technische universität
dortmund

fakultät für informatik
informatik 12

Petri Nets

Peter Marwedel
TU Dortmund,
Informatik 12

2012年 10 月 31 日

Models of computation considered in this course

Communication/ local computations	Shared memory	Message passing Synchronous | Asynchronous	
Undefined components	Plain text, use cases | (Message) sequence charts		
Communicating finite state machines	StateCharts		SDL
Data flow	Scoreboarding + Tomasulo Algorithm (☞ Comp.Archict.)		Kahn networks, SDF
Petri nets		**C/E nets, P/T nets, …**	
Discrete event (DE) model	VHDL*, Verilog*, SystemC*, …	Only experimental systems, e.g. distributed DE in Ptolemy	
Von Neumann model	C, C++, Java	C, C++, Java with libraries CSP, ADA |	

Introduction

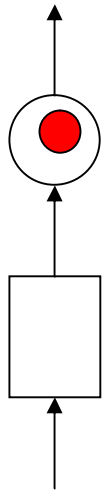
Introduced in 1962 by Carl Adam Petri in his PhD thesis.

Focus on modeling causal dependencies;

no global synchronization assumed (message passing only).

Key elements:

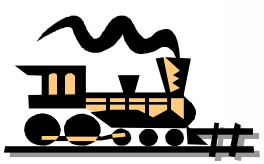
- **Conditions**
  Either met or not met.
- **Events**
  May take place if certain conditions are met**.**
- **Flow relation**
  Relates conditions and events**.**



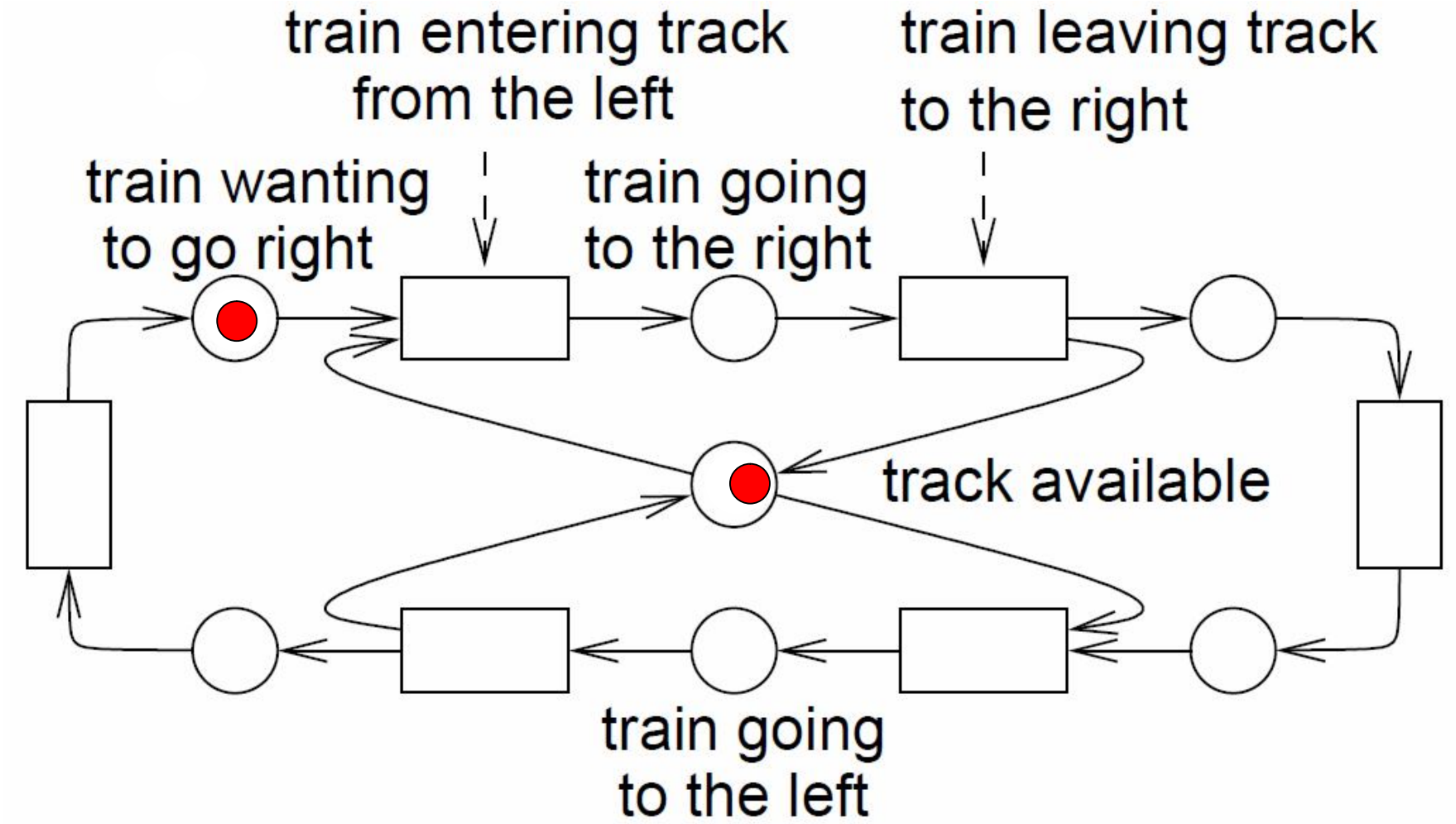
Conditions, events and the flow relation form

a **bipartite graph** (graph with two kinds of nodes).

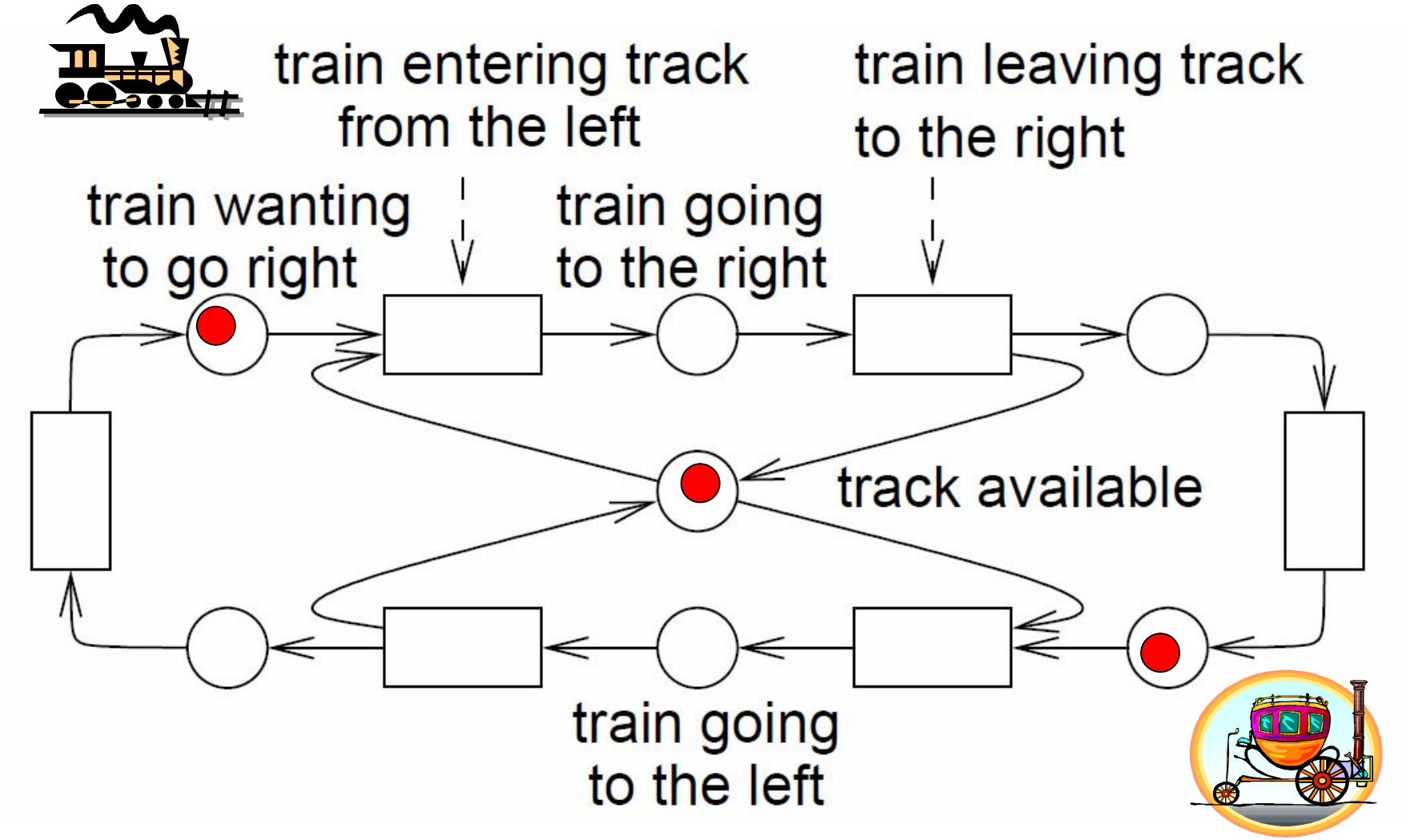
Example:
Synchronization at single track rail segment



Playing the “token game“



Conflict for resource “track“

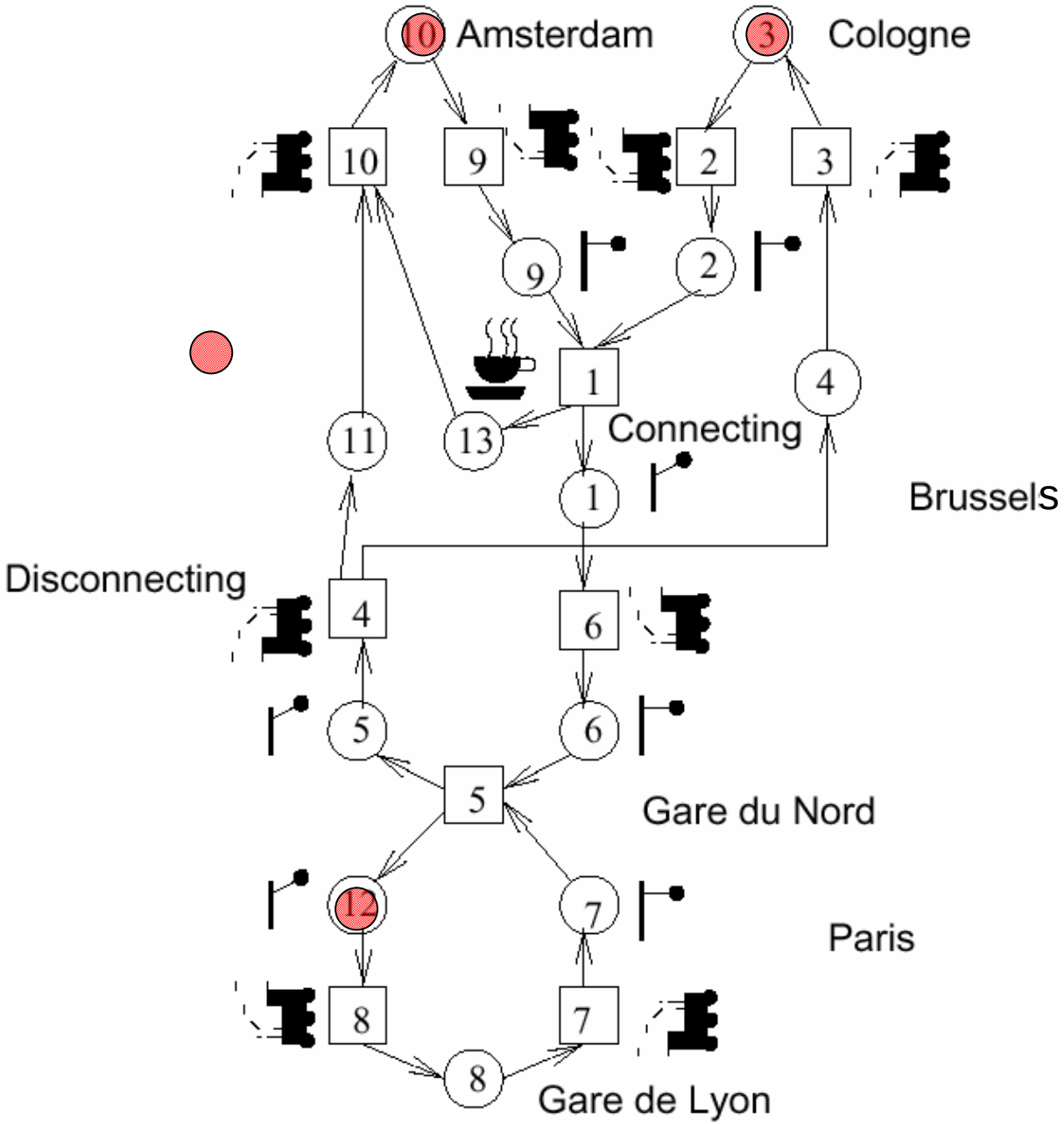


More complex example (1)

Thalys trains between Cologne, Amsterdam, Brussels and Paris.

[http://www.thalys.com/be/en]

More complex example (2)



Slightly simplified: Synchronization at Brussels and Paris, using stations “Gare du Nord” and “Gare de Lyon” at Paris

Condition/event nets

Def.: $N=(C,E,F)$ is called a **net**, iff the following holds

1. C and E are disjoint sets

2. $F \subseteq (C \times E) \cup (E \times C)$; is binary relation,
("**flow relation**")

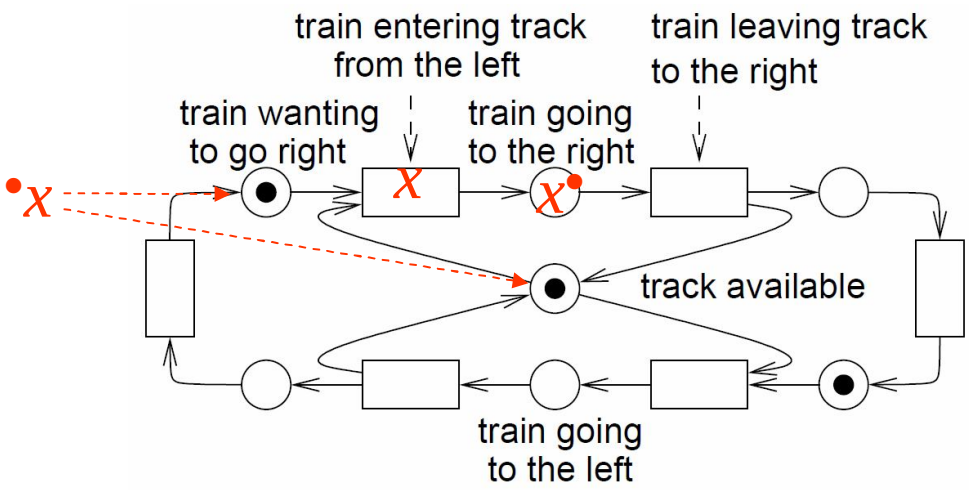
Pre- and post-sets

Def.: Let N be a net and let $x \in (C \cup E)$.

$^\bullet x := \{y \mid y\ F\ x\}$ is called the **pre-set** of x,
(or **preconditions** if $x \in E$)

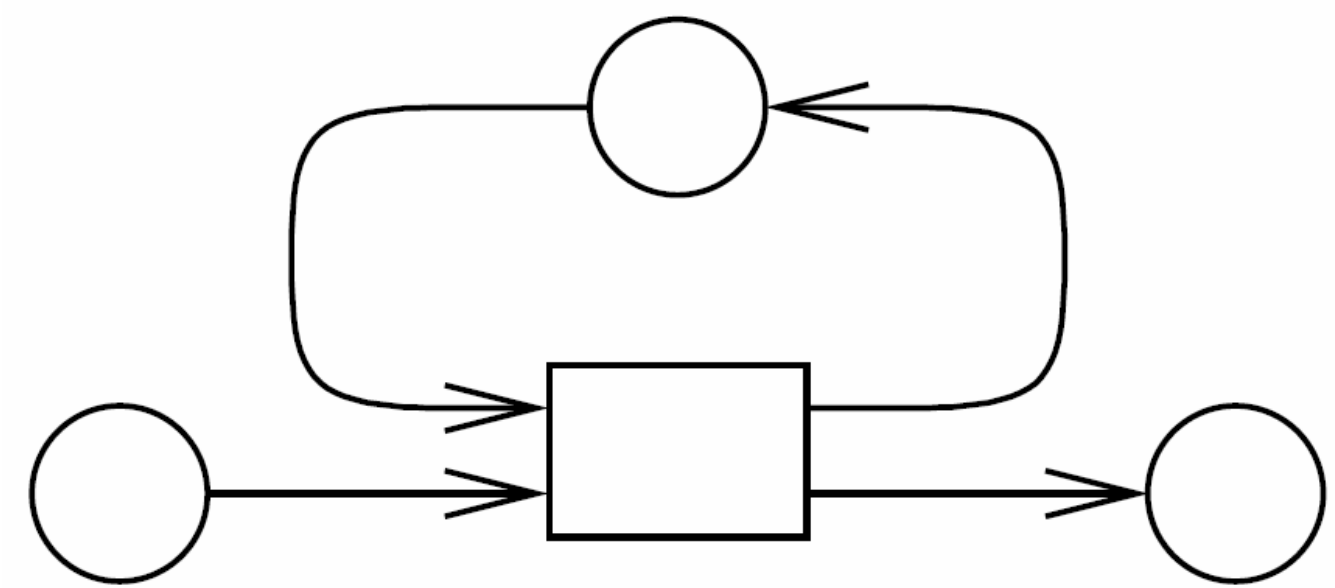
$x^\bullet := \{y \mid x\ F\ y\}$ is called the set of **post-set** of x,
(or **postconditions** if $x \in E$)

Example:



Loops and pure nets

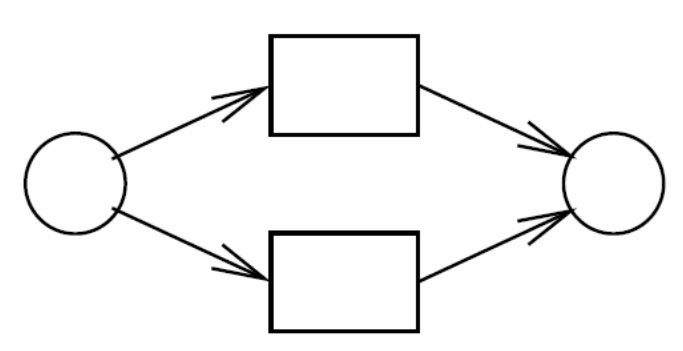
Def.: Let $(c,e) \in C \times E$. (c, e) is called a **loop** iff $cFe \wedge eFc$.

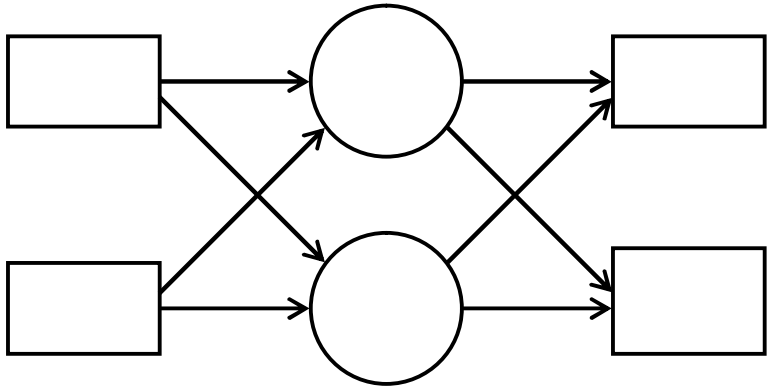


Def.: Net $N=(C,E,F)$ is called **pure**, if F does not contain any loops.

Simple nets

Def.: A net is called **simple** if no two nodes
*n*1 and *n*2 have the same pre-set and post-set.
Example (not simple nets):



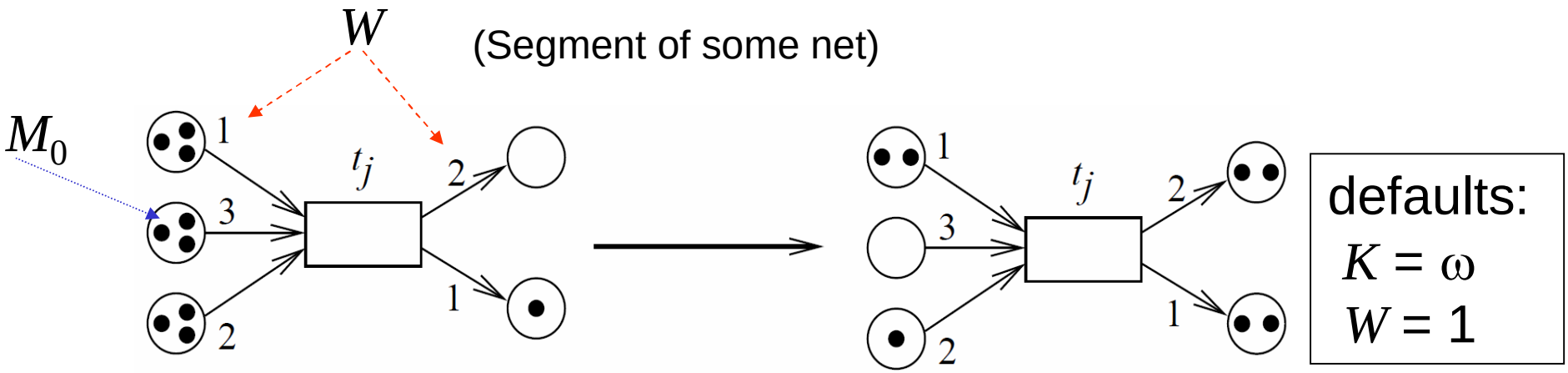


Def.: Simple nets with no isolated elements meeting some additional restrictions are called **condition/event nets (C/E nets)**.

Place/transition nets

Def.: (P, T, F, K, W, M_0) is called a **place/transition net** iff

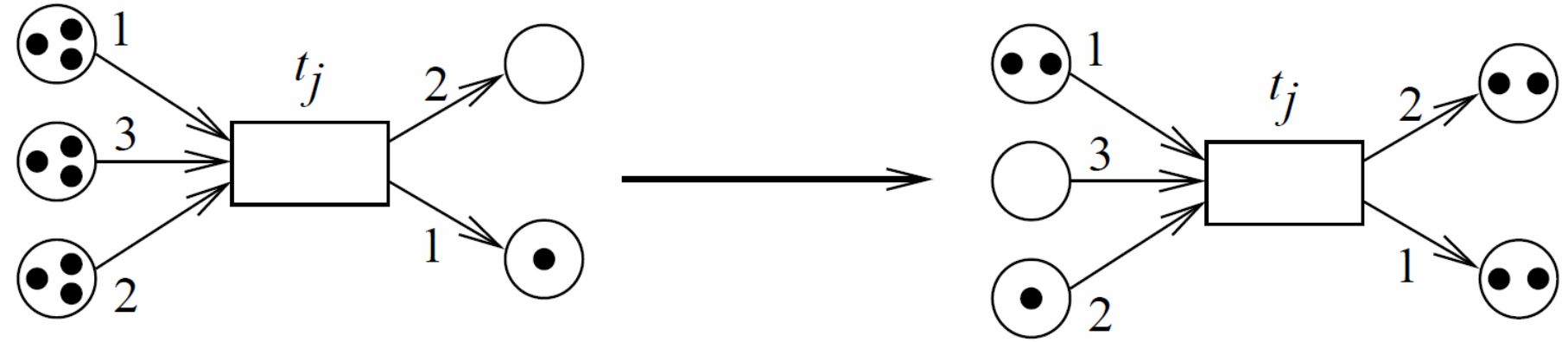
1. $N=(P, T, F)$ is a **net** with places $p \in P$ and transitions $t \in T$
2. $K: P \rightarrow (\mathbb{N}_0 \cup \{\omega\}) \setminus \{0\}$ denotes the **capacity** of places (ω symbolizes infinite capacity)
3. $W: F \rightarrow (\mathbb{N}_0 \setminus \{0\})$ denotes the **weight of graph edges**
4. $M_0: P \rightarrow \mathbb{N}_0 \cup \{\omega\}$ represents the **initial marking** of places



Computing changes of markings

"Firing" transitions t generate new markings on each of the places p according to the following rules:

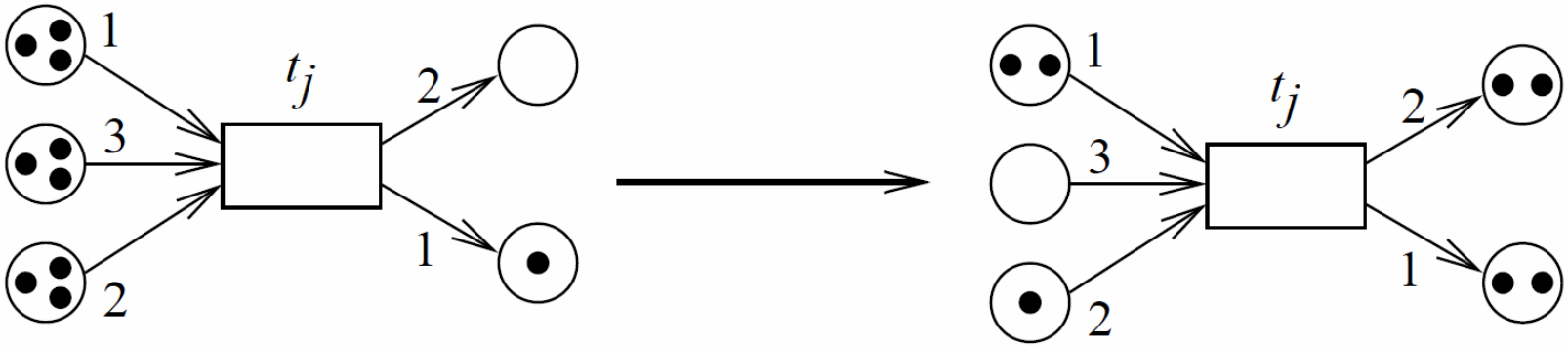
$$M'(p) = \begin{cases} M(p) - W(p,t), & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ M(p) + W(t,p), & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ M(p) - W(p,t) + W(t,p), & \text{if } p \in {}^\bullet t \cap t^\bullet \\ M(p) & \text{otherwise} \end{cases}$$



Activated transitions

Transition t is “activated“ iff

$$(\forall p \in {}^\bullet t : M(p) \geq W(p,t)) \wedge (\forall p \in t^\bullet : M(p) + W(t,p) \leq K(p))$$



Activated transitions can “take place“ or “fire“,
but don‘t have to.
We never talk about “time“ in the context of Petri nets.
The order in which activated transitions fire, is not fixed
(it is non-deterministic).

Shorthand for changes of markings

Slide 14:

$$M'(p) = \begin{cases} M(p) - W(p,t), & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ M(p) + W(t,p), & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ M(p) - W(p,t) + W(t,p), & \text{if } p \in {}^\bullet t \cap t^\bullet \\ M(p) & \text{otherwise} \end{cases}$$

Let

$$\underline{t}(p) = \begin{cases} -W(p,t) \text{ if } p \in {}^\bullet t \setminus t^\bullet \\ +W(t,p) \text{ if } p \in t^\bullet \setminus {}^\bullet t \\ -W(p,t) + W(t,p) \text{ if } p \in t^\bullet \cap {}^\bullet t \\ 0 \end{cases}$$

$\Rightarrow$ $\forall\, p \in P$: $M'(p) = M(p) + \underline{t}(p)$

$\Rightarrow$ $M' = M + \underline{t}$ +: vector add

Matrix $\underline{N}$ describing all changes of markings

$$\underline{t}(p) = \begin{cases} -W(p,t) \text{ if } p \in {}^\bullet t \setminus t^\bullet \\ +W(t,p) \text{ if } p \in t^\bullet \setminus {}^\bullet t \\ -W(p,t) + W(t,p) \text{ if } p \in t^\bullet \cap {}^\bullet t \\ 0 \end{cases}$$

Def.: Matrix $\underline{N}$ of net N is a mapping

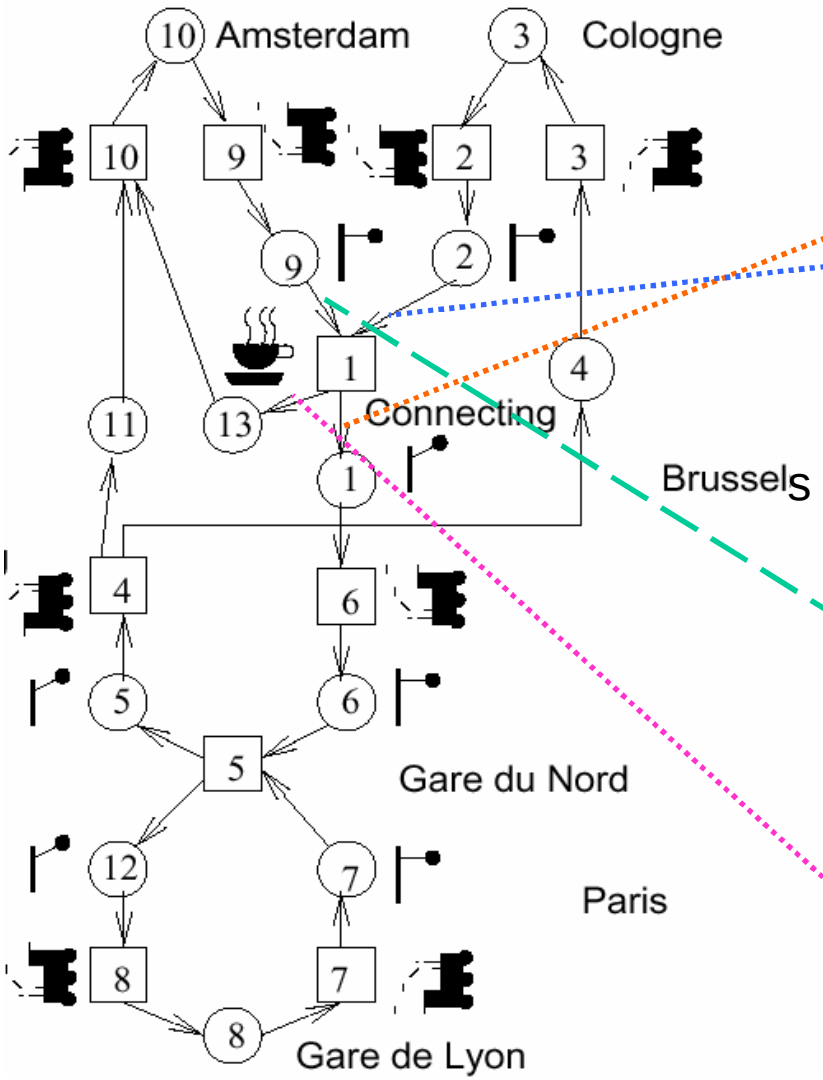
$$\underline{N}: P \times T \rightarrow \mathbb{Z} \text{ (integers)}$$

such that $\forall\ t \in T : \underline{N}(p,t) = \underline{t}(p)$

Component in column t and row p indicates the change of the marking of place p if transition t takes place.

For pure nets, $(\underline{N}, M_0)$ is a complete representation of a net.

Example: N =



	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
p_1	1					−1				
p_2	−1	1								
p_3		−1	1							
p_4			−1	1						
p_5				−1	1					
p_6					−1	1				
p_7					−1		1			
p_8							−1	1		
p_9	−1							1	1	
p_{10}									−1	1
p_{11}				1						−1
p_{12}					1			−1		
p_{13}	1									−1

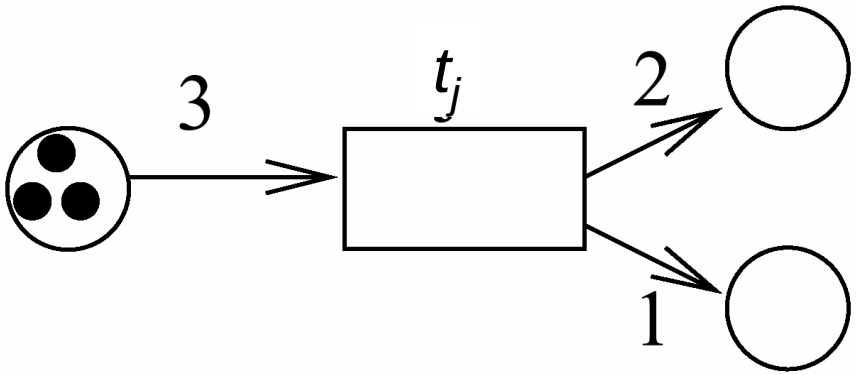
Place - invariants

Standardized technique for proving properties of system models

For any transition $t_j \in T$ we are looking for sets $R \subseteq P$ of places for which the accumulated marking is constant:

$$\sum_{p \in R} \underline{t}_j(p) = 0$$

Example:



Characteristic Vector

$$\sum_{p \in R} \underline{t}_j(p) = 0$$

Let: $$\underline{c}_R(p) = \begin{cases} 1 \text{ if } p \in R \\ 0 \text{ if } p \notin R \end{cases}$$

$$\Rightarrow \; 0 = \sum_{p \in R} \underline{t}_j(p) = \sum_{p \in P} \underline{t}_j(p)\, \underline{c}_R(p) = \underline{t}_j \cdot \underline{c}_R$$

Scalar product

Condition for place invariants

$$\sum_{p \in R} \underline{t}_j(p) = \sum_{p \in P} \underline{t}_j(p)\, \underline{c}_R(p) = \underline{t}_j \cdot \underline{c}_R = 0$$

Accumulated marking constant for **all** transitions if

$$\begin{aligned} \underline{t}_1 \cdot \underline{c}_R &= 0 \\ \ldots \quad & \ldots \quad \ldots \\ \underline{t}_n \cdot \underline{c}_R &= 0 \end{aligned}$$

Equivalent to $\underline{N}^T \, \boldsymbol{c_R} = \mathbf{0}$ where $\underline{N}^T$ is the transposed of $\underline{N}$

More detailed view of computations

$$\begin{pmatrix} \underline{t}_1(p_1) \ldots \underline{t}_1(p_n) \\ \underline{t}_2(p_1) \ldots \underline{t}_2(p_n) \\ \ldots \\ \underline{t}_m(p_1) \ldots \underline{t}_m(p_n) \end{pmatrix} \begin{pmatrix} \underline{c}_R(p_1) \\ \underline{c}_R(p_2) \\ \ldots \\ \underline{c}_R(p_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

System of linear equations.

Solution vectors must consist of zeros and ones.

Equations with multiple unknowns that must be integers called **diophantic** (☞ Greek mathematician Diophantos, ~300 B.C.).

Diophantic linear equation system more complex to solve than standard system of linear equations (actually NP-hard))

Different techniques for solving equation system (manual, ..)

Application to Thalys example

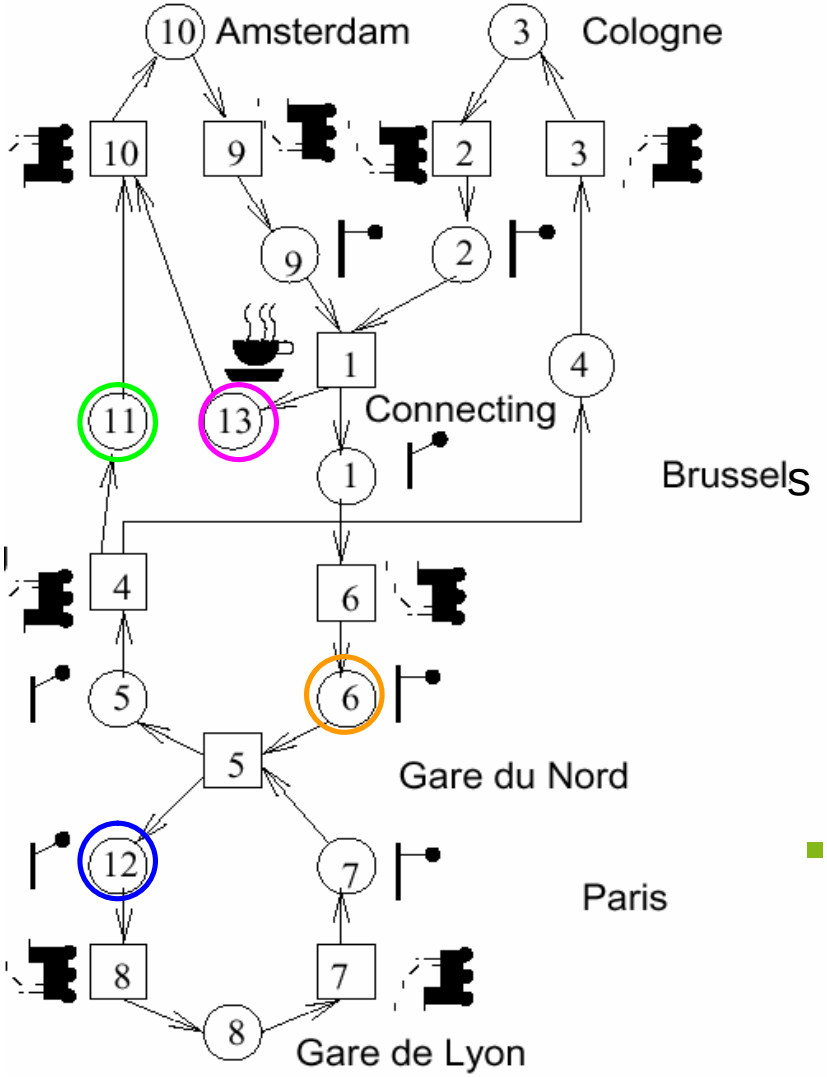
$\underline{N}^T \, c_R = 0$,
with $\underline{N}^T$ =

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

- $\sum_{rows}=0 \rightarrow$ 1 linear dependency among rows
- $\rightarrow$ rank = 10-1 = 9
- Dimension of solution space = 13 - rank = 4

Solutions? Educated guessing

Manually finding generating vectors for 4-dimensional space



Set 1 of 4 components = 1
others = 0 to obtain generating vectors

- Find independent sets components in the matrix after deleting one row (7, 8, 12 easy to identify)

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

- or assume that a particular place is included in R (e.g. p_6), whereas places along the path of other objects (e.g. p_{11}, p_{12}, p_{13}) are not.

1st basis vector

Set one of components (6, 11, 12, 13) to 1, others to 0.

→ **1st basis** b_1:

$b_1(p_6)=1$, $b_1(p_{11})=0$, $b_1(p_{12})=0$, $b_1(p_{13})=0$

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

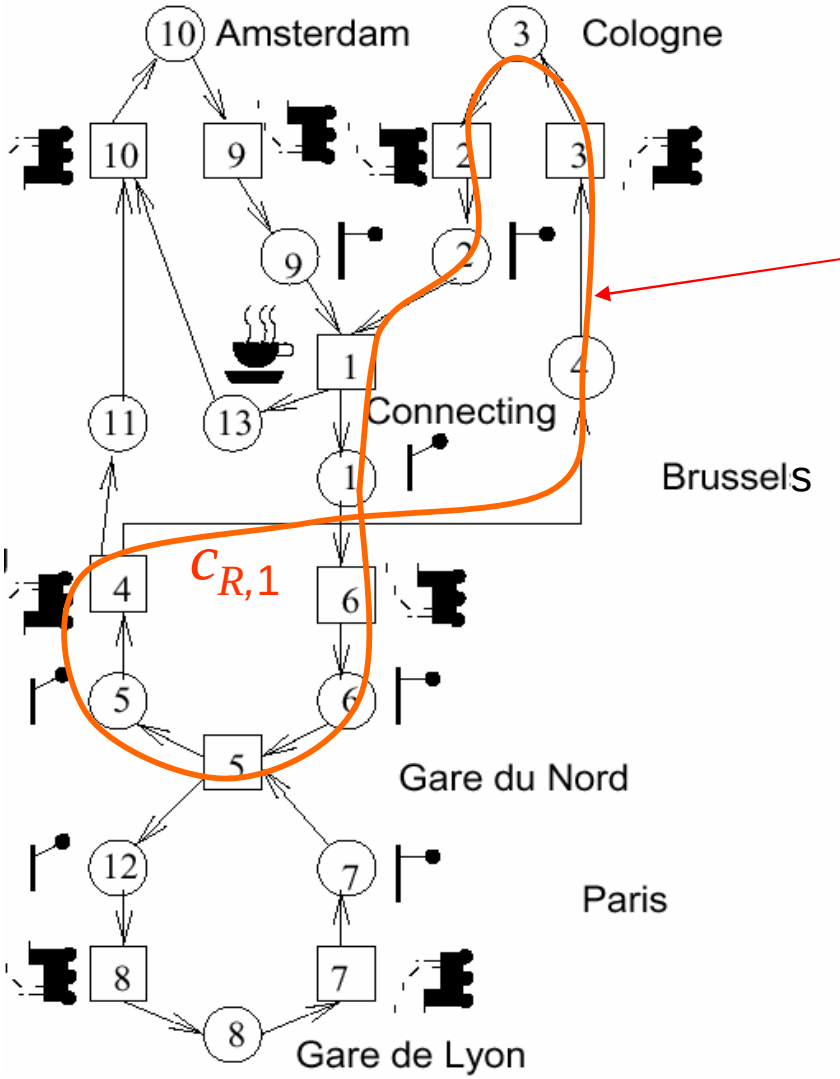
- $t_{10}(p_{10})\ b_1(p_{10}) + t_{10}(p_{11})\ b_1(p_{11}) + t_{10}(p_{13})\ b_1(p_{13}) = 0$
  → $b_1(p_{10}) = 0$
- $t_9(p_9)\ b_1(p_9) + t_9(p_{10})\ b_1(p_{10}) = 0$
  → $b_1(p_9)=0$
- …

b_1 = (1,1,1,1,1,1,0,0,0,0,0,0,0)

All components ∈ {0, 1}

→ $c_{R1} = b_1$

Interpretation of the 1st invariant



$$c_{R,1} = (1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0)$$

Characteristic vector describes places for Cologne train.

We proved that: the number of trains along the path remains constant ☺.

2nd basis vector

Set one of components (6, 11, 12, 13) to 1, others to 0.

→ **2nd basis** b_2:

$b_2(p_6)=0$, $b_2(p_{11})=1$,

$b_2(p_{12})=0$, $b_2(p_{13})=0$

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

- $t_{10}(p_{10})\ b_2(p_{10}) + t_{10}(p_{11})\ b_2(p_{11}) + t_{10}(p_{13})\ b_2(p_{13}) = 0$
  → $b_2(p_{10}) = 1$
- $t_9(p_9)\ b_2(p_9) + t_9(p_{10})\ b_2(p_{10}) = 0$
  → $b_2(p_9)=1$
- …

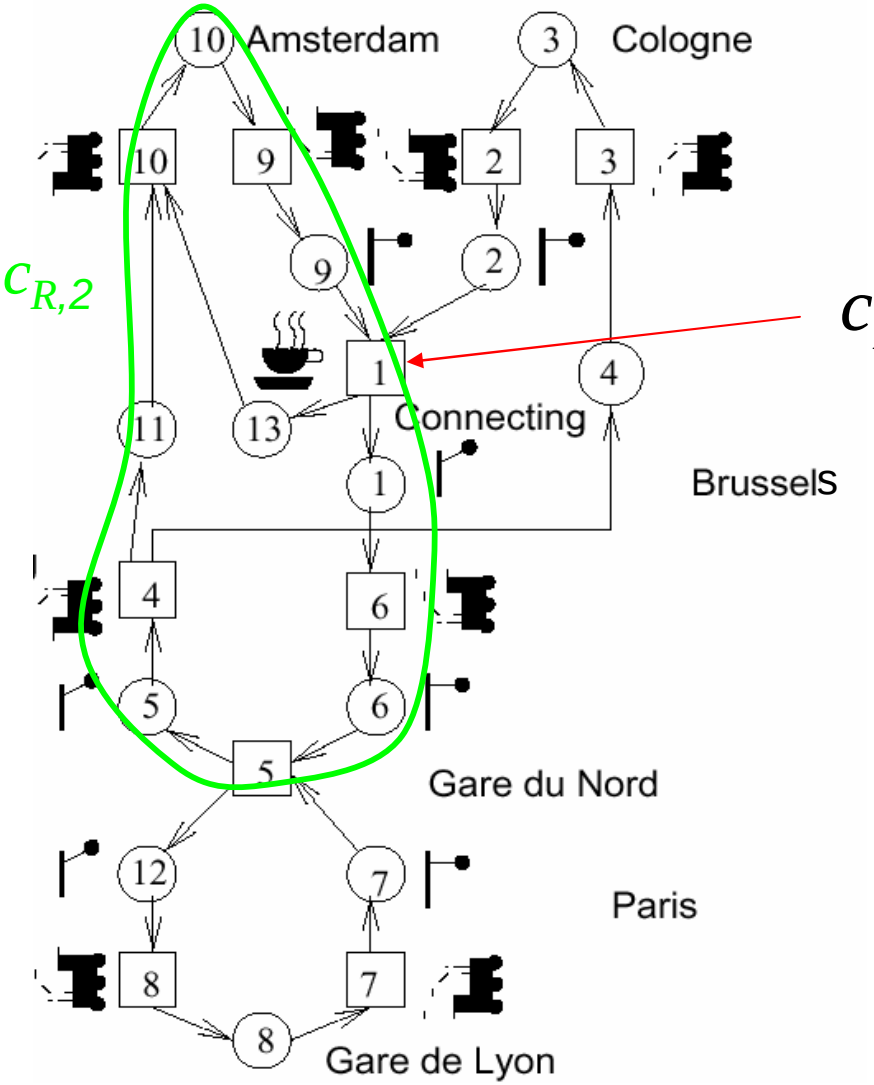
b_2 = (0,-1,-1,-1,0,0,0,0,1,1,1,0,0)

b_1 = (1, 1, 1, 1,1,1,0,0,0,0,0,0,0)

b_2 not a characteristic vector, but $\boldsymbol{c_{R,2}}=b_1+b_2$ is

→ $\boldsymbol{c_{R,2}}$ = (1,0,0,0,1,1,0,0,1,1,1,0,0)

Interpretation of the 2nd invariant



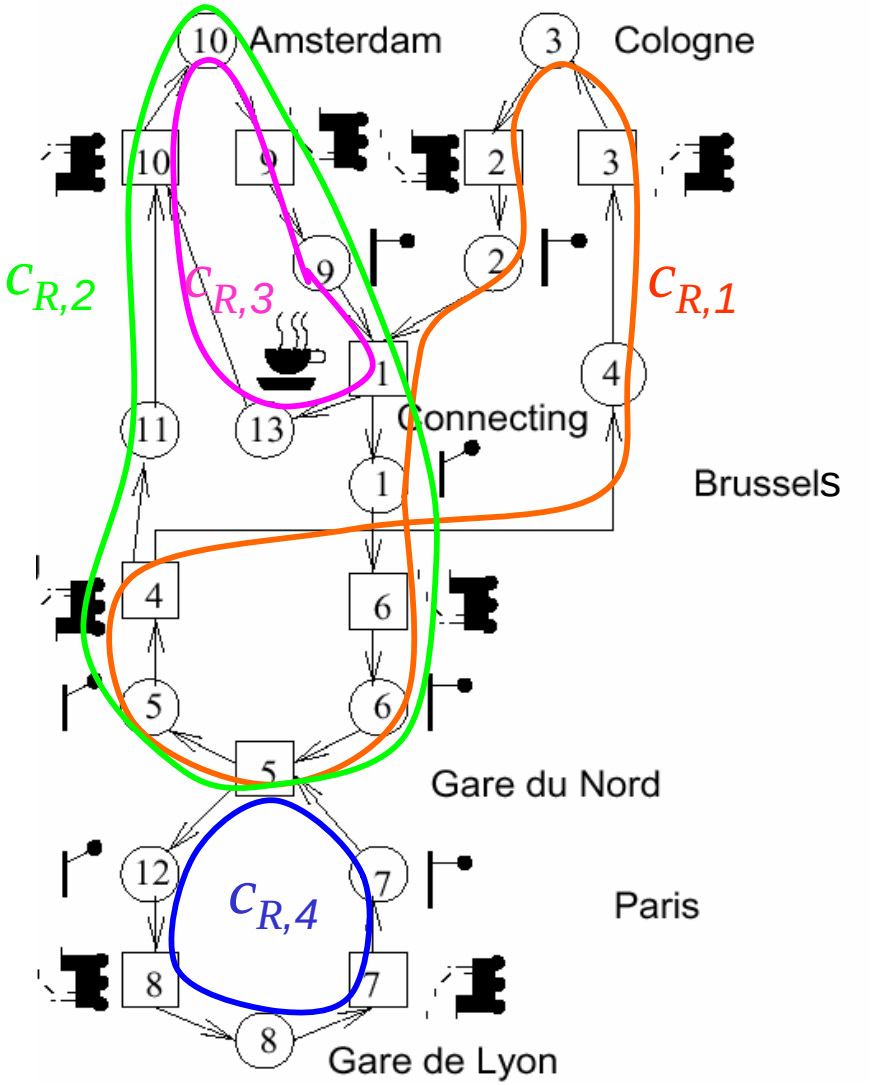
$c_{R,2}$

$$c_{R,2} = (1,0,0,0,1,1,0,0,1,1,1,0,0)$$

We proved that:

None of the Amsterdam trains gets lost (nice to know ☺).

Setting $b_3(p_{12})$ to 1 and $b_4(p_{13})$ to 1 leads to an additional 2 invariants



$$c_{R,1} = (1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0)$$

$$c_{R,2} = (1\,0\,0\,0\,1\,1\,0\,0\,1\,1\,1\,0\,0)$$

$$c_{R,3} = (0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,1)$$

$$c_{R,4} = (0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,0\,1\,0)$$

We proved that:

- the number of trains serving Amsterdam, Cologne and Paris remains constant.
- the number of train drivers remains constant.

Applications

- Modeling of resources;
- modeling of mutual exclusion;
- modeling of synchronization.

Predicate/transition nets

Goal: compact representation of complex systems.

Key changes:

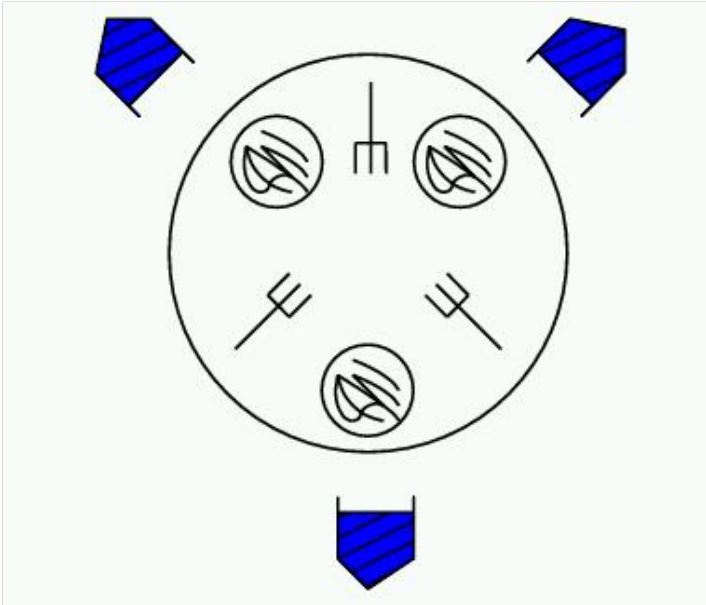
- Tokens are becoming individuals;
- Transitions enabled if functions at incoming edges true;
- Individuals generated by firing transitions defined through functions

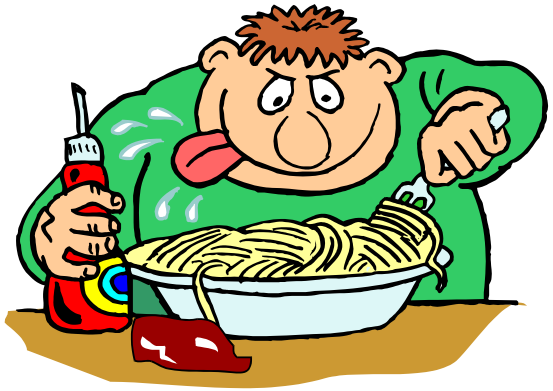
Changes can be explained by folding and unfolding C/E nets,

☞ semantics can be defined by C/E nets.

Example: Dining philosophers problem

n>1 philosophers sitting at a round table;
n forks,
n plates with spaghetti;
philosophers either thinking or eating spaghetti (using left and right fork).



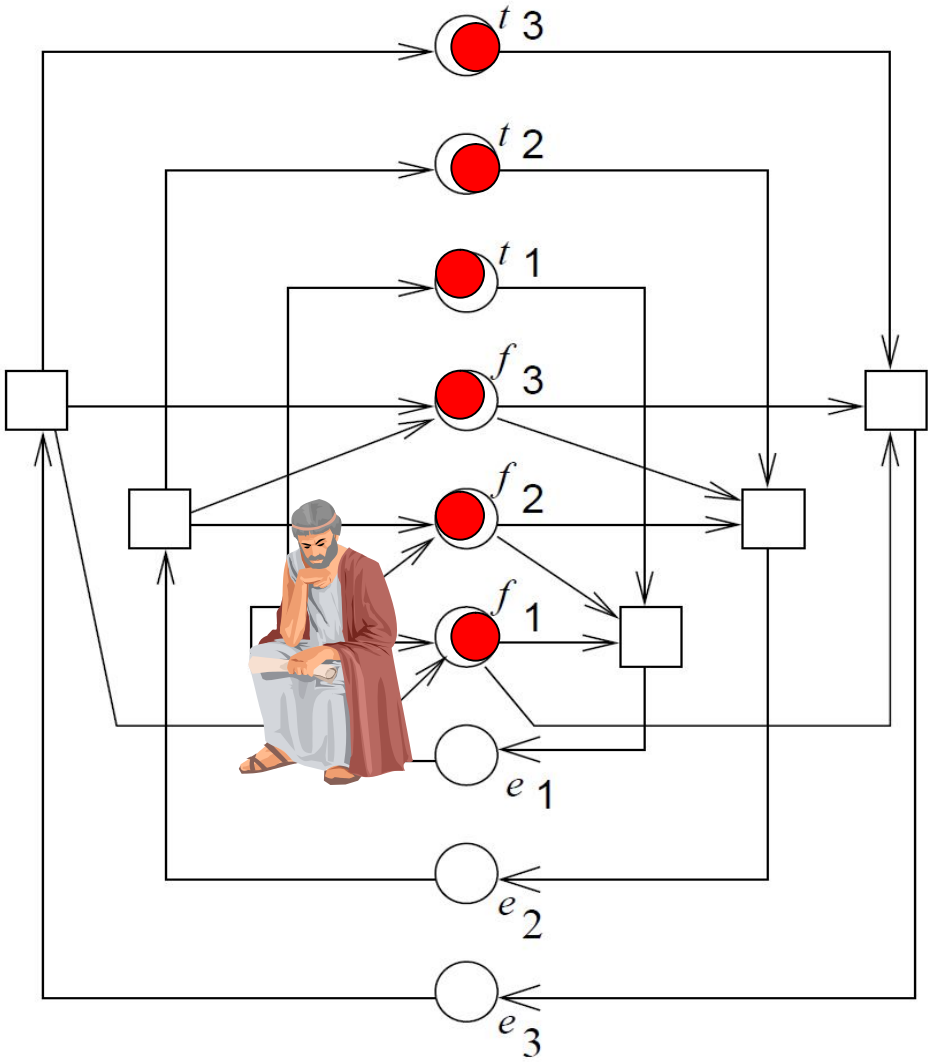


2 forks needed!

How to model conflict for forks?

How to guarantee avoiding starvation?

Condition/event net model of the dining philosophers problem



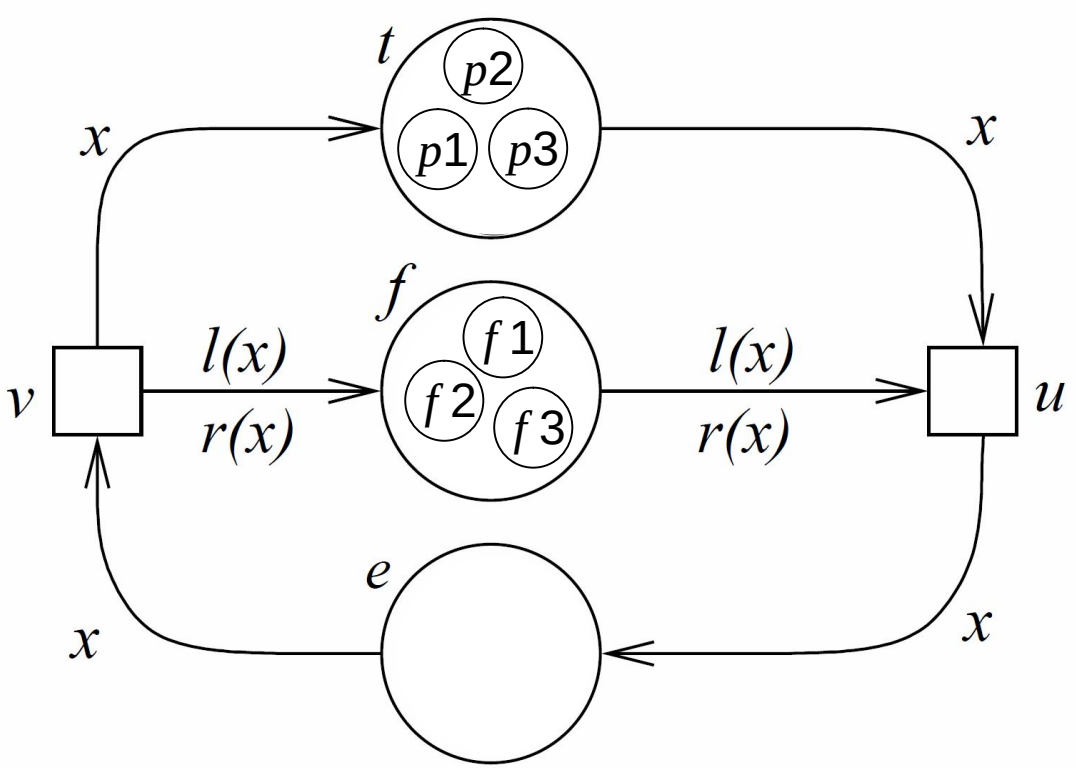
Let $x \in \{1..3\}$
t_x: x is thinking
e_x: x is eating
f_x: fork x is available

Model quite clumsy.

Difficult to extend to more philosophers.

Predicate/transition model of the dining philosophers problem (1)

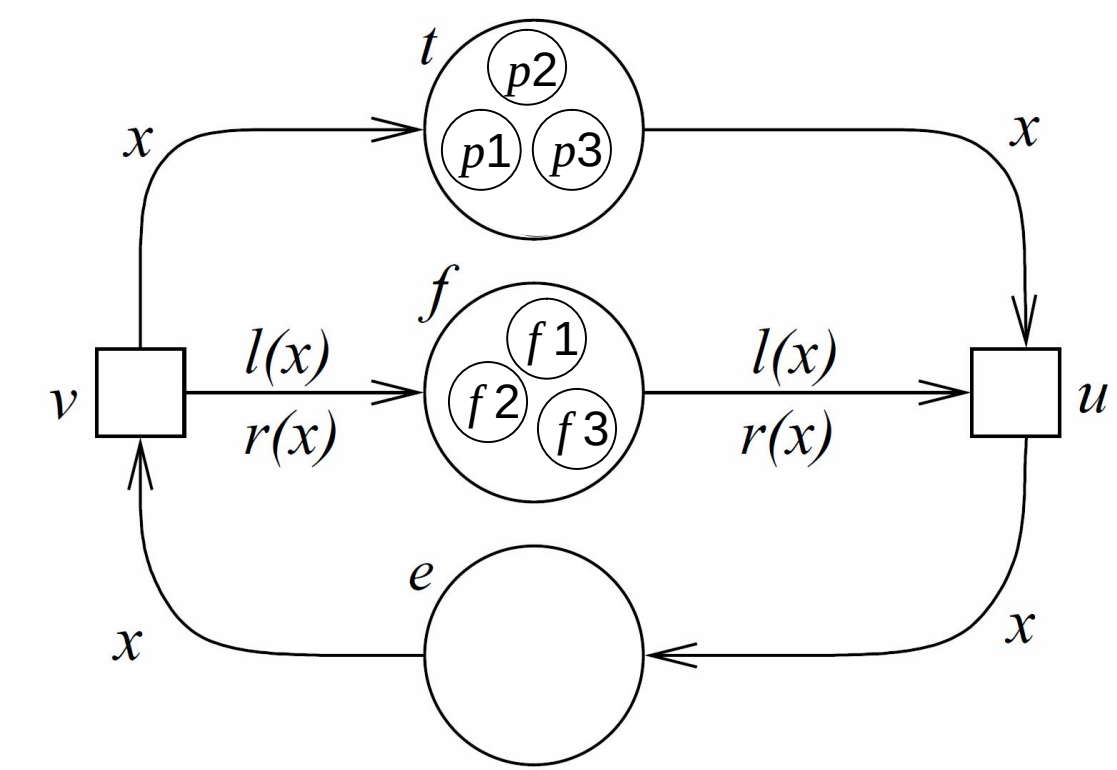
Let x be one of the philosophers,
let $l(x)$ be the left spoon of x,
let $r(x)$ be the right spoon of x.



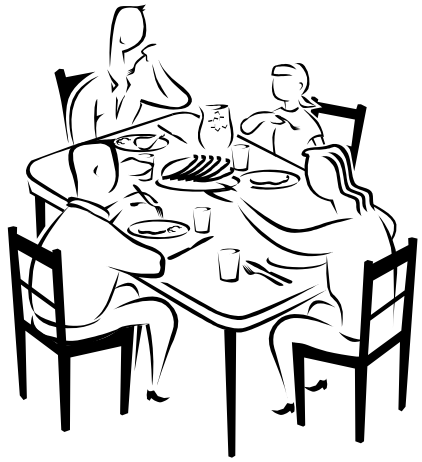
Tokens: individuals.

Semantics can be defined by replacing net by equivalent condition/event net.

Predicate/transition model of the dining philosophers problem (2)



Model can be extended to arbitrary numbers of people.



Evaluation

Pros:

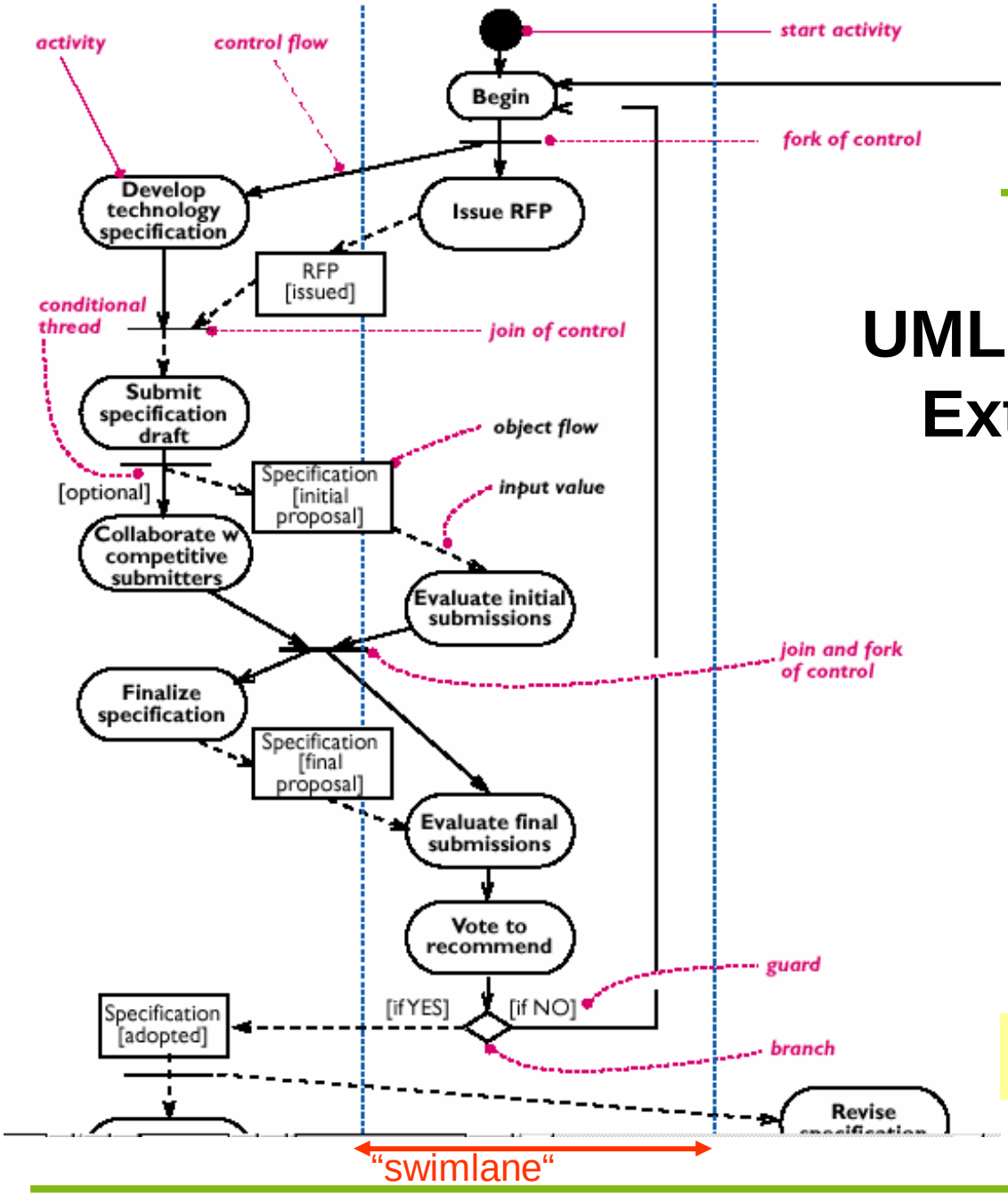
- Appropriate for distributed applications,
- Well-known theory for formally proving properties,
- Initially a quite bizarre topic, but now accepted due to increasing number of distributed applications.

Cons (for the nets presented) **:**

- problems with modeling timing,
- no programming elements,
- no hierarchy.

Extensions:

- Enormous amounts of efforts on removing limitations.



UML Activity diagrams: Extended Petri nets

Include decisions (like in flow charts). Graphical notation similar to SDL.

© Cris Kobryn: UML 2001: A Standardization Odyssey, CACM, October, 1999

Summary

Petri nets: focus on causal dependencies

- Condition/event nets
  - Single token per place
- Place/transition nets
  - Multiple tokens per place
- Predicate/transition nets
  - Tokens become individuals
  - Dining philosophers used as an example
- Extensions required to get around limitations

Activity diagrams in UML are extended Petri nets