

Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

Zeno Adrian Weil

Supervised by Dr Giovanna Varricchio

30th July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

What is the issue?



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We need to distribute goods amongst recipients



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We need to distribute goods amongst recipients *efficiently*



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Where is this encountered?

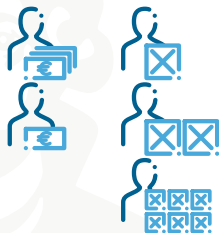


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- procurement

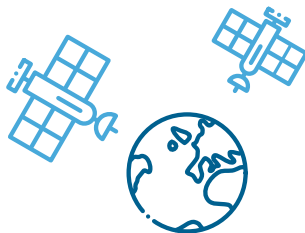
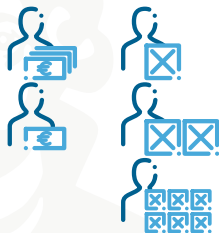


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- procurement
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- water withdrawal

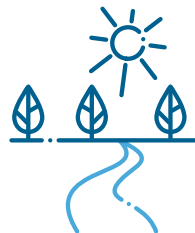
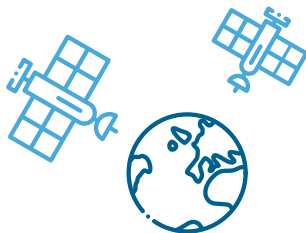
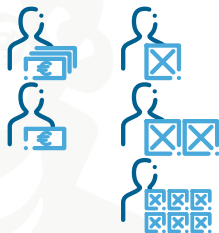


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1

Preliminaries



Preliminaries

Allocations



Setting:



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- recipients: set \mathcal{A} of n agents



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But how to measure its efficiency and fairness?

Valuation Functions



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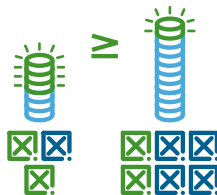
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 - diminishing returns



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Problem

$$\mathbf{x}^* \stackrel{!}{=} \arg \max_{\mathbf{x} \in X_{\mathcal{A}}(\mathcal{G})} \{\text{NSW}(\mathbf{x})\}$$

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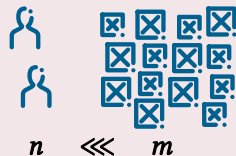
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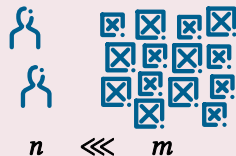
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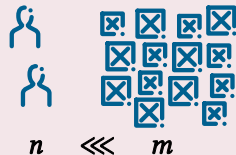
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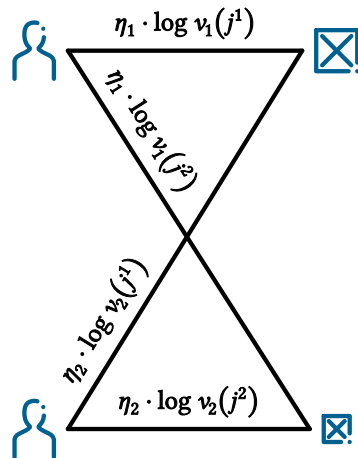
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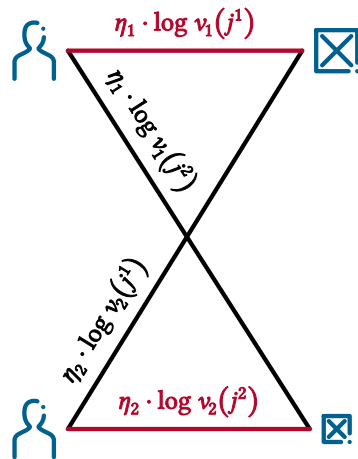
RepReMatch



Naïve Approach







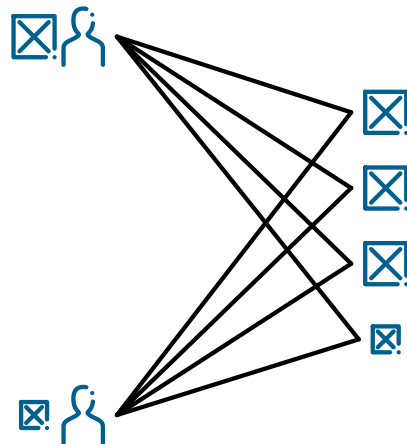
Naïve Approach



Naïve Approach

Greedy algorithm:

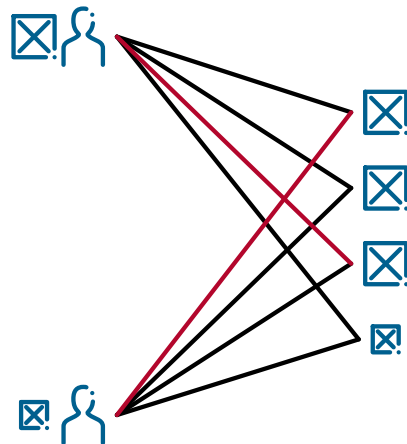
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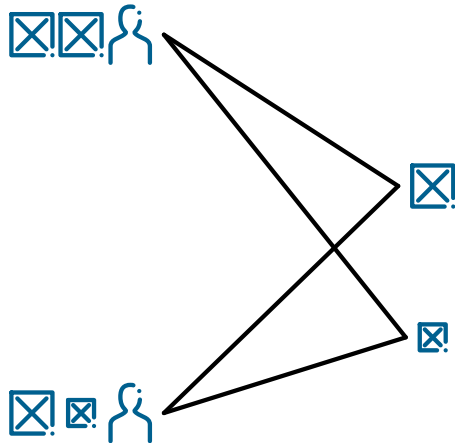
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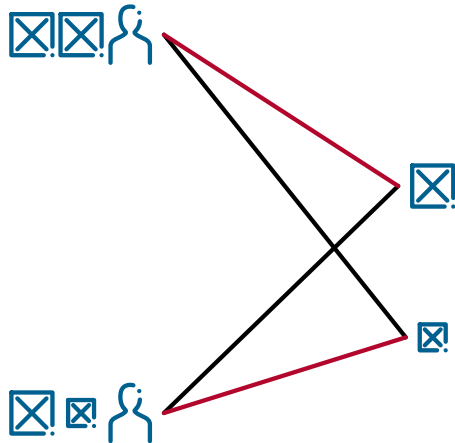
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 $\Rightarrow 2n$ -approximation (*SMatch*)



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approximable only by $\Omega(\sqrt{m/\ln m})$



Naïve Approach

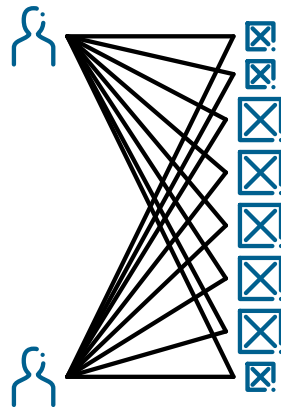
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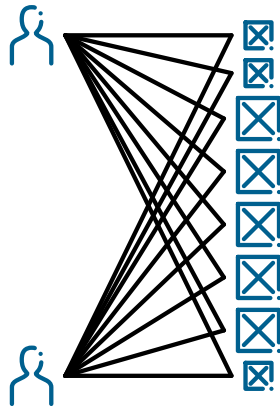
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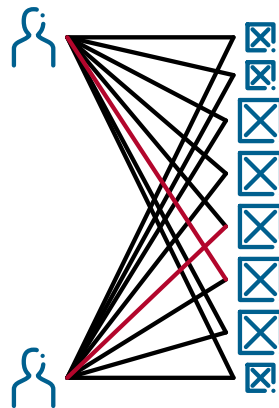
Phase I Assign enough high-value items **temporarily**.



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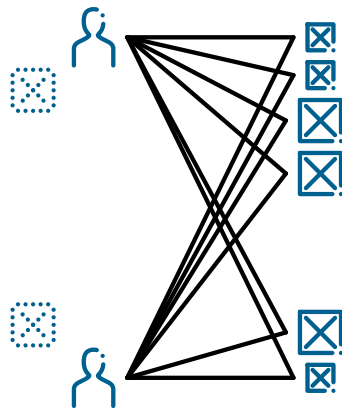
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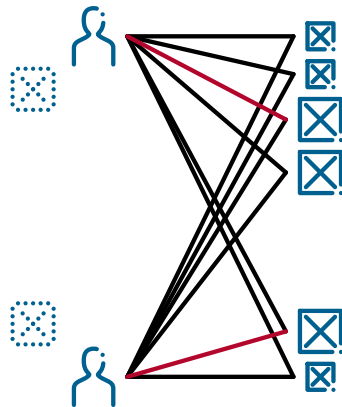
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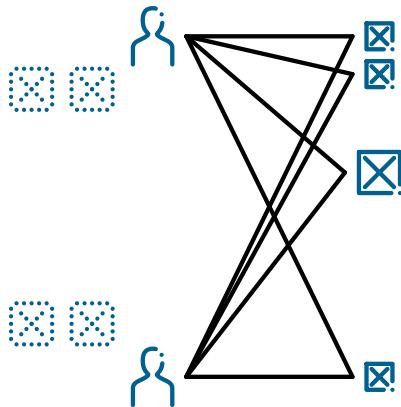
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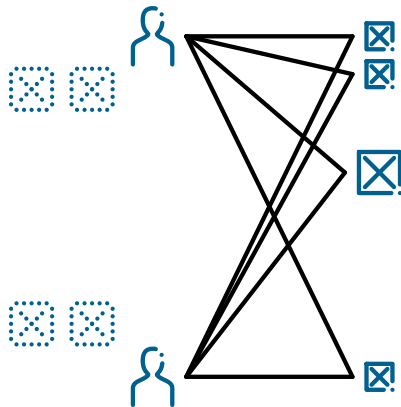


Key Ideas of the Algorithm

We need change the past in three phases:

Phase I Assign enough high-value items temporarily.

Phase II Assign the remaining items **definitely**.

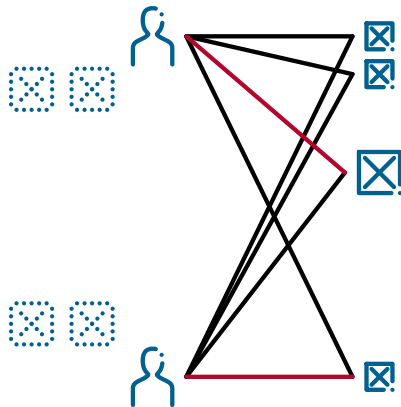


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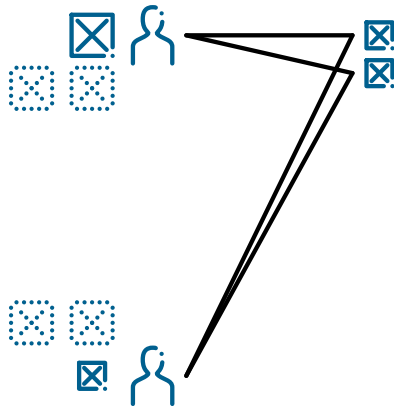


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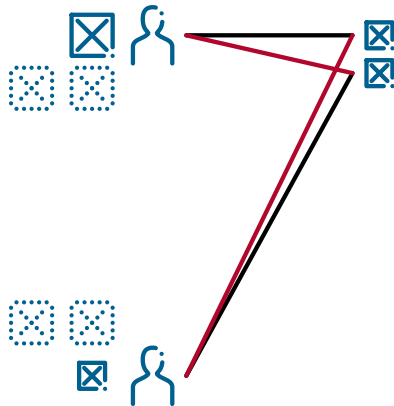


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Phase III Re-assign the items of phase I **definitely**.



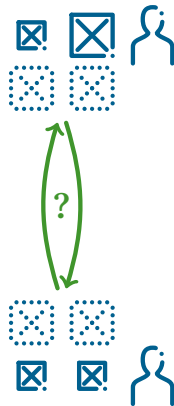
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Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$ -approximation under submodular valuations.



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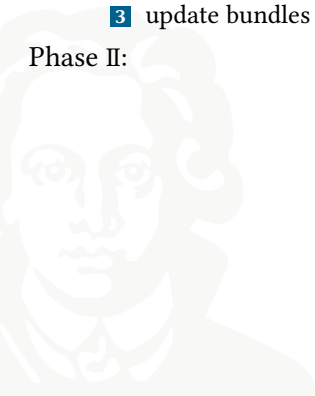


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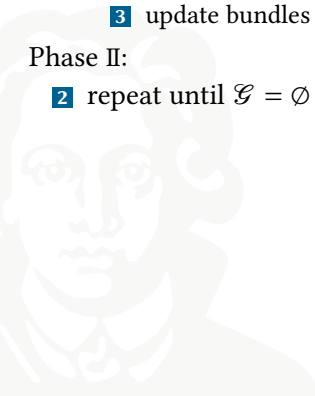
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- 2 repeat until $\mathcal{G} = \emptyset$



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Phase III:

The Algorithm

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 - 1 create bipartite graph $(\mathcal{A}, \mathcal{G}, E)$ with edge weights $\eta_i \cdot \log v_i(\mathbf{x}_i^I \cup \{j\})$
 - 2 compute maximum weight matching
 - 3 update bundles \mathbf{x}_i^I & remove assigned items

Phase III:

- 3 create bipartite graph $(\mathcal{A}, \bigcup_{i \in \mathcal{A}} \mathbf{x}_i^I, E)$ with edge weights $\eta_i \cdot \log v_i(\mathbf{x}_i^I \cup \{j\})$
- 4 compute maximum weight matching
- 5 create bundles \mathbf{x}_i^I

Analysing Phases I & III (1/2)



Analysing Phases I & III (1/2)

Phase I reserves 'high-value' items.



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Phase I reserves 'high-value' items. But what qualifies as 'high-value'?



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Let $\mathbf{x}_i^* = \{o_i^1, o_i^2, \dots\}$ be an optimal bundle.



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Definition

Let $\mathbf{x}_i^* = \{o_i^1, o_i^2, \dots\}$ be an optimal bundle. An item $j \in \mathcal{G}$ is **outstanding** if $v_i(j) \geq v_i(o_i^1)$.



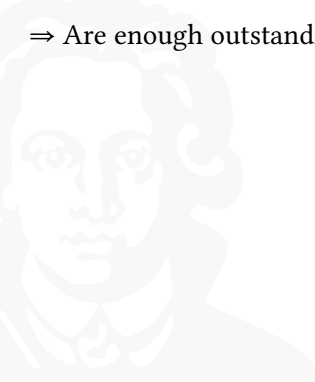
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⇒ Are enough outstanding items reserved?



Analysing Phases I & III (2/2)

Lemma

Each agent can be matched with an outstanding item in phase III.

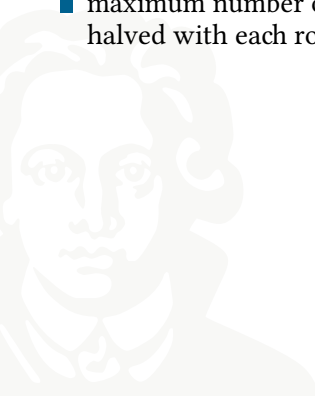


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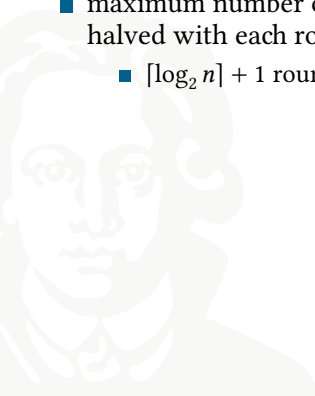


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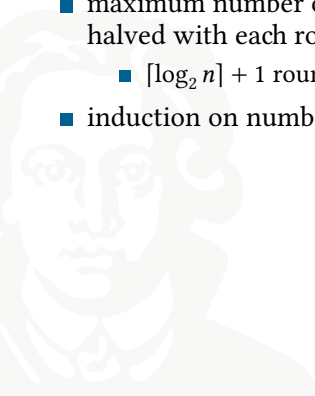


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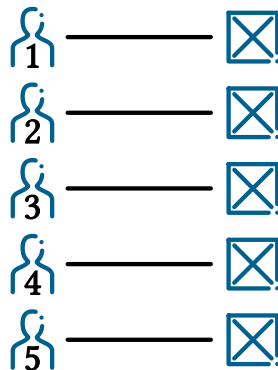
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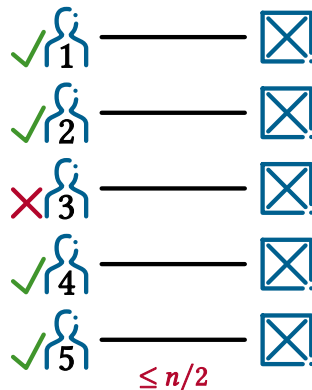
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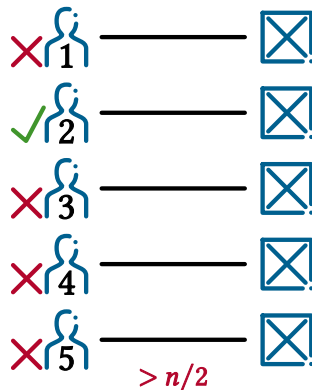
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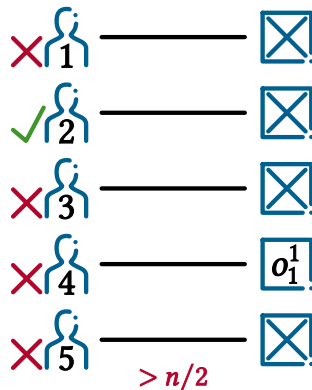
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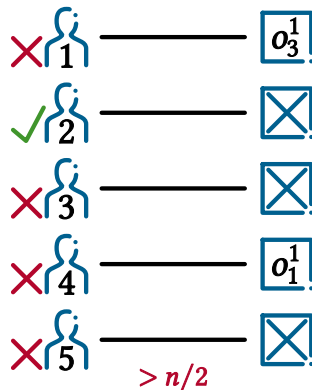
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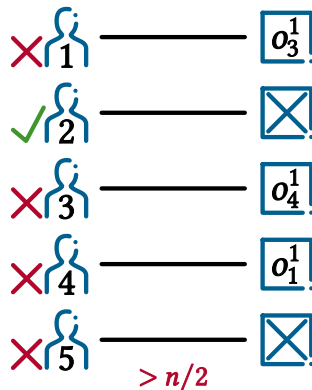
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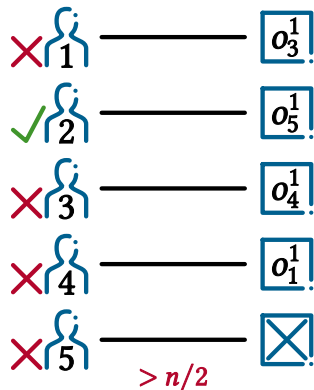
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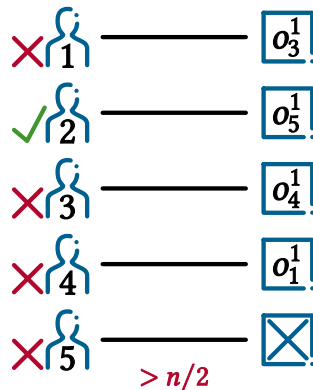
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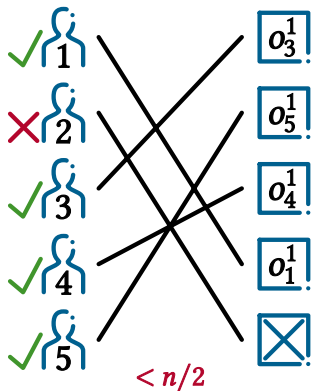
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 - $> n/2$ many agents matched with outstanding item upon release in phase III



Analysing Phase II (1/2)



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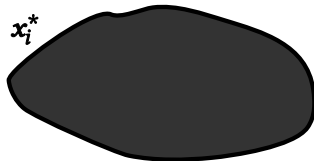
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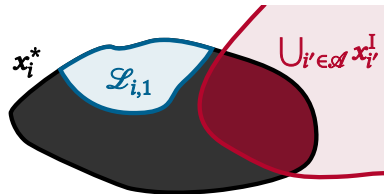
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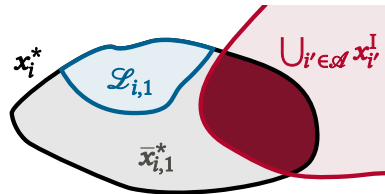
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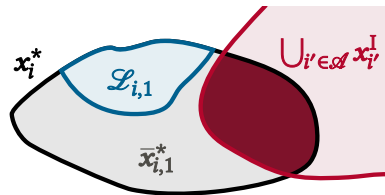
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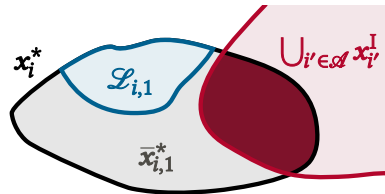
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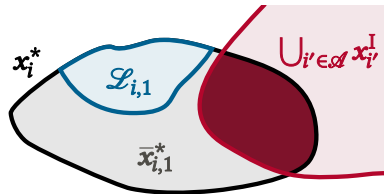
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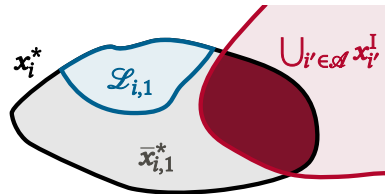
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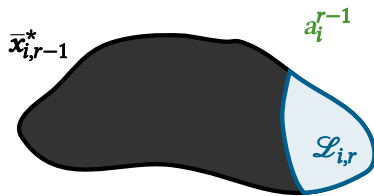
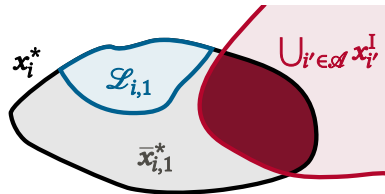
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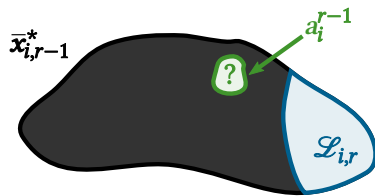
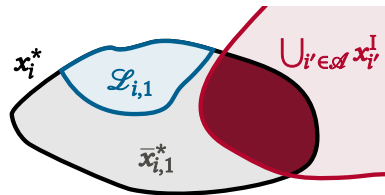
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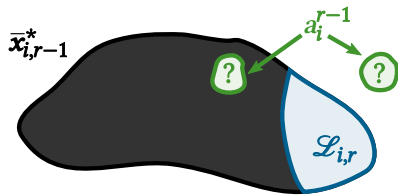
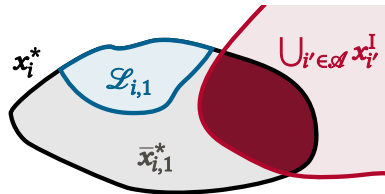
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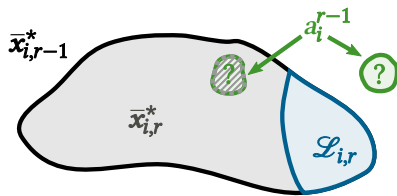
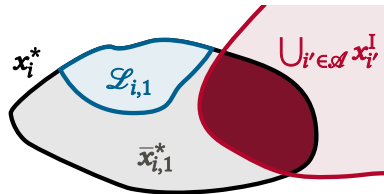
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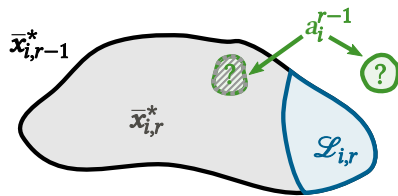
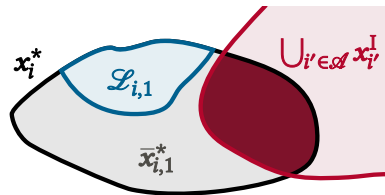
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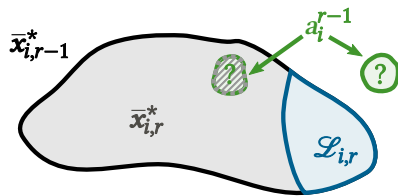
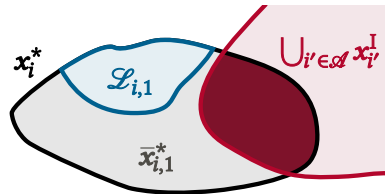
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⇒ What is the valuation of the remaining items?



Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

if $r \geq 2$



Analysing Phase II (2/2)

Lemma

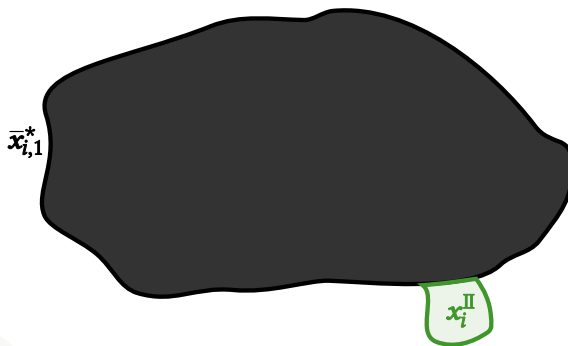
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

if $r \geq 2$ 
$$\bar{x}_{i,1}^*$$

Analysing Phase II (2/2)

Lemma

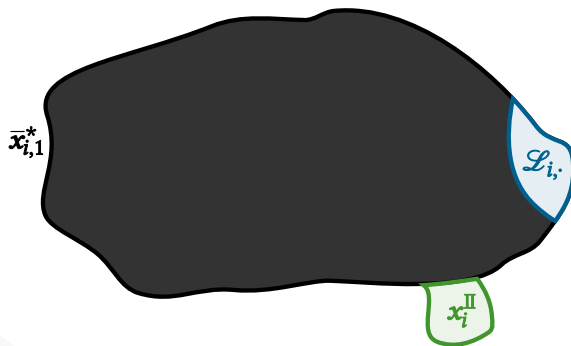
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

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Analysing Phase II (2/2)

Lemma

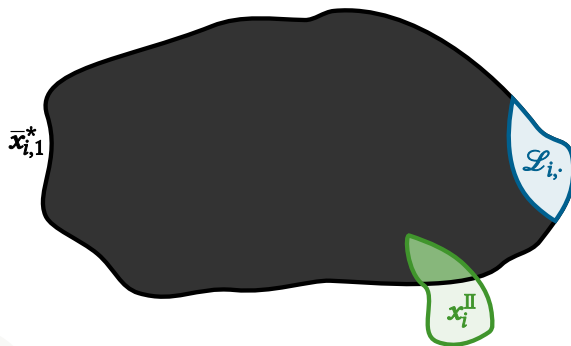
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Analysing Phase II (2/2)

Lemma

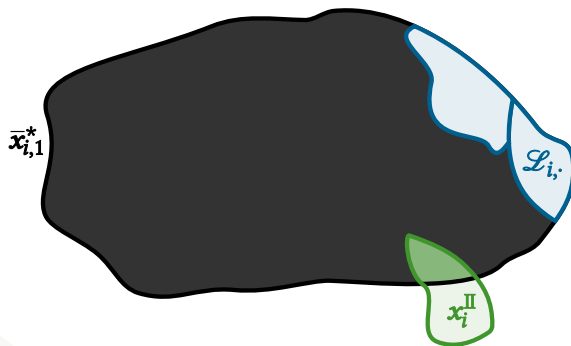
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

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Analysing Phase II (2/2)

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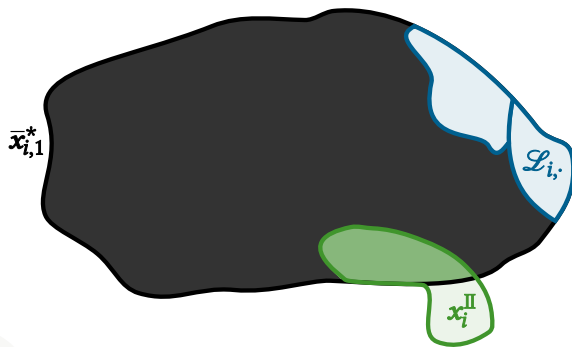
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Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

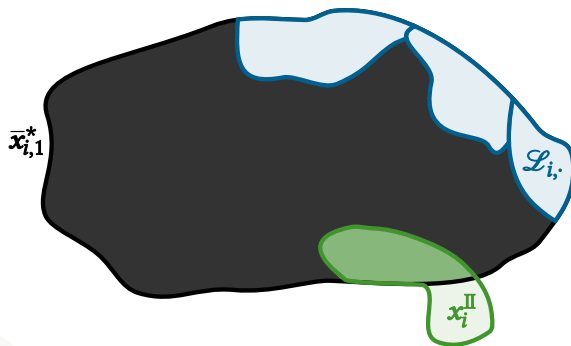
if $r \geq 2$



Analysing Phase II (2/2)

Lemma

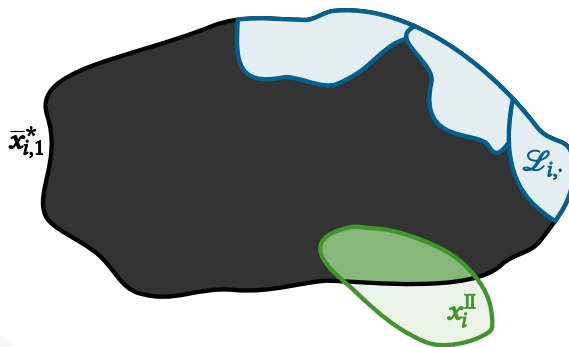
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

if $r \geq 2$ 

Analysing Phase II (2/2)

Lemma

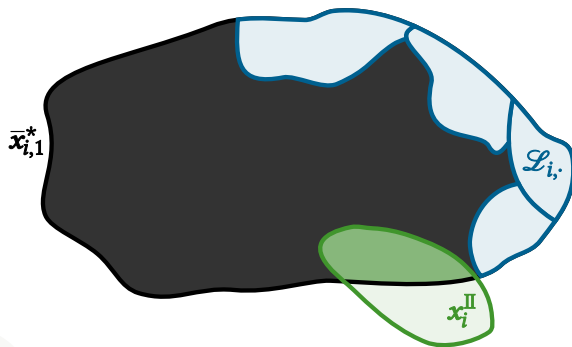
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

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Analysing Phase II (2/2)

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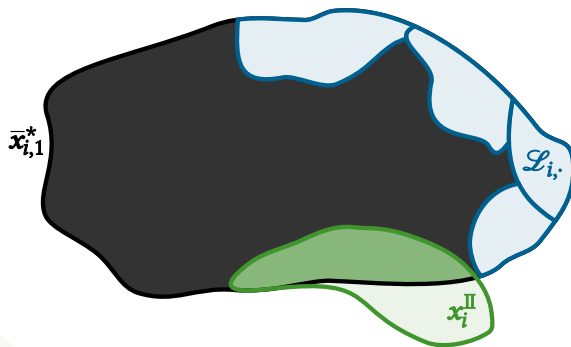
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Analysing Phase II (2/2)

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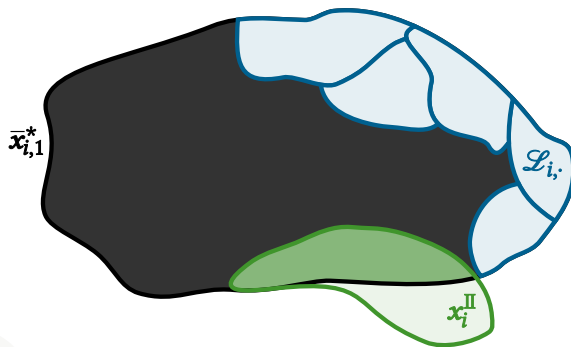


Analysing Phase II (2/2)

Lemma

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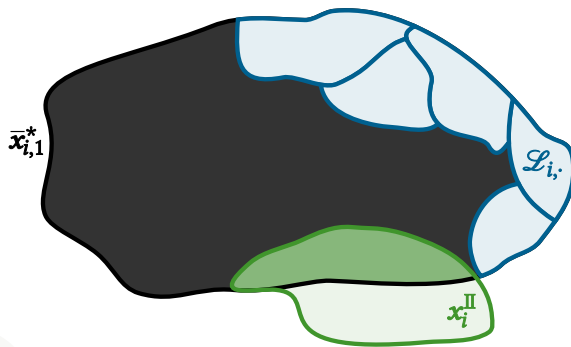


Analysing Phase II (2/2)

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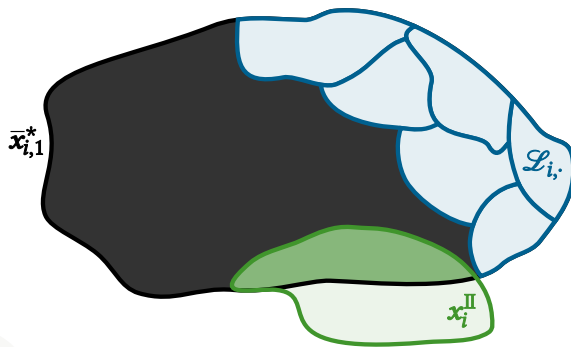


Analysing Phase II (2/2)

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$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

if $r \geq 2$

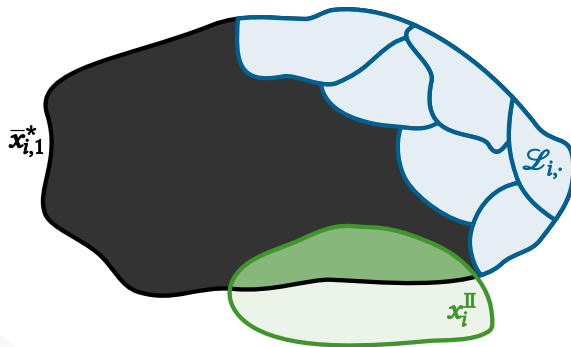


Analysing Phase II (2/2)

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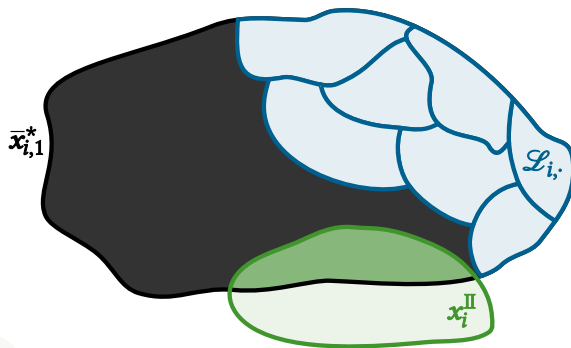


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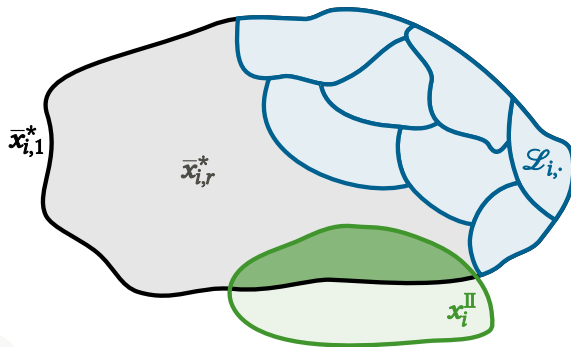


Analysing Phase II (2/2)

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$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1})$$

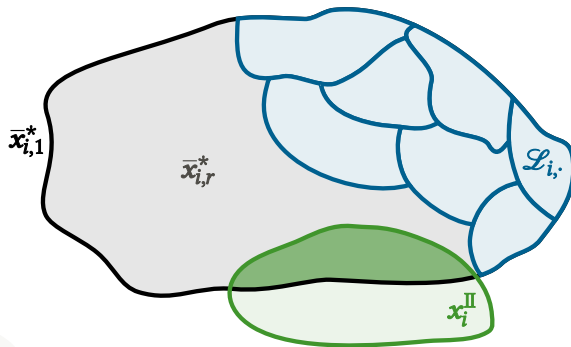
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Analysing Phase II (2/2)

Lemma

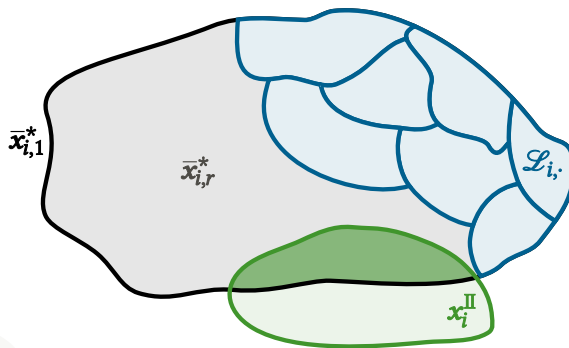
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) - v_i(a_i^1, \dots, a_i^{r-1}) \quad \text{if } r \geq 2$$



Analysing Phase II (2/2)

Lemma

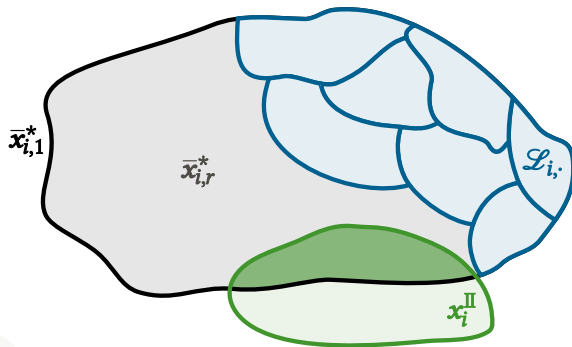
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) - v_i(a_i^1, \dots, a_i^{r-1}) \quad \text{if } r \geq 2$$



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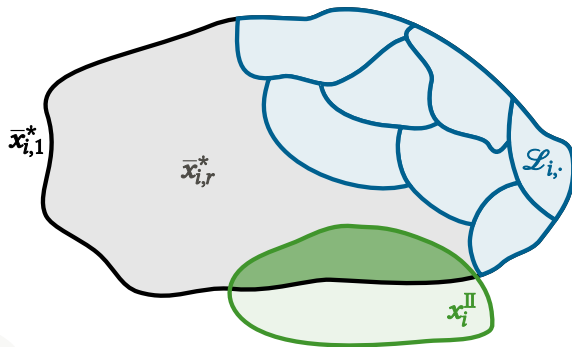
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) = -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,r}^* \cup \{a_i^1, \dots, a_i^{r-1}\}) \quad \text{if } r \geq 2$$



Analysing Phase II (2/2)

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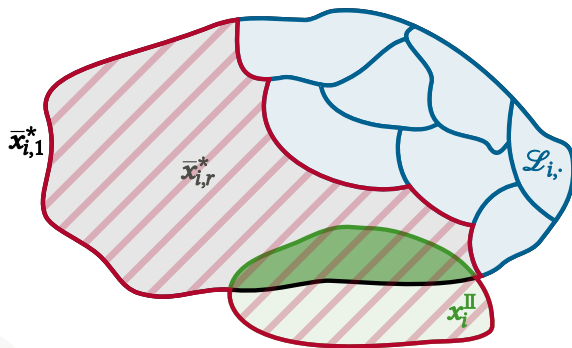
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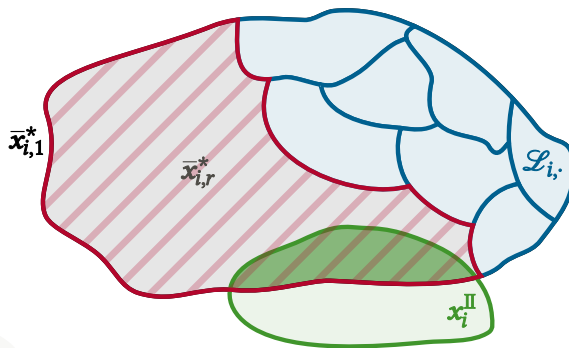
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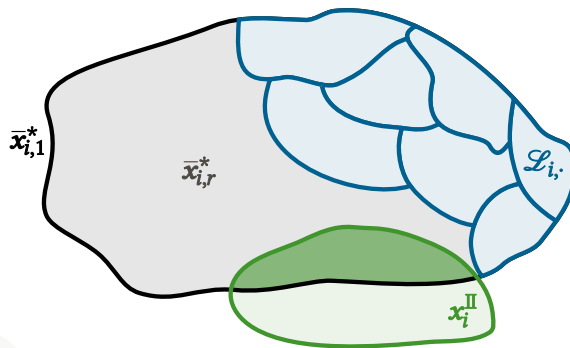


Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*)$$

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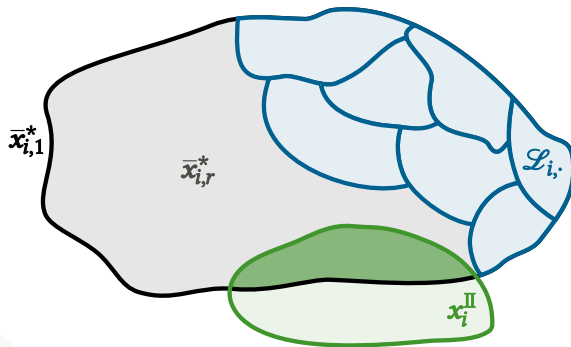


Analysing Phase II (2/2)

Lemma

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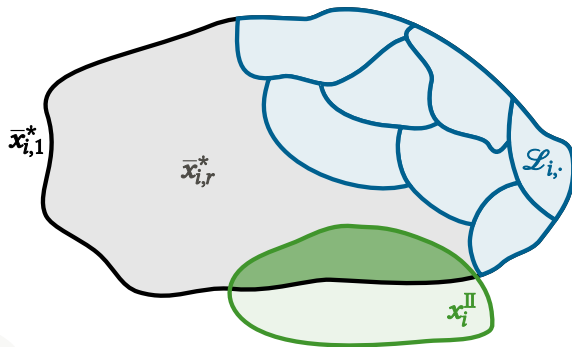
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Analysing Phase II (2/2)

Lemma

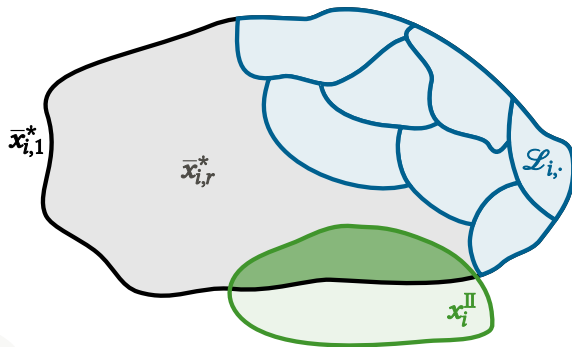
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - v_i(\mathcal{L}_{i,2} \mid a_i^1) - v_i(\mathcal{L}_{i,3} \mid a_i^1, a_i^2) \quad \text{if } r \geq 2$$



Analysing Phase II (2/2)

Lemma

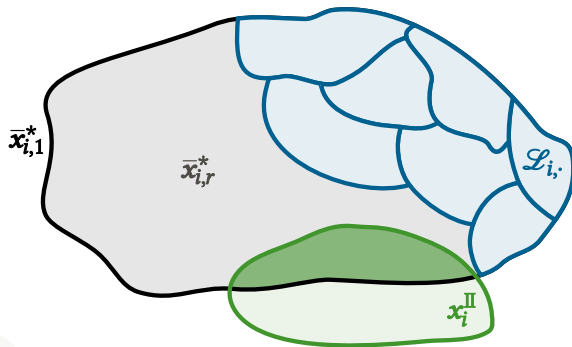
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Analysing Phase II (2/2)

Lemma

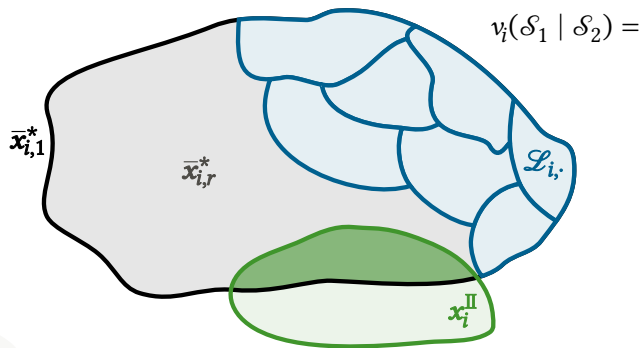
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r v_i(\mathcal{L}_{i,l} \mid a_i^1, \dots, a_i^{l-1}) \quad \text{if } r \geq 2$$



Analysing Phase II (2/2)

Lemma

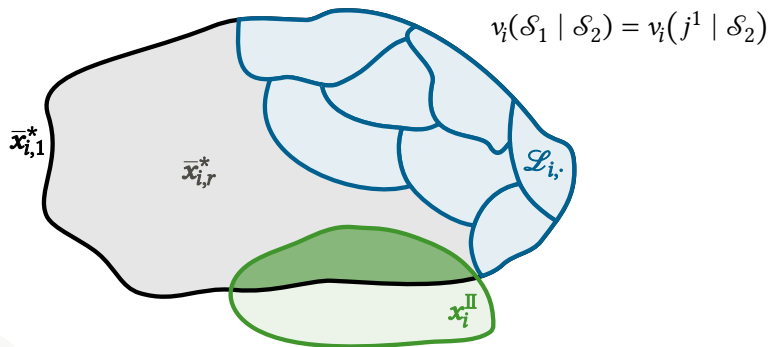
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Analysing Phase II (2/2)

Lemma

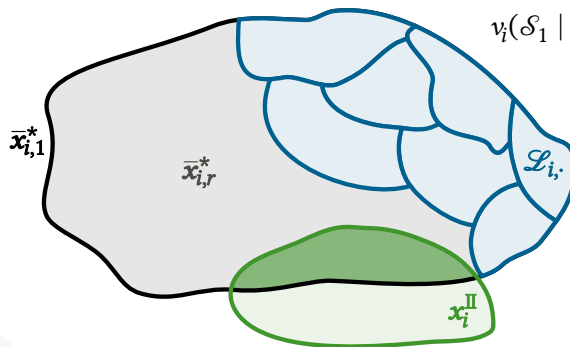
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Analysing Phase II (2/2)

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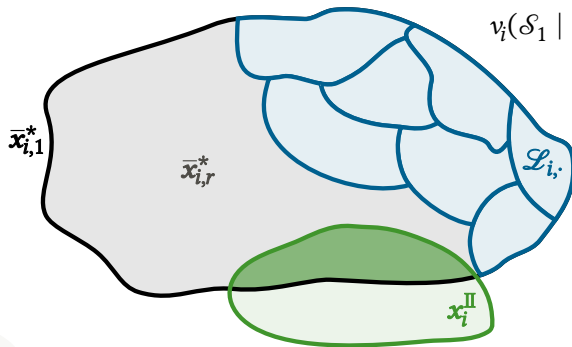


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i(j^1 \mid \mathcal{S}_2) + v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\})$$

Analysing Phase II (2/2)

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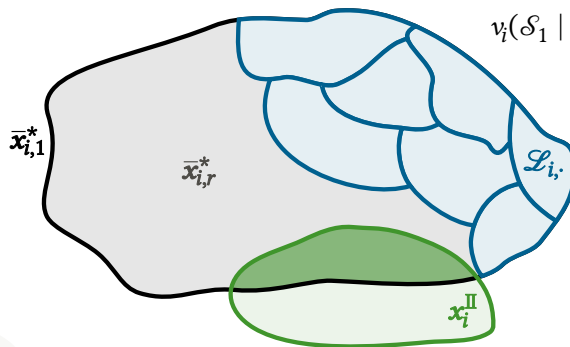


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Analysing Phase II (2/2)

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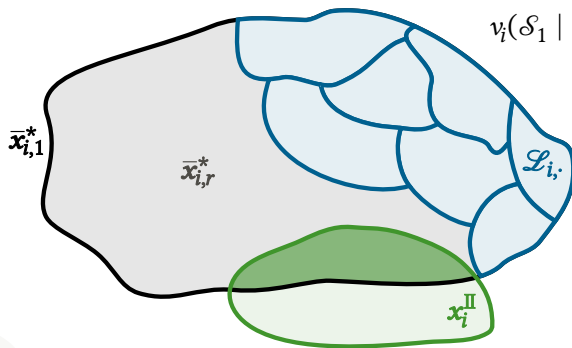


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i(j^1 \mid \mathcal{S}_2) + \\ v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\}) + \\ v_i(j^3 \mid \mathcal{S}_2 \cup \{j^1, j^2\}) + \\ \vdots$$

Analysing Phase II (2/2)

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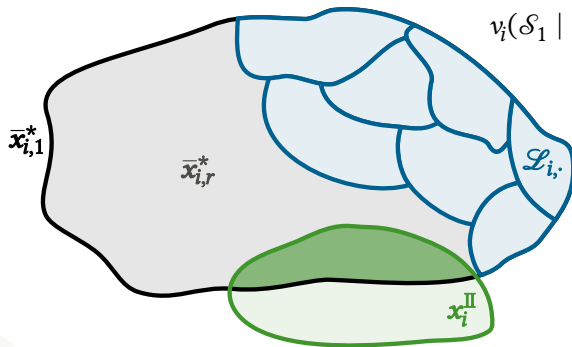


$$\begin{aligned}
 v_i(\mathcal{S}_1 \mid \mathcal{S}_2) &= v_i(j^1 \mid \mathcal{S}_2) + \\
 &\quad v_i(j^2 \mid \mathcal{S}_2 \cup \{j^1\}) + \\
 &\quad v_i(j^3 \mid \mathcal{S}_2 \cup \{j^1, j^2\}) + \\
 &\quad \vdots
 \end{aligned}$$

Analysing Phase II (2/2)

Lemma

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$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq v_i(j^1 \mid \mathcal{S}_2) +$$

$$v_i(j^2 \mid \mathcal{S}_2) +$$

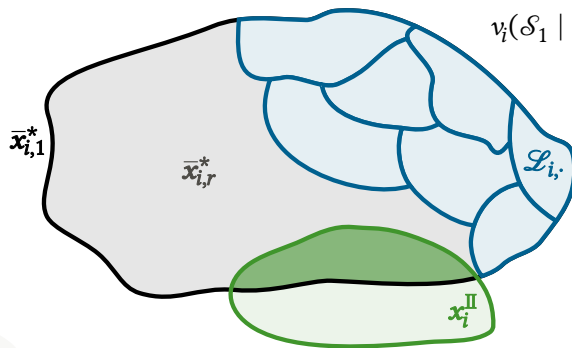
$$v_i(j^3 \mid \mathcal{S}_2) +$$

$$\vdots$$

Analysing Phase II (2/2)

Lemma

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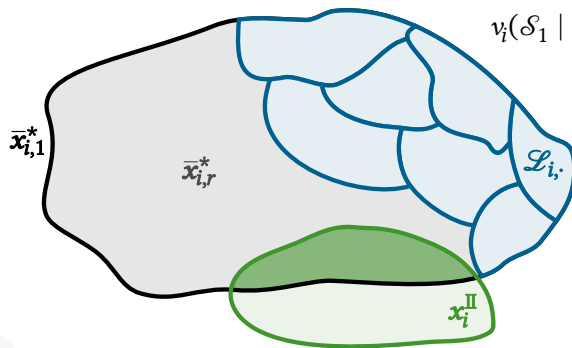


$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) \leq \sum_{j \in \mathcal{S}_1} v_i(j \mid \mathcal{S}_2)$$

Analysing Phase II (2/2)

Lemma

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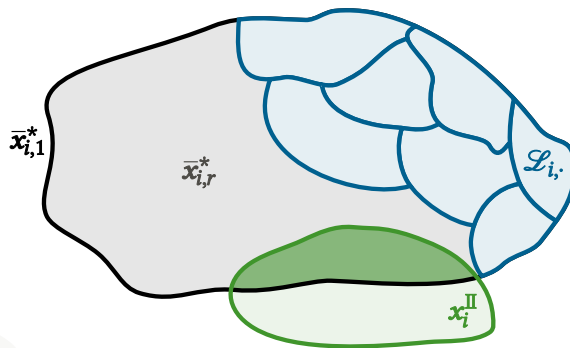


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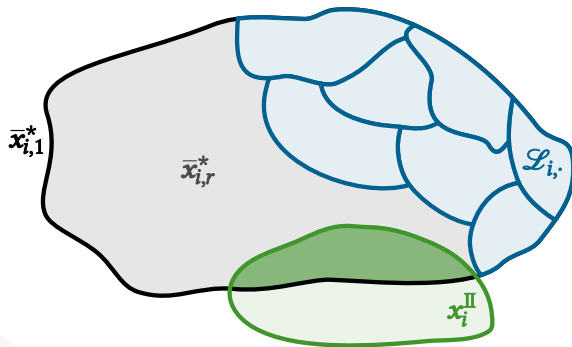
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Analysing Phase II (2/2)

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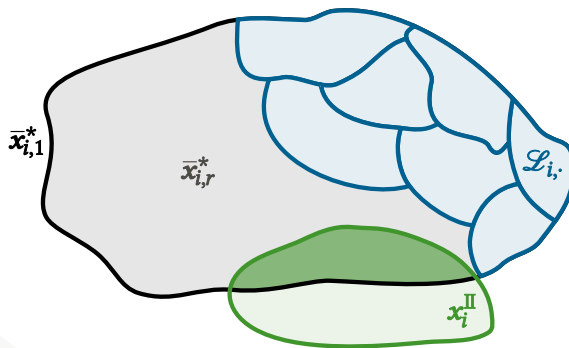
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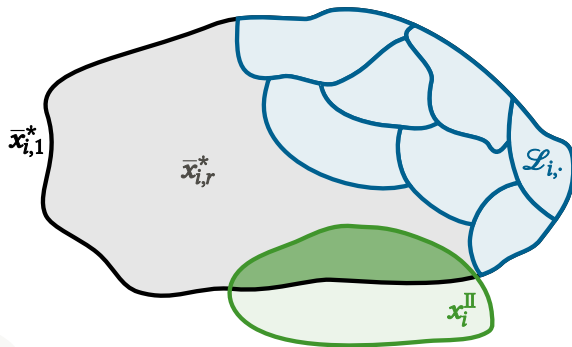
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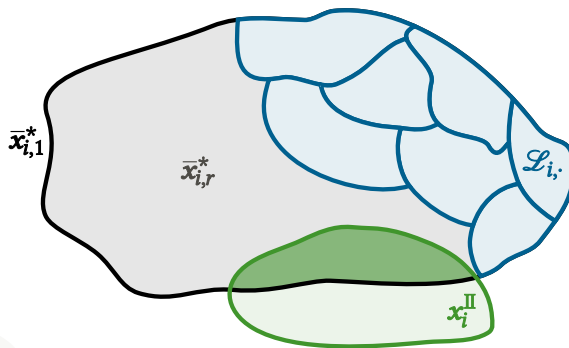
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Lemma

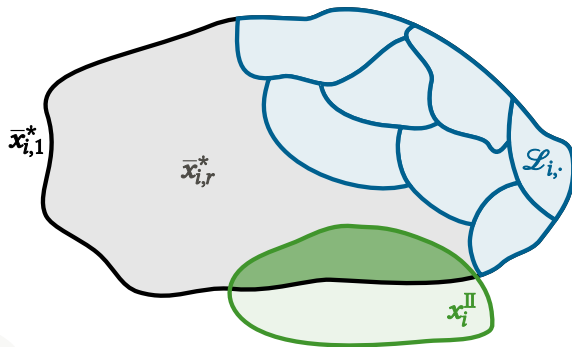
$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r \sum_{j \in \mathcal{L}_{i,l}} v_i(a_i^{l-1} \mid a_i^1, \dots, a_i^{l-2}) \quad \text{if } r \geq 2$$



Analysing Phase II (2/2)

Lemma

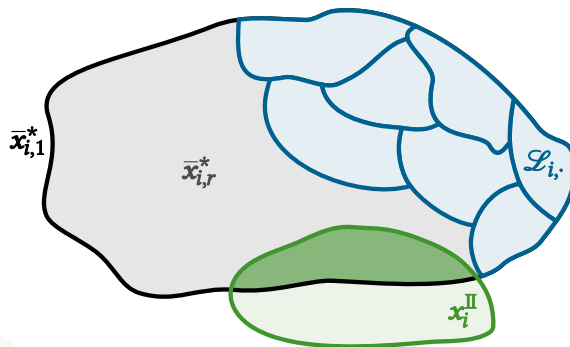
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Analysing Phase II (2/2)

Lemma

$$v_i(\bar{x}_{i,r}^* \mid a_i^1, \dots, a_i^{r-1}) \geq -v_i(a_i^1, \dots, a_i^{r-1}) + v_i(\bar{x}_{i,1}^*) - \sum_{l=2}^r (n-1) \cdot v_i(a_i^{l-1} \mid a_i^1, \dots, a_i^{l-2}) \quad \text{if } r \geq 2$$



3

Conclusion



Summary & Outlook



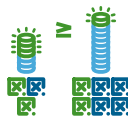
Summary & Outlook

- allocation \triangleq tuple of item sets



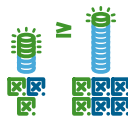
Summary & Outlook

- allocation \triangleq tuple of item sets
- bundles valued using submodular valuation functions



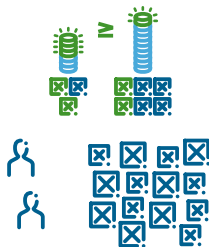
Summary & Outlook

- allocation \triangleq tuple of item sets
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- Nash social welfare \triangleq weighted geometric mean of valuations



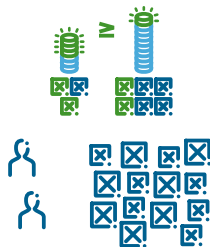
Summary & Outlook

- allocation \triangleq tuple of item sets
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- approximation factor independent from m ?



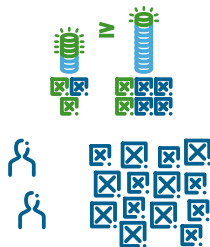
Summary & Outlook

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Summary & Outlook

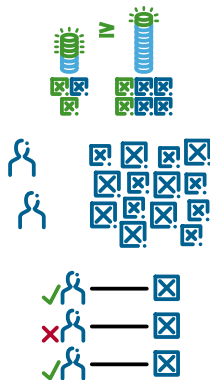
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Phase I finding enough outstanding items

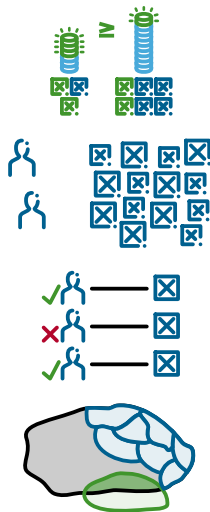


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Phase I finding enough outstanding items

Phase II assigning remaining items



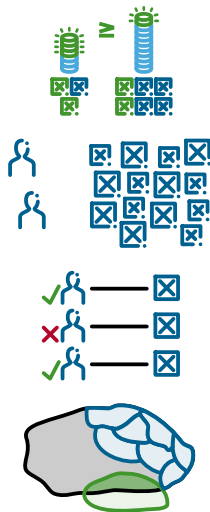
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Phase III re-assigning outstanding items



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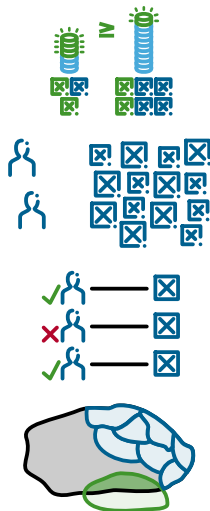
Phase I finding enough outstanding items

Phase II assigning remaining items

Phase III re-assigning outstanding items

Any Room for Improvement?

Possibly! Lower bound of $\frac{e}{e-1} \approx 1.58$



End of Talk

