

# Math 516: Linear Analysis

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## 1 Quotient Spaces

**Definition 1.1.** Let  $X$  be a vector space and  $Y$  be a subspace. Define  $\hat{x}$  be  $\{z : z = x + y, y \in Y\}$ . Then  $X/Y = \{\hat{x} : x \in X\}$  is a vector space.  $Q : X \rightarrow X/Y$  is a canonical quotient map. Suppose in addition that  $X$  is a normed space and  $Y$  is a closed subspace, then we can make  $X/Y$  into a normed space:

$$\|\hat{x}\| = \inf_{z \in \hat{x}} \|z\| = \inf_x \|x - y\| = \text{dist}(x, Y) \quad (1)$$

$X/Y$  is indeed a vector space. If  $\|\hat{x}\| = 0$  then  $\text{dist}(x, Y) = 0$  and  $Y$  is closed implies  $x \in Y$ . Therefore  $\hat{x} = Y$  which is the 0 element in quotient space.  $\forall a, b \in X$ ,  $\hat{a} + \hat{b} = Qa + Qb = Q(a + b) = \{z : z = a + b + y, y \in Y\}$ . For all  $\epsilon > 0$ , there exists  $y_0, y_1$  such that  $\|a - y_0\| \leq \|\hat{a}\| + \epsilon/2$  and  $\|b - y_1\| \leq \|\hat{b}\| + \epsilon/2$ . Then

$$\|\hat{a} + \hat{b}\| = \inf_{y \in Y} \|a + b - y\| \leq \|a + b - (y_0 + y_1)\| \leq \|a - y_0\| + \|b - y_1\| \leq \|\hat{a}\| + \epsilon/2 + \|\hat{b}\| + \epsilon/2 \leq \|\hat{a}\| + \|\hat{b}\| + \epsilon \quad (2)$$

Since  $\epsilon$  can be arbitrarily small, we have  $\|\hat{a} + \hat{b}\| \leq \|\hat{a}\| + \|\hat{b}\|$ .  $\|\alpha \hat{a}\| = |\alpha| \|\hat{a}\|$  is trivial.

**Exercise 1.2.**  $\|Q\| = 1, Q(B_x^\circ) = B_Y^\circ$

**Theorem 1.3.** If  $X$  is a Banach space and  $Y$  is a closed subspace of  $X$ . Then  $X/Y$  is a Banach space.

*Proof.* It is sufficient to prove that all absolutely convergent series are all convergent. Let  $\sum_{n=1}^\infty \hat{x}_n$  be an absolutely convergent subsequence, i.e.  $\sum_{n=1}^\infty \|\hat{x}_n\| < \infty$ . For all  $\hat{x}_n$ , pick a representative  $z_n$  such that  $\|z_n\| \leq 2\|\hat{x}_n\|$  and note that  $Qz_n = \hat{x}_n$ . Then  $\sum_{n=1}^\infty \|z_n\| \leq 2\sum_{n=1}^\infty \|\hat{x}_n\| < \infty$ . Since  $X$  is Banach,  $z := \sum_{n=1}^\infty z_n$  is a convergent sequence. By exercise 1.2, we know that  $Q$  is a bounded operator therefore continuous. Then  $\sum_{n=1}^\infty \hat{x}_n = \sum_{n=1}^\infty Qz_n = Qz$  converges  $\square$

**Theorem 1.4** (Almost Perpendicular Lemma/Reisz Lemma). Let  $X$  be a normed space and  $Y$  be a proper subspace of  $X$ , then for all  $\epsilon > 0$ , there exists  $x \in S_x$  such that  $\text{dist}(x, Y) \geq 1 - \epsilon$

*Proof.* Let  $\epsilon > 0$ . Pick a non-zero element  $\hat{z} \in X/Y$  and rescale it such that  $1 - \epsilon \leq \|\hat{z}\| < 1$ . Then pick  $z \in \hat{z}$  such that  $\|z\| \leq 1$ . Let  $x = \frac{\hat{z}}{\|\hat{z}\|}$  such that  $x \in S_X$ . Then  $\hat{x} = \{a = y + \frac{z}{\|\hat{z}\|} : y \in Y\} = \frac{\hat{z}}{\|\hat{z}\|}$ . By definition,  $\text{dist}(x, Y) = \|\hat{x}\| = \|\frac{\hat{z}}{\|\hat{z}\|}\| \in [1 - \epsilon, 1)$   $\square$

**Example 1.5.** Consider  $X = l_p$  and a sequence  $\{e_i\}$ . For  $i \neq j$ ,  $\|e_i - e_j\| \geq 1$  which means no subsequence of  $\{e_i\}$  converges, then  $B_X$  is not compact.

**Example 1.6.**  $X$  is a normed space.  $B_X$  is compact iff  $X$  is finite dimensional.

*Proof.* Assume  $X$  is finite dimensional. Then  $X \simeq l_2^N$  where  $N = \dim(X)$ .  $\simeq$  stands for isomorphic, which is surjective isomorphism, in normed space, it means the norm of two spaces are equivalent. Then since the topology in metric spaces are induced by metrics, the topology are the same. Then the unit ball in  $l_2^N$  is bounded and closed, which means it is compact. Since the topologies induced are the same, the unit ball is also compact in  $X$ .

Assume  $X$  is infinite dimensional and we are to prove  $B_X$  is not compact. Pick a point on  $S_X$ , then  $\text{span}\{x_1\}$  is a closed proper subspace of  $X$ . By Reisz's lemma, there exists  $x_2$  on  $S_X$  such that  $\|x_1 - x_2\| \geq 1/2$ .  $\text{span}\{x_1, x_2\}$  is a closed proper subspace of  $X$ , apply the lemma again, there exists  $x_3$  on  $S_X$  such that  $\|x_1 - x_3\| \geq 1/2$ ,  $\|x_2 - x_3\| \geq 1/2$ . Proceed this process inductively, then  $\{x_n\}$  has no convergent subsequence, then  $B_X$  is not compact.  $\square$

## 2 Separable Space

**Definition 2.1.** A space  $X$  is separable if it contains a countable dense subset. A subset  $E$  of  $X$  is dense if for all open set  $O \in X$ ,  $O \cap E \neq \emptyset$ .

$c_0$  is separable. Let  $A = \{(q_1, q_2, \dots, q_m, 0, \dots) : q_i \in \mathbb{Q}\}$ .  $A$  is countable. Left to show  $A$  is dense. Pick  $x = (x_1, \dots, x_n, \dots) \in c_0$  and  $\epsilon > 0$ , we have  $x_n \rightarrow 0$ . Then there exists  $n_0 \in \mathbb{N}$  such that  $|x_i| < \epsilon/2$  for all  $i \geq n_0$ . Let  $y = (x_1, \dots, x_{n_0}, 0, 0, \dots)$ . Then  $\|x - y\|_\infty < \epsilon/2$ . Let  $z = \{q_1, \dots, q_{n_0}\}$  where  $|x_i - q_i| < \epsilon/2$ . Then  $\|x - z\| \leq \|x - y\| + \|y - z\| < \epsilon$ , i.e.,  $z \in B_\epsilon(x)$  and also  $z \in A$  by construction. Therefore  $A$  is dense.

$c_0$  is isomorphic to  $c$  so  $c$  is also separable.

$l_\infty$  is not separable. Let  $\alpha$  be a subset of  $\mathbb{N}$  and  $\mathbb{I}_\alpha$  be the indicator function. Then  $\{\mathbb{I}_\alpha : \alpha \subset \mathbb{N}\}$  is an uncountable set. If there exists an  $A \subset l_\infty$  be countable and dense. Then for each  $\mathbb{I}_\alpha$ , there exists an  $a_\alpha$  such that  $\|a_\alpha - \mathbb{I}_\alpha\| < 1/3$ . Then for  $\alpha \neq \beta$ , we have

$$\begin{aligned} 1 &= \|\mathbb{I}_\alpha - \mathbb{I}_\beta\| = \|\mathbb{I}_\alpha - a_\alpha + a_\beta - \mathbb{I}_\beta + a_\alpha - a_\beta\| \\ &\leq \|\mathbb{I}_\alpha - a_\alpha\| + \|a_\beta - \mathbb{I}_\beta\| + \|a_\alpha - a_\beta\| \\ &\leq 2/3 + \|a_\alpha - a_\beta\| \end{aligned}$$

So  $\|a_\alpha - a_\beta\| \neq 0 \Rightarrow a_\alpha \neq a_\beta$ . Then  $\{a_\alpha : \alpha \subset \mathbb{N}\}$  is an uncountable subset of  $A$ .