Seminar Approximation Algorithms

ANSWuSVþ(U)M

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Abstract

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Todo list

1 Introduction

- problem introduction, motivation, applications

- formal problem definition (incl. why geometric mean?)

- short literature review: What is known, what not? New findings?

- content & structure of paper

Definition 1. Let $\mathcal{G}$ be a set of m indivisible *items* and $\mathcal{A}$ be a set of n *agents*. An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of *bundles* $\boldsymbol{x}_i \subset \mathcal{G}$ such that each item is element of exactly one bundle, that is $\bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i = \mathcal{G}$ and $\boldsymbol{x}_i \cap \boldsymbol{x}_{i'} = \emptyset$ for all $i \neq i'$. An item $j \in \mathcal{G}$ is *assigned* to agent $i \in \mathcal{A}$ if $j \in \boldsymbol{x}_i$ holds.

> redefine or rather 'allocated'?

$\vdots$

Problem 1. Given a set $\mathcal{G}$ of indivisible items and a set $\mathcal{A}$ of agents with *valuation functions* $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ and *agent weights* η_i for all agents $i \in \mathcal{A}$, the *Nash Social Welfare problem* (NSW) is to find an allocation maximising the weighted geometric mean of valuations, that is

$$\arg\max_{\boldsymbol{x} \in \boldsymbol{X}_n(\mathcal{G})} \left\{ \left(\prod_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i)^{\eta_i} \right)^{1/\sum_{i \in \mathcal{A}} \eta_i} \right\}$$

where $\boldsymbol{X}_n(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst n agents. The problem is called *symmetric* if all agent weights η_i are equal, and *asymmetric* otherwise.

> agents without items assigned have valuation zero $\to$ prevent

$\vdots$

In a slight abuse of notation, we omit curly braces delimiting a set in the arguments of a valuation function, so, for example, we write $v(j_1, j_2, \dots)$ but mean $v(\{j_1, j_2, \dots\})$.

$\vdots$

> definition of approximation factor [def environment or in-text?]

$\vdots$

Garg, Kulkarni and Kulkarni [1] consider five different types of non-negative monotonically non-decreasing valuation functions of which we are going to consider only the following two due to space constraints:

Additive The valuation $v_i(\mathcal{S})$ of an agent i for a set $\mathcal{S} \subset \mathcal{G}$ of items j is the sum of individual valuations $v_i(j)$, that is $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(j)$.

Submodular Let $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$ denote the marginal valuation of agent i for a set $\mathcal{S}_1 \subset \mathcal{G}$ of items over a *disjoint* set $\mathcal{S}_2 \subset \mathcal{G}$. This valuation functions satisfies the submodularity constraint $v_i(j \mid \mathcal{S}_1 \cup \mathcal{S}_2) \leq v_i(j \mid \mathcal{S}_1)$ for all agents $i \in \mathcal{A}$, items $j \in \mathcal{G}$ and sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$ of items.

> or rather utility?

> What is the motivation for submodular functions? [IRfCAwSUF]

We use *additive NSW* and *submodular NSW* as shorthands for the Nash social welfare problems with additive and submodular valuation functions, respectively.

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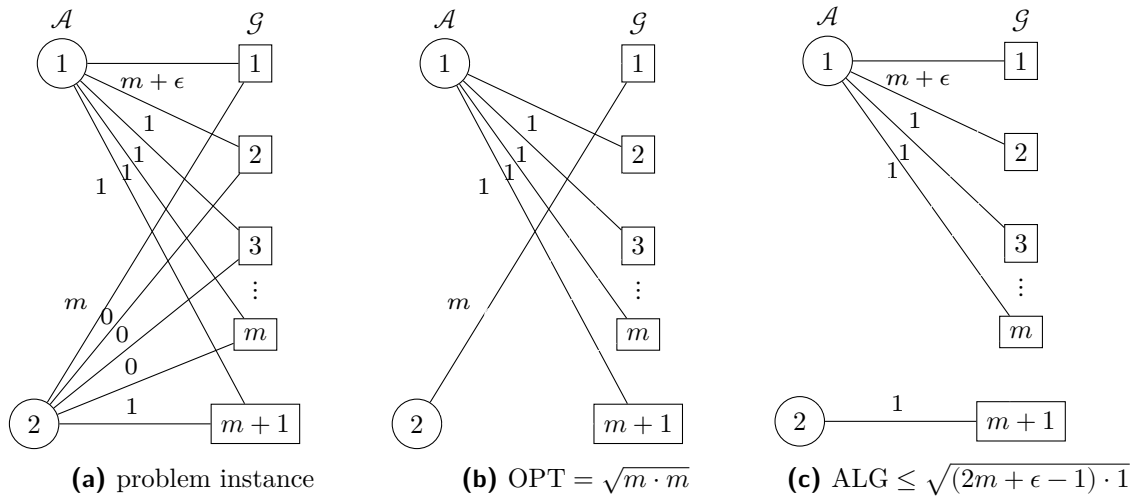
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(a) problem instance   **(b)** OPT $= \sqrt{m \cdot m}$ **(c)** ALG $\leq \sqrt{(2m + \epsilon - 1) \cdot 1}$

Figure 1: An example showing that simple repeated matching without consideration of the future leads to an approximation factor dependent on the number of items. Agent 1 values item 1 at $m + \epsilon$ and all other items at 1. Agent 2 values item 1 at m, item $m + 1$ at 1 and all other items at 0. In an optimal allocation, item 1 would be assigned to agent 2 and all other items to agent 1, resulting in a NSW of $\sqrt{m \cdot m} = m$. A repeated maximum matching algorithm would greedily assign item 1 to agent 1 and item $m + 1$ to agent 2 in the first round. Even if all remaining items were going to be assigned to agent 1, the NSW will never surpass $\sqrt{(2m + \epsilon - 1) \cdot 1} < \sqrt{2m}$. The approximation factor $\alpha \approx \sqrt{m/2}$ therefore depends on the number of items.

> fix pos of subcaptions & of agent 2

2 SMatch

In the case of an equal number of agents and items, i. e., $n = m$, the *additive* NSW can be solved exactly by finding a maximum matching on a bipartite graph with the sets of agents and of items as its parts; as weight of the edge between agent i and item j, use $\eta_i \log v_i(j)$, that is the weighted valuation of item j by agent i in the *logarithmic* Nash social welfare. Should there be more items than agents, then it would be obvious at first to just repeatedly find a maximum matching and assign the items accordingly until all items are assigned. The flaw of this idea is that such a greedy algorithm only considers the valuations of items in the current matching and perhaps the valuations of items already assigned. As the example in fig. 1 demonstrates, this leads to an algorithm with an approximation factor dependent on the number m of items. The geometric mean of the NSW favours allocations with similarly valued bundles, wherefore it may be beneficial to give items to agents who cannot expect many more valuable items in the future instead of to agents who value the item a bit more but do so for other items as well.

> What is the factor?

The algorithm SMatch, described in algorithm 1, eliminates the flaw by first gaining foresight of the valuations of items assigned after the first matching, achieving an approximation factor of $2n$ (cf. theorem 1 later on). For a fixed agent i, order the items in descending order of the valuations by agent i and denote the j-th most liked item by $\mathcal{G}_i^j$. To obtain a well-defined order, items of equal rank are further ordered numerically. SMatch, too, does repeatedly match items. During the first matching, however, the edge weights are defined as $\eta_i \log(v_i(j) + u_i/n)$ for an edge between agent i and item j. The addend u_i serves as

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Algorithm 1: SMatch for the Asymmetric Additive NSW problem

Input : set $\mathcal{A}$ of n agents with weights η_i for all agents $i \in \mathcal{A}$, set $\mathcal{G}$ of indivisible m items, additive valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each set $\mathcal{S} \subset \mathcal{G}$ of items

Output: $\frac{1}{2n}$-approximation $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of an optimal allocation

1 $\boldsymbol{x}_i \leftarrow \emptyset \quad \forall i \in \mathcal{A}$
2 $u_i \leftarrow v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m) \quad \forall i \in \mathcal{A}$ ▷ *estimation of future valuations*
3 $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(j) + u_i/n) \mid i \in \mathcal{A}, j \in \mathcal{G} \}$ ▷ *edge weights*
4 $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ ▷ *bipartite graph*
5 $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$
6 $\boldsymbol{x}_i \leftarrow \{ j \mid (i, j) \in \mathcal{M} \} \quad \forall i \in \mathcal{A}$ ▷ *assign according to matching*
7 $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{ j \mid (i, j) \in \mathcal{M} \}$ ▷ *remove assigned items*
8 while $\mathcal{G}^{\text{rem}} \neq \emptyset$ **do**
9 $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(j) + v_i(\boldsymbol{x}_i)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}} \}$
10 $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathcal{W})$
11 $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$
12 $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i \cup \{ j \mid (i, j) \in \mathcal{M} \} \quad \forall i \in \mathcal{A}$
13 $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{ j \mid (i, j) \in \mathcal{M} \}$
14 end while
15 return $\boldsymbol{x}$

---

estimation of the valuation of items assigned after the first matching and is defined as

$$u_i := \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{ v_i(\mathcal{G} \setminus \mathcal{S}) \} = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m). \tag{1}$$

The set $\mathcal{S}$ has less than $2n$ elements only if there are less than $2n$ items in total. From the second matching onwards, the edge weights are defined as $\eta_i \log(v_i(j) + v_i(\boldsymbol{x}_i))$, where $\boldsymbol{x}_i$ is the continuously updated bundle of agent i. The addend $v_i(\boldsymbol{x}_i)$ could lead to better allocations in applications, but does not improve the approximation factor asymptotically.

To calculate the approximation factor of SMatch, we first need to establish a lower bound on the valuation of single items. For convenience, we order the items in the final bundle $\boldsymbol{x}_i = \{ h_i^1, \dots, h_i^{\tau_i} \}$ of agent i by the order in which they were assigned, so that item h_i^t is assigned according to the t-th matching. Note that it holds $v_i(h_i^t) \geq v_i(h_i^{t'})$ for all $t' \geq t$.

Lemma 1. *For each agent $i \in \mathcal{A}$, her final bundle $\boldsymbol{x}_i = \{ h_i^1, \dots, h_i^{\tau_i} \}$, and her tn-th most highly valued item $\mathcal{G}_i^{tn}$, it holds $v_i(h_i^t) \geq v_i(\mathcal{G}_i^{tn})$ for all $t = 1, \dots, \tau_i$.*

Proof. At the start of the t-th round, no more than $(t-1)n$ items out of the tn most highly valued items $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ have been assigned in previous rounds since at most n items are assigned in each iteration. During the t-th round, at most $n-1$ more of those highly valued items could be assigned to all other agents $i' \neq i$, leaving at least one item in $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ unassigned. Since $v_i(\mathcal{G}_i^k) \geq v_i(\mathcal{G}_i^{tn})$ for all $k \leq tn$ by definition of $\mathcal{G}_i^{\cdot}$, the lemma follows. $\qquad\square$

We can now establish $u_i/n = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m)/n$ as lower bound on the valuations of items assigned after the first matching.

Lemma 2. *For each agent $i \in \mathcal{A}$ and her final bundle $\boldsymbol{x}_i = \{ h_i^1, \dots, h_i^{\tau_i} \}$, it holds $v_i(h_i^2, \dots, h_i^{\tau_i}) \geq u_i/n$, where n is the number of items and $u_i = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m)$.*

Proof. By lemma 1 and definition of $\mathcal{G}_i^\cdot$, every item h_i^t is worth at least as much as each item $\mathcal{G}_i^{tn+k}$ with $k \in \{0, \dots, n-1\}$ and, consequently, its valuation $v_i(h_i^t)$ is at least as high as the mean valuation $\frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1})$. Further, it holds $\tau_i n + n \geq m$ since each agent receives items for at least $\lfloor \frac{m}{n} \rfloor \geq \frac{m}{n} - 1$ rounds. Together, this yields

$$v_i(h_i^2, \dots, h_i^{\tau_i}) = \sum_{t=2}^{\tau_i} v_i(h_i^t) \geq \sum_{t=2}^{\tau_i} \frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1}) \tag{2}$$

$$\geq \frac{1}{n} v_i(\mathcal{G}_i^{2n}, \dots, \mathcal{G}_i^{m-1}) \geq \frac{1}{n} v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m) = \frac{u_i}{n} \tag{3}$$

with the last inequality stemming from $v_i(\mathcal{G}_i^{2n}) \geq v_i(\mathcal{G}_i^m)$. $\qquad \square$

Remark 1. In lemma 2, we assumed non-zero valuations for all items, hence the bundle lengths of $\tau_i \geq \lfloor \frac{m}{n} \rfloor$. Of course in an actual program, one would not assign items to agents who value them at zero. Inasmuch as additional zeros in eq. (3) do not change the sum, lemma 2 still holds nevertheless.

This allows us to calculate an approximation factor for SMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

Theorem 1. *SMatch has an approximation factor of* $2n$.

Proof. Lemma 2 can be plugged into the logarithmic NSW:

$$\log \mathrm{NSW}(\boldsymbol{x}) = \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log v_i(h_i^1, \dots, h_i^{\tau_i}) \tag{4}$$

$$= \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log(v_i(h_i^1) + v_i(h_i^2, \dots, h_i^{\tau_i})) \tag{5}$$

$$\geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log(v_i(h_i^1) + u_i/n) \tag{6}$$

Notice that the first matching of SMatch maximises the sum in eq. (6). Thus, assigning all agents i their respective most highly valued item g_i^1 in an optimal bundle $\boldsymbol{x}_i^* = \{g_i^1, \dots, g_i^{\tau_i}\}$ yields the even lower bound

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log(v_i(g_i^1) + u_i/n). \tag{7}$$

Recall the definition of u_i from eq. (1). Consider a slightly modified variant:

$$u_i = \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \quad \text{or, alternatively,} \quad u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \text{ with } \mathcal{S}_i := \operatorname*{arg\,min}_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \tag{8}$$

Moreover, consider the set $\mathcal{S}_i^*$ of the (at most) $2n$ most highly valued items in the optimal bundle $\boldsymbol{x}_i^*$, i.e.

$$\mathcal{S}_i^* := \operatorname*{arg\,min}_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S})\}. \tag{9}$$

We get the lower bound $v_i(g_i^1) \geq \frac{1}{2n} v_i(\mathcal{S}^*)$ from a similar argument as in the proof of lemma 2. Further, it holds $u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)$. We can insert these two inequalities into eq. (7) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\mathcal{S}_i^*)}{2n} + \frac{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)}{n}\right) \tag{10}$$

$$\geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n}\right) = \log\left(\frac{\mathrm{NSW}(\boldsymbol{x}^*)}{2n}\right) \tag{11}$$

$\qquad \square$

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The analysis is asymptotically tight. It is possible to design an instance for the asymmetric NSW such that SMatch achieves an approximation ratio approaching $2/n$. It remains to be shown whether this is also true for the symmetric NSW. [1, Section 6.3]

Remark 2. SMatch produces *fair* allocations which are *envy-free up to one item* (EF1). An allocation $\boldsymbol{x}$ is EF1 if, for every pair $(i_1, i_2) \in \mathcal{A}^2$ of agents, one needs to remove at most one item from the bundle $\boldsymbol{x}_{i_2}$ of agent i_2 so that agent i_1 does not envy her. In other words, either it holds $v_{i_1}(\boldsymbol{x}_{i_1}) \geq v_{i_2}(\boldsymbol{x}_{i_2})$ or there is an item $j \in \boldsymbol{x}_{i_2}$ such that $v_{i_1}(\boldsymbol{x}_{i_1}) \geq v_{i_2}(\boldsymbol{x}_{i_2} \setminus \{j\})$. [1, Section 5.2]

remove indices from allocations

3 RepReMatch

The algorithm SMatch estimates the valuation of the lowest-value items by determining the set of highest-value items and then valuing the remaining items. Unfortunately, this approach does not work for general submodular valuations because taking the set of highest-value items away does not necessarily leave a set of lowest-value items. In fact, it can be shown [3] that determining the set of lowest-value items is approximable only within a factor of $\Omega(\sqrt{m/\ln m})$.

For this reason, the algorithm RepReMatch, described in algorithm 2, relies on an approach with three phases, achieving an approximation factor of $2n(\log_2 n + 3)$ (cf. theorem 2). In phase I, a sufficiently big set of high-value items is determined through

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Algorithm 2: RepReMatch for the Asymmetric Submodular NSW problem

Input : set $\mathcal{A}$ of n agents with weights η_i for all agents $i \in \mathcal{A}$, set $\mathcal{G}$ of indivisible m items, submodular valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each set $\mathcal{S} \subset \mathcal{G}$ of items

Output: $\frac{1}{2n(\log_2 n+3)}$-approximation $\boldsymbol{x}^{\mathrm{III}} = (\boldsymbol{x}_1^{\mathrm{III}}, \ldots, \boldsymbol{x}_n^{\mathrm{III}})$ of an optimal allocation

Phase I:

1   $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \emptyset$
2   $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}$
3   **for** $t \leftarrow 1, \ldots, \lceil \log_2 n \rceil + 1$ **do**
4   **if** $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **then**
5         $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(j)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \}$ $\triangleright$ *valuation of single item*
6         $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \mathcal{W})$
7         $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$
8         $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \boldsymbol{x}_i^{\mathrm{I}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$
9         $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$
10   **end if**
11   **end for**

Phase II:

12   $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ $\triangleright$ *put allocation $\boldsymbol{x}^{\mathrm{I}}$ away and start a new one*
13   **while** $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **do**
14      $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \}$ $\triangleright$ *val. of item & cur. bundle*
15      $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \mathcal{W})$
16      $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$
17      $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$
18      $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$
19   **end while**

Phase III:

20   $\mathcal{G}^{\mathrm{rem}} \leftarrow \bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i^{\mathrm{I}}$ $\triangleright$ *release items assigned in phase I*
21   $\mathcal{W} \leftarrow \{ \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \}$ $\triangleright$ *val. of item & cur. bundle*
22   $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \mathcal{W})$
23   $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$
24   $\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$
25   $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{ j \mid (i,j) \in \mathcal{M} \}$
26   $\boldsymbol{x}^{\mathrm{III}} \leftarrow \mathrm{arbitrary\_allocation}(\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \boldsymbol{x}^{\mathrm{III}}, (v_i)_{i \in \mathcal{A}})$
27   **return** $\boldsymbol{x}^{\mathrm{III}}$

---

repeated matchings. This phase serves merley to determine this set, so items are assigned temporarily only. The edge weights reflect this by taking the valuations of just single items into account.

In phase II, the remaining items are assigned normally through repeated matchings. Consequently, each edge weight is updated in each round to be the weighted logarithm of the valuation of both the respective item and the items assigned so far.

In phase III, the high-value items assigned in phase I are released. With the knowledge of items assigned in phase II, one maximum weight matching is calculated, and the matched items are assigned accordingly. Again each edge weight is the weighted logarithm of the valuation of both the respective item and the respective agent's bundle from phase II. The remaining released items are assigned arbitrarily.

We start by analysing phase II as it is the first phase with definitive assignments. To this end, we introduce two types of item sets. Note that we use the term *round* to refer to the iterations of the loops in the phases I and II. For ease of notation, we refer to the moment before the first iteration in phase II as round 0.

Definition 2. Let $\boldsymbol{x}_i^*$ be an optimal allocation of some agent $i \in \mathcal{A}$. For any round $r \geq 1$ in phase II, the set $\mathcal{L}_{i,r} \subset \boldsymbol{x}_i^*$ of *lost* items is the set of all items $j \in \boldsymbol{x}_i^*$ assigned to other agents $i' \neq i$ in that round.

Definition 3. Let $\boldsymbol{x}_i^*$ be an optimal allocation of some agent $i \in \mathcal{A}$ and $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}}\}$ be her bundle in phase II. The set $\bar{\boldsymbol{x}}_{i,r}^*$ of *optimal and attainable* items is defined as $\bar{\boldsymbol{x}}_{i,0}^* := \boldsymbol{x}_i^* \setminus \bigcup_{i' \in \mathcal{A}} \boldsymbol{x}_{i'}^{\mathrm{I}}$ in round 0 and as $\bar{\boldsymbol{x}}_{i,r}^* := \bar{\boldsymbol{x}}_{i,r-1}^* \setminus (\mathcal{L}_{i,r} \cup \{h_i^{r-1}\}$ in round $r \in [1, \tau_i^{\mathrm{II}}]$.

We denote their sizes by $\ell_{i,r} := |\mathcal{L}_{i,r}|$ and $\bar{\tau}_{i,r}^* := |\bar{\boldsymbol{x}}_{i,r}^*|$, respectively. First, we give a lower bound on the valuations of optimal and attainable items.

> **i:** Would it be 'dirty' to include notation in the definitions?

Lemma 3. *For each agent $i \in \mathcal{A}$ and her bundle $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}}\}$, it holds in all rounds $r = 2, \ldots, \tau_i^{\mathrm{II}}$ of phase II that*

$$v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \ldots, h_i^{r-1}) \geq v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} \cdot v_i(h_i^1) - \sum_{r'=2}^{r-1} \ell_{i,r'} \cdot v_i(h_i^{r'} \mid h_i^1, \ldots, h_i^{r'-1}) - v_i(h_i^1, \ldots, h_i^{r-1}).$$

Proof. We prove the lemma by induction on the number r of rounds. In the beginning of the base case $r = 2$, agent i has already been assigned item h_i^1. For each of the optimal and attainable items $j \in \bar{\boldsymbol{x}}_{i,1}^*$ in round 1, the marginal valuation $v_i(j \mid \emptyset)$ over the empty set was at most $v_i(h_i^1 \mid \emptyset)$, as otherwise item h_i^1 would not have been assigned first. The marginal valuation $v_i(j \mid h_i^1)$ over $\{h_i^1\}$ is upper-bounded by $v_i(h_i^1 \mid \emptyset)$, too, due to the submodularity of valuations. During round 2, a further $\ell_{i,2}$ of these items j are assigned to other agents, and item h_i^2 is assigned to agent i. We can bound the marginal valuation of the remaining optimal and attainable items in round 2 in the following way:

Case 1 — $h_i^1 \in \bar{\boldsymbol{x}}_{i,1}^*$: It holds $v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,2}^* \cup \{h_i^1\}) - v_i(h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1)$.

Case 2 — $h_i^1 \notin \bar{\boldsymbol{x}}_{i,1}^*$: Due to the monotonicity of valuations, it holds $v_i(\bar{\boldsymbol{x}}_{i,2}^* \cup \{h_i^1\}) \geq v_i(\bar{\boldsymbol{x}}_{i,2}^*)$ and, therefore, $v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) \geq v_i(\bar{\boldsymbol{x}}_{i,2}^*) - v_i(h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1)$.

In both cases, the base case is proven because

$$v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) \geq v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1) \tag{12}$$
$$\geq v_i(\bar{\boldsymbol{x}}_{i,1}^*) - v_i(\mathcal{L}_{i,2}) - v_i(h_i^1) \tag{13}$$
$$\geq v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} v_i(h_i^1) - v_i(h_i^1), \tag{14}$$

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where the second inequality can be shown inductively with the definition of submodularity, and the third inequality is due all $\ell_{i,2}$ items j in set $\mathcal{L}_{i,2}$ not being assigned in round 1 although attainable, implying $v_i(j) \leq v_i(h_i^1)$.

For the induction hypothesis, we assume that the lemma holds true for all rounds up to some r. In the induction step $r \to r+1$, we differentiate the same two cases again:

Case 1—$h_i^r \in \bar{\boldsymbol{x}}_{i,r}^*$: Again we exploit the submodularity of valuations to obtain a lower bound on the marginal valuation of $\bar{\boldsymbol{x}}_{i,r+1}^*$.

$$
\begin{align}
v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \mid h_i^1, \dots, h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \cup \{h_i^r\} \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) \tag{15}\\
&= v_i(\bar{\boldsymbol{x}}_{i,r}^* \setminus \mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) \tag{16}\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) \tag{17}\\
&\quad - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1})
\end{align}
$$

Case 2—$h_i^r \notin \bar{\boldsymbol{x}}_{i,r}^*$: At first, we use the monotonicity of valuations to get the inequality

$$
\begin{align}
v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1, \dots, h_i^r\}) - v_i(h_i^1, \dots, h_i^r) \tag{18}\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1, \dots, h_i^{r-1}\}) - v_i(h_i^1, \dots, h_i^r) \tag{19}\\
&= \big(v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1, \dots, h_i^{r-1}\}) - v_i(h_i^1, \dots, h_i^{r-1})\big) \tag{20}\\
&\quad - \big(v_i(h_i^1, \dots, h_i^r) - v_i(h_i^1, \dots, h_i^{r-1})\big)\\
&= v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}). \tag{21}
\end{align}
$$

Together with the submodularity of valuation, we obtain the same lower bound again:

$$
\begin{align}
v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \mid h_i^1, \dots, h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \setminus \mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^r) \tag{22}\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^r) - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^r) \tag{23}\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^r) - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1}) \tag{24}\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) \tag{25}\\
&\quad - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1})
\end{align}
$$

In both cases, we can replace $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1})$ by the induction hypothesis and $v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1})$ by $\ell_{i,r+1} \cdot v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1})$ to prove the lemma. For a detailed calculation we refer to Garg, Kulkarni and Kulkarni [1, p. 14]. $\qquad\square$

The lemma can be used to find a lower bound on the marginal valuation of the items assigned in each round r.

Corollary 1. *From lemma 3 follows*

$$
v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) \geq \left(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} \cdot v_i(h_i^1) - \sum_{r'=2}^{r-1} \ell_{i,r'+1} \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) \right.
$$
$$
\left. - v_i(h_i^1, \dots, h_i^{r-1}) \right) \Big/ \bar{\tau}_{i,r}^*.
$$

Proof. The valuations are monotonic, i.e., $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ for all item sets $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{G}$. Induction shows that there must be an item $j \in \bar{\boldsymbol{x}}_{i,r}^*$ with a marginal valuation of at least $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1})/\bar{\tau}_{i,r}^*$. As item h_i^r was the one to be assigned, the marginal valuation of it cannot be smaller. Using lemma 3 for the value of $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1})$ proves the corollary. $\qquad\square$

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This, finally, enables us to give a lower bound on the valuation of the whole bundle assigned in phase II.

Lemma 4. *For each agent $i \in \mathcal{A}$ and her bundle $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}}\}$, it holds*

$$v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}}) \geq v_i(\bar{\boldsymbol{x}}_{i,1}^*)/n.$$

Proof. In each round $r = 1, \ldots, \tau_i^{\mathrm{II}}$, $\ell_{i,r}$ optimal and attainable items of agent i are assigned to other agents. As there are n agents in total, $n-1$ is an upper bound on $\ell_{i,r}$. Furthermore, after τ_i^{II} rounds, the number $\bar{\tau}_{i,\tau_i^{\mathrm{II}}}^*$ of optimal and attainable items is at most $n-1 \leq n$ elsewise agent i would have been assigned yet another item. Together with corollary 1, this proves the lemma:

$$v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}}) = v_i(h_i^{\tau_i^{\mathrm{II}}} \mid h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}}) + v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1}) \tag{26}$$

$$\geq \left(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} \cdot v_i(h_i^1) - \sum_{r'=2}^{\tau_i^{\mathrm{II}}-1} \ell_{i,r'+1} \cdot v_i(h_i^{r'} \mid h_i^1, \ldots, h_i^{r'-1}) \right. \tag{27}$$

$$\left. - v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1}) \right) \Big/ \bar{\tau}_{i,r}^* + v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1})$$

$$\geq \left(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - (n-1)v_i(h_i^1) - \sum_{r'=2}^{\tau_i^{\mathrm{II}}-1}(n-1)v_i(h_i^{r'} \mid h_i^1, \ldots, h_i^{r'-1}) \right. \tag{28}$$

$$\left. - v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1}) \right) \Big/ n + v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1})$$

$$\geq \left(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - (n-1)v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1}) - v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1}) \right) \Big/ n \tag{29}$$

$$+ v_i(h_i^1, \ldots, h_i^{\tau_i^{\mathrm{II}}-1})$$

$$= v_i(\bar{\boldsymbol{x}}_{i,1}^*)/n \tag{30}$$

$$\square$$

After having obtained a lower bound on the valuation of items assigned in phase II, we need a lower bound for phase III as well. Therefor we introduce a third type of item set.

Definition 4. Let $\boldsymbol{x}_i^* = \{g_i^1, \ldots, h_i^{\tau_i^*}\}$ be an optimal allocation of some agent $i \in \mathcal{A}$. The set $\mathcal{G}_i^+$ of *overly good* items is defined as $\mathcal{G}_i^+ := \{ j \in \mathcal{G} \mid v_i(j) \geq v_i(g_i^1) \}$.

Lemma 5. *In phase III, there exists a matching such that each agent $i \in \mathcal{A}$ is matched to one of her overly good items in the set $\bigcup_{i' \in \mathcal{A}} \boldsymbol{x}_{i'}^{\mathrm{I}}$ of released items.*

Proof. If all items were matched in phase I, i.e., $\bigcup_{i' \in \mathcal{A}} \boldsymbol{x}_{i'}^{\mathrm{I}} = \mathcal{G}$, then all optimal items are released in phase III and each agent can be matched to one; the lemma is proven immediately. If not, imagine for some t that only the items assigned in the first t rounds of phase I were released. Now choose some matching $\mathcal{M}_t$ with the following properties:
1. If for an agent i all overly good items were amongst the released items, she gets matched with an overly good item $j \in \mathcal{G}_i^+$.
2. The number of agents matched with one of their overly good items is maximal amongst all matchings fulfilling property 1.

Property 1 is always satisfiable as each set $\mathcal{G}_i^+$ is the only one to contain the item g_i^1, which can be matched with agent i. Property 2 leads to all agents being matched with an overly good item for $t = \lceil \log_2 n \rceil + 1$, i.e. the number of rounds in phase I, whence the lemma follows. To prove this, we denote by $\mathcal{A}_t^-$ the set of agents who are *not* matched with one of their overly good items, and show by induction on t that it holds $|\mathcal{A}_t^-| \leq n/2^t$.

> wrong! the union contains it though

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In the base case $t = 1$, none of the items are assigned initially. Denote by α the number of agents who were not assigned an overly good item in the first round of phase I. If $\alpha \leq n/2$, then a matching $\mathcal{M}_1$ obviously exists and the base case is immediately proven. Otherwise, all items from at least α many sets $\mathcal{G}_i^+$ got assigned to someone. Again: Each set $\mathcal{G}_i^+$ is the only one containing the item g_i^1, so the union of these sets contains at least α items which can be matched with at least α agents upon release. This then leaves at most $n - \alpha < n/2$ agents not matched with an overly good item.

For the induction hypothesis, we assume that the statement holds true for all rounds up to some t. In the induction step $t \to t+1$, by property 1, there is at least one unassigned item in each set $\mathcal{G}_{i'}^+$ for all agents $i' \in \mathcal{A}_t^-$ at the start of round $t+1$. Analogously to the base case, for at least half of those agents i' these unassigned items will be assigned to them or someone else and it can be argued accordingly. By the induction hypothesis, it holds $|\mathcal{A}_{t+1}^-| \leq |\mathcal{A}_t^-|/2 \leq (n/2^t)/2 = n/2^{t+1}$. $\qquad\square$

This allows us to calculate an approximation factor for RepReMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

Theorem 2. *RepReMatch has an approximation factor of* $2n(\log_2 n + 3)$.

Proof. By lemma 5, we can assign each agent i an overly good item $j_i^+ \in \mathcal{G}_i^+$ in the beginning of phase III. RepReMatch maximises the logarithmic Nash social welfare, so

$$\log \mathrm{NSW}(\boldsymbol{x}^{\mathrm{III}}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \tag{31}$$

is a lower bound on the logarithmic NSW after the first matching in phase III, whereby $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$ is the bundle of agent i from phase II.

Item j_i^+ was released in phase III, which means it was assigned in phase I, implying $j_i^+ \in \boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*$ and, subsequently, $j_i^+ \in (\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1}$. Phase I runs for at most $\lceil \log_2 n \rceil + 1$ rounds, and at most n items are assigned in each iteration. Therefore, at most $n(\log_2 n + 2)$ optimal items are assigned in that phase, i.e., $|\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*| \leq n(\log_2 n + 2)$. Furthermore, it holds $n \geq \ell_{i,1} = |\mathcal{L}_{i,1}|$ as in lemma 4. Together with the monotonicity of valuations, this yields

i: Error in paper? see also lemma 5

$$v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq v_i(j_i^+) \geq \frac{v_i((\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1})}{n(\log_2 n + 3)} \tag{32}$$

as lower bound on the valuations of bundles. Moreover, lemma 2 and the monotonicity of valuations yield

$$v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq \frac{v_i(\bar{\boldsymbol{x}}_{i,1}^*)}{n} \geq \frac{v_i(\bar{\boldsymbol{x}}_{i,1}^*)}{n(\log_2 n + 3)} = \frac{v_i(\bar{\boldsymbol{x}}_{i,0}^* \setminus \mathcal{L}_{i,1})}{n(\log_2 n + 3)} \tag{33}$$

as yet another lower bound. The mean of eqs. (32) to (33) and the monotonicity of valuations give the concise lower bound

$$v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq \frac{1}{2}\left(\frac{v_i((\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1})}{n(\log_2 n + 3)} + \frac{v_i(\bar{\boldsymbol{x}}_{i,0}^* \setminus \mathcal{L}_{i,1})}{n(\log_2 n + 3)}\right) \tag{34}$$

$$\geq \frac{1}{2} \cdot \frac{v_i(((\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1}) \cup (\bar{\boldsymbol{x}}_{i,0}^* \setminus \mathcal{L}_{i,1}))}{n(\log_2 n + 3)} \tag{35}$$

$$= \frac{v_i(\boldsymbol{x}_i^*)}{2n(\log_2 n + 3)}. \tag{36}$$

We can insert this lower bound into eq. (31) and prove the theorem thereby:

$$\log \text{NSW}(\boldsymbol{x}^{\text{III}}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n(\log_2 n + 3)}\right) = \log\left(\frac{\text{NSW}(\boldsymbol{x}^*)}{2n(\log_2 n + 3)}\right) \qquad (37)$$

$\square$

4 Hardness of Approximation

Garg, Kulkarni and Kulkarni [1, Sction 4] provide the following hardness result.

Theorem 3. *The submodular NSW is not approximable within a factor of $\frac{e}{e-1}$ in polynomial time unless $\mathcal{P} = \mathcal{NP}$.*

> **i:** *Should it not be $\frac{e-1}{e} - \epsilon$?*

Proof. Consider the related SW problem[1].

Problem 2. Given a set $\mathcal{G}$ of indivisible items and a set $\mathcal{A}$ of agents with monotonic, submodular valuation functions $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ for all agents $i \in \mathcal{A}$, the *symmetric submodular Social Welfare problem* (SW) is to find an allocation maximising the sum of valuations, that is

$$\underset{\boldsymbol{x} \in \boldsymbol{X}_n(\mathcal{G})}{\arg\max} \left\{ \sum_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i) \right\}$$

where $\boldsymbol{X}_n(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst n agents.

Note that this problem is identical to the symmetric submodular NSW except for the sum in the target function instead of a product. We will exploit this fact to calculate the NSW of instances for SW. Khot et al. [2] supply a polynomial-time reduction of SW from the following problem:

Problem 3. Given a graph $G = (V, E)$ and a constant $c \leq 1$, the *c-Gap-Max-3-Colouring problem* is to decide whether, for any 3-colouring of graph G which maximises the number of edges with different coloured endpoints, the number of such edges is $|E|$ (*Yes instance*) or $c|E|$ and below (*No instance*).

Proposition 1. *There exists a constant $c \leq 1$ such that the c-Gap-Max-3-Colouring problem is $\mathcal{NP}$-hard.*

Reducing an instance of the c-Gap-Max-3-Colouring problem yields an instance of the symmetric submodular SW problem with identical valuation functions. Its properties are as follows:

Yes instance The SW is nC because every agent values her bundle at n, whereby C is a constant depending on the input graph. The NSW of the instance would be C.

No instance The SW is $\frac{e-1}{e} nC$. Applying the inequality of arithmetic and geometric means, i.e., $(x_1 + \cdots + x_n)/n \geq \sqrt[n]{x_1 \cdots x_n}$ for all nonnegative numbers $x_1, \ldots, x_n \in \mathbb{R}_0^+$, reveals that the NSW of the instance is at most $\frac{e-1}{e} C$.

Thereout follows that the submodular NSW problem, even when symmetric and with identical valuation functions, cannot be approximated within a factor better than $\frac{e}{e-1}$; otherwise one could decide the c-Gap-Max-3-Colouring problem in polynomial time by checking whether the corresponding NSW instance has a value above $\frac{e-1}{e} C$. $\qquad \square$

For a constant number of agents, Garg, Kulkarni and Kulkarni [1, Section 5.1] describe a family $(A_\epsilon)_{\epsilon > 0}$ of algorithms for the asymmetric submodular NSW problem where each algorithm A_ϵ achieves an approximation factor of $\frac{e}{e-1} + \epsilon$.

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[1] Garg, Kulkarni and Kulkarni call it the 'Allocation problem'. We changed the name to match the naming scheme of the NSW problem and to avoid confusion, as both problems are about allocations.

5 conclusion

- Of course a short rehearsal of the results for the now knowledgeable reader.

- An outlook would be nice to have. Its content would mostly depend on what recent research has not yet answered.

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