Phase transitions and all that

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1 Introduction

Since the experimental paper of Cheeseman, Kanefsky and Taylor [1] *phase transitions in combinatorial problems* held the promise to shed light on the "practical" algorithmic complexity of combinatorial problems. However, the connection conjectured in [1] was easily seen to be inaccurate. A much more realistic possible connection has been highlighted by the results (based on experimental evidence and nonrigorous arguments from Statistical Mechanics) of Monasson et al. [2] (see also [3]). These results supported the conjecture that it is *first-order phase transitions* that have algorithmic implications for the complexity of restricted classes of algorithms, including the important class of *Davis-Putnam-Longman-Loveland (DPLL) algorithms* [4].

There exists, indeed, a nonrigorous argument supporting this conjecture: phase transitions amount to nonanalytical behavior of a certain *order parameter*; the phase transition is *first order* if the order parameter is actually discontinuous. At least for random k-SAT [5] the order parameter suggested by Statistical Mechanics considerations has a purely combinatorial interpretation: it is the *backbone* of the formula, the set of literals that assume the same value in all optimal assignments. But intuitively one can relate (see e.g. the presentation of this argument by Achlioptas, Beame and Molloy [6]) the size of the backbone to the complexity of DPLL algorithms, when run on random k-SAT instances slightly above the phase transition: All literals in the backbone require well-defined values in order to satisfy the formula. But a DPLL algorithm has very few ways to know what those "right" values are. If w.h.p. the backbone of formulas above the transition contains a positive fraction of the literals that is bounded away from zero as we approach the transition (which happens in a case of a first-order phase transition) then, intuitively, DPLL will misassign a variable having $\Omega(n)$ height in the tree representing the behavior of the algorithm, and will be forced to backtrack on the given variable. The conclusion of this intuitive argument is that a first-order phase transition implies a $2^{\Omega(n)}$ lower bound for the running time of any DPLL algorithm, valid with high probability for random instances located slightly above the transition.

While previous rigorous results [7, 8, 9], supported these intuitions, to date, the extent of a connection between first-order phase transitions and algorithmic complexity was unclear.

The goals of this paper are

- 1. To remedy this, and formally establish a connection between first-order phase transitions and the resolution complexity of random satisfiability problems, and
- 2. To take steps towards obtaining a complete classification of the order of phase transition in generalized satisfiability problems.

To accomplish these goals

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- 1. we obtain (Theorem 1) a complete characterization of sharp/coarse thresholds in the random generalized satisfiability model due to Molloy [10]. "Physical" arguments (see discussion below) imply that it makes no sense to study the order of the phase transition unless the problem has a sharp threshold.
- 2. we rigorously prove (Theorem 2) that random 3-SAT has a first-order phase transition. We extend this result in several ways: first (Theorem 6) to random (2 + p)-satisfiability, the original problem from [2], obtaining further theoretical support to the heuristic results of [2]. Second we give a sufficient condition (Theorem 4) for the existence of a first-order phase transition. We then show (Theorem 5) that all problems whose constraints have no implicates of size at most two satisfy this condition.
- 3. we show that in all the cases where we can prove the existence of a first-order phase transition, such problems have a $2^{\Omega(n)}$ lower bound on their resolution complexity (and hence the complexity of DPLL algorithms as well [4]). Indeed, the two phenomena $(2^{\Omega(n)})$ resolution complexity and the existence of a first-order phase transition) have common causes.
- 4. in contrast, we show (Theorem 3) that, for any generalized satisfiability problem, a second-order phase transition implies, for every $\alpha > 0$, a $O(2^{\alpha \cdot n})$ upper bound on the resolution complexity of their random instances (in the region where most formulas are unsatisfiable).

2 Preliminaries

Throughout the paper we will assume familiarity with the general concepts of phase transitions in combinatorial problems (see e.g. [11]), random structures [12], proof complexity [13]. Some papers whose concepts and methods we use in detail (and we assume greater familiarity with) include [14], [15], [16].

Consider a monotonically increasing problem $A=(A_n)$, under the constant probability model $\Gamma(n,p)$, that independently sets to 1 with probability p each bit of the random string. As usual, for $\epsilon>0$ let $p_\epsilon=p_\epsilon(n)$ define the canonical probability such that $\operatorname{Prob}_{x\in\Gamma(n,p_\epsilon(n))}[x\in A]=\epsilon$.

The probability that a random sample x satisfies property A (i.e. $x \in A$) is a monotonically increasing function of p. Sharp thresholds are those for which this function has a "sudden jump" from value 0 to 1:

Definition 1 Problem A has a sharp threshold iff for every
$$0 < \epsilon < 1/2$$
, we have $\lim_{n \to \infty} \frac{p_{1-\epsilon}(n) - p_{\epsilon}(n)}{p_{1/2}(n)} = 0$. A has a coarse threshold if for some $\epsilon > 0$ it holds that $\underline{\lim}_{n \to \infty} \frac{p_{1-\epsilon}(n) - p_{\epsilon}(n)}{p_{1/2}(n)} > 0$.

For satisfiability problems (whose complements are monotonically increasing) the constant probability model amounts to adding every constraint (among those allowed by the syntactic specification of the model) to the random formula independently with probability p. Related definitions can be given for the other two models for generating random structures, the *counting model* and *the multiset model* [12]. Under reasonable conditions [12] these models are equivalent, and we will liberally switch between them. In particular, for satisfiability problem A, and an instance Φ of A, $c_A(\Phi)$ will denote its *constraint density*, the ratio between the number of clauses and the number of variables of Φ . To specify the random model in this latter cases we have to specify the constraint density as a function of n, the number of variables. We will use c_A to denote the value of the constraint density $c_A(\Phi)$ (in the counting/multiset models) corresponding to taking $p = p_{1/2}$ in the constant probability model. c_A is a function on n that is believed to tend to a constant limit as $n \to \infty$. However, Friedgut's proof [14] of a sharp threshold in k-SAT (and our results) leave this issue open.

The original investigation of the order of the phase transition in k-SAT used an order parameter called *the backbone*. Bollobás et al. [8] have investigated the order of the phase transition in 2-SAT under a different order parameter, a "monotonic version" of the backbone called *the spine*.

$$Spine(\Phi) = \{ x \in Lit | (\exists)\Xi \subseteq \Phi, \Xi \in SAT, \Xi \land \{\overline{x}\} \in \overline{SAT} \}. \tag{1}$$

They showed that random 2-SAT has a continuous (second-order) phase transition: the size of the spine, normalized by dividing it by the number of variables, approaches zero (as $n \to \infty$) for $c < c_{2-SAT} = 1$, and is continuous at $c = c_{2-SAT}$. By contrast, nonrigorous arguments from Statistical Mechanics [5] imply the fact that for 3 - SAT the spine jumps discontinuously from zero to positive values at the transition point $c = c_{3-SAT}$ (a first-order phase transition).

It is easy to see that the intuition concerning the connection between the complexity of DPLL algorithms and the size of the backbone (discussed briefly in the introduction) extends to the spine as well. In this paper whenever we will discuss the order of a phase transition we will do it with respect to this latter order parameter.

We would like to obtain a complete classification of the order of the phase transition in random satisfiability problems. A preliminary problem we have to deal with is characterizing those problems that have a sharp threshold: indeed, Physics considerations require that, in order that the study of the (order of the) phase transition to be meaningful, the order parameter (in the case of k-SAT the spine) has to be, w.h.p., concentrated around its expected value (in Physics parlance it is a *self averaging quantity*), and it is zero up to a certain value of the control parameter (in our case constraint density c) and positive above it. In the case of k-SAT these conditions imply the fact that k-SAT has a sharp threshold. The argument (a "folklore" one) can be formally expressed by the following

Lemma 2.1 Let c be an arbitrary constant value for the constraint density function.

1. If
$$c < \underline{\lim}_{n \to \infty} c_{k-SAT}(n)$$
 then $\lim_{n \to \infty} \frac{|Spine(\Phi)|}{n} = 0$.

2. If for some c there exists $\delta > 0$ such that w.h.p. (as $n \to \infty$) $\frac{|Spine(\Phi)|}{n} > \delta$ then $\lim_{n \to \infty} \text{Prob}[\Phi \in SAT] = 0$, that is $c > \overline{\lim_{n \to \infty} c_{k-SAT}(n)}$.

The argument is generic enough to extend to *all* constraint satisfaction problems. So a necessary condition for the study of the phase transition to be meaningful is that the problem have a sharp threshold.

3 Coarse and sharp thresholds of random generalized satisfiability problems

In this section we obtain a complete classification of thresholds of random satisfiability problems, under Molloy's recent model of random constraint satisfaction problems from [10] (specialized to satisfiability problems, i.e. problems with domain $\{0,1\}$). This affirmatively solves an open problem raised in [17].

Definition 2 Consider the set of all $2^{2^k}-1$ potential nonempty binary constraints on k variables X_1, \ldots, X_k . We specify a probability distribution \mathcal{P} which selects a single random constraint, and let $\mathcal{C} = supp(\mathcal{P})$ be the set of constraints on which \mathcal{P} assigns positive probability.

A random formula from $SAT_{n,M}(\mathcal{P})$ is specified by the following procedure:

- *n is the number of variables.*
- M is the number of clauses, chosen by the following procedure: first select, uniformly at random and with repetition m hyperedges of the uniform hypergraph on n variables.
- for each hyperedge choose a random ordering of the variables involved. Choose a random constraint according to \mathcal{P} and apply it on the list of (ordered) variables.

 $SAT(\mathcal{C})$ refers to the random model corresponding to \mathcal{P} being the uniform distribution on \mathcal{C} .

It turns out we face a technical difficulty when studying sharp and coarse thresholds in Molloy's model; it cannot be directly mapped onto the constant probability model for which the notion of a sharp threshold in Definition 1 works. The definition of a sharp threshold we need to employ is the one from [10]

Definition 3 $SAT(\mathcal{P})$ is said to have a sharp threshold of satisfiability if there exists a function c(n) bounded away from 0 such that, for any $\epsilon > 0$ if $M < (c(n) - \epsilon)n$ then $SAT_{n,M}(\mathcal{P})$ is a.s. satisfiable and if $M > (c(n) + \epsilon)n$ then $SAT_{n,M}(\mathcal{P})$ is a.s. unsatisfiable. On the other hand, if there exist two functions $M_1(n), M_2(n)$ such that $M_1(n)/M_2(n)$ is bounded away from zero, and the satisfaction probability of random instances from $SAT_{n,M_1}(\mathcal{P}), SAT_{n,M_2}(\mathcal{P})$ is bounded away from both 0 and 1 then $SAT(\mathcal{P})$ is said to have a coarse threshold.

However, just as in [10] (where this was done in the case when \mathcal{P} is the uniform distribution), for $k \geq 3$ one can map Molloy's model onto a modified version of the constant probability model, defined as follows: Let $p_1, \ldots p_r$ be positive numbers between zero and 1. A random sample x from the model $\Gamma_{p_1,\ldots p_r}(n,p)$ is obtained in the following way: divide the bits of x into r equal groups. Set each of the bits in the i'th group to 1 independently with probability $p \cdot p_i$. For this model the definitions of p_ϵ and sharp/coarse threshold from Definition 1 carry over, and are equivalent to those from Definition 3.

Indeed, let r be the cardinality of the support of distribution \mathcal{P} , and p_1, \ldots, p_r be the associated positive probabilities.

In its general setting Molloy's model is specified as follows: divide the potential constraints into groups of rk! constraints, corresponding to all possible applications of the r constraint templates on a fixed set of k variables. For each such group, independently with probability p, we make the decision to include at least one of the constraints in the group with probability $p = \frac{M}{rk!\binom{n}{k}}$ (going from M clauses to including each potential edge independently with probability p can be done just as in the uniform case from [10]).

Each realization of constraint constraint template i is chosen with probability probability $p_i/k!$. Denote this model by $M(n, p, p_1, \dots, p_r)$.

Defining $f(x) = [(1 + xpp_1/k!) \cdot (1 + xpp_2/k!) \cdot \ldots \cdot (1 + xpp_r)/k!]^{k!} - x$ we have f(1) > 0 and, since (by a simple calculus argument) the minimum of f(x) over the choices of $p_i \ge 0$, $\sum p_i = 1$ is obtained when one of them is 1 and the others are zero,

$$f(\frac{1}{1-p}) \ge [1 + \frac{p}{k!(1-p)}]^{k!} - \frac{1}{1-p} \ge 0.$$

Let $\alpha = \alpha(n) > 0$ be the smallest solution of the equation $f(\alpha) = 0$. Thus $\frac{1}{1-p} \le \alpha$.

Define, for i = 1, r,

$$p_i' = \frac{1/k! \cdot \alpha \cdot p_i}{1 + 1/k! \cdot \alpha pp_i}$$

Claim 1 The following hold for any $p = \theta(n^{1-k})$:

1. For every formula Φ such that no two constraints on the same set of variables appear in it,

$$\underset{M(n,p,p_1,...,p_r)}{\operatorname{Prob}}(\Phi) \geq \underset{\Gamma_{p'_1,...,p'_r}(n,p)}{\operatorname{Prob}}[\Phi].$$

Consequently

$$\underset{M(n,p,p_1,...,p_r)}{\operatorname{Prob}}[SAT(\mathcal{P})] \geq \underset{\Gamma_{p'_1,...,p'_r}(n,p)}{\operatorname{Prob}}[SAT(\mathcal{P})|$$

no two constraints on the same set of variables appear in Φ].

2. On the other hand, there exists f(n) = 1 + o(1) such that for every formula Φ such that no two constraints on the same set of variables appear in it,

$$\underset{M(n,p,p_1,\ldots,p_r)}{\operatorname{Prob}}(\Phi) \leq f(n) \underset{\Gamma_{p'_1,\ldots,p'_r}(n,p)}{\operatorname{Prob}}[\Phi].$$

Consequently

$$\underset{M(n,p,p_1,...,p_r)}{\operatorname{Prob}}[SAT(\mathcal{P})] \leq (1+o(1)) \underset{\Gamma_{p_1',...,p_r'}(n,p)}{\operatorname{Prob}}[SAT(\mathcal{P})|$$

no two constraints on the same set of variables appear in Φ].

Indeed, consider the set of constraints on a fixed set of given variables. The probability (under $\Gamma_{p'_1,\dots,p'_r}$) that a given clause of type i is included, and none of the others are is equal to $\frac{pp'_i}{1-pp'_i} \cdot [(1-pp'_1)\dots(1-pp'_r)]^{k!}$. But

$$\frac{pp_i'}{1 - pp_i'} = \alpha pp_i/k!.$$

Also

$$1 - pp_i' = \frac{1}{1 + \alpha pp_i/k!},$$

so, by the definition of α ,

$$[(1 - pp'_1) \dots (1 - pp'_r)]^{k!} = \frac{1}{\alpha}.$$

This means that the probability that in a given set of constraints exactly one constraint (of type i) is chosen is equal to $pp_i/k!$, the same as in model M. On the other hand the probability that no constraint is chosen is equal to $[(1-pp_1')\dots(1-pp_r')]^{k!}=\frac{1}{\alpha}$. But the same probability in model M is 1-p, and we know that $1-p\geq \frac{1}{\alpha}$. In both model decisions on different sets of k variables are independent. The conclusion is that M assigns a larger probability than $\Gamma_{p_1',\dots,p_r'}$ to any sample x to which it assigns positive probability. Point (1) follows.

Point (2) has a similar proof: by calculus the maximum value of f(x) is obtained when the p_i 's are equal, so

$$0 = f(\alpha) \le (1 + \frac{\alpha p}{rk!})^{rk!} - \alpha \le e^{\alpha p} - \alpha.$$

Since p=o(1), for large enough n $e^{\alpha p}\leq 1+\alpha p+(\alpha p)^2$, so $(\alpha p)^2+\alpha(p-1)+1>0$, in other words

$$\alpha(1-p) \le 1 + (\alpha p)^2.$$

But the ratio of the probabilities associated to any given Φ by M and $\Gamma_{p'_1,\dots,p'_r}(n,p)$ verifies

$$\frac{\operatorname{Prob}_{M(n,p,p_1,\ldots,p_r)}(\Phi)}{\operatorname{Prob}_{\Gamma_{p'_1,\ldots,p'_r}(n,p)}[\Phi]} \le [\alpha(1-p)]^{rk!\binom{n}{k}} \le (1+(\alpha p)^2)^{rk!\binom{n}{k}}.$$

Since $p = \theta(n^{1-k})$ and $k \ge 3$ the right-hand side is a function of n that is 1 + o(1).

To prove the result it is enough to observe that for $k \geq 3$ the expected number of times a random formula in $\Gamma_{p'_1,\dots,p'_r}(n,p)$ contains two different clauses on the same set of variables is o(1), since that will imply that the satisfaction probabilities in the two models are related by a 1-o(1) factor. Indeed, this number is

$$\binom{n}{k} \cdot \left[\sum_{\alpha,\beta} \frac{pp'_{\alpha}pp'_{\beta}}{(k!)^2} \right],$$

where indices α, β span the set of different pairs of clauses from a group. Since each group is finite (contains rk! clauses) and $p = \theta(n^{1-k})$, this expected value is $\theta(n^{2-k})$, which is o(1) for $k \ge 3$.

Definition 4 Constraint C_2 is an implicate of C_1 iff every satisfying assignment for C_1 satisfies C_2 .

Definition 5 A boolean constraint C strongly depends on a literal if it has an unit clause as an implicate.

Definition 6 A boolean constraint C strongly depends on a 2-XOR relation if $\exists i, j \in \overline{1, k}$ such that constraint " $x_i \neq x_j$ " is an implicate of C.

Our result is:

Theorem 1 Consider a generalized satisfiability problem $SAT(\mathcal{P})$ (that is not trivially satisfiable by the "all zeros" or "all ones" assignment). Let $\mathcal{C} = supp(\mathcal{P})$.

- 1. if some constraint in C strongly depends on one component then $SAT(\mathcal{P})$ has a coarse threshold.
- 2. if some constraint in C strongly depends on a 2XOR-relation then SAT(P) has a coarse threshold.
- 3. in all other cases $SAT(\mathcal{P})$ has a sharp threshold.

Proof.

1. Suppose some clause C implies a unit clause. We claim that $SAT(\mathcal{P})$ has a coarse threshold in the region where the expected number of clauses is $\theta(\sqrt{n})$.

That the probability that such a formula is bounded away from zero in this region it is easy to see: consider a random formula with $c\sqrt{n}$ constraints, and let H be the k-uniform formula hypergraph. The expected number of pairs of edges C_i , C_j that have nonempty intersection is

$$\binom{c \cdot \sqrt{n}}{2} \cdot \left(1 - \frac{\binom{n-k}{k}}{\binom{n}{k}}\right) \le \frac{(ck)^2}{2}$$

Therefore with constant positive probability (that depends on c), all vertices will have degree at most 1 in the hypergraph H_n , and the formula will be satisfiable.

If, on the other hand, both positive and negative unit clauses are implicates of constraints in $\mathcal P$ then one can adapt the well-known lower bound on the probability of intersection of two random sets of size $\theta(\sqrt{n})$ to show that, with constant probability a random formula will contain two contradictory unit clauses as implicates, and be unsatisfiable.

The proof is similar in the case when only one type of unit clauses (w.l.o.g assume it's the positive unit clauses) are implicates of constraints in \mathcal{C} . Since $SAT(\mathcal{C})$ is not trivial there exists a constraint $C_1 \in \mathcal{C}$ with an implicate of the type $\overline{x_1} \vee \ldots \vee \overline{x_b}$, $b \geq 2$. We deal first with the case when there exists a constraint $C_2 \neq C_1$ such that C_2 has an unit clause as implicate. Then it is easy to construct a formula F consisting of F copies of F and one copy of F that implies the (unsatisfiable) formula F consisting of F in a random instance of F with F clauses is constant, so the probability that the instance is unsatisfiable is bounded away from zero.

Finally, in the case when the only constraint in C that has an unit implicate is C_1 . In this case one can use a trick similar to that used in the last paragraph of subsection 3.3: we use half of the random copies of C_1 to imply (random) unit clauses, and the other half to imply (random) copies of $\overline{x_1} \vee \ldots \vee \overline{x_b}$. This way we can produce, with constant probability, a copy of the formula F.

2. Suppose now that C does not fall in the first case but has a 2XOR implicate. In this case Creignou and Daudé have shown when \mathcal{P} is the uniform distribution (and this extends directly to the case of a general probability distribution as well) that $p_{1/2} = \Omega(n^{1-k})$ and the expected number of constraints is θn . Let $c_{SAT(\mathcal{P})} \cdot n$ be the expected number of constraints corresponding to $p_{1/2}$. Then there exists $\delta > 0$ such that, for every n, $c_{SAT(\mathcal{P})} = c_{SAT(\mathcal{P})}(n) > \delta$.

Let us consider a random formula with $c \cdot n$ of constraints. By the well known result on triangles in random graphs it follows that with positive probability one can use C to create a "contradictory triangle". Therefore it is easy to see that for every c>0 the satisfaction probability is bounded away from 1. It is easy to see than this statement, together with the fact that $c_{SAT(\mathcal{P})}(n) > \delta$ together imply that $SAT(\mathcal{P})$ has a coarse threshold.

3. We will concentrate in the sequel on the last one. As discussed previously, for $k \geq 3$ Molloy's model can be mapped onto a version of the constraint probability model. In the case k=2 we can establish the existence of a sharp threshold in a direct manner, by the same method as the one used by Chvátal and Reed for 2-SAT [18] (the complete proof of this case will be presented in the full version). Indeed, by the first two points of the Theorem, and the assumption k=2 the only possible constraints in \mathcal{P} can be constraints $x \vee y$, $\overline{x} \vee \overline{y}$, $\overline{x} \vee y$, $x \vee \overline{y}$, x = y, and the first two are always present.

Let us now consider the case $k \geq 3$, using the modified version of the constant probability model. We note first that there exists a simple observation that allows us to reduce the problem to the case when $\mathcal P$ is the uniform probability: the Friedgut-Bourgain result on sharp/coarse threshold properties in monotone problems [14] uses the following result, an easy consequence of the Mean Value Theorem: if a monotonic property A does *not* have a sharp threshold (under model $\Gamma(n.p)$) then there exists $p^* = p(n)$ and a constant C > 0 such that (for infinitely many n)

$$p^* \cdot I(p^*) < C, \tag{2}$$

where $I(p^*) = \frac{d\mu_p(A)}{dp}|_{p=p^*}$.

The same argument works when A is considered under model $\Gamma_{p_1,\dots p_r}(n,p)$. Moreover, it is an easy consequence of Russo's Lemma for $\Gamma_{p_1,\dots p_r}(n,p)$ that if equation 2 holds for p^* and some tuple $p_1,\dots p_r$, then it also holds (with a different constant C) for p^* and tuple $p_1=\dots p_r=1/r$. In other words it is enough to obtain a contradiction to the assumption that $SAT(\mathcal{P})$ did not have a sharp threshold in the case when \mathcal{P} is the uniform probability, which is what we show next.

3.1 A base case

To prove the theorem in the uniform case we will first consider a "base case" that is easier to explain, and will be of use in solving the general case: let a,b be two integers (not necessarily equal), both greater or equal to 2. Let S be a set consisting of two constraints C_1, C_2 of arity a, respectively b, specified by $C_1 = \overline{x_1} \vee \ldots \overline{x_a}$, $C_2 = x_1 \vee \ldots x_b$. One can represent SAT(S) in the framework of Definition 2 by "simulating" C_1, C_2 by suitable constraints of arity $\max\{a,b\}$.

We first outline how to prove that SAT(S) has a sharp threshold: we apply the Friedgut-Bourgain result [14] and infer that if SAT(S) did not have a sharp threshold than, for some $\epsilon, \delta_0, K > 0$ and some probability $p = p(n) \in [p_{\epsilon}, p_{1-\epsilon}]$

- (a) either $\operatorname{Prob}_{p=p(n)}[\Phi \text{ contains some } F \in \overline{SAT} \text{ with } |F| \leq K] > \delta_0$, or
- (b) there exists a fixed satisfiable formula F_0 , $|F_0| \leq K$ such that $\operatorname{Prob}_{p=p(n)}[\Phi \wedge F_0 \in \overline{SAT}] \operatorname{Prob}[\Phi \in \overline{SAT}] > \delta_0$.

One easy observation is that in the second alternative we can always assume that F consists of a conjunction of unit clauses: if F is satisfiable and satisfies (2), then so does the conjunction of unit clauses specifying one satisfying assignment of F. The first alternative is eliminated by a result (Proposition 4.6) from [17]. The key to disproving the second alternative, in the case of k-SAT, is a geometric result, Lemma 5.7 in [14]. We restate it here for completeness.

Lemma 3.1 For a sequence $A = (A_n)$ of subsets of the n-dimensional hypercube, $A \subseteq \{0,1\}^n$, define A to be (d, m, ϵ) -coverable if the probability for a union of a random choice of d subcubes (hyperplanes) of codimension m to cover A is greater than ϵ for large enough n.

Let f(n) be any function that tends to infinity as $n \to \infty$. For fixed k, d, and ϵ any A that is $(d, 1, \epsilon)$ -coverable is $(f(n), k, \epsilon)$ -coverable.

The connection with satisfiability can be explained as follows: the set A in the application of the Lemma 3.1 will (intuitively) refer to the set of satisfying assignments of random formula Φ . Hyperplanes of codimension 1 are associated to unit clauses, more precisely to the set of assignments forbidden by a given unit clause. The fact that A can be covered with probability ϵ by a union of d random hyperplanes of codimension 1 parallels the fact that with probability ϵ , $\Phi \wedge F_0$ becomes unsatisfiable. This is what the result of Friedgut-Bourgain gives us (for $\epsilon = \delta_0$, under the assumption that k-SAT does not have a sharp threshold). Hyperplanes of codimension k correspond to the set of assignments forbidden by a given k-clause, and the conclusion of the geometric lemma is that adding any small (but unbounded) number f(n) of random k-CNF clauses to random formula Φ boosts the probability of not being satisfiable at least as much as the addition of the (constantly many) unit clauses in F_0 .

For small enough f(n) this statement can be directly refuted, by concentration results for the binomial distribution (Lemma 5.6 in [14]). A simpler and more general way to derive it is given as Lemma 3.1 in [19].

The same outline works for the case we consider. To state the geometric result we need, however, to work with two types of hyperplanes:

Definition 7 Let $H_n = \{0,1\}^n$ be the n-dimensional hypercube, and let w_i denote the value of the i'th bit of element $w \in H_n$. A positive hyperplane of codimension d is a subset of points of H_n defined by a system of equations $x_{i_1} = \ldots = x_{i_d} = 1$, where the x_i 's are distinct variables. Negative hyperplanes have a similar definition.

Our version of the geometric Lemma is

Lemma 3.2 For a sequence $A = (A_n)$ of subsets of the n-dimensional hypercube, $A_n \subseteq \{0,1\}^n$ define A to be $(n_1,d_1,d_2,m_1,m_2,\epsilon)$ -coverable if the probability of a union of a random choice of d_1 negative hyperplanes of codimension m_1 and d_2 positive hyperplanes of codimension m_2 to cover A_n is at least ϵ if $n \ge n_1$. Let f(n), g(n) be any functions that tends to infinity as $n \to \infty$. For fixed k_1,k_2 , d, and $0 < \delta < \epsilon$, there exists n_2 that depends on $k_1,k_2,d,\epsilon,\Delta,n_1$ (but not A) such that for any $n \ge n_2$ any $A_n \subseteq \{0,1\}^n$ that is $(n_1,d_1,d_2,1,1,\epsilon)$ -coverable is $(n_2,f(n),g(n),k_1,k_2,\epsilon-\delta)$ -coverable.

We will in fact prove a stronger version of the Lemma:

Lemma 3.3 For a sequence $A = (A_n)$ of subsets of the n-dimensional hypercube, assume that

$$\operatorname{Prob}[A \subseteq H_1 \cup \dots H_d] \geq \epsilon$$

for all $n \ge n_1$, where the H_i 's are random hyperplanes of codimension 1, d_1 of them negative, d_2 of them positive.

Let f(n), g(n) be any functions that tends to infinity as $n \to \infty$. For fixed k_1, k_2 , d, and $\delta > 0$, there exists $n_* = n(n_1, d_1, d_2, k_1, k_2, \epsilon, \delta, f, g)$, (however it does not depend on A) such that for any $n \ge n_*$ and any i, $0 \le i \le d$

$$Pr[A \subseteq P_1 \cup \ldots \cup P_{\frac{if(n)}{d}} \cup N_1 \ldots \cup N_{\frac{ig(n)}{d}} \cup H_{i+1} \cup \ldots \cup H_d] \ge Pr[A \subseteq H_1 \cup \ldots \cup H_d] - \frac{i\delta}{d}, \quad (3)$$

where the N_i 's are random negative hyperplanes of codimension k_1 and the P_i 's are random positive hyperplanes of codimension k_2 .

Proof.

It is easy to see that one can assume that d|f(n), d|g(n) (since it is enough to prove the lemma for $\overline{f}(n) = d\lfloor f(n)/d\rfloor$, $\overline{g}(n) = d\lfloor g(n)/d\rfloor$).

We will prove the lemma by double induction on d_1, d_2 . By symmetry we only need to consider two "base cases:"

Case 1: $d_1 = 0$, $d_2 = 1$

In this case (and the dual, $d_1 = 1$, $d_2 = 0$) we can replace, for i = 1, equation 3 by the stronger:

$$\operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{f(n)} \cup N_1 \ldots \cup N_{g(n)}] \ge 1 - \delta. \tag{4}$$

The hypothesis implies that for $n \geq n_1$ there exist $n \cdot \epsilon$ positive hyperplanes of codimension 1 such that

$$A_n \subseteq P_1^{(n)} \cap \ldots \cap P_{n \cdot \epsilon}^{(n)}$$
.

We will assume, w.l.o.g., in what follows that A is in fact *equal* to the right hand side. If KP is a random positive hyperplane of codimension k_1 then

$$\operatorname{Prob}[A \not\subseteq KP] = 1 - \frac{\binom{n \cdot \epsilon}{k_1}}{\binom{n}{k_1}}.$$

Indeed, suppose KP is specified by the (random set of) equations $x_1^{(n)} = \ldots = x_{k_1}^{(n)} = 1$. The condition that $A \subseteq KP$ is equivalent to

$$\{x_1^{(n)}, \dots x_{k_1}^{(n)}\} \subseteq \{p_1^{(n)}, \dots, p_{n \cdot \epsilon}^{(n)}\},\$$

where $\{p_1^{(n)},\dots,p_{n\cdot\epsilon}^{(n)}\}$ are the literals that specify the hyperplanes $P_1^{(n)},\dots,P_{n\cdot\epsilon}^{(n)}$.

Thus the probability that A is included in the union of g(n) random positive hyperplanes KP_i of codimension k_1 is at least $1 - \text{Prob}[(\forall i) : A \not\subseteq KP_i]$, which is equal to

$$1 - (1 - \frac{\binom{n \cdot \epsilon}{k_1}}{\binom{n}{k_1}})^{g(n)} \sim 1 - (1 - \epsilon^{k_1})^{g(n)} \to 1 \text{ as } n \to \infty.$$

It follows that there exists $n_* = n(n_1, d_1, d_2, k_1, k_2, \epsilon, \delta, f, g)$ such that for $n \ge n_*$ the left-hand side is larger than $1 - \delta$.

Case 2: $d_1 + d_2 > 1$

It is enough to prove that there exists $n_i = n(n_1, d_1, d_2, k_1, k_2, \epsilon, \delta, f, g, i)$ such that 3 holds, for a fixed value of $i, 0 \le i \le d$, when $n \ge n_i$. Then we can take $n_* = max\{n_i\}$.

We prove this on induction on i. The claim is clearly true for i = 0. Assume, therefore, that the claim is true up to i; we will prove it for i + 1.

Denote for all j

$$p_j = \operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{\frac{(j-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(j-1)g(n)}{d}} \cup H_j \cup \ldots \cup H_d].$$

To accomplish that it is enough to show that

$$p_{i+1} \ge p_i - \delta,\tag{5}$$

By the Bayes formula:

$$p_{i} = \operatorname{Prob}[A \subseteq P_{1} \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_{1} \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}} \cup H_{i} \cup \ldots \cup H_{d}] =$$

$$= \sum_{B} Pr[B \subseteq H_{i} | A \setminus (P_{1} \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_{1} \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B] \cdot$$

$$\cdot \operatorname{Prob}[A \setminus (P_{1} \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_{1} \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B]$$

Assume without loss of generality that H_i is a positive hyperplane.

Let
$$\gamma = \frac{\delta}{2d}$$
. Let

$$C_{\gamma} = \{B \subseteq \{0,1\}^n : Pr[B \subset P] > \gamma\}.$$

Let α be the sum of those terms in 6 corresponding to sets $B \in C_{\gamma}$, and let β be the sum corresponding to sets $B \in \overline{C_{\gamma}}$.

From the definition of C_{γ} it follows that

$$0 \le \beta \le \gamma$$
,

therefore

$$\operatorname{Prob}[A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) \in C_{\gamma}] \ge$$

$$\ge \frac{1}{\gamma} \cdot \left[\operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}} \cup H_i \cup \ldots \cup H_d] - \gamma\right] =$$

$$= \frac{1}{\gamma} \cdot [p_i - \gamma].$$

On the other hand

$$\begin{aligned} p_{i+1} &= \operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{\frac{if(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{ig(n)}{d}} \cup H_{i+1} \cup \ldots \cup H_d] = \\ &= \sum_{B} Pr[B \subseteq P_{\frac{(i-1)f(n)}{d}+1} \cup \ldots \cup P_{\frac{if(n)}{d}} \cup N_{\frac{(i-1)g(n)}{d}+1} \cup \ldots \cup N_{\frac{ig(n)}{d}} | \\ &\quad A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B] \cdot \\ &\quad \cdot \operatorname{Prob}[A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B] \end{aligned}$$

Let $\overline{f}(n) = f(n)/d$, $\overline{g}(n) = g(n)/d$. Since the N_i 's, P_i 's are random hyperplanes, one can rewrite the previous recurrence as

$$\begin{split} p_{i+1} &= \operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{\frac{if(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{ig(n)}{d}} \cup H_{i+1} \cup \ldots \cup H_d] = \\ &= \sum_{B} \operatorname{Prob}[B \subseteq P_1 \cup \ldots \cup P_{\overline{f(n)}} \cup N_1 \cup \ldots \cup N_{\overline{g(n)}}| \\ &A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B] \cdot \\ &\cdot \operatorname{Prob}[A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B] \end{split}$$

Since all terms are nonnegative, one can obtain a lower bound on the left-hand size of this latter terms by only considering those $B \in C_{\gamma}$.

Applying the induction hypothesis from case one for all $B \in C_{\gamma}$ and $n \ge n_i = n_*(n_1, 0, 1, k_1, k_2, \gamma, \delta, \overline{f}, \overline{g})$, we infer that for all such B,

$$Prob[B \subseteq P_1 \cup \ldots \cup P_{\overline{f(n)}} \cup N_1 \cup \ldots \cup N_{\overline{g(n)}}] \ge (1 - \gamma),$$

therefore

$$p_{i+1} = \operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{\frac{if(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{ig(n)}{d}} \cup H_{i+1} \cup \ldots \cup H_d] \ge$$

$$(1 - \gamma) \cdot \sum_{B \in C_{\gamma}} \Pr[A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) = B]$$

$$= (1 - \gamma) \cdot \Pr[A \setminus (P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}}) \in C_{\gamma}]$$

$$\ge \frac{(1 - \gamma)}{\gamma} \cdot [\operatorname{Prob}[A \subseteq P_1 \cup \ldots \cup P_{\frac{(i-1)f(n)}{d}} \cup N_1 \cup \ldots \cup N_{\frac{(i-1)g(n)}{d}} \cup H_i \cup \ldots \cup H_d] - \gamma] =$$

$$\frac{(1 - \gamma)}{\gamma} \cdot [p_i - \gamma].$$

Since $\gamma \leq 1$,

$$p_{i+1} \ge (1-\gamma) \cdot (p_i - \gamma) = p_i - \gamma \cdot [1 + p_i - \gamma] \ge p_i - 2\gamma.$$

which is precisely equation 5 (that we wanted to prove).

3.2 How to contradict Lemma 3.2

It is now easy to infer the fact that SAT(S) has a sharp threshold, by using the previous lemma with $d_1 = |F_0 \cap Var|$, $d_2 = |F_0| - d_1$, $k_1 = b$, $k_2 = a$, $\epsilon = \delta_0$, $\Delta = \delta_0/2$ and a refutation of the geometric lemma similar to the one for 3-SAT.

The conclusion of the Geometric Lemma (similar to the one for k-SAT) is that, by adding any number f(n) of copies of $x_1 \vee \ldots \vee x_a$ and g(n) copies of $\overline{x_1} \vee \ldots \vee x_b$ suffices to boost the unsatisfiability probability by a constant.

However, Lemma 3.1 from [19] asserts that adding up to $o(\sqrt(n))$ random clauses is not enough to boost the unsatisfiability probability by more than o(1).

Because of the nature of the random model, adding $H(n) = o(\sqrt{n})$ random clauses insures that w.h.p. we have $\Theta(H(n))$ copies of each type of clause, as long as H(n) grows faster than some power of n. Taking the number of such copies to be the functions f(n), g(n) contradicts the consequence of the Geometric Lemma.

Note that we do *not* make use of all the details of the random model (such as the precise number of copies of each clause in random instances at p = p(n)), but only that:

- the expected number of copies of both C_1 and C_2 in a random formula at p = p(n) is unbounded. This is enough to make the analog of Lemma 3.1 from [19] work.
- the clauses are independent.

3.3 Putting it all together

In the previous section we have proved a geometric lemma that is used to prove that the above-defined set SAT(S) has a sharp threshold.

Consider now a set \mathcal{C} of constraints that satisfies the condition (3) of the theorem. Since $SAT(\mathcal{C})$ is not trivially satisfied by the "all zero" (all ones) assignment, there exist constraints C_1 , C_2 in \mathcal{C} and $a,b \geq 1$ such that $C_1 \models x_1 \vee \ldots x_a$, $C_2 \models \overline{x_1} \vee \ldots \vee \overline{x_b}$. In fact $a,b \geq 2$, otherwise some constraint in \mathcal{C} would strongly depend on one variable.

Just as in the base case, condition (i) in the Friedgut-Bourgain result is eliminated by the result of Creignou and Daudé, and the formula F_0 in condition (ii) can be assumed to consist of a conjunction of unit clauses. Reflecting this fact, the geometric lemma needed for the general case has the same hypothesis as the one of Lemma 3.2: the set A can be covered by a union of random hyperplanes of codimension 1. However, the covering desired in the conclusion no longer consists of hyperplanes, but of (general) sets of points in the hyperplane, corresponding to sets of assignments forbidden by a certain constraint C.

The critical observation (easiest to make in the case when $C_1 \neq C_2$) is that **Lemma 3.2 implies the geometric result we need for this case**: since $C_1 \models x_1 \lor \dots x_a$, the "forbidden set" associated to C_1 contains the "forbidden set" associated to constraint $x_1 \lor \dots x_a$ in Lemma 3.2, the hyperplane $x_1 = \dots = x_a = 0$. A similar result holds for C_2 and the positive hyperplanes.

Thus each "covering set" in a conclusion of the (general case of the) geometric lemma contains a corresponding "covering set" from the conclusion of Lemma 3.2. In other words the geometric lemma for $SAT(\mathcal{C})$ follows from the geometric lemma for the base case by monotonicity.

All is left to show is that the analog of Lemma 3.1 from [19] also works in this general case. We have previously observed that this amounts to showing that the expected number of copies of C_1 , C_2 in a random formula is unbounded. But Proposition 3.5 in [17] (slightly generalized to probability

distributions \mathcal{P} that are not uniform) and the details of the random model imply that in fact this number is linear.

A simple modification of this argument holds when $C_1 = C_2$. To see this, note that in the proof of the fact that Lemma 3.2 holds for *some* small enough (but unbounded) f(n), g(n) is enough to obtain a contradiction.

The expected number of copies of C_1 in a random formula is linear, denote it by h(n). Dividing the set of such copies into two (random) sets of cardinality h(n)/2 yields infinitely many random copies of C_1 used to imply $x_1 \vee \ldots \vee x_a$ in the previous argument and infinitely many copies used to imply $\overline{x_1} \vee \ldots \vee \overline{x_b}$. We then apply the same strategy as in the first case.

To summarize: the proof follows from the corresponding argument for the base case by monotonicity. It critically uses the fact that we are *not* in cases (i) or (ii) of the Theorem, since it is only under these conditions when the first alternative in the Friedgut-Bourgain argument can be eliminated.

4 3-SAT has a first-order phase transition

Theorem 2 k-SAT, $k \ge 3$ has a first-order phase transition. In other words there exists $\eta > 0$ such that for every sequence p = p(n) we have

$$\lim_{n \to \infty} \Pr_{p=p(n)} [\Phi \in SAT] = 0 \Rightarrow \lim_{n \to \infty} \Pr_{p=p(n)} [\frac{|Spine(\Phi)|}{n} \ge \eta] = 1. \tag{6}$$

Proof.

We start by giving a simple sufficient condition for a literal to belong to the spine of the formula:

Claim 2 Let Φ be a minimally unsatisfiable formula, and let x be a literal that appears in Φ . Then $x \in Spine(\Phi)$.

Proof. Let C be a clause that contains x. By the minimal unsatisfiability of Φ , $\Phi \setminus \{C\} \in SAT$. On the other hand $\Phi \setminus \{C\} \land \{x\} \in \overline{SAT}$, otherwise Φ would also be satisfiable. Thus $x \in Spine(\Phi \setminus \{C\})$.

Thus, to show that 3-SAT has a first-order phase transition it is enough to show that a random unsatisfiable formula contains w.h.p. a minimally unsatisfiable subformula containing a linear number of literals. A way to accomplish this is by using the two ingredients of the Chvátal-Szemerédi proof [15] that random 3-SAT has exponential resolution size w.h.p. They are explicitly stated to make the argument self-contained:

Claim 3 There exists a constant $\delta > 0$ such that for every constant $c > c_{3SAT}$ with high probability (as $n \to \infty$) a random formula Φ with n variables and cn clauses has no minimally unsatisfiable subformula of size less than $\delta \cdot n$.

Claim 4 There exists $\eta > 0$ so that w.h.p. for every $c > c_{3-SAT}$ all subformulas of a random formula Φ having between $(\delta/2) \cdot n$ and $\delta \cdot n$ clauses contain at least $\eta \cdot n$ (pure) literals (corresponding to different variables).

The argument is now transparent: if Φ is unsatisfiable then w.h.p. a minimally unsatisfiable subformula Ξ of Φ has size at least δn . By the second claim, applied to an arbitrary subformula of Ξ of size $(3\delta n)/4$, we infer that w.h.p. Ξ contains at least many $\eta \cdot n$ different variables.

5 First-order phase transitions and resolution complexity of random generalized satisfiability problems

In this section we extend the previous result (and the connection between first-order phase transitions and resolution complexity) to other classes of satisfiability problems. Interestingly, we find that a condition Molloy investigated in [10] is a sufficient condition for the existence of a first-order phase transition.

It turns out that there are differences between the case of random k-SAT and the general case that force us to employ an alternative definition of the spine. The most obvious one is that formula (1) involves negations of variables, whereas Molloy's model does not. But it has more serious problems: consider for example k-uniform hypergraph 2-coloring specified as $SAT(\{C_0\})$, where $C_0(x_1, \ldots, x_k)$ has the interpretation "not all of $x_1, \ldots x_k$ are equal". Because of the built-in symmetry to permuting colors 0 and 1, the spine of any instance is empty (under definition 1). Similar phenomena have appeared before (and forced a different definition of the backbone/spine) in k-coloring [20] (symmetry = permutation of colors) or graph partition [21] (symmetry = permutation of sides). There are other ways (to be detailed in the journal version of the paper) in which the original definition of the spine behaves differently in the general case than in that of random k-SAT. Our solution is to define the concept of spine of a random instance of a satisfiability problem $SAT(\mathcal{P})$ in a slightly different way. The definition is consistent with those in [20], [21].

Definition 8
$$Spine(\Phi) = \{x \in Var | (\exists)\Xi \subseteq \Phi, \Xi \in SAT, (\exists)C \in \mathcal{C}, x \in C, \text{ such that } \Xi \land C \in \overline{SAT} \}.$$

It is easy to see that, for k-CNF formulas whose (original) spine contains at least three literals a variable x is in the (new version of the) spine if and only if either x or \overline{x} were present in the old version. In particular the new definition does not change the order of the phase transition of random k-SAT. Moreover the proof of Claim 2 carries over to the general case. The *resolution complexity* of an instance Φ of $SAT(\mathcal{P})$ is defined as the resolution complexity of the formula obtained by converting each constraint of Φ to CNF-form.

Definition 9 Let \mathcal{P} be such that $SAT(\mathcal{P})$ has a sharp threshold. Problem $SAT(\mathcal{P})$ has a first-order phase transition if there exists $\eta > 0$ such that for every sequence p = p(n) we have

$$\lim_{n \to \infty} \underset{p=p(n)}{\text{Prob}} \left[\Phi \in SAT \right] = 0 \Rightarrow \lim_{n \to \infty} \underset{p=p(n)}{\text{Prob}} \left[\frac{|Spine(\Phi)|}{n} \ge \eta \right] = 1. \tag{7}$$

If, on the other hand, for every epsilon > 0 there exists $p^{\epsilon}(n)$ with

$$\lim_{n \to \infty} \underset{p = p^{\epsilon}(n)}{\operatorname{Prob}} \left[\Phi \in SAT \right] = 0 \text{ and } \lim_{n \to \infty} \underset{p = p(n)}{\operatorname{Prob}} \left[\frac{|Spine(\Phi)|}{n} \ge \epsilon \right] = 0 \tag{8}$$

we say that $SAT(\mathcal{P})$ has a second-order phase transition¹.

A first observation is that a second-order phase transition has computational implications:

Theorem 3 Let \mathcal{P} be such $SAT(\mathcal{P})$ has a second-order phase transition. Then for every constant $c > \overline{\lim}_{n \to \infty} c_{SAT(\mathcal{P})}(n)$, and every $\alpha > 0$, random formulas of constraint density c have w.h.p. resolution complexity $O(2^{k \cdot \alpha \cdot n})$.

Proof.

By the analog of Claim 2 for the general case, if $SAT(\mathcal{P})$ has a second-order phase transition) then for every $c > c_{SAT(\mathcal{P})}$ and for every $\alpha > 0$, minimally unsatisfiable subformulas of a random formula Φ with

¹ strictly speaking the order of the phase transition is *at least two*.

constraint density c have w.h.p. size at most $\alpha \cdot n$. Consider the backtrack tree of the natural DPLL algorithm (that tries to satisfies clauses one at a time) on such a minimally unsatisfiable subformula F. By the usual correspondence between DPLL trees and resolution complexity (e.g. [4], pp. 1) it yields a resolution proof of the unsatisfiability of Φ having size at most $2^{k \cdot \alpha \cdot n + 1}$.

Definition 10 For a formula F define $c^*(F) = \max\{\frac{|Constraints(G)|}{|Var(G)|}: \emptyset \neq G \subseteq F\}.$

The next result gives a sufficient condition for a generalized satisfiability problem to have a first-order phase transition.

Theorem 4 Let C be a set of constraints such that SAT(C) has a sharp threshold. If there exists $\epsilon > 0$ such that for every minimally unsatisfiable formula F it holds that

$$c^*(F) > \frac{1+\epsilon}{k-1}$$

then for every P with supp(P) = C

 $SAT(\mathcal{P})$ has a first-order phase transition.

Proof.

We first recall the following concept from [15]:

Definition 11 Let x, y > 0. A k-uniform hypergraph with n vertices is (x,y)-sparse if every set of $s \le xn$ vertices contains at most ys edges.

We also recall Lemma 1 from the same paper.

Lemma 5.1 Let k, c > 0 and y be such that (k-1)y > 1. Then w.h.p. the random k-uniform hypegraph with n vertices and cn edges is (x, y)-sparse, where

$$\epsilon = y - 1/(k-1),$$

$$x = (\frac{1}{2e}(\frac{y}{ce})^y)^{1/\epsilon},$$

The critical observation is that the existence of a minimally unsatisfiable formula of size xn and with $c^*(F) > \frac{1+\epsilon}{k-1}$ implies that the k-uniform hypergraph associated to the given formula is not (x,y)-sparse, for $y = \frac{\epsilon}{k-1}$.

But, according to Lemma 5.1, w.h.p. a random k-uniform hypergraph with cn edges is (x_0, y) sparse, for $x_0 = (\frac{1}{2e}(\frac{y}{ce})^y)^{1/\epsilon}$ (a dirrect application of Lemma 1 in their paper). We infer that any formula with less than $x_0 \cdot n/K$ constraints is satisfiable, therefore the same is true for any formula with $x_0 \cdot n/K$ clauses picked up from the clausal representation of constraints in Φ .

The second condition (expansion of the formula hypergraph) can be proved similarly.

One can give an explicitly defined class of satisfiability problems for which the previous result applies:

Theorem 5 Let \mathcal{P} be such that $SAT(\mathcal{P})$ has a sharp threshold. If no clause $C \in \mathcal{C} = supp(\mathcal{P})$ has an implicate of length at most 2 then

1. for every minimally unsatisfiable formula F

$$c^*(F) \ge \frac{2}{2k-3}.$$

Therefore $SAT(\mathcal{P})$ satisfies the conditions of the previous theorem, i.e. it has a first-order phase transition.

2. Moreover $SAT(\mathcal{P})$ also has $2^{\Omega(n)}$ resolution complexity².

Proof.

1. For any real $r \geq 1$, formula F and set of clauses $G \subseteq F$, define the r-deficiency of G, $\delta_r(G) = r|Clauses(G)| - |Vars(G)|$.

Also define

$$\delta_r^*(F) = \max\{\delta_r(G) : \emptyset \neq G \subseteq F\}$$
(9)

We claim that for any minimally unsatisfiable F, $\delta_{2k-3}^*(F) \ge 0$. Indeed, assume this was not true. Then there exists such F such that:

$$\delta_{2k-3}(G) \le -1 \text{ for all } \emptyset \ne G \subseteq F.$$
 (10)

Proposition 1 Let F be a formula for which condition 10 holds. Then there exists an ordering $C_1, \ldots, C_{|F|}$ of constraints in F such that each constraint C_i contains at least k-2 variables that appear in no C_j , j < i.

Proof. Denote by v_i the number of variables that appear in *exactly i* constraints of F. We have

$$\sum_{i>1} i \cdot v_i = k \cdot |Constraints(F)|.$$

therefore $2|Var(F)| - v_1 \le k \cdot |Constraints(F)|$. This can be rewritten as $v_1 \ge 2|Var(F)| - k|Constraints(F)| > |Constraints(F)| \cdot (2k-3-k) = (k-3) \cdot |Constraints(F)|$ (we have used the upper bound on $c^*(F)$. Therefore there exists at least one constraint in F with at least k-2 variables that are free in F. We set $C_{|F|} = C$ and apply this argument recursively to $F \setminus C$.

Call the k-2 new variables of C_i free in C_i . Call the other two variables bound in C_i . Let us show now that F cannot be minimally unsatisfiable. Construct a satisfying assignment for F incrementally: Consider constraint C_j . At most two of the variables in C_j are bound for C_j . Since C has no implicates of size at most two, one can set the remaining variables in a way that satisfies C_j . This yields a satisfying assignment for F, a contradiction with our assumption that F was minimally unsatisfiable.

Therefore $\delta_{2k-3}^*(F) \geq 0$, a statement equivalent to our conclusion.

2. To prove the resolution complexity lower bound we use the size-width connection for resolution complexity obtained in [16]: we prove that there exists $\eta > 0$ such that w.h.p. random instances of $SAT(\mathcal{P})$ having constraint density c have resolution width at least $\eta \cdot n$.

We use the same strategy as in [16]

²this result subsumes some of the recent results in [22]

- (a) prove that w.h.p. minimally unsatisfiable subformulas are "large".
- (b) prove that any clause implied by a satisfiable formula of "intermediate" size will contain "many" literals.

Indeed, define for a unsatisfiable formula Φ and (possibly empty) clause C

$$\mu(C) = \min\{|\Xi| : \Xi \subseteq \Phi, \Xi \models C\}.$$

Claim 5 There exists $\eta_1 > 0$ such that for any c > 0, w.h.p. $\mu(\Box) \ge \eta_1 \cdot n$ (where Φ is a random instance of $SAT(\mathcal{P})$ having constraint density c).

Proof. In the proof of Theorem 4 we have shown that there exists $\eta_0 > 0$ such that w.h.p. any unsatisfiable subformula of a given formula has at least $\eta_0 \cdot n$ constraints. Therefore *any* formula made of *clauses* in the CNF-representation of constraints in Φ , and which has less than $\eta_0 \cdot n$ clauses is satisfiable, and the claim follows, by taking $\eta_1 = \eta_0$.

The only (slightly) nontrivial step of the proof, which critically uses the fact that constraints in \mathcal{P} do not have implicates of length at most two, is to prove that clause implicates of subformulas of "medium" size have "many" variables. Formally:

Claim 6 There exists d > 0 and $\eta_2 > 0$ such that w.h.p., for every clause C such that $d/2 \cdot n < \mu(C) <= dn, |C| \ge \eta_2 \cdot n$.

Proof. Take $0 < \epsilon$. It is easy to see that if $c^*(F) < \frac{2}{2k-3+\epsilon}$ then w.h.p. for every subformula G of F, at least $\frac{\epsilon}{3} \cdot |Constraints(G)|$ have at least k-2 private variables: Indeed, since $c^*(G) < \frac{2}{2k-3+\epsilon}$, by a reasoning similar to the one we made previously $v_1(G) \geq (k-3+\epsilon)|Constraints(G)|$. Since constraints in G have arity k, at least $\epsilon/3 \cdot |Constraints(G)|$ have at least k-2 "private" variables.

Choose $y = \frac{2}{2k-3+\epsilon}$ in Lemma 5.1 for $\epsilon > 0$ a small enough constant. Since the problem has a sharp threshold in the region where the number of clauses is linear,

$$d = \inf\{x(y,c) : c >= c_{SAT(\mathcal{P})}\} > 0.$$

W.h.p. all subformulas of Φ having size less than $d/k \cdot n$ have a formula hypergraph that is (x, y)-sparse, therefore fall under the scope of the previous argument.

Let Ξ be a subformula of Φ , having minimal size, such that $\Xi \models C$. We claim:

Claim 7 For every clause P of Ξ with k-2 "private" variables, (i.e. one that does not appear in any other clause), at least one of these "private" variables appears in C.

Indeed, suppose there exists a clause D of Ξ such that none of its private variables appears in C.

Because of the minimality of Ξ there exists an assignment F that satisfies $\Xi \setminus \{D\}$ but does not satisfy D or C. Since D has no implicates of size two, there exists an assignment G, that differs from F only on the private variables of D, that satisfies Ξ . But since C does not contain any of the private variables of D, F coincides with G on variables in C. The conclusion is that G does not satisfy C, which contradicts the fact that $\Xi \models C$.

The proof of Claim 6 (and of item 2. of Theorem 5) follows: since for any clause K of one of the original constraints $\mu(K) = 1$, since $\mu(\Box) > \eta_1 \cdot n$ and since w.l.o.g. $0 < d < \eta_1$ (otherwise replace d with the smaller value) there exists a clause C such that

$$\mu(C) \in [d/2k \cdot n, d/k \cdot n]. \tag{11}$$

Indeed, let C' be a clause in the resolution refutation of Φ minimal with the property that $\mu(C') > dn$. Then at least one clause C, of the two involved in deriving C' satisfies equation 11.

By the previous claim it C contains at least one "private" variable from each clause of Ξ . Therefore $|C| \ge \eta_2 \cdot n$, with $\eta_2 = d/2k \cdot \epsilon$.

It is instructive to note that the condition in the theorem is violated (as expected) by random 2-SAT, as well as by random 1-in-k SAT: the formula $C(x_1,x_2,\ldots,x_{k-1},x_k) \wedge C(\overline{x_k},x_{k+1},\ldots,x_{2k-2},x_1) \wedge C(\overline{x_1},x_{2k-1},\ldots,x_{3k-3},\overline{x_k}) \wedge C(x_k,x_{3k-2},\ldots,x_{4k-4},x_1)$ (where C is the constraint "1-in-k") is minimally unsatisfiable, but has clause/variable ratio 1/(k-1) and implicates $\overline{x_1} \vee \overline{x_k}$ and $x_1 \vee x_k$.

It would be tempting to speculate that whenever both $x \vee y$ and $\overline{x} \vee \overline{y}$ are implicates of clauses in \mathcal{C} then $SAT(\mathcal{P})$ has a second-order phase transitions for every distribution \mathcal{P} with $supp(\mathcal{P}) = \mathcal{C}$. That is, however, not true, at least for some distributions \mathcal{P} . Consider the random (2+p)-SAT model of Monasson et al. [2]. In this model p is a fixed real in [0,1]. A random instances of (2+p)-SAT with n variables and $c \cdot n$ clauses is obtained by choosing pcn random clauses of length 3 and (1-p)cn random clauses of length 2. It was shown in [2] (using the nonrigorous replica method) that

- 1. (2+p)-SAT has a second-order phase transition for $0 \le p \le p_0 \sim 0.413...$
- 2. the transition becomes first-order for $p > p_0$.
- 3. when the transition changes from second-order to first-order the complexity of a certain DPLL algorithm changes from polynomial to exponential.

Several rigorous results have complemented these findings. Achlioptas et al. [7] have shown that for $0 \le p \le 0.4$ the phase transition in (2+p)-SAT only depends on the "2-SAT part". One can perhaps use the techniques of [8] to confirm statement (i).

It is easily seen that (2+p)-SAT can be represented in Molloy's framework. On the other hand Achlioptas, Beame and Molloy [9] have shown that for those p for which (2+p)-SAT does *not* behave like 2-SAT the resolution complexity of the problem is exponential. Using Theorem 3 and and results in [9] we get:

Theorem 6 Let $p \in [0,1]$ be s.t. there exist $\epsilon > 0$ and $c < \frac{(1-\epsilon)}{1-p}$ s.t. random instances of (2+p)-SAT with n variables and $c \cdot n$ clauses are w.h.p. unsatisfiable. Then (2+p)-SAT has a first-order phase transition.

It would be interesting to obtain a complete characterization of the order of the phase transition in an arbitrary problem $SAT(\mathcal{P})$. Such a characterization, however, requires substantial advances: Exactly locating the "tricritical point" p_0 in random 2+p-SAT (or merely deciding whether it is equal or not to 0.4) is an open problem. A complete characterization would yield a solution to this problem as a byproduct.

On the other hand Theorem 6 suggests an interesting conjecture: whenever the location of the phase transition is *not* determined by the implicates of size at most two in the given formula, the phase transition in $SAT(\mathcal{P})$ is first-order. Perhaps techniques in [9] can help settle this question.

6 Discussion

We have shown that the existence of a first-order phase transition in a random satisfiability problem is often correlated with a $2^{\Omega(n)}$ peak in the complexity of resolution/DPLL algorithms at the transition point.

As for the extent of the connection it is easy to see that it does not extend to a substantially larger class of algorithms: consider random k-XOR-SAT, the problem of testing the satisfiability of random systems of linear equations of size k over \mathbb{Z}_2 . k-XOR-SAT is a version of XOR-SAT, one of the polynomial time cases of satisfiability from Schaefer's Dichotomy Theorem [23]. Indeed, it is easily solved by Gaussian elimination. But Ricci-Tersenghi et al. [24] have presented a non-rigorous argument using the replica method that supports the existence of a first-order phase transition, and we can show this formally (as a direct consequence of Theorem 5):

Proposition 2 Random k-XOR-SAT, $k \geq 3$, has a first-order phase transition.

To sum up: the intuitive argument states that a first-order phase transition correlates with a $2^{\Omega(n)}$ lower bound of the complexity of DPLL algorithms at the transition. This is true in many cases, and the underlying reason is that the two phenomena (the jump in the order parameter and the resolution complexity lower bound) have common causes. However, at least for satisfiability problems, this connection does not extend substantially beyond the class of DPLL algorithms.

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