Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

Zeno Adrian Weil
Supervised by Dr Giovanna Varricchio

28th July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

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What is the issue?

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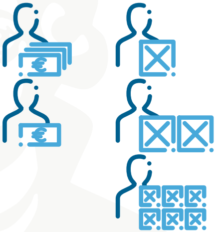

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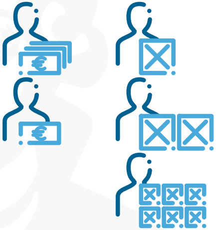

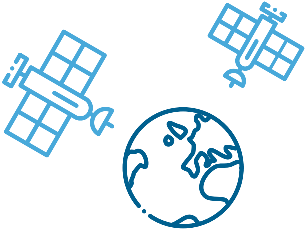

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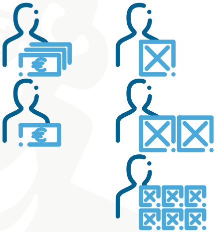

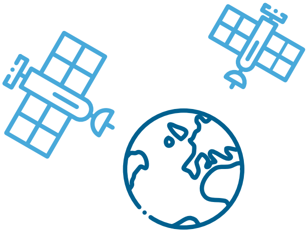

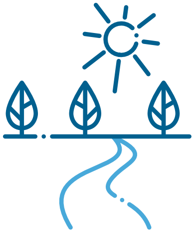

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But how to measure its efficiency and fairness?

Valuation Functions

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- **monotonically non-decreasing:** $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ if $\mathcal{S}_1 \subset \mathcal{S}_2$

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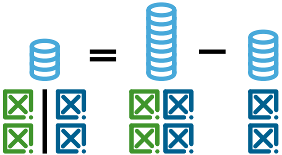

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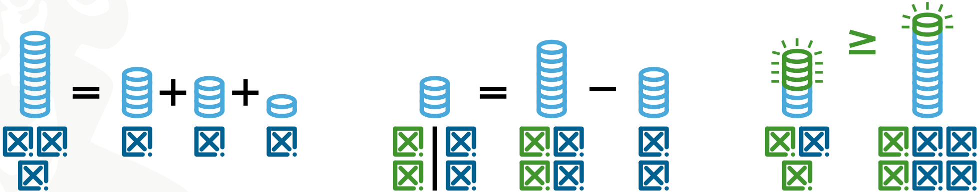
  - diminishing returns



Asymmetric Maximum Nash Social Welfare Problem

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Problem

$$\boldsymbol{x}^* \stackrel{!}{=} \underset{\boldsymbol{x} \in \boldsymbol{X}_{\mathscr{A}}(\mathscr{G})}{\arg\max} \{\mathrm{NSW}(\boldsymbol{x})\}$$

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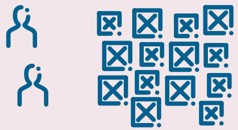

$n \lll m$

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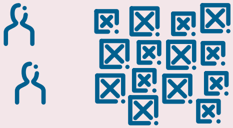

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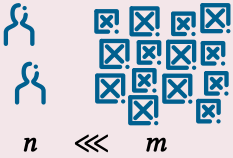
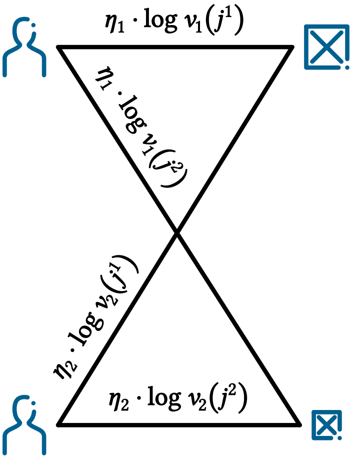


2 RepReMatch

Naïve Approach

Naïve Approach

Greedy algorithm:



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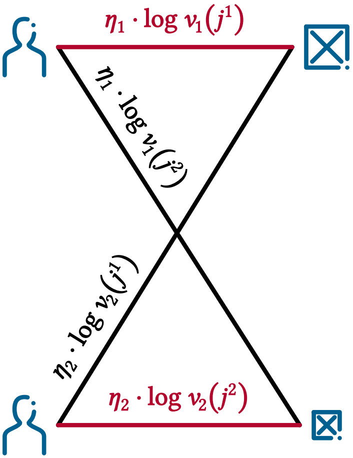

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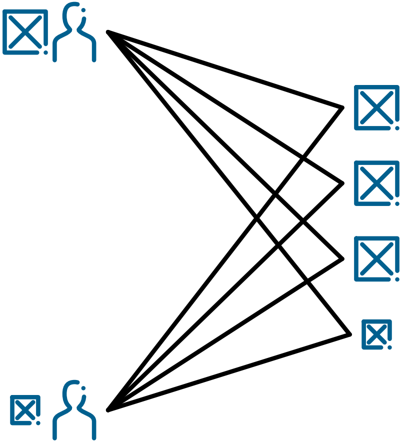
- repeatedly use maximum weight matchings



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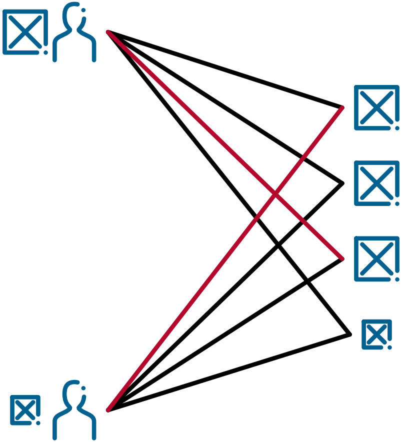

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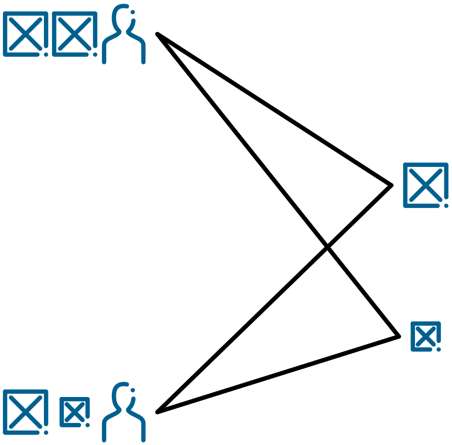

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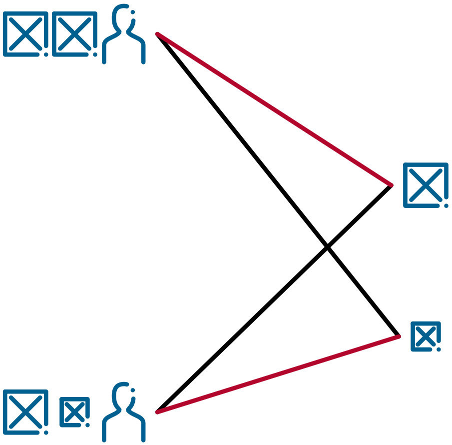

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Key Ideas of the Algorithm

We need change the past in three phases:

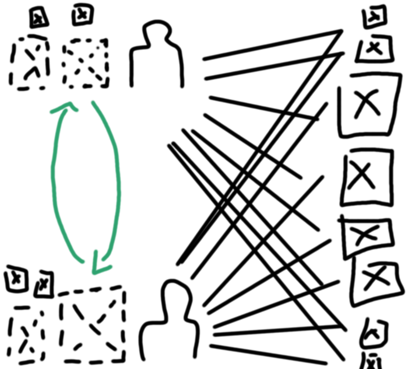
Phase I Assign enough high-value items temporarily.

Phase II Assign the remaining items definitely.

Phase III Re-assign the items of phase I definitely.

Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$-approximation under submodular valuations.



(Well, not everything on here would be shown at once.)

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1. repeat $\lceil \log_2 n \rceil + 1$ times

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4. compute maximum weight matching
5. create bundles $\boldsymbol{x}_i^{\mathrm{III}}$

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$\Rightarrow$ Are enough outstanding items reserved?

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Lemma

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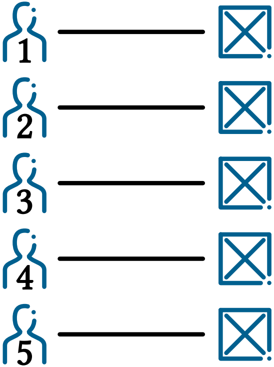
Base Case: In round 1 of phase I, either



Analysing Phases I & III (2/2)

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Each agent can be matched with an outstanding item in phase III.

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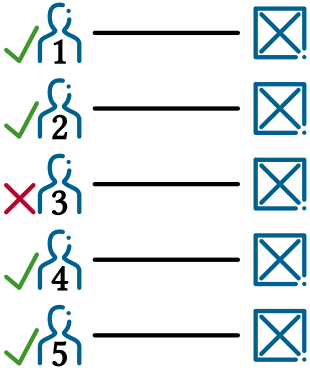

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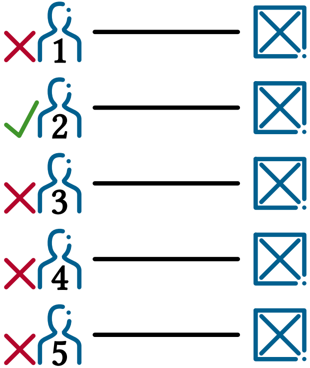

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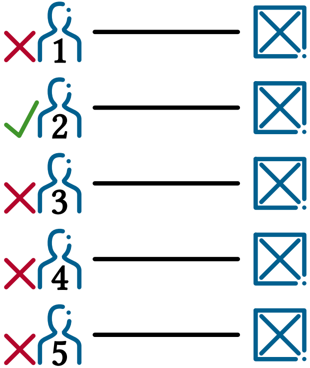

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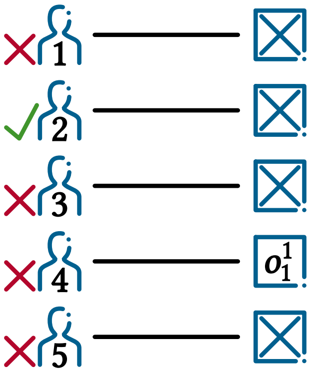

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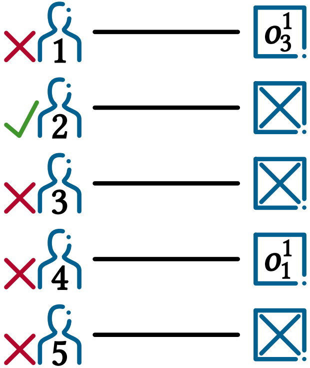

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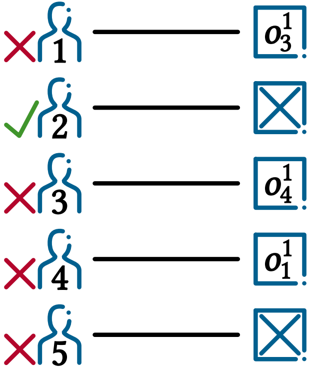

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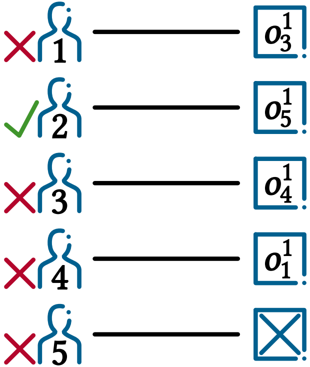

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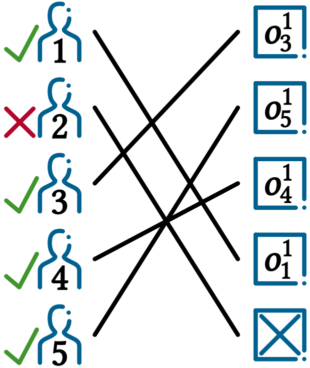

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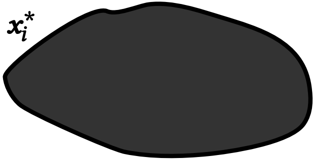

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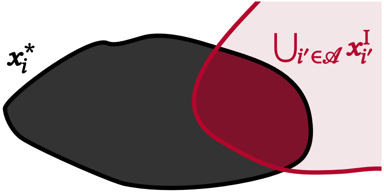

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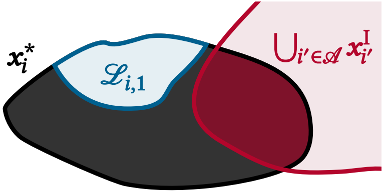

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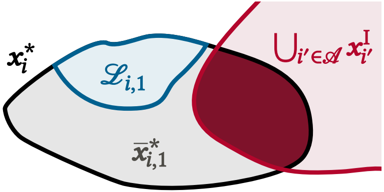

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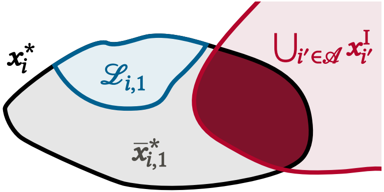

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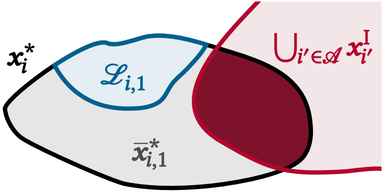

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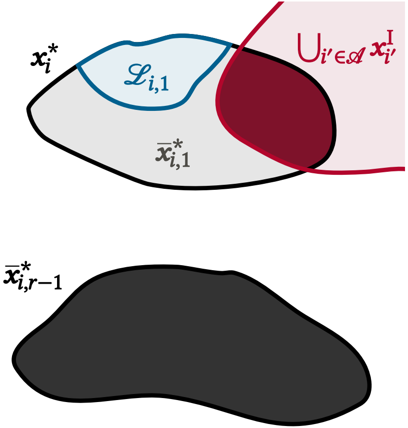

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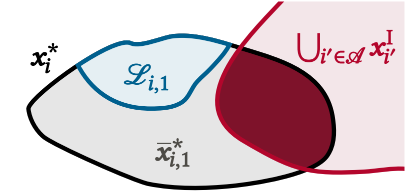

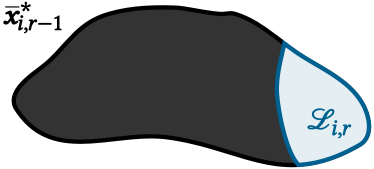

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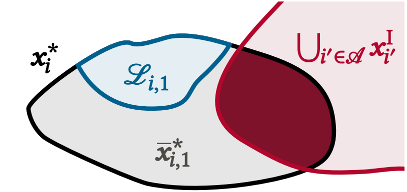

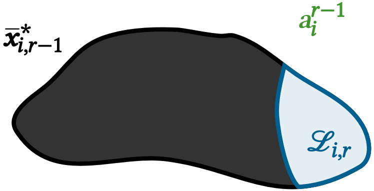

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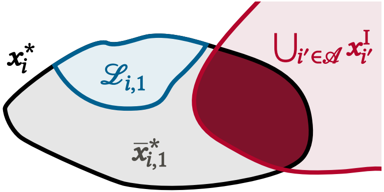

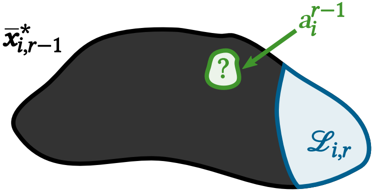

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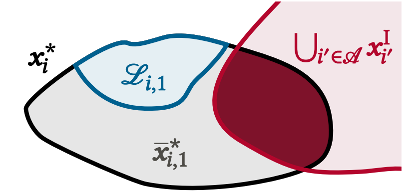

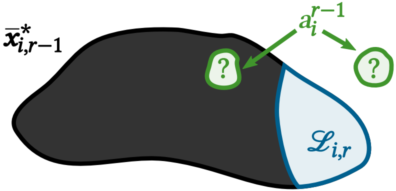

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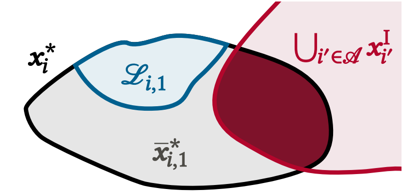

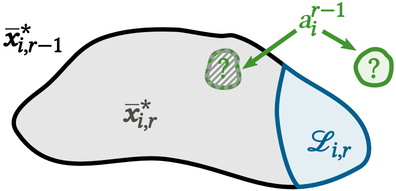

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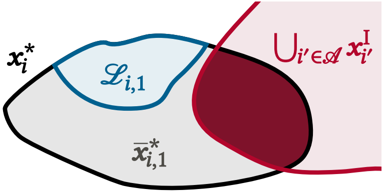

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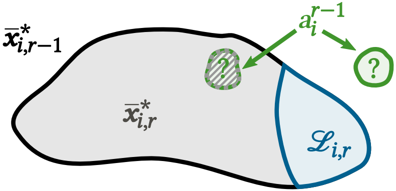

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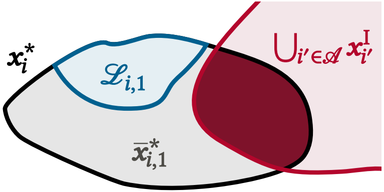
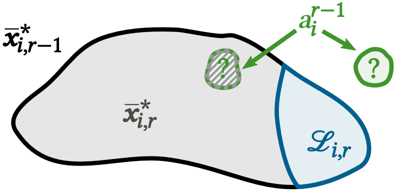
$\Rightarrow$ What is the valuation of the remaining items?





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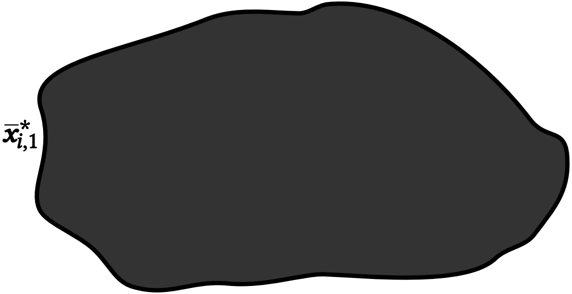

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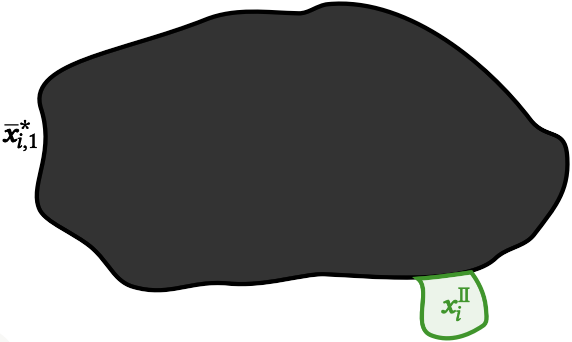

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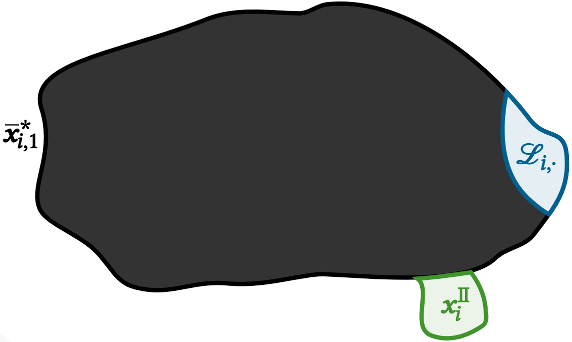

Analysing Phase II (2/2)

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Lemma

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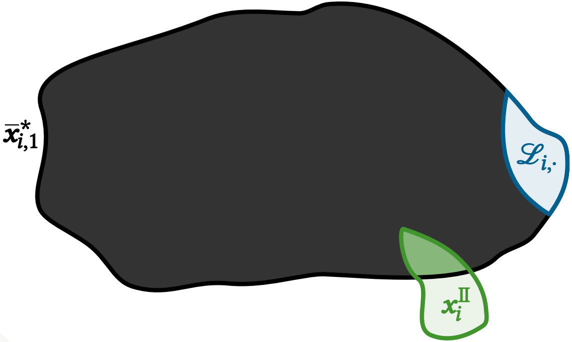

Analysing Phase II (2/2)

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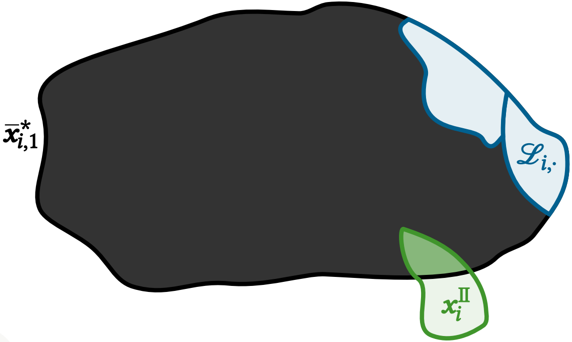

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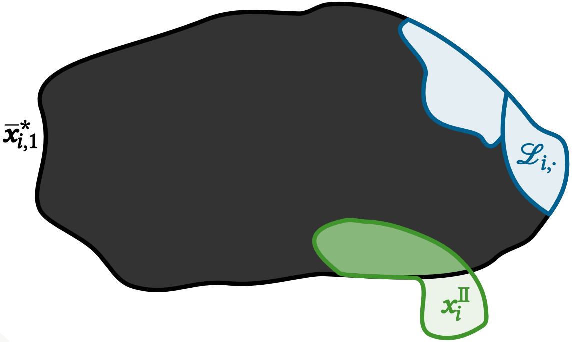

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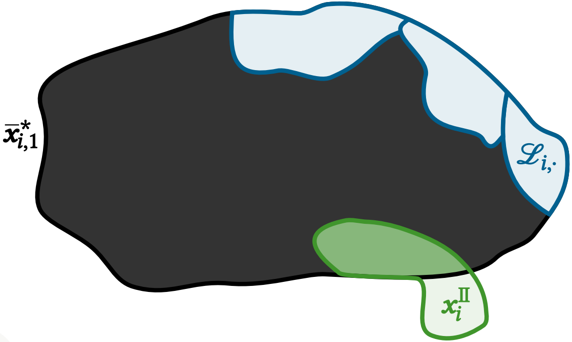

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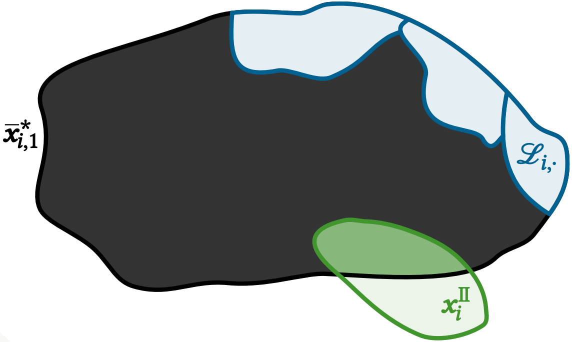

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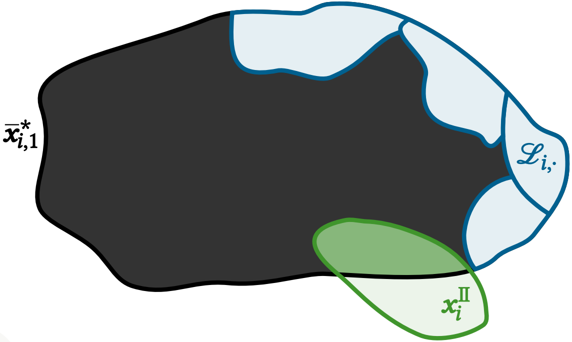

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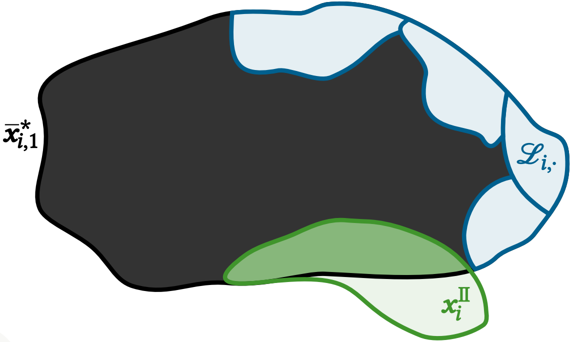

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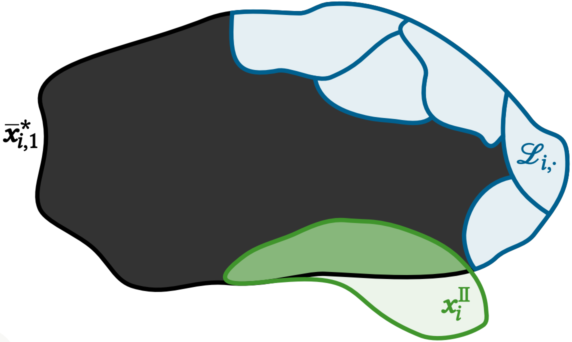

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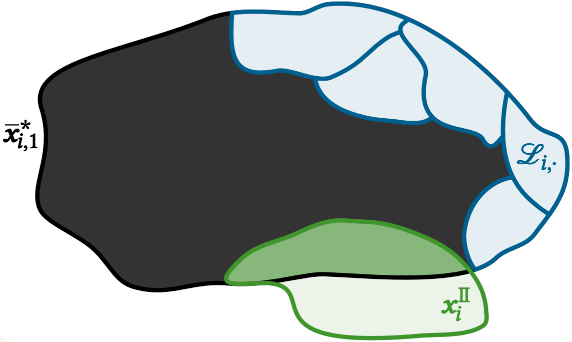

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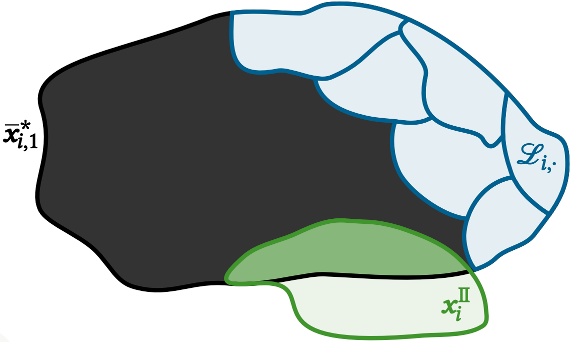

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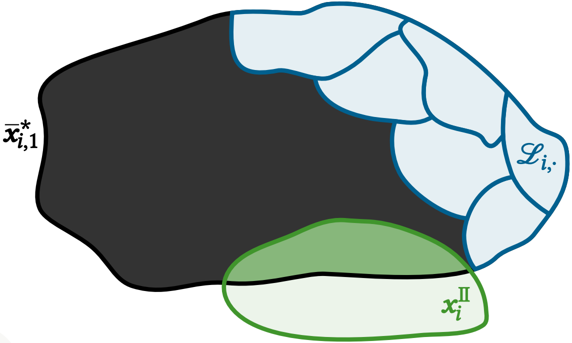

Analysing Phase II (2/2)

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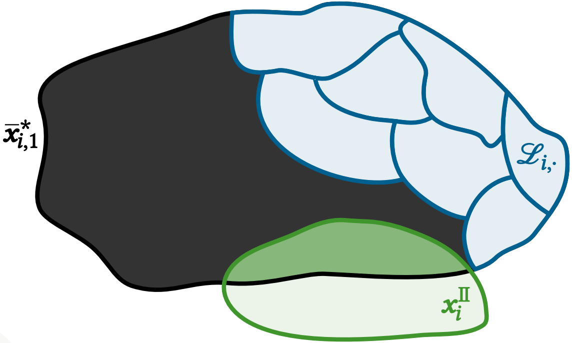

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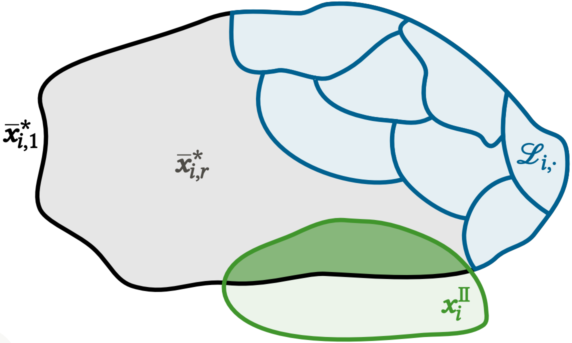
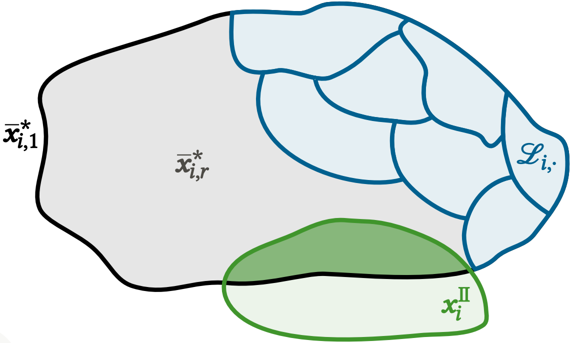


Analysing Phase II (2/2)

Lemma

$$v_i\left(\overline{x}^*_{i,r} \mid a_i^1, \dots, a_i^{r-1}\right) = v_i\left(\overline{x}^*_{i,r} \cup \left\{a_i^1, \dots, a_i^{r-1}\right\}\right) - v_i\left(a_i^1, \dots, a_i^{r-1}\right)$$



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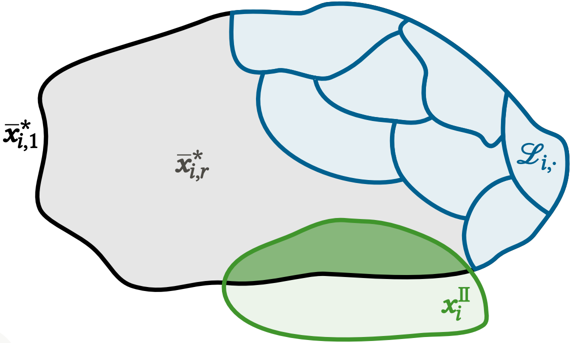

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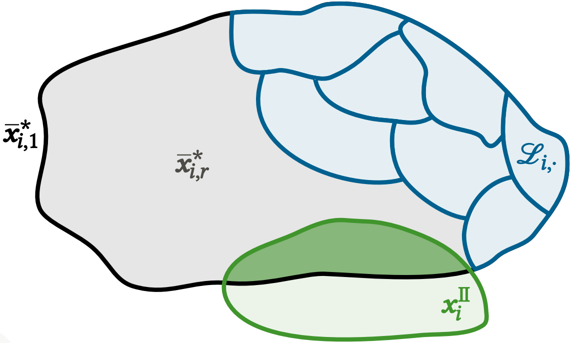

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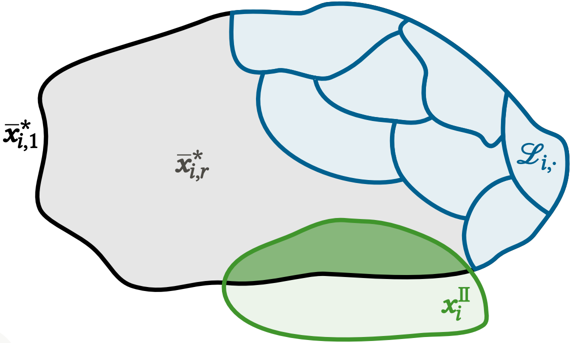

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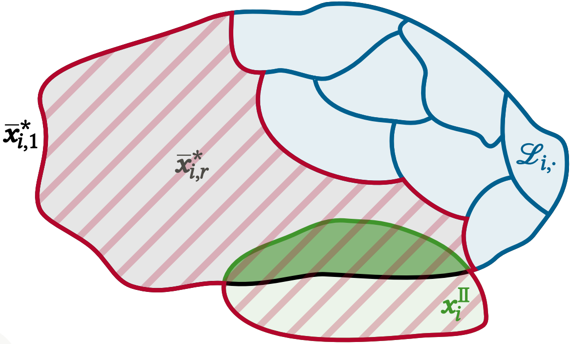

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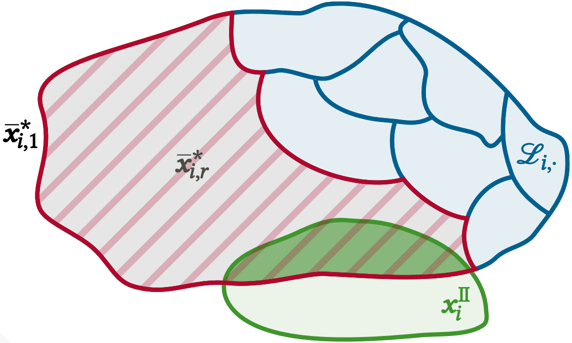

Analysing Phase II (2/2)

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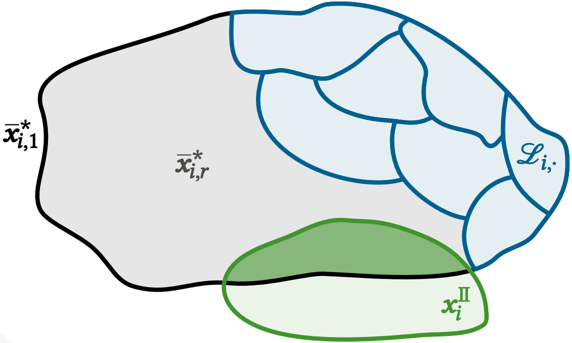

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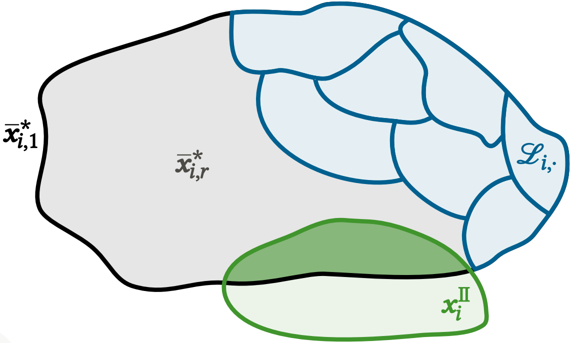

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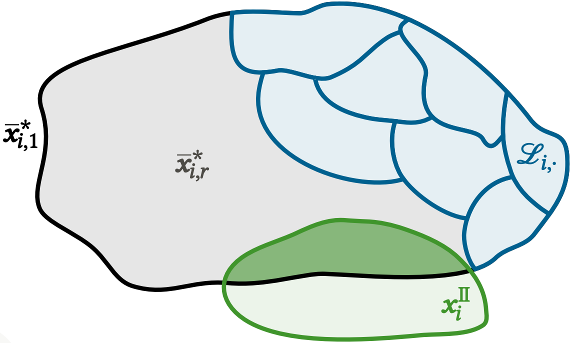

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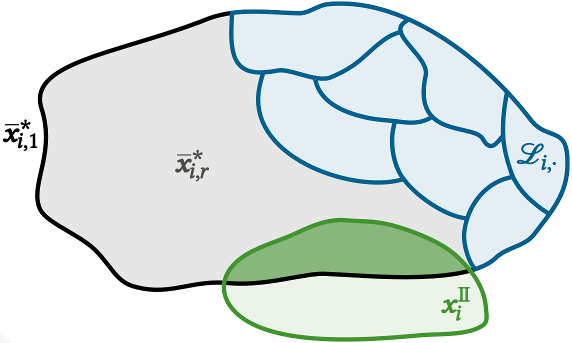

Analysing Phase II (2/2)

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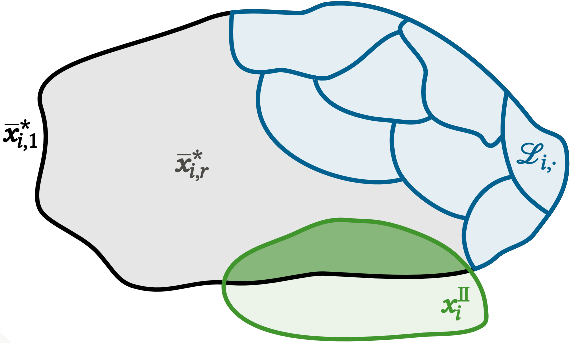
$$v_i\left(\bar{\boldsymbol{x}}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}\right) \geq -v_i\left(a_i^1, \dots, a_i^{r-1}\right) + v_i\left(\bar{\boldsymbol{x}}_{i,1}^*\right) - \sum_{l=2}^{r} v_i\left(\mathscr{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1}\right)$$



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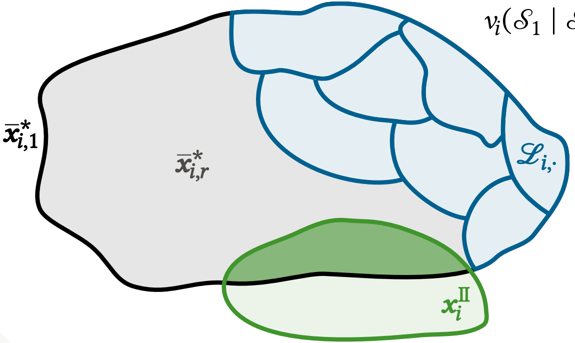
$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) =$



Analysing Phase II (2/2)

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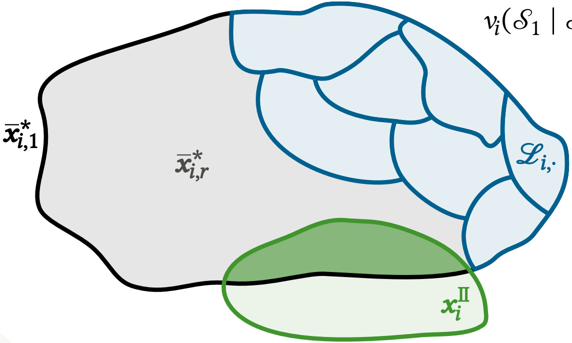
$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i\left(j^1 \mid \mathcal{S}_2\right)$$



Analysing Phase II (2/2)

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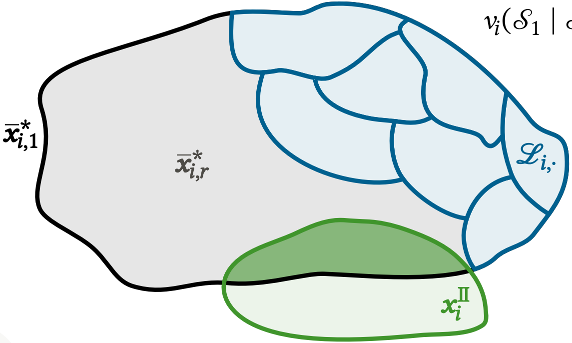

$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i\left(j^1 \mid \mathcal{S}_2\right) + v_i\left(j^2 \mid \mathcal{S}_2 \cup \{j^1\}\right)$$

Analysing Phase II (2/2)

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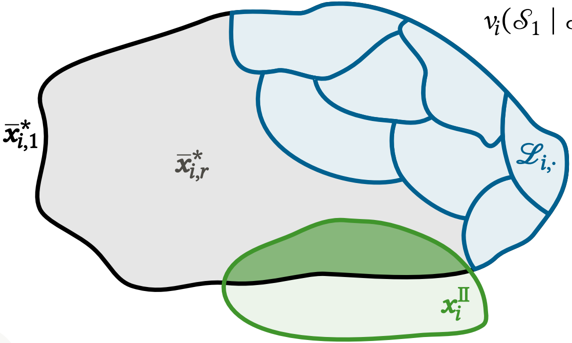

$$\begin{aligned} v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i\left(j^1 \mid \mathcal{S}_2\right) &+ \\ v_i\left(j^2 \mid \mathcal{S}_2 \cup \left\{j^1\right\}\right) &+ \\ v_i\left(j^3 \mid \mathcal{S}_2 \cup \left\{j^1, j^2\right\}\right)& \end{aligned}$$

Analysing Phase II (2/2)

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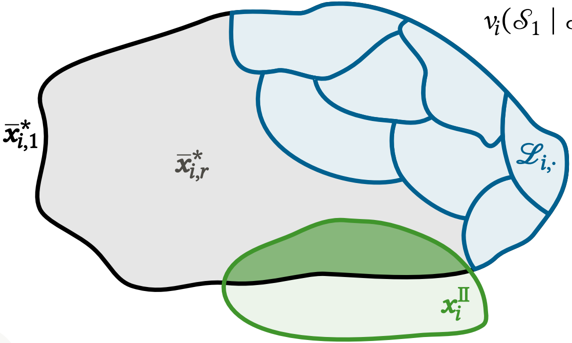

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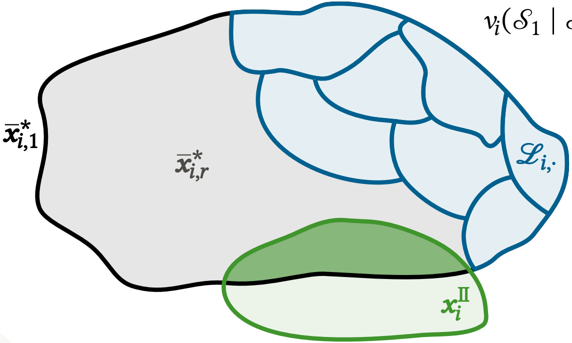

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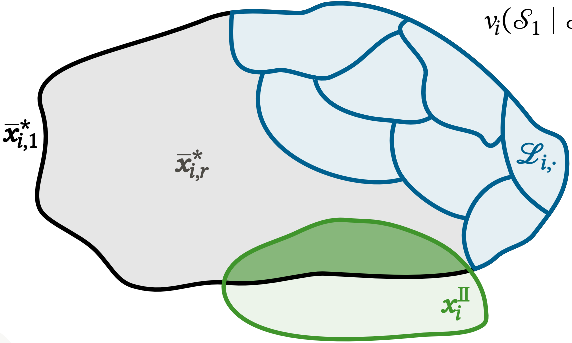


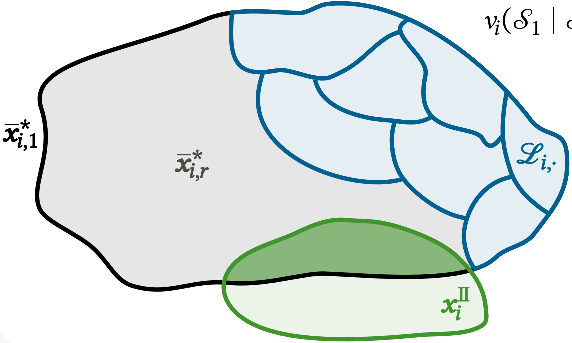
$$\begin{aligned} v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq\ & v_i\left(j^1 \mid \mathcal{S}_2\right) + \\ & v_i\left(j^2 \mid \mathcal{S}_2\right) + \\ & v_i\left(j^3 \mid \mathcal{S}_2\right) + \\ & \vdots \end{aligned}$$

Analysing Phase II (2/2)

Lemma

$$v_i\left(\overline{\boldsymbol{x}}^*_{i,r} \mid a_i^1, \dots, a_i^{r-1}\right) \geq -v_i\left(a_i^1, \dots, a_i^{r-1}\right) + v_i\left(\overline{\boldsymbol{x}}^*_{i,1}\right) - \sum_{l=2}^{r} v_i\left(\mathscr{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1}\right)$$

$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq \sum_{j \in \mathcal{S}_1} v_i(j \mid \mathcal{S}_2)$$



Analysing Phase II (2/2)

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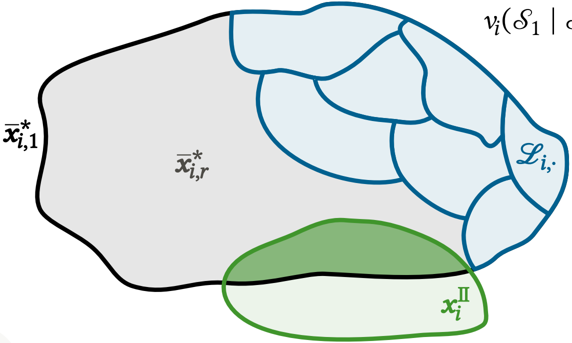

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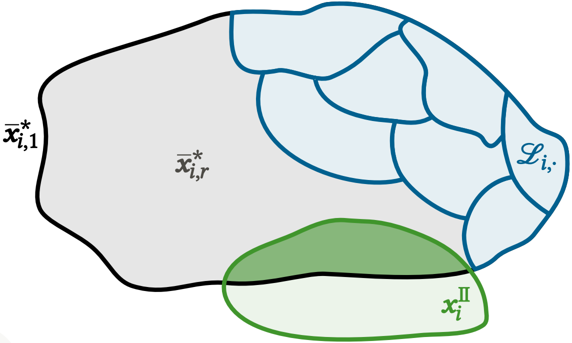

Analysing Phase II (2/2)

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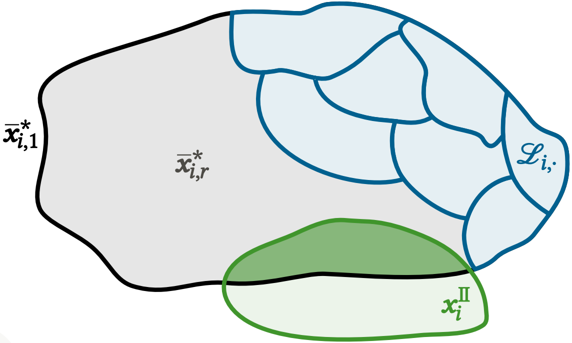

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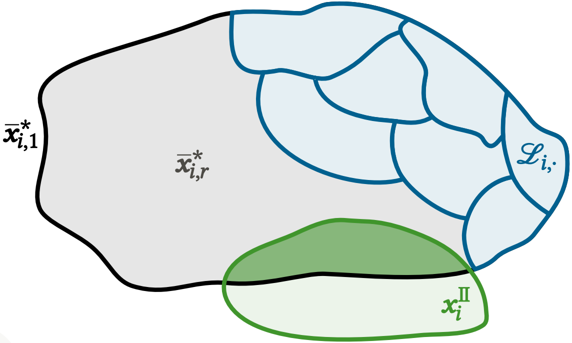

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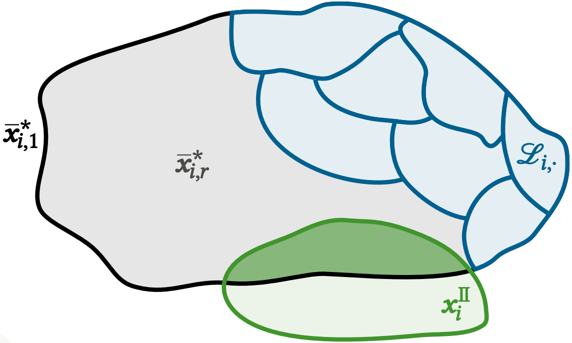

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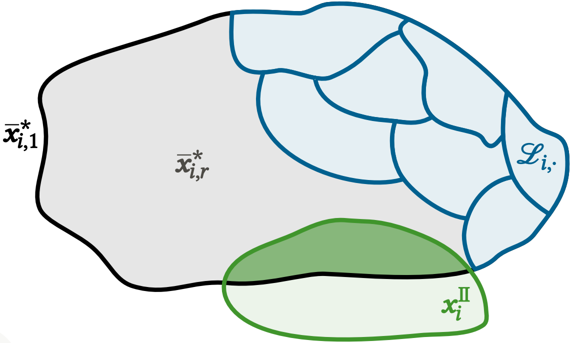

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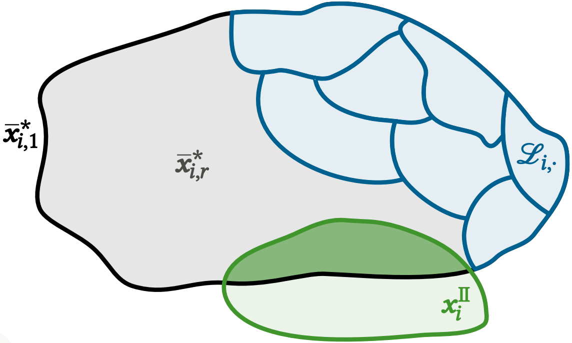

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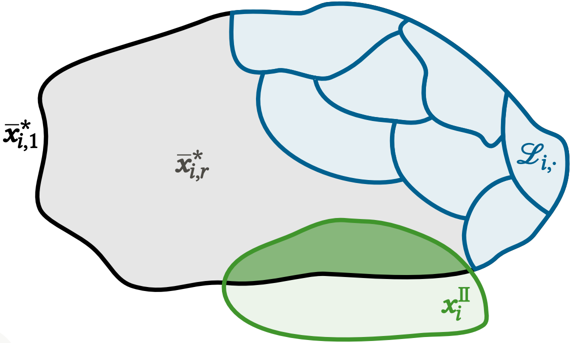

Conclusion

Summary & Outlook

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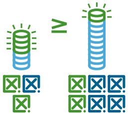

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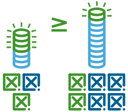

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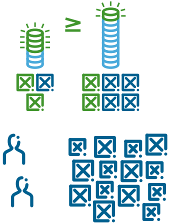

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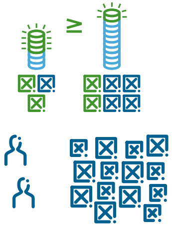

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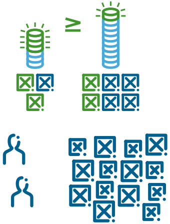

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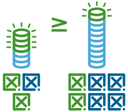

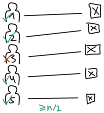

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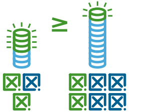

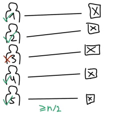
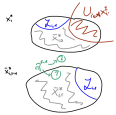

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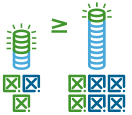
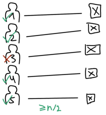
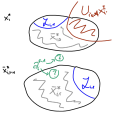
Phase II assigning remaining items
Phase III assigning outstanding items







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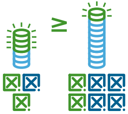

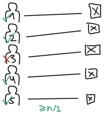

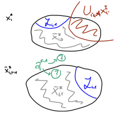

Any Room for Improvement?

Possibly! Lower bound of $\frac{e}{e-1} \approx 1.58$

End of Talk