ON SELBERG'S THEOREM C IN THE THEORY OF THE RIEMANN ZETA-FUNCTION

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ABSTRACT. In this paper we obtain new theorems about classes of exceptional sets for the Selberg's theorem C (1942). Our theorems, as based on discrete method, are not accessible for Karatsuba's theory (1984) since this theory is a continuous theory. This paper is English version of our paper [8], the results of our paper [9] are added too.

1. Introduction

- 1.1. We use the following notions. Let
 - (a) $\psi(t)$ be a positive increasing to infinity function such that

$$\psi(t) \le \sqrt{\ln t},$$

(b) S be the set of values of t

$$(1.1) t \in [T, T + T^{1/2 + \epsilon}]$$

for which there is at least one zero point of the function

$$\zeta\left(\frac{1}{2}+it\right)$$

within the interval

(1.3)
$$\left(t, t + \frac{\psi(t)}{\ln t}\right), \quad S = S(T, \epsilon, \psi).$$

Let us remind that the set of segments (1.1) for every small and fixed $\epsilon > 0$ is the minimal set for the Selberg's theory. It is the assertion of the Selberg's C theorem (see [10], p. 49) relative to segment (1.1)

(1.4)
$$m(S) \sim T^{1/2+\epsilon}, \quad T \to \infty,$$

that is, the measure of the set \bar{S} of such values

$$t \in [T, T + T^{1/2 + \epsilon}]$$

for which there is no zero of the function (1.2) in the interval (1.3) is

(1.5)
$$m(\bar{S}) = o(T^{1/2+\epsilon}), \quad T \to \infty.$$

1.2. Next, let

$$\{g_{\nu}\}$$

denote the sequence that is defined by the formula

$$\vartheta_1(g_{\nu}) = \frac{\pi}{2}\nu, \quad \nu = 1, 2, \dots$$

(see [6], $\bar{t}_{\nu} = g_{\nu}$, comp. [3], [4]), where

$$\vartheta_1(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8}.$$

Remark 1. Since the set

(1.6)
$$W = \{g_{\nu}: g_{\nu} \in [T, T + T^{1/2 + \epsilon}]\}$$

is the finite one, then

$$(1.7) m(W) = 0.$$

Consequently, we have the following: no information is contained in the Selberg's theorem C (comp. (1.5), (1.7)) about the zeros of odd order of the function (1.2) in the intervals

$$\left(g_{\nu}, g_{\nu} + \frac{\psi(g_{\nu})}{\ln g_{\nu}}\right), \ g_{\nu} \in W$$

that is the set W is the exceptional set for the Selberg's theorem C.

- 1.3. Now, let us remind the following deep methods of the English mathematicians:
 - (a) continuous method of Hardy-Littlewood (see [1]),
 - (b) discrete method of E.C. Titchmarsh (see [11]).

In our papers [6], [7], we have constructed a discrete analogue of the Hardy-Littlewood continuous method (that is, some synthesis of (a) and (b)). Especially, we have obtained the following estimate (see [7])

$$(1.8) N_0(T + T^{5/12}\psi \ln^3 T) - N_0(T) > A(\psi)T^{5/12}\psi \ln^3 T,$$

where $N_0(T)$ denotes the number of zeros of the function

$$\zeta\left(\frac{1}{2}+it\right), \quad t\in(0,T],$$

and $A(\psi)$ is the constant that depends on choice of ψ , for example, if

$$\psi = \ln \ln \ln T$$
,

then

$$A(\ln \ln \ln T)$$

is an absolute constant.

Remark 2. We notice explicitly that

(a) our improvement of the classical Hardy-Littlewood exponent $\frac{1}{2}$

$$\frac{1}{2} \longrightarrow \frac{5}{12}$$

is a 16.6% change after 61 years,

(b) the estimate (1.8) was the first step on a way to proof of the Selberg's hypothesis (see [10], p. 5, comp. [2], pp. 37,39).

Remark 3. Let us notice that I have sent the manuscripts of my papers [6], [7] to A.A. Karatsuba in the beginning of 1981.

After this analysis, it is clear that our estimate (1.8) is in need of a corresponding analogue of the Selberg's theorem C. Consequently, in this paper we shall prove an analogue of that theorem for finite set W_1 (and also for others) of values

$$g_{\nu} \in [T, T + T^{5/12}\psi \ln^3 T]; \quad m(W_1) = 0,$$

i. e. for the exceptional set in the sense of the Selberg's theorem.

2. Theorem 1

2.1. Let (see [6], (2.5))

(2.1)
$$\omega = \frac{\pi}{\ln \frac{T}{2\pi}} = \frac{\pi}{2 \ln P_0}, \ U = T^{5/12} \psi \ln^3 T,$$
$$\ln T < M < \sqrt[3]{T} \ln T.$$

Next, let

$$\bar{\psi}(t)$$

be the function of the same kind as $\psi(t)$ and fulfilling the condition

(2.2)
$$\frac{\bar{\psi}}{\sqrt[3]{\bar{\psi}}} = o(1), \ T \to \infty,$$

and let

$$G(T, \psi, \bar{\psi})$$

denote the number of such

$$g_{\nu} \in [T, T+U]$$

that the interval

$$(g_{\nu}, g_{\nu} + \bar{\psi}(g_{\nu}))$$

contains a zero of the odd order of the function

$$\zeta\left(\frac{1}{2}+it\right),\ t\in[T,T+U].$$

The following theorem holds true.

Theorem 1.

(2.4)
$$G(T, \psi, \bar{\psi}) \sim \frac{1}{\pi} U \ln T, \ T \to \infty.$$

Remark 4. Since (see [6], (8))

(2.5)
$$\sum_{T \le g_{\nu} \le T + U} 1 \sim \frac{1}{\pi} U \ln T, \ T \to \infty$$

then we have by Theorem 1 that for almost all

$$g_{\nu} \in [T, T+U]$$

the interval (2.3) contains a zero of the odd order of the function

$$\zeta\left(\frac{1}{2}+it\right)$$
.

Remark 5. Let N(T) denote the number of zeros of the function

$$\zeta(s), \ s = \sigma + it, \ \sigma \in (0,1), \ t \in (0,T].$$

It is then true that (comp. [12], p. 181)

(2.6)
$$N(T+U) - N(T) \sim \frac{1}{2\pi} U \ln T, \ T \to \infty.$$

Of course, our formula (2.4) is not in a contradiction with the formula (2.6) since many of intervals (2.3) can intersect.

Remark 6. Let us notice explicitly that also for the theory of Karatsuba giving the estimate (comp. [2], p. 39)

$$N_0(T+T^{27/82+\epsilon})-N_0(T)>A(\epsilon)T^{27/82+\epsilon},$$

we have that the set of values

$$g_{\nu} \in [T, T + T^{27/82 + \epsilon}]$$

is the exceptional set, since the theory is *continuous* as well as the classical theories of Hardy-Littlewood and Selberg. Consequently, our Theorem 1 is not improvable also by Karatsuba's theory.

2.2. Now we give the proof of Theorem 1. The basic point of the proof is the estimate (see [7], (3.16))

$$(2.7) R < A \frac{U \ln^2 T}{M}$$

where R denotes the number of such

$$g_{\nu}^* \in [T, T+U]$$

for which the sequence

$$\{Z(g_{\nu}^* + k\omega)\}_{k=1}^M$$

preserves the sign (comp. [7], (3.9), (3.11)). Next,

$$\bar{\psi}(g_{\nu}) \geq \bar{\psi}(T), \ g_{\nu} \in [T, T+U],$$

and

$$\frac{\bar{\psi}(T)}{\omega} \sim \frac{1}{\pi} \bar{\psi} \ln T > \frac{1}{2\pi} \bar{\psi} \ln T \ge \left[\frac{1}{2\pi} \bar{\psi} \ln T \right] = M_1,$$

of course,

$$M_1 \in (\ln T, \sqrt[3]{\psi} \ln T)$$

(see (2.1) - inequalities for M and (2.2)). Putting $M = M_1$ in (2.7) one obtains

$$R = o(U \ln T).$$

Now, the formula (2.4) follows from the previous by (2.5).

3. Lemmas about translations $g_{\nu} \longrightarrow g_{\nu}(\tau), \tau \in [-\pi, \pi]$

3.1. Let

$$\{g_{\nu}(\tau)\}$$

denote the infinite set of sequences which are defined (comp. [5]) by the formula

$$\vartheta_1[g_{\nu}(\tau)] = \frac{\pi}{2}\nu + \frac{\tau}{2}, \ \nu = 1, 2, \dots, \ \tau \in [-\pi, \pi],$$

where, of course,

$$g_{\nu}(0) = g_{\nu}.$$

Now, we shall study how the lemmas from the papers [6], [7] are sensitive with respect to the translations

$$g_{\nu} \longrightarrow g_{\nu}(\tau), g_{\nu} \in [T, T + U], \ \tau \in [-\pi, \pi].$$

First of all we have (comp. [6], (22) - (36)) the following

Lemma \bar{A} .

$$g_{\bar{\nu}_1+p+1}(\tau) = g_{\bar{\nu}_1}(\tau) + \bar{\omega}_0 p - \bar{\omega}_0 D(p) + \mathcal{O}\left(\frac{U^3}{T^2 \ln T}\right),$$

 $p = 0, 1, \dots, N_1 - 1,$

where (comp. [6], (11), (12))

$$\begin{split} \bar{\omega}_0 &= \frac{\pi}{\ln \frac{T}{2\pi}} - \frac{\pi^2}{2} \frac{1}{T \ln^3 \frac{T}{2\pi}} - \pi \frac{g_{\bar{\nu}_1}(\tau) - T}{T \ln^2 \frac{T}{2\pi}}, \\ Q &= Q(T) = \frac{\pi}{T \ln^2 \frac{T}{2\pi}}, \\ D(p) &= \sum_{q=1}^p \{1 - (1 - Q)^q\}, 1 \le p \le N_1 - 1, \ D(0) = 0, \\ g_{\bar{\nu}_1}(\tau) &= \min_{g_{\nu}(\tau) \in [T, T + U]} \{g_{\nu}(\tau)\}, \\ g_{\bar{\nu}_1 + N_1}(\tau) &= \max_{q_{\nu}(\tau) \in [T, T + U]} \{g_{\nu}(\tau)\}, \ \bar{\nu}_1 = \bar{\nu}_1(\tau), \ N_1 = N_1(\tau), \end{split}$$

and the \mathcal{O} is valid uniformly for $\tau \in [-\pi, \pi]$.

3.2. Next, since (see [6], (118))

(3.1)
$$\bar{\vartheta}_{1,k} = \vartheta_1[g_{\nu}(\tau) + k\omega] = \frac{\pi}{2}\nu + \frac{\tau}{2} + k\omega \ln P_0 + \mathcal{O}\left(\frac{MU}{T \ln T}\right),$$

then (comp. [6], (121))

$$Z[g_{\nu}(\tau) + k\omega] \cdot Z[g_{\nu}(\tau) + l\omega] =$$

$$= 2\sum_{m,n < P_{0}} \frac{1}{\sqrt{nm}} \cos\{g_{\nu}(\tau) \ln \frac{n}{m} + k\omega \ln \frac{P_{0}}{n} - l\omega \frac{P_{0}}{m}\} +$$

$$+ 2\sum_{m,n < P_{0}} \frac{(-1)^{\nu}}{\sqrt{nm}} \cos\{g_{\nu}(\tau) \ln(mn) - \tau - k\omega \ln \frac{P_{0}}{n} - l\omega \frac{P_{0}}{m}\} +$$

$$+ \mathcal{O}\left(\frac{MU}{\sqrt{T} \ln T}\right) + \mathcal{O}(T^{-1/12} \ln T),$$

and the \mathcal{O} -estimates in (3.1), (3.2) are valid uniformly for $\tau \in [-\pi, \pi]$.

Now, we put (see (3.2), comp. [6], (16), (17))

$$\bar{S}_1(T, U, M, \tau) = \sum_{m < n < P_0} \frac{1}{\sqrt{mn}} \sum_{T \le g_{\nu}(\tau) \le T + U} \cos\left\{g_{\nu}(\tau) \ln\frac{n}{m} + \varphi_1\right\},\,$$

where

$$\varphi_1 = k\omega \ln \frac{P_0}{m} - l\omega \ln \frac{P_0}{n},$$

and also (comp. [6], (19), (20))

$$\bar{S}_2(T, U, M, \tau) = \sum_{m < n < P_0} \frac{1}{\sqrt{mn}} \sum_{T < q_{\nu}(\tau) < T + U} (-1)^{\nu} \cos\{g_{\nu}(\tau) \ln(mn) + \bar{\varphi}_2\},$$

where

$$\bar{\varphi}_2 = -k\omega \ln \frac{P_0}{n} - l\omega \ln \frac{P_0}{m} - \tau = \varphi_2 - \tau.$$

Since

$$T \le g_{\nu}(\tau) \le T + U,$$

and

$$\bar{S}_2(T, U, M, \tau) = \operatorname{Re} \left\{ e^{-i\tau} \sum_{m,n < P_0} \frac{1}{\sqrt{mn}} \times \sum_{T \le g_{\nu}(\tau) \le T + U} (-1)^{\nu} \exp\{i[g_{\nu}(\tau) \ln(mn) + \varphi_2]\} \right\}$$

then the following estimates (comp. [6], (18), (21), (37) – (93)) hold true

Lemma \bar{B} .

$$\bar{S}_1(T, U, M, \tau) = \mathcal{O}(MT^{5/12} \ln^3 T)$$

uniformly for $\tau \in [-\pi, \pi]$.

Lemma \bar{C} .

$$\bar{S}_2(T, U, M, \tau) = \mathcal{O}(T^{5/12} \ln^2 T)$$

uniformly for $\tau \in [-\pi, \pi]$.

3.3. Let (see [6], (3))

$$\bar{J} = \bar{J}(T, U, M, \tau) = \sum_{T \le g_{\nu}(\tau) \le T + U} \left\{ \sum_{k=0}^{M} Z[g_{\nu}(\tau) + k\omega] \right\}^{2}.$$

Now we obtain by method [6], (94) - (127) the following

Lemma $\bar{\alpha}$.

$$\bar{J} = AMU \ln^2 T + o(MU \ln^2 T),$$

(A > 0 is an absolute constant) uniformly for $\tau \in [-\pi, \pi]$.

Next, we have, instead of [7], (5.1), (5.2), the following

$$4\cos\bar{\vartheta}_k\cos\bar{\vartheta}_l\cos(\bar{\vartheta}_k-\bar{\vartheta}_l) =$$

$$= 1 + (-1)^{k+l} + (-1)^{\nu+k} \cos \tau + (-1)^{\nu+l} \cos \tau + \mathcal{O}\left(\frac{MU}{T \ln T}\right),$$
$$-4\cos^2 \bar{\vartheta}_k = -2 - 2(-1)^{\nu+k} \cos \tau + \mathcal{O}\left(\frac{MU}{T \ln T}\right).$$

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Now, putting (comp. [7], (3.4), (3.5))

$$\begin{split} \bar{N} &= \sum_{T \leq g_{\nu}(\tau) \leq T + U} |\bar{K}|^2, \\ \bar{K} &= \sum_{k=0}^{M} \left\{ e^{-i\vartheta[g_{\nu}(\tau) + k\omega]} Z[g_{\nu}(\tau) + k\omega] - 1 \right\}, \end{split}$$

we obtain by method [7], (4.1) - (7.3) the following

Lemma $\bar{\beta}$.

$$\bar{N} = \mathcal{O}(MU \ln^2 T)$$

uniformly for $\tau \in [-\pi, \pi]$.

- 4. Two theorems connected with translations $g_{\nu} \longrightarrow g_{\nu}(\tau), \tau \in [-\pi, \pi]$
- 4.1. First of all we give (comp. [7], (2.6) (2.8)) the following

Definition 1. We shall call the segment

$$[g_{\nu}(\tau) + k(\nu)\omega, g_{\nu}(\tau) + (k(\nu) + 1)\omega]$$

where

$$g_{\nu}(\tau) \in [T, T + U], \ \tau \in [-\pi, \pi], \ 0 \le k(\nu) \le M_2 = [\delta \ln T], \ \delta > 1,$$

and $k(\nu) \in \mathbb{N}_0$ as the good segment (comp. [7], [11]) if

$$Z[g_{\nu}(\tau) + k(\nu)\omega] \cdot Z[g_{\nu}(\tau) + (k(\nu) + 1)\omega] < 0.$$

Next, let

$$G_1(T, U, \delta, \tau)$$

denote the number of non-intersecting good segments within the interval [T, T+U]. Then we obtain, similarly to [7], (3.7), (3.20), the following result

Theorem 2. There are

$$\delta_0 > 1$$
, $A(\psi, \delta_0) > 0$, $T_0(\psi, \delta_0) > 0$

such that

$$(4.1) G_1(T, U, \delta_0, \tau) > A(\psi, \delta_0)U, \ T \ge T_0(\psi, \delta_0)$$

for all $\tau \in [-\pi, \pi]$.

Remark 7. We notice explicitly that the estimate [7], (2.9) concerning the number of good segments (relatively to $\{g_{\nu}\}$) is invariant with respect to translations

$$g_{\nu} \longrightarrow g_{\nu}(\tau), \ \tau \in [-\pi, \pi], \ g_{\nu} \in [T, T + U].$$

4.2. Above listed facts make clear that we have obtained a kind of generalization of our Theorem 1. Namely, let

$$G_2(T, \psi, \bar{\psi}, \tau)$$

stand for the number of values

$$q_{\nu}(\tau) \in [T, T+U]$$

such that the interval

$$(4.2) (g_{\nu}(\tau), g_{\nu}(\tau) + \bar{\psi}[g_{\nu}(\tau)])$$

contains a zero of the odd order of the function

$$\zeta\left(\frac{1}{2}+it\right).$$

Then the following theorem holds true.

Theorem 3.

$$G_2(T, \psi, \bar{\psi}, \tau) \sim \frac{1}{\pi} U \ln T, \ T \to \infty, \ \tau \in [-\pi, \pi].$$

5. Remarks on Selberg's theorems about zeros of function $\zeta\left(\frac{1}{2}+it\right)$

We have given a discrete commentary to fundamental Selberg's memoir [10] in our paper [9]. Here we put two results from our paper [9].

5.1. Let

$$(5.1) H_1 \in [a_1, a_2\sqrt{\ln P_0}],$$

where

$$a_1 = \frac{10}{\pi \epsilon}, \quad a_2 = a_1 \sqrt{\frac{2}{\pi}}, \ H_1 \in \mathbb{N}.$$

The origin of (5.1) is as follows: we put

$$\omega = \frac{\pi}{2 \ln P_0}, \ H_1 \omega = H, \ \xi = \left(\frac{T}{2\pi}\right)^{\epsilon/10} = P_0^{\epsilon/5}, \ \epsilon \le \frac{1}{10},$$

(see [9], (9)), and further, we assume that

$$\frac{1}{\ln \xi} \le H \le \frac{1}{\sqrt{\ln \xi}},$$

(see [9], (10)).

Definition 2. We shall call the segment

$$[g_{\nu}(\tau) + (k(\nu,\tau) - 1)\omega, g_{\nu}(\tau) + k(\nu,\tau)\omega],$$

where

$$g_{\nu}(\tau) \in [T, T + U], \ 1 \le k(\nu, \tau) \le N_1,$$

and $k(\nu, \tau) \in \mathbb{N}$ as the good segment (see [9], p. 113) if

$$Z[q_{\nu}(\tau) + (k(\nu, \tau) - 1)\omega] \cdot Z[q_{\nu}(\tau) + k(\nu, \tau)\omega] < 0.$$

Next, let

$$G_3(T, U, H_1, \tau)$$

denote the number of non-intersecting good segments within the interval [T, T+U]. Then the following theorem holds true.

Theorem 4. There are

$$\bar{H}_1 \in [a_1, a_2\sqrt{\ln P_0}], \ A(\epsilon) > 0, \ T_0(\epsilon) > 0$$

such that

(5.2)
$$G_3(T, U, \tau) > A(\epsilon)U \ln T, \ T \ge T_0(\epsilon),$$

where, of course,

$$G_3(T, U, \tau) = G_3(T, U, \bar{H}_1, \tau)$$

Remark 8. Since

$$(5.3) G_3(T, U, \tau) < AU \ln T,$$

then the order of G_3 is $U \ln T$ for every fixed $\tau \in [-\pi, \pi]$ (comp. (5.2), (5.3)). We shall call this property as generalized Gram's law for the set of sequences $\{g_{\nu}(\tau)\}$.

Now, we obtain the following from our Theorem 4.

Corollary.

$$N_0(T + T^{1/2+\epsilon}) - N_0(T) > A(\epsilon)T^{1/2+\epsilon} \ln T$$

i. e. the Selberg's theorem A (see [10], p. 46, $a = 1/2 + \epsilon$).

Remark 9. Consequently, the generalized Gram's law is the discrete basis of fundamental Selberg's theorem A.

Remark 10. Our Theorem 4 is also not improvable by the Karatsuba theory.

Remark 11. Finally, we mention that our Theorem 3 is valid also for the intervals of the following form

$$\left(g_{\nu}(\tau), g_{\nu}(\tau) + \frac{\psi[g_{\nu}(\tau)]}{\ln g_{\nu}(\tau)}\right), \ g_{\nu}(\tau) \in [T, T + T^{1/2 + \epsilon} \ln T),$$

(comp. [9], p. 112). i.e. for our system of exceptional sets, where

$$a_1 \le \left\lceil \frac{\psi(\tau)}{2\pi} \right\rceil \le a_2 \sqrt{\ln P_0},$$

as a discrete analogue of the Selberg's theorem C.

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