

## Seminar Approximation Algorithms

# ANSWuSVp(U)M

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## Abstract

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## Todo list

or rather ‘allocated’?	4
$i$ : less strict def of valuations; restriction for our case later on	4
definition of approximation factor [def environment or in-text?]	4
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decide on $x$ or $x^{\text{III}}$ ; discrepancies in orig paper in def of $x_{\text{III}}$ !	8

# 1 Introduction

- problem introduction, motivation, applications
- formal problem definition
- short literature review: What is known, what not? New findings?
- content & structure of paper

**Definition 1.** Let  $\mathcal{G} := \{1, \dots, m\}$  be a set of indivisible *items* and  $\mathcal{A} := \{1, \dots, n\}$  be a set of *agents*. An *allocation* is a tuple  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{P}(\mathcal{G})^n$  such that each item is element of exactly one  $\mathbf{x}_i$ , that is  $\bigcup_{i \in \mathcal{A}} \mathbf{x}_i = \mathcal{G}$  and  $\mathbf{x}_i \cap \mathbf{x}_{i'} = \emptyset$  for all  $i \neq i'$ . An item  $j \in \mathcal{G}$  is *assigned* to agent  $i \in \mathcal{A}$  if  $j \in \mathbf{x}_i$  holds.

⋮

**Definition 2.** Given a set  $\mathcal{G}$  of items and a set  $\mathcal{A}$  of agents with *valuations*  $v_i: \mathcal{P}(\mathcal{G}) \rightarrow \mathbb{R}$  and *agent weights*  $\eta_i$  for all agents  $i \in \mathcal{A}$ , the *Nash Social Welfare problem* (NSW) is to find an allocation maximising the weighted geometric mean of valuations, that is

$$\arg \max_{\mathbf{x} \in \Pi_n(\mathcal{G})} \left\{ \left( \prod_{i \in \mathcal{A}} v_i(\mathbf{x}_i)^{\eta_i} \right)^{1/\sum_{i \in \mathcal{A}} \eta_i} \right\}$$

where  $\Pi_n(\mathcal{G})$  is the set of all possible allocations of the items in  $\mathcal{G}$  amongst  $n$  agents. The problem is called *symmetric* if all agent weights  $\eta_i$  are equal, and *asymmetric* otherwise.

⋮

In a slight abuse of notation, we omit the brackets in a valuation function if the set of items contains only one item, that is  $v_i(j) = v_i(\{j\})$ .

⋮

definition of approximation factor [def environment or in-text?]

⋮

Garg, Kulkarni and Kulkarni consider five different types of non-negative monotonically non-decreasing valuation functions of which we are going to consider only the following two due to space constraints:

**Additive** The valuation  $v_i(\mathcal{S})$  of an agent  $i$  for a set  $\mathcal{S} \subset \mathcal{G}$  of items  $j$  is the sum of individual valuations  $v_i(j)$ , that is  $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(j)$ .

**Submodular** Let  $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$  denote the marginal utility of agent  $i$  for a set  $\mathcal{S}_1 \subset \mathcal{G}$  of items over the disjoint set  $\mathcal{S}_2 \subset \mathcal{G}$ . This valuation functions satisfies the submodularity constraint  $v_i(j \mid \mathcal{S}_1 \cup \mathcal{S}_2) \leq v_i(j \mid \mathcal{S}_1)$  for all agents  $i \in \mathcal{A}$ , items  $j \in \mathcal{G}$  and sets  $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$  of items.

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## 2 SMatch

- giving intuition why naïve repeated maximum matching is not independent of  $m$  (cf. Figure 1)
- textual walk-through of algorithm
- giving intuition of lemmas used and theorem proof

Proofs are short so this section might not be much shorter than the original one.

On second thought, this section could possibly be shortened even more by omitting the theory and getting merged with the introduction, depending on the length of the rest of the document.

For a fixed agent  $i$ , order the items in descending order of the valuations by agent  $i$  and denote the  $j$ -th most liked item by  $\mathcal{G}_i^{(j)}$ .

**Lemma 1.**  $v_i(h_i^t) \geq v_i(\mathcal{G}_i^{(tn)})$ .

*Proof.* At the start of the  $t$ -th iteration, at most  $(t-1)n$  items out of the  $tn$  most highly valued items  $\mathcal{G}_i^{(1)}, \dots, \mathcal{G}_i^{(tn)}$  have been assigned in previous iterations since at most  $n$  items are assigned in each iteration. During the  $t$ -th iteration, at most  $n-1$  more of those highly valued items could be assigned to all other agents  $i' \neq i$ , leaving at least one item in  $\mathcal{G}_i^{(1)}, \dots, \mathcal{G}_i^{(tn)}$  unassigned. Since  $v_i(\mathcal{G}_i^{(k)}) \geq v_i(\mathcal{G}_i^{(tn)})$  for all  $k \leq i$  by definition, the lemma follows.  $\square$

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to be men-  
tioned  
earlier be-  
cause of  $u_i$

**Lemma 2.**  $v_i(h_i^2, \dots, h_i^{\tau_i}) \geq \frac{u_i}{n}$ .

*Proof Sketch.*  $\square$

Ändere Parameterreihenfolge ab und ergänze sinnvolle Säumniswerte

Begrenzer mit Unicode

Begrenzer in Makros

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**Algorithm 1:** SMatch for the Asymmetric Additive NSW problem
 

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**Input:** set  $\mathcal{A} = \{1, \dots, n\}$  of agents with weights  $\eta_i \forall i \in \mathcal{A}$ , set  $\mathcal{G} = \{1, \dots, m\}$  indivisible items, additive valuations  $v_i: \mathcal{P}(\mathcal{G}) \rightarrow \mathbb{R}_{>0}^+$  where  $v_i(\mathcal{S})$  is the valuation of agent  $i \in \mathcal{A}$  for each item set  $\mathcal{S} \subset \mathcal{G}$

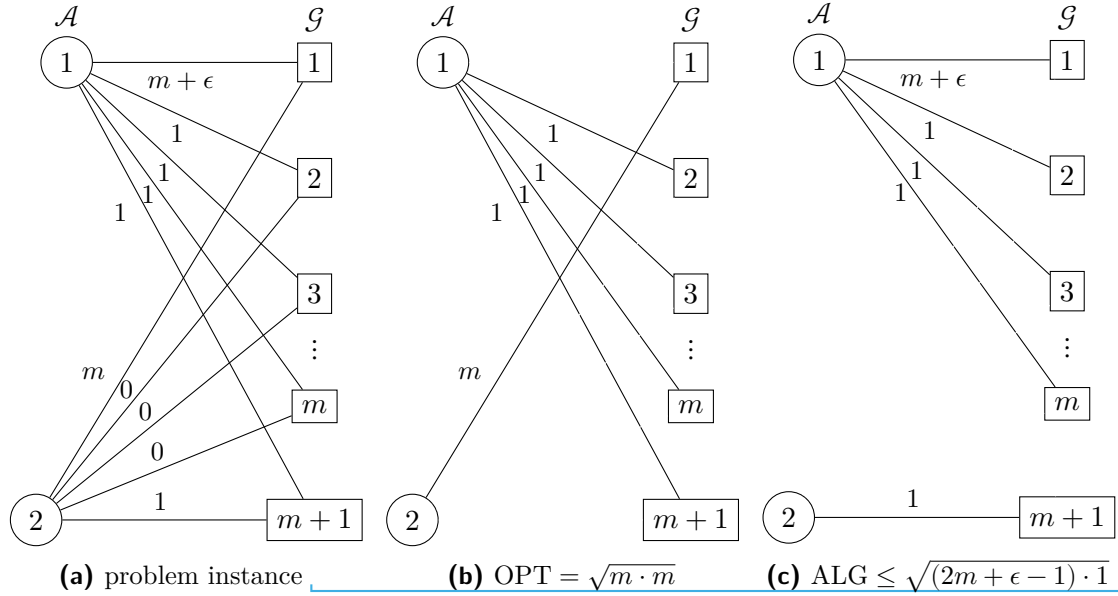
**Output:**  $\frac{1}{2n}$ -approximation  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  of an optimal allocation

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1  $\mathbf{x}_i \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ 
2  $u_i \leftarrow v_i(\mathcal{G}_{i, [2n+1:m]}) \quad \forall i \in \mathcal{A}$ 
3  $\mathcal{W} \leftarrow \{\eta_i \cdot \log(v_i(j) + \frac{u_i}{n}) \mid i \in \mathcal{A}, j \in \mathcal{G}\}$   $\triangleright$  edge weights
4  $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$   $\triangleright$  bipartite graph
5  $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$ 
6  $\mathbf{x}_i \leftarrow \{j \mid (i, j) \in \mathcal{M}\} \quad \forall i \in \mathcal{A}$   $\triangleright$  allocate according to matching
7  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{j \mid (i, j) \in \mathcal{M}\}$   $\triangleright$  remove allocated goods
8 while  $\mathcal{G}^{\text{rem}} \neq \emptyset$  do
9    $\mathcal{W} \leftarrow \{\eta_i \cdot \log(v_i(j) + v_i(\mathbf{x}_i)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}}\}$ 
10   $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathcal{W})$ 
11   $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$ 
12   $\mathbf{x}_i \leftarrow \mathbf{x}_i \cup \{j \mid (i, j) \in \mathcal{M}\} \quad \forall i \in \mathcal{A}$ 
13   $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{j \mid (i, j) \in \mathcal{M}\}$ 
14 end while
15 return  $\mathbf{x}$ 

```

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**Figure 1:** Agent 1 values item 1 at  $m + \epsilon$ , and all other items at 1. Agent 2 values item 1 at  $m$ , item  $m + 1$  at 1, and all other items at 0. In an optimal allocation, item 1 would be assigned to agent 2 and all other items to agent 1, resulting in a NSW of  $\sqrt{m \cdot m} = m$ . A repeated maximum matching algorithm would greedily assign item 1 to agent 1 and item  $m + 1$  to agent 2 in the first round. Even if all remaining items were going to be assigned to agent 1, the NSW will never surpass  $\sqrt{(2m + \epsilon - 1) \cdot 1} < \sqrt{2m}$ . The approximation factor  $\alpha \approx \sqrt{m/2}$  therefore depends on the number of items.

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**Algorithm 2:** RepReMatch for the Asymmetric Submodular NSW problem
 

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**Input:** set  $\mathcal{A} = \{1, \dots, n\}$  of agents with weights  $\eta_i \forall i \in \mathcal{A}$ , set  $\mathcal{G} = \{1, \dots, m\}$  indivisible items, additive valuations  $v_i: \mathcal{P}(\mathcal{G}) \rightarrow \mathbb{R}_{>0}^+$  where  $v_i(\mathcal{S})$  is the valuation of agent  $i \in \mathcal{A}$  for each item set  $\mathcal{S} \subset \mathcal{G}$

**Output:**  $\frac{1}{2n \log n}$ -approximation  $\mathbf{x}^{\text{III}} = (\mathbf{x}_1^{\text{III}}, \dots, \mathbf{x}_n^{\text{III}})$  of an optimal allocation

*Phase I:*

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1  $\mathbf{x}_i^{\text{I}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ 
2  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}$ 
3 for  $t = 0, \dots, \lceil \log n \rceil - 1$  do
4   if  $\mathcal{G}^{\text{rem}} \neq \emptyset$  then
5      $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(j)) \mid i \in \mathcal{A}, j \in \mathcal{G} \}$ 
6      $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ 
7      $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$ 
8      $\mathbf{x}_i^{\text{I}} \leftarrow \mathbf{x}_i^{\text{I}} \cup \{j\} \quad \forall (i, j) \in \mathcal{M}$ 
9      $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{j \mid (i, j) \in \mathcal{M}\}$ 
10  end if
11 end for

```

*Phase II:*

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12  $\mathbf{x}_i^{\text{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ 
13 while  $\mathcal{G}^{\text{rem}} \neq \emptyset$  do
14    $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(\mathbf{x}_i^{\text{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \}$ 
15    $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ 
16    $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$ 
17    $\mathbf{x}_i^{\text{II}} \leftarrow \mathbf{x}_i^{\text{II}} \cup \{j\} \quad \forall (i, j) \in \mathcal{M}$ 
18    $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{j \mid (i, j) \in \mathcal{M}\}$ 
19 end while

```

*Phase III:*

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20  $\mathcal{G}^{\text{rem}} \leftarrow \bigcup_{i \in \mathcal{A}} \mathbf{x}_i^{\text{I}} \quad \triangleright \text{release items allocated in first phase}$ 
21  $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(\mathbf{x}_i^{\text{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \}$ 
22  $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ 
23  $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$ 
24  $\mathbf{x}_i^{\text{III}} \leftarrow \mathbf{x}_i^{\text{II}} \cup \{j\} \quad \forall (i, j) \in \mathcal{M}$ 
25  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{j \mid (i, j) \in \mathcal{M}\}$ 
26  $\mathbf{x}_i^{\text{III}} \leftarrow \text{arbitrary\_allocation}(\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathbf{x}_i^{\text{III}}, (v_i)_{i \in \mathcal{A}})$ 
27 return  $\mathbf{x}^{\text{III}}$ 

```

decide on  $\mathbf{x}$  or  $\mathbf{x}^{\text{III}}$ ; discrepancies in orig paper in def of  $\mathbf{x}^{\text{III}}$ !