a_i^2, \dots\}$ be the bundle of agent i . The set of *optimal and attainable items* is defined as

$$\bar{\mathbf{x}}_{i,r}^* := \begin{cases} \mathbf{x}_i^* \setminus (\bigcup_{i' \in \mathcal{A}} \mathbf{x}_{i'}^{\text{I}} \cup \mathcal{L}_{i,1}) & \text{in round } r = 1, \\ & \text{in round } r \geq 2. \end{cases}$$



Analysing Phase II (1/2)

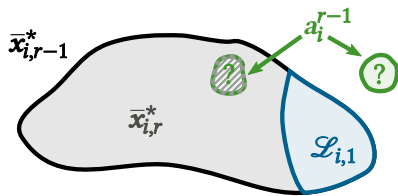
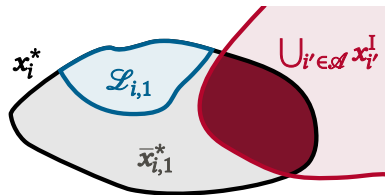
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Analysing Phase II (1/2)

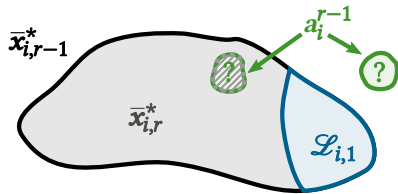
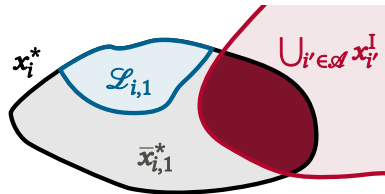
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Analysing Phase II (1/2)

Definition

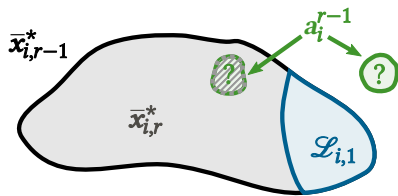
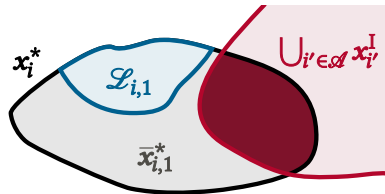
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⇒ What is the valuation of the remaining items?



Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

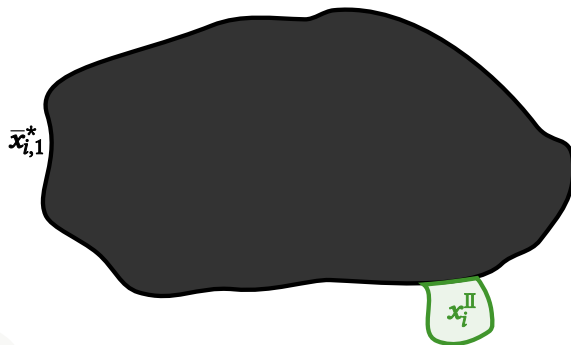


$\bar{x}_{i,1}^*$

Analysing Phase II (2/2)

Lemma

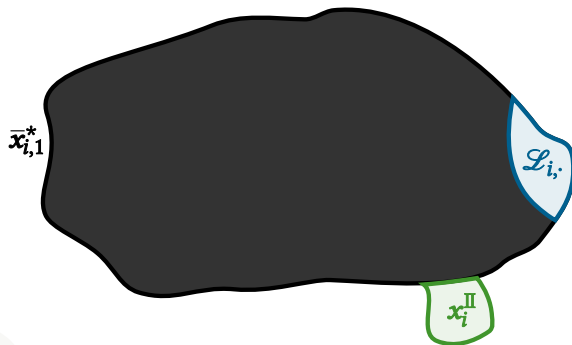
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

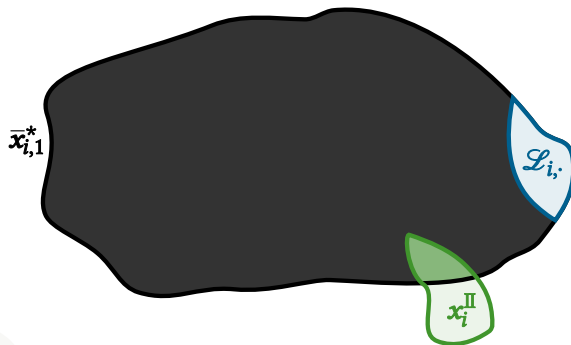
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

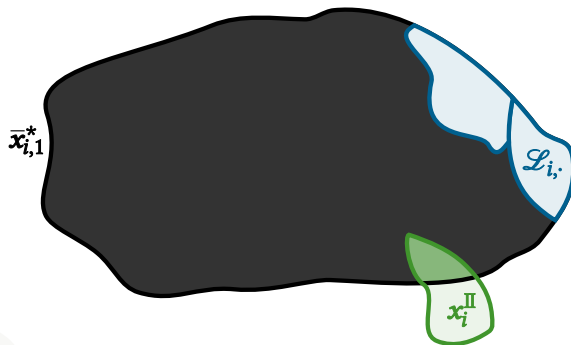
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

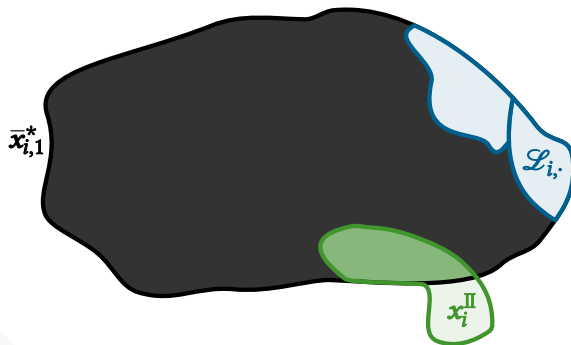
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

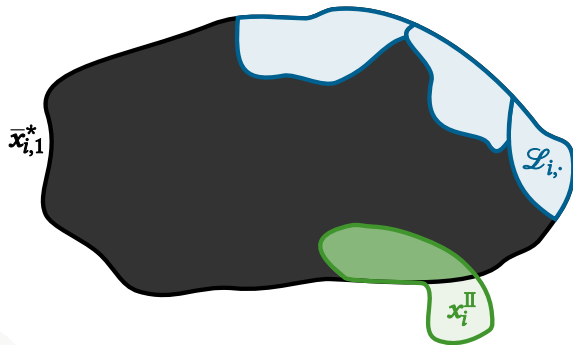
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

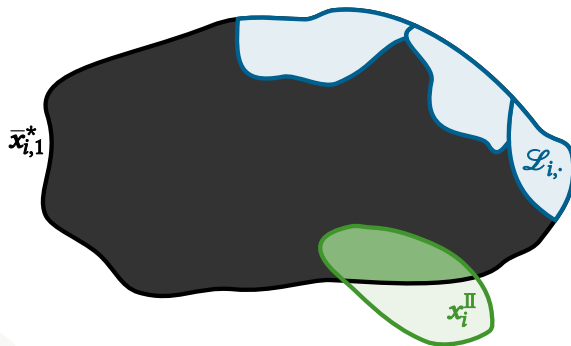
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

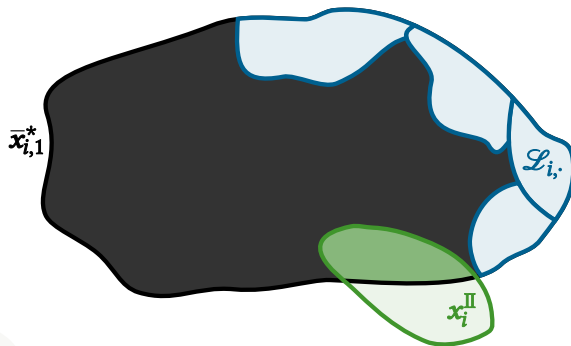
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

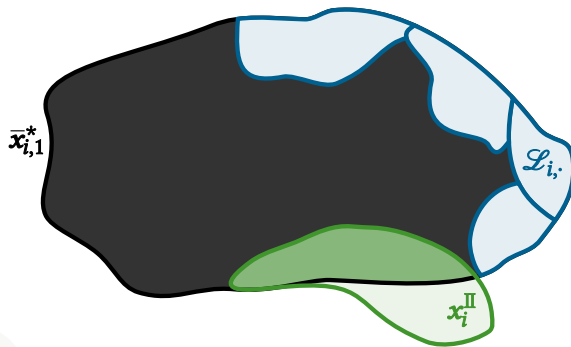
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

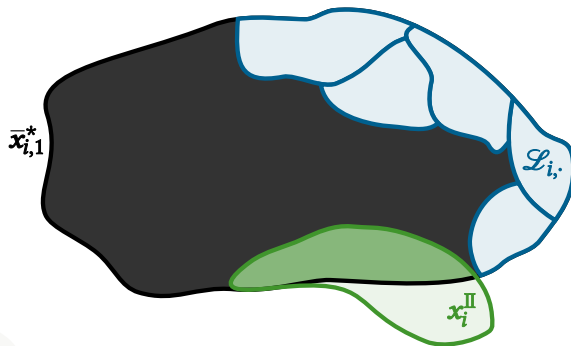
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

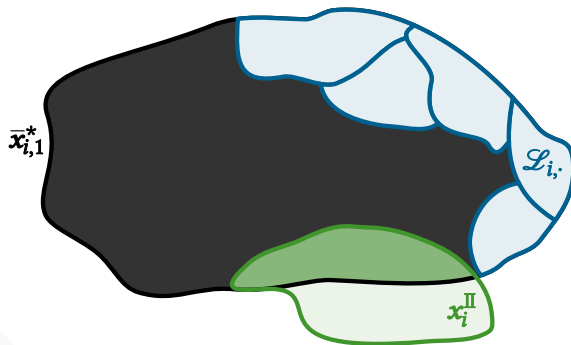
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

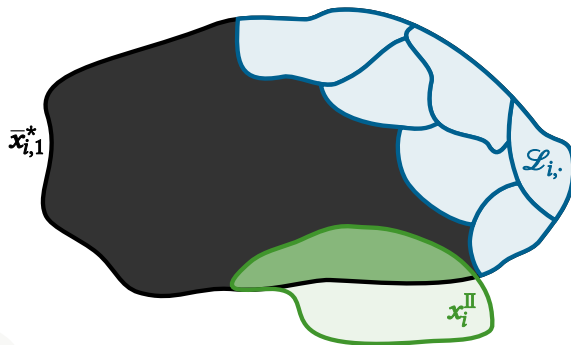
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

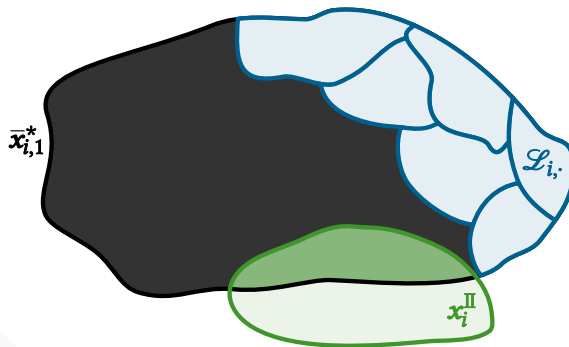
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

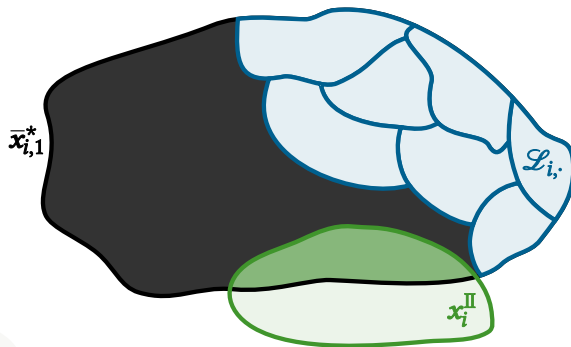
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

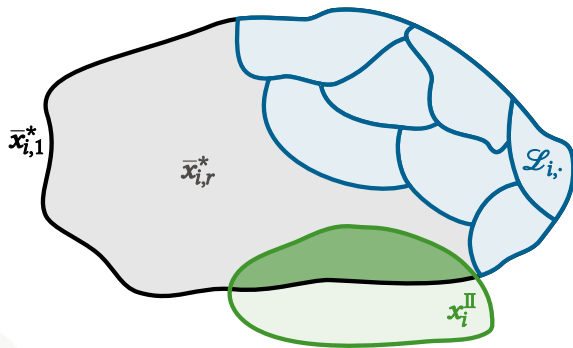
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

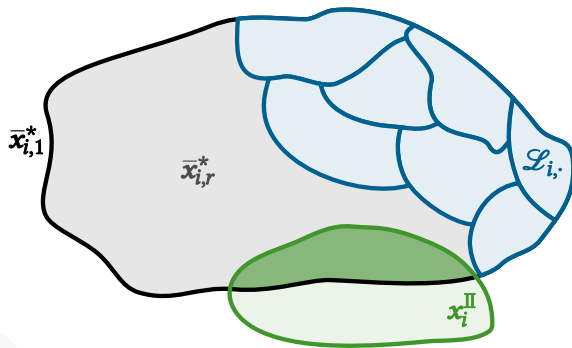
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

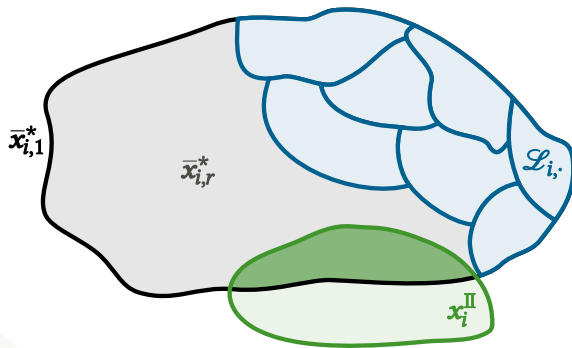
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) - v_i(a_i^1, \dots, a_i^{r-1})$$



Analysing Phase II (2/2)

Lemma

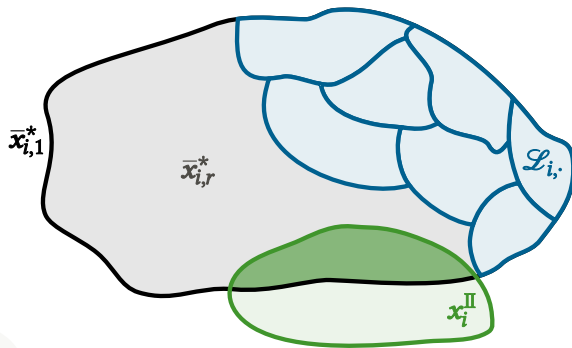
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Analysing Phase II (2/2)

Lemma

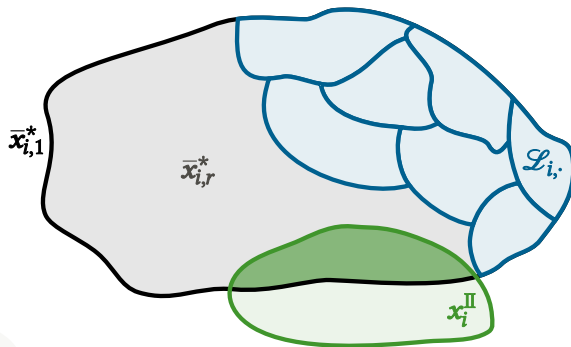
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\})$$



Analysing Phase II (2/2)

Lemma

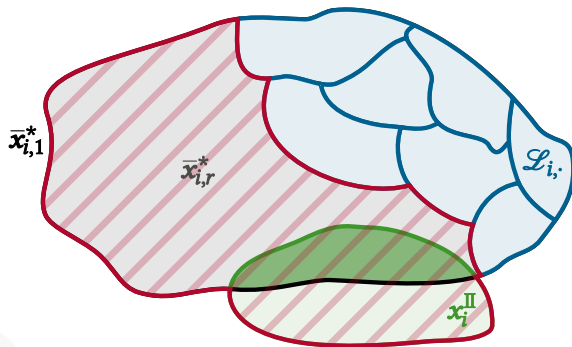
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\})$$



Analysing Phase II (2/2)

Lemma

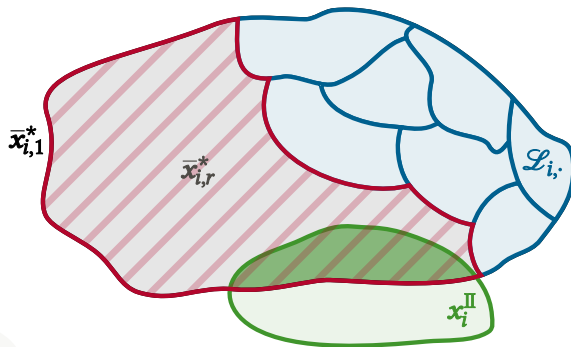
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Analysing Phase II (2/2)

Lemma

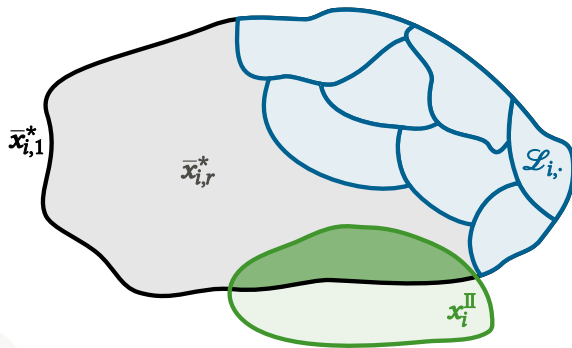
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Analysing Phase II (2/2)

Lemma

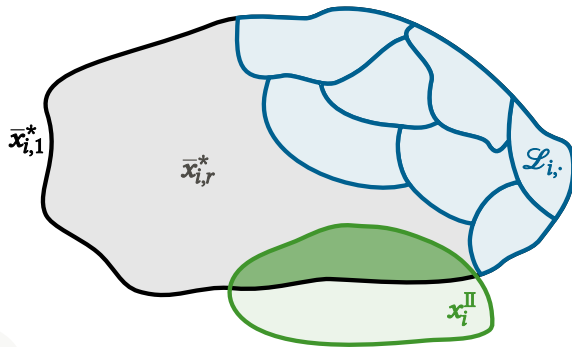
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*)$$



Analysing Phase II (2/2)

Lemma

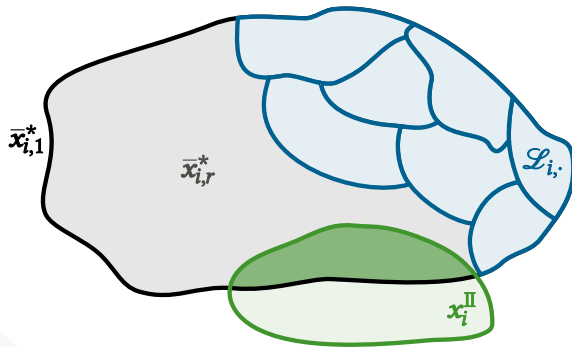
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - v_i(\mathcal{L}_{i,2} \mid a_i^1)$$



Analysing Phase II (2/2)

Lemma

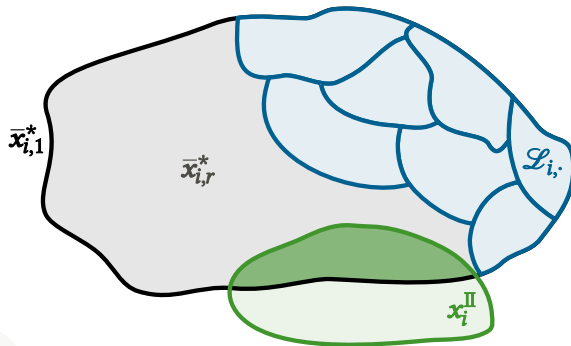
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - v_i(\mathcal{L}_{i,2} \mid a_i^1) - v_i(\mathcal{L}_{i,3} \mid a_i^1, a_i^2)$$



Analysing Phase II (2/2)

Lemma

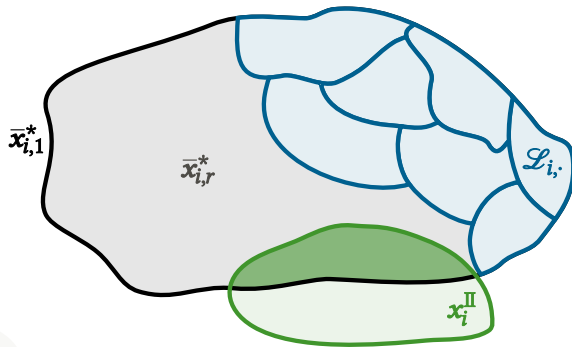
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Analysing Phase II (2/2)

Lemma

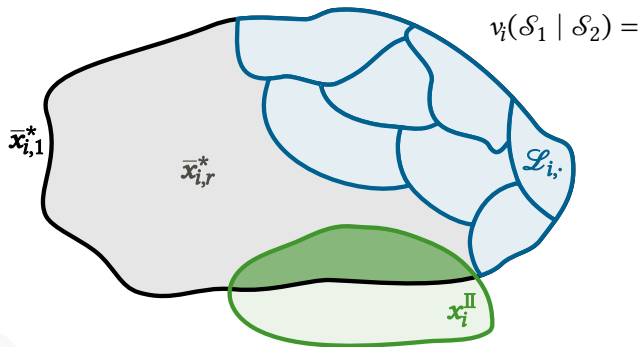
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Analysing Phase II (2/2)

Lemma

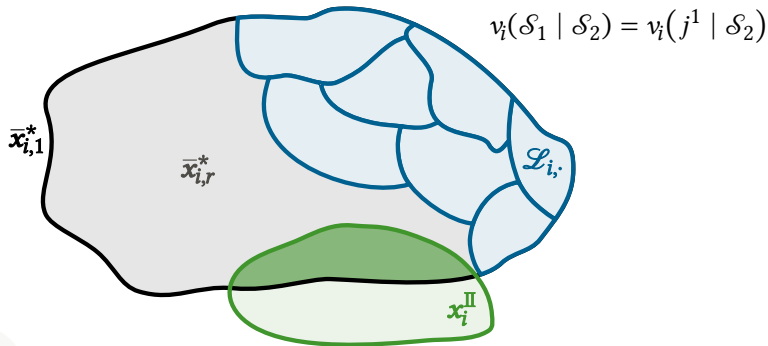
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Analysing Phase II (2/2)

Lemma

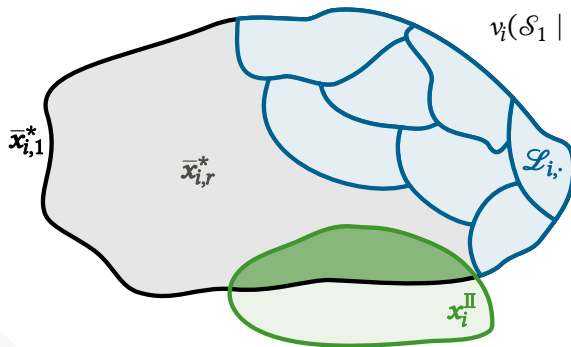
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1})$$



Analysing Phase II (2/2)

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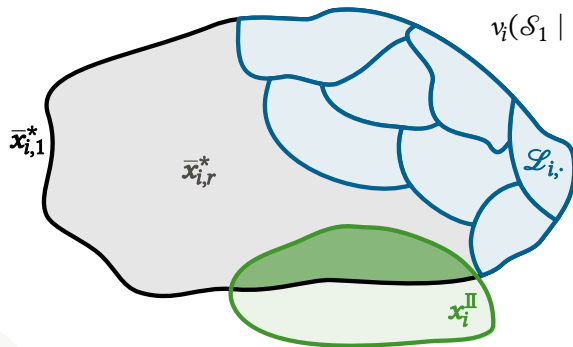


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i(j^1 \mid \mathcal{S}_2) + v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\})$$

Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1})$$

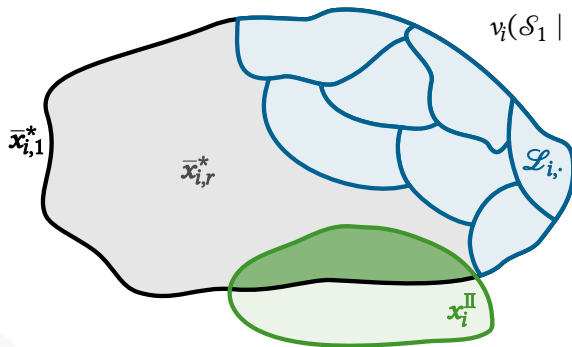


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i(j^1 \mid \mathcal{S}_2) + v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\}) + v_i(j^3 \mid \mathcal{S}_2 \cup \{j^1, j^2\})$$

Analysing Phase II (2/2)

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$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1})$$

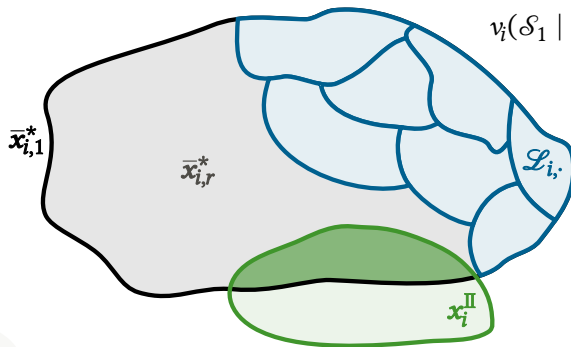


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i(j^1 \mid \mathcal{S}_2) + \\ v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\}) + \\ v_i(j^3 \mid \mathcal{S}_2 \cup \{j^1, j^2\}) + \\ \vdots$$

Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1})$$

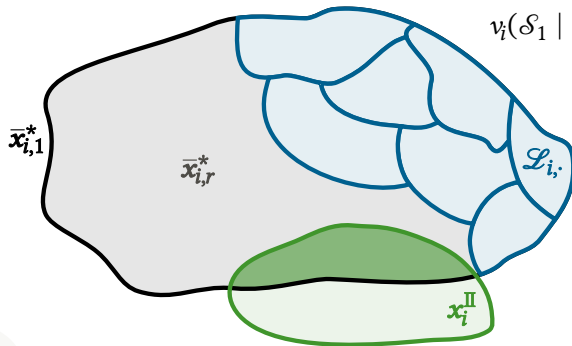


$$\begin{aligned} v_i(\mathcal{S}_1 \mid \mathcal{S}_2) &= v_i(j^1 \mid \mathcal{S}_2) + \\ &v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\}) + \\ &v_i(j^3 \mid \mathcal{S}_2 \cup \{j^1, j^2\}) + \\ &\vdots \end{aligned}$$

Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1})$$



$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq v_i(j^1 \mid \mathcal{S}_2) +$$

$$v_i(j^2 \mid \mathcal{S}_2) +$$

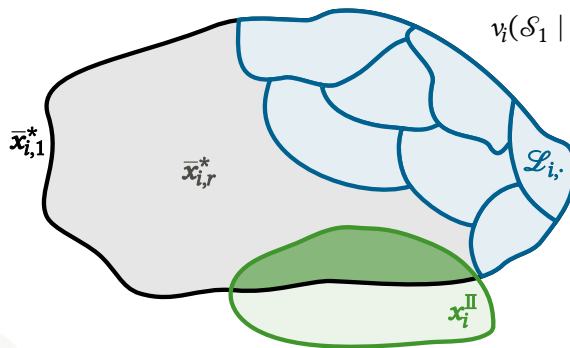
$$v_i(j^3 \mid \mathcal{S}_2) +$$

$$\vdots$$

Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1})$$

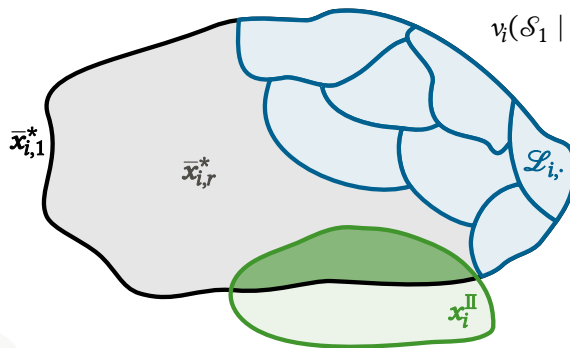


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq \sum_{j \in \mathcal{S}_1} v_i(j \mid \mathcal{S}_2)$$

Analysing Phase II (2/2)

Lemma

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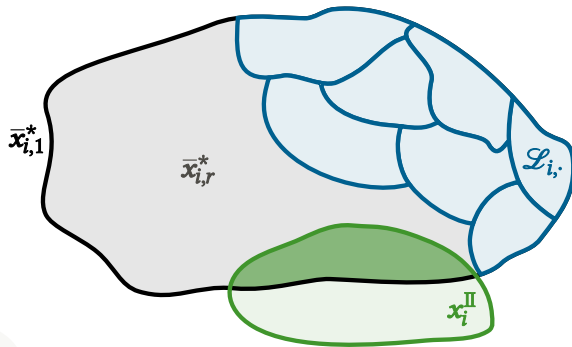


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Analysing Phase II (2/2)

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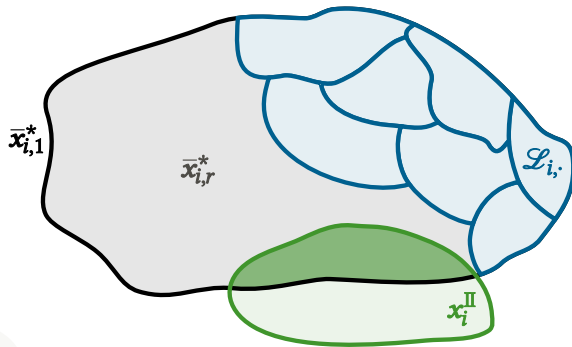
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Analysing Phase II (2/2)

Lemma

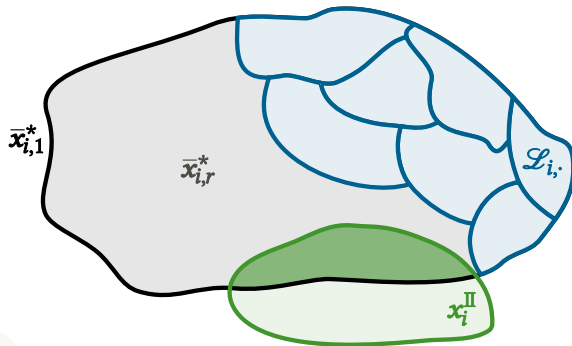
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Analysing Phase II (2/2)

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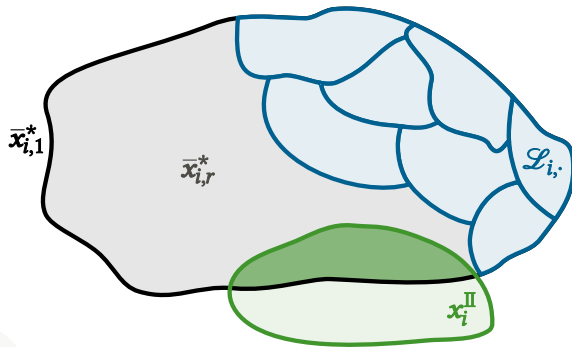
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Analysing Phase II (2/2)

Lemma

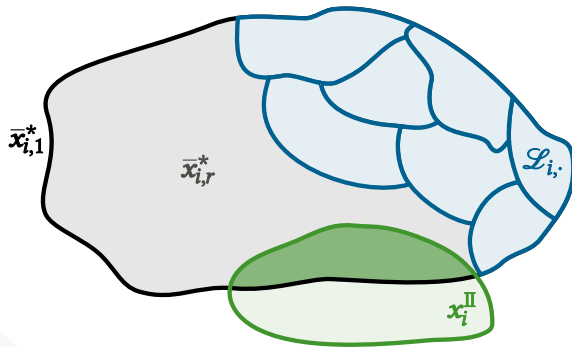
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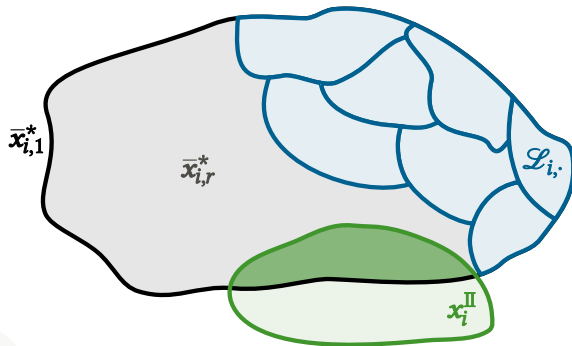
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Analysing Phase II (2/2)

Lemma

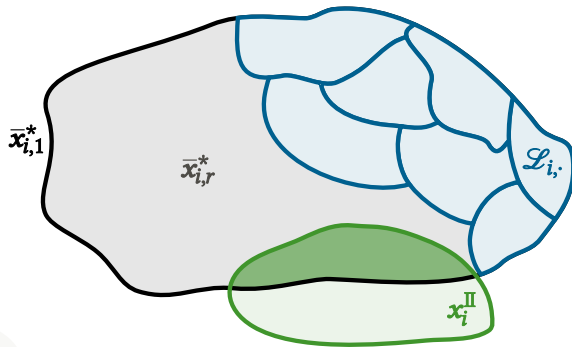
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Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r (n-1) \cdot v_i(a_i^{l-1} \mid a_i^1, \dots, a_i^{l-2})$$



3

Conclusion



Summary & Outlook



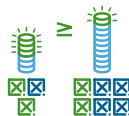
Summary & Outlook

- allocation: partition of items amongst agents



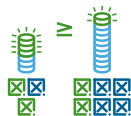
Summary & Outlook

- allocation: partition of items amongst agents
- bundles valued using submodular valuation functions



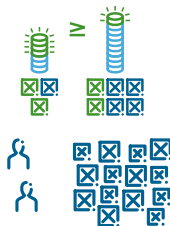
Summary & Outlook

- allocation: partition of items amongst agents
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- Nash social welfare: weighted geometric mean of valuations



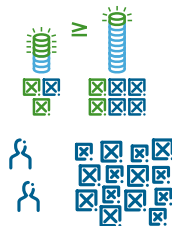
Summary & Outlook

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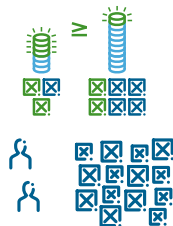
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Summary & Outlook

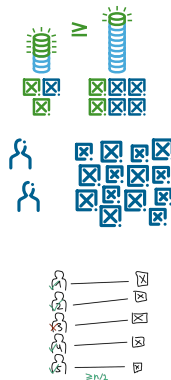
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Phase I finding enough outstanding items

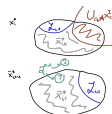
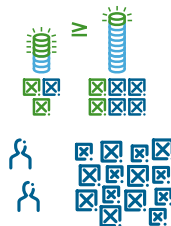


Summary & Outlook

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Phase I finding enough outstanding items

Phase II assigning remaining item



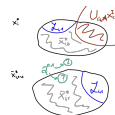
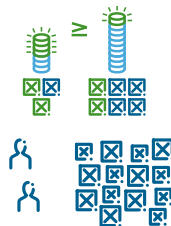
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Phase I finding enough outstanding items

Phase II assigning remaining item

Phase III assigning outstanding items



Summary & Outlook

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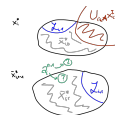
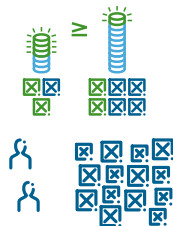
Phase I finding enough outstanding items

Phase II assigning remaining item

Phase III assigning outstanding items

Any Room for Improvement?

Possibly! Lower bound of $\frac{e}{e-1} \approx 1.58$





End of Talk

