Coalescence Model of Rock-Paper-Scissors Particles

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Abstract The rock-paper-scissors game, commonly played in East Asia, gives a simple model to understand physical, biological, psychological and other problems. The interacting rock-paper-scissors particle system is a point of contact between the kinetic theory of gases by Maxwell and Boltzmann (collision model) and the coagulation theory by Smoluchowski (coalescence model). A 2s+1 types extended rock-paper-scissors collision model naturally introduces a nonlinear integrable system. The time evolution of the 2s+1 types extended rock-paper-scissors coalescence model is obtained from the logarithmic time change of the nonlinear integrable system. We also discuss the behavior of a discrete rock-paper-scissors coalescence model.

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1 Introduction

The rock-paper-scissors game is commonly played in East Asia. The cyclic dominance systems naturally occurs in biological systems as color morphisms of the side-blotched lizard [23] and strains of Escherichia coli [19]. The types, rock paper and scissors, can also represent social groups, opinions, or survival strategies of organisms in a preferential attachment graph model [9].

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Considering simple models [18] for kinetic theory of gases by Maxwell and Boltzmann, we introduced a collision model of rock (type 1), paper (type 2), and scissors (type 3) particles with cyclic dominance Fig. 1, where 2 dominates 1, 3 dominates 2, and 1 dominates 3, and obtained a Lotka-Volterra equation [10] (the Boltzmann equation for the rock-paper-scissors particles). We can extend the argument to the 2s + 1 types rock-paper-scissors particles [13], which gives a nonlinear integrable system [3, 15, 17, 22] with s + 1 conserved quantities like the Toda lattice [28] and the Calogero system [4]. For the case of finite number of particles, the probability of coexistence of types is obtained by using s + 1 martingales, which are stochastic version of the s + 1 conserved quantities [11, 14]. The rock-paper-scissors lattice model greatly enriches the dynamics as studied in the physics literatures [7, 8, 20, 25, 26, 27].

We introduce a coalescence model of rock-paper-scissors particles as in Fig. 2. For a given initial distribution of the particles of rock (type 1), paper (type 2) and scissors (type 3), what type of the particle will finally survive? As we see in Fig. 3 and Fig. 4 it gives a leader selection problem, which is for another aspect of the previously studied problem[6]. We carried out simulation studies for the coalescence model of rock-paper-scissors particles [12] for finite size fluctuation, where the total number of particles decreases at each step. Cyclic trapping reactions for finite size fluctuations [2] gives insights to the behavior of our model for finite size.

Here we study the time evolution of the coalescence model of rock-paperscissors particles with sufficiently large number of particles. We apply the study of coalescence of clusters [1, 21, 24] to our problem. Let us follow the argument [21] for the simplest case. Infinite set of master equations that describe how the cluster mass distribution $n_k(t)$ evolves by the rule

$$A_i + A_j \to A_{i+j},\tag{1}$$

in which clusters of mass i and j irreversibly join to form a cluster of mass i + j, is given by

$$\frac{d n_k(t)}{dt} = 1/2 \sum_{i+j=k} n_i n_j - \sum_{j=1}^{\infty} n_j n_k.$$
 (2)

The first term on the right-hand side is for the creation of a k-mers due to the coalescence of two clusters of mass i and j, we sum over all such pairs

with i + j = k. The factor 1/2 in the gain term is needed to avoid double counting.

Instead of Eq. (1), the rock-paper-scissors coalescence in Fig. 2 is represented as,

$$A_i + A_j \to \begin{cases} A_i & \text{if } i - j \equiv 0, 1 \pmod{3}, \\ A_j & \text{if } i - j \equiv 2 \pmod{3}. \end{cases}$$
 (3)

There are several solvable cases for the Smoluchowski coalescence equations [1] for the size of clusters. The "Boltzmann equation" for the collision model of the extended 2s+1 types rock-paper-scissors particles is a nonlinear integrable system [3, 15, 17, 22]. Consider the time evolution of the relative abundance (concentration ratio), as in [21], for the "Smoluchowski equation" of the coalescence of extended 2s+1 types rock-paper-scissors particles. It is obtained by a logarithmic time change of the nonlinear integrable system, as will be shown in Section 4.

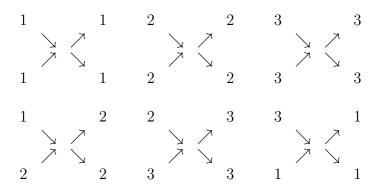


Figure 1: Rock-paper-scissors collision model [10]

2 Smoluchowski equation for coalescence of rock-paper-scissors particles

The Smoluchowski coagulation model is for the time evolution of distribution of cluster size as Eq. (1), while our coalescence model given by the following

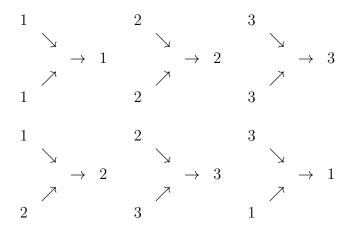


Figure 2: Rock-paper-scissors coalescence model [11].

1), 2), 3) and 4) is for the distribution of particle types of rock, paper and scissors.

Coalescence model of rock-paper-scissors particles:

- 1) There are 3 types (rock, paper, scissors) of particles 1, 2, 3 whose numbers of particles at time t, are $n_1(t), n_2(t), n_3(t)$ respectively, for which $n_1(t) + n_2(t) + n_3(t) = n(t)$.
- 2) Each of n(t) particles coalescence with other particles **n(t) dt** times on the average in [t, t + dt].
- 3) Each particle is in a chaotic bath of particles. Each coalescence pair is equally likely to be chosen.
- 4) By a coalescence of a particle of type i and a particle of type j, the two particles become one particle of type i, if $i j \equiv 0, 1 \pmod{3}$, otherwise become one particle of type j as Eq.(3), as also shown in Fig. 2.

Examples of trees generated by our coalescence model are shown in Fig. 3 and Fig. 4.

Now we derive the master equation. One of particles of type 1 and the other particle of type 1 are chosen at random with probability $(\frac{n_1(t)}{n}dt)n_1(t)$ and coalescent with each other and then become one particle of type 1. One particle of type 1 and one particle of type 3 are chosen at random

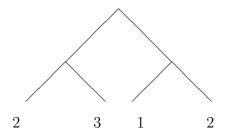


Figure 3: Possible tree of coalescence model: starting from n(0) = 4 particle. By the coalescence of a particle of 2 (paper) and the particle of 3 (scissors) the particle of 3 survives. By the coalescence of a particle of 1 (rock) and the particle of 2 the particle of 2 survives. By the coalescence of the two survivors the particle of type 3 (scissors) survives finally.

 $(\frac{n_3(t)}{n}dt)n_1(t)$ and coalescent with each other and then become a particle of type 1. Consider

$$dn_1(t) = n_1(t + dt) - n_1(t). (4)$$

We have

$$n_1(t+dt) = \left(\frac{1}{2}\frac{n_1(t)}{n}n(t)\ dt\right)n_1(t) + \frac{n_3(t)}{n}n(t)\ dt\right)n_1(t) + (1-n(t)\ dt)n_1(t). \tag{5}$$

Hence we have

$$dn_1(t) = \left(\frac{1}{2}\frac{n_1(t)}{n}n(t) dt\right)n_1(t) + \frac{n_3(t)}{n}n(t) dt)n_1(t) - n(t) dtn_1(t)$$
(6)
= $-\frac{1}{2}(n_1(t)n_1(t) + 2n_1(t)n_2(t))dt,$ (7)

which shows the master equation

$$\begin{cases} \frac{\partial n_1(t)}{\partial t} &= -\frac{1}{2}n_1(t)(2n_2(t) + n_1(t))\\ \frac{\partial n_2(t)}{\partial t} &= -\frac{1}{2}n_2(t)(2n_3(t) + n_2(t))\\ \frac{\partial n_3(t)}{\partial t} &= -\frac{1}{2}n_3(t)(2n_1(t) + n_3(t)), \end{cases}$$
(8)

which is written for Eq. (3) as,

$$\frac{dn_k(t)}{dt} = 1/2 \sum_{A_i + A_j \to A_k} n_i n_j - \sum_{j=1}^3 n_j n_k,$$
 (9)

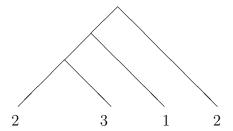


Figure 4: Possible tree of coalescence model: starting from n(0) = 4 particle, by the coalescence of a particle of 2 (paper) and a particle of 3 (scissors), the particle of 3 survives, then by the coalescence with the particle of 1 (rock), the particle of 1 survives. By the coalescence of the particle of 1 and the particle of 2 the particle of 2 (paper) survives finally.

which will be studied for a general 2s + 1 hands rock-paper-scissors particles in Section 4.

Since the coalescence, given in Fig. 2, is a binary interaction, from the above 2) it occurs $\frac{n(t)}{2}n(t)$ dt times in [t, t+dt], as will be mentioned in later Section 5.

3 Collision model of 2s + 1types rock-paperscissors particles

Consider the model defined by the following 1), 2), 3), and 4). Collision model of 2s + 1 types rock-paper-scissors particles:

- 1) There are 2s+1 types rock-paper-scissors particles 1, 2, ..., 2s+1 whose abundances at time t, are $n_1(t), n_2(t), ..., n_{2s+1}(t)$ respectively, for which $n_1(t) + n_2(t) + ... + n_{2s+1}(t) = n(t)$.
- 2) Each particle collides with other particles dt times on the average per time length dt.
- 3) Each particle is in a chaotic bath of particles. Each colliding pair is equally likely to be chosen.
- 4) By a collision a particle of type i and a particle of type j become two particles of type i, if $i j \equiv 0, 1, ..., s \pmod{2s+1}$, otherwise become two particles of type j as Eq.(10), as also shown in Fig. 1 for the case s = 1.

For the collision model of 2s+1 types of rock-paper-scissors particles, we extend the rule of the rock-paper-scissors particles by the cyclic dominance rule,

$$A_i + A_j \to \begin{cases} 2A_i & \text{if } i - j \equiv 0, ..., s \text{ (mod } 2s + 1) \\ 2A_j & \text{if } i - j \equiv s + 1, ..., 2s \text{ (mod } 2s + 1), \end{cases}$$
 (10)

as shown in Fig. 1 for the case s = 1.

The total number of particles n(t) is time invariant. The relative abundance for each type of particles [10, 13] is given by the master equation

$$\frac{\partial}{\partial u} P_i(u) = P_i(u) \left(\sum_{j=1}^s P_{i-j}(u) - \sum_{j=1}^s P_{i+j}(u) \right). \tag{11}$$

Consider 2r+1 types out of the 2s+1 types $A_1, ..., A_{2s+1}$. If each of the 2r+1 types dominates the other r types and is dominated by the other remained r types, then we say the 2r+1 types are in a regular tournament.

Take 2r+1 particles at random from our system of the 2s+1 types $A_1,...,A_{2s+1}$ of particles with Eq. (11). Let $H_r(P(t))$ for $P(t) \equiv (P_1(t),...,P_{2s+1}(t))$, be the probability that the 2r+1 particles whose corresponding 2r+1 types are in a regular tournament. Then we have the s+1 conserved quantities [15]

$$H_r(\vec{P(t)}) = H_r(\vec{P(0)}), \text{ for } r = 0, ..., s.$$
 (12)

For example, for the case 2s + 1 = 5, we have the conserved quantities

$$H_0(\vec{P(t)}) = \sum_{i=1}^{5} P_i(t)$$
 (13)

$$H_1(\vec{P(t)}) = \sum_{i=1}^{5} P_i(t) P_{i+1} P_{i+3}$$
(14)

$$H_2(\vec{P(t)}) = P_1(t)P_2(t)P_3(t)P_4(t)P_5(t). \tag{15}$$

Eq. (11) is a nonlinear integrable system [3, 15, 17, 22].

4 Coalescence model of 2s+1-types rock-paperscissors particles

We consider the coalescence model defined by the following 1), 2), 3) and 4).

Coalescence model of 2s + 1-types rock-paper-scissors particles:

- 1) There are 2s+1 types rock-paper-scissors particles 1, 2, ..., 2s+1 whose abundances at time t, are $n_1(t), n_2(t), ..., n_{2s+1}(t)$ respectively, for which $n_1(t) + n_2(t) + ... + n_{2s+1}(t) = n(t)$.
- 2) Each particle coalescence with other particles n(t) dt times on the average in [t, t + dt].
- 3) Each particle is in a chaotic bath of particles. Each coalescence pair is equally likely to be chosen.
- 4) By a coalescence a particle of type i and a particle of type j become one particles of type i, if $i-j \equiv 0, 1, ..., s \pmod{2s+1}$, otherwise become one particle of type j as

$$A_i + A_j \to \begin{cases} A_i & \text{if } i - j \equiv 0, ..., s \pmod{2s+1} \\ A_j & \text{if } i - j \equiv s+1, ..., 2s \pmod{2s+1}, \end{cases}$$
 (16)

We have for i = 1, 2, ..., 2s + 1,

$$\frac{\partial n_i(t)}{\partial t} = -\frac{1}{2}n_i(t)(n_i(t) + 2\sum_{j=1}^s n_{i+j}(t)),\tag{17}$$

which shows

$$\frac{\partial n_i(t)}{\partial t} = \frac{1}{2} n_i(t) \left(\sum_{i=1}^s n_{i-j}(t) - \sum_{i=1}^s n_{i+j}(t) \right) - \frac{1}{2} n_i(t) n(t)$$
 (18)

where $n(t) = n_1(t) + n_2(t) + \dots + n_{2s+1}(t)$

Since

$$\frac{\partial n(t)}{\partial t} = -\frac{1}{2}n(t)^2, \tag{19}$$

we have

$$n(t) = \frac{2}{t + 2/n(0)}. (20)$$

As in [21], "one useful trick that often simplifies master equations is to eliminate the loss terms by considering concentration ratios, rather than the concentrations themselves". We have from Eq. (18)

$$\frac{\partial}{\partial t} \frac{n_i(t)}{n(t)} = \frac{1}{2} n(t) \frac{n_i(t)}{n(t)} \left(\sum_{i=1}^s \frac{n_{i-j}(t)}{n(t)} - \sum_{i=1}^s \frac{n_{i+j}(t)}{n(t)} \right). \tag{21}$$

Putting $\frac{n_i(t)}{n(t)} = Q_i(t)$, i = 1, ..., 2s + 1, we have

$$\frac{\partial}{\partial t}Q_i(t) = \frac{1}{t + \frac{2}{n(0)}}Q_i(t)(\sum_{j=1}^s Q_{i-j}(t) - \sum_{j=1}^s Q_{i+j}(t)). \tag{22}$$

For $u = \log[t + \frac{2}{n(0)}]$, considering

$$e^u = t + \frac{2}{n(0)} \tag{23}$$

we have

$$\frac{\partial}{\partial t}Q_i(e^u - \frac{2}{n(0)})\tag{24}$$

$$= \frac{1}{e^u} Q_i (e^u - \frac{2}{n(0)}) \left(\sum_{i=1}^s Q_{i-j} (e^u - \frac{2}{n(0)})\right)$$
 (25)

$$-\sum_{j=1}^{s} Q_{i+j} \left(e^{u} - \frac{2}{n(0)}\right). \tag{26}$$

Since

$$\frac{\partial u}{\partial t} = \frac{1}{t + \frac{2}{n(0)}} = \frac{1}{e^u} \text{ and } \frac{\partial}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial}{\partial u} = \frac{1}{e^u} \frac{\partial}{\partial u}, \tag{27}$$

we have

$$P_i(u) = Q_i(e^u - \frac{2}{n(0)}), \text{ for } i = 1, ..., 2s + 1.$$
 (28)

Hence for the solution $P_i(u)$ to the nonlinear integrable system Eq. (11), we have

$$Q_i(t) = P_i(\log[t + \frac{2}{n(0)}]), \text{ for } i = 1, ..., 2s + 1.$$
 (29)

Thus we see the solution of the dynamical system of the 2s + 1 types rock-paper-scissors coalescence model is obtained by a logarithmic time change of the solution to Eq. (11) for the 2s + 1 types rock-paper-scissors collision model.

We can extend our argument to the infinitely many types rock-paperscissors particles coalescence model. Eq. (11) for the collision model is extended to

$$\frac{d}{dt}P(x,t) = P(x,t)\left(\int_{x-\pi}^{x} P(y,t)dy - \int_{x}^{x+\pi} P(y,t)dy\right)$$
(30)

where $P(x,t) = P(x+2\pi,t)$ for each x, P(x,t) being the probability density for a point on the unit circle [5, 16]. The logarithmic time change to Eq. (30) gives the dynamics of infinitely many types rock-paper-scissors particles coalescence model.

5 Simulation of a discrete time model

Consider the coalescence model of rock-paper-scissors particles (Section 2) with the following 2)' instead of 2 in Section 2.

2)' One of $n(t_l) = n - l$ particles coalescence with another particle at the discrete time t_l for l = 0, 1, ..., n - 1, where

$$t_{n-l} = \frac{2}{(n-l)} - \frac{2}{n}$$
 for $l = 0, 1, ..., n-1$. (31)

In Section 2, from 2) coalescence occurs $\frac{n(t)}{2}n(t)$ dt times in (t, t + dt) for the total number of particles n(t) at time t. Hence for our discrete time model it is natural to take the above time scale Eq. (31).

Since the solution $(n_1(t), n_2(t), n_3(t))$ of Eq. (8) approaches to 0 very quickly, here we show the relative abundances $(\frac{n_1(t)}{n(t)}, \frac{n_2(t)}{n(t)}, \frac{n_3(t)}{n(t)})$ in Fig. 5. Eq. (31) gives the reasonable time scale to compare the simulation results with the solution of the "Smoluchowski equation" Eq. (8). Starting from the system with 30000 rock particles, 20000 paper particles and 10000 scissors particles, a paper particle finally survives as shown in Fig. 7.

We write $N_i(t)$ for the abundance i = 1, 2, 3 for our discrete model where $N_1(t) + N_2(t) + N_3(t) = n(t)$ at time t. As we see in Fig. 5 and Fig. 6, for the period $[0, t_{60000-50}]$, where 50 particles exist at $t_{60000-50} = \frac{2}{50} - \frac{2}{60000}$, the trajectory $(\frac{N_1(t)}{n(t)}, \frac{N_2(t)}{n(t)}, \frac{N_3(t)}{n(t)})$ seems to be approximated by the numerical solution of Eq. (22) qualitatively. As the decrease of the total number of particles, the fluctuation of the relative abundance of each type increases as

we see in Fig. 7. It is remarkable that at $t_{60000-8} = \frac{2}{8} - \frac{2}{60000}$ all of the three types coexist. Finally one paper particle survives at time $t_{60000-1} = \frac{2}{1} - \frac{2}{60000}$ as shown in Fig. 7.

For the system for the collision model (Fig. 1) without deterministic approximation, we have a stochastic version of each conserved quantity, which is shown to be a martingale in probability theory. Considering the second largest eigen value and its corresponding eigen vector, we obtain the asymptotic probability of coexistence [14] by using the martingale. For our coalescence model (Fig. 2) to obtain the asymptotic probability of coexistence is our next problem. Our simulation result (Fig. 7) may be an extreme case. However the three types coexist until very final stage in our many other simulations. The conserved quantity for the deterministic approximation makes the coexistence of rock particles, paper particles and scissors particles for the system with sufficiently large number of particles.

6 Conclusion

The "Boltzmann equation" of the collision model of 2s+1-types rock-paper-scissors particles is shown to be a nonlinear integrable system in the previous papers. The relative abundance (concentration ratio) for the solution of the "Smoluchowski equation" of the coalescence of 2s+1 types rock-paper-scissors particles is obtained by a logarithmic time change of t the solution of the "Boltzmann equation". For the simulation of the rock-paper-scissors coalescence model (Fig. 6), the system with sufficiently large number of particles, the conserved quantities of our "Smoluchowski equation" works for the coexistence of three types. The system with finite number of particles, without deterministic approximation, gives interesting probabilistic questions, "which type of particle finally survives?", "when the coexistence of three types of particles ends?", etc. , for our next study.

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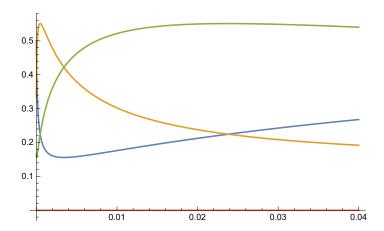


Figure 5: The numerical solution of Eq. (22) for s=1 starting from the system with 30000 rock particles (blue), 20000 paper particles (mustard yellow) and 10000 scissors particles (green).

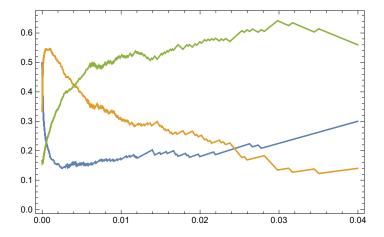


Figure 6: The trajectory $(\frac{N_1(t)}{n(t)}, \frac{N_2(t)}{n(t)}, \frac{N_3(t)}{n(t)})$ for the coalescence model of rock-paper-scissors particles seems to be approximated by the numerical solution of Eq. (22) for s=1.

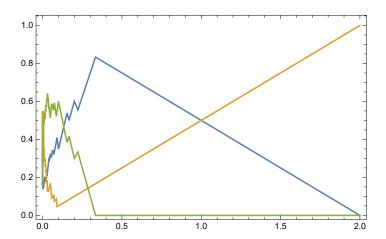


Figure 7: Starting from the system with 30000 rock particles, 20000 paper particles and 10000 scissors particles, a paper particle finally survives.

References

- [1] Aldous, D. J. 1999 Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. Bernoulli, 5(1), 3-48.
- [2] Ben-Naim, E., and Krapivsky P. L. 2004 Finite-size fluctuations in interacting particle systems, Physical Review E 69.4: 046113.
- [3] Bogoyavlensky O I 1988 Integrable discretizations of the KdV equation, Phys. Lett. A 134 34-38.
- [4] Calogero, F. 1975 Exactly solvable one dimensional many-body problems, Lett. Nuovo Cimento 13 411-416
- [5] Evans, S. N. 1999 Infinitely-Many-Species Lotka-Volterra Equations Arising from Systems of Coalescing Masses, Journal of the London Mathematical Society, 60 171-186.
- [6] Fuchs, M., Hwang, H-K. and Itoh, Y. 2017 From coin tossing to rock-paper-scissors and beyond: a log-exp gap theorem for selecting a leader, Journal of Applied Probability 54 213-235.

- [7] Frachebourg, L. and Krapivsky, P. L. 1998 Fixation in a cyclic Lotka-Volterra model, J. Phys. A: Math. Gen., 31, L287-L293.
- [8] Feldager, C. W., Mitarai, N., and Ohta, H. 2017 Deterministic extinction by mixing in cyclically competing species, Physical Review E 95.3 032318.
- [9] Haslegrave, J. and Jordan, J. 2018 Non-convergence of proportions of types in a preferential attachment graph with three co-existing types, Electronic Communications in Probability, 23 (54), 1-12.
- [10] Itoh, Y. 1971 Boltzmann equation on some algebraic structure concerning struggle for existence. Proceedings of the Japan Academy, 47 854-858.
- [11] Itoh, Y. 1973 On a ruin problem with interaction, Annals of the Institute of Statistical Mathematics, 25 635-641.
- [12] Itoh, Y. 1973 Model of struggle for existence, JUSE Symposium on Mathematical Programing 27, Model building and control problem of Ecosystem, Edited by T. Kitagwa, K. Kunisawa, S. Moriguti, 19-40 (in Japanese).
- [13] Itoh, Y. 1975 An H-theorem for a system of competing species, Proceedings of the Japan Academy, 51 374-379.
- [14] Itoh, Y. 1979 Random collision models in oriented graphs. Journal of Applied Probability, 16 36-44.
- [15] Itoh, Y. 1987 Integrals of a Lotka-Volterra system of odd number of variables, Progress of theoretical physics, 78 507-510.
- [16] Itoh, Y. 1988 Integrals of a Lotka-Volterra system of infinite species, Progress of Theoretical Physics, 80, 749-751.
- [17] Itoh, Y. 2008 A combinatorial method for the vanishing of the Poisson brackets of an integrable Lotka-Volterra system, Journal of Physics A: Mathematical and Theoretical, 42(2), 025201.
- [18] Kac, M. (Ed.) 1959 Probability and related topics in physical sciences (Vol. 1), American Mathematical Soc.

- [19] Kerr, B., Riley, M. A., Feldman, M. W., and Bohannan, B. J. 2002 Local dispersal promotes biodiversity in a real-life game of rock-paper-scissors, Nature, 418 171-174.
- [20] Knebel, J., Kruger, T., Weber, M. F., and Frey, E. 2013 Coexistence and survival in conservative Lotka-Volterra networks, Physical Review Letters, 110 168106.
- [21] Krapivsky, P. L., Redner, S., Ben-Naim, E. 2010 A kinetic view of statistical physics, Cambridge University Press.
- [22] Narita, K. 1982 Soliton solution to extended Volterra equation, Journal of the Physical Society of Japan, 51 1682-1685.
- [23] Sinervo, B. and Lively, C. M. 1996 The rock-paper-scissors game and the evolution of alternative male strategies, Nature, 380 240-243.
- [24] Smoluchowski, M. V. 1917 Mathematical theory of the kinetics of the coagulation of colloidal solutions, Z. Phys. Chem. 92 129-168.
- [25] Szabo, G., Szolnoki, A., and Izsak, R. 2004 Rock-scissors-paper game on regular small-world networks, Journal of physics A: Mathematical and General, 37(7), 2599.
- [26] Tainaka, K. I. 1988 Lattice model for the Lotka-Volterra system, Journal of the Physical Society of Japan, 57 2588-2590.
- [27] Tainaka, K., and Itoh, Y. 1991 Topological phase transition in biological ecosystems, EPL (Europhysics Letters), 15 399-404.
- [28] Toda M. 1967 Vibration of a chain with nonlinear interaction, Journal of the Physical Society of Japan, 22 431-436.