Algorithms and Complexity Group
Institute for Computer Science
Goethe University Frankfurt

Summer Semester 2023

**Seminar Approximation Algorithms**

# ANSWuSVþ(U)M

Zeno Adrian Weil

5th June 2023

Examiner: Prof Dr Martin Hoefer
Supervisor: Dr Giovanna Varricchio

## Abstract

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.

## Todo list

# 1 Introduction

- problem introduction, motivation, applications
- formal problem definition (incl. why geometric mean?)
- short literature review: What is known, what not? New findings?
- content & structure of paper

**Definition 1.** Let $\mathcal{G}$ be a set of $m$ indivisible *items* and $\mathcal{A}$ be a set of $n$ *agents*. An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of *bundles* $\boldsymbol{x}_i \subset \mathcal{G}$ such that each item is element of exactly one bundle, that is $\bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i = \mathcal{G}$ and $\boldsymbol{x}_i \cap \boldsymbol{x}_{i'} = \emptyset$ for all $i \neq i'$. An item $j \in \mathcal{G}$ is *assigned* to agent $i \in \mathcal{A}$ if $j \in \boldsymbol{x}_i$ holds.

or rather 'allocated'?

⋮

**Definition 2.** Given a set $\mathcal{G}$ of items and a set $\mathcal{A}$ of agents with *valuations* $v_i : \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ and *agent weights* $\eta_i$ for all agents $i \in \mathcal{A}$, the *Nash Social Welfare problem* (NSW) is to find an allocation maximising the weighted geometric mean of valuations, that is

$$\operatorname*{arg\,max}_{\boldsymbol{x} \in \Pi_n(\mathcal{G})} \left\{ \left( \prod_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i)^{\eta_i} \right)^{1/\sum_{i \in \mathcal{A}} \eta_i} \right\}$$

where $\Pi_n(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst $n$ agents. The problem is called *symmetric* if all agent weights $\eta_i$ are equal, and *asymmetric* otherwise.

agents without items assigned have valuation zero $\to$ prevent

⋮

In a slight abuse of notation, we omit curly braces delimiting a set in the arguments of a valuation function, so, for example, we write $v(j_1, j_2, \dots)$ but mean $v(\{j_1, j_2, \dots\})$.

⋮

definition of approximation factor [def environment or in-text?]

⋮

Garg, Kulkarni and Kulkarni [1] consider five different types of non-negative monotonically non-decreasing valuation functions of which we are going to consider only the following two due to space constraints:

**Additive** The valuation $v_i(\mathcal{S})$ of an agent $i$ for a set $\mathcal{S} \subset \mathcal{G}$ of items $j$ is the sum of individual valuations $v_i(j)$, that is $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(j)$.

**Submodular** Let $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$ denote the marginal valuation of agent $i$ for a set $\mathcal{S}_1 \subset \mathcal{G}$ of items over a *disjoint* set $\mathcal{S}_2 \subset \mathcal{G}$. This valuation functions satisfies the submodularity constraint $v_i(j \mid \mathcal{S}_1 \cup \mathcal{S}_2) \leq v_i(j \mid \mathcal{S}_1)$ for all agents $i \in \mathcal{A}$, items $j \in \mathcal{G}$ and sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$ of items.

or rather utility?

What is the motivation for submodular functions?

We use *additive NSW* and *submodular NSW* as shorthands for the Nash social welfare problems with additive and submodular valuation functions, respectively.

Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

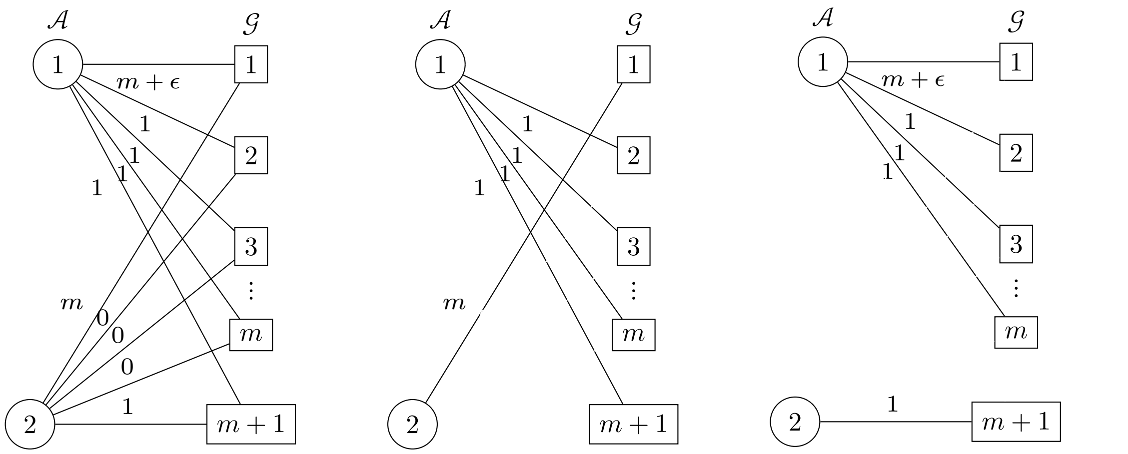
Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.

Aliquam lectus. Vivamus leo. Quisque ornare tellus ullamcorper nulla. Mauris porttitor pharetra tortor. Sed fringilla justo sed mauris. Mauris tellus. Sed non leo. Nullam elementum, magna in cursus sodales, augue est scelerisque sapien, venenatis congue nulla arcu et pede. Ut suscipit enim vel sapien. Donec congue. Maecenas urna mi, suscipit in, placerat ut, vestibulum ut, massa. Fusce ultrices nulla et nisl.

Etiam ac leo a risus tristique nonummy. Donec dignissim tincidunt nulla. Vestibulum rhoncus molestie odio. Sed lobortis, justo et pretium lobortis, mauris turpis condimentum augue, nec ultricies nibh arcu pretium enim. Nunc purus neque, placerat id, imperdiet sed, pellentesque nec, nisl. Vestibulum imperdiet neque non sem accumsan laoreet. In hac habitasse platea dictumst. Etiam condimentum facilisis libero. Suspendisse in elit quis nisl aliquam dapibus. Pellentesque auctor sapien. Sed egestas sapien nec lectus. Pellentesque vel dui vel neque bibendum viverra. Aliquam porttitor nisl nec pede. Proin mattis libero vel turpis. Donec rutrum mauris et libero. Proin euismod porta felis. Nam lobortis, metus quis elementum commodo, nunc lectus elementum mauris, eget vulputate ligula tellus eu neque. Vivamus eu dolor.

Nulla in ipsum. Praesent eros nulla, congue vitae, euismod ut, commodo a, wisi. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Aenean nonummy magna non leo. Sed felis erat, ullamcorper in, dictum non, ultricies ut, lectus. Proin vel arcu a odio lobortis euismod. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Proin ut est. Aliquam odio. Pellentesque massa turpis, cursus eu, euismod nec, tempor congue, nulla. Duis viverra gravida mauris. Cras tincidunt. Curabitur eros ligula, varius ut, pulvinar in, cursus faucibus, augue.




**(a)** problem instance **(b)** OPT $= \sqrt{m \cdot m}$ **(c)** ALG $\leq \sqrt{(2m+\epsilon-1)\cdot 1}$

**Figure 1:** An example showing that simple repeated matching without consideration of the future leads to an approximation factor dependent on the number of items. Agent 1 values item 1 at $m+\epsilon$ and all other items at 1. Agent 2 values item 1 at $m$, item $m+1$ at 1 and all other items at 0. In an optimal allocation, item 1 would be assigned to agent 2 and all other items to agent 1, resulting in a NSW of $\sqrt{m \cdot m} = m$. A repeated maximum matching algorithm would greedily assign item 1 to agent 1 and item $m+1$ to agent 2 in the first round. Even if all remaining items were going to be assigned to agent 1, the NSW will never surpass $\sqrt{(2m+\epsilon-1)\cdot 1} < \sqrt{2m}$. The approximation factor $\alpha \approx \sqrt{m/2}$ therefore depends on the number of items.

fix pos of subcaptions & of agent 2

## 2 SMatch

In the case of an equal number of agents and items, i. e., $n = m$, the *additive* NSW can be solved exactly by finding a maximum matching on a bipartite graph with the sets of agents and of items as its parts; as weight of the edge between agent $i$ and item $j$, use $\eta_i \log v_i(j)$, that is the weighted valuation of item $j$ by agent $i$ in the *logarithmic* Nash social welfare. Should there be more items than agents, then it would be obvious at first to just repeatedly find a maximum matching and assign the items accordingly until all items are assigned. The flaw of this idea is that such a greedy algorithm only considers the valuations of items in the current matching and perhaps the valuations of items already assigned. As the example in fig. 1 demonstrates, this leads to an algorithm with an approximation factor dependent on the number $m$ of items. The geometric mean of the NSW favours allocations with similarly valued bundles, wherefore it may be beneficial to give items to agents who cannot expect many more valuable items in the future instead of to agents who value the item a bit more but do so for other items as well.

What is the factor?

The algorithm SMatch, described in algorithm 1, eliminates the flaw by first gaining foresight of the valuations of items assigned after the first matching, achieving an approximation factor of $2n$ (cf. theorem 1 later on). For a fixed agent $i$, order the items in descending order of the valuations by agent $i$ and denote the $j$-th most liked item by $\mathcal{G}_i^j$. To obtain a well-defined order, items of equal rank are further ordered numerically. SMatch, too, does repeatedly match items. During the first matching, however, the edge weights are defined as $\eta_i \log(v_i(j) + u_i/n)$ for an edge between agent $i$ and item $j$. The addend $u_i$ serves as

**Algorithm 1:** SMatch for the Asymmetric Additive NSW problem

**Input :** set $\mathcal{A}$ of $n$ agents with weights $\eta_i$ for all agents $i \in \mathcal{A}$, set $\mathcal{G}$ of indivisible $m$ items, additive valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each set $\mathcal{S} \subset \mathcal{G}$ of items

**Output:** $\frac{1}{2n}$-approximation $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of an optimal allocation

$\boldsymbol{x}_i \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

$u_i \leftarrow v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m) \quad \forall i \in \mathcal{A}$ ▷ *estimation of future valuations*

$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j) + u_i/n) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$ ▷ *edge weights*

$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ ▷ *bipartite graph*

$\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$

$\boldsymbol{x}_i \leftarrow \{\, j \mid (i,j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$ ▷ *assign according to matching*

$\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$ ▷ *remove assigned items*

**while** $\mathcal{G}^{\text{rem}} \neq \emptyset$ **do**

  $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j) + v_i(\boldsymbol{x}_i)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}} \,\}$

  $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathcal{W})$

  $\mathcal{M} \leftarrow \text{max\_weight\_matching}(G)$

  $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i \cup \{\, j \mid (i,j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$

  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$

**end while**

**return** $\boldsymbol{x}$

estimation of the valuation of items assigned after the first matching and is defined as

$$u_i := \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m). \tag{1}$$

The set $\mathcal{S}$ has less than $2n$ elements only if there are less than $2n$ items in total. From the second matching onwards, the edge weights are defined as $\eta_i \log(v_i(j) + v_i(\boldsymbol{x}_i))$, where $\boldsymbol{x}_i$ is the continuously updated bundle of agent $i$. The addend $v_i(\boldsymbol{x}_i)$ could lead to better allocations in applications, but does not improve the approximation factor asymptotically.

To calculate the approximation factor of SMatch, we first need to establish a lower bound on the valuation of single items. For convenience, we order the items in the final bundle $\boldsymbol{x}_i = \{h_i^1, \dots, h_i^{\tau_i}\}$ of agent $i$ by the order in which they were assigned, so that item $h_i^t$ is assigned according to the $t$-th matching. Note that it holds $v_i(h_i^t) \geq v_i(h_i^{t'})$ for all $t' \geq t$.

**Lemma 1.** *For each agent $i \in \mathcal{A}$, her final bundle $\boldsymbol{x}_i = \{h_i^1, \dots, h_i^{\tau_i}\}$, and her $tn$-th most highly valued item $\mathcal{G}_i^{tn}$, it holds $v_i(h_i^t) \geq v_i(\mathcal{G}_i^{tn})$ for all $t = 1, \dots, \tau_i$.*

Proof. At the start of the $t$-th round, no more than $(t-1)n$ items out of the $tn$ most highly valued items $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ have been assigned in previous rounds since at most $n$ items are assigned in each iteration. During the $t$-th round, at most $n-1$ more of those highly valued items could be assigned to all other agents $i' \neq i$, leaving at least one item in $\mathcal{G}_i^1, \dots, \mathcal{G}_i^{tn}$ unassigned. Since $v_i(\mathcal{G}_i^k) \geq v_i(\mathcal{G}_i^{tn})$ for all $k \leq tn$ by definition of $\mathcal{G}_i^{\cdot}$, the lemma follows. □

We can now establish $u_i/n = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m)/n$ as lower bound on the valuations of items assigned after the first matching.

**Lemma 2.** *For each agent $i \in \mathcal{A}$ and her final bundle $\boldsymbol{x}_i = \{h_i^1, \dots, h_i^{\tau_i}\}$, it holds $v_i(h_i^2, \dots, h_i^{\tau_i}) \geq u_i/n$, where $n$ is the number of items and $u_i = v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^m)$.*

Proof. By lemma 1 and definition of $\mathcal{G}_i^{\cdot}$, every item $h_i^t$ is worth at least as much as each item $\mathcal{G}_i^{tn+k}$ with $k \in \{0, \dots, n-1\}$ and, consequently, its valuation $v_i(h_i^t)$ is at least as high as the mean valuation $\frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1})$. Further, it holds $\tau_i n + n \geq m$ since each agent receives items for at least $\lfloor \frac{m}{n} \rfloor \geq \frac{m}{n} - 1$ rounds. Together, this yields

$$v_i(h_i^2, \dots, h_i^{\tau_i}) = \sum_{t=2}^{\tau_i} v_i(h_i^t) \geq \sum_{t=2}^{\tau_i} \frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1}) \tag{2}$$

$$\geq \frac{1}{n} v_i(\mathcal{G}_i^{2n}, \dots, \mathcal{G}_i^{m-1}) \geq \frac{1}{n} v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^{m}) = \frac{u_i}{n} \tag{3}$$

with the last inequality stemming from $v_i(\mathcal{G}_i^{2n}) \geq v_i(\mathcal{G}_i^{m})$. □

*Remark* 1. In lemma 2, we assumed non-zero valuations for all items, hence the bundle lengths of $\tau_i \geq \lfloor \frac{m}{n} \rfloor$. Of course in an actual program, one would not assign items to agents who value them at zero. Inasmuch as additional zeros in eq. (3) do not change the sum, lemma 2 still holds nevertheless.

This allows us to calculate an approximation factor for SMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

**Theorem 1.** *SMatch has an approximation factor of* $2n$.

Proof. Lemma 2 can be plugged into the logarithmic NSW:

$$\log \mathrm{NSW}(\boldsymbol{x}) = \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log v_i(h_i^1, \dots, h_i^{\tau_i}) \tag{4}$$

$$= \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\big(v_i(h_i^1) + v_i(h_i^2, \dots, h_i^{\tau_i})\big) \tag{5}$$

$$\geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\big(v_i(h_i^1) + u_i/n\big) \tag{6}$$

Notice that the first matching of SMatch maximises the sum in eq. (6). Thus, assigning all agents $i$ their respective most highly valued item $g_i^1$ in an optimal bundle $\boldsymbol{x}_i^* = \{g_i^1, \dots, g_i^{\tau_i^*}\}$ yields the even lower bound

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\big(v_i(g_i^1) + u_i/n\big). \tag{7}$$

Recall the definition of $u_i$ from eq. (1). Consider a slightly modified variant:

$$u_i = \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \quad \text{or, alternatively,} \quad u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \text{ with } \mathcal{S}_i := \arg\min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \tag{8}$$

Moreover, consider the set $\mathcal{S}_i^*$ of the (at most) $2n$ most highly valued items in the optimal bundle $\boldsymbol{x}_i^*$, i. e.

$$\mathcal{S}_i^* := \arg\min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S})\}. \tag{9}$$

We get the lower bound $v_i(g_i^1) \geq \frac{1}{2n} v_i(\mathcal{S}^*)$ from a similar argument as in the proof of lemma 2. Further, it holds $u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)$. We can insert these two inequalities into eq. (7) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\mathcal{S}_i^*)}{2n} + \frac{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)}{n}\right) \tag{10}$$

$$\geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n}\right) = \log\left(\frac{\mathrm{NSW}(\boldsymbol{x}^*)}{2n}\right) \tag{11}$$

□

tightness

envy-freeness?

## 3 RepReMatch

The algorithm SMatch estimates the valuation of the lowest-value items by determining the set of highest-value items and then valuing the remaining items. Unfortunately, this approach does not work for general submodular valuations because taking the set of highest-value items away does not necessarily leave a set of lowest-value items. In fact, it can be shown [2] that determining the set of lowest-value items is approximable only within a factor of $\Omega(\sqrt{m/\ln m})$.

For this reason, the algorithm RepReMatch, described in algorithm 2, relies on an approach with three phases, achieving an approximation factor of $2n(\log_2 n + 3)$ (cf. theorem 2). In phase I, a sufficiently big set of high-value items is determined through

---

**Algorithm 2:** RepReMatch for the Asymmetric Submodular NSW problem

---

**Input :** set $\mathcal{A}$ of $n$ agents with weights $\eta_i$ for all agents $i \in \mathcal{A}$, set $\mathcal{G}$ of indivisible $m$ items, submodular valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each set $\mathcal{S} \subset \mathcal{G}$ of items

**Output:** $\frac{1}{2n(\log_2 n+3)}$-approximation $\boldsymbol{x}^{\mathrm{III}} = (\boldsymbol{x}_1^{\mathrm{III}}, \dots, \boldsymbol{x}_n^{\mathrm{III}})$ of an optimal allocation

*Phase I:*

$\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \emptyset$<br>
$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}$<br>
**for** $t \leftarrow 1, \dots, \lceil \log_2 n \rceil + 1$ **do**<br>
    **if** $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **then**<br>
        $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \,\}$ ▷ *valuation of single item*<br>
        $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \mathcal{W})$<br>
        $\mathcal{M} \leftarrow$ max_weight_matching$(G)$<br>
        $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \boldsymbol{x}_i^{\mathrm{I}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$<br>
        $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$<br>
    **end if**<br>
**end for**

*Phase II:*

$\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ ▷ *put allocation $\boldsymbol{x}^{\mathrm{I}}$ away and start a new one*<br>
**while** $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **do**<br>
    $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \,\}$ ▷ *val. of item & cur. bundle*<br>
    $G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \mathcal{W})$<br>
    $\mathcal{M} \leftarrow$ max_weight_matching$(G)$<br>
    $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$<br>
    $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$<br>
**end while**

*Phase III:*

$\mathcal{G}^{\mathrm{rem}} \leftarrow \bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i^{\mathrm{I}}$ ▷ *release items assigned in phase I*<br>
$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\mathrm{rem}} \,\}$ ▷ *val. of item & cur. bundle*<br>
$G \leftarrow (\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \mathcal{W})$<br>
$\mathcal{M} \leftarrow$ max_weight_matching$(G)$<br>
$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$<br>
$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$<br>
$\boldsymbol{x}^{\mathrm{III}} \leftarrow$ arbitrary_allocation$(\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \boldsymbol{x}^{\mathrm{III}}, (v_i)_{i \in \mathcal{A}})$<br>
**return** $\boldsymbol{x}^{\mathrm{III}}$

---

repeated matchings. This phase serves merley to determine this set, so items are assigned temporarily only. The edge weights reflect this by taking the valuations of just single items into account.

In phase II, the remaining items are assigned normally through repeated matchings. Consequently, each edge weight is updated in each round to be the weighted logarithm of the valuation of both the respective item and the items assigned so far.

In phase III, the high-value items assigned in phase I are released. With the knowledge of items assigned in phase II, one maximum weight matching is calculated, and the matched items are assigned accordingly. Again each edge weight is the weighted logarithm of the valuation of both the respective item and the respective agent's bundle from phase II. The remaining released items are assigned arbitrarily.

We start by analysing phase II as it is the first phase with definitive assignments. To this end, we introduce two types of item sets. Note that we use the term *round* to refer to the iterations of the loops in the phases I and II. For ease of notation, we refer to the moment before the first iteration in phase II as round 0.

**Definition 3.** Let $\boldsymbol{x}_i^*$ be an optimal allocation of some agent $i \in \mathcal{A}$. For any round $r \geq 1$ in phase II, the set $\mathcal{L}_{i,r} \subset \boldsymbol{x}_i^*$ of *lost* items is the set of all items $j \in \boldsymbol{x}_i^*$ assigned to other agents $i' \neq i$ in that round.

**Definition 4.** Let $\boldsymbol{x}_i^*$ be an optimal allocation of some agent $i \in \mathcal{A}$ and $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$ be her bundle in phase II. The set $\bar{\boldsymbol{x}}_{i,r}^*$ of *optimal and attainable* items is defined as $\bar{\boldsymbol{x}}_{i,0}^* := \boldsymbol{x}_i^* \setminus \bigcup_{i' \in \mathcal{A}} \boldsymbol{x}_{i'}^{\mathrm{I}}$ in round 0 and as $\bar{\boldsymbol{x}}_{i,r}^* := \bar{\boldsymbol{x}}_{i,r-1}^* \setminus (\mathcal{L}_{i,r} \cup \{h_i^{r-1}\}$ in round $r \in [1, \tau_i^{\mathrm{II}}]$.

We denote their sizes by $\ell_{i,r} := |\mathcal{L}_{i,r}|$ and $\bar{\tau}_{i,r}^* := |\bar{\boldsymbol{x}}_{i,r}^*|$, respectively. First, we give a lower bound on the valuations of optimal and attainable items.

*i*: Would it be 'dirty' to include notation in the definitions?

**Lemma 3.** *For each agent $i \in \mathcal{A}$ and her bundle $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$, it holds in all rounds $r = 2, \dots, \tau_i^{\mathrm{II}}$ of phase II that*

$$v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) \geq v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} \cdot v_i(h_i^1) - \sum_{r'=2}^{r-1} \ell_{i,r'} \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) - v_i(h_i^1, \dots, h_i^{r-1}).$$

Proof. We prove the lemma by induction on the number $r$ of rounds. In the beginning of the base case $r = 2$, agent $i$ has already been assigned item $h_i^1$. For each of the optimal and attainable items $j \in \bar{\boldsymbol{x}}_{i,1}^*$ in round 1, the marginal valuation $v_i(j \mid \emptyset)$ over the empty set was at most $v_i(h_i^1 \mid \emptyset)$, as otherwise item $h_i^1$ would not have been assigned first. The marginal valuation $v_i(j \mid h_i^1)$ over $\{h_i^1\}$ is upper-bounded by $v_i(h_i^1 \mid \emptyset)$, too, due to the submodularity of valuations. During round 2, a further $\ell_{i,2}$ of these items $j$ are assigned to other agents, and item $h_i^2$ is assigned to agent $i$. We can bound the marginal valuation of the remaining optimal and attainable items in round 2 in the following way:

*Case 1* — $h_i^1 \in \bar{\boldsymbol{x}}_{i,1}^*$: It holds $v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,2}^* \cup \{h_i^1\}) - v_i(h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1)$.

*Case 2* — $h_i^1 \notin \bar{\boldsymbol{x}}_{i,1}^*$: Due to the monotonicity of valuations, it holds $v_i(\bar{\boldsymbol{x}}_{i,2}^* \cup \{h_i^1\}) \geq v_i(\bar{\boldsymbol{x}}_{i,2}^*)$ and, therefore, $v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) \geq v_i(\bar{\boldsymbol{x}}_{i,2}^*) - v_i(h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1)$.

In both cases, the base case is proven because

$$\begin{aligned} v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) &\geq v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1) && (12)\\ &\geq v_i(\bar{\boldsymbol{x}}_{i,1}^*) - v_i(\mathcal{L}_{i,2}) - v_i(h_i^1) && (13)\\ &\geq v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} v_i(h_i^1) - v_i(h_i^1), && (14) \end{aligned}$$

where the second inequality can be shown inductively with the definition of submodularity, and the third inequality is due all $\ell_{i,2}$ items $j$ in set $\mathcal{L}_{i,2}$ not being assigned in round 1 although attainable, implying $v_i(j) \le v_i(h_i^1)$.

Do that or, alternatively, find a paper show-

ditto other def.

For the induction hypothesis, we assume that the lemma holds true for all rounds up to some $r$. In the induction step $r \to r+1$, we differentiate the same two cases again:

*Case 1—$h_i^r \in \bar{\boldsymbol{x}}_{i,r}^*$:* Again we exploit the submodularity of valuations to obtain a lower bound on the marginal valuation of $\bar{\boldsymbol{x}}_{i,r+1}^*$.

$$\begin{aligned}
v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \mid h_i^1,\dots,h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \cup \{h_i^r\} \mid h_i^1,\dots,h_i^{r-1}) - v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1}) && (15)\\
&= v_i(\bar{\boldsymbol{x}}_{i,r}^* \setminus \mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^{r-1}) - v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1}) && (16)\\
&\ge v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^{r-1}) - v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1}) && (17)\\
&\qquad - v_i(\mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^{r-1})
\end{aligned}$$

*Case 2—$h_i^r \notin \bar{\boldsymbol{x}}_{i,r}^*$:* At first, we use the monotonicity of valuations to get the inequality

$$\begin{aligned}
v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1,\dots,h_i^r\}) - v_i(h_i^1,\dots,h_i^r) && (18)\\
&\ge v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1,\dots,h_i^{r-1}\}) - v_i(h_i^1,\dots,h_i^r) && (19)\\
&= \left(v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1,\dots,h_i^{r-1}\}) - v_i(h_i^1,\dots,h_i^{r-1})\right) && (20)\\
&\qquad - \left(v_i(h_i^1,\dots,h_i^r) - v_i(h_i^1,\dots,h_i^{r-1})\right)\\
&= v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^{r-1}) - v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1}). && (21)
\end{aligned}$$

Together with the submodularity of valuation, we obtain the same lower bound again:

$$\begin{aligned}
v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \mid h_i^1,\dots,h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \setminus \mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^r) && (22)\\
&\ge v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^r) - v_i(\mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^r) && (23)\\
&\ge v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^r) - v_i(\mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^{r-1}) && (24)\\
&\ge v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^{r-1}) - v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1}) && (25)\\
&\qquad - v_i(\mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^{r-1})
\end{aligned}$$

In both cases, we can replace $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^{r-1})$ by the induction hypothesis and $v_i(\mathcal{L}_{i,r+1} \mid h_i^1,\dots,h_i^{r-1})$ by $\ell_{i,r+1} \cdot v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1})$ to prove the lemma. For a detailed calculation we refer to Garg, Kulkarni and Kulkarni [1, p. 14]. □

include if space enough

The lemma can be used to find a lower bound on the marginal valuation of the items assigned in each round $r$.

**Corollary 1.** *From lemma 3 follows*

$$\begin{aligned}
v_i(h_i^r \mid h_i^1,\dots,h_i^{r-1}) \ge \Bigg(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} \cdot v_i(h_i^1) - \sum_{r'=2}^{r-1} \ell_{i,r'+1} \cdot v_i(h_i^{r'} \mid h_i^1,\dots,h_i^{r'-1})&\\
- v_i(h_i^1,\dots,h_i^{r-1})\Bigg) \Big/ \bar{\tau}_{i,r}^*.&
\end{aligned}$$

Proof. The valuations are monotonic, i. e., $v_i(\mathcal{S}_1) \le v_i(\mathcal{S}_2)$ for all item sets $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{G}$. Induction shows that there must be an item $j \in \bar{\boldsymbol{x}}_{i,r}^*$ with a marginal valuation of at least $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^{r-1})/\bar{\tau}_{i,r}^*$. As item $h_i^r$ was the one to be assigned, the marginal valuation of it cannot be smaller. Using lemma 3 for the value of $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1,\dots,h_i^{r-1})$ proves the corollary. □

change proof when intro is finished

This, finally, enables us to give a lower bound on the valuation of the whole bundle assigned in phase II.

**Lemma 4.** *For each agent $i \in \mathcal{A}$ and her bundle $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$, it holds*

$$v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq v_i(\bar{\boldsymbol{x}}_{i,1}^*)/n.$$

*Proof.* In each round $r = 1, \dots, \tau_i^{\mathrm{II}}$, $\ell_{i,r}$ optimal and attainable items of agent $i$ are assigned to other agents. As there are $n$ agents in total, $n-1$ is an upper bound on $\ell_{i,r}$. Furthermore, after $\tau_i^{\mathrm{II}}$ rounds, the number $\bar{\tau}_{i,\tau_i^{\mathrm{II}}}^*$ of optimal and attainable items is at most $n-1 \leq n$ elsewise agent $i$ would have been assigned yet another item. Together with corollary 1, this proves the lemma:

$$\begin{aligned}
v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) &= v_i(h_i^{\tau_i^{\mathrm{II}}} \mid h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) + v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1}) && (26)\\
&\geq \Big(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - \ell_{i,2} \cdot v_i(h_i^1) - \sum_{r'=2}^{\tau_i^{\mathrm{II}}-1} \ell_{i,r'+1} \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) && (27)\\
&\qquad - v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1})\Big) \Big/ \bar{\tau}_{i,r}^* + v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1}) \\
&\geq \Big(v_i(\bar{\boldsymbol{x}}_{i,1}^*) - (n-1) v_i(h_i^1) - \sum_{r'=2}^{\tau_i^{\mathrm{II}}-1} (n-1) v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) && (28)\\
&\qquad - v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1})\Big) \Big/ n + v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1}) \\
&\geq (v_i(\bar{\boldsymbol{x}}_{i,1}^*) - (n-1) v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1}) - v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1}))/n && (29)\\
&\qquad + v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}-1}) \\
&= v_i(\bar{\boldsymbol{x}}_{i,1}^*)/n && (30)
\end{aligned}$$

□

After having obtained a lower bound on the valuation of items assigned in phase II, we need a lower bound for phase III as well. Therefor we introduce a third type of item set.

**Definition 5.** Let $\boldsymbol{x}_i^* = \{g_i^1, \dots, h_i^{\tau_i^*}\}$ be an optimal allocation of some agent $i \in \mathcal{A}$. The set $\mathcal{G}_i^+$ of *overly good* items is defined as $\mathcal{G}_i^+ \coloneqq \{\, j \in \mathcal{G} \mid v_i(j) \geq v_i(g_i^1) \,\}$.

**Lemma 5.** *In the beginning of phase III, there exists a matching such that each agent $i \in \mathcal{A}$ is matched to one of her overly good items in the set $\bigcup_{i' \in \mathcal{A}} \boldsymbol{x}_{i'}^{\mathrm{I}}$ of released items.*

*Proof.* If all items were matched in phase I, i. e., $\bigcup_{i' \in \mathcal{A}} \boldsymbol{x}_{i'}^{\mathrm{I}} = \mathcal{G}$, then all optimal items are released in phase III and each agent can be matched to one; the lemma is proven immediately. If not, imagine for some $t$ that only the items assigned in the first $t$ rounds of phase I were released. Now choose some matching $\mathcal{M}_t$ with the following properties:

1. If for an agent $i$ all overly good items were amongst the released items, she gets matched with an overly good item $j \in \mathcal{G}_i^+$.
2. The number of agents matched with one of their overly good items is maximal amongst all matchings fulfilling property 1.

Property 1 is always satisfiable as each set $\mathcal{G}_i^+$ is the only one to contain the item $g_i^1$, which can be matched with agent $i$. Property 2 leads to all agents being matched with an overly good item for $t = \lceil \log_2 n \rceil + 1$, i. e. the number of rounds in phase I, whence the lemma follows. To prove this, we denote by $\mathcal{A}_t^-$ the set of agents who are *not* matched with one of their overly good items, and show by induction on $t$ that it holds $|\mathcal{A}_t^-| \leq n/2^t$.

In the base case $t = 1$, none of the items are assigned initially. Denote by $\alpha$ the number of agents who were not assigned an overly good item in the first round of phase I. If $\alpha \leq n/2$, then a matching $\mathcal{M}_1$ obviously exists and the base case is immediately proven. Otherwise, all items from at least $\alpha$ many sets $\mathcal{G}_i^+$ got assigned to someone. Again: Each set $\mathcal{G}_i^+$ is the only one containing the item $g_i^1$, so the union of these sets contains at least $\alpha$ items which can be matched with at least $\alpha$ agents upon release. This then leaves at most $n - \alpha < n/2$ agents not matched with an overly good item.

For the induction hypothesis, we assume that the statement holds true for all rounds up to some $t$. In the induction step $t \to t+1$, by property 1, there is at least one unassigned item in each set $\mathcal{G}_{i'}^+$ for all agents $i' \in \mathcal{A}_t^-$ at the start of round $t+1$. Analogously to the base case, for at least half of those agents $i'$ these unassigned items will be assigned to them or someone else and it can be argued accordingly. By the induction hypothesis, it holds $|\mathcal{A}_{t+1}^-| \leq |\mathcal{A}_t^-|/2 \leq (n/2^t)/2 = n/2^{t+1}$. □

This allows us to calculate an approximation factor for RepReMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

**Theorem 2.** *RepReMatch has an approximation factor of* $2n(\log_2 n + 3)$*.*

*Proof.* By lemma 5, we can assign each agent $i$ an overly good item $j_i^+ \in \mathcal{G}_i^+$ in the beginning of phase III. RepReMatch maximises the logarithmic Nash social welfare, so

$$\log \mathrm{NSW}(\boldsymbol{x}^{\mathrm{III}}) \geq \frac{1}{\sum_{i\in\mathcal{A}} \eta_i} \cdot \sum_{i\in\mathcal{A}} \eta_i \log v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \tag{31}$$

is a lower bound on the logarithmic NSW after the first matching in phase III, whereby $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$ is the bundle of agent $i$ from phase II.

Item $j_i^+$ was released in phase III, which means it was assigned in phase I, implying $j_i^+ \in \boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*$ and, subsequently, $j_i^+ \in (\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1}$. Phase I runs for at most $\lceil \log_2 n \rceil + 1$ rounds, and at most $n$ items are assigned in each iteration. Therefore, at most $n(\log_2 n + 2)$ optimal items are assigned in that phase, i. e., $|\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*| \leq n(\log_2 n + 2)$. Furthermore, it holds $n \geq \ell_{i,1} = |\mathcal{L}_{i,1}|$ as in lemma 4. Together with the monotonicity of valuations, this yields

$i$: Error in paper? see also lemma 5

$$v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq v_i(j_i^+) \geq \frac{v_i((\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1})}{n(\log_2 n + 3)} \tag{32}$$

as lower bound on the valuations of bundles. Moreover, lemma 2 and the monotonicity of valuations yield

$$v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq \frac{v_i(\bar{\boldsymbol{x}}_{i,1}^*)}{n} \geq \frac{v_i(\bar{\boldsymbol{x}}_{i,1}^*)}{n(\log_2 n + 3)} = \frac{v_i(\bar{\boldsymbol{x}}_{i,0}^* \setminus \mathcal{L}_{i,1})}{n(\log_2 n + 3)} \tag{33}$$

as yet another lower bound. The mean of eqs. (32) to (33) and the monotonicity of valuations give the concise lower bound

$$\begin{aligned} v_i(j_i^+, h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) &\geq \frac{1}{2}\left(\frac{v_i((\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1})}{n(\log_2 n + 3)} + \frac{v_i(\bar{\boldsymbol{x}}_{i,0}^* \setminus \mathcal{L}_{i,1})}{n(\log_2 n + 3)}\right) &(34)\\ &\geq \frac{1}{2} \cdot \frac{v_i(((\boldsymbol{x}_i^* \setminus \bar{\boldsymbol{x}}_{i,0}^*) \cup \mathcal{L}_{i,1}) \cup (\bar{\boldsymbol{x}}_{i,0}^* \setminus \mathcal{L}_{i,1}))}{n(\log_2 n + 3)} &(35)\\ &= \frac{v_i(\boldsymbol{x}_i^*)}{2n(\log_2 n + 3)}. &(36) \end{aligned}$$

We can insert this lower bound into eq. (31) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}^{\mathrm{III}}) \geq \frac{1}{\sum_{i \in \mathcal{A}} \eta_i} \cdot \sum_{i \in \mathcal{A}} \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n(\log_2 n + 3)}\right) = \log\left(\frac{\mathrm{NSW}(\boldsymbol{x}^*)}{2n(\log_2 n + 3)}\right) \tag{37}$$

$\square$

# 4 Hardness of Approximation

- probably close to the original section

Cras dapibus, augue quis scelerisque ultricies, felis dolor placerat sem, id porta velit odio eu elit. Aenean interdum nibh sed wisi. Praesent sollicitudin vulputate dui. Praesent iaculis viverra augue. Quisque in libero. Aenean gravida lorem vitae sem ullamcorper cursus. Nunc adipiscing rutrum ante. Nunc ipsum massa, faucibus sit amet, viverra vel, elementum semper, orci. Cras eros sem, vulputate et, tincidunt id, ultrices eget, magna. Nulla varius ornare odio. Donec accumsan mauris sit amet augue. Sed ligula lacus, laoreet non, aliquam sit amet, iaculis tempor, lorem. Suspendisse eros. Nam porta, leo sed congue tempor, felis est ultrices eros, id mattis velit felis non metus. Curabitur vitae elit non mauris varius pretium. Aenean lacus sem, tincidunt ut, consequat quis, porta vitae, turpis. Nullam laoreet fermentum urna. Proin iaculis lectus.

Sed mattis, erat sit amet gravida malesuada, elit augue egestas diam, tempus scelerisque nunc nisl vitae libero. Sed consequat feugiat massa. Nunc porta, eros in eleifend varius, erat leo rutrum dui, non convallis lectus orci ut nibh. Sed lorem massa, nonummy quis, egestas id, condimentum at, nisl. Maecenas at nibh. Aliquam et augue at nunc pellentesque ullamcorper. Duis nisl nibh, laoreet suscipit, convallis ut, rutrum id, enim. Phasellus odio. Nulla nulla elit, molestie non, scelerisque at, vestibulum eu, nulla. Ut odio nisl, facilisis id, mollis et, scelerisque nec, enim. Aenean sem leo, pellentesque sit amet, scelerisque sit amet, vehicula pellentesque, sapien.

Sed consequat tellus et tortor. Ut tempor laoreet quam. Nullam id wisi a libero tristique semper. Nullam nisl massa, rutrum ut, egestas semper, mollis id, leo. Nulla ac massa eu risus blandit mattis. Mauris ut nunc. In hac habitasse platea dictumst. Aliquam eget tortor. Quisque dapibus pede in erat. Nunc enim. In dui nulla, commodo at, consectetuer nec, malesuada nec, elit. Aliquam ornare tellus eu urna. Sed nec metus. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas.

Phasellus id magna. Duis malesuada interdum arcu. Integer metus. Morbi pulvinar pellentesque mi. Suspendisse sed est eu magna molestie egestas. Quisque mi lorem, pulvinar eget, egestas quis, luctus at, ante. Proin auctor vehicula purus. Fusce ac nisl aliquam ante hendrerit pellentesque. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Morbi wisi. Etiam arcu mauris, facilisis sed, eleifend non, nonummy ut, pede. Cras ut lacus tempor metus mollis placerat. Vivamus eu tortor vel metus interdum malesuada.

Sed eleifend, eros sit amet faucibus elementum, urna sapien consectetuer mauris, quis egestas leo justo non risus. Morbi non felis ac libero vulputate fringilla. Mauris libero eros, lacinia non, sodales quis, dapibus porttitor, pede. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Morbi dapibus mauris condimentum nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Etiam sit amet erat. Nulla varius. Etiam tincidunt dui vitae turpis. Donec leo. Morbi vulputate convallis est. Integer aliquet. Pellentesque aliquet sodales urna.

Nullam eleifend justo in nisl. In hac habitasse platea dictumst. Morbi nonummy. Aliquam ut felis. In velit leo, dictum vitae, posuere id, vulputate nec, ante. Maecenas vitae pede nec dui dignissim suscipit. Morbi magna. Vestibulum id purus eget velit laoreet laoreet. Praesent sed leo vel nibh convallis blandit. Ut rutrum. Donec nibh. Donec interdum. Fusce sed pede sit amet elit rhoncus ultrices. Nullam at enim vitae pede vehicula iaculis.

Remark section 5.3

# 5 conclusion

- Of course a short rehearsal of the results for the now knowledgeable reader.
- An outlook would be nice to have. Its content would mostly depend on what recent research has not yet answered.

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.