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ANSWuSVþ(U)M

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Abstract

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Todo list

1 Introduction

- problem introduction, motivation, applications
- formal problem definition (incl. why geometric mean?)
- short literature review: What is known, what not? New findings?
- content & structure of paper

Definition 1. Let $\mathcal{G} := \{1, \dots, m\}$ be a set of indivisible *items* and $\mathcal{A} := \{1, \dots, n\}$ be a set of *agents*. An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathcal{P}(G)^n$ of *bundles* $\boldsymbol{x}_i$ such that each item is element of exactly one bundle, that is $\bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i = \mathcal{G}$ and $\boldsymbol{x}_i \cap \boldsymbol{x}_{i'} = \emptyset$ for all $i \neq i'$. An item $j \in \mathcal{G}$ is *assigned* to agent $i \in \mathcal{A}$ if $j \in \boldsymbol{x}_i$ holds.

or rather 'allocated'?

⋮

Definition 2. Given a set $\mathcal{G}$ of items and a set $\mathcal{A}$ of agents with *valuations* $v_i : \mathcal{P}(\mathcal{G}) \to \mathbb{R}$ and *agent weights* η_i for all agents $i \in \mathcal{A}$, the *Nash Social Welfare problem* (NSW) is to find an allocation maximising the weighted geometric mean of valuations, that is

$\boldsymbol{i}$: less strict def of valuations; restriction for our case later on

$$\operatorname*{arg\,max}_{\boldsymbol{x} \in \Pi_n(\mathcal{G})} \left\{ \left(\prod_{i \in \mathcal{A}} v_i(\boldsymbol{x}_i)^{\eta_i} \right)^{1/\sum_{i \in \mathcal{A}} \eta_i} \right\}$$

where $\Pi_n(\mathcal{G})$ is the set of all possible allocations of the items in $\mathcal{G}$ amongst n agents. The problem is called *symmetric* if all agent weights η_i are equal, and *asymmetric* otherwise.

agents without items assigned have valuation zero → prevent

⋮

In a slight abuse of notation, we omit curly braces delimiting a set in the arguments of a valuation function, so for example we write $v(j_1, j_2, \dots)$ to denote $v(\{j_1, j_2, \dots\})$.

⋮

definition of approximation factor [def environment or in-text?]

⋮

Garg, Kulkarni and Kulkarni [1] consider five different types of non-negative monotonically non-decreasing valuation functions of which we are going to consider only the following two due to space constraints:

Additive The valuation $v_i(\mathcal{S})$ of an agent i for a set $\mathcal{S} \subset \mathcal{G}$ of items j is the sum of individual valuations $v_i(j)$, that is $v_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} v_i(j)$.

Submodular Let $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$ denote the marginal valuation of agent i for a set $\mathcal{S}_1 \subset \mathcal{G}$ of items over a *disjoint* set $\mathcal{S}_2 \subset \mathcal{G}$. This valuation functions satisfies the submodularity constraint $v_i(j \mid \mathcal{S}_1 \cup \mathcal{S}_2) \leq v_i(j \mid \mathcal{S}_1)$ for all agents $i \in \mathcal{A}$, items $j \in \mathcal{G}$ and sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$ of items.

or rather utility?

We use *additive NSW* and *submodular NSW* as shorthands for the Nash social welfare problems with additive and submodular valuation functions, respectively.

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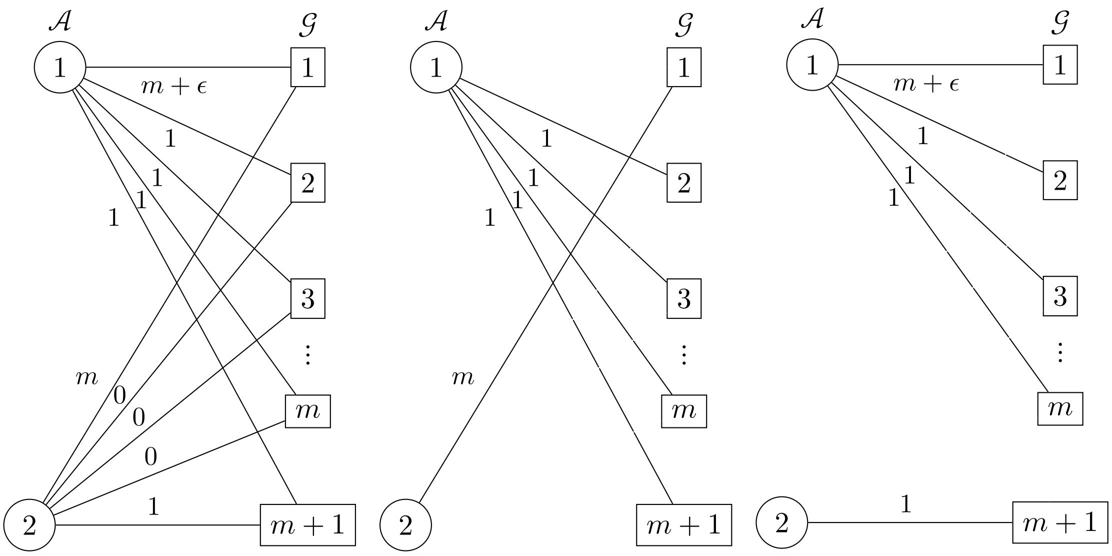

(a) problem instance **(b)** OPT $= \sqrt{m \cdot m}$ **(c)** ALG $\leq \sqrt{(2m+\epsilon-1) \cdot 1}$

i: might delete if document in two columns → unused edges in grey

Figure 1: Agent 1 values item 1 at $m+\epsilon$ and all other items at 1. Agent 2 values item 1 at m, item $m+1$ at 1 and all other items at 0. In an optimal allocation, item 1 would be assigned to agent 2 and all other items to agent 1, resulting in a NSW of $\sqrt{m \cdot m} = m$. A repeated maximum matching algorithm would greedily assign item 1 to agent 1 and item $m+1$ to agent 2 in the first round. Even if all remaining items were going to be assigned to agent 1, the NSW will never surpass $\sqrt{(2m+\epsilon-1) \cdot 1} < \sqrt{2m}$. The approximation factor $\alpha \approx \sqrt{m/2}$ therefore depends on the number of items.

2 SMatch

In the case of an equal number of agents and items, i. e., $|\mathcal{A}| = n = m = |\mathcal{G}|$, the *additive* NSW can be solved exactly by finding a maximum matching on a bipartite graph with the sets of agents and of items as its parts; as weight of the edge between agent i and item j, use $\eta_i \log v_i(j)$, that is the weighted valuation of item j by agent i in the *logarithmic* Nash social welfare. Should there be more items than agents, then it would be obvious to just repeatedly find a maximum matching and assign the items accordingly until all items are assigned. The flaw of this idea is that such a greedy algorithm only considers the valuations of items in the current matching and perhaps the valuations of items already assigned. As the example in fig. 1 demonstrates, this leads to an algorithm with an approximation factor dependent on the number m of items. The geometric mean of the NSW favours allocations with similarly valued bundles, wherefore it may be beneficial for an agent to leave items to other agents if those agents cannot expect many more valuable items in the future.

What is the factor?

The algorithm SMatch, described in algorithm 1, eliminates the flaw by first gaining foresight of the valuations of items assigned after the first matching, achieving an approximation factor of $2n$ (cf. theorem 1 later on). For a fixed agent i, order the items in descending order of the valuations by agent i and denote the j-th most liked item by $\mathcal{G}_i^j$. To obtain a well-defined order, items of equal rank are further ordered numerically. SMatch, too, does repeatedly match items. During the first matching, however, the edge weights are defined as $\eta_i \log(v_i(j) + u_i/n)$ for an edge between agent i and item j. The addend u_i serves as estimation of the valuation of items assigned after the first matching

Algorithm 1: SMatch for the Asymmetric Additive NSW problem

Input : set $\mathcal{A} = \{1, \ldots, n\}$ of agents with weights $\eta_i \forall i \in \mathcal{A}$, set $\mathcal{G} = \{1, \ldots, m\}$ of indivisible items, additive valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each set $\mathcal{S} \subset \mathcal{G}$ of items

Output: $\frac{1}{2n}$-approximation $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ of an optimal allocation

$\boldsymbol{x}_i \leftarrow \emptyset \quad \forall i \in \mathcal{A}$

$u_i \leftarrow v_i(\mathcal{G}_i^{2n+1}, \ldots, \mathcal{G}_i^m) \quad \forall i \in \mathcal{A}$ ▷ *est. valuations after the 1st matching*

$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j) + u_i/n) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$ ▷ *edge weights*

$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$ ▷ *bipartite graph*

$\mathcal{M} \leftarrow$ max_weight_matching(G)

$\boldsymbol{x}_i \leftarrow \{\, j \mid (i, j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$ ▷ *assign according to matching*

$\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G} \setminus \{\, j \mid (i, j) \in \mathcal{M} \,\}$ ▷ *remove assigned items*

while $\mathcal{G}^{\text{rem}} \neq \emptyset$ **do**

  $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j) + v_i(\boldsymbol{x}_i)) \mid i \in \mathcal{A}, j \in \mathcal{G}^{\text{rem}} \,\}$

  $G \leftarrow (\mathcal{A}, \mathcal{G}^{\text{rem}}, \mathcal{W})$

  $\mathcal{M} \leftarrow$ max_weight_matching(G)

  $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i \cup \{\, j \mid (i, j) \in \mathcal{M} \,\} \quad \forall i \in \mathcal{A}$

  $\mathcal{G}^{\text{rem}} \leftarrow \mathcal{G}^{\text{rem}} \setminus \{\, j \mid (i, j) \in \mathcal{M} \,\}$

end while

return $\boldsymbol{x}$

and is defined as

$$u_i := \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} = v_i(\mathcal{G}_i^{2n+1}, \ldots, \mathcal{G}_i^m). \tag{1}$$

The set $\mathcal{S}$ has less than $2n$ elements only if there are less than $2n$ items in total. From the second matching onwards, the edge weights are defined as $\eta_i \log(v_i(j) + v_i(\boldsymbol{x}_i))$, where $\boldsymbol{x}_i$ is the continuously updated bundle of agent i. The addend $v_i(\boldsymbol{x}_i)$ could lead to better allocations in applications, but does not improve the approximation factor asymptotically.

For convenience, we order the items in the final bundle $\boldsymbol{x}_i = \{h_i^1, \ldots, h_i^{\tau_i}\}$ of agent i by the order in which they were assigned, so that item h_i^t is assigned according to the t-th matching. To calculate the approximation factor of SMatch, we first need to establish a lower bound on the valuation of single items.

Lemma 1. *For each agent $i \in \mathcal{A}$ and her final bundle $\boldsymbol{x}_i = \{h_i^1, \ldots, h_i^{\tau_i}\}$, the item h_i^t is worth at least as much as the overall tn-th most highly valued item $\mathcal{G}_i^{tn}$, i. e. $v_i(h_i^t) \geq v_i(\mathcal{G}_i^{tn})$, for all $t = 1, \ldots, \tau_i$.*

Proof. At the start of the t-th round, no more than $(t-1)n$ items out of the tn most highly valued items $\mathcal{G}_i^1, \ldots, \mathcal{G}_i^{tn}$ have been assigned in previous rounds since at most n items are assigned in each iteration. During the t-th round, at most $n-1$ more of those highly valued items could be assigned to all other agents $i' \neq i$, leaving at least one item in $\mathcal{G}_i^1, \ldots, \mathcal{G}_i^{tn}$ unassigned. Since $v_i(\mathcal{G}_i^k) \geq v_i(\mathcal{G}_i^{tn})$ for all $k \leq tn$ by definition of $\mathcal{G}_i^{\cdot}$, the lemma follows. □

We can now establish $u_i/n = v_i(\mathcal{G}_i^{2n+1}, \ldots, \mathcal{G}_i^m)/n$ as lower bound on the valuations of items assigned after the first matching.

Lemma 2. *For each agent $i \in \mathcal{A}$ and her final bundle $\boldsymbol{x}_i = \{h_i^1, \ldots, h_i^{\tau_i}\}$, it holds $v_i(h_i^2, \ldots, h_i^{\tau_i}) \geq u_i/n$, where n is the number of items and $u_i = v_i(\mathcal{G}_i^{2n+1}, \ldots, \mathcal{G}_i^m)$.*

Proof. By lemma 1 and definition of $\mathcal{G}_i^{\cdot}$, every item h_i^t is worth at least as much as each item $\mathcal{G}_i^{tn+k}$ with $k \in \{0, \dots, n-1\}$ and, consequently, its valuation $v_i(h_i^t)$ is at least as high as the mean valuation $\frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1})$. Further, it holds $\tau_i n + n \geq m$ since each agent receives items for at least $\lfloor \frac{m}{n} \rfloor \geq \frac{m}{n} - 1$ rounds. Together, this yields

$$v_i(h_i^2, \dots, h_i^{\tau_i}) = \sum_{t=2}^{\tau_i} v_i(h_i^t) \geq \sum_{t=2}^{\tau_i} \frac{1}{n} v_i(\mathcal{G}_i^{tn}, \dots, \mathcal{G}_i^{tn+n-1}) \tag{2}$$

$$\geq \frac{1}{n} v_i(\mathcal{G}_i^{2n}, \dots, \mathcal{G}_i^{m-1}) \geq \frac{1}{n} v_i(\mathcal{G}_i^{2n+1}, \dots, \mathcal{G}_i^{m}) = \frac{u_i}{n} \tag{3}$$

with the last inequality stemming from $v_i(\mathcal{G}_i^{2n}) \geq v_i(\mathcal{G}_i^m)$. □

Remark 1. In lemma 2, we assumed non-zero valuations for all items, hence the bundle lengths of $\tau_i \geq \lfloor \frac{m}{n} \rfloor$. Of course in an actual program, one would not assign items to agents who value them at zero. Inasmuch as additional zeros in eq. (3) do not change the sum, lemma 2 still holds nevertheless.

This allows us to calculate an approximation factor for SMatch by comparing its output with an optimal allocation $\boldsymbol{x}^*$.

Theorem 1. *SMatch has an approximation factor of* $2n$.

Proof. Lemma 2 can be plugged into the logarithmic NSW:

$$\log \mathrm{NSW}(\boldsymbol{x}) = \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log v_i(h_i^1, \dots, h_i^{\tau_i}) \tag{4}$$

$$= \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log(v_i(h_i^1) + v_i(h_i^2, \dots, h_i^{\tau_i})) \tag{5}$$

$$\geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log(v_i(h_i^1) + u_i/n) \tag{6}$$

Notice that the first matching of SMatch maximises the sum in eq. (6). Thus, assigning all agents i their respective most highly valued item g_i^1 in an optimal bundle $\boldsymbol{x}_i^* = \{g_i^1, \dots, g_i^{\tau_i^*}\}$ yields the even lower bound

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log(v_i(g_i^1) + u_i/n). \tag{7}$$

Recall the definition of u_i from eq. (1). Consider a slightly modified variant:

$$u_i = \min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \quad \text{or, alternatively,} \quad u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \text{ with } \mathcal{S}_i := \arg\min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\mathcal{G} \setminus \mathcal{S})\} \tag{8}$$

Moreover, consider the set $\mathcal{S}_i^*$ of the (at most) $2n$ most highly valued items in the optimal bundle $\boldsymbol{x}_i^*$, i. e.

$$\mathcal{S}_i^* := \arg\min_{\substack{\mathcal{S} \subset \mathcal{G} \\ |\mathcal{S}| \leq 2n}} \{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S})\}. \tag{9}$$

We get the lower bound $v_i(g_i^1) \geq \frac{1}{2n} v_i(\mathcal{S}^*)$ from a similar argument as in the proof of lemma 2. Further, it holds $u_i = v_i(\mathcal{G} \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i) \geq v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)$. We can substitute these two inequalities into eq. (7) and prove the theorem thereby:

$$\log \mathrm{NSW}(\boldsymbol{x}) \geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log\left(\frac{v_i(\mathcal{S}_i^*)}{2n} + \frac{v_i(\boldsymbol{x}_i^* \setminus \mathcal{S}_i^*)}{n}\right) \tag{10}$$

$$\geq \frac{1}{\sum_{i=1}^n \eta_i} \cdot \sum_{i=1}^n \eta_i \log\left(\frac{v_i(\boldsymbol{x}_i^*)}{2n}\right) = \log\left(\frac{\mathrm{OPT}}{2n}\right) \tag{11}$$

□

3 RepReMatch

- SMatch does not work for general submodular valuations since we need to detect the set of lowest valuation. This is not possible independent of m [SF11].

what exactly

First, we bound the valuation of an agent i for her optimal items which still may be assigned to her in phase II.

i: Would 'to bound' (strongly) imply giving an upper bound as well?

Lemma 3. *During phase II, each agent $i \in \mathcal{A}$ gets assigned a bundle $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$. For all rounds $r = 2, \dots, \tau_i^{\mathrm{II}}$, the following inequality about her set $\bar{\boldsymbol{x}}_{i,r}^*$ of optimal and attainable items holds true:*

$$v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) \geq u_i^* - \ell_{i,2} v_i(h_i^1) - \sum_{r'=2}^{r-1} \ell_{i,r'} \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) - v_i(h_i^1, \dots, h_i^{r-1}),$$

where $\ell_{i,l}$ is the size of her set $\mathcal{L}_{i,l} \subset \bar{\boldsymbol{x}}_{i,l-1}^*$ of optimal items which were attainable in round $l-1$ and were assigned to other agents in round l, and $u_i^* = v_i(\bar{\boldsymbol{x}}_{i,1}^*)$ is her valuation of attainable and optimal items during the first round.

Proof. We prove the lemma by induction on the number r of rounds. In the beginning of the base case $r = 2$, agent i has already been assigned item h_i^1 but not the items in set $\mathcal{L}_{i,1}$. For each of the remaining optimal and attainable items j in round 1, the marginal valuation $v_i(j \mid \emptyset)$ over the empty set is at most $v_i(h_i^1 \mid \emptyset)$, as otherwise item h_i^1 would not have been assigned first. Furthermore, the marginal valuation $v_i(j \mid h_i^1)$ over $\{h_i^1\}$ is upper-bounded by $v_i(h_i^1 \mid \emptyset)$ due to the submodularity of valuations. During the round, a further $\ell_{i,2}$ out of these items are assigned to other agents, and item h_i^2 is assigned to agent i. We can bound the marginal valuation of the remaining optimal and attainable items in round 2 in the following way:

Case 1 — $h_i^1 \in \bar{\boldsymbol{x}}_{i,1}^*$: It holds $v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,2}^* \cup \{h_i^1\}) - v_i(h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1)$.

Case 2 — $h_i^1 \notin \bar{\boldsymbol{x}}_{i,1}^*$: Due to the monotonicity of valuations, it holds $v_i(\bar{\boldsymbol{x}}_{i,2}^* \cup \{h_i^1\}) \geq v_i(\bar{\boldsymbol{x}}_{i,2}^*)$ and, therefore, $v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) \geq v_i(\bar{\boldsymbol{x}}_{i,2}^*) - v_i(h_i^1) = v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1)$.

In both cases, the base case is proved because

$$\begin{aligned} v_i(\bar{\boldsymbol{x}}_{i,2}^* \mid h_i^1) &\geq v_i(\bar{\boldsymbol{x}}_{i,1}^* \setminus \mathcal{L}_{i,2}) - v_i(h_i^1) && (12)\\ &\geq v_i(\bar{\boldsymbol{x}}_{i,2}^*) - v_i(\mathcal{L}_{i,2}) - v_i(h_i^1) && (13)\\ &\geq u_i^* - \ell_{i,2} v_i(h_i^1) - v_i(h_i^1), && (14) \end{aligned}$$

where the second inequality can be shown inductively with the definition of submodularity, and the third inequality is due all $\ell_{i,2}$ items j in $\mathcal{L}_{i,2}$ being attainable but not assigned in round 1, implying $v_i(j) \leq v_i(h_i^1)$.

Do that or, alternatively, find a paper show-

As induction hypothesis, we assume that the lemma is true for all rounds up to some r. For the induction step $r \to r+1$, we differentiate the same two cases again:

Case 1 — $h_i^r \in \bar{\boldsymbol{x}}_{i,r}^*$: Again we use the submodularity of valuations to obtain a lower bound on the marginal valuation of $\bar{\boldsymbol{x}}_{i,r+1}^*$.

ditto

other def.

$$\begin{aligned} v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \mid h_i^1, \dots, h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \cup \{h_i^r\} \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) && (15)\\ &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \setminus \mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) && (16)\\ &\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) && (17)\\ &\qquad - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1}) \end{aligned}$$

Case 2 — $h_i^r \notin \bar{\boldsymbol{x}}_{i,r}^*$: We use the monotonicity of valuations for the inequality

$$\begin{aligned}
v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1, \dots, h_i^r\}) - v_i(h_i^1, \dots, h_i^r) && (18)\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1, \dots, h_i^{r-1}\}) - v_i(h_i^1, \dots, h_i^r) && (19)\\
&= \left(v_i(\bar{\boldsymbol{x}}_{i,r}^* \cup \{h_i^1, \dots, h_i^{r-1}\}) - v_i(h_i^1, \dots, h_i^{r-1})\right) && (20)\\
&\qquad - \left(v_i(h_i^1, \dots, h_i^r) - v_i(h_i^1, \dots, h_i^{r-1})\right)\\
&= v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) && (21)
\end{aligned}$$

after first using the submodularity twice to obtain the same lower bound again:

rethink formulation

$$\begin{aligned}
v_i(\bar{\boldsymbol{x}}_{i,r+1}^* \mid h_i^1, \dots, h_i^r) &= v_i(\bar{\boldsymbol{x}}_{i,r}^* \setminus \mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^r) && (22)\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^r) - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^r) && (23)\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^r) - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1}) && (24)\\
&\geq v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) - v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) && (25)\\
&\qquad - v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1})
\end{aligned}$$

In both cases, we can replace $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1})$ by the induction hypothesis and $v_i(\mathcal{L}_{i,r+1} \mid h_i^1, \dots, h_i^{r-1})$ by $\ell_{i,r+1} \cdot v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1})$ to prove the lemma. For a more detailed formula simplification, we refer to Garg, Kulkarni and Kulkarni [1, p. 14]. □

i: I hope the shortcut is allowed. Might remove it if rest of document not too long.

The lemma can be used to bound the marginal valuation of the items assigned in each round r.

Corollary 1. *From lemma 3 follows*

$$v_i(h_i^r \mid h_i^1, \dots, h_i^{r-1}) \geq \left(u_i^* - \ell_{i,2} v_i(h_i^1) - \sum_{r'=2}^{r-1} \ell_{i,r'+1} \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) - v_i(h_i^1, \dots, h_i^{r-1})\right) \Big/ \left(\bar{\tau}_{i,0}^* - \sum_{l=1}^{r} \ell_{i,l}\right)$$

where $\bar{\tau}_{i,0}^* := |\bar{\boldsymbol{x}}_{i,0}^*|$ *denotes the number of optimal and attainable items after phase I.*

Proof. There are $\bar{\tau}_{i,0}^*$ optimal and attainable items in $\bar{\boldsymbol{x}}_{i,0}^*$ at the start of phase II. Of those, $\ell_{i,l}$ many are assigned to other agents in each round $l \leq r$, and also some items h_i^l assigned to agent i may be optimal, whence an upper bound of $\bar{\tau}_{i,0}^* - \sum_{l=1}^{r} \ell_{i,l}$ on the number $\bar{\tau}_{i,r}^*$ of items in the set $\bar{\boldsymbol{x}}_{i,r}^*$.

The valuations are monotonic, i. e., $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ for all sets $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{G}$ of items. Induction shows that there must be an item $j \in \bar{\boldsymbol{x}}_{i,r}^*$ with a marginal valuation of at least $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1}) / \bar{\tau}_{i,r}^*$. As item h_i^r was the one to be assigned, the marginal valuation of it cannot be smaller. Using lemma 3 for the value of $v_i(\bar{\boldsymbol{x}}_{i,r}^* \mid h_i^1, \dots, h_i^{r-1})$ proves the corollary. □

Possibly this whole estimation can be omitted as we ditch it in the following lemma.

This, finally, enables us to give a lower bound on the valuation of the whole bundle assigned in phase II.

Lemma 4. *For each agent $i \in \mathcal{A}$ and her bundle $\boldsymbol{x}_i^{\mathrm{II}} = \{h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}\}$ assigned in phase II, her valuation $v_i(\boldsymbol{x}_i^{\mathrm{II}})$ of her bundle is lower-bounded by her valuation $u_i^* = v_i(\bar{\boldsymbol{x}}_{i,1}^*)$ of the optimal and attainable items in $\bar{\boldsymbol{x}}_{i,1}^*$ after the first round divided by the number n of agents, i. e.,*

$$v_i(h_i^1, \dots, h_i^{\tau_i^{\mathrm{II}}}) \geq u_i^* / n.$$

Proof. In each round $r = 1, \dots, \tau_i^{\text{II}}$, $\ell_{i,r}$ optimal and attainable items of agent i are assigned to other agents. As there are n agents in total, $n-1$ is an upper bound on $\ell_{i,r}$. Furthermore, after τ_i^{II} rounds, the number $\bar{\tau}^*_{i,\tau_i^{\text{II}}} \leq \bar{\tau}^*_{i,0} - \sum_{l=1}^{r} \ell_{i,l}$ of optimal and attainable items is at most $n-1 \leq n$ else she would have been assigned yet another item. Together with corollary 1, this proves the lemma:

i: swapping the summands' positions reduces the number of lines used

$$
\begin{aligned}
v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}}) &= v_i(h_i^{\tau_i^{\text{II}}} \mid h_i^1, \dots, h_i^{\tau_i^{\text{II}}}) + v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1}) && (26)\\
&\geq \Big(u_i^* - \ell_{i,2} v_i(h_i^1) - \sum_{r'=2}^{\tau_i^{\text{II}}-1} \ell_{i,r'+1} \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) && (27)\\
&\qquad - v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1})\Big) \Big/ \Big(\bar{\tau}^*_{i,0} - \sum_{l=1}^{\tau_i^{\text{II}}} \ell_{i,l}\Big) + v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1})\\
&\geq \Big(u_i^* - (n-1) v_i(h_i^1) - \sum_{r'=2}^{\tau_i^{\text{II}}-1} (n-1) \cdot v_i(h_i^{r'} \mid h_i^1, \dots, h_i^{r'-1}) && (28)\\
&\qquad - v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1})\Big) \Big/ n + v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1})\\
&\geq (u_i^* - (n-1) v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1}) - v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1})) \big/ n && (29)\\
&\qquad + v_i(h_i^1, \dots, h_i^{\tau_i^{\text{II}}-1})\\
&= u_i^*/n && (30)
\end{aligned}
$$

□

Remark 2. possibly not or shorter

Lemma 5. *about phase I & III*

Theorem 2. *RepReMatch has an approximation factor of* $2n(\log n + 2)$.

Proof. □

Remark 3. possibly not or shorter

Algorithm 2: RepReMatch for the Asymmetric Submodular NSW problem

Input : set $\mathcal{A} = \{1, \dots, n\}$ of agents with weights $\eta_i \forall i \in \mathcal{A}$, set $\mathcal{G} = \{1, \dots, m\}$ of indivisible items, submodular valuations $v_i \colon \mathcal{P}(\mathcal{G}) \to \mathbb{R}_0^+$ where $v_i(\mathcal{S})$ is the valuation of agent $i \in \mathcal{A}$ for each set $\mathcal{S} \subset \mathcal{G}$ of items

Output: $\frac{1}{2n(\log n+2)}$-approximation $\boldsymbol{x}^{\mathrm{III}} = (\boldsymbol{x}_1^{\mathrm{III}}, \dots, \boldsymbol{x}_n^{\mathrm{III}})$ of an optimal allocation

Phase I:

$\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$
$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}$
for $t \leftarrow 0, \dots, \lceil \log n \rceil - 1$ **do**
  if $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **then**
    $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(j)) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$
    $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$
    $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$
    $\boldsymbol{x}_i^{\mathrm{I}} \leftarrow \boldsymbol{x}_i^{\mathrm{I}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$
    $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$
  end if
end for

Phase II:

$\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \emptyset \quad \forall i \in \mathcal{A}$ ▷ *put allocation $\boldsymbol{x}^{\mathrm{I}}$ aside and start a new one*
while $\mathcal{G}^{\mathrm{rem}} \neq \emptyset$ **do**
  $\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$
  $G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$
  $\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$
  $\boldsymbol{x}_i^{\mathrm{II}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$
  $\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$
end while

Phase III:

$\mathcal{G}^{\mathrm{rem}} \leftarrow \bigcup_{i \in \mathcal{A}} \boldsymbol{x}_i^{\mathrm{I}}$ ▷ *release items assigned in phase I*
$\mathcal{W} \leftarrow \{\, \eta_i \cdot \log(v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})) \mid i \in \mathcal{A}, j \in \mathcal{G} \,\}$
$G \leftarrow (\mathcal{A}, \mathcal{G}, \mathcal{W})$
$\mathcal{M} \leftarrow \mathrm{max\_weight\_matching}(G)$
$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \boldsymbol{x}_i^{\mathrm{II}} \cup \{j\} \quad \forall (i,j) \in \mathcal{M}$
$\mathcal{G}^{\mathrm{rem}} \leftarrow \mathcal{G}^{\mathrm{rem}} \setminus \{\, j \mid (i,j) \in \mathcal{M} \,\}$
$\boldsymbol{x}_i^{\mathrm{III}} \leftarrow \mathrm{arbitrary\_allocation}(\mathcal{A}, \mathcal{G}^{\mathrm{rem}}, \boldsymbol{x}_i^{\mathrm{III}}, (v_i)_{i \in \mathcal{A}})$
return $\boldsymbol{x}^{\mathrm{III}}$

decide on $\boldsymbol{x}$ or $\boldsymbol{x}^{\mathrm{III}}$; discrepancies in orig paper in def of $\boldsymbol{x}^{\mathrm{III}}$!