Supplement 2

to the paper "Floating bundles and their applications"

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This paper is the supplement to the section 2 of the paper "Floating bundles and their applications" [1]. Below we study some properties of category, connected with cobordism rings of FBSP. In particular, we shall show that it is the tensor category.

In [1] the series $\mathfrak{G}(x,y) \in H[[x,y]] = \Omega_U^*(\widetilde{Gr})$, where $H = \Omega_U^*(Gr)$, was defined. Recall that it corresponds to the direct limit κ of the maps $\kappa_{k,l} \colon \widetilde{Gr}_{k,kl} \to \mathbb{C}P^{kl-1}$, where $\widetilde{Gr}_{k,kl}$ is the canonical FBSP over $Gr_{k,kl}$ ((k,l)=1). In [1] some properties of $\mathfrak{G}(x,y)$ were studied. In particular, it was shown that

$$(\varepsilon \mathfrak{G})(x,y) = F(x,y),$$

where $\varepsilon \colon H \to R = \Omega_U^*(\mathrm{pt})$ is the counit of the Hopf algebra H and $F(x,y) \in R[[x,y]]$ is the formal group of geometric cobordisms.

Let $\varphi_{k,l}$ be the map

$$\kappa_{k,l} \times \operatorname{id}_{\widetilde{G}r_{k,kl}} : \widetilde{G}r_{k,kl} \to \mathbb{C}P^{kl-1} \times \widetilde{G}r_{k,kl}.$$

The commutativity of the following diagram

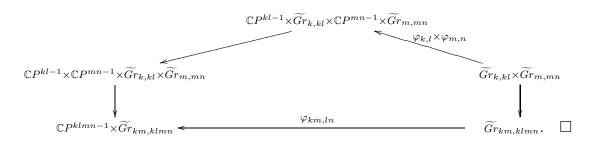
$$\widetilde{Gr}_{k,kl} \xrightarrow{\varphi_{k,l}} \mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl}
\varphi_{k,l} \downarrow \qquad \qquad \downarrow \operatorname{id}_{\mathbb{C}P} \times \varphi_{k,l}
\mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl} \xrightarrow{\operatorname{diag}_{\mathbb{C}P} \times \operatorname{id}_{\widetilde{Gr}}} \mathbb{C}P^{kl-1} \times \mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl}$$
(1)

allows us to define on the algebra H[[x,y]] the structure of $R[[z]] = \Omega_U^*(\mathbb{C}P^{\infty})$ -module such that z acts as the multiplication by $\mathfrak{G}(x,y)$. Let us denote this R[[z]]-module by $(H[[x,y]]; \mathfrak{G}(x,y))$.

Let us consider $R[[z]] = \Omega_U^*(\mathbb{C}P^{\infty})$ as a Hopf algebra. Recall that $\Delta_{R[[z]]}(z) = F(z \otimes 1, 1 \otimes z).$

Proposition 1. H[[x,y]] is the module coalgebra over R[[z]], i. e. $R[[z]] \widehat{\otimes}_R H[[x,y]] \to H[[x,y]]$ is the homomorphism of coalgebras.

Proof. The proof follows from the following commutative diagram ((km, ln) = 1):



Let us consider the next commutative diagram ((km, ln) = 1):

$$\widetilde{Gr_{k,kl} \times Gr_{m,mn}} \xrightarrow{\psi_{kl,mn}} \widetilde{Gr_{km,klmn}}
\downarrow \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad (2)$$

$$Gr_{k,kl} \times Gr_{m,mn} \xrightarrow{\phi_{kl,mn}} Gr_{km,klmn} ,$$

where $Gr_{k,kl} \times Gr_{m,mn}$ is the FBSP over $Gr_{k,kl} \times Gr_{m,mn}$, induced by the map $\phi_{kl,mn}$ (the definition of $\phi_{kl,mn}$ was given in [1]). Clearly that the bundle $Gr_{k,kl} \times Gr_{m,mn}$ (with fiber $\mathbb{C}P^{km-1} \times \mathbb{C}P^{ln-1}$) is ("external") Segre's product of the canonical FBSP over $Gr_{k,kl}$ and $Gr_{m,mn}$. By definition, put

$$\widetilde{Gr \times Gr} = \varinjlim_{(km,ln)=1} Gr_{k,kl} \times Gr_{m,mn} ,$$

$$\psi = \varinjlim_{(km,ln)=1} \psi_{km,ln} : \widetilde{Gr \times Gr} \to \widetilde{Gr} .$$

$$\psi = \lim_{\substack{\longrightarrow \\ (km,ln)=1}} \psi_{km,ln} : \quad \widetilde{Gr \times Gr} \to \widetilde{Gr}.$$

We have the homomorphism of R[[z]]-modules

$$\Psi \colon (H[[x,y]]; \ \mathfrak{G}(x,y)) \to (H \widehat{\otimes}_R H[[x,y]]; \ (\Delta \mathfrak{G})(x,y)) \ ,$$

defined by the fiber map ψ (recall that Δ is the comultiplication in the Hopf algebra $H = \Omega_U^*(Gr)$). Clearly that the restriction $\Psi|_H$ coincides with Δ .

Let $\mathcal{P}^{k-1} \times \mathcal{Q}^{l-1}$ be a FBSP over a finite CW-complex X with fiber $\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1}$. Recall ([1]) that if k and l are sufficiently large then there exist a classifying map $f_{k,l}$ and the corresponding fiber map

$$\mathcal{P}^{k-1} \underset{X}{\times} \mathcal{Q}^{l-1} \rightarrow \widetilde{Gr}_{k,kl}
\downarrow \qquad \downarrow
X \stackrel{f_{k,l}}{\rightarrow} Gr_{k,kl}$$
(3)

which are unique up to homotopy and up to fiber homotopy respectively. Let $\mathcal{P}^{km-1} \underset{X}{\times} \mathcal{Q}^{ln-1}$, (km, ln) = 1 be Segre's product of $\mathcal{P}^{k-1} \underset{X}{\times} \mathcal{Q}^{l-1}$ with the trivial FBSP $X \times \mathbb{C}P^{m-1} \times \mathbb{C}P^{n-1}$. Let us pass to the direct limit

$$\mathcal{P} \underset{X}{\times} \mathcal{Q} = \underset{\stackrel{\longrightarrow}{i}}{\lim} (\mathcal{P}^{km_i-1} \underset{X}{\times} \mathcal{Q}^{ln_i-1}),$$

where $(km_i, ln_i) = 1$, $m_i \mid m_{i+1}, n_i \mid n_{i+1}, m_i, n_i \to \infty$, as $i \to \infty$. The stable equivalence class of FBSP (see [1]) $\mathcal{P}_{X}^{k-1} \times \mathcal{Q}^{l-1}$ may be unique restored by the direct limit $\mathcal{P}_{X} \times \mathcal{Q}$. We have also a classifying map $f = \lim_{(k,l)=1} f_{k,l}$ and the corresponding fiber map

$$\begin{array}{ccc}
\mathcal{P} \underset{X}{\times} \mathcal{Q} & \to & \widetilde{Gr} \\
\downarrow & & \downarrow \\
X & \stackrel{f}{\to} & Gr .
\end{array} \tag{4}$$

Let us define the category \mathfrak{FBSP}_f by the following way.

- (i) $Ob(\mathfrak{FBSP}_f)$ is the class of direct limits $\mathcal{P} \times \mathcal{Q}$ of FBSP over finite CWcomplexes X (in other words, the class of stable equivalence classes of FBSP);
- (ii) $\operatorname{Mor}_{\mathfrak{FBGp}_f}(\mathcal{P} \underset{X}{\times} \mathcal{Q}, \ \mathcal{P}' \underset{Y}{\times} \mathcal{Q}')$ is the set of fiber maps

$$\begin{array}{cccc}
\mathcal{P}_{X} & \mathcal{Q} & \rightarrow & \mathcal{P}'_{Y} & \mathcal{Q}' \\
\downarrow & & \downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array} \tag{5}$$

such that its restrictions to any fiber ($\cong \mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty}$) are isomorphisms.

Applying the functor of unitary cobordisms Ω_U^* to an object $\mathcal{P} \times \mathcal{Q} \in \mathrm{Ob}(\mathfrak{FBSP}_f)$, we get the R[[z]]-module $(A[[x,y]]; (f^*\mathfrak{G})(x,y)) \in \mathrm{Ob}(\Omega_U^*(\mathfrak{FBSP}_f))$, where $A = \Omega_U^*(X)$ and $f \colon X \to Gr$ is a classifying map for $\mathcal{P} \times \mathcal{Q}$. It is clear that $((\varepsilon_A \circ f^*)\mathfrak{G})(x,y) = F(x,y)$, where $\varepsilon_A \colon A \to R$ is the homomorphism, induced by an embedding of a point pt $\hookrightarrow X$. In other words, for any object in the category $\Omega_U^*(\mathfrak{FBSP}_f)$ there exists the canonical morphism $(A[[x,y]]; (f^*\mathfrak{G})(x,y)) \to (R[[x,y]]; F(x,y))$.

Hence there exist the initial object $(H[[x,y]]; \mathfrak{G}(x,y))$ and the final object (R[[x,y]]; F(x,y)) in the category $\Omega_U^*(\mathfrak{FBSP})$.

Let's consider a pair $(A[[x,y]]; (f^*\mathfrak{G})(x,y)), (B[[x,y]]; (g^*\mathfrak{G})(x,y)) \in Ob(\Omega_U^*(\mathfrak{FBGp}_{\mathfrak{f}})),$ where $(B[[x,y]]; (g^*\mathfrak{G})(x,y)) = \Omega_U^*(\mathcal{P}' \times \mathcal{Q}').$ Let's define their "tensor product" as the object $((A \otimes B)[[x,y]]; (((f^* \otimes g^*) \circ \Delta)\mathfrak{G})(x,y)) \in Ob(\Omega_U^*(\mathfrak{FBGp}_{\mathfrak{f}}))$ (recall that $\Delta \colon H \to H \widehat{\otimes} H$ is the comultiplication in the Hopf algebra H).

Proposition 2. The category $\Omega_U^*(\mathfrak{FBSP}_{\mathfrak{f}})$ is the tensor category with the just defined tensor product and the unit I = (R[[x,y]]; F(x,y)).

Proof. The proof is trivial. For example, the associativity axiom follows from the identity $(((\Delta \otimes id_H) \circ \Delta)\mathfrak{G})(x,y) = (((id_H \otimes \Delta) \circ \Delta)\mathfrak{G})(x,y)$ which follows from the next commutative diagram ((kmt, lnu) = 1):

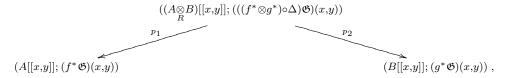
$$Gr_{km,klmn} \times Gr_{t,tu} \rightarrow \widetilde{Gr}_{kmt,klmntu}$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$Gr_{k,kl} \times \widetilde{Gr}_{m,mn} \times Gr_{t,tu} \rightarrow Gr_{k,kl} \times Gr_{mt,mntu},$$
(6)

where $Gr_{k,kl} \times Gr_{m,mn} \times Gr_{t,tu}$ is external Segre's product of the canonical FBSP over $Gr_{k,kl}$, $Gr_{m,mn}$ and $Gr_{t,tu}$ (it is the bundle over $Gr_{k,kl} \times Gr_{m,mn} \times Gr_{t,tu}$ with fiber $\mathbb{C}P^{kmt-1} \times \mathbb{C}P^{lnu-1}$). \square

Note that there exist the canonical homomorphisms p_1 , p_2 :



such that $p_1 \mid_{A \underset{R}{\otimes} B} = \mathrm{id}_A \otimes \varepsilon_B$, $p_2 \mid_{A \underset{R}{\otimes} B} = \varepsilon_A \otimes \mathrm{id}_B$.

References

[1] A. V. Ershov Floating bundles and their applications.— ${\rm arXiv:} {\rm math.AT/} 0102054$