Seminar Approximation Algorithms

Approximating Nash Social Welfare under Submodular Valuations through (Un)Matchings

Based on a paper of the same title by J. Garg, P. Kulkarni, and R. Kulkarni

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Supervised by Dr Giovanna Varricchio

26th July 2023 · Algorithms and Complexity (Prof. Dr Martin Hoefer)

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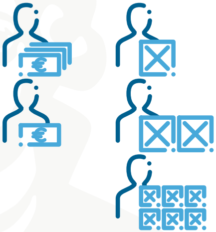

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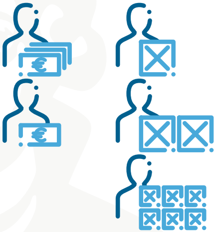

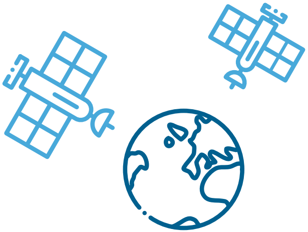

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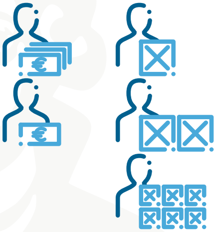
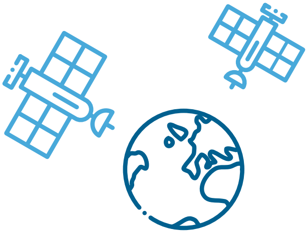
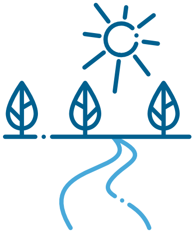
- industrial procurement
- satellites
- water withdrawal







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Preliminaries

Allocations

Setting:

- recipients: set $\mathscr{A}$ of n agents
- goods: set $\mathscr{G}$ of m items

Definition

An *allocation* is a tuple $\boldsymbol{x} = (\boldsymbol{x}_i)_{i\in\mathscr{A}}$ of bundles $\boldsymbol{x}_i \subset \mathscr{G}$ such that each item is element of precisely one bundle.
Item j is *assigned* to agent i if $j \in \boldsymbol{x}_i$.

But how to measure its efficiency and fairness?

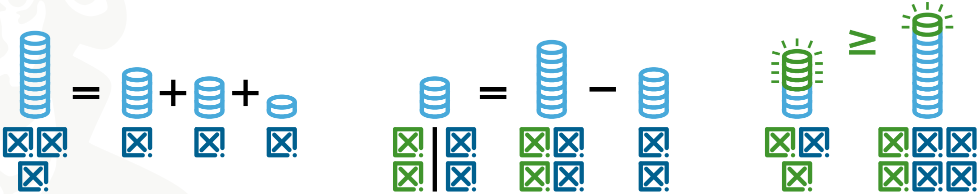
Valuation Functions

Requirements:

- monotonically non-decreasing: $v_i(\mathcal{S}_1) \leq v_i(\mathcal{S}_2)$ if $\mathcal{S}_1 \subset \mathcal{S}_2$
- normalised: $v_i(\emptyset) = 0$

Types:

- additive: $v_i(\mathcal{S}) := \sum_{j \in \mathcal{S}} v_i(j)$
- submodular: $v_i(\mathcal{S}_1 \mid \mathcal{S}_2) := v_i(\mathcal{S}_1 \cup \mathcal{S}_2) - v_i(\mathcal{S}_2)$
  - diminishing returns



Asymmetric Maximum Nash Social Welfare Problem

Problem

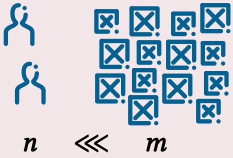
$$x^* \stackrel{!}{=} \operatorname*{arg\,max}_{x \in X_{\mathscr{A}}(\mathscr{G})} \{\mathrm{NSW}(x)\} \quad \text{with } \mathrm{NSW}(x) := \Big(\prod_{i \in \mathscr{A}} v_i(x_i)^{\eta_i}\Big)^{1/\sum_{i\in\mathscr{A}} \eta_i}$$

- $X_{\mathscr{A}}(\mathscr{G})$: all possible allocations
- η_i: agent weight

The NSW strikes a middle ground between efficiency and fairness!

Is there a polynomial-time algorithm with an approximation factor …

- … dependent on n?
- … independent from m?

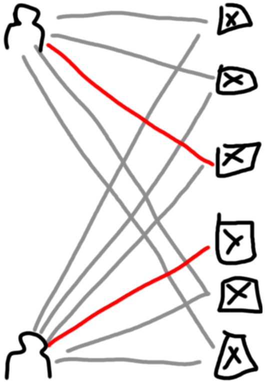


2 RepReMatch

Naïve Approach

Naïve approach:

- repeatedly use maximum matchings
- fails because of missing foresight
  - additive valuations: sort items by valuation
    $\Rightarrow 2n$-approximation (SMatch)
  - submodular valuations: lowest valuation
    approximable only by $\Omega(\sqrt{m/\ln m})$ ↯



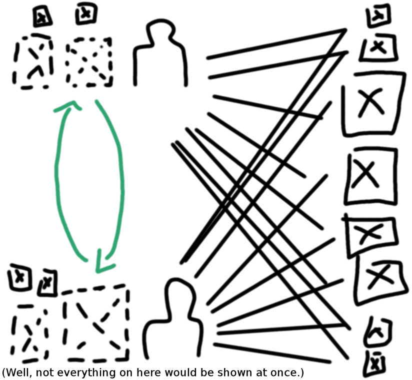
Key Ideas of the Algorithm

We need change the past in three phases:

Phase I Assign enough high-value items temporarily.

Phase II Assign the remaining items definitely.

Phase III Re-assign the items of phase I definitely.



(Well, not everything on here would be shown at once.)

Theorem

RepReMatch guarantees a $2n(\log_2 n + 3)$-approximation under submodular valuations.

The Algorithm

Phase I:

1. repeat $\lceil \log_2 n \rceil + 1$ times
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\log v_i(j)^{\eta_i}$
   2. compute maximum weight matching
   3. update bundles $\boldsymbol{x}_i^{\mathrm{I}}$ & remove assigned items

Phase II:

2. repeat until $\mathscr{G} = \emptyset$
   1. create bipartite graph $(\mathscr{A}, \mathscr{G}, E)$ with edge weights $\log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})^{\eta_i}$
   2. compute maximum weight matching
   3. update bundles $\boldsymbol{x}_i^{\mathrm{II}}$ & remove assigned items

Phase III:

3. create bipartite graph $\left(\mathscr{A}, \bigcup_{i \in \mathscr{A}} \boldsymbol{x}_i^{\mathrm{I}}, E\right)$ with edge weights $\log v_i(\boldsymbol{x}_i^{\mathrm{II}} \cup \{j\})^{\eta_i}$
4. compute maximum weight matching
5. create bundles $\boldsymbol{x}_i^{\mathrm{III}}$

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$\Rightarrow$ Are enough outstanding items reserved?

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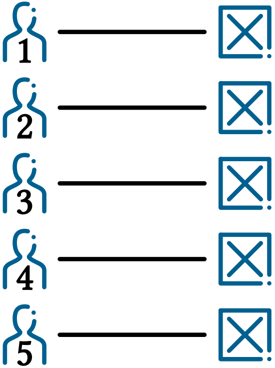

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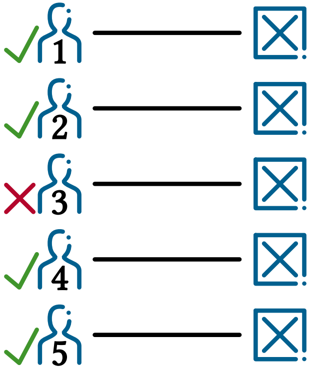

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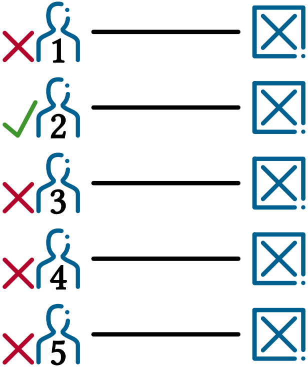

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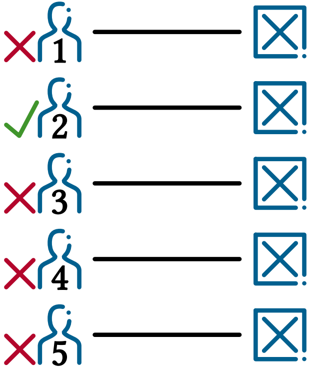

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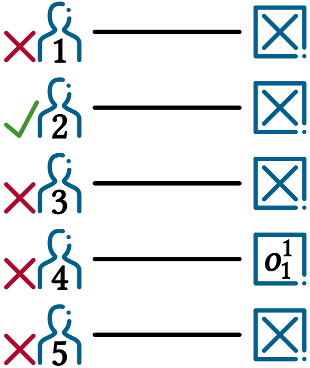

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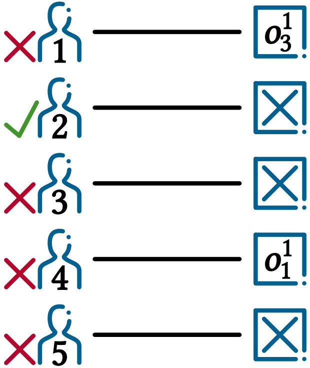

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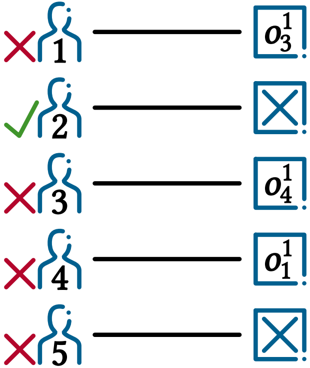

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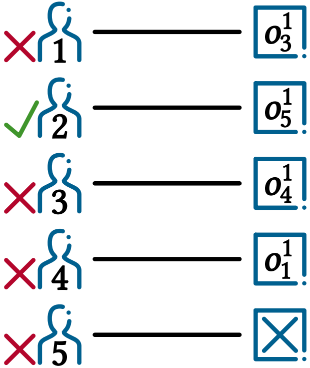

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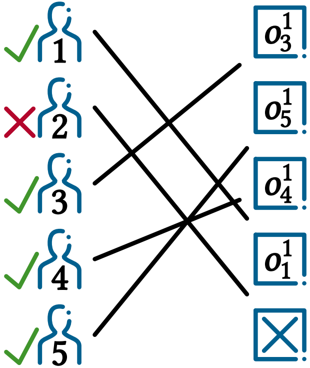

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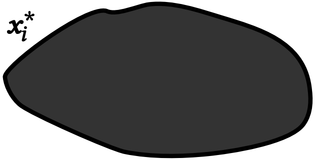

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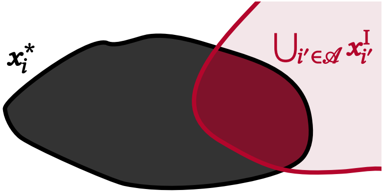

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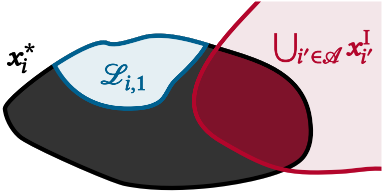

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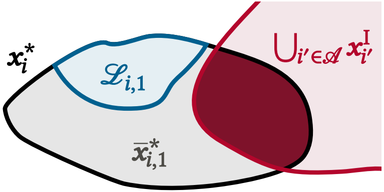

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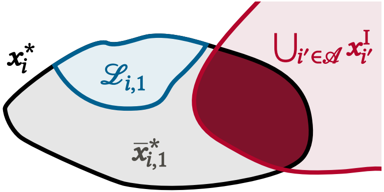

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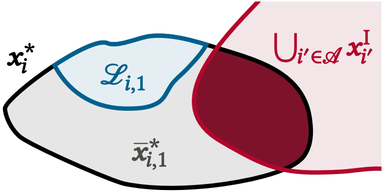

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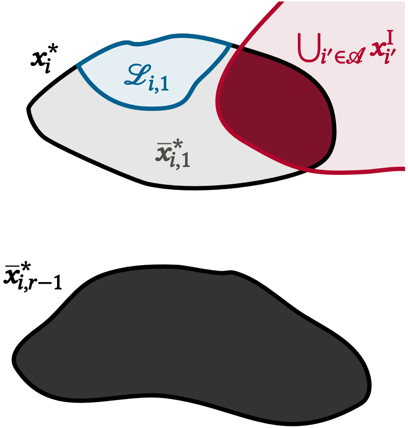

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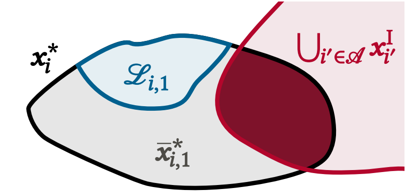

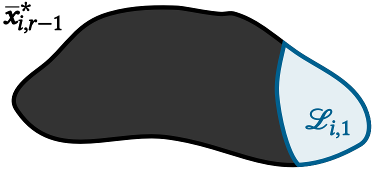

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Definition

Let $\boldsymbol{x}_i^{\mathrm{II}} = \{a_i^1, a_i^2, \ldots\}$ be the bundle of agent i. The set of *optimal and attainable items* is defined as

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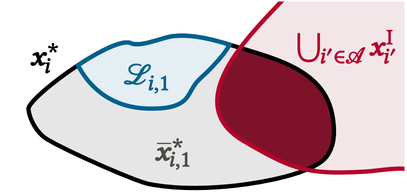

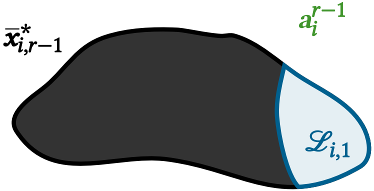

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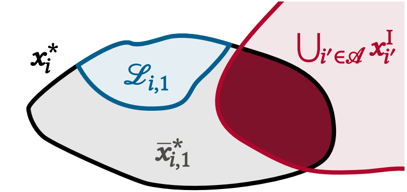

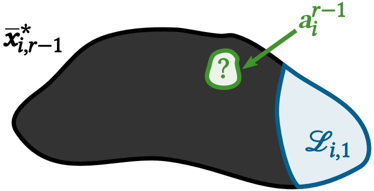

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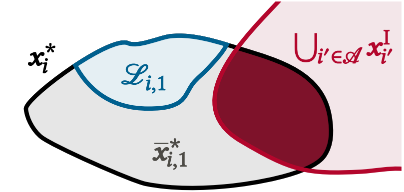

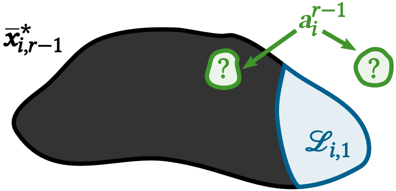

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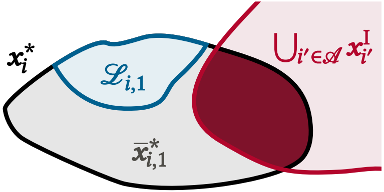

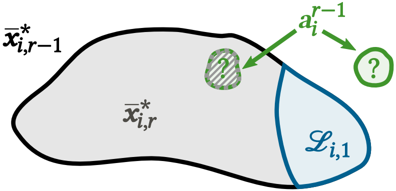

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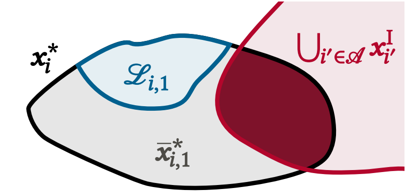

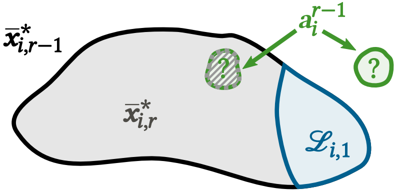

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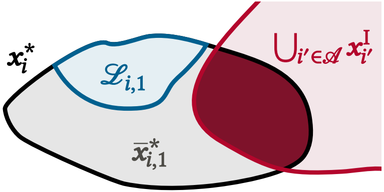
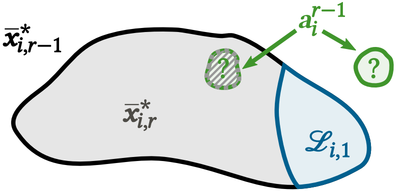
$\Rightarrow$ What is the valuation of the remaining items?





Analysing Phase II (2/2)

Lemma

$$v_i\left(\overline{x}^*_{i,r} \mid a_i^1, \dots, a_i^{r-1}\right)$$

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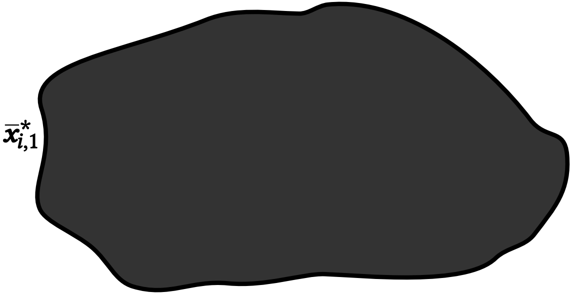

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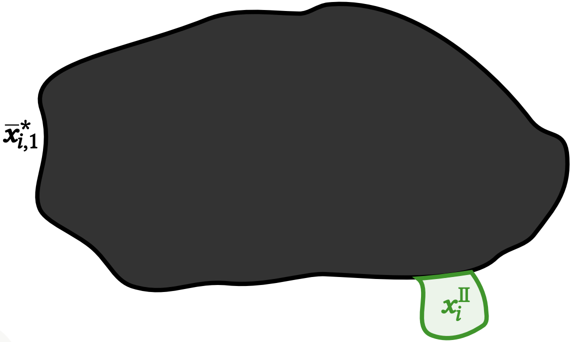

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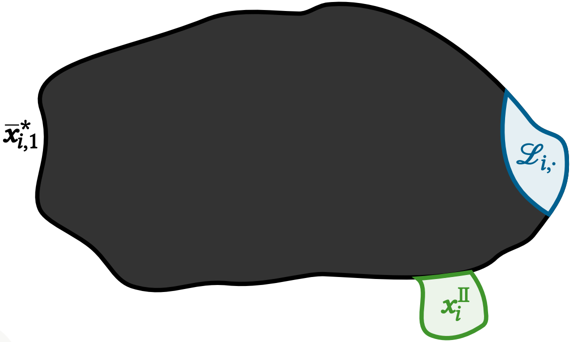

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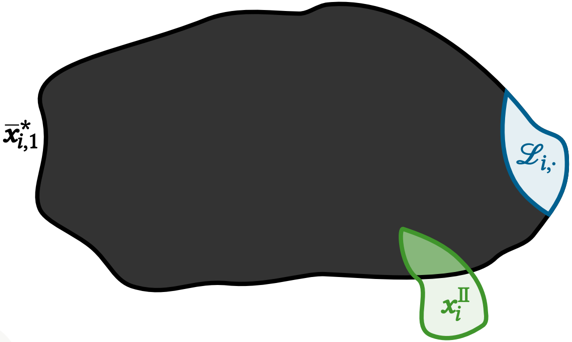

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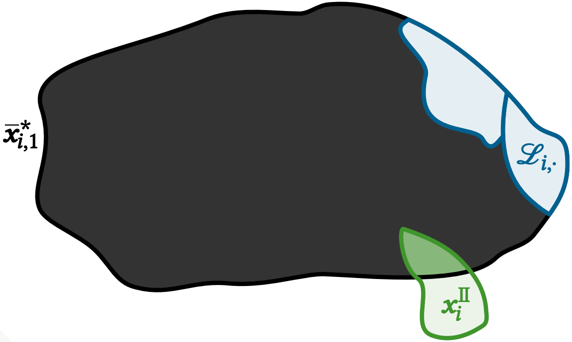

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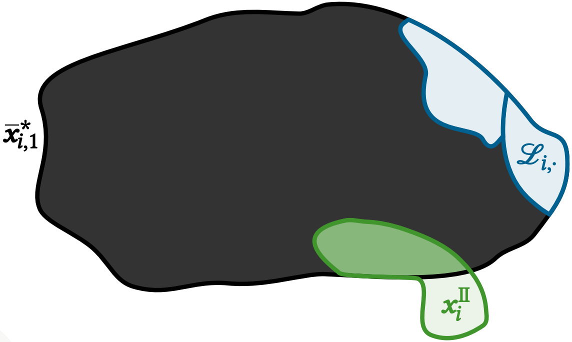

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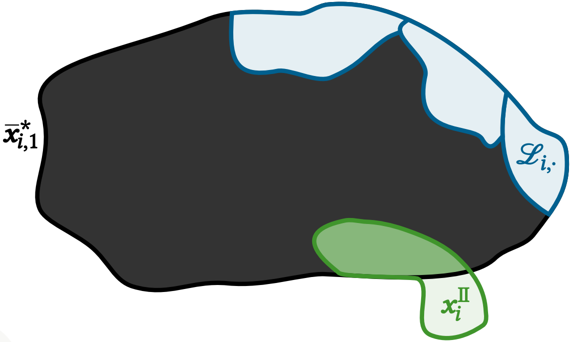

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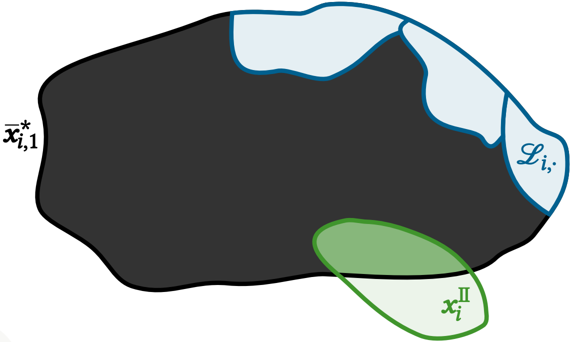

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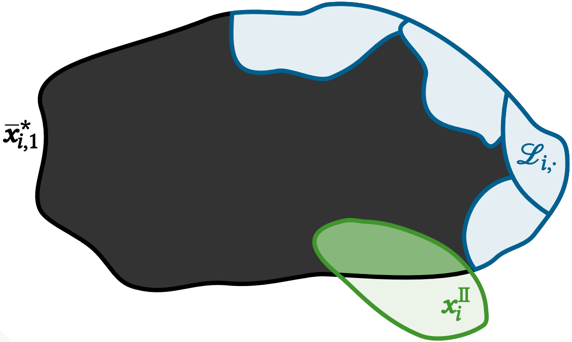

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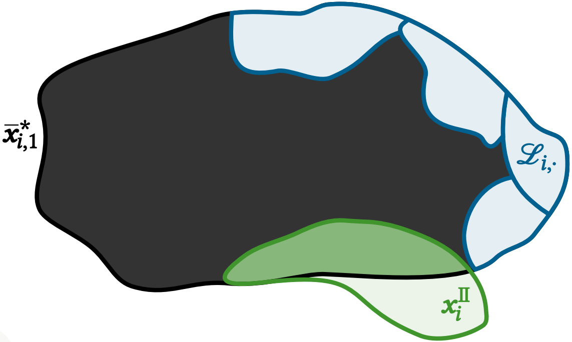

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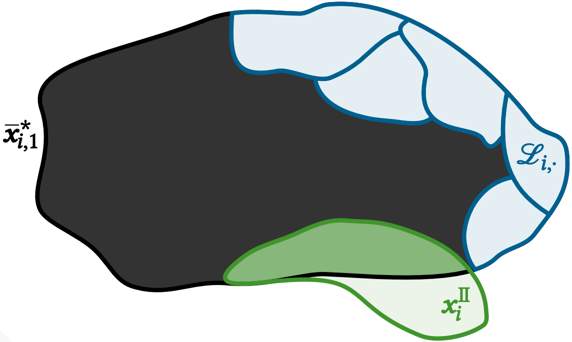

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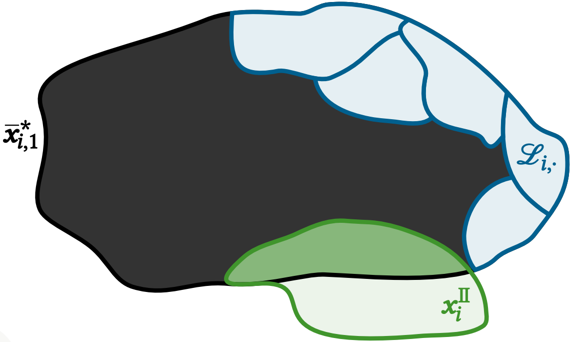

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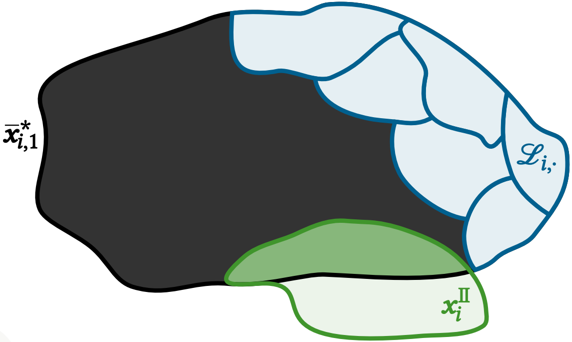

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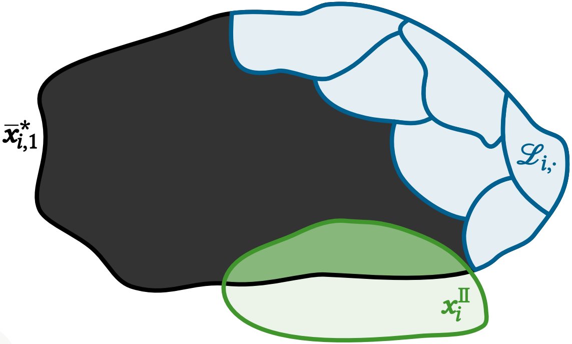

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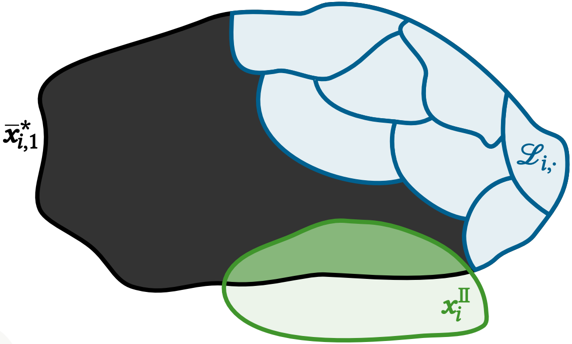

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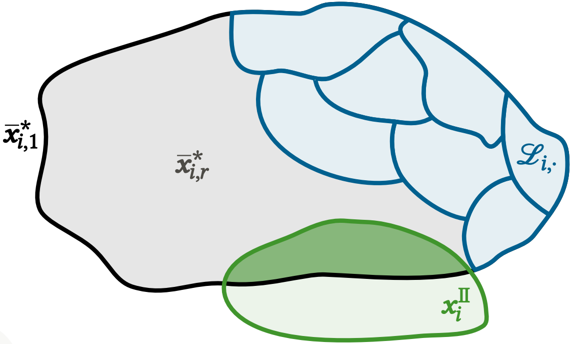

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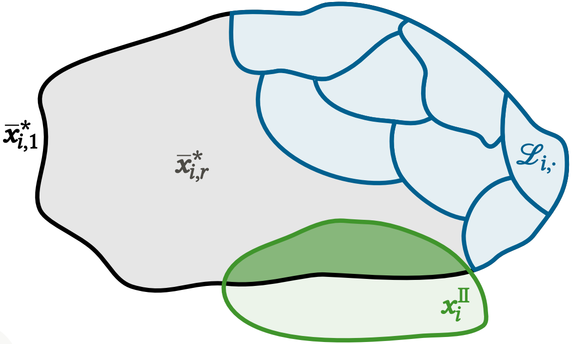

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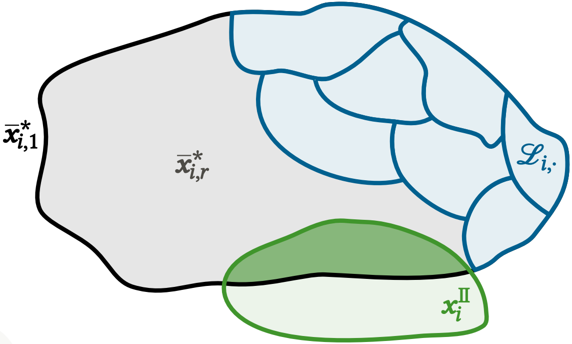

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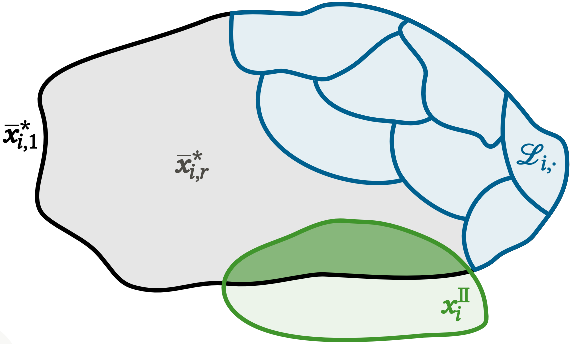

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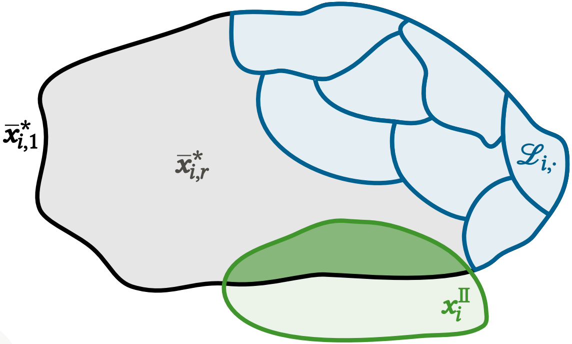

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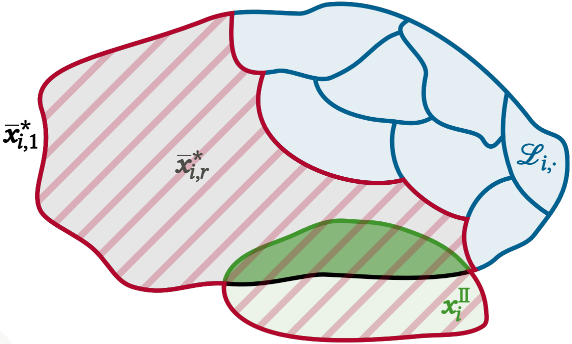

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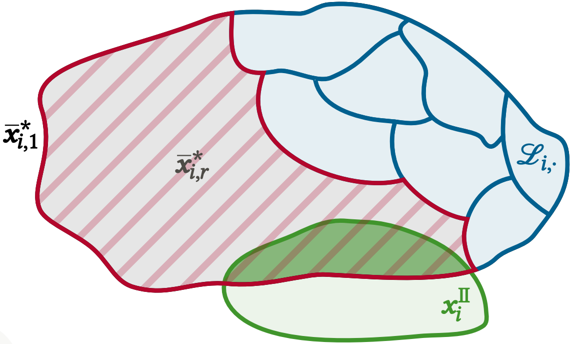

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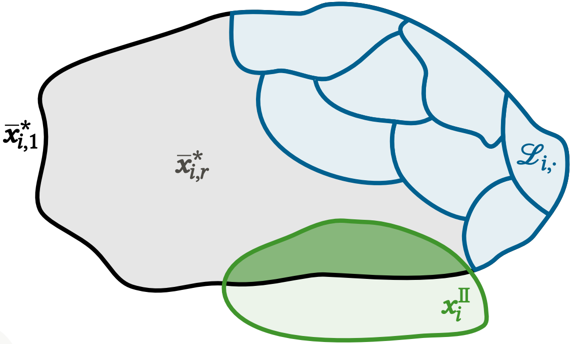

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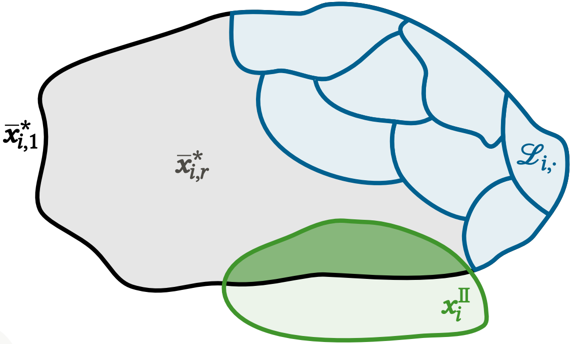

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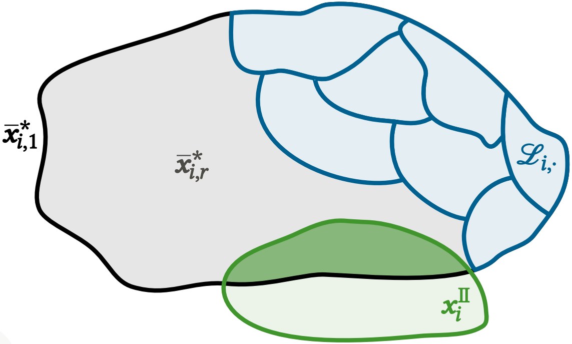

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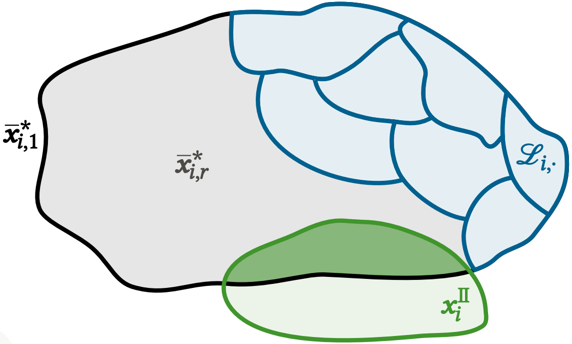

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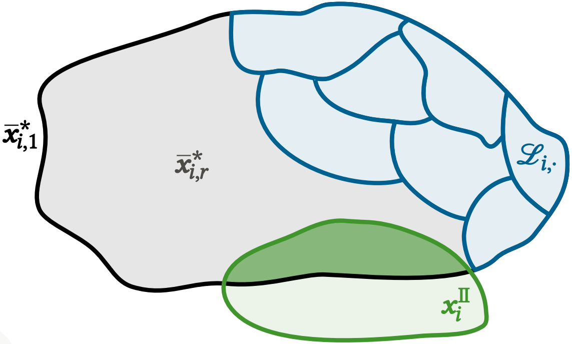

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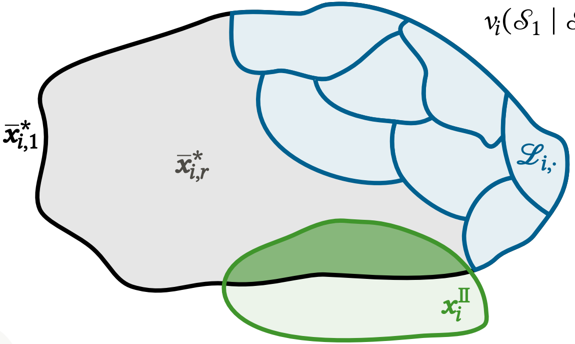
$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) =$



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$$v_i\left(\overline{\boldsymbol{x}}_{i,r}^* \mid a_i^1, \ldots, a_i^{r-1}\right) \geq -v_i\left(a_i^1, \ldots, a_i^{r-1}\right) + v_i\left(\overline{\boldsymbol{x}}_{i,1}^*\right) - \sum_{l=2}^{r} v_i\left(\mathscr{L}_{i,l} \mid a_i^1, \ldots, a_i^{l-1}\right)$$

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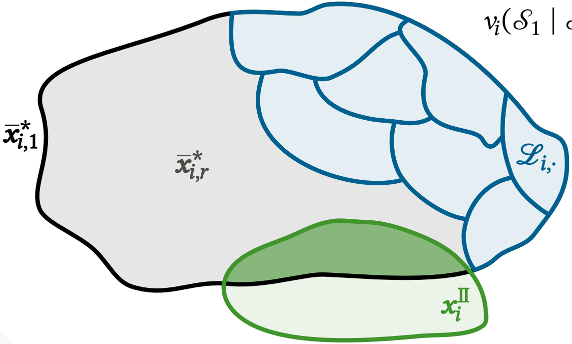

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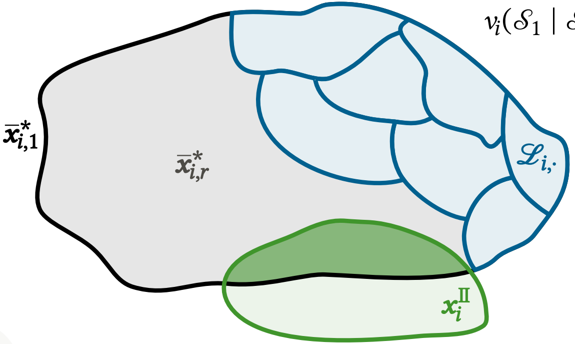

$$v_i(\mathcal{S}_1 \mid \mathcal{S}_2) = v_i\left(j^1 \mid \mathcal{S}_2\right) + v_i\left(j^2 \mid \mathcal{S}_2 \cup \left\{j^1\right\}\right)$$

Analysing Phase II (2/2)

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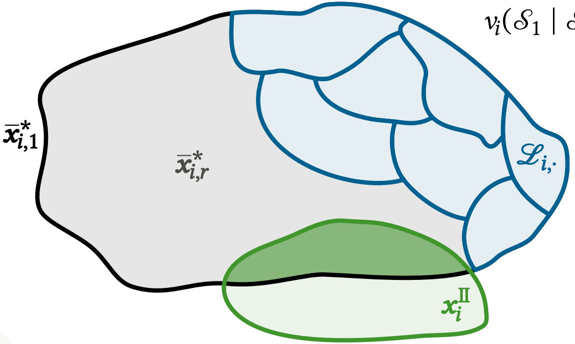

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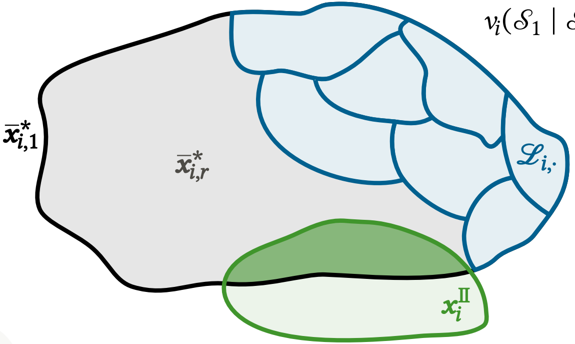

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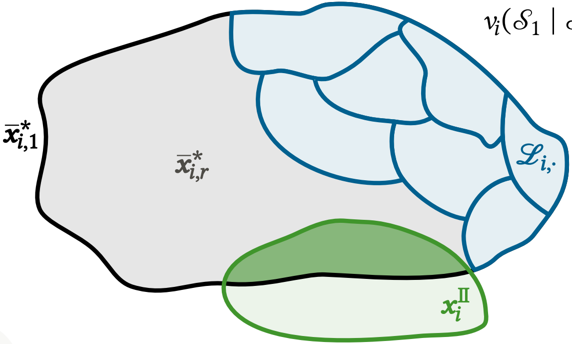

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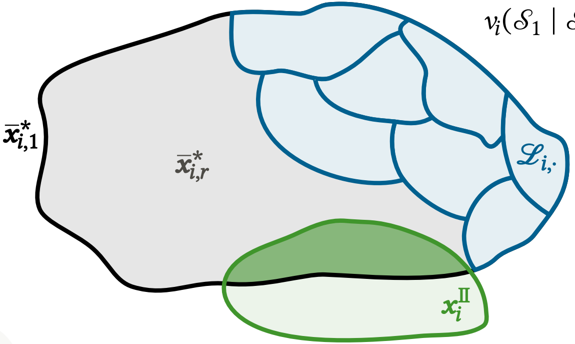

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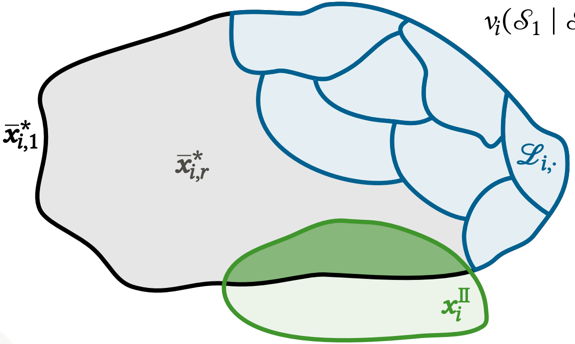

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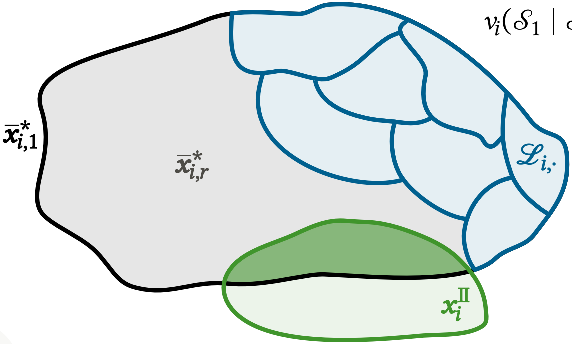

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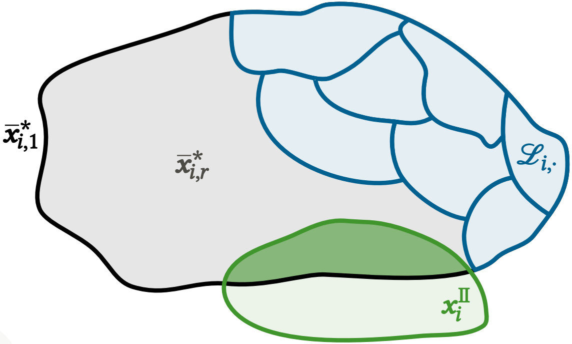

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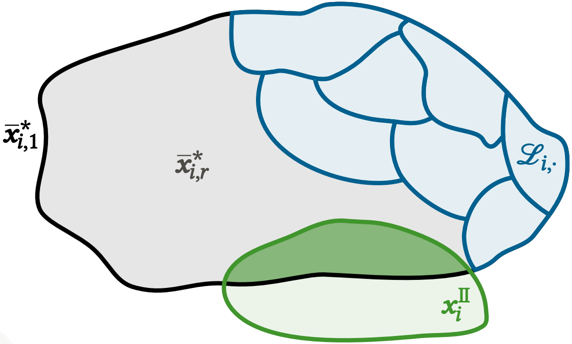

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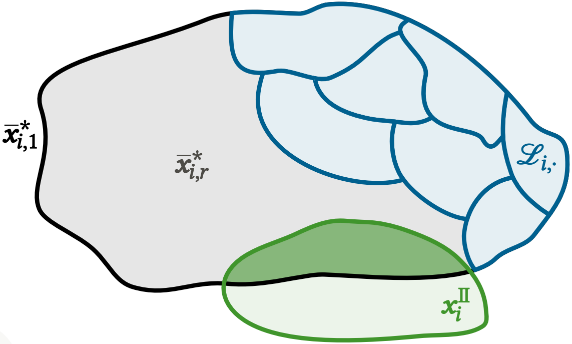

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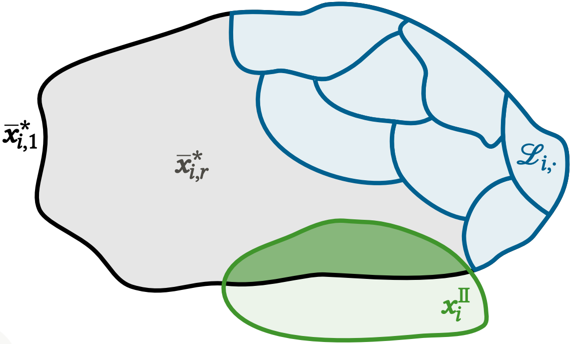

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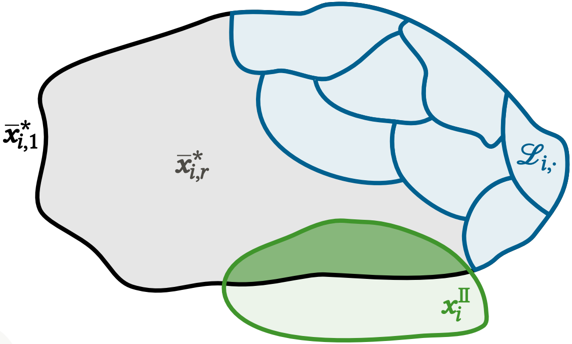

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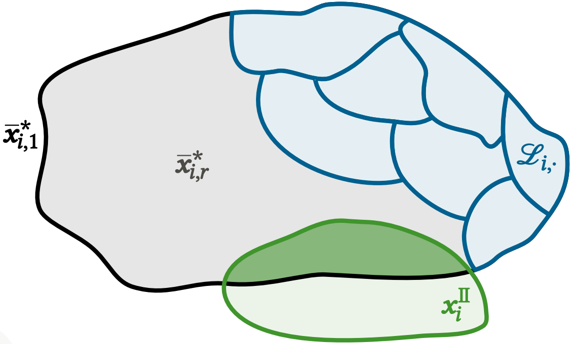

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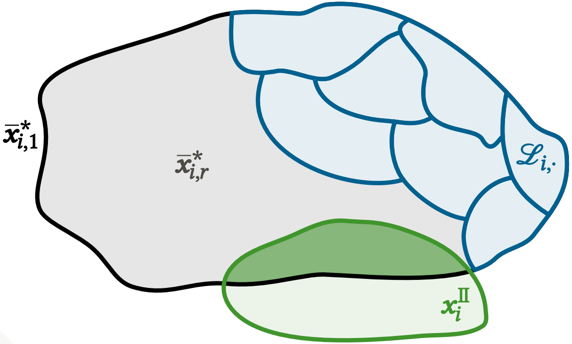

Conclusion

Summary & Outlook

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- allocation: partition of items amongst agents

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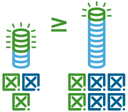

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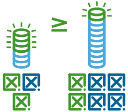

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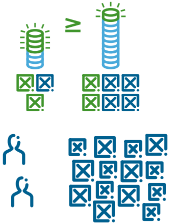

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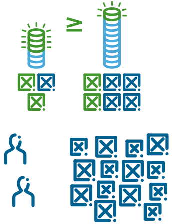

Summary & Outlook

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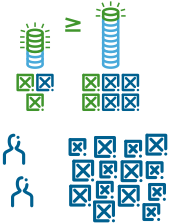

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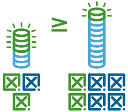
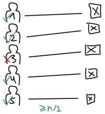
  Phase I finding enough outstanding items





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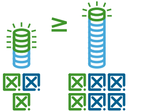

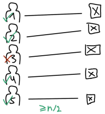

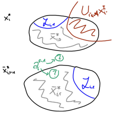

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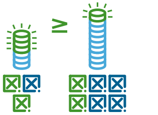
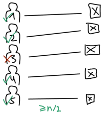
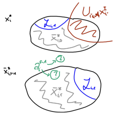
Phase II assigning remaining item
Phase III assigning outstanding items







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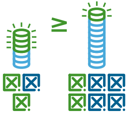

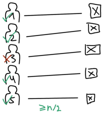
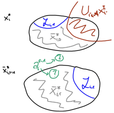

Any Room for Improvement?

Possibly! Lower bound of $\frac{\mathrm{e}}{\mathrm{e}-1} \approx 1.58$

End of Talk