Math 516: Linear Analysis

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1 Quotient Spaces

Definition 1.1. Let X be a vector space and Y be a subspace. Define $\hat{x}$ be $\{z : z = x+y, y \in Y\}$. Then $X/Y = \{\hat{x} : x \in X\}$ is a vector space. $Q : X \to X/Y$ is a canonical quotient map. Suppose in addition that X is a normed space and Y is a closed subspace, then we can make X/Y into a normed space:

$$\|\hat{x}\| = \inf_{z\in\hat{x}} \|z\| = \inf_{x} \|x - y\| = \mathrm{dist}(x, Y) \tag{1}$$

X/Y is indeed a vector space. If $\|\hat{x}\| = 0$ then $\mathrm{dist}(x, Y) = 0$ and Y is closed implies $x \in Y$. Therefore $\hat{x} = Y$ which is the 0 element in quotient space. $\forall a, b \in X$, $\hat{a} + \hat{b} = Qa + Qb = Q(a + b) = \{z : z = a + b + y, y \in Y\}$. For all $\epsilon > 0$, there exists y_0, y_1 such that $\|a - y_0\| \le \|\hat{a}\| + \epsilon/2$ and $\|b - y_0\| \le \|\hat{b}\| + \epsilon/2$. Then

$$\|\hat{a}+\hat{b}\| = \inf_{y\in Y} \|a+b-y\| \le \|a+b-(y_0+y_1)\| \le \|a-y_0\|+\|b-y_1\| \le \|\hat{a}\|+\epsilon/2+\|\hat{b}\|+\epsilon/2 \le \|\hat{a}\|+\|\hat{b}\|+\epsilon \tag{2}$$

Since ϵ can be arbitrarily small, we have $\|\hat{a} + \hat{b}\| \le \|\hat{a}\| + \|\hat{b}\|$. $\|\alpha\hat{a}\| = |\alpha|\|\hat{a}\|$ is trivial.

Exercise 1.2. $\|Q\| = 1, Q(B^\circ_x) = B^\circ_Y$

Theorem 1.3. *If X is a Banach space and Y is a closed subspace of X. Then X/Y is a Banach space.*

Proof. It is sufficient to prove that all absolutely convergent series are all convergent. Let $\sum_{n=1}^\infty \hat{x}_n$ be an absolutely convergent subsequence, i.e. $\sum_{n=1}^\infty \|\hat{x}_n\| < \infty$. For all $\hat{x}_n$, pick a representative z_n such that $\|z_n\| \le 2\|\hat{x}_n\|$ and note that $Qz_n = \hat{x}_n$. Then $\sum_{n=1}^\infty \|z_n\| \le 2\sum_{n=1}^\infty \|x_n\| < \infty$. Since X is Banach, $z := \sum_{n=1}^\infty z_n$ is a convergent sequence. By exercise 1.2, we know that Q is a bounded operator therefore continuous. Then $\sum_{n=1}^\infty \hat{x}_n = \sum_{n=1}^\infty Qz_n = Qz$ converges □

Theorem 1.4 (Almost Perpendicular Lemma/Reisz Lemma)**.** *Let X be a normed space and Y be a proper subspace of X, then for all $\epsilon > 0$, there exists $x \in S_x$ such that dist$(x, Y) \ge 1 - \epsilon$*

Proof. Let $\epsilon > 0$. Pick a non-zero element $\hat{z} \in X/Y$ and rescale it such that $1 - \epsilon \le \|\hat{z}\| < 1$. Then pick $z \in \hat{z}$ such that $\|z\| \le 1$. Let $x = \frac{z}{\|z\|}$ such that $x \in S_X$. Then $\hat{x} = \{a = y + \frac{z}{\|z\|} : y \in Y\} = \frac{\hat{z}}{\|z\|}$. By definition, $\mathrm{dist}(x, Y) = \|\hat{x}\| = \|\frac{\hat{z}}{\|z\|}\| \in [1 - \epsilon, 1)$ □

Example 1.5. Consider $X = l_p$ and a sequence $\{e_i\}$. For $i \ne j$, $\|e_i - e_j\| \ge 1$ which means no subsequence of $\{e_i\}$ converges, then B_X is not compact.

Example 1.6. X is a normed space. B_X is compact iff X is finite dimensional.

Proof. Assume X is finite dimensional. Then $X \simeq l_2^N$ where $N = dim(X)$. $\simeq$ stands for isomorphic, which is surjective isomorphism, in normed space, it means the norm of two spaces are equivalent. Then since the topology in metric spaces are induced by metrics, the topology are the same. Then the unit ball in l_2^N is bounded and closed, which means it is compact. Since the topologies induced are the same, the unit ball is also compact in X.
Assume X is infinite dimensional and we are to prove B_X is not compact. Pick a point on S_x, then $span\{x_1\}$ is a closed proper subspace of X. By Reisz's lemma, there exists x_2 on S_X such that $\|x_1 - x_2\| \geq 1/2$. $span\{x_1, x_2\}$ is a closed proper subspace of X, apply the lemma again, there exists x_3 on S_X such that $\|x_1 - x_3\| \geq 1/2$, $\|x_2 - x_3\| \geq 1/2$. Proceed this process inductively, then $\{x_n\}$ has no convergent subsequence, then B_X is not compact. $\square$

2 Separable Space

Definition 2.1. A space X is separable if it contains a countable dense subset. A subset E of X is dense if for all open set $O \in X$, $O \cap E \neq \emptyset$.

c_0 is separable. Let $A = \{(q_1, q_2, ..., q_m, 0, ...) : q_i \in \mathbb{Q}\}$. A is countable. Left to show A is dense. Pick $x = (x_1, ..., x_n, ...) \in c_0$ and $\epsilon > 0$, we have $x_n \to 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|x_i| < \epsilon/2$ for all $i \geq n_0$. Let $y = (x_1, .., x_{n_0}, 0, 0, ...)$. Then $\|x - y\|_\infty < \epsilon/2$. Let $z = \{q_1, ..., q_{n_0}\}$ where $|x_i - q_i| < \epsilon/2$. Then $\|x - z\| \leq \|x - y\| + \|y - z\| < \epsilon$,i.e., $z \in B_\epsilon(x)$ and also $z \in A$ by construction. Therefore A is dense.

c_0 is isomorphic to c so c is also separable.

l_∞ is not separable. Let α be a subset of $\mathbb{N}$ and $\mathbb{I}_\alpha$ be the indicator function. Then $\{\mathbb{I}_\alpha : \alpha \subset \mathbb{N}\}$ is an uncountable set. If there exists an $A \subset l_\infty$ be countable and dense. Then for each $\mathbb{I}_\alpha$, there exists an a_α such that $\|a_\alpha - \mathbb{I}_\alpha\| < 1/3$. Then for $\alpha \neq \beta$, we have

$$\begin{aligned}
1 = \|I_\alpha - I_\beta\| &= \|I_\alpha - a_\alpha + a_\beta - I_\beta + a_\alpha - a_\beta\| \\
&\leq \|I_\alpha - a_\alpha\| + \|a_\beta - I_\beta\| + \|a_\alpha - a_\beta\| \\
&\leq 2/3 + \|a_\alpha - a_\beta\|
\end{aligned}$$

So $\|a_\alpha - a_\beta\| \neq 0 \Rightarrow a_\alpha \neq a_\beta$. Then $\{a_\alpha : \alpha \subset N\}$ is an uncountable subset of A.

$L_\infty([0,1])$ is also not separable. We can map every point in l_∞ into $L_\infty([0,1])$. That is define $f_n = \mathbb{I}_{[\frac{1}{2^n}, \frac{1}{2^{n-1}}]}$. We then define $T : e_n \to f_n$, i.e., $\forall x \in l_\infty, (Tx)(s) = x_n$ for $s \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. Then the uncoutable equidistant set in l_∞ is mapped to $L_\infty([0,1])$. Then if a set is dense, then it should be uncountable. $l_1 \subset l_2 \subset\subset ... \subset c \subset l_\infty$ and all of them are separable except l_∞.

$c([0,1])$ is separable. Let $A = \{$all peice-wise linear functions with rational nodes$\}$. For all $f \in c([0,1])$, f is uniformly continuous. For all $\epsilon > 0$, there exists n sufficiently large such that all intervals taking the form $[\frac{i}{n}, \frac{i+1}{n}]$ for $i \leq n$, the variation of f is less than ϵ. Then there exists $h \in A$ such that for all $i \in [n]$ and $t \in [\frac{i}{n}, \frac{i+1}{n}]$, $|f(t) - h(t)| < \epsilon$. Then $||f - h|| < \epsilon$ and A is dense.

Lemma 2.2. *c([0,1]) is dense in* $L_p([0,1])$.

Proof. WLOG, we assume $f \geq 0$, otherwise we can replace $f = f^+ - f^-$. For all $f \geq 0 \in c([0,1])$, let g_n be $f \wedge n$, then $g_n \to f$. Note that $|f - g_n|^p \leq 2^p|f|^p$ therefore in $L_p([0,1])$. By Dominated Convergence Theorem, $\lim_{n\to\infty} \int |f - g_n|^p = \int \lim_{n\to\infty} |f - g_n|^p = 0$. Hence for all $\epsilon > 0$, there exists $M \in \mathbb{R}$ such that $\|f - g\| = \int |f - g| < \epsilon$. By Lusin's theorem, which states that every finite measurable function is an almost continuous function, i.e., there exists $h \in c([0,1])$ such that g agrees with h on a set A with $\mu(A) < \frac{\epsilon}{M}$. WLOG, we assume $h \leq M$ otherwise we pay a price of ϵ as before to truncate it in. Then $\|g - h\| = \int |g - h| \leq M\mu(A) = \epsilon$. Finally $\|f - h\| < \|f - g\| + \|g - h\| < 2\epsilon$ and $c([0,1])$ is thus dense. $\square$

Theorem 2.3. *For $1 \leq p < \infty, L_p([0,1])$ is separable.*

Proof. Since $c([0,1])$ is separable in $\|\cdot\|_\infty$ we pick a countable dense set A. We claim that A is still a dense set in $L_p([0,1])$ in $\|\cdot\|_p$. Fix $\epsilon > 0$. For all $f \in L_p([0,1])$, by previous lemma, there exists $g \in c([0,1])$ such that $\|f - g\| < \epsilon$. Then we find h in A such that $\|g - h\|_\infty < \epsilon$. Then $\|f - g\| < \|f - g\| + \|g - h\| < \|f - g\| + \|g - h\|_\infty < 2\epsilon$. The last inequality is due to $\|f\|_p < \|f\|_\infty$ in $L_p([0,1])$. $\square$

Proposition 2.4 (exercise)**.** *X is separable iff S_X is separable*

Proof. Assume X is separable. Find A to be the dense and countable subset. Then $B = \{x = \frac{y}{\|y\|} : y \in A\}$ is dense in S_X. For all $\tilde{x} \in S_X$, there exists $x \in X$ such that $\tilde{x} = \frac{x}{\|x\|}$. For x and $\epsilon' > 0$, $\exists y \in A$ where $\|x - y\| < \epsilon'$. Note that in order for all $\epsilon > 0$, we have

$$
\begin{aligned}
\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| &= \frac{1}{\|x\|}\left\|x - \frac{\|x\|}{\|y\|}y\right\| \\
&= \frac{1}{\|x\|}\|x - y + \frac{\|y\| - \|x\|}{\|y\|}y\| < \epsilon \\
\|x - y\| &< \|x\|\epsilon + \big|\|y\| - \|x\|\big|
\end{aligned}
$$

Then by setting $\epsilon' = \|x\|\epsilon + \big|\|y\| - \|x\|\big|$ we show that A is dense.

Assume S_X is separable. Find B to be the dense and countable subset. Then $A = \{qx : q \in \mathbb{Q}, x \in B\}$ is dense in X. Fix $\epsilon > 0$. For all $\tilde{x} \in X$, $x = \frac{\tilde{x}}{\|\tilde{x}\|} \in S_X$. Then by definition of B, there exists $y \in B$ such that for all $\epsilon' > 0$ $\|x - y\| < \epsilon'$. There exists a convergent sequence $\{q_n\}$ in $\mathbb{Q}$ such that $q_n \to \|\tilde{x}\|$. Hence there exists $n_0 \in \mathbb{N}$ s.t. for all $n \geq n_0$, $\|x \cdot \|\tilde{x}\| - yq_n\| - \|x \cdot \|\tilde{x}\| - y\|\tilde{x}\|\| < \epsilon/2$. By setting $\epsilon' = \frac{\epsilon}{2\|\tilde{x}\|}$, we obtain $\|x \cdot \|\tilde{x}\| - y\|\tilde{x}\|\| < \epsilon/2$ therefore $|x \cdot \|\tilde{x}\| - yq_n\| = \|x - yq_n\| < \epsilon$ and A is thus dense. $\square$