Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

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Supervised by Dr Giovanna Varricchio

27th July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

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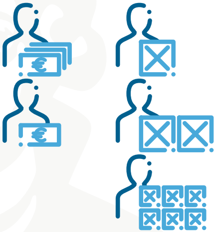

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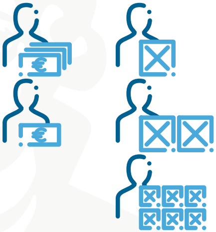

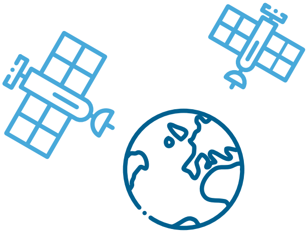

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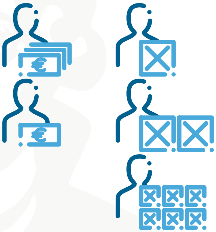

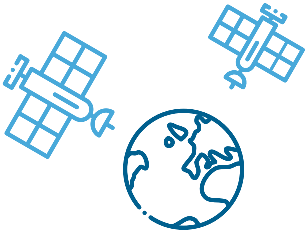

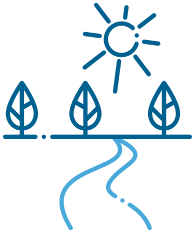

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But how to measure its efficiency and fairness?

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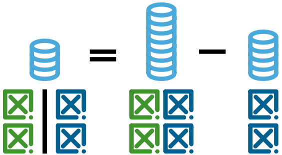

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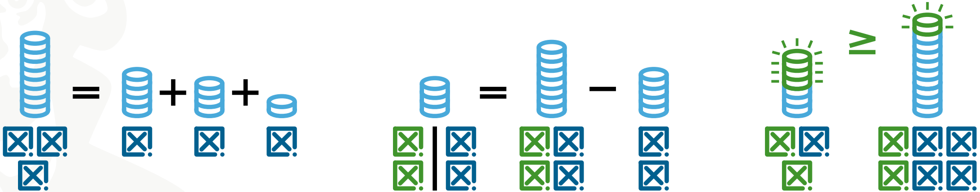
  - diminishing returns



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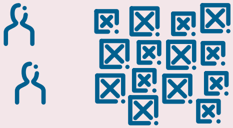

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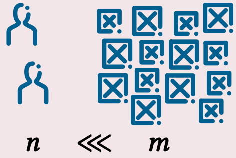

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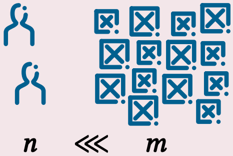
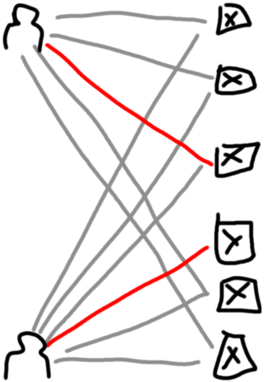


2 RepReMatch

Naïve Approach

Naïve approach:

- repeatedly use maximum matchings
- fails because of missing foresight
    - additive valuations: sort items by valuation
      $\Rightarrow 2n$-approximation (SMatch)
    - submodular valuations: lowest valuation
      approximable only by $\Omega(\sqrt{m/\ln m})$ ⚡



Key Ideas of the Algorithm

We need change the past in three phases:

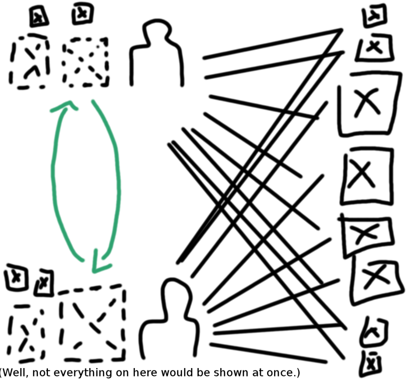
Phase I Assign enough high-value items temporarily.

Phase II Assign the remaining items definitely.

Phase III Re-assign the items of phase I definitely.

Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$-approximation under submodular valuations.



(Well, not everything on here would be shown at once.)

The Algorithm

Phase I:

1. repeat $\lceil \log_2 n \rceil + 1$ times
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\log v_i(j)^{\eta_i}$
   2. compute maximum weight matching
   3. update bundles $\boldsymbol{x}_i^{\mathrm{I}}$ & remove assigned items

Phase II:

2. repeat until $\mathscr{G} = \emptyset$
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})^{\eta_i}$
   2. compute maximum weight matching
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Phase III:

3. create bipartite graph $\left(\mathscr{A}, \bigcup_{i \in \mathscr{A}} \boldsymbol{x}_i^{\mathrm{I}}, E\right)$ with edge weights $\log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})^{\eta_i}$
4. compute maximum weight matching
5. create bundles $\boldsymbol{x}_i^{\mathrm{III}}$

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$\Rightarrow$ Are enough outstanding items reserved?

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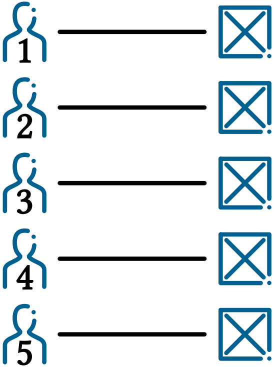

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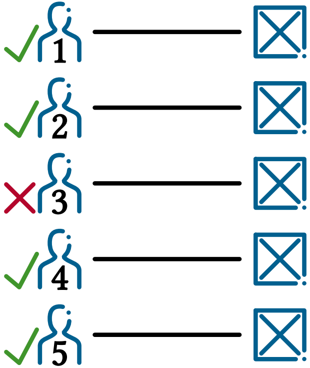

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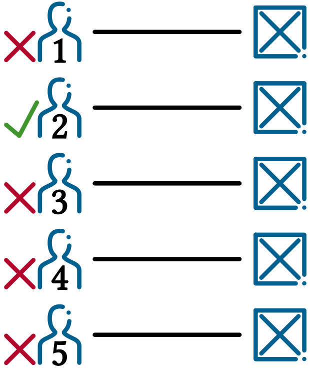

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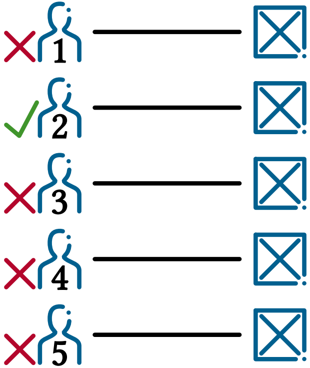

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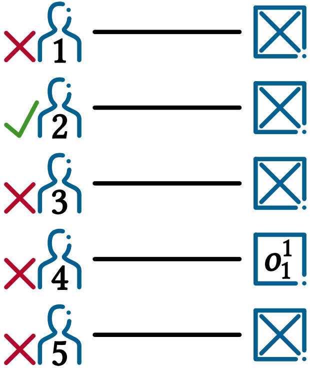

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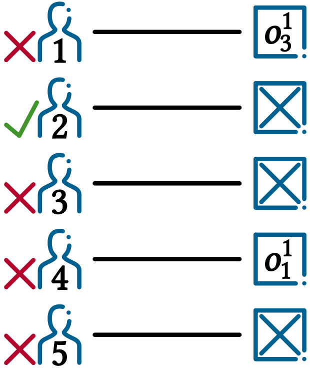

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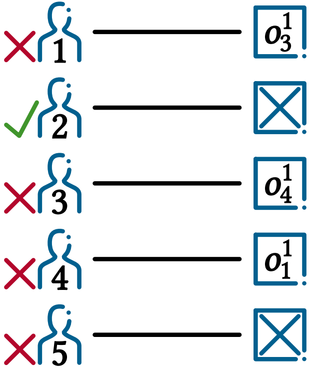

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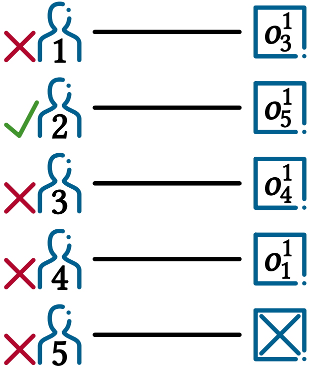

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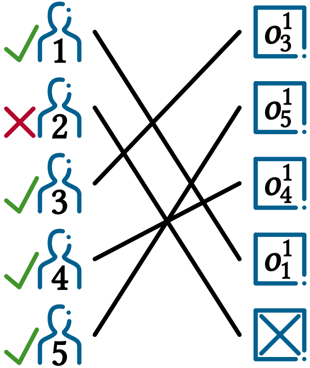

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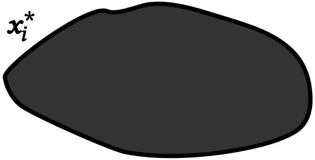

Analysing Phase II (1/2)

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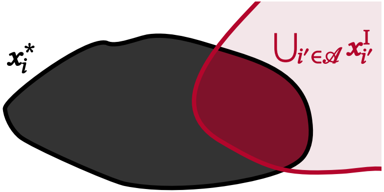

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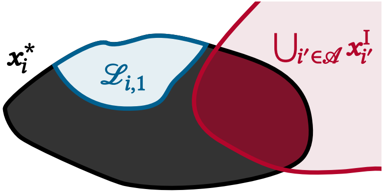

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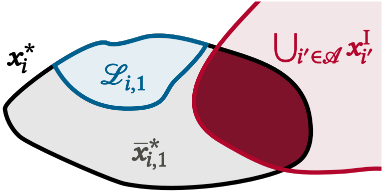

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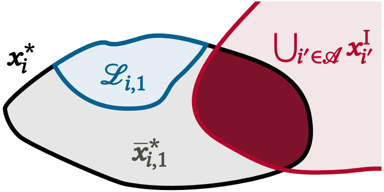

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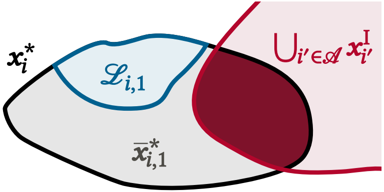

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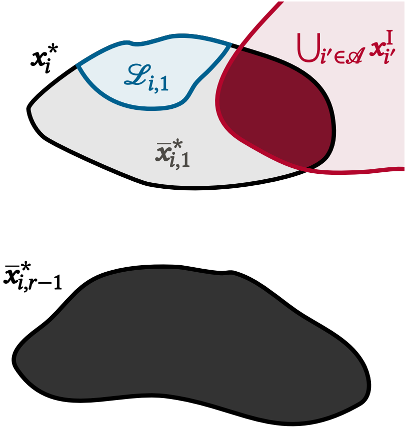

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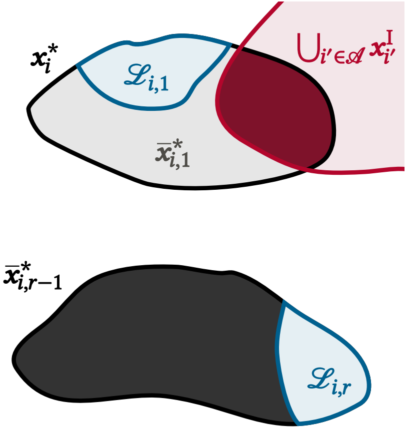

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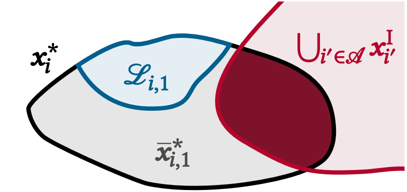

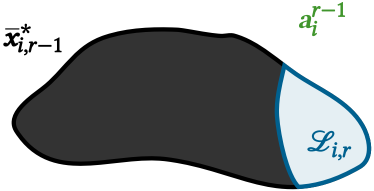

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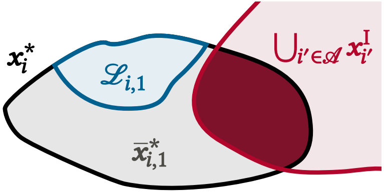

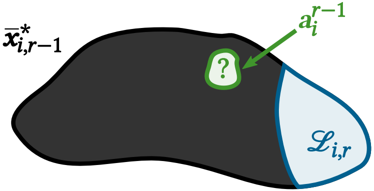

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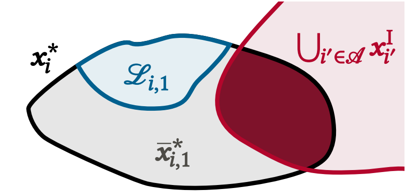

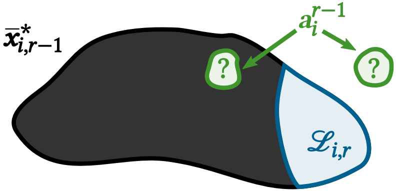

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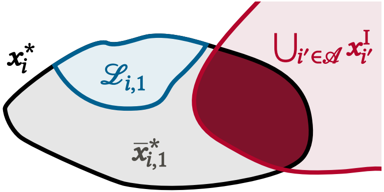

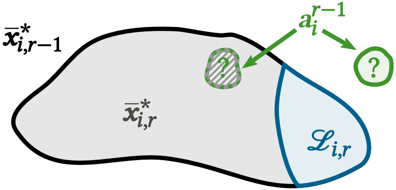

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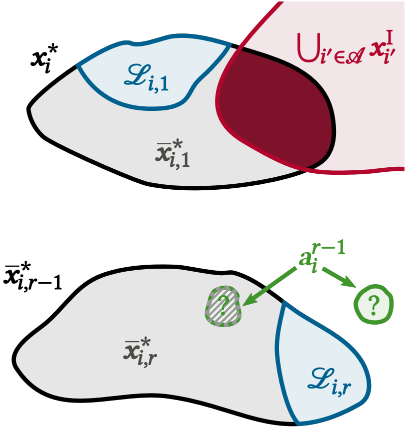

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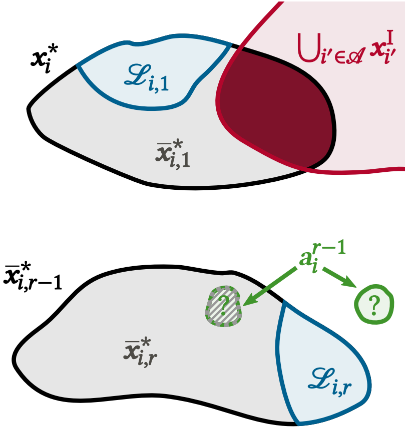
$\Rightarrow$ What is the valuation of the remaining items?



Analysing Phase II (2/2)

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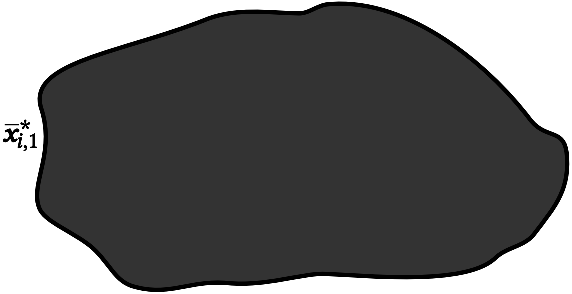

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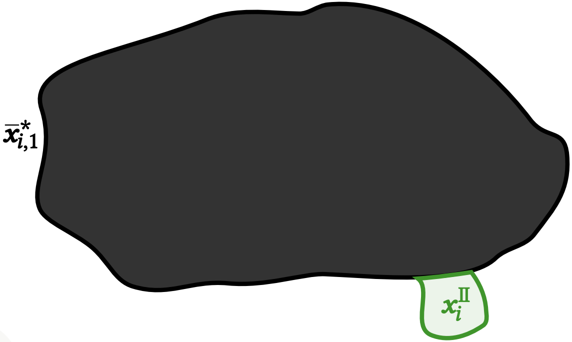

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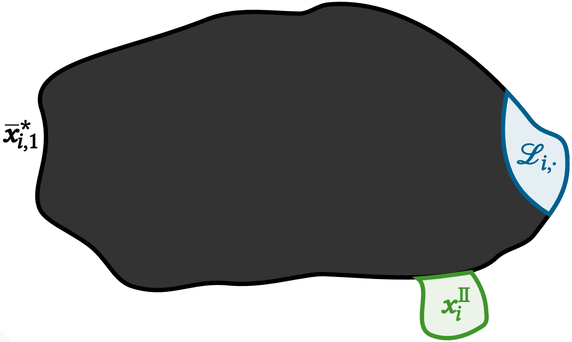

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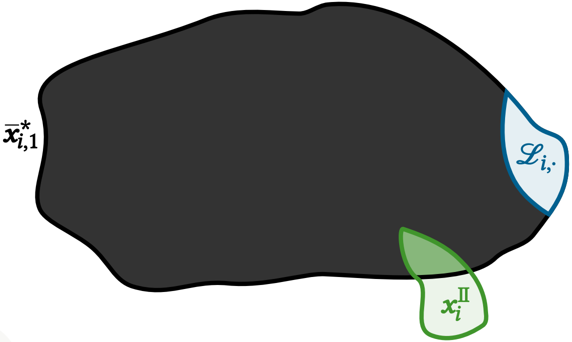

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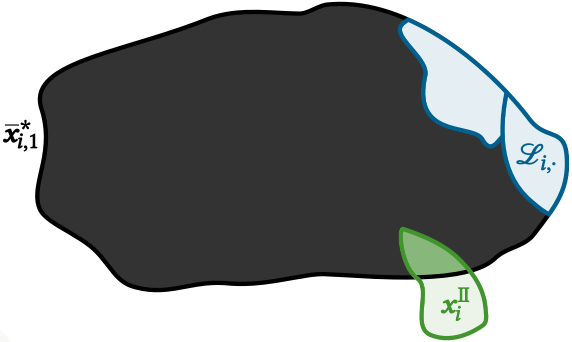

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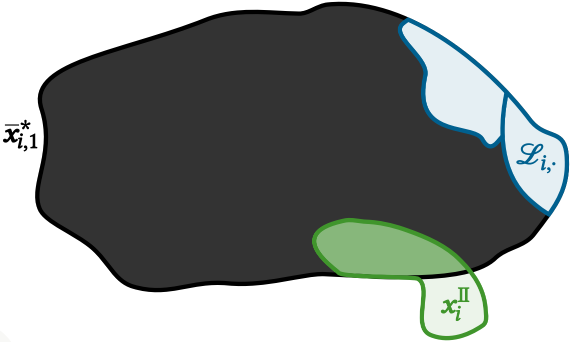

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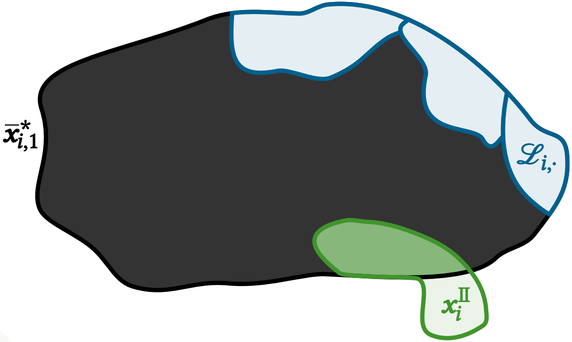

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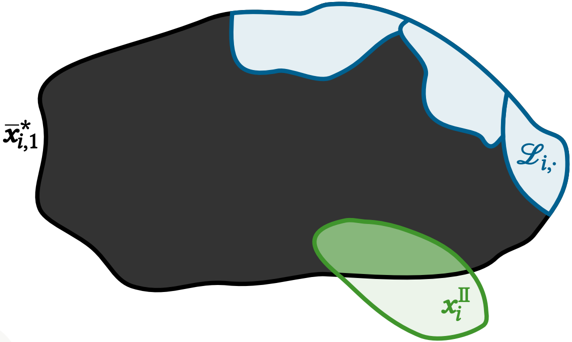

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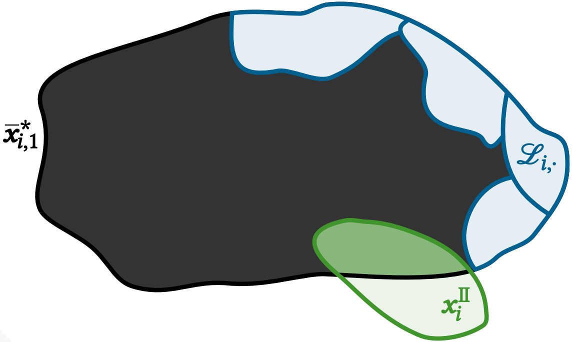

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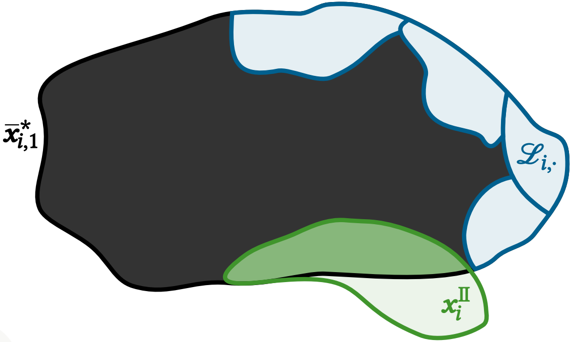

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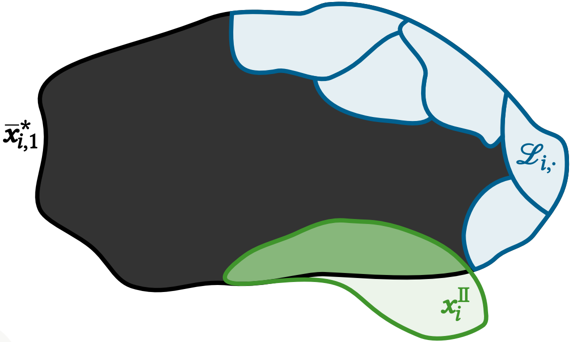

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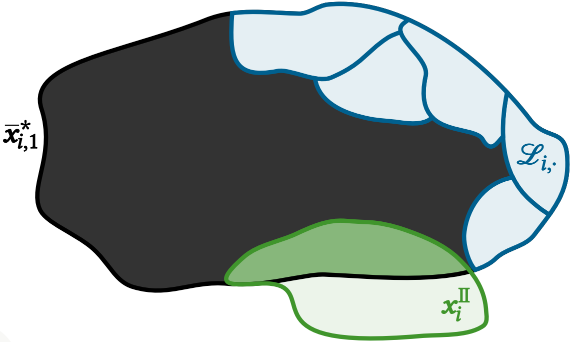

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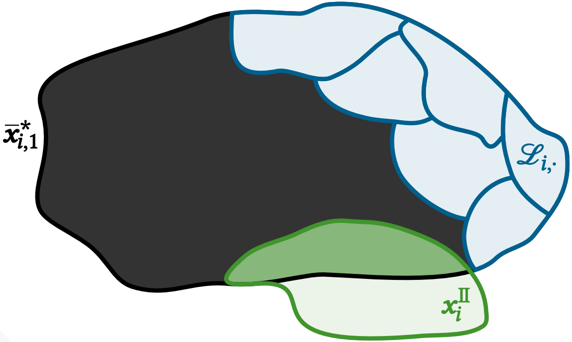

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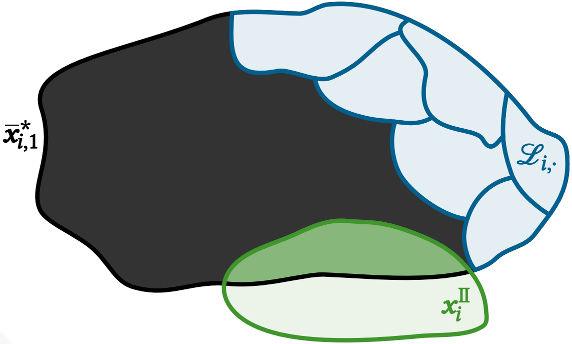

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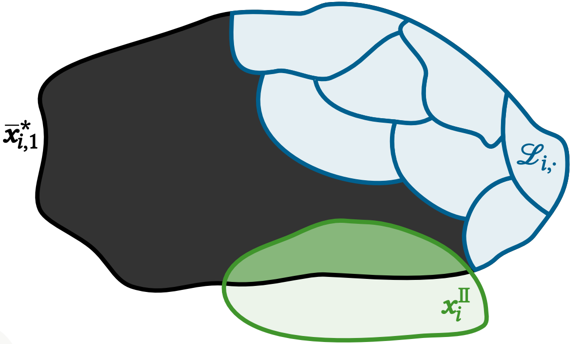

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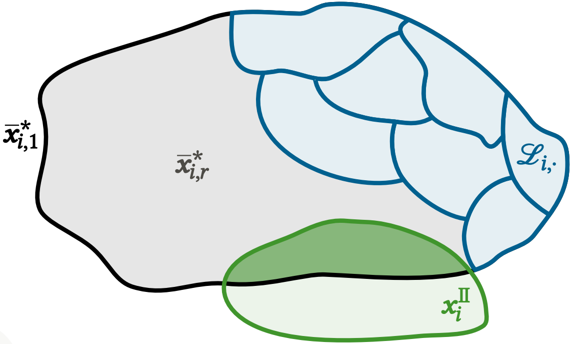

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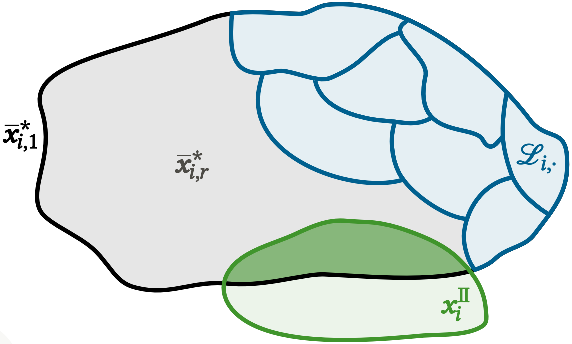

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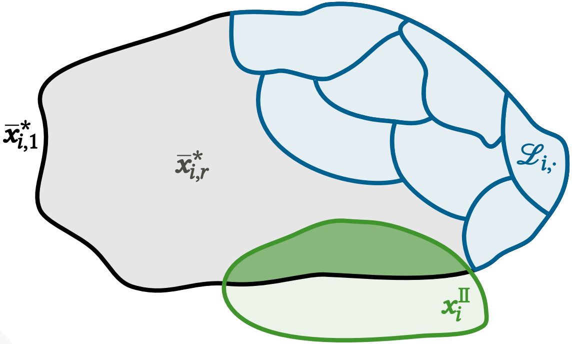

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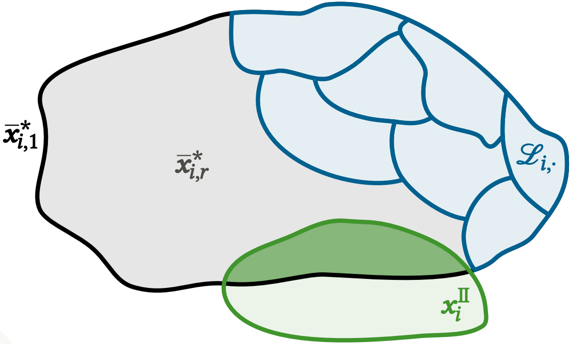

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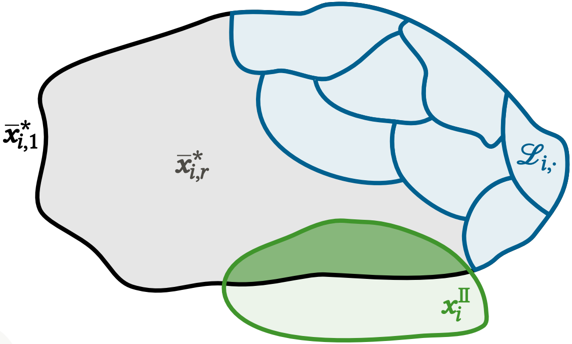

Analysing Phase II (2/2)

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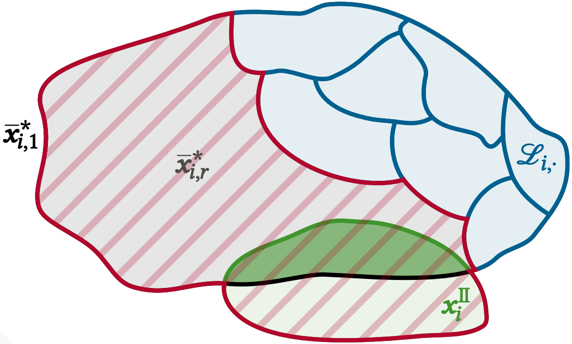

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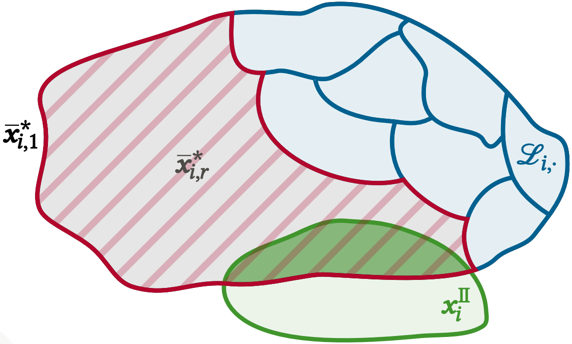

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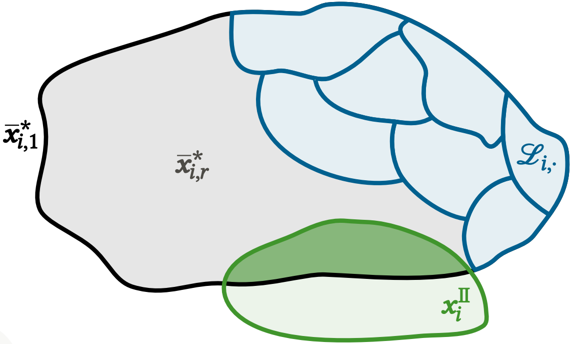

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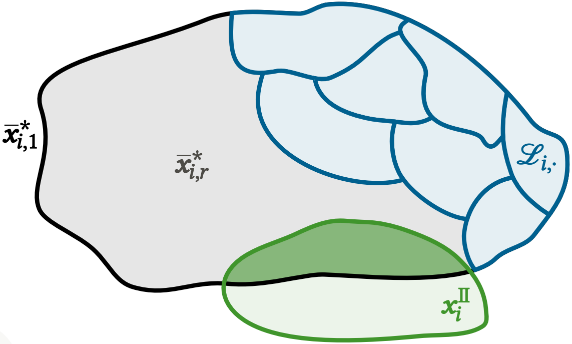

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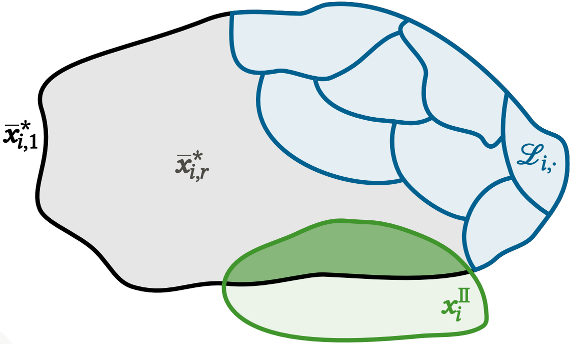

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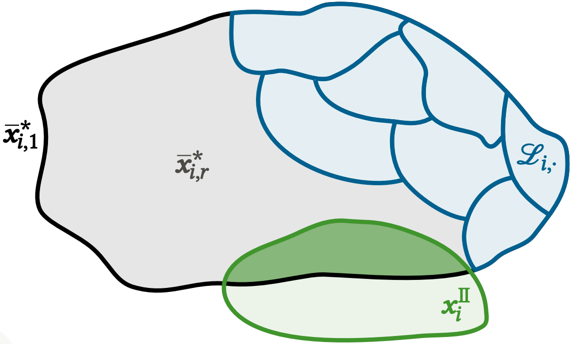

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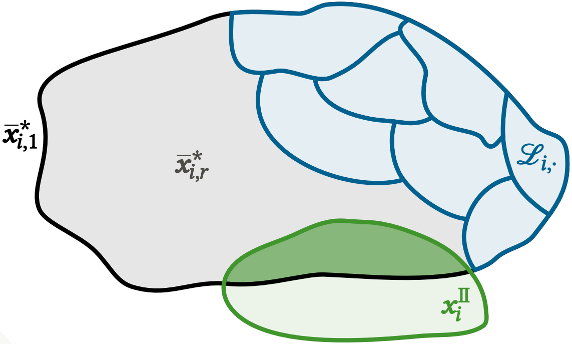

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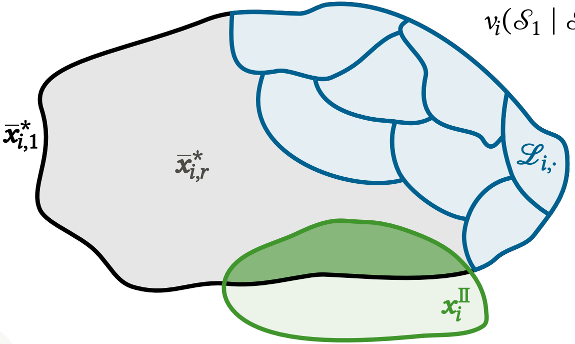

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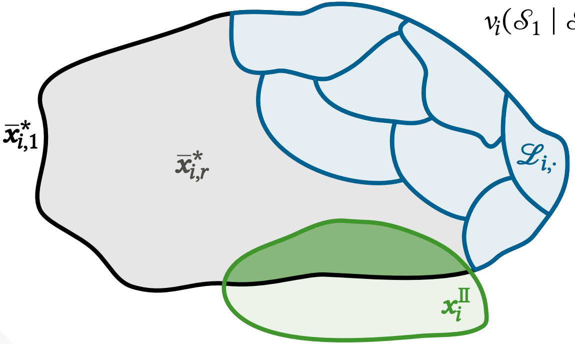

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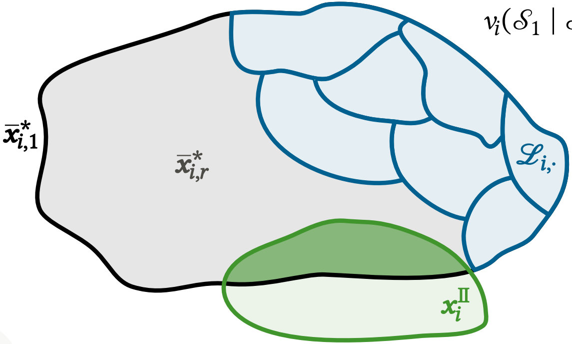

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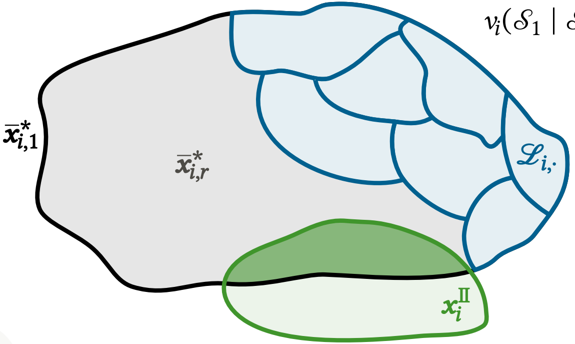

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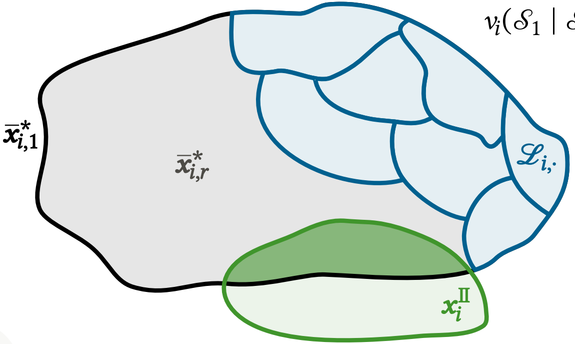

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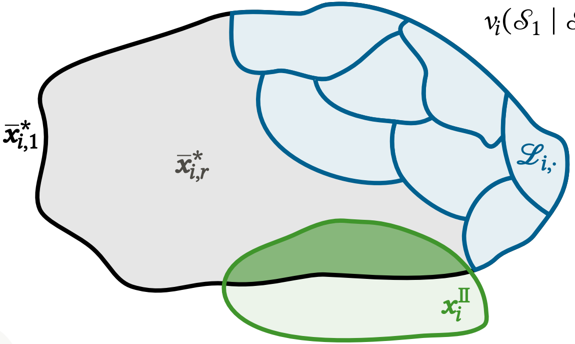

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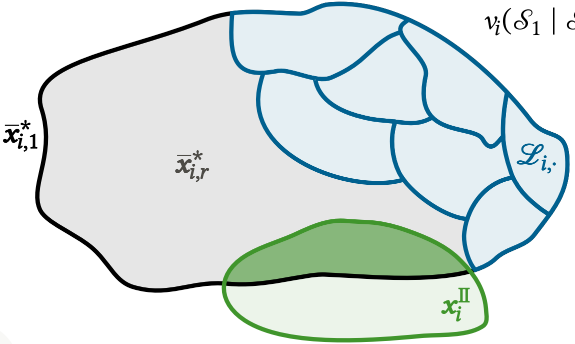

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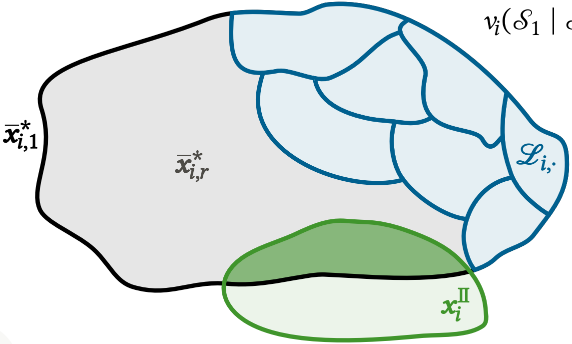

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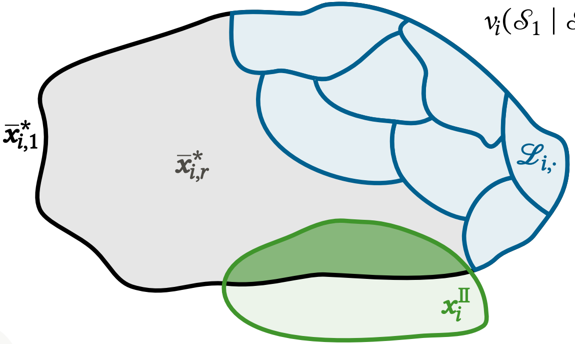

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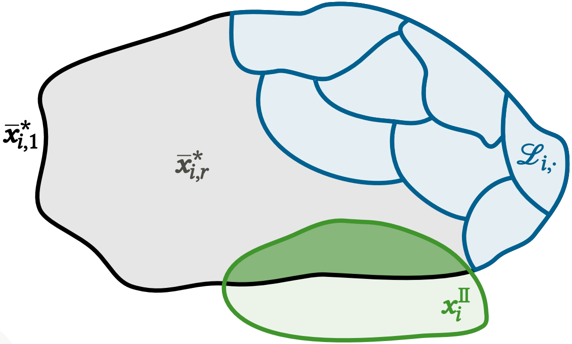

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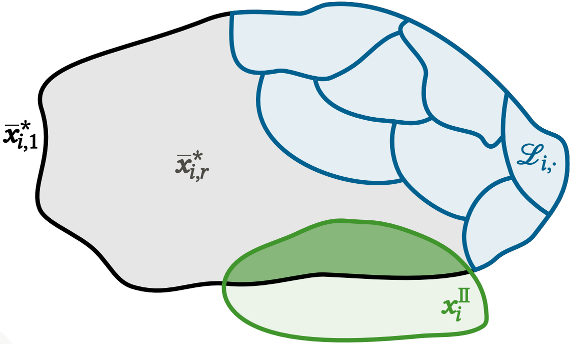

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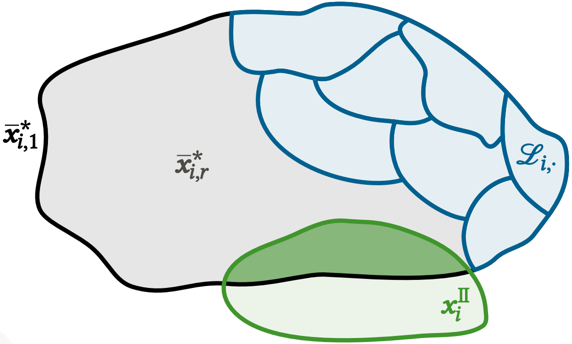

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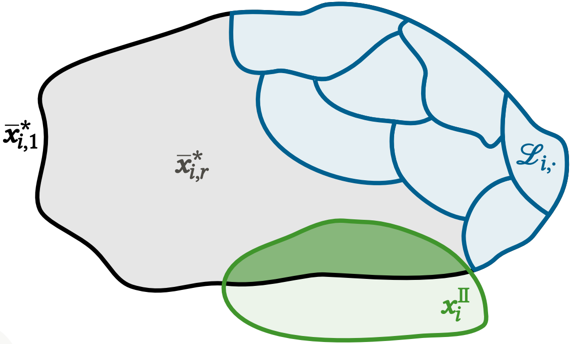

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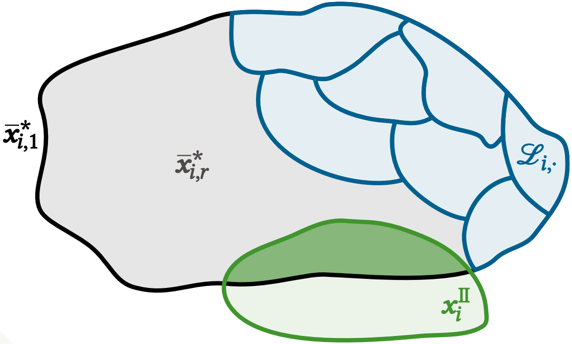

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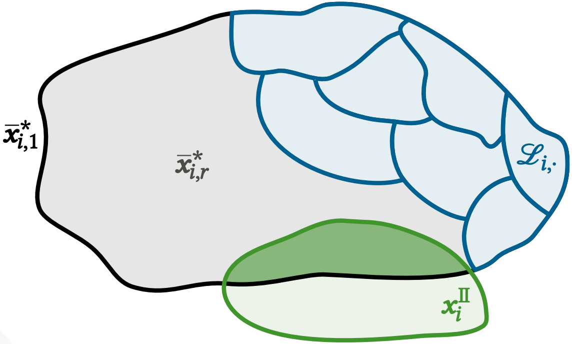

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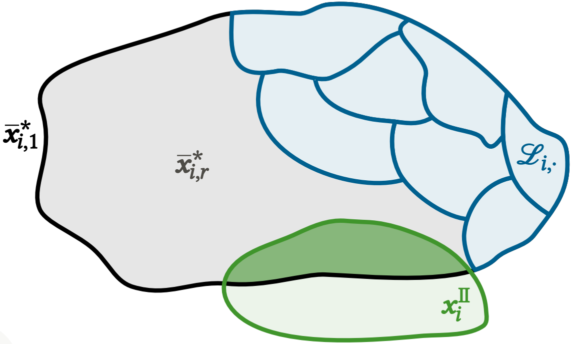

3

Conclusion

Summary & Outlook

Summary & Outlook

- allocation: partition of items amongst agents

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- allocation: partition of items amongst agents
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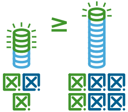

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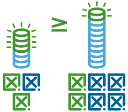

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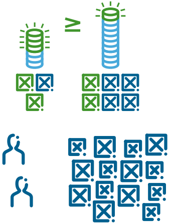

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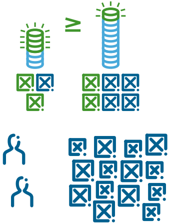

Summary & Outlook

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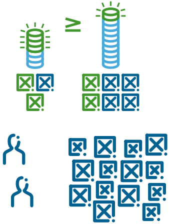

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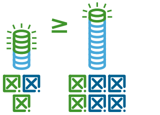
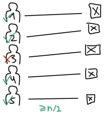
  Phase I finding enough outstanding items





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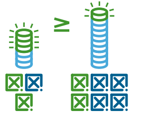
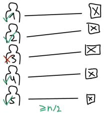
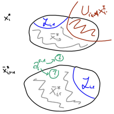
Phase I finding enough outstanding items

Phase II assigning remaining item







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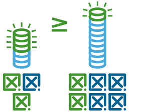
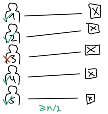
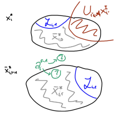
- Nash social welfare: weighted geometric mean of valuations
- approximation factor independent from m?
- simple, repeated matching fails because of missing foresight
- RepReMatch: $2n(\log n + 3)$-approximative

Phase I finding enough outstanding items
Phase II assigning remaining item
Phase III assigning outstanding items







Summary & Outlook

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- bundles valued using submodular valuation functions
- Nash social welfare: weighted geometric mean of valuations
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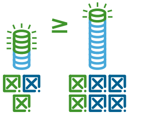

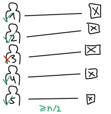

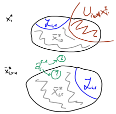

Any Room for Improvement?

Possibly! Lower bound of $\frac{e}{e-1} \approx 1.58$

End of Talk