INTERSECTION PAIRINGS IN THE N-FOLD REDUCED PRODUCT OF ADJOINT ORBITS

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ABSTRACT. In previous work we computed the symplectic volume of the symplectic reduced space of the product of N adjoint orbits of a compact Lie group. In this paper we compute the intersection pairings in the same object.

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1. Introduction

Let G be a compact connected Lie group with maximal torus T. As a vector space, the equivariant cohomology of a Hamiltonian G-space M is isomorphic to the tensor product of the ordinary cohomology of M and the G-equivariant cohomology of a point. Here $S(\mathfrak{t})$ is the polynomial ring on the Lie algebra of the maximal torus T, which is denoted \mathfrak{t} . This result comes from [20] (Proposition 5.8). The above isomorphism is only an isomorphism of vector spaces, not of rings.

When M and G are as above, there is a surjective ring homomorphism κ (the Kirwan map) from the equivariant cohomology of M to

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the ordinary cohomology of the symplectic reduced space or symplectic quotient M_{red} , which is defined as

$$M_{\rm red} = \mu^{-1}(0)/G$$

where μ is the moment map. The ordinary cohomology of the reduced space is the quotient of the equivariant cohomology of M by the kernel of κ .

Provided the reduced space is a smooth manifold, it satisfies Poincaré duality, so its cohomology ring is determined by the intersection pairings (in other words the evaluation of cohomology classes against the fundamental class).

Let M be the product of a collection of adjoint orbits of G. In this situation, the above isomorphism is an isomorphism of $H_G^*(pt)$ -modules. We give a formula for the intersection pairings in $M_{\rm red}$ using the same methods as in our earlier paper [15], in other words the localization theorem of Atiyah-Bott and Berline-Vergne and the residue formula of [16] (Theorem 8.1).

2. Notation and Conventions

Let G be a compact connected Lie group. Let \mathfrak{g} be the Lie algebra of G. Let \mathfrak{g}^* be the dual vector space of \mathfrak{g} .

We choose a maximal torus T in G. Let \mathfrak{t} be the Lie algebra of T. Let \mathfrak{t}^* be the dual vector space of \mathfrak{t} . Let $W = N_G(T)/T$ be the corresponding Weyl group.

Let $Ad: G \to Aut(\mathfrak{g})$ be the adjoint representation of G. Let $Cd: G \to Aut(\mathfrak{g}^*)$ be the coadjoint representation of G. More explicitly,

(1)
$$\langle \operatorname{Cd}(g)\xi, X \rangle = \langle \xi, \operatorname{Ad}(g^{-1})X \rangle$$

for all $g \in G$, $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between a covector and a vector.

Remark. Note that for all $g, h \in G$, $Ad(g) \circ Ad(h) = Ad(gh)$ and $Cd(g) \circ Cd(h) = Cd(gh)$. That is, both Ad and Cd are left actions.

Let $ad : \mathfrak{g} \to End(\mathfrak{g})$ be the adjoint representation of \mathfrak{g} . Let $cd : \mathfrak{g} \to End(\mathfrak{g}^*)$ denote the coadjoint representation of the Lie algebra \mathfrak{g} . Thus, $cd(X) = -ad(X)^*$.

Remark. Note that both ad : $\mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ and cd : $\mathfrak{g} \to \operatorname{End}(\mathfrak{g}^*)$ are Lie algebra homomorphisms.

For convenience we work with orbits of the adjoint action rather than the coadjoint action, so our orbits are subsets of \mathfrak{g} instead of \mathfrak{g}^* . The invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} (invariant under the adjoint

action) gives a G-equivariant isomorphism between \mathfrak{g} (equipped with the adjoint action) and \mathfrak{g}^* (with the coadjoint action).

Let $\mathcal{O}(\xi)$ denote the adjoint orbit through $\xi \in \mathfrak{g}$. The following theorem is well known.

Theorem 1 (Kirillov-Kostant-Souriau). [19] Given any $\xi \in \mathfrak{g}$, the adjoint orbit $\mathcal{O}(\xi)$ is a smooth compact connected submanifold in \mathfrak{g} and there exists a natural G-invariant (under the adjoint action) symplectic structure on $\mathcal{O}(\xi)$. In other words, there exists a closed non-degenerate G-invariant real 2-form $\omega_{\mathcal{O}(\xi)} \in \Omega^2(\mathcal{O}(\xi); \mathbb{R})$ on $\mathcal{O}(\xi)$. More explicitly, $\omega_{\mathcal{O}(\xi)}$ can be constructed in the following way.

For all $\eta \in \mathcal{O}(\xi)$, let B_{η} be the antisymmetric bilinear form on \mathfrak{g} defined by

(2)
$$B_{\eta}(X,Y) := \langle \eta, [X,Y] \rangle$$

for all $X, Y \in \mathfrak{g}$. Then $\omega_{\mathcal{O}(\xi)}$ can be defined by

(3)
$$\omega_{\mathcal{O}(\xi)}(\eta)([X,\eta],[Y,\eta]) = \langle \eta,[X,Y] \rangle$$

for all $X, Y \in \mathfrak{g}, \eta \in \mathcal{O}(\xi)$.

Note that for all $\eta \in \mathcal{O}(\xi) \subseteq \mathfrak{g}$, $T_{\eta}\mathcal{O}(\xi) = \{[X, \eta] : X \in \mathfrak{g}\}.$

This natural 2-form $\omega_{\mathcal{O}(\xi)}$ is sometimes referred to as the Kirillov-Kostant-Souriau symplectic form on the adjoint orbit $\mathcal{O}(\xi)$.

Therefore, an adjoint orbit $\mathcal{O}(\xi)$ becomes a symplectic manifold when it is equipped with its Kirillov-Kostant-Souriau symplectic form $\omega_{\mathcal{O}(\xi)}$. In addition, we have the following:

Proposition 2. The adjoint action of G on $(\mathcal{O}(\xi), \omega_{\mathcal{O}(\xi)})$ is a Hamiltonian G-action with the moment map given by the inclusion map $\mu_{\mathcal{O}(\xi)}: \mathcal{O}(\xi) \hookrightarrow \mathfrak{g}$. In other words, $\mu_{\mathcal{O}(\xi)}$ is equivariant with respect to the adjoint action of G on $\mathcal{O}(\xi)$ and the adjoint action of G on \mathfrak{g} , and for all $X \in \mathfrak{g}$,

(4)
$$d\mu_{\mathcal{O}(\xi)}^X = \iota_{X^{\sharp}} \omega_{\mathcal{O}(\xi)}$$

where $\mu_{\mathcal{O}(\xi)}^X : \mathcal{O}(\xi) \to \mathbb{R}$ is defined by $\mu_{\mathcal{O}(\xi)}^X(\eta) = \langle \mu_{\mathcal{O}(\xi)}(\eta), X \rangle$ for all $\eta \in \mathcal{O}(\xi)$ and X^{\sharp} is the vector field on $\mathcal{O}(\xi)$ such that for all $\eta \in \mathcal{O}(\xi)$, the tangent vector $X^{\sharp}(\eta) \in T_{\eta}\mathcal{O}(\xi)$ is

(5)
$$\frac{d}{dt}\Big|_{t=0} \left(\operatorname{Ad}(\exp(tX)) \eta \right).$$

Let $\mathcal{O}(\xi_1), \dots, \mathcal{O}(\xi_N)$ be N adjoint orbits. Then we can form their Cartesian product:

(6)
$$\mathcal{M}(\xi) := \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$$

where

(7)
$$\underline{\xi} := (\xi_1, \cdots, \xi_N) \in \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}^{N}.$$

We assume the following:

Assumption 3. All of $\mathcal{O}(\xi_1), \dots, \mathcal{O}(\xi_N)$ are diffeomorphic to the homogeneous space G/T. This assumption is equivalent to the assumption that all of the stabilizer groups $\operatorname{Stab}_G(\xi_1), \dots, \operatorname{Stab}_G(\xi_N)$ are conjugate to the chosen maximal torus T. If all of ξ_1, \dots, ξ_N are contained in $\mathfrak{t} \subseteq \mathfrak{g}$, then this assumption is saying that

$$\operatorname{Stab}_G(\xi_1) = \cdots = \operatorname{Stab}_G(\xi_N) = T.$$

Remark. Since every adjoint orbit $\mathcal{O}(\xi)$ can be written as $\mathcal{O}(\xi')$ for some $\xi' \in \mathfrak{t} \subseteq \mathfrak{g}$, we will always assume that $\underline{\xi} = (\xi_1, \dots, \xi_N)$ satisfies that $\xi_j \in \mathfrak{t} \subseteq \mathfrak{g}$ for all j.

The Cartesian product $\mathcal{M}(\underline{\xi}) = \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$ carries a natural symplectic structure ω_{ξ} defined by:

(8)
$$\omega_{\xi} := \operatorname{pr}_{1}^{*} \omega_{\mathcal{O}(\xi_{1})} + \dots + \operatorname{pr}_{N}^{*} \omega_{\mathcal{O}(\xi_{N})}$$

where $\operatorname{pr}_j: \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N) \to \mathcal{O}(\xi_j)$ is the projection onto the *j*-th component.

Let G act on $\mathcal{M}(\underline{\xi}) = \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$ by the diagonal action Δ :

(9)
$$\Delta(g)(\eta_1, \dots, \eta_N) := (\mathrm{Ad}(g)(\eta_1), \dots, \mathrm{Ad}(g)(\eta_N))$$

for all $g \in G$, $\eta_j \in \mathcal{O}(\xi_j)$.

We mentioned above that the symplectic form $\omega_{\underline{\xi}}$ is invariant under this action of G. We also have the following:

Proposition 4. The diagonal action Δ of G on $(\mathcal{M}(\underline{\xi}), \omega_{\underline{\xi}})$ is a Hamiltonian G-action with the moment map $\mu_{\xi} : \mathcal{M}(\underline{\xi}) \to \mathfrak{g}$ being:

(10)
$$\mu_{\underline{\xi}}(\underline{\eta}) = \sum_{j=1}^{N} \eta_j$$

for all $\underline{\eta} := (\eta_1, \cdots, \eta_N) \in \mathcal{M}(\underline{\xi}).$

We assume that:

Assumption 5. $0 \in \mathfrak{g}$ is a regular value for $\mu_{\underline{\xi}} : \mathcal{M}(\underline{\xi}) \to \mathfrak{g}$ and $\mu_{\underline{\xi}}^{-1}(0) \neq \emptyset$.

Remark. By Sard's theorem, the set

(11)
$$\mathcal{A} := \left\{ \underline{\xi} \in \overbrace{\mathfrak{t} \times \cdots \times \mathfrak{t}}^{N} : \text{ Assumptions 3, 5 hold} \right\}$$

has nonempty interior in $\mathfrak{t} \times \cdots \times \mathfrak{t}$.

Then, the level set

$$\mathcal{M}_0(\underline{\xi}) := \mu_{\xi}^{-1}(0)$$

is a closed, thus compact, submanifold of $\mathcal{M}(\underline{\xi})$ and the diagonal action Δ of G restricts to an action on $\mathcal{M}_0(\underline{\xi})$. Therefore, we can form the quotient space (or symplectic reduction) with respect to this action of G on $\mathcal{M}_0(\xi)$:

(12)
$$\mathcal{M}_{\text{red}}(\xi) := \mathcal{M}_0(\xi)/G.$$

The quotient space is also compact.

If the G-action on $\mathcal{M}_0(\underline{\xi})$ is free and proper (in our situation, properness is automatically satisfied), then the quotient space $\mathcal{M}_{\text{red}}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ is a smooth manifold. However, in our situation, the G-action on $\mathcal{M}_0(\underline{\xi})$ is in general not free. Hence in general it follows from the treatment in [12] that the quotient space is an orbifold [13] rather than a smooth manifold. To avoid this complication, we will make the following assumption.

Assumption 6. The quotient space $\mathcal{M}_{\text{red}}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ is a smooth compact manifold.

Assumption 6 is satisfied provided the stabilizer of the action of G at all points in $\mathcal{M}_0(\xi)$ is the identity.

Remark. The above assumption will put further restrictions on which $\underline{\xi} \in \mathfrak{t} \times \cdots \times \mathfrak{t}$ we can choose as initial data. Thus we only choose initial data from the following set:

(13)
$$\mathcal{A}' := \left\{ \underline{\xi} \in \overbrace{\mathfrak{t} \times \cdots \times \mathfrak{t}}^{N} : \text{ Assumptions 3, 5, and 6 hold} \right\}$$

Notice that since the elements in the center of G always act trivially on $\mathcal{M}(\underline{\xi})$ and $\mathcal{M}_0(\underline{\xi})$, Assumption 6 is valid if PG = G/Z(G) acts freely on $\mathcal{M}_0(\underline{\xi})$. This happens for G = SU(n) if all the coadjoint orbits $\mathcal{O}(\xi_i)$ are generic.

Then, we have the following well known theorem:

Theorem 7 (Marsden-Weinstein). The smooth compact manifold

$$\mathcal{M}_{\rm red}(\xi) = \mathcal{M}_0(\xi)/G$$

carries a unique symplectic structure $\omega_{\rm red}(\xi)$ such that

(14)
$$i^* \omega_{\underline{\xi}} = \pi^* \omega_{\text{red}}(\underline{\xi})$$

where $i: \mathcal{M}_0(\underline{\xi}) \hookrightarrow \mathcal{M}(\underline{\xi})$ is the inclusion map and $\pi: \mathcal{M}_0(\underline{\xi}) \rightarrow \mathcal{M}_{\mathrm{red}}(\underline{\xi})$ is the associated projection map.

Definition 8. We call this compact symplectic manifold

$$(\mathcal{M}_{\mathrm{red}}(\xi), \omega_{\mathrm{red}}(\xi))$$

an N-fold reduced product.

Remark. The dimension of an N-fold reduced product is

(15)
$$N(\dim G - \dim T) - 2\dim G = (N-2)\dim G - N\dim T$$

when all orbits are generic. In the case $G=\mathrm{SU}(3)$ and N=3, this is $\dim G-3\dim T=8-6=2$. These reduced products are diffeomorphic to the 2-sphere [18].

Remark. If the initial point $\underline{\xi}$ is clear from the context, we will suppress the inclusion of the point $\underline{\xi}$ in our notations and write, for example, $\mathcal{M}, \mathcal{M}_0, \mathcal{M}_{\text{red}}$ instead of $\mathcal{M}(\underline{\xi}), \mathcal{M}_0(\underline{\xi}), \mathcal{M}_{\text{red}}(\underline{\xi})$, respectively. Similarly, this is done for the notations of the symplectic structures and so on.

3. Intersection pairings of N-fold reduced products

3.1. **Introduction.** In our previous paper [15], we investigated the symplectic volume of N-fold reduced products and derived the following formula for all generic N-fold reduced products:

Theorem 9. In the notation introdued earlier, and under the hypotheses imposed in the previous section, we have

(16)
$$\int_{\mathcal{M}_{\text{red}}} e^{i\omega_{\text{red}}} = \sum_{\underline{w} \in W^N} \operatorname{sgn}(\underline{w}) \int_{X \in \mathfrak{t}} \frac{e^{i\langle \mu_T(\underline{w} \cdot \underline{\xi}), X \rangle}}{\overline{\omega}^{N-2}(X)} dX$$

where $\mu_T : \mathcal{M} \to \mathfrak{t}$ is the moment map for the T-action on \mathcal{M} , $\underline{\xi} = (\xi_1, \dots, \xi_N) \in (\mathfrak{t})^N$ is generic, $\underline{w} = (w_1, \dots, w_N) \in W^N$ and

(17)
$$\varpi(X) = \prod_{\gamma} \langle \gamma, X \rangle$$

where γ runs over all the positive roots of G.

3.2. Equivariant cohomology and the Cartan model. The main tool we used to prove Theorem 9 is the Atiyah-Bott-Berline-Vergne localization formula. (See [16].) We make use of the Cartan model for equivariant cohomology (see for example [24]). In this model, an equivariant differential form is represented by a linear combination of differential forms α_i with polynomial dependence on a parameter

$$X \in \mathfrak{t}$$
.

We assume α_j has degree j in X. The grading is the sum of the differential form grading and two times the degree as a polynomial in X. The differential is

$$d_X = d - \iota_{X^{\sharp}}$$

where ι denotes interior product. Recall that X^{\sharp} is the fundamental vector field generated by the action of X. For example, the extension of the symplectic form to an equivariantly closed form is

$$\bar{\omega}(X) = \omega + \mu_X$$

where μ_X is the moment map associated to X (in other words the function whose Hamiltonian vector field is X^{\sharp}).

An equivariant m-form α in the Cartan model is a sum of terms α_j for $2j \leq m$, where the degree of α_j as a differential form is m-2j. If the differential form degree is 0, then j=m/2 where m is the (real) dimension of the manifold.

The restriction of α to a fixed point of the T action is $\alpha_{m/2}$ (the term of degree 0 as a differential form). If the form α is equivariantly closed, it follows that

$$d\alpha_i = \iota_{X^{\sharp}} \alpha_{i-1}$$

for all j.

Let M be a Hamiltonian G-manifold. The Kirwan map, which we shall denote by κ , is a map from $H_G^*(M)$ to $H_G^*(M_0)$, where M_0 is defined as the zero level set of the moment map on M. It is the restriction map to a level set of the moment map. If 0 is a regular value of the moment map, then $H_G^*(M_0) \cong H^*(M_0/G)$. When 0 is a regular value of the moment map, Kirwan proved that the map κ is surjective [20].

3.3. Cohomology of orbits. For an adjoint orbit homeomorphic to G/T, we see (for example from [9], Chap. 10.2 Proposition 3) that the cohomology is generated multiplicatively by the first Chern classes of line bundles L_{β} over the orbit, where

(18)
$$L_{\beta} = G \times_{T,\beta} \mathbf{C}$$

where we write the orbit as G/T and the equivalence relation is

$$(g,z) \sim (gt,\beta(t)^{-1}z)$$

for $g \in G$, $t \in T$, $z \in \mathbf{C}$ and for a weight $\beta \in \text{Hom}(T, U(1))$. For example, for G = SU(n), the collection of β comprising the simple roots of G gives rise to a basis for the cohomology of G/T. For G = SU(n), a proof of this result can be found in Fulton's book [9] (Chapter 10.2, Proposition 3). For general Lie groups this is Theorem 5 in Section 4 in the article by Tu [26].

We can write each weight β as

$$\beta(\exp X) = \exp(2\pi B(X))$$

for a linear map $B: \mathfrak{t} \to \mathbf{R}$ which sends the integer lattice (the kernel of the exponential map) to \mathbf{Z} . Here we have used the exponential map $\exp: \mathfrak{t} \to T$. The equivariant first Chern class of the line bundle L_{β} is denoted

$$c_1^{\text{eq}}(L_b).$$

Its restriction to an isolated fixed point F is

$$c_1^{\text{eq}}(L_b)|_F = c_1(L_\beta)|_F + B(X).$$

The restriction of this equivariant first Chern class to a component F of the fixed point set is B(X). By naturality, we have that

(19)
$$\pi_j^* \left(c_1(L_j) \right) = c_1 \left(\pi_j^* L_j \right)$$

where

$$\pi_j: \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N} \to \mathcal{O}_{\xi_j}$$

is projection on the j-th orbit, and L_j is a line bundle over \mathcal{O}_{ξ_j} .

3.4. **Localization.** The Atiyah-Bott-Berline-Vergne localization formula leads to the following (see [16], Theorem 8.1):

$$\int_{M_{\text{red}}} \kappa(\alpha) = \operatorname{Res} \sum_{F} \alpha_{m/2}(X) \frac{e^{i\mu_X(F)}}{e_F(X)}.$$

In the case when M is the product of N adjoint orbits when

$$\alpha = \exp(i\bar{\omega})$$

is the equivariant extension of the symplectic volume form, and

$$\kappa(\alpha) = e^{i\omega_{\rm red}}$$

is the symplectic volume form on M_{red} . Theorem 9 may be expressed as follows.

(20)
$$\int_{M_{\text{red}}} \kappa(\alpha) = \operatorname{Res} \sum_{w \in W} e^{i(w\lambda, X)} \operatorname{sgn}(w) \frac{1}{(\varpi(X))^{N-2}}.$$

Equation (20) is the meaning of the integral over t in equation (16) whose definition is given in [11] and elaborated in [16]. The symbol Res (the residue) is defined in [16], Theorem 8.1. See also [17], Proposition 3.2. The residue has several equivalent definitions (as outlined in [17]). One of these definitions characterizes the residue as an iteration of one-variable residues.

Remark. One feature that is special to our situation (Cartesian products of adjoint orbits) is that all the equivariant Euler classes are the same, except for the sign (which is $\operatorname{sgn}(\underline{w})$, the product of the signatures of the permutations). Up to sign, the equivariant Euler class is a power $\varpi(X)^N$ of $\varpi(X)$ where ϖ is the product of positive roots.

In the above notation, we have the following generalization of Theorem 9:

Theorem 10. Let \mathcal{M} be as above, and let ζ be a G-equivariant cohomology class on \mathcal{M} . Let $\kappa: H_G^*(\mathcal{M}) \to H^*(\mathcal{M}_{red})$ be the Kirwan map. We have

(21)
$$\int_{\mathcal{M}_{\text{red}}} e^{i\omega_{\text{red}}} \kappa(\zeta) = \sum_{w \in W^N} \operatorname{sgn}(\underline{w}) \int_{X \in \mathfrak{t}} \frac{e^{i\langle \mu_T(\underline{w} \cdot \underline{\xi}), X \rangle} \zeta(X)|_{\underline{w} \cdot \underline{\xi}}}{\varpi(X)^{N-2}} dX.$$

(22)
$$= \operatorname{Res} \sum_{w \in W^{N}} e^{i(\underline{w} \cdot \underline{\xi}, X)} \operatorname{sgn}(\underline{w}) \frac{\zeta(X)|_{\underline{w} \cdot \underline{\xi}}}{(\varpi(X))^{N-2}}.$$

Here $\zeta(X)$ is a product of powers of a collection of equivariant first Chern classes $(c_1^{eq}(L_{\beta_\ell}(X))^{n_\ell})$ where the index ℓ runs from 1 to N if we are considering the reduced space of the product of N orbits and n_ℓ is a nonnegative integer, and the weight of the ℓ -th line bundle is β_ℓ with associated linear map B_ℓ . The restriction of ζ to the fixed point set of the T action is

$$\prod_{\ell} \left(B_{\ell}(X) \right)^{n_{\ell}}.$$

Remark. Theorem 10 describes all intersection pairings between cohomology classes of \mathcal{M}_{red} .

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