Cohomology of group theoretic Dehn fillings III: Applications

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Abstract

This is the third paper in a series of three papers studying cohomology of group theoretic Dehn fillings. In the present paper, we apply the spectral sequence constructed in the previous two papers [Sun18, Sun19] to prove several results concerning cohomological properties of Dehn fillings. As a further application, we improve known results on SQ-universality and common quotients of acylindrically hyperbolic groups by adding cohomological finiteness conditions.

1 Introduction

1.1 Dehn surgery in 3-manifolds. In 3-dimensional topology, *Dehn surgery* is a method of modifiying a 3-manifold by cutting off a solid torus and then gluing the torus back in a different way. This notion is partially motivated by the Lickorish-Wallace theorem, which states that every closed connected orientable 3-manifold can be constructed from the 3-sphere by using finitely many Dehn surgeries.

The second step of the surgery, called *Dehn filling*, starts with a 3-manifold M with toral boundary and constructs a new manifold by gluing a solid torus to M by identifying their boundaries. Topologically distinct ways of gluing a solid torus are parametrized by free homotopy classes of essential simple closed curves on ∂M, called *slopes*. For a slope s, the new manifold constructed by the corresponding Dehn filling is denoted as M_s. A celebrated result of Thurston asserts that most Dehn fillings preserve hyperbolicity. More precisely,

Theorem 1.1 ([Thu82, Theorem [TH1]]). *Let M be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite volume hyperbolic structure. Then for all but finitely many slopes s on ∂M, M_s admits a hyperbolic structure.*

1.2 Group theoretic Dehn fillings. There is an analogous construction in group theory, called *(group theoretic) Dehn filling*, which can be formalized as follows. Given a group G with a subgroup H and a normal subgroup N of H, the Dehn filling associated with the triple (G, H, N) is the quotient $G/\langle\langle N\rangle\rangle$, where $\langle\langle N\rangle\rangle$ is the normal closure of N in G.

The relation between these two versions of Dehn fillings can be seen via the following example: Under the assumptions of Theorem 1.1, the natural homomorphism $\pi_1(\partial M) \to \pi_1(M)$ is injective and thus $\pi_1(M)$ can be thought of as a subgroup of $\pi_1(M)$. Let $G =$

$\pi_1(M)$ and $H = \pi_1(\partial M)$. Every slope s on ∂M generates a normal subgroup $N_s \lhd H$ and we have $\pi_1(M_s) = G/\langle\langle N_s \rangle\rangle$ by the Seifert-van Kampen theorem.

Dehn filling is a fundamental tool in group theory. It appears, for instance, in the solution of the virtually Haken conjecture [AGM13], the study of Farrell-Jones conjecture and isomorphism problem of relatively hyperbolic groups [ACG18, DG18], and the construction of purely pseudo-Anosov normal subgroups of mapping class groups [DGO17]. Other applications of Dehn fillings can be found, for example, in [AGM16, GMS16].

An algebraic analog of Theorem 1.1 can be proved for groups satisfying certain negative curvature conditions. The first result of this kind was for relatively hyperbolic groups by Osin [Osi07] and independently, by Groves-Manning [GM08]. Later, [DGO17] introduced a generalization of relative hyperbolicity based on the notion of a hyperbolically embedded subgroup and proved a generalization of the main results of [Osi07, GM08]. We postpone the definition of hyperbolically embedded subgroups to Section 3.3 and only discuss several examples for the moment. For other examples of hyperbolically embedded subgroups, the reader is referred to the survey [Osi18]. Below, we use $H \hookrightarrow_h G$ to indicate that H is a hyperbolically embedded subgroup of G.

Example 1.2. If G is a group hyperbolic relative to its subgroup H, then $H \hookrightarrow_h G$ [DGO17, Proposition 2.4]. In particular, if M is a compact orientable manifold with one boundary component and $M \setminus \partial M$ admits a complete finite volume hyperbolic structure, then $\pi_1(\partial M) \hookrightarrow_h \pi_1(M)$ [Bow12, Far98].

Example 1.3. If a group G acts on a Gromov hyperbolic space S acylindrically by isometries and $g \in G$ is a loxodromic element, then there exists a maximal virtually-cyclic subgroup $E(g) \leqslant G$ containing g such that $E(g) \hookrightarrow_h G$ [DGO17, Corollary 2.9]. In particular, if G is a hyperbolic group (resp. mapping class group of a finite type surface [DGO17, Theorem 2.19], outer automorphism group of a finite rank free group [DGO17, Theorem 2.20]) and g is an infinite order (resp. a pseudo-Anosov, a fully irreducible) element, then $E(g) \hookrightarrow_h G$.

Theorem 1.4 ([DGO17, Theorem 2.27])**.** *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then there exists a finite set $\mathcal{F} \subset H \setminus \{1\}$ such that if $N \lhd H$ and $N \cap \mathcal{F} = \emptyset$, then the natural homomorphism $H/N \to G/\langle\langle N \rangle\rangle$ maps H/N injectively onto a hyperbolically embedded subgroup of $G/\langle\langle N \rangle\rangle$.*

Under the assumptions of Theorem 1.1, Theorem 1.4 and some basic facts about relatively hyperbolic groups imply that $\pi_1(M_s)$ is non-virtually-cyclic and word-hyperbolic for all but finitely many slopes s. The geometrization conjecture, proved by Perelman, implies that this algebraic statement about $\pi_1(M_s)$ is equivalent to the hyperbolicity of M_s. Thus, Theorem 1.4 generalizes Theorem 1.1.

1.3 Motivation: a question on cohomology. Theorem 1.1 asserts that M_s is often hyperbolic and thus its universal cover is $\mathbb{H}^3$, which is contractible. It follows that M_s is a model of the classifying space of $\pi_1(M_s)$ and thus the group cohomology $\mathrm{H}^*(\pi_1(M_s); \cdot)$ can be computed using M_s.

On the other hand, Theorem 1.4 cannot be used to obtain any information about cohomology as there is no geometry involved in it. Therefore, it is natural to ask the following.

Question A. *For a group G with a subgroup $H \hookrightarrow_h G$ and a normal subgroup $N \lhd H$, what can be said about $\mathrm{H}^*(G/\langle\langle N \rangle\rangle; \cdot)$?*

The main goal of this series of three papers is to answer the above question. In the previous two papers [Sun18, Sun19], we have already derived a spectral sequence for Dehn fillings. In the present paper, we apply this spectral sequence to prove several results concerning cohomological properties of Dehn fillings. As a further application, we improve known results on SQ-universality and common quotients of acylindrically hyperbolic groups by adding cohomological finiteness conditions.

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2 Main results

2.1 A spectral sequence for Dehn fillings. We start by recalling the main result of [Sun19]. Given a group G with a subgroup H and a normal subgroup N of H, we introduce the following notation

$$\overline{G} = G/\langle\langle N \rangle\rangle, \quad \overline{H} = H/N.$$

Definition 2.1. Let G be a group with a subgroup $H \hookrightarrow_h G$. We say that a property P holds *for every sufficiently deep* normal subgroup $N \lhd H$ if there exists a finite set $\mathcal{F} \subset H \setminus \{1\}$ such that P holds for every normal subgroup $N \lhd H$ with $N \cap \mathcal{F} = \emptyset$.

Theorem 2.2 ([Sun19, Theorem 2.5])**.** *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then for every sufficiently deep normal subgroup $N \lhd H$ and every $\mathbb{Z}\overline{G}$-module A, there is a spectral sequence of cohomological type*

$$E_2^{p,q} = \begin{cases} \mathrm{H}^p(\overline{H}; \mathrm{H}^q(N; A)) & , \textit{ if } q \neq 0 \\ \mathrm{H}^p(\overline{G}; A) & , \textit{ if } q = 0 \end{cases} \Rightarrow \mathrm{H}^{p+q}(G; A). \tag{1}$$

Here, the action of G on A factors through $\overline{G}$. In particular, the action of N on A fixes every element of A.

Remark 2.3. Historically, spectral sequences were first introduced by [Ler46] in order to compute sheaf cohomology. The spectral sequence (1) can be thought of as a refined version of the classical Lyndon-Hochschild-Serre spectral sequence [HS53, Lyn48] in the settings of Dehn fillings.

Given certain cohomological properties of the groups $G, H, \overline{H}$, we apply spectral sequence (1) to derive the corresponding properties of $\overline{G}$.

2.2 Cohomological properties of Dehn fillings. Recall that the *cohomological dimension* of a group G is

$$\mathrm{cd}(G) = \sup\{\ell \in \mathbb{N} \mid \mathrm{H}^\ell(G; A) \neq \{0\} \text{ for some } \mathbb{Z}G\text{-module } A\}.$$

(In this paper, the set $\mathbb{N}$ of natural numbers contains 0, while the set of positive natural numbers is denoted as $\mathbb{N}^+$.)

Theorem 2.4. *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then for every sufficiently deep normal subgroup $N \lhd H$, every $\ell \geqslant \mathrm{cd}(H) + 2$, and every $\mathbb{Z}\overline{G}$-module A, we have*

$$\mathrm{H}^\ell(\overline{G}; A) \cong \mathrm{H}^\ell(G; A) \oplus \mathrm{H}^\ell(\overline{H}; A). \tag{2}$$

In particular,

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(H) + 1, \mathrm{cd}(\overline{H})\}.$$

Notice that, the direct sum decomposition (2) does not hold for $\ell \leqslant \mathrm{cd}(H)+1$, as shown by the following.

Example 2.5. Let G be a group freely generated by two elements x, y and let $H = \langle h \rangle \leqslant G$ where $h = xyx^{-1}y^{-1}$. Then $H \hookrightarrow_h G$ by Example 1.3 and $\mathrm{cd}(H)+1 = 2$. Let $N = \langle h^k \rangle \lhd H$. Note that we can pick k large enough so that N avoids any given finite subset of $H \setminus \{1\}$. By [Lyn50, Theorem 11.1], $\mathrm{H}^2(\overline{G}; \mathbb{Z}) \cong \mathbb{Z}$, and it is well-known that $\mathrm{H}^2(G; \mathbb{Z}) = \{0\}$ and $\mathrm{H}^2(\overline{H}; \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$. Thus, $\mathrm{H}^2(\overline{G}; \mathbb{Z}) \ncong \mathrm{H}^2(G; \mathbb{Z}) \oplus \mathrm{H}^2(\overline{H}; \mathbb{Z})$.

In case G is a free group and $H \leqslant G$ is a maximal cyclic subgroup, the direct sum decomposition (2) is proved by [Lyn50, Theorem 11.1]. In case $G = G_1 * G_2$ is a free product of locally indicable groups G_1, G_2 and $H \leqslant G$ is the cyclic subgroup generated by an element $g \in G$ such that g is not a proper power and does not conjugate into either G_1 or G_2, the direct sum decomposition (2) is proved by [How84, Theorem 3]. Note that in these two cases, we have $H \hookrightarrow_h G$ by Example 1.3. Thus, Theorem 2.4 recovers the results of [Lyn50, How84] for sufficiently deep (but not all) normal subgroups.

If a group G is hyperbolic relative to its subgroup H (in particular, $H \hookrightarrow_h G$, by Example 1.2) and both G and H are of type FP_∞, then for any sufficiently deep normal subgroup $N \lhd H$, [Wan18, Theorem 1.1] provides a spectral sequence of homological type which computes the relative cohomology $\mathrm{H}^*(\overline{G}, \overline{H}; \mathbb{Z}\overline{G})$ from certain combination of homology and cohomology. One should not confuse the spectral sequence (1) with the spectral sequence of [Wan18], as only cohomology is involved in (1).

It is worth noting that one can also use (1) to compute relative cohomology. In fact, as a byproduct of the proof of Theorem 2.4, we show that, for $\ell \geqslant \mathrm{cd}(H) + 3$, we have $\mathrm{H}^\ell(\overline{G}, \overline{H}; A) \cong \mathrm{H}^\ell(G; A)$ (see Corollary 4.11).

Also recall that for $k \in \mathbb{N}^+ \cup \{\infty\}$, a group G is of *type FP_k* if there is a projective resolution

$$\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

over $\mathbb{Z}G$ such that P_ℓ is finitely generated for every $\ell \in \mathbb{N}$ with $\ell \leqslant k$. If a group G is of type FP_∞ and $\mathrm{cd}(G) < \infty$, then one says that G is of *type FP*.

Theorem 2.6. *Let G be a group of type FP_k for some $k \in \mathbb{N}^+ \cup \{\infty\}$ (resp. FP) with a subgroup $H \hookrightarrow_h G$. Then $\overline{G}$ is of type FP_k (resp. FP) for every sufficiently deep normal subgroup $N \lhd H$ such that $\overline{H}$ is of type FP_k (resp. FP).*

In the above theorem, the condition that $\overline{H}$ is of type FP_k is necessary. Indeed, for every sufficiently deep normal subgroup $N \lhd H$, we have $\overline{H} \hookrightarrow_h \overline{G}$ by Theorem 1.4. If $\overline{G}$ is of type FP_k, then so is $\overline{H}$ [DGO17, Theorem 2.11].

Remark 2.7. In fact, the present paper proves more general results for families of weakly hyperbolically embedded subgroups. The corresponding general versions of Theorems 2.4 and 2.6 can be applied to graph of groups (see Example 3.14 and Theorems 3.23, 4.8,4.13).

2.3 Quotients of acylindrically hyperbolic groups. The notion of an acylindrically hyperbolic group was introduced by Osin [Osi16] as a generalization of non-elementary hyperbolic and non-elementary relatively hyperbolic groups. Examples of acylindrically hyperbolic groups can be found in many classes of group that interest group theorists for years, e.g., mapping class groups of surfaces [MM99, Bow08], outer automorphism groups of free groups [BF10], small cancellation groups [GS18], convergence groups [Sun17], the Cremona group (see [DGO17] and references therein; the main contribution towards this result is due to [CL13]), tame automorphism groups of three-dimensional affine spaces [LP19], etc. We refer to [Osi18] for details and other examples.

Every acylindrically hyperbolic group G contains hyperbolically embedded subgroups [DGO17, Theorem 6.14] and Dehn fillings can often be applied to construct useful quotients of G. For instance, [DGO17, Theorem 2.33] proves that G is *SQ-universal*, i.e., every countable group can be embedded into a quotient of G. As an application of our main results, we study cohomological properties of those quotients.

Recall that every acylindrically hyperbolic group G has a maximal finite normal subgroup denoted as $K(G)$ [DGO17, Theorem 6.14].

Theorem 2.8. *Let G be an acylindrically hyperbolic group, and let C be any countable group. Then C embeds into a quotient $\overline{G}$ of $G/K(G)$ (in particular, $\overline{G}$ is a quotient of G) such that*

(a) *$\overline{G}$ is acylindrically hyperbolic;*

(b) $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(C)\}$;

(c) *for every $\ell \geqslant 3$ and every $\mathbb{Z}\overline{G}$-module A, we have*

$$\mathrm{H}^\ell(\overline{G}; A) \cong \mathrm{H}^\ell(G/K(G); A) \oplus \mathrm{H}^\ell(C; A),$$

where the action of $G/K(G)$ (resp. C) on A is induced by the quotient map $G/K(G) \to \overline{G}$ (resp. the embedding $C \hookrightarrow \overline{G}$);

(d) *if C is finitely generated, then $C \hookrightarrow_h \overline{G}$;*

(e) *if for some $k \in \mathbb{N}^+ \cup \{\infty\}$, G and C are of type FP_k, then so is $\overline{G}$.*

If two finitely generated acylindrically hyperbolic groups G_1 and G_2 are given, one can construct a common acylindrically hyperbolic quotient of G_1 and G_2 [Hul16, Corollary 7.4]. Such a construction was used in [Hul16] and [MO19] to exhibit groups with various interesting properties. Below is an improvement of [Hul16, Corollary 7.4].

Theorem 2.9. *Let G_1 and G_2 be finitely generated acylindrically hyperbolic groups. Then there exists a common quotient G of $G_1/K(G_1)$ and $G_2/K(G_2)$ (in particular, G is a common quotient of G_1 and G_2) such that*

(a) *G is acylindrically hyperbolic;*

(b) $\mathrm{cd}(G) \leqslant \max\{\mathrm{cd}(G_1), \mathrm{cd}(G_2)\}$;

(c) *for every $\ell \geqslant 3$ and every $\mathbb{Z}G$-module A, we have*

$$\mathrm{H}^\ell(G; A) \cong \mathrm{H}^\ell(G_1/K(G_1); A) \oplus \mathrm{H}^\ell(G_2/K(G_2); A),$$

where the actions of $G_1/K(G_1)$ and $G_2/K(G_2)$ on A factor through G;

(d) *if for some $k \in \{2, 3, ..., \infty\}$, G_1 and G_2 are of type FP_k, then so is G.*

Except for the cohomological conditions, Theorems 2.8 and 2.9 are proved by [DGO17, Theorem 2.33] and [Hul16, Corollary 7.4], respectively. The benefit of Theorems 2.8 and 2.9 is that they allow constructions of various acylindrically hyperbolic groups satisfying certain cohomological properties.

2.4 Organization of the paper. We start by recalling preliminary definitions and results of the previous two papers in Section 3, which reviews spectral sequences and (weakly) hyperbolically embedded subgroups. The reader is referred to [Sun18, Sun19] for details. The proofs of Theorems 2.4, 2.6, 2.8, and 2.9 rely on computations with spectral sequences. In Section 4, we first perform such a computation and then prove the main results.

3 Preliminaries

We start with conventions and notations. Throughout this paper, all group actions (resp. modules) are left actions (resp. modules). Let G be a group. If G is the free product of its subgroups $G_\lambda, \lambda \in \Lambda$, then we write $G = \prod^*_{\lambda\in\Lambda} G_\lambda$. If S is a subset of G, then $\langle\langle S \rangle\rangle$ is the normal closure of S in G. Let H be a subgroup of G. Then $LT(H, G)$ is the set of left transversals of H in G.

Let X be a generating set of G and consider the Cayley graph $\Gamma(G, X)$. If p is a path in $\Gamma(G, X)$, then the label of p is denoted as $\mathbf{Lab}(p)$, the length p is denoted as $\ell_X(p)$, and the initial (resp. terminal) vertex of p is denoted as p^- (resp. p^+). If w is a word over X, then $\|w\|$ denotes the number of letters in w. In certain cases, it might be possible to view

w as a word over another alphabet Y. In such a case, we use $\|w\|_X$ (resp. $\|w\|_Y$) to denote the number of letters of X (resp. Y) in w.

We briefly review the conjugation action of a group on cohomology of its normal subgroups. For details, the reader is referred to [Bro94, Chapter III.8]. Suppose that A is a $\mathbb{Z}G$-module, H is a normal subgroup of G, and $P \to \mathbb{Z}$ is a projective resolution over $\mathbb{Z}G$. Then $\mathrm{Hom}_{\mathbb{Z}H}(P, A)$ admits a G/H-action induced by the following G-action

$$(g \circ f)(x) = g \cdot f(g^{-1} \cdot x) \text{ for all } g \in G, x \in P, f \in \mathrm{Hom}_{\mathbb{Z}H}(P, A). \tag{3}$$

Formula (3) induces an action of G/H on $\mathrm{H}^*(H; A)$.

3.1 Spectral sequences

The proofs in this paper rely on computations with spectral sequences. For details the reader is referred to [Rot09, Wei94]. In this section, we only clarify notations.

We exclusively work with first quadrant spectral sequences of cohomological type. Such a spectral sequence is denoted as $E = (E_r, d_r)_{r\geqslant a}$, where E_r is the E_r-page of E and d_r is the differential of E_r.

Given two spectral sequences $E_1 = (E_{1,r}, d_{1,r})_{r\geqslant a}, E_2 = (E_{2,r}, d_{2,r})_{r\geqslant a}$, we denote a morphism of spectral sequences from E_1 to E_2 as

$$\phi = (\phi_r)_{r\geqslant b} : E_1 \to E_2,$$

where $\phi_r : E_{1,r} \to E_{2,r}$ is the restriction of ϕ and it is understood that $b \geqslant a$ and ϕ_r is defined only for $r \geqslant b$. For $p, q \in \mathbb{Z}$, we write

$$\phi_r^{p,q} : E_{1,r}^{p,q} \to E_{2,r}^{p,q}, \qquad \phi_r^{*,q} : E_{1,r}^{*,q} \to E_{2,r}^{*,q}, \qquad \phi_r^{p,*} : E_{1,r}^{p,*} \to E_{2,r}^{p,*}$$

for the maps induced by ϕ. Similarly,

$$d_{1,r}^{p,q} : E_{1,r}^{p,q} \to E_{1,r}^{p+r,q+r-1}$$

is the map induced by $d_{1,r}$. Moreover, if

$$f : H_1 = \bigoplus_{\ell\geqslant 0} H_1^\ell \to H_2 = \bigoplus_{\ell\geqslant 0} H_2^\ell$$

is a morphism of graded abelian groups of degree 0, then

$$f^\ell : H_1^\ell \to H_2^\ell$$

denotes the map induced by f.

We recall the notions of convergence and compatibility.

Definition 3.1. A spectral sequence $E = (E_r, d_r)_{r\geqslant a}$ *converges* to a graded abelian group $H = \bigoplus_{\ell\geqslant 0} H^\ell$, denoted as $E_a^{p,q} \Rightarrow H^{p+q}$, if for every $\ell \geqslant 0$, there exist $R > 0$ and a filtration

$$\{0\} = F_{\ell+1}H^\ell \subset \cdots \subset F_0 H^\ell = H^\ell$$

of H^ℓ such that $F_k H^\ell / F_{k+1} H^\ell \cong E_r^{l-k,k}$ as abelian groups for $r \geqslant R$.

Definition 3.2. Let $\phi = (\phi_r)_{r\geqslant b} : E_1 \to E_2$ (resp. $f : H_1 \to H_2$) be a morphism between spectral sequences $E_1 = (E_{1,r}, d_{1,r})_{r\geqslant a}$ and $E_2 = (E_{2,r}, d_{2,r})_{r\geqslant a}$ (resp. graded abelian groups $H_1 = \bigoplus_{\ell\geqslant 0} H_1^\ell$ and $H_2 = \bigoplus_{\ell\geqslant 0} H_2^\ell$ of degree 0). Suppose

$$E_{1,a}^{p,q} \Rightarrow H_1^{p+q}, \quad E_{2,a}^{p,q} \Rightarrow H_2^{p+q}.$$

Then ϕ and f are *compatible* if for every $\ell \geqslant 0$, there exists a number $R > 0$ and filtrations

$$\{0\} = F_{\ell+1}H_1^\ell \subset \cdots \subset F_0 H_1^\ell = H_1^\ell, \quad \{0\} = F_{\ell+1}H_2^\ell \subset \cdots \subset F_0 H_2^\ell = H_2^\ell$$

such that

$$f(F_k H_1^\ell) \subset F_k H_2^\ell$$

for $k = 0, ..., \ell + 1$ and for every $r \geqslant R$, there exist isomorphisms

$$\sigma : F_k H_1^\ell / F_{k+1} H_1^\ell \to E_{1,r}^{l-k,k}, \quad \tau : F_k H_2^\ell / F_{k+1} H_2^\ell \to E_{2,r}^{l-k,k}$$

with $\phi^{l-k,k} \circ \sigma = \tau \circ \overline{f}$ for $k = 0, ..., \ell$, where

$$\overline{f} : F_k H_1^\ell / F_{k+1} H_1^\ell \to F_k H_2^\ell / F_{k+1} H_2^\ell$$

is the map induced by f.

The following simple observation is useful in Section 4.1 when we perform computations with spectral sequences.

Remark 3.3. If $p, q \geqslant 0, p + q = \ell$, and $r \geqslant \max\{a, \ell + 2\}$, then the target of $d_r^{p,q}$ is $E_r^{p+r,q-r+1} = \{0\}$ and the domain of $d_r^{p-r,q-r+1}$ is $E_r^{p-r,q+r-1} = \{0\}$. Therefore,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q-r+1}) = E_r^{p,q}.$$

Hence, it suffices to let $R = \max\{a, \ell + 2\}$ and $R = \max\{b, \ell + 2\}$ in the above Definitions 3.1 and 3.2, respectively.

3.2 Cohen-Lyndon triples

Let G be a group and let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of G. For every $\lambda \in \Lambda$, let N_λ be a normal subgroup of H_λ. For future reference, we specify the following notations.

Notation 3.4. Let

$$\mathcal{N} = \bigcup_{\lambda\in\Lambda} N_\lambda, \quad \overline{G} = G/\langle\langle \mathcal{N} \rangle\rangle, \quad \overline{H}_\lambda = H_\lambda / N_\lambda.$$

Given a $\mathbb{Z}\overline{G}$-module A, there is a natural action of G on A which factors through $\overline{G}$. In particular, the action of N_λ on A is trivial (i.e., fixes every element of A). In the sequel, the cohomology groups $\mathrm{H}^*(G; A), \mathrm{H}^*(H_\lambda; A)$, etc will be computed using this natural action.

The inclusions $H_\lambda \leqslant G$ and $N_\lambda \leqslant \langle\langle \mathcal{N} \rangle\rangle$ induce restriction maps

$$r_{H_\lambda} : \mathrm{H}^*(G; A) \to \mathrm{H}^*(H_\lambda; A), \quad r_{N_\lambda} : \mathrm{H}^*(\langle\langle \mathcal{N} \rangle\rangle; A) \to \mathrm{H}^*(N_\lambda; A).$$

Let

$$r_G : \mathrm{H}^*(G; A) \to \prod_{\lambda \in \Lambda} \mathrm{H}^*(H_\lambda; A)$$

be the map induced by the maps r_{H_λ}. We think of r_G as a map of graded abelian groups and write

$$r_G^\ell : \mathrm{H}^\ell(G; A) \to \prod_{\lambda \in \Lambda} \mathrm{H}^\ell(H_\lambda; A)$$

for the maps induced by r_G.

For each $\lambda \in \Lambda$, the map r_{N_λ} and the natural homomorphism $\overline{H}_\lambda \to \overline{G}$ induce a cohomology map

$$\psi_\lambda : \mathrm{H}^*(\overline{G}; \mathrm{H}^*(\langle\langle \mathcal{N} \rangle\rangle; A)) \to \mathrm{H}^*(\overline{H}_\lambda; \mathrm{H}^*(N_\lambda; A)).$$

Let

$$\psi : \mathrm{H}^*(\overline{G}; \mathrm{H}^*(\langle\langle \mathcal{N} \rangle\rangle; A)) \to \prod_{\lambda \in \Lambda} \mathrm{H}^*(\overline{H}_\lambda; \mathrm{H}^*(N_\lambda; A))$$

be the map induced by the maps ψ_λ.

For $p, q \in \mathbb{Z}$ and $\lambda \in \Lambda$, we write

$$\psi_\lambda^{p,q} : \mathrm{H}^p(\overline{G}; \mathrm{H}^q(\langle\langle \mathcal{N} \rangle\rangle; A)) \to \mathrm{H}^p(\overline{H}_\lambda; \mathrm{H}^q(N_\lambda; A)),$$

$$\psi^{p,q} : \mathrm{H}^p(\overline{G}; \mathrm{H}^q(\langle\langle \mathcal{N} \rangle\rangle; A)) \to \prod_{\lambda \in \Lambda} \mathrm{H}^p(\overline{H}_\lambda; \mathrm{H}^q(N_\lambda; A))$$

for the maps induced by ψ_λ and ψ, respectively.

Remark 3.5. It is well-known that $\mathrm{H}^0(\langle\langle \mathcal{N} \rangle\rangle; A) = A^{\langle\langle \mathcal{N} \rangle\rangle} = A$ as the action of $\langle\langle \mathcal{N} \rangle\rangle$ on A is trivial, where $A^{\langle\langle \mathcal{N} \rangle\rangle}$ is the $\langle\langle \mathcal{N} \rangle\rangle$-fixed points of A. Therefore, for $p \in \mathbb{Z}$ and $\lambda \in \Lambda$, $\psi_\lambda^{p,0} : \mathrm{H}^p(\overline{G}; A) \to \mathrm{H}^p(\overline{H}_\lambda; A)$ is the cohomology map induced by the natural homomorphism $\overline{H}_\lambda \to \overline{G}$.

Definition 3.6. We call $(G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})$ a *Cohen-Lyndon triple* if there exist left transversals $T_\lambda \in LT(H_\lambda \langle\langle \mathcal{N} \rangle\rangle, G), \lambda \in \Lambda$, such that

$$\langle\langle \mathcal{N} \rangle\rangle = \prod_{\lambda \in \Lambda, t \in T_\lambda}^{*} t N_\lambda t^{-1}.$$

The above property was first considered by Cohen-Lyndon [CL63], hence the name "Cohen-Lyndon triple". The interested reader is referred to [EH87, GMS16, Sun18] for more results about such triples.

The following are some of the cohomological properties of Cohen-Lyndon triples. For simplicity, we use notations defined in Notation 3.4 in the statements.

Proposition 3.7 ([Sun19, Corollary 4.5])**.** *Suppose that* $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ *is a Cohen-Lyndon triple. Then for all* $q \in \mathbb{Z}\setminus\{0\}$ *and every* $\mathbb{Z}\overline{G}$*-module* A*,* ψ *maps* $\mathrm{H}^*(\overline{G}; \mathrm{H}^q(\langle\langle\mathcal{N}\rangle\rangle; A))$ *isomorphically onto* $\prod_{\lambda\in\Lambda} \mathrm{H}^*(\overline{H}_\lambda; \mathrm{H}^q(N_\lambda; A))$*.*

Theorem 3.8 ([Sun19, Theorem 4.1])**.** *Let* $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ *be a Cohen-Lyndon triple. Then for every* $\mathbb{Z}\overline{G}$*-module* A*, there are spectral sequences of cohomological type*

$$E^{p,q}_{G,2} = \mathrm{H}^p(\overline{G}; \mathrm{H}^q(\langle\langle\mathcal{N}\rangle\rangle; A)) \Rightarrow \mathrm{H}^{p+q}(G; A), \tag{4}$$

$$E^{p,q}_{\mathcal{H},2} = \prod_{\lambda\in\Lambda} \mathrm{H}^p(\overline{H}_\lambda; \mathrm{H}^q(N_\lambda; A)) \Rightarrow \prod_{\lambda\in\Lambda} \mathrm{H}^{p+q}(H_\lambda; A), \tag{5}$$

and a morphism between spectral sequences $\phi = (\phi_r)_{r\geqslant 2}$ *from* (4) *to* (5) *such that*

(a) The maps ϕ *and* r_G *are compatible.*

(b) The map ϕ_2 *can be identified with the map* ψ *and thus* $\phi_2^{*,q}$ *is an isomorphism for all* $q \in \mathbb{Z}\setminus\{0\}$*.*

3.3 (Weakly) hyperbolically embedded subgroups and acylindrically hyperbolic groups

The notion of a family of (weakly) hyperbolically embedded subgroups were introduced by [DGO17]. In this section, we recall definitions and results necessary to the present paper.

Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of G, let X be a subset of G such that G is generated by X together with the union of all $H_\lambda, \lambda \in \Lambda$ (in which case X is called a *relative generating set* of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$), and let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$. Consider the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$. Note that, for $\lambda \in \Lambda$, there is a natural embedding $\Gamma(H_\lambda, H_\lambda) \hookrightarrow \Gamma(G, X \sqcup \mathcal{H})$.

Remark 3.9. We do allow $X \cap H_\lambda \neq \emptyset$ and $H_\lambda \cap H_\mu \neq \emptyset$ for $\lambda, \mu \in \Lambda$, in which case there will be multiple edges between some pairs of vertices of $\Gamma(G, X \sqcup \mathcal{H})$.

For $\lambda \in \Lambda$, an edge path in $\Gamma(G, X \sqcup \mathcal{H})$ between vertices of $\Gamma(H_\lambda, H_\lambda)$ is called H_λ-*admissible* if it does not contain any edge of $\Gamma(H_\lambda, H_\lambda)$. Note that an H_λ-admissible path is allowed to pass through vertices of $\Gamma(H_\lambda, H_\lambda)$.

Definition 3.10. For every pair of elements $h, k \in H_\lambda$, let $\widehat{d}_\lambda(h, k) \in [0, +\infty]$ be the length of a shortest H_λ-admissible path connecting the vertices labeled by h and k. If no such path exists, set $\widehat{d}_\lambda(h, k) = +\infty$. The laws of summation on $[0, +\infty)$ extend naturally to $[0, +\infty]$ and it is easy to verify that $\widehat{d}_\lambda : H_\lambda \times H_\lambda \to [0, +\infty]$ defines a metric on H_λ, which is called the *relative metric on* H_λ *with respect to* X.

Definition 3.11. We say that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ *weakly hyperbolically embeds into* (G, X) (denoted as $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$) if G is generated by the set X together with union of all $H_\lambda, \lambda \in \Lambda$, and the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is a Gromov hyperbolic space.

If $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ and for each $\lambda \in \Lambda$, the metric space $(H_\lambda, \widehat{d}_\lambda)$ is proper, i.e., every ball of finite radius contains only finitely many elements, then $\{H_\lambda\}_{\lambda\in\Lambda}$ *hyperbolically embeds into* (G, X) (denoted as $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$).

Further, we say that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ *hyperbolically embeds into* G, denoted as $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$, if there exists some subset $X \subset G$ such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$.

Notation 3.12. Let G, H be groups and let $X \subset G$. If $\{H\} \hookrightarrow_h (G, X)$, then we drop braces and write $H \hookrightarrow_h (G, X)$ and $H \hookrightarrow_h G$.

Example 3.13. If a group G can be decomposed as a free product $G = \prod^*_{\lambda\in\Lambda} G_\lambda$, then $\{G_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, \emptyset)$ [DGO17, Example 4.12].

Example 3.14. Suppose that $\mathcal{G}$ is a graph of groups. Let $\pi_1(\mathcal{G})$ be the fundamental group of $\mathcal{G}$, $\{G_v\}_{v\in V\mathcal{G}}$ the collection of vertex subgroups, and $\{G_e\}_{e\in E\mathcal{G}}$ the collection of edge subgroups. By [DGO17, Example 4.12], $\{G_v\}_{v\in V\mathcal{G}} \hookrightarrow_{wh} (\pi_1(\mathcal{G}), X)$ for any set $X \subset G$ consisting of stable letters (i.e., generators corresponding to edges of $\mathcal{G} \setminus T\mathcal{G}$, where $T\mathcal{G}$ is a spanning tree of $\mathcal{G}$).

Peripheral subgroups are also examples of hyperbolically embedded subgroups. The following definition and lemma come from [DGO17, Definition 3.6 and Proposition 4.28].

Definition 3.15. A group G is *hyperbolic relative* to its subgroup H if there exists a finite subset $X \subset G$ such that $H \hookrightarrow_h (G, X)$.

Lemma 3.16 ([DGO17])**.** *A group G is hyperbolic relative to its subgroup H if and only if G has a finite relative presentation with respect to H with a linear relative isoperimetric function.*

The reader is referred to [DGO17] for the definition of a linear relative isoperimetric function.

The following lemma, which is a combination of [DGO17, Remark 4.41 and Theorem 4.42], provides a convenient way to construct hyperbolically embedded subgroups and will be used in the proof of Theorem 2.9.

Lemma 3.17 ([DGO17, Lemma 4.21])**.** *Suppose that* $\operatorname{card}(\Lambda) < \infty$*, G acts on a Gromov hyperbolic space (S, d) by isometries, and the following three conditions are satisfied, then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$.*

(C_1) Every H_λ acts on S properly.

(C_2) There exists $s \in S$ such that for every $\lambda \in \Lambda$, the H_λ-orbit $H_\lambda(s)$ of s is quasi-convex in S.

(C_3) For every $\epsilon > 0$ and some $s \in S$, there exists $R > 0$ such that the following holds. Suppose that for some $g \in G$ and $\lambda, \mu \in \Lambda$, we have

$$\operatorname{diam}\left(H_\mu(s) \cap (gH_\lambda(s))^{+\epsilon}\right) \geqslant R,$$

then $\lambda = \mu$ and $g \in H_\lambda$, where $(gH_\lambda(s))^{+\epsilon}$ denotes the ϵ-neighborhood of $gH_\lambda(s)$ in S.

We recall two useful properties of hyperbolically embedded subgroups. The next lemma will be used to alter relative generating sets.

Lemma 3.18 ([DGO17, Corollary 4.27])**.** *Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of G, and let $X_1, X_2 \subset G$ be relative generating sets of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$. Suppose* $\operatorname{card}(X_1 \Delta X_2) < \infty$. *Then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X_1)$ if and only if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X_2)$.*

"Being a hyperbolically embedded subgroup" is a transitive property. More precisely:

Proposition 3.19 ([DGO17, Proposition 4.35])**.** *Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite family of subgroups of G, let $X \subset G$, and let $Y_\lambda \subset H_\lambda$ for every $\lambda \in \Lambda$. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$ and for every $\lambda \in \Lambda$, there is a family of subgroups $\{K_{\lambda,\mu}\}_{\mu\in M_\lambda} \hookrightarrow_h (H_\lambda, Y_\lambda)$. Then*

$$\bigcup_{\lambda\in\Lambda} \{K_{\lambda,\mu}\}_{\mu\in M_\lambda} \hookrightarrow_h \left(G, X \cup \left(\bigcup_{\lambda\in\Lambda} Y_\lambda\right)\right).$$

Definition 3.20. Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some subset $X \subset G$. For every $\lambda \in \Lambda$, let $\widehat{d}_\lambda$ be the relative metric on H_λ with respect to X. A property P *holds for all sufficiently deep normal subgroups* $N_\lambda \lhd H_\lambda, \lambda \in \Lambda$, if there exists a number $C > 0$ such that if $N_\lambda \lhd H_\lambda$ and $\widehat{d}_\lambda(1, n) > C$ for all $n \in N_\lambda \setminus \{1\}$ and $\lambda \in \Lambda$, then P holds.

In Definition 3.20, if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$, then the relative metrics $\widehat{d}_\lambda$ are locally finite. Thus,

$$\operatorname{card}\left(\{h \in H_\lambda \mid \widehat{d}_\lambda(1, h) \leqslant C\}\right) < \infty$$

for all $C > 0$. Therefore,

Lemma 3.21. *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Suppose that a property P holds for all sufficiently deep normal subgroups $N_\lambda \lhd H_\lambda$. Then there exist finite sets $\mathcal{F}_\lambda \subset H_\lambda \setminus \{1\}, \lambda \in \Lambda$, such that P holds whenever $N_\lambda \cap \mathcal{F}_\lambda = \emptyset$ for all $\lambda \in \Lambda$.*

Remark 3.22. The converse of the above lemma is also true, provided $\operatorname{card}(\Lambda) < \infty$.

The following theorem was partially proved by [DGO17, Theorem 7.19], which was later improved by [Sun18].

Theorem 3.23 ([Sun18, Theorem 5.1])**.** *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some $X \subset G$. Then for all sufficient deep normal subgroups $N_\lambda \lhd H_\lambda$, $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$is a Cohen-Lyndon triple.*

Combining the above theorem with Theorem 3.8, we obtain:

Corollary 3.24 ([Sun19, Corollary 4.3])**.** *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some $X \subset G$. Then for all sufficiently deep normal subgroups $N_\lambda \lhd H_\lambda$ and every $\mathbb{Z}\overline{G}$-module A, there is a morphism between spectral sequences from* (4) *to* (5) *which satisfies (a) and (b) of Theorem 3.8.*

Another property of sufficiently deep Dehn fillings is that they preserve acylindrical hyperbolicity. We first recall the definition.

Definition 3.25. A group G is *acylindrically hyperbolic* if G admits a non-elementary acylindrical action on some Gromov hyperbolic space by isometries.

For the definition an acylindrical action, the reader is referred to [Osi18]. Intuitively, one can think of acylindricity as an analog of properness. An acylindrical action of a group G is *non-elementary* if its orbits are unbounded and G is not virtually-cyclic [Osi16, Theorem 1.1].

Theorem 3.26 ([DGO17, Theorem 7.19]-[Osi16, Theorem 1.2])**.** *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$ for some $X \subset G$. Then for all sufficiently deep normal subgroups $N_\lambda \lhd H_\lambda$, the natural homomorphism $\overline{H}_\lambda \to \overline{G}$ is injective for $\lambda \in \Lambda$ and we have $\{\overline{H}_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h \overline{G}$ (see Notation 3.4 for the meaning of $\overline{H}_\lambda$ and $\overline{G}$). Moreover, if for some $\lambda \in \Lambda$, $\mathrm{card}(\overline{H}_\lambda) = \infty$ and $\overline{H}_\lambda$ is a proper subgroup of $\overline{G}$, then $\overline{G}$ is acylindrically hyperbolic.*

Every acylincrically hyperbolic group contains lots of (in fact, infinitely many) hyperbolically embedded subgroups.

Theorem 3.27 ([DGO17, Theorem 6.14])**.** *Let G be an acylindrically hyperbolic group. Then G has a maximal finite normal subgroup denoted as $K(G)$. Moreover, for every $n \in \mathbb{N}$, there exists a free group F of rank n such that $F \times K(G) \hookrightarrow_h G$.*

3.4 Isolated components

In the proof of Theorem 2.9, we need to construct specific hyperbolically embedded subgroups. A tool to do this is the notion of an isolated component. In this section, we recall the definition and collect several results.

Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of G, let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$, and let X be a relative generating set of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$. For $\lambda \in \Lambda$, let $\widehat{d}_\lambda$ be the relative metric on H_λ with respect to X. The following terminology goes back to [Osi06].

Definition 3.28. Let p be a path in $\Gamma(G, X \sqcup \mathcal{H})$. For every $\lambda \in \Lambda$, an *H_λ-subpath* q of p is a nontrivial subpath of p such that $\mathbf{Lab}(q)$ is a word over the alphabet H_λ (if p is a cycle, we allow q to be a subpath of some cyclic shift of p). An H_λ-subpath q of p is an *H_λ-component* if q is not properly contained in any other H_λ-subpath. Two H_λ-components q_1 and q_2 of p are *connected* if there exists a path t in $\Gamma(G, X \sqcup \mathcal{H})$ such that t connects a vertex of q_1 to a vertex of q_2, and that $\mathbf{Lab}(t)$ is a letter from H_λ. An H_λ-component q of p is *isolated* if it is not connected to any other H_λ-component of p.

Suppose that q is an H_λ-component of a path $p \subset \Gamma(G, X \sqcup \mathcal{H})$. Then q^- (resp. q^+) is labeled by an element $g \in G$ (resp $h \in G$) and we have $g^{-1}h \in H_\lambda$. In this case, let

$$\widehat{\ell}_\lambda(q) = \widehat{d}_\lambda(1, g^{-1}h).$$

A nice property of isolated components is that in a geodesic polygon p, the total $\widehat{\ell}$-length of isolated components of p is bounded linearly by the number of sides of p. More precisely:

Proposition 3.29 ([DGO17, Proposition 4.14] (see also [Osi07, Proposition 3.2]))**.** *If $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G,X)$, then there exists a number $D > 0$ satisfying the following property: Let p be an n-gon in $\Gamma(G, X\sqcup\mathcal{H})$ with geodesic sides $p_1, ..., p_n$ and let I be a subset of the set of sides of p such that every side $p_i \in I$ is an isolated H_{λ_i}-component of p for some $\lambda_i \in \Lambda$. Then*

$$\sum_{p_i\in I} \widehat{\ell}_{\lambda_i}(p_i) \leqslant Dn.$$

The next technical lemma can be used to show that certain subgroups satisfy Lemma 3.17 and thus form a hyperbolically embedded family.

Lemma 3.30 ([DGO17, Lemma 4.21])**.** *Suppose $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G,X)$. Let W be the set consisting of all words w over $X\sqcup\mathcal{H}$ such that*

- *(W1) w contains no subwords of type xy, where $x, y \in X$;*
- *(W2) if w contains a letter $h \in H_\lambda$ for some $\lambda\in\Lambda$, then $\widehat{d}_\lambda(1,h) > 50D$, where D is given by Lemma 3.29;*
- *(W3) if h_1xh_2 (resp. h_1h_2) is a subword of w, where $x\in X, h_1 \in H_\lambda, h_2 \in H_\mu$, then either $\lambda \neq \mu$ or the element represented by x in G does not belong to H_λ (resp. $\lambda\neq\mu$).*

Then the following hold.

- *(a) Every path in the Cayley graph $\Gamma(G, X\sqcup\mathcal{H})$ labeled by a word from W is a $(4,1)$-quasi-geodesic.*
- *(b) If p is a path in $\Gamma(G, X\sqcup\mathcal{H})$ labeled by a word from W, then for every $\lambda\in\Lambda$, every H_λ-component of p is isolated.*
- *(c) For every $\epsilon > 0$, there exists $R > 0$ satisfying the following condition. Let p, q be two paths in $\Gamma(G, X\sqcup\mathcal{H})$ such that*

  $$\ell_{X\sqcup\mathcal{H}}(p) \geqslant R, \qquad \boldsymbol{Lab}(p), \boldsymbol{Lab}(q) \in W,$$

  and p, q are oriented ϵ-close, i.e.,

  $$\max\{d_{X\sqcup\mathcal{H}}(p^-, q^-), d_{X\sqcup\mathcal{H}}(p^+, q^+)\} \leqslant \epsilon,$$

  where $d_{X\sqcup\mathcal{H}}$ is the combinatorial metric of $\Gamma(G, X\sqcup\mathcal{H})$. Then there exist five consecutive components of p which are respectively connected to five consecutive components of q. In other words,

  $$p = x_0a_1x_1a_2x_2a_3x_3, \qquad q = y_0b_0y_1b_1y_2b_2y_3b_3,$$

  where x_i's (resp. y_i's) are either trivial paths or subpaths of p (resp. q) labeled by letters of X, and for $j = 1,2,3$, a_j, b_j are connected H_{λ_j}-components for some $\lambda_j\in\Lambda$.

Remark 3.31. Conclusion (b) of Lemma 3.30 is not stated in [DGO17, Lemma 4.21], but it is proved in the second paragraph of the proof of [DGO17, Lemma 4.21].

4 Proof of main results

Theorem 3.8 provides us with a morphism ϕ between spectral sequences such that $\phi_2^{*,q}$ is an isomorphism unless $q = 0$. In this section, we extract certain information from such a morphism. And then we use the extracted information to prove Theorems 2.4, 2.6, 2.8, and 2.9.

4.1 Computations with spectral sequences

Let $E_1 = (E_{1,r}, d_{1,r})_{r\geqslant 2}$ and $E_2 = (E_{2,r}, d_{2,r})_{r\geqslant 2}$ be two spectral sequences such that

$$E_{1,2}^{p,q} \Rightarrow H_1^{p+q}, \qquad E_{2,2}^{p,q} \Rightarrow H_2^{p+q}$$

for some graded abelian groups $H_1 = H_1^\ell$ and $H_2 = H_2^\ell$. Suppose that there is a morphism between spectral sequences

$$\phi = (\phi_r)_{r\geqslant 2} : E_1 \to E_2$$

and a 0-degree morphism

$$f : H_1 \to H_2$$

such that ϕ and f are compatible and for all $q \in \mathbb{Z} \setminus \{0\}$, ϕ maps $E_{1,2}^{*,q}$ isomorphically onto $E_{2,2}^{*,q}$. The goal of this section is to compute $E_{1,2}^{*,0}$ from the information above.

Recall that $d_{1,r}^{p,q}, d_{2,r}^{p,q}, \phi_r^{p,q}, f^\ell$, etc are the maps induced by $d_{1,r}, d_{2,r}, \phi, f$. We first observe:

Lemma 4.1. *For $p, q \in \mathbb{Z}$ and $r \geqslant 2$,*

$$\phi_r^{p,q}\left(\ker(d_{1,r}^{p,q})\right) \subset \ker(d_{2,r}^{p,q}), \qquad \phi_r^{p,q}\left(\mathrm{im}(d_{1,r}^{p-r,q+r-1})\right) \subset \mathrm{im}(d_{2,r}^{p-r,q+r-1}).$$

As ϕ_{r+1} is the cohomology map induced by ϕ_r, it follows that

(a) $\phi_{r+1}^{p,q}$ is surjective if and only if

$$\phi_r^{p,q}\left(\ker(d_{1,r}^{p,q})\right) + \mathrm{im}(d_{2,r}^{p-r,q+r-1}) = \ker(d_{2,r}^{p,q});$$

(b) $\phi_{r+1}^{p,q}$ is injective if and only if the preimage of $\mathrm{im}(d_{2,r}^{p-r,q+r-1})$ under $\phi_r^{p,q}$ is $\mathrm{im}(d_{1,r}^{p-r,q+r-1})$.

Main ideas of this section are illustrated by the example below.

Example 4.2. Let us consider the simple case where

$$E_{1,2}^{*,q} = E_{2,2}^{*,q} = \{0\} \text{ whenever } q \neq 0, 1. \tag{6}$$

The only possibly nontrivial differentials at pages $E_{1,2}$ and $E_{2,2}$ are the ones going from the first rows to the 0-th rows. Two such differentials are shown in Figure 1, where the unlabeled arrows are $d_{1,2}^{p-2,1}$ and $d_{2,2}^{p-2,1}$, respectively. After finishing the computations at pages $E_{1,2}$

and $E_{2,2}$, we obtain pages $E_{1,3}$ and $E_{2,3}$, which are shown by Figure 2. In Figure 2, the line segment connecting $\operatorname{coker}(d_{1,2}^{p-3,0}), \ker(d_{1,2}^{p-2,1})$, and H_1^{p-1} indicates the exact sequence

$$1 \to \operatorname{coker}(d_{1,2}^{p-3,0}) \to H_1^{p-1} \to \ker(d_{1,2}^{p-2,1}) \to 1,$$

which is a consequence of $E_{1,2}^{p,q} \Rightarrow H_1^{p+q}$. Similarly, other line segments in Figure 2 indicate different consequences of the limits of E_1 and E_2.

$$\begin{tikzcd}
E_{1,2}^{p-2,1} \arrow[ddrr] & E_{1,2}^{p-1,1} & E_{1,2}^{p,1} & & & E_{2,2}^{p-2,1} \arrow[ddrr] & E_{2,2}^{p-1,1} & E_{2,2}^{p,1} \\
 & & & {} \arrow[r, "\phi_2"] & {} & & & \\
E_{1,2}^{p-2,0} & E_{1,2}^{p-1,0} & E_{1,2}^{p,0} & & & E_{2,2}^{p-2,0} & E_{2,2}^{p-1,0} & E_{2,2}^{p,0}
\end{tikzcd}$$

Figure 1: Pages $E_{1,2}$ and $E_{2,2}$

$$\begin{tikzcd}
H_1^{p-1} \arrow[dr, dash] & & H_1^{p} \arrow[dr, dash] & & \\
 & \ker(d_{1,2}^{p-2,1}) \arrow[drr, dash] & & \ker(d_{1,2}^{p-1,1}) \arrow[dr, dash] & \ker(d_{1,2}^{p,1}) \\
 & \operatorname{coker}(d_{1,2}^{p-4,1}) & & \operatorname{coker}(d_{1,2}^{p-3,1}) \arrow[dd, "\phi_3"] & \operatorname{coker}(d_{1,2}^{p-2,1}) \\
H_2^{p-1} \arrow[dr, dash] & & H_2^{p} \arrow[dr, dash] & & \\
 & \ker(d_{2,2}^{p-2,1}) \arrow[drr, dash] & & \ker(d_{2,2}^{p-1,1}) \arrow[dr, dash] & \ker(d_{2,2}^{p,1}) \\
 & \operatorname{coker}(d_{2,2}^{p-4,1}) & & \operatorname{coker}(d_{2,2}^{p-3,1}) & \operatorname{coker}(d_{2,2}^{p-2,1})
\end{tikzcd}$$

Figure 2: Pages $E_{1,3}$ and $E_{2,3}$

For $p \in \mathbb{Z}$, the map $\phi_3^{p-2,1}$ results from $\phi_2^{p-2,1}$ by restricting the domain to $\ker(d_{1,2}^{p-2,1})$ and restricting the target to $\ker(d_{2,2}^{p-2,1})$. Thus,

Observation 4.2.1. *$\phi_3^{p-2,1}$ is injective as $\phi_2^{p-2,1}$ is.*

In general, $\phi_3^{p-2,1}$ need not be surjective, although $\phi_2^{p-2,1}$ is surjective. For instance, if

$$\ker(\phi_2^{p,0}) \cap \operatorname{im}(d_{1,2}^{p-2,1}) \neq \{0\},$$

then there exists $x \in E_{1,2}^{p-2,1}$ such that

$$d_{1,2}^{p-2,1}(x) \in \ker(\phi_2^{p,0}) \setminus \{0\}.$$

Let $y = \phi_2^{p-2,1}(x)$. Then

$$d_{2,2}^{p-2,1}(y) = d_{2,2}^{p-2,1} \circ \phi_2^{p-2,1}(x) = \phi_2^{p,0} \circ d_{1,2}^{p-2,1}(x) = 0.$$

Thus, $y \in \ker(d_{2,2}^{p-2,1})$. We claim that y has no preimage under $\phi_3^{p-2,1}$. Indeed, $\phi_3^{p-2,1}$ is a restriction of $\phi_2^{p-2,1}$, which is injective. Therefore, the only candidate for the preimage of y under $\phi_3^{p-2,1}$ is x. But $x \notin \ker(d_{1,2}^{p-2,1})$ and thus x is not in the domain of $\phi_3^{p-2,1}$.

Observation 4.2.2. *By the above argument, if $\phi_3^{p-2,1}$ is surjective (for example, if f^{p-1} is surjective), then $\ker(\phi_2^{p,0}) \cap \operatorname{im}(d_{1,2}^{p-2,1}) = \{0\}$, that is, $\phi_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ injectively into $E_{2,2}^{p,0}$.*

Let us make some other observations. Note that $\phi_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ onto

$$\operatorname{im}(\phi_2^{p,0} \circ d_{1,2}^{p-2,1}) = \operatorname{im}(d_{2,2}^{p-2,1} \circ \phi_2^{p-2,1}).$$

By assumption, $\phi_2^{p-2,1}$ is an isomorphism. In particular, $\phi_2^{p-2,1}$ is surjective. If $d_{2,2}^{p-2,1}$ is also surjective (for example, if $H_2^p = \{0\}$ and thus $\operatorname{coker}(d_{2,2}^{p-2,1}) = \{0\}$), then $d_{2,2}^{p-2,1} \circ \phi_2^{p-2,1}$ will be surjective, which will imply the surjectivity of $\phi_2^{p,0} \circ d_{1,2}^{p-2,1}$. Therefore,

Observation 4.2.3. *If $H_2^p = \{0\}$, then $\phi_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ surjectively onto $E_{2,2}^{p,0}$.*

Now suppose that for some p, f^{p-1} is surjective and $H_2^p = \{0\}$. By Observations 4.2.2 and 4.2.3, $\phi_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ isomorphically onto $E_{2,2}^{p,0}$. It follows that

(1) $1 \to \ker(\phi_2^{p,0}) \to E_{1,2}^{p,0} \to E_{2,2}^{p,0} \to 1$ is a split exact sequence;

(2) $E_{1,2}^{p,0} = \ker(\phi_2^{p,0}) \oplus \operatorname{im}(d_{1,2}^{p-2,1})$ and thus $\operatorname{coker}(d_{1,2}^{p-2,1}) \cong \ker(\phi_2^{p,0})$.

As $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$, another implication of $H_2^p = \{0\}$ is $\ker(d_{2,2}^{p-1,1}) = \{0\}$. By Observation 4.2.1, $\phi_3^{p-1,1}$ is injecitve. Thus, a consequence of $\ker(d_{2,2}^{p-1,1}) = \{0\}$ is $\ker(d_{1,2}^{p-1,1}) = \{0\}$, which, together with $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, implies $H_1^p \cong \operatorname{coker}(d_{1,2}^{p-2,1})$. Thus,

Observation 4.2.4. *If for some p, f^{p-1} is surjective and $H_2^p = \{0\}$, then*

$$E_{1,2}^{p,0} = \ker(\phi_2^{p,0}) \oplus \operatorname{im}(d_{1,2}^{p-2,1}) \cong \operatorname{coker}(d_{1,2}^{p-2,1}) \oplus E_{2,2}^{p,0} \cong H_1^p \oplus E_{2,2}^{p,0}.$$

The rest of this section aims to prove Observation 4.2.4 without assuming (6). The following Lemma 4.3 is a generalization of Observation 4.2.1.

Lemma 4.3. *For $r \geqslant 2$,*

(a) $\phi_r^{*,q}$ is injective for all $q \in \mathbb{Z} \setminus \{0\}$;

(b) $\phi_r^{*,q}$ is an isomorphism if $q \geqslant r-1$.

Proof. We prove these statements by an induction on r. The base case $r = 2$ follows from the assumptions.

Suppose that (a) and (b) hold for $r = R \geqslant 2$. Consider the case $r = R+1$. The following Claims 4.3.1 and 4.3.2 follow directly from the induction hypothesis and Lemma 4.1.

Claim 4.3.1. *For all $q \geqslant 1$, $\phi_R^{*,q}$ maps* $\ker(d_{1,R}^{*,q})$ *injectively into* $\ker(d_{2,R}^{*,q})$*. If $q \geqslant R$, then $\phi_R^{*,q}$ maps* $\ker(d_{1,R}^{*,q})$ *isomorphically onto* $\ker(d_{2,R}^{*,q})$.

Claim 4.3.2. *For all $p \in \mathbb{Z}$ and $q \geqslant R$, $\phi_R^{p+R,q-R+1}$ maps* $\operatorname{im}(d_{1,R}^{p,q})$ *isomorphically onto* $\operatorname{im}(d_{2,R}^{p,q})$.

Statements (a) and (b) are immediate consequences of Claims 4.3.1 and 4.3.2 and Lemma 4.1. □

Lemma 4.4. *Suppose $p \geqslant 1$. Let $r \in \{0, ..., p-1\}$ and let $R \geqslant r+2$. If $\phi_{R+1}^{p-r-1,r}$ is surjective, then $\phi_R^{p-r-1,r}$ is also surjective.*

Proof. Suppose that $\phi_R^{p-r-1,r}$ is not surjective. Note that the target of $d_{1,R}^{p-r-1,r}$ is $E_{1,R}^{p+R-r-1,r-R+1} = \{0\}$. Thus,

$$\ker(d_{1,R}^{p-r-1,r}) = E_{1,R}^{p-r-1,r}.$$

Similarly,

$$\ker(d_{2,R}^{p-r-1,r}) = E_{2,R}^{p-r-1,r}.$$

As $\phi_R^{p-r-1,r}$ is not surjective, we have

$$\phi_R^{p-r-1,r}\left(\ker(d_{1,R}^{p-r-1,r})\right) \neq \ker(d_{2,R}^{p-r-1,r}). \tag{7}$$

By Lemma 4.3, $\phi_R^{p-r-R-1,r+R-1}$ is an isomorphism. It follows that

$$\begin{aligned}
&\phi_R^{p-r-1,r}\left(\mathrm{im}(d_{1,R}^{p-r-R-1,r+R-1})\right) \\
=\,&\mathrm{im}(\phi_R^{p-r-1,r}\circ d_{1,R}^{p-r-R-1,r+R-1}) \\
=\,&\mathrm{im}(d_{2,R}^{p-r-R-1,r+R-1}\circ \phi_R^{p-r-R-1,r+R-1}) \quad \text{as } \phi \text{ is a morphism of spectral sequences} \\
=\,&\mathrm{im}(d_{2,R}^{p-r-R-1,r+R-1}) \quad \text{as } \phi_R^{p-r-R-1,r+R-1} \text{ is an isomorphism.}
\end{aligned} \tag{8}$$

Formulas (7) and (8) and Lemma 4.1 imply that $\phi_{R+1}^{p-r-1,r}$ is not surjective, contradicting our assumption. ☐

Suppose $p \geqslant 2$. Note that for all $r \geqslant 2$, $d_{1,r}^{p,0}$ is a map from $E_{1,r}^{p,0}$ to $E_{1,r}^{p+r,1-r} = \{0\}$. It follows that $\ker(d_{1,r}^{p,0}) = E_{1,r}^{p,0}$ and thus $E_{1,r+1}^{p,0}$ is a quotient of $E_{1,r}^{p,0}$. Similarly, $E_{2,r+1}^{p,0}$ is a quotient of $E_{2,r}^{p,0}$. For $r = 2, ..., p+1$, let

$$\alpha_{1,r} : E_{1,r}^{p,0} \to E_{1,r+1}^{p,0}, \qquad \alpha_{2,r} : E_{2,r}^{p,0} \to E_{2,r+1}^{p,0}$$

be the corresponding quotient maps.

For simplicity, we also let

$$\alpha_{1,1} : E_{1,2}^{p,0} \to E_{1,2}^{p,0}, \qquad \alpha_{2,1} : E_{2,2}^{p,0} \to E_{2,2}^{p,0}$$

be the identity maps. For $r = 1, ..., p+1$, let $\beta_{1,r}$ (resp. $\beta_{2,r}$) be the composition of $\alpha_{1,i}$ (resp. $\alpha_{2,i}$) for $1 \leqslant i \leqslant r$, i.e.,

$$\beta_{1,r} = \alpha_{1,r}\circ\cdots\alpha_{1,1} : E_{1,2}^{p,0} \to E_{1,r+1}^{p,0}, \qquad \beta_{2,r} = \alpha_{2,r}\circ\cdots\alpha_{2,1} : E_{2,2}^{p,0} \to E_{2,r+1}^{p,0}.$$

Remark 4.5. For $r = 2, ..., p+1$, $\alpha_{1,r}$ (resp. $\alpha_{2,r}$) is the cohomology map sending every $x \in E_{1,r}^{p,0}$ (resp. $y \in E_{2,r}^{p,0}$) to the cohomology class in $E_{1,r+1}^{p,0}$ (resp. $E_{2,r+1}^{p,0}$) represented by x (resp. y). Thus,

$$\begin{aligned}
&\ker(\alpha_{1,r}) = \mathrm{im}(d_{1,r}^{p-r,r-1}), && \ker(\alpha_{2,r}) = \mathrm{im}(d_{2,r}^{p-r,r-1}), \\
&\phi_{r+1}^{p,0}\circ\alpha_{1,r} = \alpha_{2,r}\circ\phi_r^{p,0}, && \phi_{r+1}^{p,0}\circ\beta_{1,r} = \beta_{2,r}\circ\phi_2^{p,0}.
\end{aligned}$$

Lemma 4.6. *Suppose $p \geqslant 2$. Then*

(a) If the maps

$$\phi_{r+2}^{p-r-1,r} : E_{1,r+2}^{p-r-1,r} \to E_{2,r+2}^{p-r-1,r}$$

are surjective for $r = 1, ..., p-1$, then $\beta_{1,p+1}$ maps $\ker(\phi_2^{p,0})$ injectively into $\ker(\phi_{p+2}^{p,0})$.

(b) If $E_{2,p+2}^{p,0} = \{0\}$, then $\phi_2^{p,0}$ maps $\ker(\beta_{1,p+1})$ surjectively onto $E_{2,2}^{p,0}$.

Proof.

(a) By Remark 4.5, $\beta_{1,p+1}$ maps $\ker(\phi_2^{p,0})$ into $\ker(\phi_{p+2}^{p,0})$. It remains to prove that $\beta_{1,p+1}$ maps $\ker(\phi_2^{p,0})$ injectively into $E_{1,p+2}^{p,0}$. Suppose that this is not true. As $\beta_{1,p+1}$ is the composition of $\alpha_{1,r}$, there exists $1 \leqslant r \leqslant p$ such that $\alpha_{1,r+1}$ does not map $\beta_{1,r}\left(\ker(\phi_2^{p,0})\right)$ injectively into $E_{1,r+2}^{p,0}$. We prove that $\phi_{r+2}^{p-r-1,r}$ is not surjective, which contradicts our assumption. By Lemma 4.3, $\phi_{r+1}^{p-2r-2,2r}$ is an isomorphism. It follows that

$$\begin{aligned}
&\phi_{r+1}^{p-r-1,r}\left(\operatorname{im}(d_{1,r+1}^{p-2r-2,2r})\right)\\
=&\operatorname{im}(\phi_{r+1}^{p-r-1,r}\circ d_{1,r+1}^{p-2r-2,2r})\\
=&\operatorname{im}(d_{2,r+1}^{p-2r-2,2r}\circ \phi_{r+1}^{p-2r-2,2r}) && \text{as } \phi \text{ is a morphism of spectral sequences}\\
=&\operatorname{im}(d_{2,r+1}^{p-2r-2,2r}) && \text{as } \phi_{r+1}^{p-2r-2,2r} \text{ is an isomorphism.}
\end{aligned}$$

In view of Lemma 4.1, it suffices to show

$$\phi_{r+1}^{p-r-1,r}\left(\ker(d_{1,r+1}^{p-r-1,r})\right) \neq \ker(d_{2,r+1}^{p-r-1,r}).$$

By the Remark 4.5, we have $\ker(\alpha_{1,r+1}) = \operatorname{im}(d_{1,r+1}^{p-r-1,r})$. This, together with the assumption that $\alpha_{1,r+1}$ does not map $\beta_{1,r}\left(\ker(\phi_2^{p,0})\right)$ injectively into $E_{1,r+2}^{p,0}$, implies

$$\beta_{1,r}\left(\ker(\phi_2^{p,0})\right)\cap \operatorname{im}(d_{1,r+1}^{p-r-1,r}) \neq \{0\}. \tag{9}$$

Let W be the preimage of $\beta_{1,r}\left(\ker(\phi_2^{p,0})\right)$ under $d_{1,r+1}^{p-r-1,r}$. Note that

$$\begin{aligned}
&d_{2,r+1}^{p-r-1,r}\circ\phi_{r+1}^{p-r-1,r}(W)\\
=&\phi_{r+1}^{p,0}\circ d_{1,r+1}^{p-r-1,r}(W) && \text{as } \phi \text{ is a morphism of spectral sequences}\\
\subset&\phi_{r+1}^{p,0}\circ\beta_{1,r}\left(\ker(\phi_2^{p,0})\right)\\
=&\beta_{2,r}\circ\phi_2^{p,0}\left(\ker(\phi_2^{p,0})\right) && \text{by Remark 4.5}\\
=&\{0\}.
\end{aligned}$$

Thus,

$$\phi_{r+1}^{p-r-1,r}(W)\subset \ker(d_{2,r+1}^{p-r-1,r}).$$

Formula (9) implies

$$\ker(d_{1,r+1}^{p-r-1,r}) \subsetneq W.$$

By Lemma 4.3, $\phi_{r+1}^{p-r-1,r}$ is injective. Thus,

$$\phi_{r+1}^{p-r-1,r}\left(\ker(d_{1,r+1}^{p-r-1,r})\right) \subsetneq \phi_{r+1}^{p-r-1,r}(W)\subset \ker(d_{2,r+1}^{p-r-1,r}).$$

(b) Suppose, for the contrary, that

$$\phi_2^{p,0}\left(\ker(\beta_{1,p+1})\right) \neq E_{2,2}^{p,0}.$$

Compare the following two sequences

$$\{\phi_{r+1}^{p,0} \circ \beta_{1,r}\left(\ker(\beta_{1,p+2})\right)\}_{r=1}^{p+1}, \qquad \{E_{2,r+1}^{p,0}\}_{r=1}^{p+1}.$$

Note that

$$\phi_2^{p,0} \circ \beta_{1,1}\left(\ker(\beta_{1,p+1})\right) = \phi_2^{p,0}\left(\ker(\beta_{1,p+1})\right) \neq E_{2,2}^{p,0},$$

but

$$\phi_{p+2}^{p,0} \circ \beta_{1,p+1}\left(\ker(\beta_{1,p+1})\right) = E_{2,p+2}^{p,0} = \{0\}.$$

Thus, there exists $1 \leqslant r \leqslant p$ such that

$$\phi_{r+1}^{p,0} \circ \beta_{1,r}\left(\ker(\beta_{1,p+1})\right) \neq E_{2,r+1}^{p,0}, \tag{10}$$

$$\phi_{r+2}^{p,0} \circ \beta_{1,r+1}\left(\ker(\beta_{1,p+1})\right) = E_{2,r+2}^{p,0}. \tag{11}$$

Let $x \in E_{2,r+1}^{p,0}$. Then $\alpha_{2,r+1}(x) \in E_{2,r+2}^{p,0}$. By (11), there exists $y \in \ker(\beta_{1,p+2})$ such that

$$\phi_{r+2}^{p,0} \circ \beta_{1,r+1}(y) = \alpha_{2,r+1}(x).$$

Note that

$$\begin{aligned} 0 =& \phi_{r+2}^{p,0} \circ \beta_{1,r+1}(y) - \alpha_{2,r+1}(x) \\ =& \phi_{r+2}^{p,0} \circ \alpha_{1,r+1} \circ \beta_{1,r}(y) - \alpha_{2,r+1}(x) && \text{by the definition of } \beta_{1,r+1} \\ =& \alpha_{2,r+1} \circ \phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - \alpha_{2,r+1}(x) && \text{by Remark 4.5} \\ =& \alpha_{2,r+1}(\phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - x). \end{aligned}$$

In other words,

$$\phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - x \in \ker(\alpha_{2,r+1}).$$

By Remark 4.5, there exists $z \in E_{2,r+1}^{p-r-1,r}$ such that

$$d_{2,r+1}^{p-r-1,r}(z) = \phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - x.$$

By Lemma 4.3, $\phi_{r+1}^{p-r-1,r}$ is an isomorphism. Thus, there exists $t \in E_{1,r+1}^{p-r-1,r}$ such that $\phi_{r+1}^{p-r-1,r}(t) = z$. By Remark 4.5 again,

$$d_{1,r+1}^{p-r-1,r}(t) \in \ker(\alpha_{1,r+1}) \subset \beta_{1,r}\left(\ker(\beta_{1,p+1})\right).$$

Thus,

$$\begin{aligned}
x =&\phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - d_{2,r+1}^{p-r-1,r}(z) \\
=&\phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - d_{2,r+1}^{p-r-1,r} \circ \phi_{r+1}^{p-r-1,r}(t) \\
=&\phi_{r+1}^{p,0} \circ \beta_{1,r}(y) - \phi_{r+1}^{p,0} \circ d_{1,r+1}^{p-r-1,r}(t) \\
=&\phi_{r+1}^{p,0} \left(\beta_{1,r}(y) - d_{1,r+1}^{p-r-1,r}(t)\right) \\
\in&\phi_{r+1}^{p,0} \circ \beta_{1,r} \left(\ker(\beta_{1,p+1})\right).
\end{aligned}$$

As x is arbitrary, we have

$$\phi_{r+1}^{p,0} \circ \beta_{1,r} \left(\ker(\beta_{1,p+1})\right) = E_{2,r+1}^{p,0},$$

contradicting (10).

□

We are now ready to prove the following generalization of Observation 4.2.4.

Lemma 4.7. *If f^{p-1} is surjective and $H_2^p = \{0\}$, then $\phi_2^{p,0}$ is surjective with* $\ker(\phi_2^{p,0}) \cong H_1^p$. *Moreover,*

$$E_{1,2}^{p,0} \cong E_{2,2}^{p,0} \oplus H_1^p.$$

Proof. If $p \leqslant -1$, then $E_{1,2}^{p,0} = E_{2,2}^{p,0} = \{0\}$. If $p = 0$, then $E_{1,2}^{0,0} \cong H_1^0$ as $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, and $E_{2,2}^{0,0} \cong H_2^0 = \{0\}$ as $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$. Thus, the lemma holds in those two cases.

Suppose $p = 1$. By assumption, $H_2^1 = \{0\}$. It follows from Remark 3.3 and $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$ that

$$E_{2,3}^{0,1} = E_{2,3}^{1,0} = \{0\}.$$

As $E_{1,2}^{-1,1} = E_{2,2}^{-1,1} = \{0\}$, the argument of Remark 3.3 shows

$$E_{1,2}^{1,0} = E_{1,3}^{1,0}, \qquad E_{2,2}^{1,0} = E_{2,3}^{1,0} = \{0\}.$$

By Lemma 4.3, $\phi_3^{1,0}$ maps $E_{1,3}^{0,1}$ injectively into $E_{2,3}^{0,1}$ and thus $E_{1,3}^{0,1} = \{0\}$. Therefore,

$$\begin{aligned}
E_{1,2}^{1,0} =&E_{1,3}^{1,0} \\
\cong&H_1^1 && \text{by } E_{1,3}^{0,1} = \{0\}, E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}, \text{ and Remark 3.3} \\
\cong&E_{2,2}^{1,0} \oplus H_1^1 && \text{as } E_{2,2}^{1,0} = \{0\}.
\end{aligned}$$

Let us assume $p \geqslant 2$. As f^{p-1} is surjective and ϕ is compatible with f, Remark 3.3 implies that the maps $\phi_{p+1}^{p-r-1,r}$ are surjective for $r = 1, ..., p-1$. By succesively applying

Lemma 4.4, we see that the maps $\phi_{r+2}^{p-r-1,r}$ are also surjective. It follows from Lemma 4.6 that $\beta_{1,p+1}$ maps $\ker(\phi_2^{p,0})$ injectively into $E_{1,p+2}^{p,0}$. Thus,

$$\ker(\beta_{1,p+1}) \cap \ker(\phi_2^{p,0}) = \{0\}. \tag{12}$$

By Remark 3.3, $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$, and $H_2^p = \{0\}$, we have $E_{2,p+2}^{p,0} = \{0\}$. It follows from Lemma 4.6 that $\phi_2^{p,0}$ maps $\ker(\beta_{1,p+1})$ surjectively onto $E_{2,2}^{p,0}$. Together with equation (12), this implies

$$\ker(\beta_{1,p+1}) \cong E_{2,2}^{p,0}$$

and

$$E_{1,2}^{p,0} = \ker(\beta_{1,p+1}) \oplus \ker(\phi_2^{p,0}). \tag{13}$$

We have already shown that $\beta_{1,p+1}$ maps $\ker(\phi_2^{p,0})$ injectively into $E_{1,p+2}^{p,0}$. Thus, equation (13) implies that $\beta_{1,p+1}$ maps $\ker(\phi_2^{p,0})$ isomorphically onto $E_{1,p+2}^{p,0}$.

For $r = 1, ..., p$, as $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$ and $H_2^p = \{0\}$, we have $E_{2,p+2}^{p-r,r} = \{0\}$ by Remark 3.3. By Lemma 4.3, $\phi_{p+2}^{p-r,r}$ maps $E_{1,p+2}^{p-r,r}$ injectively into $E_{2,p+2}^{p-r,r}$. Thus, $E_{1,p+2}^{p-r,r} = \{0\}$. As $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, Remark 3.3 implies

$$H_1^p \cong E_{1,p+2}^{p,0} \cong \ker(\phi_2^{p,0}).$$

Therefore,

$$E_{1,2}^{p,0} \cong \ker(\beta_{1,p+2}) \oplus \ker(\phi_2^{p,0}) \cong E_{2,2}^{p,0} \oplus E_{1,p+2}^{p,0} \cong E_{2,2}^{p,0} \oplus H_1^p.$$

□

4.2 Cohomology of Dehn filling quotients

We prove Theorems 2.4 and 2.6 in this section. In the proof, we will frequently use notations defined in Notation 3.4.

Theorem 4.8. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple and let A be a $\mathbb{Z}\overline{G}$-module. Suppose that for some $p \in \mathbb{N}$, $\prod_{\lambda\in\Lambda} \mathrm{H}^p(H_\lambda; A) = \{0\}$ and r_G^{p-1} is surjective. Then $\psi^{p,0}$ is also surjective and* $\ker(\psi^{p,0}) \cong \mathrm{H}^p(G; A)$. *Moreover,*

$$\mathrm{H}^p(\overline{G}; A) \cong \left(\prod_{\lambda\in\Lambda} \mathrm{H}^p(\overline{H}_\lambda; A)\right) \oplus \mathrm{H}^p(G; A). \tag{14}$$

Proof. Let ϕ be as in Theorem 3.8. Note that ϕ and r_G satisfy the assumptions of Lemma 4.7, which yields the direct sum decomposition (14) and shows that $\phi_2^{p,0}$ is surjective. By Remark 3.5, $\phi_2^{p,0}$ can be identified with $\psi^{p,0}$ and thus $\psi^{p,0}$ is surjective. □

Recall that the *cohomological dimension* of a group G is

$$\mathrm{cd}(G) = \sup\{\ell \in \mathbb{N} \mid \mathrm{H}^\ell(G, A) \neq \{0\} \text{ for some } \mathbb{Z}G\text{-module } A\}.$$

If $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is a Cohen-Lyndon triple, let

$$\mathrm{cd}(\mathcal{H}) = \sup_{\lambda\in\Lambda}\{\mathrm{cd}(H_\lambda)\}, \qquad \mathrm{cd}(\overline{\mathcal{H}}) = \sup_{\lambda\in\Lambda}\{\mathrm{cd}(\overline{H}_\lambda)\}.$$

Corollary 4.9. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. Then*

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(\mathcal{H}) + 1, \mathrm{cd}(\overline{\mathcal{H}})\}.$$

Proof. If $\mathrm{cd}(\overline{G}) \leqslant \mathrm{cd}(\mathcal{H})+1$, then the desired conclusion already holds. Thus, let us assume $\mathrm{cd}(G) \geqslant \mathrm{cd}(\mathcal{H}) + 2$. Let $\ell \geqslant \mathrm{cd}(\mathcal{H}) + 2$. It follows from Theorem 4.8 that

$$\mathrm{H}^\ell(\overline{G}; A) \cong \left(\prod_{\lambda\in\Lambda} \mathrm{H}^\ell(\overline{H}_\lambda; A)\right) \oplus \mathrm{H}^\ell(G; A),$$

which implies $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(\overline{\mathcal{H}})\}$. □

Besides absolute cohomology, Theorem 4.8 can also be used to study relative cohomology of groups, which was introduced by [BE78]. Let us first recall the following result of [BE78].

Proposition 4.10 ([BE78, Proposition 1.1])**.** *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then for every $\mathbb{Z}G$-module A, there is a long exact sequence*

$$\cdots \to \mathrm{H}^\ell(G, \{H_\lambda\}_{\lambda\in\Lambda}; A) \to \mathrm{H}^\ell(G; A) \xrightarrow{r_G} \prod_{\lambda\in\Lambda} \mathrm{H}^\ell(H_\lambda; A) \to \mathrm{H}^{\ell+1}(G, \{H_\lambda\}_{\lambda\in\Lambda}; A) \to \cdots$$

Corollary 4.11. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. Then for all $\ell \geqslant \mathrm{cd}(\mathcal{H}) + 3$ and every $\mathbb{Z}\overline{G}$-module A, there is an isomorphism*

$$\mathrm{H}^\ell(\overline{G}, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \cong \mathrm{H}^\ell(G; A).$$

For $\ell = \mathrm{cd}(\mathcal{H}) + 2$, there is a surjection $\mathrm{H}^\ell(\overline{G}, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \twoheadrightarrow \mathrm{H}^\ell(G; A)$.

Proof. By Proposition 4.10 and Remark 3.5, there is a long exact sequence

$$\cdots \to \mathrm{H}^\ell(\overline{G}, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \to \mathrm{H}^\ell(\overline{G}; A) \xrightarrow{\psi^{\ell,0}} \prod_{\lambda\in\Lambda} \mathrm{H}^\ell(\overline{H}_\lambda; A) \to \mathrm{H}^{\ell+1}(\overline{G}, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \to \cdots$$

By Theorem 4.9, if $\ell \geqslant \mathrm{cd}(\mathcal{H}) + 2$, then $\psi^{\ell,0}$ is surjective and $\ker(\psi^{\ell,0}) \cong \mathrm{H}^\ell(G; A)$, which implies the desired result. □

Our next result concerns another cohomological finiteness property. Recall that a group G is of *type FP_k* for some $k \in \mathbb{N}^+ \cup \{\infty\}$ if there is a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z}$$

over $\mathbb{Z}G$ such that P_ℓ is finitely generated for all $\ell \in \mathbb{N}$ with $\ell \leqslant k$. The following characterization of FP_k is a special case of [Bie81, Theorem 1.3] and [Bro75, Theorem 2].

Theorem 4.12 ([Bie81, Bro75]). *For a group G, the following are equivalent.*

(a) G is of type FP_k for some $k \in \mathbb{N}^+ \cup \{\infty\}$.

(b) For every $\ell \in \mathbb{N}$ with $\ell \leqslant k$ and every direct system $\{A_i\}_{i\in I}$ of $\mathbb{Z}G$-modules such that $\varinjlim A_i = \{0\}$, we have $\varinjlim \mathrm{H}^\ell(G; A_i) = \{0\}$.

Theorem 4.13. *Let Λ be a finite index set and let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. If for some $k \in \mathbb{N}^+ \cup \{\infty\}$, the groups $G, H_\lambda, \overline{H}_\lambda, \lambda \in \Lambda$, are of type FP_k, then so is $\overline{G}$.*

Proof. Let $\{A_i\}_{i\in I}$ be a direct system of $\mathbb{Z}\overline{G}$-modules such that $\varinjlim A_i = \{0\}$. For $i \in I$, let

$$E^{p+q}_{G,i,2} = \mathrm{H}^p\left(\overline{G}; \mathrm{H}^q(\langle\!\langle \mathcal{N} \rangle\!\rangle; A_i)\right) \Rightarrow \mathrm{H}^{p+q}(G; A_i),$$

$$E^{p+q}_{\mathcal{H},i,2} = \prod_{\lambda\in\Lambda} \mathrm{H}^p\left(\overline{H}_\lambda; \mathrm{H}^q(N_\lambda; A_i)\right) \Rightarrow \prod_{\lambda\in\Lambda} \mathrm{H}^{p+q}(H_\lambda; A_i)$$

be the spectral sequences in Theorem 3.8 and let $\phi_i : E_{G,i} \to E_{\mathcal{H},i}$ be the morphism provided by that theorem. Also let E_G (resp. $E_\mathcal{H}$) be the direct limit of $E_{G,i}$ (resp. $E_{\mathcal{H},i}$), and let

$$\phi = \varinjlim \phi_i : E_G \to E_\mathcal{H}.$$

For every $p \in \mathbb{N}$ with $p \leqslant k$, we have

$$\prod_{\lambda\in\Lambda} \varinjlim \mathrm{H}^{p-1}(H_\lambda; A_i) = \prod_{\lambda\in\Lambda} \varinjlim \mathrm{H}^p(H_\lambda; A_i) = \{0\}$$

by Theorem 4.12 and the assumption that the groups H_λ are of type FP_∞. As $\mathrm{card}(\Lambda) < \infty$, direct limits commute with direct products and we have

$$\varinjlim \prod_{\lambda\in\Lambda} \mathrm{H}^{p-1}(H_\lambda; A_i) = \varinjlim \prod_{\lambda\in\Lambda} \mathrm{H}^p(H_\lambda; A_i) = \{0\}.$$

Therefore, the map ϕ and the index p satisfy the assumptions of Lemma 4.7, which yields an isomorphism

$$\varinjlim \mathrm{H}^p(\overline{G}; A_i) \cong \left(\varinjlim \prod_{\lambda\in\Lambda} \mathrm{H}^p(\overline{H}_\lambda; A_i)\right) \oplus \varinjlim \mathrm{H}^p(G; A_i). \tag{15}$$

By assumption, the groups G and $\overline{H}_\lambda$ are of type FP_k. As $\mathrm{card}(\Lambda) < \infty$, Theorem 4.12 implies that the right-hand side of isomorphism (15) is $\{0\}$, which implies $\varinjlim \mathrm{H}^p(\overline{G}; A_i) = \{0\}$. Applying Theorem 4.12 once again, we see that $\overline{G}$ is of type FP_k. □

Recall that a group G is of *type FP* if $\mathrm{cd}(G) < \infty$ and G is of type FP_∞. As $\mathrm{cd}(\mathcal{H}) \leqslant \mathrm{cd}(G)$, the following is a consequence of Corollary 4.9 and Theorem 4.13.

Corollary 4.14. *Let Λ be a finite index set and let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. If the groups $G, H_\lambda, \overline{H}_\lambda, \lambda \in \Lambda$, are of type FP, then so is $\overline{G}$.*

Proof of Theorem 2.4 and 2.6. By Lemma 3.21 and Theorem 3.23, (G, H, N) is a Cohen-Lyndon triple for every sufficiently deep normal subgroup $N \lhd H$. Theorem 2.4 follows directly from Theorem 4.8 and Corollary 4.9. Suppose that the group G is of type FP_k for some $k \in \mathbb{N}^+ \cup \{\infty\}$ (resp. FP). Then H is also of type FP_k (resp. FP) by [DGO17, Theorem 2.11]. Therefore, Theorem 2.6 follows from Corollary 4.9 and Theorem 4.13. □

4.3 Cohomology and embedding theorems

We prove Theorem 2.8 in this section. Given any acylindrically hyperbolic group G, G has a maximal finite normal subgroup $K(G)$ by Theorem 3.27. $G_0 = G/K(G)$ is again acylindrically hyperbolic [Hul16, Lemma 5.10] and $K(G_0) = \{1\}$. By Theorem 3.27, there is a non-cyclic free group $F \hookrightarrow_h G_0$. It is well-known that F is SQ-universal and thus given any countable group C, there is a normal subgroup $N \lhd F$ such that $C \hookrightarrow F/N$. The main idea of the proof of Theorem 2.8 is to choose a particular N so that all conclusions of Theorem 2.8 hold for $\overline{G} = G_0/\langle\langle N \rangle\rangle$.

Lemma 4.15. *Let F_4 be a free group of rank 4, let $\mathcal{F} \subset F_4$ be a finite set, and let C be a countable group with $\mathrm{cd}(C) \geqslant 2$. Then there exists a quotient R of F_4 such that the following hold.*

(1) R can be decomposed as a free product $R = \mathbb{Z} * R_0$ with $\mathrm{card}(R_0) = \infty$. Moreover, C embeds into R_0.

(2) The quotient map $F_4 \to R$ is injective on $\mathcal{F}$.

(3) $\mathrm{cd}(R) \leqslant \mathrm{cd}(C)$.

(4) For every $\ell \geqslant 3$ and every $\mathbb{Z}R$-module A, we have $\mathrm{H}^\ell(R; A) \cong \mathrm{H}^\ell(C; A)$, where the action of C on A is induced by the embedding $C \hookrightarrow R$.

(5) If C is finitely generated, then R_0 is hyperbolic relative to C.

(6) If C is of type FP_k for some $k \in \mathbb{N}^+ \cup \{\infty\}$, then so is R.

Remark 4.16. Except for assertions (3), (4), and (6), Lemma 4.15 is proved in [DGO17, Lemma 8.4]. We refine the method of [DGO17] so as to impose cohomological conditions.

The proof of Lemma 4.15 relies on small cancellation theory, the reader is referred to [LS01, Chapter V] for a treatment.

Proof. Let $\{x, y, z, t\}$ be a free basis of F_4, let $F_3 < F_4$ be the subgroup generated by x, y, t, and let $\{c_i\}_{i\in I}$ be a generating set of C. There exist freely reduced words $w_i, v_i, i \in I$, over the alphabet $\{x, y\}$ such that

(a) the words $c_i w_i, i \in I$, satisfy the $C'(1/2)$ small cancellation condition over the free product $\langle x\rangle * \langle y\rangle * C$;

(b) the words $v_i, i \in I$, satisfy the $C'(1/2)$ small cancellation condition over the alphabet $\{x, y\}$;

(c) the words $tc_i w_i t^{-1} v_i, i \in I$, satisfy the $C'(1/6)$ small cancellation condition over the free product $\langle x\rangle * \langle y\rangle * \langle t\rangle * C$.

Indeed, we can first construct words w_i satisfying condition (a), and then pick sufficiently long words v_i to ensure conditions (b) and (c).

Let N (resp. N_0) be the normal subgroup of $F_4 * C$ (resp. $F_3 * C$) generated by $tc_i w_i t^{-1} v_i, i \in I$, and let

$$R_0 = (F_3 * C)/N_0, \qquad R = (F_4 * C)/N.$$

For $i \in I$, let $\overline{t}$ (resp. $\overline{c}_i, \overline{w}_i, \overline{v}_i, \overline{z}$) be the image of t (resp. c_i, w_i, v_i, z) under the quotient map $F_4 * C \to R$. Then we have

$$R = \langle \overline{z}\rangle * R_0 = \mathbb{Z} * R_0.$$

Note that $\overline{t}\overline{c}_i\overline{w}_i\overline{t}^{-1}\overline{v}_i = 1$ and we can rewrite this equation as $\overline{c}_i = \overline{t}^{-1}\overline{v}_i^{-1}\overline{t}\overline{w}_i^{-1}$. Thus, R is generated by $\overline{t}, \overline{z}, \overline{w}_i, \overline{v}_i, i \in I$, and hence is a quotient of F_4.

Let

$$\alpha : F_4 \to R$$

be the corresponding quotient map. We can also think of α as the restriction of the quotient map $F_4 * C \to R$ to F_4. It follows from the Greendlinger's lemma for free products [LS01, Chapter V Theorem 9.3] that if $\|w_i\|, \|v_i\|, i \in I$, are sufficiently large, then α is injective on $\mathcal{F}$ and thus statement (2) is guaranteed.

Let $L = \langle x\rangle * \langle y\rangle * C$, let $W \leqslant L$ be the subgroup generated by the elements $c_i w_i, i \in I$, and let $V \leqslant L$ be the subgroup generated by the elements $v_i, i \in I$.

Claim 4.16.1. *W (resp. V) is freely generated by $c_i w_i$, (resp. v_i,) $i \in I$. In particular, W and V are free groups of rank* $\operatorname{card}(I)$.

Proof of Claim 4.16.1. We prove the claim for W. The proof for V is similar. Let

$$u \equiv \prod_{k=1}^{n} (c_{i_k} w_{i_k})^{\epsilon_k}$$

be a nonempty freely reduced word over the alphabet $\{c_i w_i\}_{i\in I}$, where $i_k \in I$ and $\epsilon_k = \pm 1$ for $k = 1, ..., n$. Think of u as a word over the alphabet $\langle x\rangle \cup \langle y\rangle \cup C$ and then reduce u to its normal form $\overline{u}$ corresponding to the free product $\langle x\rangle * \langle y\rangle * C$ (see [LS01, Chapter IV] for the definition of normal forms). By condition (a), for each factor $(c_{i_k} w_{i_k})^{\epsilon_k}$ of u, a non-empty subword of $(c_{i_k} w_{i_k})^{\epsilon_k}$ survives in $\overline{u}$. In particular, $\overline{u}$ is nonempty and thus u does not represent 1 in L. $\square$

Note that the relations $\bar{t}\bar{c}_i\bar{w}_i\bar{t}^{-1}\bar{v}_i = 1, i \in I$, can be rewritten as $\bar{t}\bar{c}_i\bar{w}_i\bar{t}^{-1} = \bar{v}_i^{-1}, i \in I$. Thus, R_0 is the HNN-extension of L with associated subgroups W and V. In particular, L embeds into R_0. As $\mathrm{card}(L) = \infty$, we have $\mathrm{card}(R_0) = \infty$. Since C embeds into L, C embeds into R_0. Thus, statement (1) holds.

By [Bie75, Theorem 3.1], there is a long exact sequence

$$\cdots \to \mathrm{H}^{\ell-1}(W; A) \to \mathrm{H}^{\ell}(R_0; A) \to \mathrm{H}^{\ell}(L; A) \to \mathrm{H}^{\ell}(W; A) \to \cdots \tag{16}$$

for any $\mathbb{Z}R$-module A.

As W is free, for $\ell \geqslant 3$, exact sequence (16) implies

$$\mathrm{H}^{\ell}(R_0; A) \cong \mathrm{H}^{\ell}(L; A) \cong \mathrm{H}^{\ell}(C; A).$$

As $R = \mathbb{Z} * R_0$, statement (4) holds. Combining statement (4) with $\mathrm{cd}(C) \geqslant 2$, we see that $\mathrm{cd}(R) \leqslant \mathrm{cd}(C)$. Hence, statement (3) holds.

If C is finitely generated, then we can construct R using a finite generating set of C. Then R_0 is the quotient of $F_3 * C$ by adding finitely many relations $tc_iw_it^{-1}v_i, i \in I$, and thus has a finite relative presentation over C. The Greendlinger's lemma for free products implies that the relative isoperimetric function of R_0 with respect to C is linear. Thus, Lemma 3.16 implies that R_0 is hyperbolic relative to C, which is statement (5).

If C is of type FP_k for some $k \in \mathbb{N}^+ \cup \{\infty\}$, then C is finitely generated and we can construct R using a finite generating set of C, that is, $\mathrm{card}(I) < \infty$. Note that the rank of the free group W is $\mathrm{card}(I)$. Thus, W is of type FP_∞. Note also that L is the free product of a finite rank free group with C and thus is of type FP_k. Let $\{A_i\}_{i\in I}$ be a direct system of $\mathbb{Z}R_0$-modules such that $\varinjlim A_i = \{0\}$. For every $\ell \in \mathbb{N}$ with $\ell \leqslant k$, Theorem 4.12 implies

$$\varinjlim \mathrm{H}^{\ell}(W; A_i) = \varinjlim \mathrm{H}^{\ell}(L; A_i) = \{0\},$$

where the actions of W and L on A_i are induced by their embeddings into R_0. It follows from the five lemma and exact sequence (16) that $\varinjlim \mathrm{H}^{\ell}(R_0; A_i) = \{0\}$. Therefore, R_0 is of type FP_k by Theorem 4.12. As $R = \mathbb{Z} * R_0$, R is of type FP_k, which is statement (6). $\square$

Proof of Theorem 2.8. By Theorem 3.27, G has a maximal finite normal subgroup $K(G)$. By [Hul16, Lemma 5.10], $G_0 = G/K(G)$ is acylindrically hyperbolic.

If $\mathrm{cd}(C) = 0$, then $C = \{1\}$. Let $\overline{G} = G_0$. By Theorem 3.27, $C \hookrightarrow_h \overline{G}$. Conclusions (a), (b), (c), and (d) hold trivially. As $\overline{G}$ and G are quasi-isometric, [Alo94, Corollary 9] implies (e).

If $\mathrm{cd}(C) = 1$, then by the Stallings-Swan theorem [Swa69, corollary to Theorem 1], C is free. By Theorem 3.27, there exists a finitely generated non-cyclic free group F such that $F \hookrightarrow_h G_0$. Let $\overline{G} = G_0$. It is well-known that the free group C embeds into F. Thus, C also embeds into $\overline{G}$. Once again, conclusions (a), (b), (c), and (e) hold trivially. If, in addition, C is finitely generated, then C is a finite rank free group and we can let $F = C$. Thus, (d) also holds.

Let us assume $\mathrm{cd}(C) \geqslant 2$. By Theorem 3.27, there exists a free subgroup $F_4 \hookrightarrow_h G_0$ of rank 4. By Lemma 3.21 and Theorem 3.26, there exists a finite set $\mathcal{F} \subset F_4 \setminus \{1\}$ such that if $N \lhd F_4$ satisfies $N \cap \mathcal{F} = \emptyset$, then

(1) $F_4/N \hookrightarrow_h G_0/\langle\langle N \rangle\rangle$, where $\langle\langle N \rangle\rangle$ is the normal closure of N in G_0;

(2) (G_0, F_4, N) is a Cohen-Lyndon triple and thus Theorems 4.8 and 4.13 and Corollary 4.9 can be applied to it.

By Lemma 4.15, C embeds into an infinite quotient $R = \mathbb{Z} * R_0$ of F_4 such that the conclusions of Lemma 4.15 hold and the quotient map $F_4 \to R$ is injective on $\mathcal{F}$. Let N be the kernel of $F_4 \to R$. Then $N \cap \mathcal{F} = \emptyset$ and thus statements (1) and (2) hold. Let $\overline{G} = G/\langle\langle N \rangle\rangle$.

As $R = \mathbb{Z} * R_0$, R_0 is a proper subgroup of R and in particular, R_0 is a proper subgroup of G. Example 3.13 implies $R_0 \hookrightarrow_h R$. By Proposition 3.19 and statement (1), we have $R_0 \hookrightarrow_h G$. As $\mathrm{card}(R_0) = \infty$, Theorem 3.26 implies that $\overline{G}$ is acylindrically hyperbolic, that is, statement (a) holds. As C embeds into R_0, C also embeds into $\overline{G}$.

Consider statement (b). Corollary 4.9 implies

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G_0), \mathrm{cd}(F_3) + 1, \mathrm{cd}(R)\}.$$

If $K(G) \neq \{1\}$, then G has torsion and thus $\mathrm{cd}(G) = \infty$ by [Bro94, Chapter VIII Corollary 2.5], in which case (b) is a void statement. Thus, let us assume $K(G) = \{1\}$ and thus $G_0 = G$. As $\mathrm{cd}(R) \leqslant \mathrm{cd}(C)$ and $\mathrm{cd}(C) \geqslant 2$, we have

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(F_3) + 1, \mathrm{cd}(R)\} \leqslant \max\{\mathrm{cd}(G), 2, \mathrm{cd}(C)\} = \max\{\mathrm{cd}(G), \mathrm{cd}(C)\}.$$

Thus, (b) holds. Moreover, (c) follows from Theorem 4.8 and statement (4) of Lemma 4.15.

If C is finitely generated, then Lemma 4.15 implies that R_0 is hyperbolic relative to C. By Example 1.2, we have $C \hookrightarrow_h R_0$. As $R_0 \hookrightarrow_h \overline{G}$, we have $C \hookrightarrow_h \overline{G}$ by Proposition 3.19. Thus, statement (d) holds.

If C is of type FP_k for some $k \in \mathbb{N}^+ \cup \{\infty\}$, then Lemma 4.15 implies that so is R. As G and G_0 are quasi-isometric, G_0 is of type FP_k by [Alo94, Corollary 9]. As F_4 has finite rank and thus is of type FP_∞, Theorem 4.13 implies that $\overline{G}$ is of type FP_k. Thus, statement (e) also holds. □

4.4 Constructing hyperbolically embedded subgroups

In this section, we construct a special type of hyperbolically embedded subgroups, which will be useful in the next section when we prove Theorem 2.9.

Suppose that G is a group, $\{a_{\lambda,1}, ..., a_{\lambda,k_\lambda}\}_{\lambda \in \Lambda}$ is a finite subset of G (in other words, $\mathrm{card}(\Lambda), \mathrm{card}(k_\lambda) < \infty$), and there is a finite family of subgroups $\{F_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$ such that each F_λ is a free group of rank $2k_\lambda$.

If we perform Dehn fillings on $\{F_\lambda\}_{\lambda\in\Lambda}$, then it is unclear how the set $\{a_{\lambda,1},...,a_{\lambda,k_\lambda}\}_{\lambda\in\Lambda}$ will be affected. Therefore, we combine $\{a_{\lambda,1},...,a_{\lambda,k_\lambda}\}$ with F_λ to construct new subgroups $H_\lambda \leqslant G$. We are going to show that, under mild conditions, we have $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$, which fits into our framework of Dehn fillings.

For $\lambda\in\Lambda$, let $\{f_{\lambda,i}, g_{\lambda,i}\}_{i=1}^{k_\lambda}$ be a basis of F_λ. As F_λ is the free product of $\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle, i = 1,...,k_\lambda$, we have $\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{i=1}^{k_\lambda} \hookrightarrow_h F_\lambda$ by Example 3.13. It follows from Theorem 3.19 that there exists a set $X\subset G$ such that

$$\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{\lambda\in\Lambda, i\in\{1,...,k_\lambda\}} \hookrightarrow_h (G,X). \tag{17}$$

By Lemma 3.18, we may assume that $a_{\lambda,i}\in X$ for all $\lambda\in\Lambda$ and all $i\in\{1,...,k_\lambda\}$, as Λ is finite.

For $\lambda\in\Lambda$ and $i\in\{1,...,k_\lambda\}$, let

$$\widehat{d}_{\lambda,i,f} : \langle f_{\lambda,i}\rangle\times\langle f_{\lambda,i}\rangle \to [0,+\infty], \qquad \widehat{d}_{\lambda,i,g} : \langle g_{\lambda,i}\rangle\times\langle g_{\lambda,i}\rangle \to [0,+\infty]$$

be the relative metrics corresponding to (17), and let

$$\mathcal{K}_\lambda = \left(\bigsqcup_{i=1}^{k_\lambda}\langle f_{\lambda,i}\rangle\right) \sqcup \left(\bigsqcup_{i=1}^{k_\lambda}\langle g_{\lambda,i}\rangle\right), \qquad \mathcal{K} = \bigsqcup_{\lambda\in\Lambda}\mathcal{K}_\lambda.$$

The metrics $\widehat{d}_{\lambda,i,f}$ and $\widehat{d}_{\lambda,i,g}$ are locally finite. As $\mathrm{card}(\Lambda), \mathrm{card}(k_\lambda) < \infty$, for sufficiently large n, we will have

$$\widehat{d}_{\lambda,i,f}(1, f^n_{\lambda,i}), \widehat{d}_{\lambda,i,g}(1, g^n_{\lambda,i}) > 50D$$

for all λ and i, where $D>0$ is given by Lemma 3.29.

Fix such an n. Let $H_\lambda\leqslant G$ be the subgroup generated by

$$U_\lambda = \{f^n_{\lambda,i}a_{\lambda,i}g^n_{\lambda,i}\}_{i=1}^{k_\lambda}.$$

The above reasoning does not involve the intersections $F_\lambda\cap F_\mu$. Let us assume, in addtion, that $F_\lambda\cap F_\mu = \{1\}$ whenever $\lambda\neq\mu$. Then we have

$$\langle f_{\lambda,i}\rangle\cap\langle g_{\mu,j}\rangle = \{1\} \quad \text{for all } \lambda,\mu\in\Lambda, \tag{18}$$

$$\langle f_{\lambda,i}\rangle\cap\langle f_{\mu,j}\rangle = \langle g_{\lambda,i}\rangle\cap\langle g_{\mu,j}\rangle = \{1\} \quad \text{for all } \lambda,\mu\in\Lambda \text{ with } \lambda\neq\mu \tag{19}$$

$$\langle f_{\lambda,i}\rangle\cap\langle f_{\lambda,j}\rangle = \langle g_{\lambda,i}\rangle\cap\langle g_{\lambda,j}\rangle = \{1\} \quad \text{for all } \lambda\in\Lambda \text{ and } i,j\in\{1,...,k_\lambda\} \text{ with } i\neq j. \tag{20}$$

In particular, $\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{\lambda\in\Lambda, i\in\{1,...,k_\lambda\}}$ is a distinct family of hyperbolically embedded subgroups of G.

Remark 4.17. For every $\lambda\in\Lambda$ and every freely reduced word w over U_λ, we can think of w as a word over the alphabet $X\sqcup\mathcal{K}$, i.e., regard every $f^n_{\lambda,i}$ (resp. $g^n_{\lambda,i}$) as a letter from $\langle f_{\lambda,i}\rangle$ (resp. $\langle g_{\lambda,i}\rangle$) and regard every $a_{\lambda,i}$ as a letter from X. In this sense, w satisfies the conditions (W1), (W2), and (W3) of Lemma 3.30.

Lemma 4.18. *For all $\lambda \in \Lambda$, H_λ is freely generated by U_λ.*

Proof. We need to show that for every $\lambda \in \Lambda$, every nonempty freely reduced word over U_λ does not represent 1 in G. Suppose that

$$w = \prod_{k=1}^{\ell} (f_{\lambda,i_k}^n a_{\lambda,i_k} g_{\lambda,i_k}^n)^{\epsilon_k}$$

is a freely reduced word over U_λ for some $\lambda \in \Lambda$ such that w represents 1 in G, where $\epsilon_k = \pm 1$ for $k = 1, ..., \ell$. As Remark 4.17, we think of w as a word over $X \sqcup \mathcal{K}$. Then w labels a 3ℓ-gon p in $\Gamma(G, X \sqcup \mathcal{K})$ with geodesic sides. Notice that p has 2ℓ components. By Lemma 3.30, each of these components is isolated. Lemma 3.29 then implies

$$2n \cdot 50D \leqslant 3nD,$$

which is absurd. Therefore, such a word w does not exist. □

Consider the action $G \curvearrowright \Gamma(G, X \sqcup \mathcal{K})$. Let $d_{X \sqcup \mathcal{K}}$ be the combinatorial metric of $\Gamma(G, X \sqcup \mathcal{K})$.

Lemma 4.19. *For every $\lambda \in \Lambda$, the action $H_\lambda \curvearrowright \Gamma(G, X \sqcup \mathcal{K})$ is proper.*

Proof. Fix $\lambda \in \Lambda$. It suffices to prove that for every $R > 0$, there are only finitely many $h \in H_\lambda$ such that $d_{X \sqcup \mathcal{K}}(1, h) \leqslant R$. Let $h \in H_\lambda$ such that $d_{X \sqcup \mathcal{K}}(1, h) \leqslant R$ and let w be a freely reduced word over U_λ representing h in G. As in Remark 4.17, think of w as a word over $X \sqcup \mathcal{K}$. By Lemma 3.30, w labels a $(4, 1)$-quasi-geodesic in $\Gamma(G, X \sqcup \mathcal{K})$. Thus,

$$\|w\|_{U_\lambda} = \frac{\|w\|_{X \sqcup \mathcal{K}}}{3} \leqslant \frac{4R + 1}{3}.$$

There are only finitely many words w satisfying the above inequality. It follows that the number of $h \in H_\lambda$ such that $d_{X \sqcup \mathcal{K}}(1, h) \leqslant R$ is finite. □

For every $\lambda \in \Lambda$, we identify H_λ with the subset of $\Gamma(G, X \sqcup \mathcal{K})$ labeled by elements of H_λ. Equivalently, we identify H_λ with the H_λ-orbit of the identity vertex of $\Gamma(G, X \sqcup \mathcal{K})$.

Lemma 4.20. *For every $\lambda \in \Lambda$, the orbit H_λ is quasi-convex in $\Gamma(G, X \sqcup \mathcal{K})$.*

Proof. Fix $\lambda \in \Lambda$. Let $h \in H_\lambda$ and let γ be a geodesic in $\Gamma(G, X \sqcup \mathcal{K})$ from the vertex 1 to the vertex h. As $\Gamma(G, X \sqcup \mathcal{K})$ is a Gromov hyperbolic space, there exists $R > 0$ such that if α and β are $(4, 1)$-quasi-geodesics with the same endpoint, then $d_{Hau}(\alpha, \beta) \leqslant R$, where d_{Hau} is the Haudorff metric with respect to $d_{X \sqcup \mathcal{K}}$.

Let w be a freely reduced word over U_λ representing h in G. As in Remark 4.17, think of w as a word over $X \sqcup \mathcal{K}$. By Lemma 3.30, w labels a $(4, 1)$-quasi-geodesic α in $\Gamma(G, X \sqcup \mathcal{K})$. Note that α lies in the 2-neighborhood of the orbit H_λ, and γ lies in the R-neighborhood of α. Thus, γ lies in the $(R + 2)$-neighborhood of H_λ. □

For $\lambda, \mu \in \Lambda$, the orbits H_λ and H_μ are subsets of $\Gamma(G, X \sqcup \mathcal{K})$. Thus, it makes sense to talk about the diameter of $H_\mu \cap (gH_\lambda)^{+\epsilon}$ in $\Gamma(X \sqcup \mathcal{K})$, which is denoted as $\operatorname{diam}\left(H_\mu \cap (gH_\lambda)^{+\epsilon}\right)$.

Lemma 4.21. *Suppose that for every $\lambda \in \Lambda$, we have*

$$(F_\lambda \setminus \{1\}) \cap \{a_{\lambda,1}, ..., a_{\lambda,k_\lambda}\} = \emptyset. \tag{21}$$

Then for every $\epsilon > 0$, there exists $R > 0$ such that the following holds. Suppose that for some $g \in G$ and $\lambda, \mu \in \Lambda$, we have

$$\operatorname{diam}\left(H_\mu \cap (gH_\lambda)^{+\epsilon}\right) \geqslant R.$$

Then $\lambda = \mu$ and $g \in H_\lambda$.

Proof. Fix $\epsilon > 0$. By Lemma 3.30, there exists $R > 0$ such that if p, q are two paths in $\Gamma(G, X \sqcup \mathcal{K})$ such that if $\ell(p) \geqslant R$, $\mathbf{Lab}(p)$ (resp. $\mathbf{Lab}(q)$) is a freely reduced word over U_μ (resp. U_λ) for some $\mu \in \Lambda$ (resp. $\lambda \in \Lambda$), and p, q are oriented ϵ-close, then there exist five consecutive components of p which are connected to five consecutive components of q.

Suppose that $\operatorname{diam}\left(H_\mu \cap (gH_\lambda)^{+\epsilon}\right) \geqslant R$ for some $g \in G$ and $\lambda, \mu \in \Lambda$. Then there exist vertices $v_1, v_2 \in H_\mu$ and $v_3, v_4 \in gH_\lambda$ such that

$$d(v_1, v_2) \geqslant R, \quad d(v_1, v_3), d(v_2, v_4) \leqslant \epsilon.$$

Let p (resp. q) be a path from v_1 (resp. v_3) to v_2 (resp. v_4) such that $\mathbf{Lab}(p)$ (resp. $\mathbf{Lab}(q)$) is a freely reduced word over U_μ (resp. U_λ). Then $\ell(p) \geqslant R$ and p, q are oriented ϵ-close. Thus, there exist five consecutive components of p which are connected to five consecutive components of q. In particular, there exist two pairs of adjacent components of p which are connected to four consecutive components of q. Some of possible configurations of these two pairs of adjacent components are shown in Figure 3, where each horizontal line represents one possible configuration, the red and blue segments represent the two pairs of adjacent components of p, and the corresponding labels are written on top of the subpaths.

Below, we assume, without loss of generality, that these two pairs of adjacent components are of the form

$$f^{-n}_{\mu,i} g^{-n}_{\mu,j}, f^{-n}_{\mu,j} g^{-n}_{\mu,r}.$$

Other possible configurations can be analyzed similarly. We distinguish two cases.

Case 1. The first pair of adjacent components of p are respectively connected to a pair of adjacent components of q.

Case 1 is displayed in Figure 4, where the red (resp. blue) dash line represents a path with label in $\langle f^n_{\mu,i} \rangle$ (resp. $\langle g^n_{\mu,j} \rangle$) connecting the corresponding red (resp. blue) components. Equations (18), (19), and (20) imply that $\lambda = \mu$ and the red (resp. blue) component of q is labeled by $f^{-n}_{\mu,i}$ (resp. $g^{-n}_{\mu,i}$). As the red and blue dash lines form a loop, another consequence of (18) is that both of these dash lines are labeled by 1.

Let p_1 (resp. q_1) be the subpath of p (resp. q) labeled by $uf^{-n}_{\mu,i}$ (resp. $vf^{-n}_{\mu,i}$). Then $p_1^+ = q_1^+$. By the structure of U_μ (resp. U_λ), $\mathbf{Lab}(p_1)$ (resp. $\mathbf{Lab}(q_1)$) is a word over U_μ

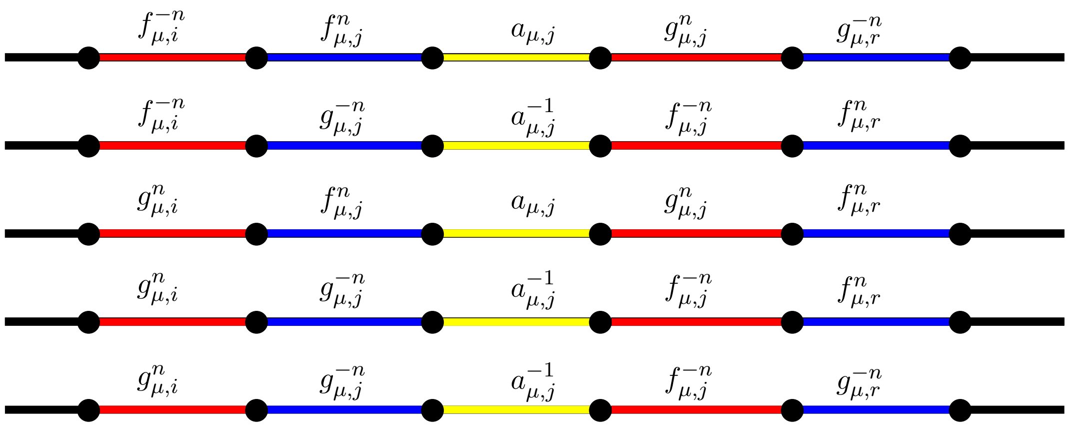

Figure 3: Some possible configurations of the two pairs of adjacent components of p

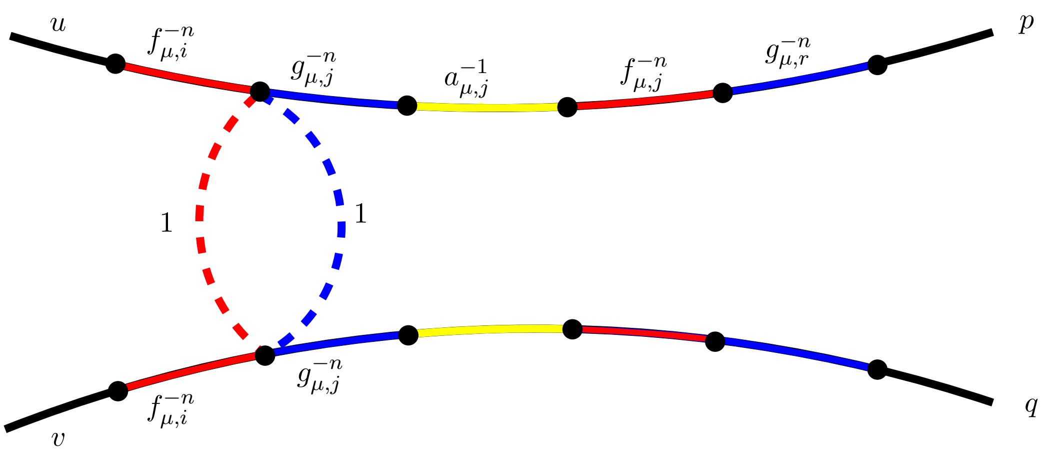

Figure 4: Case 1

(resp. U_λ) and thus represents an element in H_μ (resp. H_λ). Therefore, $p_1^+ \in H_\mu \cap gH_\lambda = H_\mu \cap gH_\mu$. It follows that $g \in H_\mu$.

Case 2. The first pair of adjacent components of p are respectively connected to two consecutive, but not adjacent, components of q.

Case 2 is displayed in Figure 5. Once again, Equations (18), (19), and (20) imply $\lambda = \mu$. The structures of U_μ imply $i = j = r$. The red (resp. blue) dash line on the left is labeled by an element in $\langle f_{\mu,i}^n \rangle$ (resp. $\langle g_{\mu,i}^n \rangle$). As these dash segments and the yellow segment labeled by $a_{\mu,i}$ form a loop, formula (21) implies that both of these dash segments are labeled by

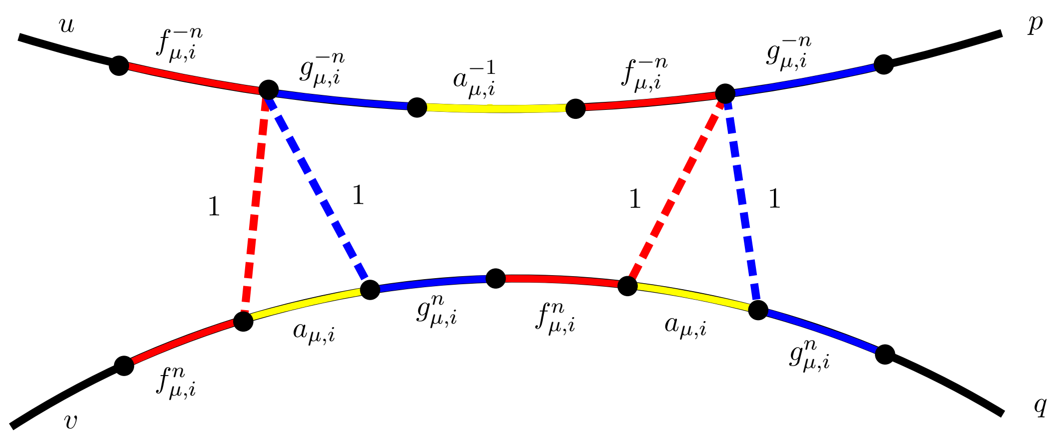

Figure 5: Case 2

1. Similarly, the red and blue dash segments on the right are both labeled by 1.

Therefore, the word $g^{2n}_{\mu,i} f^{2n}_{\mu,i} a_{\mu,i}$ labels a loop in $\Gamma(G, X \sqcup \mathcal{K})$ and thus represents 1 in G, which is in contradiction with formula (21). Hence, Case 2 is in fact impossible. $\square$

Proposition 4.22. *Suppose that G is a group, $\{a_{\lambda,1}, ..., a_{\lambda,k_\lambda}\}_{\lambda\in\Lambda}$ is a finite subset of G (in other words,* $\mathrm{card}(\Lambda), \mathrm{card}(k_\lambda) < \infty$*), and there is a finite family of subgroups $\{F_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ such that*

(a) each F_λ is freely generated by a finite set $\{f_{\lambda,i}, g_{\lambda,i}\}_{i=1}^{k_\lambda} \subset G$;

(b) $F_\lambda \cap F_\mu = \{1\}$ whenever $\lambda \neq \mu$;

(c) $(F_\lambda \setminus \{1\}) \cap \{a_{\lambda,1}, ..., a_{\lambda,k_\lambda}\} = \emptyset$.

Then for sufficiently large $n \in \mathbb{N}^+$, the set

$$\{f^n_{\lambda,i} a_{\lambda,i} g^n_{\lambda,i}\}_{i=1}^{k_\lambda} \subset G$$

freely generetes a subgroup $H_\lambda \leqslant G$ and

$$\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G. \tag{22}$$

Proof. The first assertion follows from Lemma 4.18 and formula (22) follows Lemmas 3.17, 4.19, 4.20, and 4.21. $\square$

4.5 Common quotients of acylindrically hyperbolic groups

In this section, we prove Theorem 2.9. Given finitely generated acylindrically hyperbolic groups G_1 and G_2, we construct a common quotient G of $G_{10} = G_1/K(G_1)$ and $G_{20} = G_2/K(G_2)$ satisfying the conclusions of that theorem, where $K(G_1)$ (resp. $K(G_2)$) is the maximal finite normal subgroup of G_1 (resp. G_2).

The idea is to consider $\widetilde{G} = G_{10} * G_{20}$ and pick a finite generating set A (resp. B) of G_{10} (resp. G_{20}). The quotient G is constructed by adding particular relations (which will be done by Dehn filling) to $\widetilde{G}$ which identify elements of A (resp. B) with certain elements of G_{20} (resp. G_{10}).

As G_1 and G_2 are infinite, G_{10} and G_{20} are also infinite and thus there exists a finite generating set $A = \{a_1, ..., a_k\}$ (resp. $B = \{b_1, ..., b_k\}$) of G_{10} (resp. G_{20}) for some $k \in \mathbb{N}^+$. For simplicity, let $a_{k+1} = a_{k+2} = b_{k+1} = b_{k+2} = 1$.

To perform Dehn filling on $\widetilde{G}$, the first step is to find hyperbolically embedded subgroups. By [Hul16, Lemma 5.10], G_{10} and G_{20} are acylindrically hyperbolic with $K(G_{10}) = K(G_{20}) = \{1\}$. Thus, Theorem 3.27 implies that there exist free groups

$$F_1 \hookrightarrow_h G_{10}, \quad F_2 \hookrightarrow_h G_{20},$$

each of which has rank $2k + 4$. By Example 3.13, we have $\{G_{10}, G_{20}\} \hookrightarrow_h \widetilde{G}$. Thus, Proposition 3.19 implies

$$\{F_1, F_2\} \hookrightarrow_h \widetilde{G}.$$

It is unclear how A, B will be affected if we perform Dehn fillings on $\{F_1, F_2\}$, so we instead apply Proposition 4.22 to construct other hyperbolically embedded subgroups from A, B, F_1, F_2. Note that

$$(F_2 \setminus \{1\}) \cap \{a_1, ..., a_{k+2}\} = (F_1 \setminus \{1\}) \cap \{b_1, ..., b_{k+2}\} = \emptyset.$$

Let $\{f_{1,i}, g_{1,i}\}_{i=1}^{k+2}$ (resp. $\{f_{2,i}, g_{2,i}\}_{i=1}^{k+2}$) be a basis of the free group F_1 (resp. F_2). Then Proposition 4.22 implies:

Lemma 4.23. *For sufficiently large $n \in \mathbb{N}^+$, $\{f_{1,i}^n b_i g_{1,i}^n\}_{i=1}^{k+2}$ (resp. $\{f_{2,i}^n a_i g_{2,i}^n\}_{i=1}^{k+2}$) freely generates a subgroup $H_1 \leqslant \widetilde{G}$ (resp. $H_2 \leqslant \widetilde{G}$) and*

$$\{H_1, H_2\} \hookrightarrow_h \widetilde{G}.$$

Proof of Theorem 2.9. As G_{10} and G_{20} are acylindrically hyperbolic, we have $|G_{10}| = |G_{20}| = \infty$ and thus $\mathrm{cd}(G_{10}), \mathrm{cd}(G_{20}) \geqslant 1$. Suppose $\mathrm{cd}(G_{10}) = \mathrm{cd}(G_{20}) = 1$. Then G_{10} and G_{20} are free by the Stallings-Swan theorem [Swa69, corollary to Theorem 1]. Without loss of generality, we may assume that the rank of G_{10} is greater than or equal to the rank of G_{20}. It follows that G_{20} is a quotient of G_{10}. Let $G = G_{20}$. Statements (a), (b), and (c) follow trivially. Statement (d) also holds because if G_2 is of type FP_k for some $k \in \{2, 3, ..., \infty\}$, then G_{20} is also of type FP_k by [Alo94, Corollary 9].

Thus, let us assume $\max\{\mathrm{cd}(G_{10}), \mathrm{cd}(G_{20})\} \geqslant 2$. Fix a sufficiently large $n \in \mathbb{N}^+$ and let H_1 and H_2 be the subgroups constructed above. By Lemmas 3.21 and 4.23 and Theorem 3.26, there exist finite sets $\mathcal{F}_1 \subset H_1 \setminus \{1\}$ and $\mathcal{F}_2 \subset H_2 \setminus \{1\}$ such that if $N_1 \lhd H_1, N_2 \lhd H_2$ and $N_1 \cap \mathcal{F}_1 = N_2 \cap \mathcal{F}_2 = \emptyset$, then the following hold.

(1) $\{H_1/N_1, H_2/N_2\} \hookrightarrow_h G/\langle\langle N_1 \cup N_2 \rangle\rangle$.

(2) $(G, \{H_1, H_2\}, \{N_1, N_2\})$ is a Cohen-Lyndon triple and thus Theorems 4.8 and 4.13 and Corollary 4.9 can be applied to it.

Let u_i, (resp. v_i,) $1 \leqslant i \leqslant k$, be freely reduced words over the alphabet $\{f^n_{1,k+1}b_{k+1}g^n_{1,k+1}, f^n_{1,k+2}b_{k+2}g^n_{1,k+2}\}$ (resp. $\{f^n_{2,k+1}a_{k+1}g^n_{2,k+1}, f^n_{2,k+2}a_{k+2}g^n_{2,k+2}\}$) satisfying the $C'(1/6)$ small cancellation condition, and let N_1 (resp. N_2) be the normal subgroup of H_1 (resp. H_2) generated by $\{f^n_{1,i}b_ig^n_{1,i}u_i\}^k_{i=1}$ (resp. $\{f^n_{2,i}a_ig^n_{2,i}v_i\}^k_{i=1}$).

By Lemma 4.23, H_1 and H_2 are freely generated by $\{f^n_{1,i}b_ig^n_{1,i}\}^{k+2}_{i=1}$ and $\{f^n_{2,i}a_ig^n_{2,i}\}^{k+2}_{i=1}$, respectively. Thus, H_1/N_1 and H_2/N_2 can be presented as

$$\begin{aligned} H_1/N_1 &= \langle f^n_{1,i}b_ig^n_{1,i}, i = 1, ..., k+2 \mid f^n_{1,j}b_jg^n_{1,j}u_j, j = 1, ..., k\rangle \\ &= \langle f^n_{1,k+1}b_{k+1}g^n_{1,k+1}, f^n_{1,k+2}b_{k+2}g^n_{1,k+2}\rangle, \end{aligned} \tag{23}$$

$$\begin{aligned} H_2/N_2 &= \langle f^n_{2,i}a_ig^n_{2,i}, i = 1, ..., k+2 \mid f^n_{2,j}a_jg^n_{2,j}v_j, j = 1, ..., k\rangle \\ &= \langle f^n_{2,k+1}a_{k+1}g^n_{2,k+1}, f^n_{2,k+2}a_{k+2}g^n_{2,k+2}\rangle, \end{aligned} \tag{24}$$

where the last equality of (23) (resp. (24)) follows from eliminating $f^n_{1,i}b_ig^n_{1,i}, i = 1, ..., k$ (resp. $f^n_{2,i}a_ig^n_{2,i}, i = 1, ..., k$) by Tietze transformations [LS01, Chapter II].

Thus, H_1 and H_2 are free groups of rank 2. In particular,

$$\mathrm{card}(H_1/N_1) = \infty \tag{25}$$

By the Greendlinger's lemma for free groups [LS01, Chapter V Theorem 4.5], if $\|u_i\|, \|v_i\|, 1 \leqslant i \leqslant k$, are sufficiently large, then

$$N_1 \cap \mathcal{F}_1 = N_2 \cap \mathcal{F}_2 = \emptyset.$$

Let

$$G = \widetilde{G}/\langle\langle N_1 \cup N_2 \rangle\rangle.$$

As $a_{k+1} = a_{k+2} = b_{k+1} = b_{k+2} = 1$, G is a common quotient of G_{10} and G_{20}. In particular, G is a common quotient of G_1 and G_2.

If $H_1/N_1 = G$, then G is a non-cyclic free group and thus is acylindrically hyperbolic. If H_1/N_2 is a proper subgroup of G, then equation (25), statement (1), and Theorem 3.26 imply that G is acylindrically hyperbolic. Statement (a) is proved.

Consider statement (b). If either $K(G_1)$ or $K(G_2)$ is not the trivial group $\{1\}$, then (b) is trivial. Suppose $K(G_1) = K(G_2) = \{1\}$. Then Corollary 4.9 implies

$$\begin{aligned}
\mathrm{cd}(G) &\leqslant \max\{\mathrm{cd}(\widetilde{G}), \mathrm{cd}(H_1/N_1), \mathrm{cd}(H_2/N_2), \mathrm{cd}(H_1)+1, \mathrm{cd}(H_2)+1\} \\
&= \max\{\mathrm{cd}(\widetilde{G}), 1, 2\} && \text{as } H_1, H_2, H_1/N_1, \text{ and } H_2/N_2 \text{ are free groups} \\
&= \max\{\mathrm{cd}(G_{10}), \mathrm{cd}(G_{20})\} && \text{as } \mathrm{cd}(G_{10}), \mathrm{cd}(G_{20}) \geqslant 2 \\
&= \max\{\mathrm{cd}(G_1), \mathrm{cd}(G_2)\},
\end{aligned}$$

which proves (b).

Consider statement (c). For every $\ell \geqslant 3$ and every $\mathbb{Z}G$-module A, we have

$$\begin{aligned}
&\mathrm{H}^\ell(G; A) \\
\cong\, &\mathrm{H}^\ell(\widetilde{G}; A) \oplus \mathrm{H}^\ell(H_1/N_1; A) \oplus \mathrm{H}^\ell(H_2/N_2; A) && \text{by Theorem 4.8} \\
\cong\, &\mathrm{H}^\ell(\widetilde{G}; A) && \text{as } H_1/N_1 \text{ and } H_2/N_2 \text{ are free groups} \\
\cong\, &\mathrm{H}^\ell(G_{10}; A) \oplus \mathrm{H}^\ell(G_{20}; A) && \text{as } \widetilde{G} = G_{10} * G_{20},
\end{aligned}$$

which is (c).

Suppose that for some $k \in \{2, 3, ..., \infty\}$, G_1 and G_2 are of type FP_k. Then so are G_{10} and G_{20} by [Alo94, Corollary 9]. As H_1/N_1 and H_2/N_2 are free groups of finite rank, they are of type FP_∞. Therefore, Theorem 4.13 implies that G is of type FP_k and thus statement (d) holds. □

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