Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

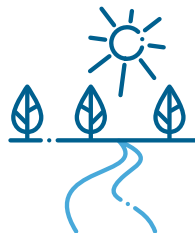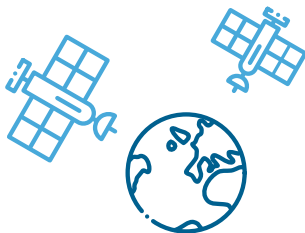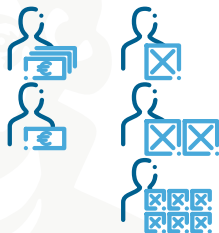Zeno Adrian Weil
Supervised by Dr Giovanna Varricchio

23rd July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

What is the issue?

We need to distribute goods amongst recipients *efficiently* and *fairly*.

Where is this encountered?

- industrial procurement
- satellites
- water withdrawal

Table of Contents

Preliminaries

1

Allocations

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Setting:

- recipients: set $\mathscr{A}$ of n agents
- goods: set $\mathscr{G}$ of m items

Definition

An *allocation* is a tuple $x = (x_i)_{i \in \mathscr{A}}$ of bundles $x_i \subset \mathscr{G}$ such that each item is element of precisely one bundle.
Item j is *assigned* to agent i if $j \in x_i$.
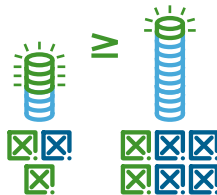
But how to measure its efficiency and fairness?

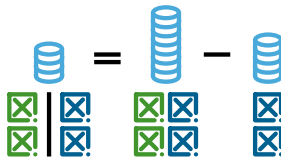# Valuation Functions

Requirements:

- monotonically non-decreasing: $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ if $\mathcal{S}_1 \subset \mathcal{S}_2$
- normalised: $v_i(\emptyset) = 0$

Types:

- additive: $v_i(\mathcal{S}) := \sum_{j \in \mathcal{S}} v_i(j)$
- submodular: $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$
  - diminishing returns

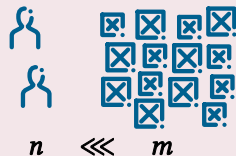# Asymmetric Maximum Nash Social Welfare Problem

Problem

$$x^* \overset{!}{=} \underset{x \in X_{\mathscr{A}}(\mathscr{G})}{\arg\max}\{\text{NSW}(x)\} \quad \text{with NSW}(x) := \Big(\prod_{i \in \mathscr{A}} v_i(x_i)^{\eta_i}\Big)^{1/\sum_{i \in \mathscr{A}} \eta_i}$$

- $X_{\mathscr{A}}(\mathscr{G})$: all possible allocations
- η_i: agent weight

The NSW strikes a middle ground between efficiency and fairness!

Is there a polynomial-time algorithm with an approximation factor ...

- ... dependent on n?
- ... independent from m?

$n \lll m$

RepReMatch

Naïve Approach

Naïve approach:

- repeatedly use maximum matchings
- fails because of missing foresight
  - additive valuations: sort items by valuation
    $\Rightarrow 2n$-approximation (SMatch)
  - submodular valuations: lowest valuation
    approximable only by $\Omega(\sqrt{m/\ln m})$ ⚡
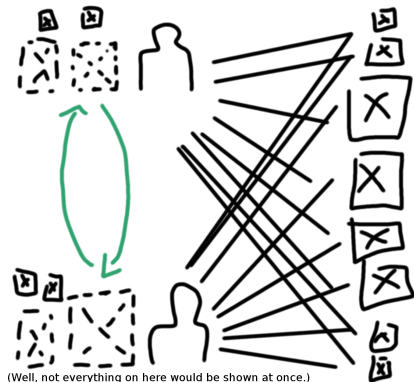
Key Ideas of the Algorithm

We need change the past in three phases:

Phase I Assign enough high-value items temporarily.

Phase II Assign the remaining items definitely.

Phase III Re-assign the items of phase I definitely.

Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$-approximation under submodular valuations.

(Well, not everything on here would be shown at once.)

The Algorithm

Phase I:

1. repeat $\lceil \log_2 n \rceil + 1$ times
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\log v_i(j)^{\eta_i}$
   2. compute maximum weight matching
   3. update bundles x_i^{I} & remove assigned items

Phase II:

2. repeat until $\mathscr{G} = \varnothing$
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\log v_i(x_i^{\mathrm{II}} \cup \{j\})^{\eta_i}$
   2. compute maximum weight matching
   3. update bundles x_i^{II} & remove assigned items

Phase III:

3. create bipartite graph $\left(\mathscr{A}, \bigcup_{i \in \mathscr{A}} x_i^{\mathrm{I}}, E\right)$ with edge weights $\log v_i(x_i^{\mathrm{II}} \cup \{j\})^{\eta_i}$
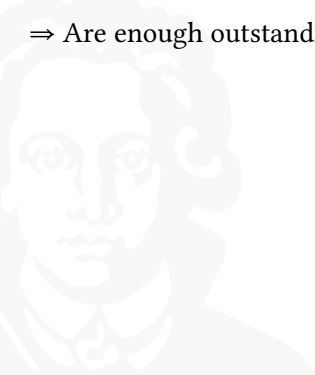4. compute maximum weight matching
5. create bundles x_i^{III}

Analysing Phases I & III (1/2)

Phase I reserves 'high-value' items. But what qualifies as 'high-value'?

Definition

Let $x_i^* = \{o_i^1, o_i^2, \dots\}$ be an optimal bundle. An item $j \in \mathcal{G}$ is *outstanding* if $v_i(j) \geq v_i(o_i^1)$.

$\Rightarrow$ Are enough outstanding items reserved?

Analysing Phases I & III (2/2)

Lemma

Each agent can be matched with an outstanding item in phase III.

- maximum number of unmatched agents halved with each round of phase I
    - $\lceil \log_2 n \rceil + 1$ rounds in phase I are enough
- induction on number of rounds in phase I

Base Case: In round 1 of phase I, either

- $\geq n/2$ many agents matched with an outstanding item
- $< n/2$ many agents matched with an outstanding item
    - $> n/2$ many items o_i^1 assigned to someone else
    - $> n/2$ many agents matched upon release in phase III

Analysing Phase II (1/2)

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The set $\mathscr{L}_{i,r}$ of *lost items* is the set of all optimal items $j \in \boldsymbol{x}_i^*$

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Let $\boldsymbol{x}_i^{\mathrm{II}} = \{a_i^1, a_i^2, \dots\}$ be the bundle of agent i.

$$\Bigg\{$$

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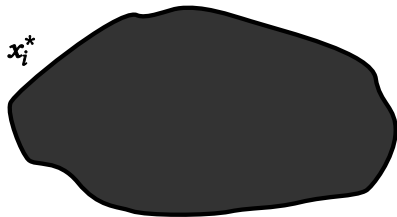
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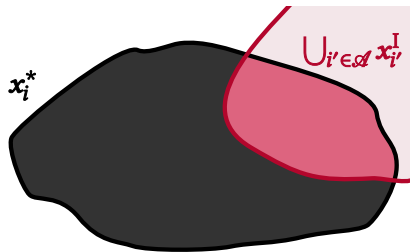

$$x_i^*$$

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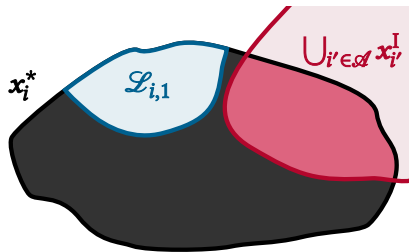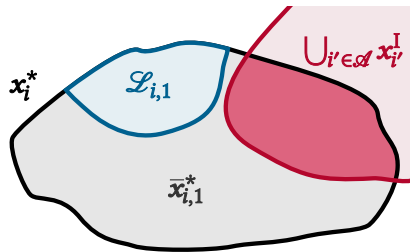
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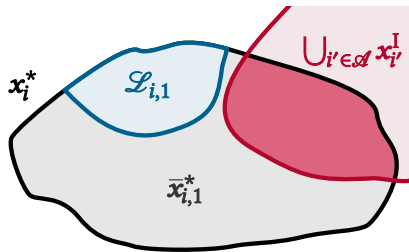
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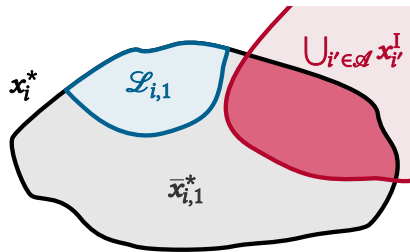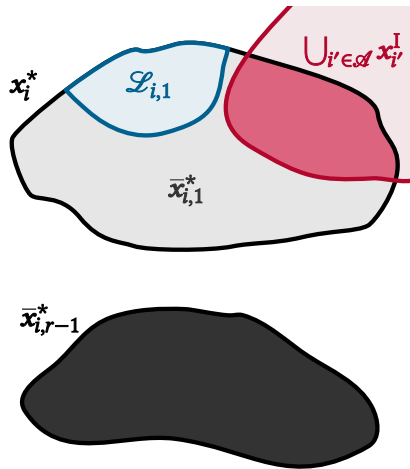
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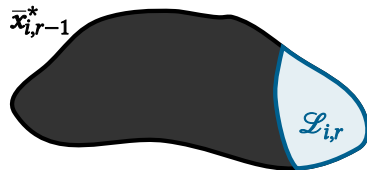
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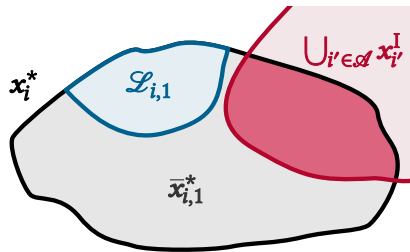
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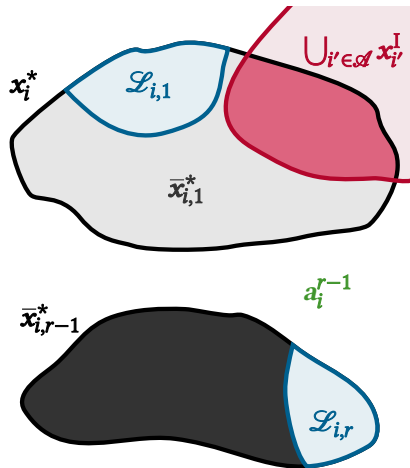
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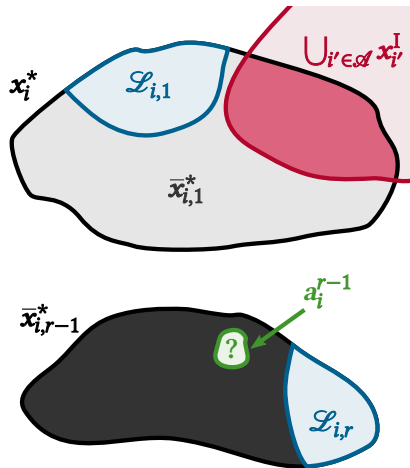
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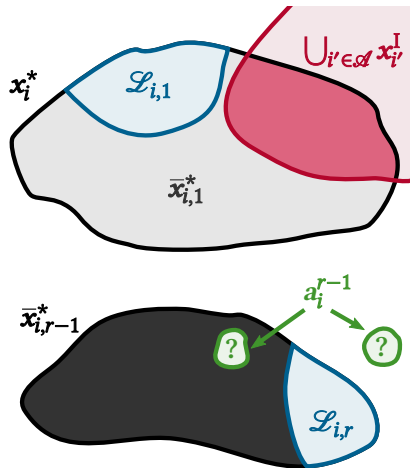
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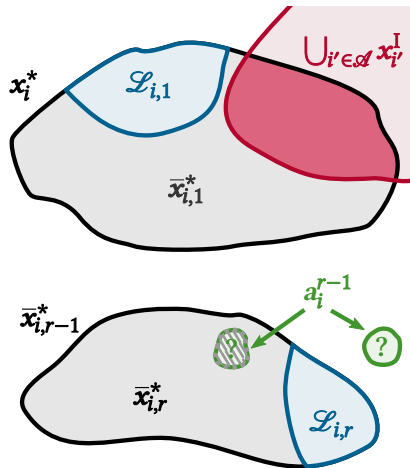
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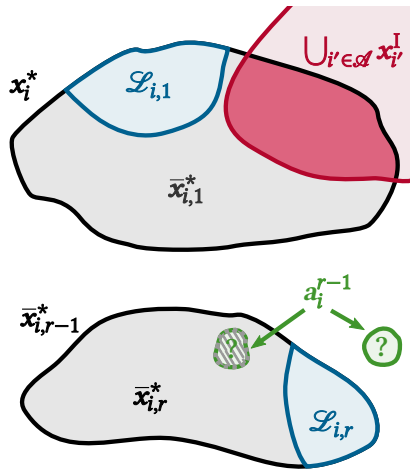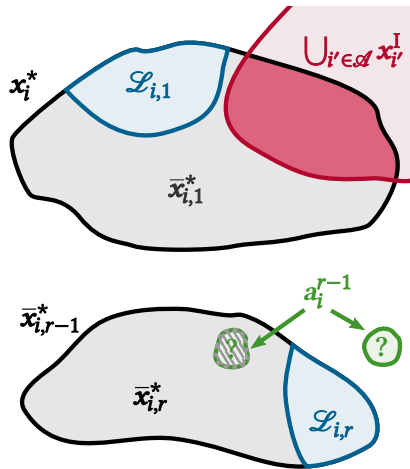
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$\Rightarrow$ What is the valuation of the remaining items?

$\overline{x}_{i,1}^*$

Lemma

$$v_i(\overline{x}_{i,r}^* \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^*) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
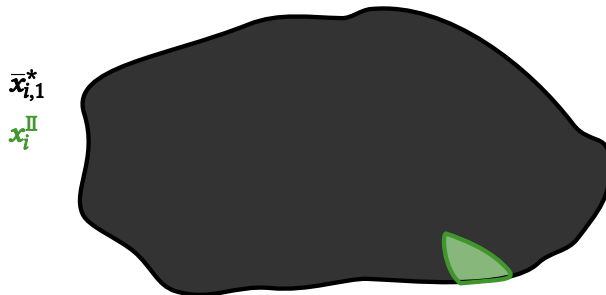
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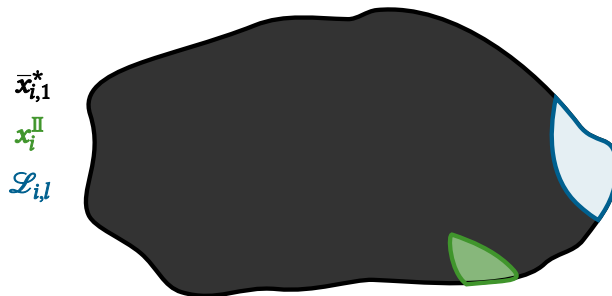
$\overline{x}^*_{i,1}$

x^{II}_i

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}^*_{i,r} \mid a^1_i, \ldots, a^{r-1}_i) \geq v_i(\overline{x}^*_{i,r}) - v_i(a^1_i, \ldots, a^{r-1}_i) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a^{l-1}_i \mid a^1_i, \ldots, a^{l-2}_i)$$

Analysing Phase II (2/2)

$\overline{x}_{i,1}^{*}$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
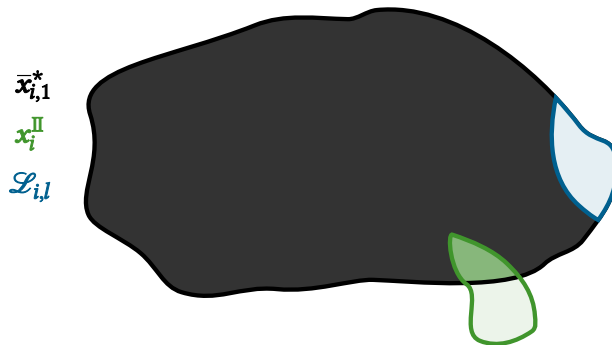
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$\overline{x}_{i,1}^{*}$

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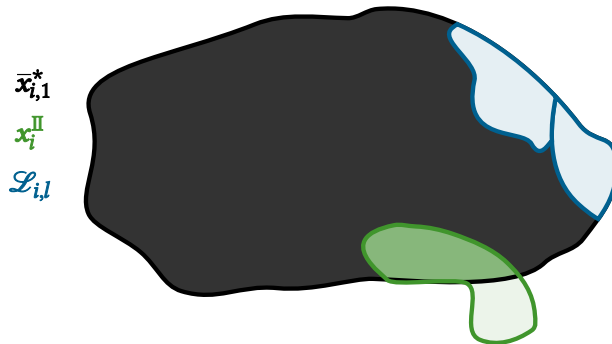
$\overline{\boldsymbol{x}}_{i,1}^{*}$

$\boldsymbol{x}_i^{\mathrm{II}}$

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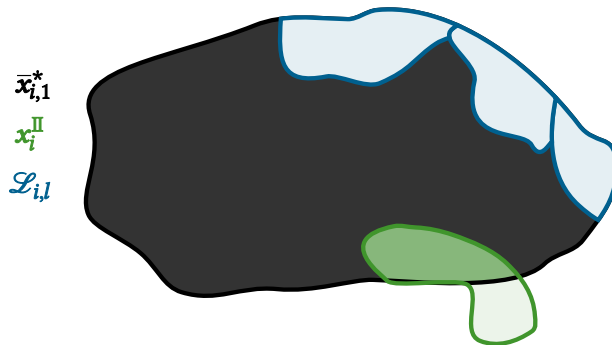
$\overline{x}_{i,1}^*$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

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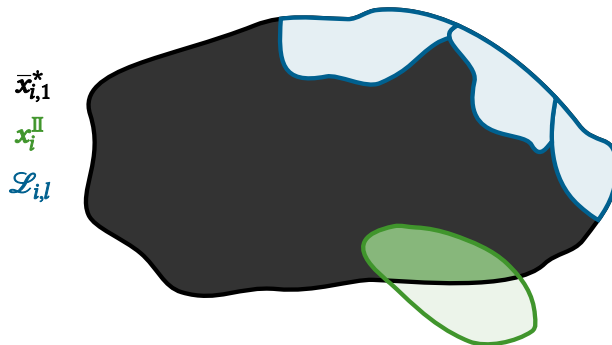
Analysing Phase II (2/2)

$\overline{x}_{i,1}^{*}$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \dots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \dots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \dots, a_i^{l-2})$$
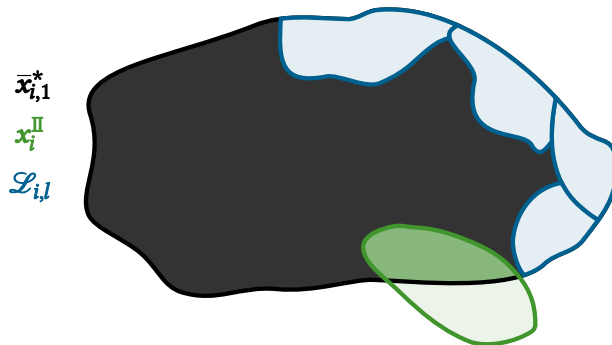
$\overline{x}_{i,1}^*$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^*) - v_i(a_i^1, \dots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \dots, a_i^{l-2})$$
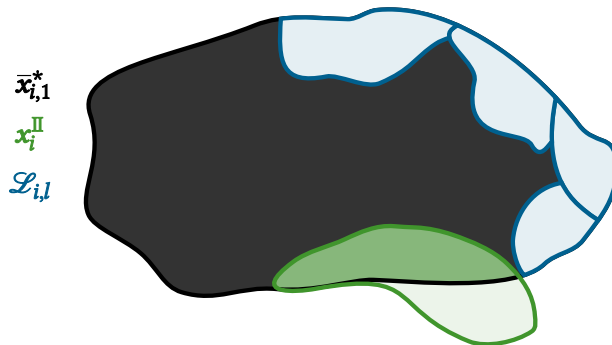
$\overline{\boldsymbol{x}}^*_{i,1}$

$\boldsymbol{x}^{\mathrm{II}}_i$

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}^*_{i,r} \mid a^1_i, \ldots, a^{r-1}_i) \geq v_i(\overline{x}^*_{i,r}) - v_i(a^1_i, \ldots, a^{r-1}_i) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a^{l-1}_i \mid a^1_i, \ldots, a^{l-2}_i)$$

$\overline{\boldsymbol{x}}_{i,1}^{*}$

$\boldsymbol{x}_i^{\mathrm{II}}$

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
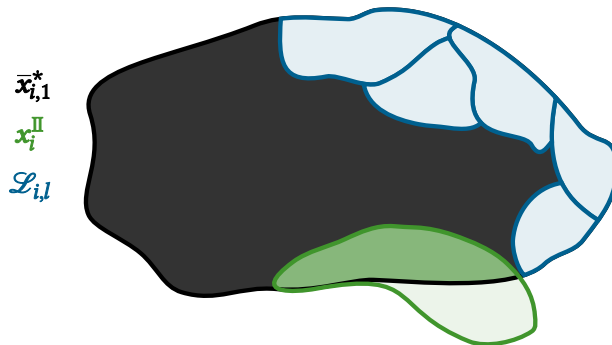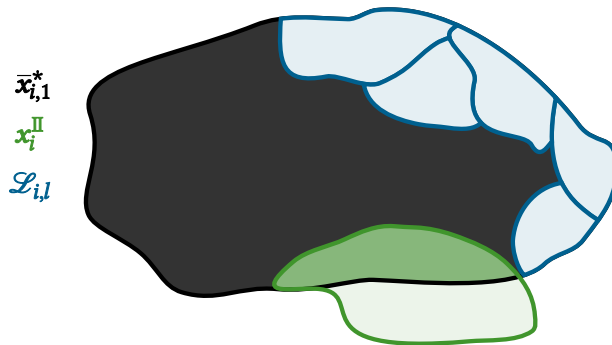
Analysing Phase II (2/2)

$\overline{x}_{i,1}^{*}$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \dots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \dots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \dots, a_i^{l-2})$$
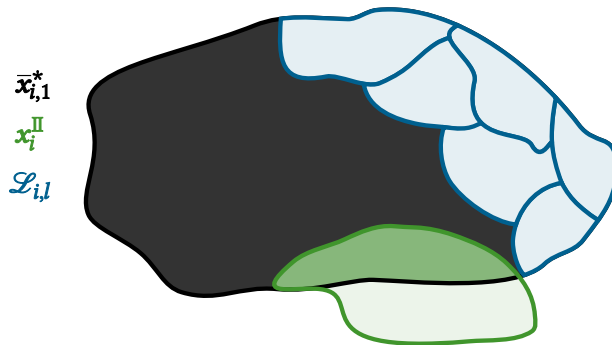
Analysing Phase II (2/2)

$\overline{x}_{i,1}^{*}$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
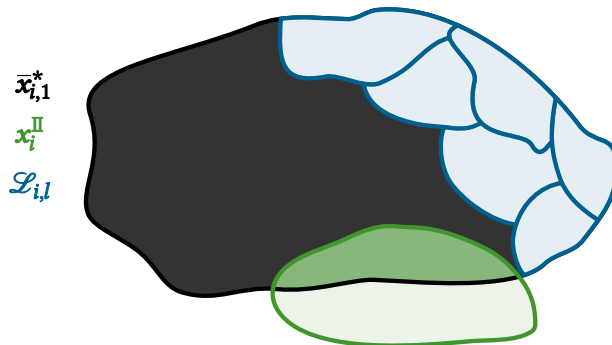
$\overline{x}_{i,1}^{*}$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^* \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^*) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
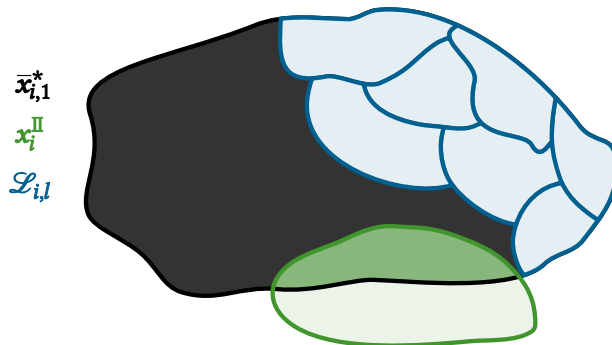
Analysing Phase II (2/2)

$\overline{x}_{i,1}^{*}$

x_i^{II}

$\mathscr{L}_{i,l}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
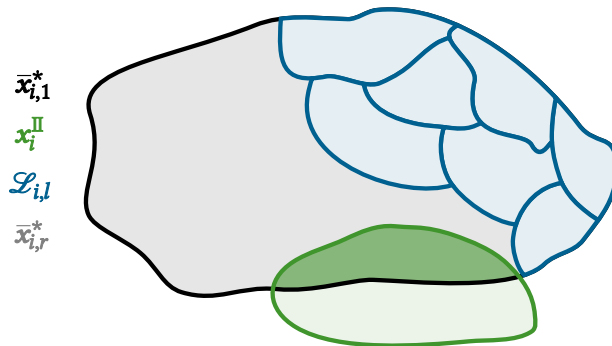
Analysing Phase II (2/2)

$\overline{\boldsymbol{x}}_{i,1}^{*}$

$\boldsymbol{x}_i^{\mathrm{II}}$

$\mathscr{L}_{i,l}$

$\overline{\boldsymbol{x}}_{i,r}^{*}$

Lemma

$$v_i(\overline{x}_{i,r}^{*} \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\overline{x}_{i,r}^{*}) - v_i(a_i^1, \ldots, a_i^{r-1}) - \sum_{l=2}^{r} |\mathscr{L}_{i,l}| \cdot v_i(a_i^{l-1} \mid a_i^1, \ldots, a_i^{l-2})$$
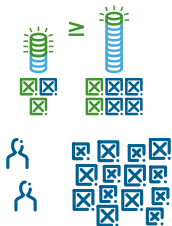
3

Conclusion

Summary & Outlook

- allocation: partition of items amongst agents
- bundles valued using submodular valuation functions
  - diminishing returns
- Nash social welfare: weighted geometric mean of valuations
- approximation factor independent from m?
- simple, repeated matching fails because of missing foresight
- RepReMatch: $2n(\log n + 3)$-approximative

   Phase I finding enough outstanding items
  Phase II assigning remaining item
  Phase III assigning outstanding items

Any Room for Improvement?

Possibly! Lower bound of $\frac{e}{e-1} \approx 1.58$

End of Talk