Mathematical Analysis of a Triple Integral using Gamma Function and Binomial Series

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$$\int_{\theta_1}^{\theta_2}\int_{y_1}^{y_2}\int_{0}^{\infty}\frac{t^{a-1}\sum_{p=0}^{n}\frac{(n)p}{p!}e^{(n-p-b)^t}r^p\cos[(m-p)\theta]}{(e^{2t}+2re^t\cos\theta+r^2)^r}\,dt\,dr\,d\theta$$

Some notation, formulas and theorems used in this paper are introduced below.

0.1 Notations

0.2 Gamma Function

Suppose that a is a postive real number, then $\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}$

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$(s)_k = s(s-1)\cdots(s-k+1)$, where s is a real number.

1 Formulas

1.1 Euler's Formula

$e^{ix} = cos(x) + isin(x),$ where x is any real number.

1.1.1 Demoivre's Formula

$(cos(x)+sin(x)^2 = cos(nx)+isin(nx),$ where n is any interger, and x is any real number.

2 Theorems

Two important theorems used in this study are introduced

2.1 Bionomial Series

$(1+z)^s = \sum_{k=0}^{\infty}\frac{(s)_k}{k!}z^k$, where z is a complex number, [z]<1, and s is a real number.

2.2 Integration Terms by Term Theorem

Suppose that $\{g_n\}_{n=0}^{\infty}$ is a sequence of Lebesgue interable function defined on I.
If $\sum_{n=0}^{\infty} \int_I |g_n|$ is convergent, then
$\int_I \sum_{n=1}^{\infty} g_n = \sum_{n=0}^{\infty} \int_I g_n$ Before deriving the major results of this study, wee need a lemma.

3 Lemma

3.1 Lemma

Suppose that z is a complex number, [z]<1, a,b are real numbers, a¿0, b≥0, and m.n are postive integers. Th

$$\int_0^{\infty} \frac{t^{a-1} e^{-bt} z^m}{(e^t+z)^n} dt = \Gamma(a) \sum_{k=0}^{\infty} \frac{(-n)_k}{k!(k+n+b)^a} z^{k+m}$$

$\frac{t^{a+1} e^{-bt} z^m}{(e^t+z)^n} = t^{a-1} e^{-bt} e^{-nt} z^m \cdot \frac{1}{\left(1+\frac{z}{e^t}\right)^n} = t^{a+1} e^{-(n+b)t} z^m \cdot \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} e^{-kt} z^k$
(by binomial series)
$= \sum_{k=0}^{\infty} \frac{(-n)k}{k!} t^{a-1} e^{-(k+n+b)t} z^{k+m}$

$$\int_0^{\infty} \frac{t^{a-1} e^{-bt} z^m}{(e^t+z)^n} dt \tag{1}$$

$$= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} t^{a-1} e^{-(k+n+b)t} z^{k+m} dt \tag{2}$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left(\int_0^{\infty} t^{a-1} e^{(-k+n+n)t} dt \right) z^{k+m} \tag{3}$$

$$\Gamma(a) \cdot \sum_{k=0}^{\infty} \frac{(-n)_k}{k!(k+n+b)a} z^{k+m} \tag{4}$$

Firstly, we determine the infinete series form of the triple integral(1).

3.2 Theorem

Assume that r_1, r_2, $theta_1$, θ_2 are real numbers, $[r_1]$<1, $[r_2]$1, a,b are real numbers, a > 0, b ≥0, m.n are positive intefers. Then the triple integral:

$$\int_{\theta_1}^{\theta_2} \int_{y_1}^{y_2} \int_0^{\infty} \frac{t^{a-1} \sum_{p=0}^{n} \frac{(n)p}{p!} e^{(n-p-b)^t} r^p cos[(m-p)\theta]}{(e^{2t} + 2re^t cos\theta + r^2)^r} dtdrd\theta$$

See page 3 for the solution and explanation

3.2 Contiuned

$$\int_{\theta_1}^{\theta_2}\int_{y_1}^{y_2}\int_0^{\infty}\frac{t^{a-1}\sum_{p=0}^{n}\frac{(n)p}{p!}e^{(n-p-b)^t}r^p cos[(m-p)\theta]}{(e^{2t}+2re^t cos\theta+r^2)^r}dtdrd\theta$$

$$(-n)_k(r2^{k+1}-r1^{k+1})$$

$$=\Gamma(a)\cdot\sum_{k=0}^{\infty}\frac{[sin(k+m)\theta 2-sin(k+m)\theta 1]}{k!(k+1)(k+m)(k+n+b)^a}$$

Proof Let z= re$^i\theta$

$$\int_0^{\infty}\frac{t^{a-1}e^{-bt}(re^{i\theta})^m}{(e^t+re^{i\theta})^n}dt$$

$$=\Gamma(a)\cdot\sum_{k=0}^{\infty}\frac{(-n)_k}{k!(k+n+b)^a}(re^{i\theta})^{k+m}$$

By Euler's formula and DeMoivre's formula, we obtain:

$$\int_0^{\infty}\frac{t^{a-1}e^{-bt}e^{im(\theta)}(e^{+}re^{-i\theta})^n}{(e^2t+2re^t cos\theta+r^2)^a}dt=\Gamma(a)\cdot\sum_{k=0}^{\infty}\frac{(-n)_k}{k!(k+n+b)^a}r^k e^i(k+m)\theta$$

Therefore,

$$\int_0^{\infty}\frac{t^{a-1}\sum_{p=0}^{n}e^{(n-p-b)^t}r^p e^{i(m-p)\theta}}{(e^2t+2re^t cos\theta+r^2)^n}dt$$

$$=\Gamma(a)\cdot\sum_{k=0}^{\infty}\frac{(-n)_k}{k!(k+n+b)^a}r^k e^{i(k+m)\theta}$$