Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same name by Garg, Kulkarni and Kulkarni

Zeno Adrian Weil
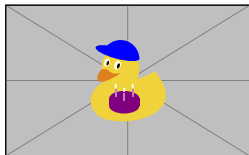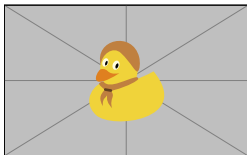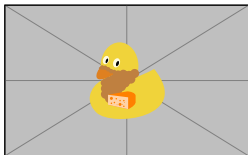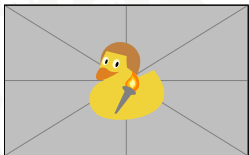Supervised by Dr Giovanna Varricchio

13th July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

What is the issue?

We need to distribute goods amongst recipients *fast*, *efficient* and *fairly*.

Where is this encountered?

- industrial procurement
- mobile edge computing
- satellites
- water withdrawal

Table of Contents

Preliminaries

Allocations

Setting:

- goods: set $\mathscr{G}$ of m items
  - unsharable
  - indivisible
- recipients: set $\mathscr{A}$ of n agents

Definition (1)

An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_i)_{i \in \mathscr{A}}$ of bundles $\boldsymbol{x}_i \subset \mathscr{G}$ such that each item is element of precisely one bundle.

Item j is *assigned* to agent i if $j \in \boldsymbol{x}_i$.

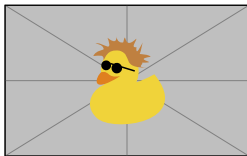But how to measure its efficiency and fairness?

Valuation Functions

Requirements:

- monotonically non-decreasing: $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2) \quad \forall \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{G}$
- normalised: $v_i(\emptyset) = 0$
- non-negative: $v_i(\mathcal{S}) \geq 0 \quad \forall \mathcal{S} \subset \mathcal{G}$

Types:

- additive: $v_i(\mathcal{S}) := \sum_{j \in \mathcal{S}} v_i(j) \quad \forall \mathcal{S} \subset \mathcal{G}$
- submodular: $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2) \quad \forall \mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$ with $\mathcal{S}_1, \mathcal{S}_2$ disjoint
  - more general (encompasses additivity)
  - diminishing returns
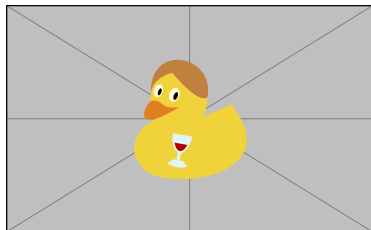
Asymmetric Maximum Nash Social Welfare Problem

GOETHE
UNIVERSITÄT
FRANKFURT AM MAIN

Problem (2)

$$x^* \overset{!}{=} \underset{x \in X_{\mathscr{A}}(\mathscr{G})}{\arg\max}\{\mathrm{NSW}(x)\} \quad \text{with } \mathrm{NSW}(x) := \Big(\prod_{i \in \mathscr{A}} v_i(x_i)^{\eta_i}\Big)^{1/\sum_{i \in \mathscr{A}} \eta_i}$$

- $X_{\mathscr{A}}(\mathscr{G})$: all possible allocations
- η_i: agent weight

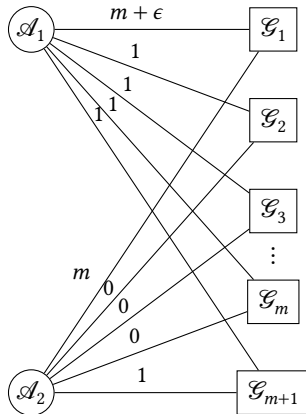- middle ground between efficiency and fairness

Challenge

Algorithm with approximation factor *independent from m*!

RepReMatch

Naïve Approach

What are the items of lowest valuation?

Looking into the Future? Changing the Past!

Under submodular valuations, the set of lowest valuation is approximable only by $\Omega(\sqrt{m/\ln m})$. ⚡
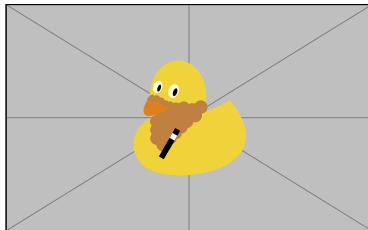
We need change the past in three phases:

Phase I Assign enough high-value items temporarily.
Phase II Assign the remaining items definitely.
Phase III Re-assign the items of phase I definitely.

A $2n(\log_2 n + 3)$-approximation is possible!

The Algorithm

Phase I

1. repeat $\lceil \log_2 n \rceil + 1$ times or until $\mathscr{G} = \varnothing$
   1. create bipartite graph $G = (\mathscr{A}, \mathscr{G}, E)$ with edge weights $w(i, j) = \eta_i \log v_i(j)$
   2. compute maximum weight matching $\mathscr{M}$
   3. update bundles x_i^{I} according to matching $\mathscr{M}$ and remove assigned items

Phase II

2. repeat until $\mathscr{G} = \varnothing$
   1. create bipartite graph $G = (\mathscr{A}, \mathscr{G}, E)$ with edge weights $w(i, j) = \eta_i \log\big(v_i(x_i^{\mathrm{II}} \cup \{j\})\big)$
   2. compute maximum weight matching $\mathscr{M}$
   3. update bundles x_i^{II} according to matching $\mathscr{M}$ and remove assigned items

Phase III

3. create bipartite graph $G = (\mathscr{A}, \bigcup_{i \in \mathscr{A}} x_i^{\mathrm{I}}, E)$ with edge weights $w(i, j) = \eta_i \log\big(v_i(x_i^{\mathrm{II}} \cup \{j\})\big)$
4. compute maximum weight matching $\mathscr{M}$
5. create bundles x_i^{III} according to matching $\mathscr{M}$ and previous bundles x_i^{II}

Definition (9)
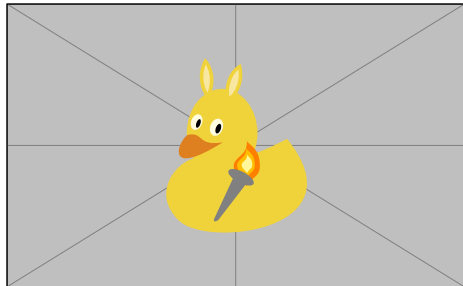
The set $\mathscr{L}_{i,r}$ of *lost items* is the set of all optimal items $j \in x_i^*$ assigned to other agents $i' \neq i$ in round r.

Definition (10)

Let $x_i^{\mathrm{II}} = \{a_i^1, a_i^2, \dots\}$ be the bundle of agent i. The set of *optimal and attainable items* is defined as

$$\bar{x}_{i,r}^* := \begin{cases} x_i^* \setminus \bigcup_{i' \in \mathscr{A}} x_{i'}^{\mathrm{I}} & \text{in round } r = 0, \\ \bar{x}_{i,0}^* \setminus \mathscr{L}_{i,1} & \text{in round } r = 1, \\ \bar{x}_{i,r-1}^* \setminus (\mathscr{L}_{i,r} \cup \{a_i^{r-1}\}) & \text{in round } r \geq 2. \end{cases}$$

⇒ What is the valuation of the remaining items?

Lemma (11)

$$v_i(\overline{x}^*_{i,r} \mid a_i^1, ..., a_i^{r-1}) \geq v_i(\overline{x}^*_{i,1}) - \sum_{r'=1}^{r-1} |\mathscr{L}_{i,r'+1}| \cdot v_i(a_i^{r'} \mid a_i^1, ..., a_i^{r'-1}) - v_i(a_i^1, ..., a_i^{r-1}) \qquad \forall r \geq 2$$

Proof

- definition of marginal valuation

$$v_i(\overline{x}^*_{i,r} \mid a_i^1, ..., a_i^{r-1}) = v_i(\overline{x}^*_{i,r} \cup \{a_i^1, ..., a_i^{r-1}\}) - v_i(a_i^1, ..., a_i^{r-1})$$

$\Rightarrow$ We need a lower bound on $v_i(\overline{x}^*_{i,r} \cup \{a_i^1, ..., a_i^{r-1}\})$!

Lemma (11)

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \ldots, a_i^{r-1}) \geq v_i(\bar{x}_{i,1}^*) - \sum_{r'=1}^{r-1} |\mathscr{L}_{i,r'+1}| \cdot v_i(a_i^{r'} \mid a_i^1, \ldots, a_i^{r'-1}) - v_i(a_i^1, \ldots, a_i^{r-1}) \qquad \forall r \geq 2$$

Proof

- $\bar{x}_{i,r}^* = \bar{x}_{i,r-1}^* \setminus (\mathscr{L}_{i,r} \cup \{a_i^{r-1}\}) \implies (\bar{x}_{i,r}^* \cup \{a_i^{r-1}\}) \supset (\bar{x}_{i,r-1}^* \setminus \mathscr{L}_{i,r})$
  - item a_i^{r-1} perhaps not element of $\bar{x}_{i,r-1}^*$
- diminishing returns $\implies v_i(\mathcal{S}_1 \mid \mathcal{S}_2 \cup \mathcal{S}_3) \leq v_i(\mathcal{S}_1 \mid \mathcal{S}_2)$

$$
\begin{aligned}
v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \ldots, a_i^{r-1}\}) &\geq v_i(\bar{x}_{i,r-1}^* \setminus \mathscr{L}_{i,r} \cup \{a_i^1, \ldots, a_i^{r-2}\}) \\
&= v_i(\bar{x}_{i,r-1}^* \cup \{a_i^1, \ldots, a_i^{r-2}\}) - v_i(\mathscr{L}_{i,r} \mid \bar{x}_{i,r-1}^* \setminus \mathscr{L}_{i,r} \cup \{a_i^1, \ldots, a_i^{r-2}\}) \\
&\geq v_i(\bar{x}_{i,r-1}^* \cup \{a_i^1, \ldots, a_i^{r-2}\}) - v_i(\mathscr{L}_{i,r} \mid a_i^1, \ldots, a_i^{r-2})
\end{aligned}
$$

Lemma (11)

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq v_i(\bar{x}_{i,1}^*) - \sum_{r'=1}^{r-1} |\mathscr{L}_{i,r'+1}| \cdot v_i(a_i^{r'} \mid a_i^1, \dots, a_i^{r'-1}) - v_i(a_i^1, \dots, a_i^{r-1}) \qquad \forall r \geq 2$$

Proof

- apply inequality recursively

$$v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) \geq v_i(\bar{x}_{i,r-1}^* \cup \{a_i^1, \dots, a_i^{r-2}\}) - v_i(\mathscr{L}_{i,r} \mid a_i^1, \dots, a_i^{r-2})$$

$$\geq v_i(\bar{x}_{i,1}^*) - \sum_{r'=1}^{r-1} v_i(\mathscr{L}_{i,r'+1} \mid a_i^1, \dots, a_i^{r'-2})$$

$\Rightarrow$ We need an upper bound on $v_i(\mathscr{L}_{i,r'+1} \mid a_i^1, \dots, a_i^{r'-2})$!

Lemma (11)

$$v_i(\overline{x}_{i,r}^* \mid a_i^1, ..., a_i^{r-1}) \geq v_i(\overline{x}_{i,1}^*) - \sum_{r'=1}^{r-1} |\mathscr{L}_{i,r'+1}| \cdot v_i(a_i^{r'} \mid a_i^1, ..., a_i^{r'-1}) - v_i(a_i^1, ..., a_i^{r-1}) \qquad \forall r \geq 2$$

Proof

- diminishing returns $\implies v_i(\mathscr{S}) \leq \sum_{j \in \mathscr{S}} v_i(j)$ for all $\mathscr{S}$
- item $a_i^{r'}$ assigned before any item $j \in \mathscr{L}_{i,r'+1} \implies v_i(a_i^{r'} \mid a_i^1, ..., a_i^{r'-1}) \geq v_i(j \mid a_i^1, ..., a_i^{r'-1})$

$$v_i(\mathscr{L}_{i,r'+1} \mid a_i^1, ..., a_i^{r'-2}) \leq \sum_{j \in \mathscr{L}_{i,r'+1}} v_i(j \mid a_i^1, ..., a_i^{r'-2})$$

$$\leq |\mathscr{L}_{i,r'+1}| \cdot v_i(a_i^{r'} \mid a_i^1, ..., a_i^{r'-1}) \qquad \square$$

Reminder

Phase I Temporary assignments. Valuations of single items as edge weights.

Phase III Items of phase I released. Valuations of items and bundles from phase II as edge weights.

Phase I reserves 'high-value' items. But what qualifies as 'high-value'?

Definition (14)

Let $x_i^* = \{o_i^1, o_i^2, \dots\}$ be an optimal bundle. An item $j \in \mathscr{G}$ is *outstanding* if $v_i(j) \geq v_i(o_i^1)$.

$\Rightarrow$ Are enough outstanding items reserved?

Analysing Phases I & III (2/2)

Lemma (15)

Each agent can be matched with an outstanding item in phase III.

Sketch Proof

- number of unmatched agents halved with each round of phase I
  - $\lceil \log_2 n \rceil + 1$ rounds in phase I are enough
- induction on number of rounds in phase I

Base Case: In round 1 of phase I, either

- $\geq n/2$ many agents matched with an outstanding item
- $< n/2$ many agents matched with an outstanding item
  - $> n/2$ many items o_i^1 assigned to someone else
  - $> n/2$ many agents matched upon release in phase III

3 Conclusion

Summary

- allocation: partition of items amongst agents
- bundles valued using submodular valuation functions
    - diminishing returns
- Nash social welfare: weighted geometric mean of valuations
- wanted: approximation factor independent from m
- naïve, repeated matching fails because of missing foresight
- RepReMatch: $2n(\log n + 3)$-approximative
    - **Phase I** finding enough outstanding items
    - **Phase II** assigning remaining item (lost vs. optimal and attainable)
    - **Phase III** assigning outstanding items
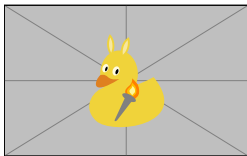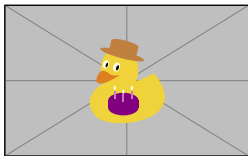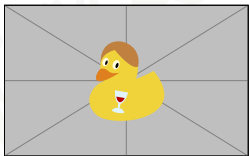
An improvement over previous results?
Yes! $(m - n + 1)$ best known approximation factor before.

Any room for improvement left?
Possibly! Lower bound of 1.72.

Outlook:

- less general valuation functions $\rightsquigarrow$ better factors
- limits on agent weights $\rightsquigarrow$ linear and even constant factors
- focus on fairness

End of Talk