AN ADDEDUM TO THE PAPER "SOME ELEMENTARY ESTIMATES FOR THE NAVIER-STOKES SYSTEM"

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ABSTRACT. In this paper we give a proof of the existence of global regular solutions to the Fourier transformed Navier-Stokes system with small initial data in Φ (2) via an iteration argument. The proof of the regularity theorem is a minor modification of the proof given in the paper "Some elementary estimates for the Navier-Stokes system", so this paper is intended to be just a complement to the afore mentioned paper.

1. Introduction

A Generalized Navier-Stokes system (with periodic boundary conditions on $[0, 1]^3$) is a system of the form

(1)
$$v^{k}(\xi, t) = \psi^{k}(\xi) \exp\left(-|\xi|^{2} t\right)$$

 $+ \int_{0}^{t} \exp\left(-|\xi|^{2} (t-s)\right) \sum_{\mathbf{q} \in \mathbb{Z}^{3}} M_{ijk}(\xi) v^{i}(q, s) v^{j}(\xi - q, s) ds,$

for $\xi \in \mathbb{Z}^3$, and where $M_{ijk}(\xi)$ satisfies the bound

$$|M_{ijk}(\xi)| \leq |\xi|$$
.

To solve this problem it is usual to consider the following iteration scheme

$$v_{n+1}^{k}(\xi,t) = \psi^{k}(\xi) \exp\left(-|\xi|^{2} t\right) + \int_{0}^{t} \exp\left(-|\xi|^{2} (t-s)\right) \sum_{\mathbf{q} \in \mathbb{Z}^{3}} M_{ijk}(\xi) v_{n}^{i}(q,s) v_{n}^{j}(\xi-q,s) ds.$$

In what follows we will show the convergence of this method for small initial conditions on $\Phi(2)$ (for the definition of the space $\Phi(2)$ see [4]). More exactly we will show that

Theorem 1. There exists an $\epsilon > 0$ such that if $\|\psi\|_2 < \epsilon$, then (1) has a global regular solution with initial condition ψ .

The main purpose on writing this note is for it to serve as a complement to our paper [4], and to show that the free divergence condition, neither the fact of considering Leray-Hopf weak solutions is an issue for the proofs presented in that paper.

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2. Existence

We start with two auxiliary results,

Lemma 1. There exists an $\epsilon > 0$ such that if $\|\psi\| < \epsilon$ then the sequence $v_n^k(\xi, t)$ is uniformly bounded on [0, T] for ξ fixed.

Proof. To proof this fact it is enough to show that if

$$\left|v_n^k\left(\xi,t\right)\right| \le \frac{\epsilon}{\left|\xi\right|^2}$$

then

(2)
$$\left| \sum_{\mathbf{q} \in \mathbb{Z}^3} M_{ijk} \left(\xi \right) v_n^i \left(q, s \right) v_n^j \left(\xi - q, s \right) \right| \le c \epsilon^2,$$

where c is a universal constant, because then we would have, for any $t \geq 0$ and $\epsilon > 0$ small enough,

$$\begin{aligned} \left| v_{n+1}^k \left(\xi, t \right) \right| & \leq \left| \left| \psi^k \left(\xi \right) \right| \exp \left(- \left| \xi \right|^2 t \right) + c \int_0^t \exp \left(t - s \right) \epsilon^2 \, ds \\ & \leq \frac{\epsilon}{\left| \xi \right|^2} \exp \left(- \left| \xi \right|^2 t \right) + \frac{c \epsilon^2}{\left| \xi \right|^2} \left(1 - \exp \left(- \left| \xi \right|^2 t \right) \right) \\ & \leq \frac{\epsilon}{\left| \xi \right|^2} \exp \left(- \left| \xi \right|^2 t \right) + \frac{\epsilon}{\left| \xi \right|^2} \left(1 - \exp \left(- \left| \xi \right|^2 t \right) \right) = \frac{\epsilon}{\left| \xi \right|^2}. \end{aligned}$$

We proceed to show the validity of (2). Write

$$\sum_{\mathbf{q}\in\mathbb{Z}^{3}}M_{ijk}\left(\xi\right)v_{n}^{i}\left(q,s\right)v_{n}^{j}\left(\xi-q,s\right)=I+II+III,$$

where

$$I = \sum_{1 \leq |q| \leq 2|\xi|, 1 \leq |\xi-q| \leq \frac{|\xi|}{2}} M_{ijk}\left(\xi\right) v_n^i\left(\xi, t\right) v_n^j\left(\xi - q, t\right),$$

$$II = \sum_{1 \leq |q| \leq 2|\xi|, |\xi-q| > \frac{|\xi|}{2}} M_{ijk}\left(\xi\right) v_n^i\left(\xi, t\right) v_n^j\left(\xi - q, t\right),$$

and

$$III = \sum_{|q|>2|\xi|} M_{ijk}\left(\xi\right) v_n^i\left(\xi,t\right) v_n^j\left(\xi-q,t\right).$$

To estimate I observe that if $|\xi - q| \le \frac{|\xi|}{2}$, then $|q| \ge \frac{|\xi|}{2}$. Therefore, using that

$$|M_{ijk}(\xi)| \le c |\xi|$$

and the elementary inequality

$$(3) \sum_{1 \le |q| \le r} \frac{1}{|q|^2} \le cr$$

(where c is a universal constant) we can bound as follows,

$$|I| \leq c|\xi| \frac{\epsilon^2}{|\xi|^2} \sum_{1 \leq |\xi - q| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - q|^2}$$
$$\leq \frac{c\epsilon^2}{|\xi|} \frac{|\xi|}{2} = c\epsilon^2.$$

II can be estimated in the same way, so we also obtain

$$|II| \le c\epsilon^2$$
.

To estimate III, first notice that $|q| > 2|\xi|$ implies that $|\xi - q| \ge \frac{1}{2}|q|$. Hence, using the inequality

$$(4) \sum_{|q|>r} \frac{1}{|q|^4} \le \frac{c}{r},$$

we can bound as follows,

$$|III| \leq c |\xi| \epsilon^2 \sum_{|q| > 2|\xi|} \frac{1}{|q|^2} \frac{1}{|\xi - q|^2}$$

$$\leq c |\xi| \epsilon^2 \sum_{|q| > 2|\xi|} \frac{1}{|q|^4}$$

$$\leq c |\xi| \epsilon^2 \frac{1}{|\xi|} = c\epsilon^2.$$

This shows the lemma.

Lemma 2. If there is an $\epsilon > 0$ such that the sequence $v_n^k\left(\xi,t\right)$ satisfies

$$\left\| v_{n}^{k}\left(t\right) \right\| _{2}<\epsilon \quad for\ all\quad t\in \left[0,T\right]$$

The sequence $v_{n}^{k}\left(\xi,t\right)$ is equicontinuous on $\left[0,T\right]$ for ξ fixed.

Proof. Let $t_1, t_2 \in (\rho, T), t_2 > t_1$. Then we estimate for ξ fixed

$$\left|v_{n+1}^{k}\left(\xi, t_{2}\right) - v_{n+1}^{k}\left(\xi, t_{1}\right)\right| \leq I + II + III$$

where

$$I = \left| \psi^k \left(\xi \right) \right| \left| \exp \left(- \left| \xi \right|^2 t_2 \right) - \exp \left(- \left| \xi \right|^2 t_1 \right) \right|,$$

$$II = \int_{0}^{t_{1}} \left| \exp\left(-\left|\xi\right|^{2} (t_{2} - s)\right) - \exp\left(-\left|\xi\right|^{2} (t_{1} - s)\right) \right|$$

$$\sum_{\mathbf{q} \in \mathbb{Z}^{3}} \left| M_{ijk} \left(\xi\right) v_{n}^{i} \left(q, s\right) v_{n}^{j} \left(\xi - q, s\right) \right| ds$$

and

$$III = \int_{t_1}^{t_2} \exp\left(-\left|\xi\right|^2 (t-s)\right) \sum_{\mathbf{q} \in \mathbb{Z}^3} \left| M_{ijk}\left(\xi\right) v_n^i\left(q,s\right) v_n^j \left(\xi - q,s\right) \right| \, ds.$$

Let us bound each of the previous expressions,

$$I = \left| \exp\left(-|\xi|^{2} t_{1}\right) \right| \left| 1 - \exp\left(-|\xi|^{2} (t_{2} - t_{1})\right) \right|$$

$$\leq \frac{\epsilon}{|\xi|^{2}} |\xi|^{2} |t_{2} - t_{1}| = \epsilon |t_{2} - t_{1}|,$$

$$II \leq \int_{0}^{t_{1}} \left| \exp\left(-\left|\xi\right|^{2} (t_{2} - s)\right) - \exp\left(-\left|\xi\right|^{2} (t_{1} - s)\right) \right| \epsilon^{2} ds$$

$$= \int_{\tau_{n}}^{t_{1}} \exp\left(-\left|\xi\right|^{2} (t_{1} - s)\right) \left|1 - \exp\left(-\left|\xi\right|^{2} (t_{2} - t_{1})\right) \right| \epsilon^{2} ds$$

$$\leq \left|\xi\right|^{2} (t_{2} - t_{1}) \frac{1}{\left|\xi\right|^{2}} \left|1 - \exp\left(-\left|\xi\right|^{2} t_{1}\right)\right|,$$

$$III \leq \int_{t_1}^{t_2} \epsilon^2 \exp\left(-|\xi|^2 (t_2 - s)\right) ds$$

$$\leq \frac{\epsilon^2}{|\xi|^2} \left| 1 - \exp\left(-|\xi|^2 (t_2 - t_1)\right) \right| \leq \frac{1}{|\xi|^2} \epsilon^2 |\xi|^2 |t_2 - t_1|,$$

and hence

$$\left| v_{n+1}^{k}\left(\xi,t_{2}\right) -v_{n+1}^{k}\left(\xi,t_{1}\right) \right| < C\left(\epsilon\right) \left(t_{2}-t_{1}\right)$$

for n > 0, and the lemma is proved.

The previous Lemmas via the theorem of Arzela-Ascoli, using Cantor's diagonal procedure, show that there is a well defined $v \in \Phi(2)$ defined on [0, T] such that,

(5)
$$v^{k}(\xi, t) = v^{k}(\xi, t) \exp\left(-|\xi|^{2} t\right)$$

 $+ \int_{0}^{t} \exp\left(-|\xi|^{2} (t-s)\right) \sum_{q \in \mathbb{Z}^{3}} M_{ijk}(\xi) v^{i}(q, s) v^{j}(\xi - q, s) ds$

Let us give a proof of this. To simplify notation, let us assume that the sequence converging uniformly on [0,T] for each ξ is the sequence $v_n(\xi,t)$. By what we have shown, there exists a D not depending on t, ξ or n such that

$$\left|v_n^j\left(\xi,t\right)\right| \le \frac{D}{\left|\xi\right|^2}.$$

Let ξ be fixed, and let $\eta>0$ arbitrary. the previous estimate allows us to choose a Q such that

$$\left| \sum_{|q| \ge Q} M_{ijk}(\xi) v_n^i(\xi, t) v_n^j(\xi, t) \right| \le \eta.$$

and also that the same inequality is valid with v_n replaced by v (this can be done since the choice of Q only depends on D). Hence we have

$$|v_{n+1}^{k}(\xi,t) - \psi^{k}(\xi) \exp\left(-|\xi|^{2} t\right) - \int_{0}^{t} \exp\left(-|\xi|^{2} (t-s)\right) \sum_{1 \leq |q| < Q} M_{ijk}(\xi) v_{n}^{i}(q,s) v_{n}^{j}(\xi-q,s) ds$$

$$\leq \eta$$

Taking $n \to \infty$, we obtain

$$|v^{k}(\xi, t) - \psi^{k}(\xi) \exp(-|\xi|^{2} t)$$

$$- \int_{0}^{t} \exp(-|\xi|^{2} (t-s)) \sum_{1 \leq |q| < Q} M_{ijk}(\xi) v^{i}(q, s) v^{j}(\xi - q, s) ds$$

$$< \eta$$

and from this follows that

$$\begin{vmatrix} v^{k}(\xi, t) & - & \psi^{k}(\xi) \exp\left(-\left|\xi\right|^{2} t\right) \\ - & \int_{0}^{t} \exp\left(-\left|\xi\right|^{2} (t-s)\right) \sum_{q \in \mathbb{Z}^{3}} M_{ijk}(\xi) v^{i}(q, s) v^{j}(\xi - q, s) ds \end{vmatrix}$$

$$\leq 2\eta.$$

Since $\eta > 0$ is arbitrary, our claim is proved.

3. Regularity

We shall show now that the solutions produced by the iteration scheme are regular under certain smallness condition. Indeed, we have

Theorem 2. Let $v \in L^{\infty}(0,T;\Phi(2))$ be a solution to (1). There exists an $\epsilon > 0$ such that if there is a k_{-1} for which v satisfies

(6)
$$\sup_{|\xi| \ge k-1} |\xi|^2 \left| v^k(\xi, t) \right| < \epsilon \quad \text{for all} \quad t \in (0, T)$$

then v is smooth.

To prove Theorem 2 we will need to estimate term

$$\sum M_{ijk}(\xi) u^{i}(q) u^{j}(\xi - q).$$

This is the content of Lemma 3. But before we state and prove Lemma 3 and in order to express our estimates in a convenient way we will define to sequences of numbers. Namely

$$\left\{ \begin{array}{ll} \mu_0 = 1 & \mu_1 = 1 \\ \mu_{n+1} = 2\mu_n - 1, & n \ge 2 \end{array} \right.$$

and

$$k_n = \frac{1}{\epsilon^{2^n}} k_0$$

where k_0 is such that

$$\frac{k_{-1}}{k_0} \cdot D < \min\left\{\epsilon, \frac{1}{2}\right\}$$

and $D = \sup_{(0,T)} ||u(t)||$.

We are now ready to estate and prove,

Lemma 3. Assume that for all ξ such that $|\xi| \geq k_{-1}$

$$\left|v^{k}\left(\xi,s\right)\right| \leq \frac{\epsilon}{\left|\xi\right|^{2}}$$

and if $|\xi| \geq k_m$

$$\left|v^{k}\left(\xi,s\right)\right| \leq \frac{\epsilon^{\mu_{m}}}{\left|\xi\right|^{2}}$$

Then for $|\xi| \geq k_{m+1}$ it holds that,

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(q,s) v^j(\xi - q,s) \right| \le \epsilon^{\mu_{m+1}}.$$

Proof. First recall that $|M_{ijk}(\xi)| \leq c |\xi|$.

$$(7)I \leq |\xi| \sum_{1 \leq |q| < k_{-1}} |v^{i}(q,s) v^{j}(\xi - q,s)| + |\xi| \sum_{k_{-1} \leq |q| < k_{m}} |v^{i}(q,s) v^{j}(\xi - q,s)| + |\xi| \sum_{|q| > k_{m}} |v^{i}(q,s) v^{j}(\xi - q,s)|$$

We estimate the first sum. Observe that $k_{-1} \leq \frac{|\xi|}{2}$, so if $|q| < k_{-1}$, we must have $|\xi - q| \geq \frac{|\xi|}{2}$. Hence, using the elementary inequality (3), we can bound

$$\begin{split} \sum_{1 \leq |q| < k_{-1}} \left| v^{i}\left(q,s\right) v^{j}\left(\xi - q,s\right) \right| & \leq & \frac{4\epsilon^{\mu_{m}}}{\left|\xi\right|^{2}} \sum_{1 \leq |q| < k_{-1}} \frac{D}{\left|q\right|^{2}} \\ & \leq & 4c\epsilon^{\mu_{m}} \frac{k_{-1}}{k_{m}} \leq 4c\epsilon^{2\mu_{m}} \end{split}$$

To estimate the second sum, notice that if $|\xi| \geq k_{m+1}$ and $|q| \leq k_m$, then $|\xi - q| \geq \frac{|\xi|}{2} \geq k_m$. All this said, using inequality (3) again we obtain,

$$\sum_{1 \le |q| < k_m} \left| v^i \left(q, s \right) v^j \left(\xi - q, s \right) \right| \le \frac{4\epsilon^{\mu_m}}{\left| \xi \right|^2} \sum_{1 \le |q| < k_m} \frac{\epsilon}{\left| q \right|^2}$$

$$\le \frac{4\epsilon^{\mu_m}}{\left| \xi \right|^2} \epsilon k_m$$

Observe now that $\frac{k_m}{k_{m+1}} \le \epsilon^{2^m} \le \epsilon^{\mu_m}$. This yields the bound,

$$\sum_{1 \leq |q| \leq k_m} \left| v^i \left(q, s \right) v^j \left(\xi - q, s \right) \right| \leq \frac{4 \epsilon^{\mu_m}}{|\xi|} \frac{k_m}{k_{m+1}} \leq \frac{4 \epsilon^{2\mu_m}}{|\xi|}$$

To estimate the second sum on the righthanside of (7) we split it into three sums, namely

(8)
$$\sum_{|q| \ge k_m} |v^i(q, s) v^j(\xi - q, s)| = \sum_{k_m \le |q| < \frac{|\xi|}{2}} |v^i(q, s) v^j(\xi - q, s)| + \sum_{\frac{|\xi|}{2} \le |q| < 2|\xi|} |v^i(q, s) v^j(\xi - q, s)| + \sum_{|q| \ge 2|\xi|} |v^i(q, s) v^j(\xi - q, s)|$$

Estimating the three sums on the right hand side separately. Observe that if $|q| \le \frac{|\xi|}{2}$ then we must have $|\xi - q| \ge \frac{|\xi|}{2} > k_m$. Therefore, using inequality (3), we get

$$\sum_{k_m \le |q| < \frac{|\xi|}{2}} \left| v^i \left(q, s \right) v^j \left(\xi - q, s \right) \right| \le \frac{4\epsilon^{2\mu_m}}{\left| \xi \right|^2} \sum_{1 \le |q| < \frac{|\xi|}{2}} \frac{1}{\left| q \right|^2} \\ \le \frac{4\epsilon^{2\mu_m}}{\left| \xi \right|}$$

To estimate the second sum we split it into two sums,

$$\sum_{\substack{|\xi| \\ \frac{1}{2} \le |q| < 2|\xi|}} |v^{i}(q,s) v^{j}(\xi - q,s)| = \sum_{\substack{|\xi| \\ \frac{1}{2} \le |q| < 2|\xi|, k_{m} \le |\xi - q|}} |v^{i}(q,s) v^{j}(\xi - q,s)| + \sum_{\substack{|\xi| \\ \frac{1}{2} \le |q| < 2|\xi|, |\xi - q| < k_{m}}} |v^{i}(q,s) v^{j}(\xi - q,s)|$$

Estimating the first sum on the righthandside of the previous equality,

$$\sum_{\substack{|\xi| \\ 2} \le |q| < 2|\xi|, |\xi - q| \ge k_m} |v^i(q, s) v^j(\xi - q, s)| \le \frac{4\epsilon^{2\mu_m}}{|\xi|^2} \sum_{1 \le |\xi - q| < 3|\xi|} \frac{1}{|\xi - q|^2} \le \frac{12\epsilon^{2\mu_m}}{|\xi|}$$

The estimation of the second sum proceeds in exactly the same way as the estimation of the first sum on the right hand side of (7), and hence we obtain

$$\sum_{\frac{|\xi|}{2} < |q| < 2|\xi|, |\xi - q| < k_m} |v^i(q, s) v^j(\xi - q, s)| \le \frac{4\epsilon^{2\mu_m}}{|\xi|}.$$

Now we estimate the third sum in the righthandside of (8). Using that $|q| \ge 2 |\xi|$ implies that $|\xi - q| \ge \frac{1}{2} |q|$, and inequality (4) we can bound,

$$\sum_{|q| \ge 2|\xi|} \left| v^i(q, s) \, v^j(\xi - q, s) \right| \le 4\epsilon^{2\mu_m} \sum_{|q| \ge 2|\xi|} \frac{1}{|q|^4} \le \frac{4\epsilon^{2\mu_m}}{|\xi|}$$

Putting all the previous estimations together, we arrive at

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk} \left(\xi \right) v^i \left(q, s \right) v^j \left(\xi - q, s \right) \right| \leq |\xi| \left(\frac{28 \epsilon^{\mu_m}}{|\xi|} \right),$$

and if we assume $0 < \epsilon < \frac{1}{28}$, the previous inequality reads as

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(q, s) v^j(\xi - q, s) \right| \le \epsilon^{2\mu_m - 1} \le \epsilon^{\mu_{m+1}},$$

and the Lemma is proved.

3.1. **Proof of Theorem 2.** Given $0 < \rho < T$, we will first show that for a constant $K(\rho)$, there exists a constant D such that if $|\xi| \ge K(\rho)$ then

$$\left|v^{k}\left(\xi,t\right)\right| \leq \frac{D}{\left|\xi\right|^{2+\frac{1}{4}}} \quad \text{if} \quad t > \rho.$$

Define

$$\tau_m = \rho - \frac{\rho}{2^m}.$$

We will show by induction that

(P)
$$v^{k}(\xi, t) \leq \frac{\epsilon^{\mu_{n}}}{|\xi|^{2}} \quad \text{if} \quad t > \tau_{n} \quad \text{and} \quad |\xi| \geq k_{n}.$$

For n = 0, our choice of k_0 guarantees that (P) holds. Assume that (P) holds for n = m. First observe that v satisfies

$$v^{k}(\xi, t) = v^{k}(\xi, \tau_{n}) \exp\left(-|\xi|^{2} (t - \tau_{n})\right) + \int_{\tau_{n}}^{t} \exp\left(-|\xi|^{2} (t - s)\right) \sum_{\mathbf{q} \in \mathbb{Z}^{3}} M_{ijk}(\xi) v^{i}(q, s) v^{j}(\xi - q, s) ds.$$

Using this identity, we bound as follows.

$$v^{k}(\xi, t) \leq v^{k}(\xi, \tau_{m}) \exp\left(-|\xi|^{2}(t - \tau_{m})\right) + \int_{\tau_{m}}^{t} \exp\left(-|\xi|^{2}(t - s)\right) e^{2\mu_{m}} ds$$

$$\leq \frac{\epsilon^{\mu_{m}}}{|\xi|^{2}} \exp\left(-k_{m+1}(\tau_{m+1} - \tau_{m})\right)$$

$$+ \frac{\epsilon^{2\mu_{m}}}{|\xi|^{2}} \left(\exp\left(-|\xi|^{2}\tau_{m}\right) - \exp\left(-|\xi|^{2}t\right)\right)$$

$$\leq \frac{\epsilon^{\mu_{m}}}{|\xi|^{2}} + \frac{\epsilon^{2\mu_{m}}}{|\xi|^{2}}.$$

From this last bound it follows that if $t \ge \rho > \rho - \frac{\rho}{2^m}$, then if $k_m \le |\xi| < k_{m+1}$ it holds that

$$\left|v^{k}\left(\xi,t\right)\right| \leq \frac{\epsilon^{\mu_{m}}}{\left|\xi\right|^{2}}.$$

Since $\mu_m \geq 2^{n-1}$ and $k_m = \frac{k_0}{\epsilon^{2m}}$, it is easy to check that $\epsilon^{\mu_m} \leq \frac{k_0^{\frac{1}{4}}}{|\xi|^{\frac{1}{4}}}$. Hence for all $t \geq \rho$ the following estimate holds,

$$\left|v^{k}\left(\xi,t\right)\right| \leq \frac{D}{\left|\xi\right|^{2+\frac{1}{4}}}.$$

The following Lemma will then finish the proof of Theorem 2.

Lemma 4. Let v be a solution to (FNS) such that for all $t \in (0,T)$ satisfies

$$\left|v^{k}\left(\xi,t\right)\right| \leq \frac{D}{\left|\xi\right|^{2+\eta}}$$

with D and $\eta > 0$ independent of t. Then v is smooth.

Proof. Let $\rho > 0$. Under the hypothesis of the Lemma, we will show that there exists a constant $K := K(\rho)$ such that if t > T and $|\xi| > K$, then for a constant E independent of time,

$$\left|v^{k}\left(\xi,t\right)\right| \leq \frac{E}{\left|\xi\right|^{2+\min\left(\frac{1}{2},\frac{3}{2}\eta\right)}}.$$

Since $\rho > 0$ is arbitrary, a finite number of applications of the previous claim shows that for any $\rho > 0$, the Fourier transform of v decays faster than any polynomial, and this shows the lemma.

First, we will estimate the term

$$S = \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(\xi, s) v^j(\xi, s)$$

under the hypotesis of the lemma. In order to do this we write,

$$S = I_a + I_b + II_a + II_b + III_a + III_b + IV_a + IV_b$$

where

$$I_{a} = \sum_{1 \leq |q| \leq \sqrt{|\xi|}} M_{ijk} (\xi) v^{i} (\xi, s) v^{j} (\xi, s) ,$$

$$II_{a} = \sum_{\sqrt{|\xi|} < q \leq \frac{|\xi|}{2}} M_{ijk} (\xi) v^{i} (\xi, s) v^{j} (\xi, s) ,$$

$$III_{a} = \sum_{|q| \geq \frac{|\xi|}{2}, 1 \leq |\xi - q| < 2|\xi|} M_{ijk}(\xi) v^{i}(\xi, s) v^{j}(\xi, s),$$

and

$$IV_{a} = \sum_{|q| \ge \frac{|\xi|}{2}, |\xi - q| \ge 2|\xi|} M_{ijk}(\xi) v^{i}(\xi, s) v^{j}(\xi, s)$$

The corresponding I_b , II_b , III_b and IV_b are the same as their a counterparts, except that the role of q and $\xi - q$ is interchanged. Noticed that by the triangular inequality not both q and $\xi - q$ can be less than $\frac{|\xi|}{2}$, and hence all possible cases are covered. Since $|q| < \sqrt{|\xi|} < \frac{|\xi|}{2}$, and hence $|\xi - q| \ge \frac{|\xi|}{2}$. Hence we have,

$$|I_a| \leq |\xi| \sum_{1 \leq |q| \leq \sqrt{|\xi|}} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi - q|^{2+\eta}}$$

$$\leq |\xi| \frac{2^{2+\eta}D^2}{|\xi|^{2+\eta}} \sum_{1 \leq |q| \leq \sqrt{|\xi|}} \frac{D}{|q|^2}$$
and by inequality (2)

and by inequality (3)

$$\leq \qquad |\xi|\,\frac{2^{2+\eta}D^2}{|\xi|^{2+\eta}}\sqrt{|\xi|} = \frac{2^{2+\eta}D^2}{|\xi|^{\frac{1}{2}+\eta}}.$$

Estimating II_a and III_a is pretty straightforward, via the inequality

$$\sum_{1 \le |q| \le r} 1 \le cr^3.$$

Indeed,

$$|II_{a}| \leq |\xi| \frac{2^{2+\eta}D}{|\xi|^{2+\eta}} \cdot \frac{D}{\left(\sqrt{|\xi|}\right)^{2+\eta}} \left(\sum_{|q| \leq \frac{|\xi|}{2}} 1\right)$$

$$\leq \frac{2^{2+\eta}D}{|\xi|^{1+\eta}} \cdot \frac{D}{|\xi|^{1+\frac{\eta}{2}}} |\xi|^{3} = \frac{2^{2+\eta}D^{2}}{|\xi|^{\frac{3}{2}\eta}}.$$

$$|III_{a}| \leq |\xi| \sum_{|q| \geq \frac{|\xi|}{2}, \frac{|\xi|}{2} \leq |\xi - q| < 2|\xi|} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi - q|^{2+\eta}}$$

$$\leq |\xi| \cdot \frac{2^{2+\eta} D^{2}}{|\xi|^{4+2\eta}} \left(\sum_{1 \leq |\xi - q| < 2|\xi|} 1 \right)$$

$$\leq \frac{2^{2+\eta}}{|\xi|^{2\eta}}$$

Finally, using that $|\xi - q| \ge 2|\xi|$ and $|q| \ge \frac{|\xi|}{2}$ imply that $|q| \ge \frac{2}{3}|\xi - q|$ and inequality (4) we can bound IV_a as follows,

$$|IV_{a}| \leq |\xi| \sum_{|q| \geq \frac{|\xi|}{2}, |\xi - q| \geq 2|\xi|} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi - q|^{2+\eta}}$$

$$\leq \frac{2^{2\eta}}{|\xi|^{2\eta}} \left(\frac{3}{2}\right)^{2+\eta} \sum_{|q| \geq \frac{|\xi|}{2}} \frac{D^{2}}{|q|^{4}}$$

$$\leq |\xi| \frac{1}{|\xi|^{2\eta}} \frac{D^{2}}{|\xi|} = \frac{D^{2}}{|\xi|^{2\eta}}.$$

The proof is now complete.

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