Algorithms and Complexity Group
Institute for Computer Science
Goethe University Frankfurt

Summer Semester 2023

Seminar Approximation Algorithms

ANSWuSVþ(U)M

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18th May 2023

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Abstract

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1 Introduction

- problem introduction & motivation
- formal problem definition
- short literature review
- results & structure of paper

Definition 1. Let $\mathcal{G} := \{1, \dots, m\}$ be a set of indivisible *items* and $\mathcal{A} := \{1, \dots, n\}$ be a set of *agents*. An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathcal{P}(G)^n$ such that each item is element of exactly one $\boldsymbol{x}_i$, that is $\bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i = \mathcal{G}$ and $\boldsymbol{x}_i \cap \boldsymbol{x}_{i'} = \emptyset$ for all $i \neq i'$. An item $j \in \mathcal{G}$ is *assigned* to agent $i \in \mathcal{A}$ if $j \in \boldsymbol{x}_i$ holds.

⋮

Definition 2. Given a set $\mathcal{G}$ of items and a set $\mathcal{A}$ of agents with *valuations* $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ and *agent weights* η_i for all agents $i \in \mathcal{A}$, the *Nash Social Welfare problem* (NSW) is to find an allocation maximising the weighted geometric mean of valuations, that is

less strict def of valuations; restriction for our case later on

$$\operatorname*{arg\,max}_{\boldsymbol{x} \in \Pi_n(\mathcal{G})} \left\{ \left(\prod_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i)^{\eta_i} \right)^{1 / \sum_{i \in \mathcal{A}} \eta_i} \right\}$$

where $\Pi_n(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst n agents. The problem is called *symmetric* if all agent weights η_i are equal, and *asymmetric* otherwise.

⋮

In a slight abuse of notation, we omit the brackets in a valuation function if the set of items contains only one item, that is $v_i(j) = v_i(\{j\})$.

⋮

Garg, Kulkarni and Kulkarni consider five different types of non-negative monotonically non-decreasing valuation functions of which we are going to consider only the following two due to space constraints:

Additive The valuation $v_i(\mathcal{S})$ of an agent i for a set $\mathcal{S} \subset \mathcal{G}$ of items j is the sum of individual valuations $v_i(j)$, that is $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(j)$.

Submodular Let $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$ denote the marginal utility of agent i for a set $\mathcal{S}_1 \subset \mathcal{G}$ of items over the disjoint set $\mathcal{S}_2 \subset \mathcal{G}$. This valuation functions satisfies the submodularity constraint $v_i(j \mid \mathcal{S}_1 \cup \mathcal{S}_2) \leq v_i(j \mid \mathcal{S}_1)$ for all agents $i \in \mathcal{A}$, items $j \in \mathcal{G}$ and sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$ of items.

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2 SMatch

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Algorithm 1: SMatch for the Asymmetric Additive NSW problem

Input: set $\mathcal{A} = \{1, \dots, n\}$ of agents with weights $\eta_i \forall i \in \mathcal{A}$, set $\mathcal{G} = \{1, \dots, m\}$ indivisible items, additive valuations $v_i : \mathcal{P}(\mathcal{G}) \to \mathbb{R}^+_{>0}$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each item set $\mathcal{S} \subset \mathcal{G}$

Output: $\frac{1}{2n}$-approximation $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of an optimal allocation

$\boldsymbol{x}_i \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

$u_i \leftarrow v_i(\mathcal{G}_{i,[2n+1:m]}) \quad \forall i \in \mathcal{A}$

$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log\big(v_i(j) + \frac{u_i}{n}\big) \;\big|\; i \in \mathcal{A}, j \in \mathcal{G} \,\}$ ▷ *edge weights*

$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ ▷ *bipartite graph*

$\mathcal{M} \leftarrow$ max_weight_matching(G)

$\boldsymbol{x}_i \leftarrow \{\, j \mid (i,j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$ ▷ *allocate according to matching*

$\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$ ▷ *remove allocated goods*

while $\mathcal{G}^{\text{rem}} \neq \emptyset$ **do**

    $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log\big(v_i(j) + v_i(\boldsymbol{x}_i)\big) \;\big|\; i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}} \,\}$

    $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathcal{W})$

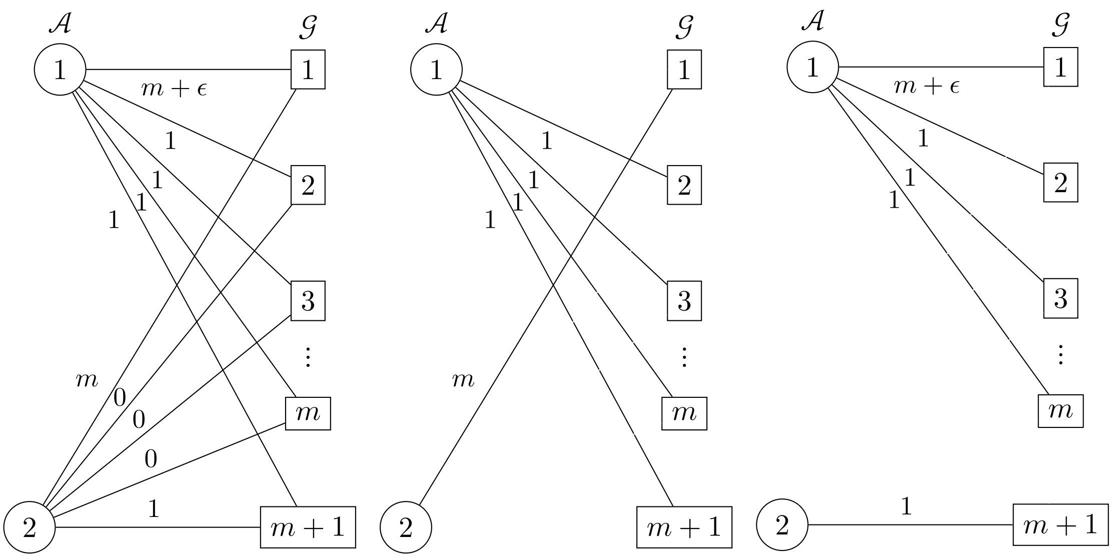
    $\mathcal{M} \leftarrow$ max_weight_matching(G)

    $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i \cup \{\, j \mid (i,j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$

    $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$

end while

return $\boldsymbol{x}$



(a) base case [might merge] **(b)** OPT $= \sqrt{m \cdot m}$ **(c)** ALG $\leq \sqrt{(2m+\epsilon-1)\cdot 1}$

Figure 1: Agent 1 values item 1 at $m+\epsilon$, and all other items at 1. Agent 2 values item 1 at m, item $m+1$ at 1, and all other items at 0. In an optimal allocation, item 1 would be assigned to agent 2 and all other items to agent 1, resulting in a NSW of $\sqrt{m \cdot m} = m$. A repeated maximum matching algorithm would greedily assign item 1 to agent 1 and item $m+1$ to agent 2 in the first round. Even if all remaining items were going to be assigned to agent 1, the NSW will never surpass $\sqrt{(2m+\epsilon-1)\cdot 1} < \sqrt{2m}$. The approximation factor $\alpha \approx \sqrt{m/2}$ is therefore dependant on the number m of items.

Algorithm 2: RepReMatch for the Asymmetric Submodular NSW problem

Input: set $\mathcal{A} = \{1, \dots, n\}$ of agents with weights $\eta_i \forall i \in \mathcal{A}$, set $\mathcal{G} = \{1, \dots, m\}$ indivisible items, additive valuations $v_i : \mathcal{P}(\mathcal{G}) \to \mathbb{R}^+_{>0}$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each item set $\mathcal{S} \subset \mathcal{G}$

Output: $\frac{1}{2n \log n}$-approximation $\boldsymbol{x}^{\mathrm{III}} = (\boldsymbol{x}_1^{\mathrm{III}}, \dots, \boldsymbol{x}_n^{\mathrm{III}})$ of an optimal allocation

Phase I:

$\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}$

for $t = 0, \dots, \lceil \log n \rceil - 1$ **do**

  if $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **then**

    $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j)) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$

    $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$

    $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$

    $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \boldsymbol{x}_i^{\mathrm{I}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$

    $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$

  end if

end for

Phase II:

$\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

while $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **do**

  $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$

  $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$

  $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$

  $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$

  $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$

end while

Phase III:

$\mathcal{G}^{\mathrm{rem}} \leftarrow \bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i^{\mathrm{I}}$ ▷ *release items allocated in first phase*

$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$

$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$

$\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$

$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$

$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$

$\boldsymbol{x}^{\mathrm{III}} \leftarrow \text{arbitrary\_allocation}(\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \boldsymbol{x}^{\mathrm{III}}, (v_i)_{i \in \mathcal{A}})$

return $\boldsymbol{x}^{\mathrm{III}}$