Algorithms and Complexity Group
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ANSWuSVþ(U)M

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Abstract

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Todo list

1 Introduction

- problem introduction, motivation, applications
- formal problem definition
- short literature review: What is known, what not? New findings?
- content & structure of paper

Definition 1. Let $\mathcal{G} := \{1, \dots, m\}$ be a set of indivisible *items* and $\mathcal{A} := \{1, \dots, n\}$ be a set of *agents*. An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathcal{P}(G)^n$ such that each item is element of exactly one set $\boldsymbol{x}_i$, that is $\bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i = \mathcal{G}$ and $\boldsymbol{x}_i \cap \boldsymbol{x}_{i'} = \emptyset$ for all $i \neq i'$. An item $j \in \mathcal{G}$ is *assigned* to agent $i \in \mathcal{A}$ if $j \in \boldsymbol{x}_i$ holds.

or rather 'allocated'?

$\vdots$

Definition 2. Given a set $\mathcal{G}$ of items and a set $\mathcal{A}$ of agents with *valuations* $v_i : \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ and *agent weights* η_i for all agents $i \in \mathcal{A}$, the *Nash Social Welfare problem* (NSW) is to find an allocation maximising the weighted geometric mean of valuations, that is

less strict def of valuations; restriction for our case later on

$$\operatorname*{arg\,max}_{\boldsymbol{x} \in \Pi_n(\mathcal{G})} \left\{ \left(\prod_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i)^{\eta_i} \right)^{1/\sum_{i \in \mathcal{A}} \eta_i} \right\}$$

where $\Pi_n(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst n agents. The problem is called *symmetric* if all agent weights η_i are equal, and *asymmetric* otherwise.

agents without items assigned have valuation zero $\to$ prevent

$\vdots$

In a slight abuse of notation, we omit curly braces delimiting a set in the arguments of a valuation function, so for example we write $v(j_1, j_2, \dots)$ to denote $v(\{j_1, j_2, \dots\})$.

$\vdots$

definition of approximation factor [def environment or in-text?]

$\vdots$

Garg, Kulkarni and Kulkarni [1] consider five different types of non-negative monotonically non-decreasing valuation functions of which we are going to consider only the following two due to space constraints:

Additive The valuation $v_i(\mathcal{S})$ of an agent i for a set $\mathcal{S} \subset \mathcal{G}$ of items j is the sum of individual valuations $v_i(j)$, that is $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(j)$.

Submodular Let $v_i(\mathcal{S} \mid \mathcal{S}) := v_i(\mathcal{S} \cup \mathcal{S}) - v_i(\mathcal{S})$ denote the marginal utility of agent i for a set $\mathcal{S} \subset \mathcal{G}$ of items over the disjoint set $\mathcal{S} \subset \mathcal{G}$. This valuation functions satisfies the submodularity constraint $v_i(j \mid \mathcal{S} \cup \mathcal{S}) \leq v_i(j \mid \mathcal{S})$ for all agents $i \in \mathcal{A}$, items $j \in \mathcal{G}$ and sets $\mathcal{S}, \mathcal{S} \subset \mathcal{G}$ of items.

We use *additive NSW* and *submodular NSW* as shorthands for the Nash social welfare problems with additive and submodular valuation functions, respectively.

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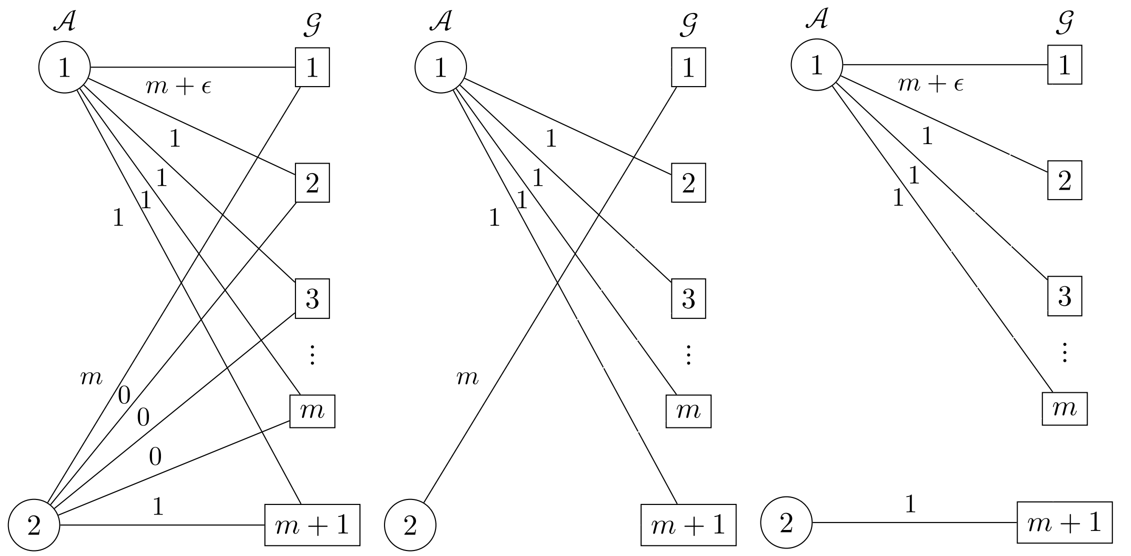

(a) problem instance **(b)** OPT $= \sqrt{m \cdot m}$ **(c)** ALG $\leq \sqrt{(2m+\epsilon-1)\cdot 1}$

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Figure 1: Agent 1 values item 1 at $m+\epsilon$, and all other items at 1. Agent 2 values item 1 at m, item $m+1$ at 1, and all other items at 0. In an optimal allocation, item 1 would be assigned to agent 2 and all other items to agent 1, resulting in a NSW of $\sqrt{m \cdot m} = m$. A repeated maximum matching algorithm would greedily assign item 1 to agent 1 and item $m+1$ to agent 2 in the first round. Even if all remaining items were going to be assigned to agent 1, the NSW will never surpass $\sqrt{(2m+\epsilon-1)\cdot 1} < \sqrt{2m}$. The approximation factor $\alpha \approx \sqrt{m/2}$ therefore depends on the number of items.

2 SMatch

In the case of an equal number of agents and items, i. e. $|\mathcal{A}| = n = m = |\mathcal{G}|$, the *additive* NSW can be solved exactly by finding a maximum matching on a bipartite graph with the sets of agents and of items as its parts; as weight of the edge between agent i and item j, use $\eta_i \log v_i(j)$, that is the weighted valuation of item j by agent i in the *logarithmic* Nash social welfare. Should there be more items than agents, then it would be obvious to just repeatedly find a maximum matching and assign the items accordingly until all items are allocated. The flaw of this idea is that such a greedy algorithm only considers the valuations of items in the current matching and perhaps the valuations of items already assigned. As the example in fig. 1 demonstrates, this leads to an algorithm with an approximation factor dependent on the number m of items. The geometric mean of the NSW favours allocations with similar valuations, wherefore it may be beneficial for an agent to leave items to other agents if those agents cannot expect many more valuable items in the future.

What is stimmt das?

The algorithm SMatch, described in algorithm 1, eliminates the flaw by first gaining foresight of the valuations of items assigned after the first matching, achieving an approximation factor of $2n$ (cf. theorem 1 later on). For a fixed agent i, order the items in descending order of the valuations by agent i and denote the j-th most liked item by $\mathcal{G}_i^j$. To obtain a well-defined order, items of equal rank are further ordered numerically. SMatch, too, does repeatedly match items. During the first matching, however, the edge weights are defined as $\eta_i \log(v_i(j) + u_i/n)$ for an edge between agent i and item j. The addend u_i serves as estimation of the valuation of items assigned after the first matching

Algorithm 1: SMatch for the Asymmetric Additive NSW problem

Input: set $\mathcal{A} = \{1, \dots, n\}$ of agents with weights $\eta_i \forall i \in \mathcal{A}$, set $\mathcal{G} = \{1, \dots, m\}$ indivisible items, additive valuations $v_i : \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each item set $\mathcal{S} \subset \mathcal{G}$

Output: $\frac{1}{2n}$-approximation $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of an optimal allocation

$\boldsymbol{x}_i \leftarrow \emptyset \quad \forall i \in \mathcal{A}$
$u_i \leftarrow v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m) \quad \forall i \in \mathcal{A}$
$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j) + u_i/n) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$ ▷ *edge weights*
$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ ▷ *bipartite graph*
$\mathcal{M} \leftarrow$ max_weight_matching(G)
$\boldsymbol{x}_i \leftarrow \{\, j \mid (i,j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$ ▷ *allocate according to matching*
$\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$ ▷ *remove allocated goods*
while $\mathcal{G}^{\text{rem}} \neq \emptyset$ **do**
  $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j) + v_i(\boldsymbol{x}_i)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}} \,\}$
  $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathcal{W})$
  $\mathcal{M} \leftarrow$ max_weight_matching(G)
  $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i \cup \{\, j \mid (i,j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$
  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$
end while
return $\boldsymbol{x}$

and is defined as

$$u_i := \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \le 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m). \tag{1}$$

The set $\mathcal{S}$ has less than $2n$ elements only if there are less than $2n$ items in total. From the second matching onwards, the edge weights are defined as $\eta_i \log(v_i(j) + v_i(\boldsymbol{x}_i))$, where $\boldsymbol{x}_i$ is the continuously updated set of items assigned to agent i in previous matchings. The addend $v_i(\boldsymbol{x}_i)$ could lead to better allocations in applications, but does not improve the approximation factor asymptotically.

For convenience, we order the items in the final allocation $\boldsymbol{x}_i = \{h_i^1, \dots, h_i^{\tau_i}\}$ of agent i by the order in which they were assigned, so that item h_i^t is assigned after the t-th matching. To calculate the approximation factor of SMatch, we first need to establish a lower bound on the valuation of single items.

Lemma 1. *For each agent i and their final allocation $\boldsymbol{x}_i = \{h_i^1, \dots, h_i^{\tau_i}\}$, the item h_i^t is worth at least as much as the overall tn-th most highly valued item $\mathcal{G}_i^{tn}$, i. e. $v_i(h_i^t) \ge v_i(\mathcal{G}_i^{tn})$, for all $t = 1, \dots, \tau_i$.*

Proof. At the start of the t-th iteration, no more than $(t-1)n$ items out of the tn most highly valued items $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ have been assigned in previous iterations since at most n items are assigned in each iteration. During the t-th iteration, at most $n-1$ more of those highly valued items could be assigned to all other agents $i' \neq i$, leaving at least one item in $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ unassigned. Since $v_i(\mathcal{G}_i^k) \ge v_i(\mathcal{G}_i^{tn})$ for all $k \le tn$ by definition of $\mathcal{G}_i^{\cdot}$, the lemma follows. □

We can now establish $u_i/n = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m)/n$ as lower bound on the valuations of items assigned after the first matching.

Lemma 2. $v_i(h_i^2, \dots, h_i^{\tau_i}) \ge u_i/n$.

Proof. By lemma 1 and definition of $\mathcal{G}_i^{\cdot}$, every item h_i^t is worth at least as much as each item $\mathcal{G}_i^{tn+k}$ with $k \in \{0, \dots, n-1\}$ and, consequently, its valuation $v_i(h_i^t)$ is at least as high as the mean valuation $\frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1})$. Further, it holds $\tau_i n + n \geq m$ since each agent receives items for at least $\lfloor \frac{m}{n} \rfloor \geq \frac{m}{n} - 1$ rounds. Together, this yields

$$v_i(h_i^2, \dots, h_i^{\tau_i}) = \sum_{t=2}^{\tau_i} v_i(h_i^t) \geq \sum_{t=2}^{\tau_i} \frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1}) \tag{2}$$

$$\geq \frac{1}{n} v_i(\mathcal{G}_i^{2n}, \dots, \mathcal{G}_i^{m-1}) \geq \frac{1}{n} v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^{m}) = \frac{u_i}{n} \tag{3}$$

with the last inequality stemming from $v_i(\mathcal{G}_i^{2n}) \geq v_i(\mathcal{G}_i^m)$. □

This allows us to calculate a lower bound on the approximation factor of SMatch.

Theorem 1. *SMatch has an approximation factor of* $2n$.

Proof. Lemma 2 can be plugged into the logarithmic NSW:

$$\log \mathrm{NSW}(\boldsymbol{x}) = \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log v_i(h_i^1, \dots, h_i^{\tau_i}) \tag{4}$$

$$= \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log(v_i(h_i^1) + v_i(h_i^2, \dots, h_i^{\tau_i})) \tag{5}$$

$$\geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log(v_i(h_i^1) + u_i/n) \tag{6}$$

Notice that the first matching of SMatch maximises the sum in eq. (6). Thus, assigning all agents i their respective most highly valued item g_i^1 in an optimal allocation $\boldsymbol{x}_i^* = \{g_i^1, \dots, g_i^{\tau_i^*}\}$ yields the even lower bound

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log(v_i(g_i^1) + u_i/n). \tag{7}$$

Recall the definition of u_i from eq. (1). Consider a slightly modified variant:

$$u_i = \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \quad \text{or, alternatively,} \quad u_i = v_i(\mathcal{G} \setminus \mathcal{S}) \text{ with } \mathcal{S} := \arg\min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \tag{8}$$

Moreover, consider the set $\mathcal{S}^*$ of the (at most) $2n$ most highly valued items in the optimal allocation $\boldsymbol{x}_i^*$, i. e.

$$\mathcal{S}^* := \arg\min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S})\}. \tag{9}$$

We get the lower bound $v_i(g_i^1) \geq \frac{1}{2n} v_i(\mathcal{S}^*)$ from a similar argument as in the proof of lemma 2. Further, it holds $u_i = v_i(\mathcal{G} \setminus \mathcal{S}) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}^*)$. We can substitute these into eq. (7) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log\left(\frac{v_i(\mathcal{S}^*)}{2n} + \frac{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}^*)}{n}\right) \tag{10}$$

$$\geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n}\right) = \log\left(\frac{\mathrm{OPT}}{2n}\right) \tag{11}$$

□

3 RepReMatch

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Algorithm 2: RepReMatch for the Asymmetric Submodular NSW problem

Input: set $\mathcal{A} = \{1, \dots, n\}$ of agents with weights $\eta_i \forall i \in \mathcal{A}$, set $\mathcal{G} = \{1, \dots, m\}$ indivisible items, additive valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each item set $\mathcal{S} \subset \mathcal{G}$

Output: $\frac{1}{2n \log n}$-approximation $\boldsymbol{x}^{\mathrm{III}} = (\boldsymbol{x}_1^{\mathrm{III}}, \dots, \boldsymbol{x}_n^{\mathrm{III}})$ of an optimal allocation

Phase I:

$\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}$

for $t = 0, \dots, \lceil \log n \rceil - 1$ **do**

  if $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **then**

    $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j)) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$

    $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$

    $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$

    $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \boldsymbol{x}_i^{\mathrm{I}} \cup \{j\} \quad \forall (i, j) \in \mathcal{M}$

    $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i, j) \in \mathcal{M} \,\}$

  end if

end for

Phase II:

$\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

while $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **do**

  $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$

  $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$

  $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$

  $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i, j) \in \mathcal{M}$

  $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i, j) \in \mathcal{M} \,\}$

end while

Phase III:

$\mathcal{G}^{\mathrm{rem}} \leftarrow \bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i^{\mathrm{I}}$ ▷ *release items allocated in first phase*

$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$

$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$

$\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$

$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i, j) \in \mathcal{M}$

$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i, j) \in \mathcal{M} \,\}$

$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \mathrm{arbitrary\_allocation}(\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \boldsymbol{x}_i^{\mathrm{III}}, (v_i)_{i \in \mathcal{A}})$

return $\boldsymbol{x}^{\mathrm{III}}$

decide on $\boldsymbol{x}$ or $\boldsymbol{x}^{\mathrm{III}}$; discrepancies in orig paper in def of $\boldsymbol{x}^{\mathrm{III}}$!