CONTINUOUS EXTENSION OF ARITHMETIC VOLUMES

ATSUSHI MORIWAKI

To the memory of the late Professor Masayoshi Nagata

ABSTRACT. This paper is the sequel of the paper [4], in which we established the arithmetic volume function of C^∞-hermitian $\mathbb{Q}$-invertible sheaves and proved its continuity. The continuity of the volume function has a lot of applications as treated in [4]. In this paper, we would like to consider its continuous extension over $\mathbb{R}$.

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INTRODUCTION

Let X be a d-dimensional projective arithmetic variety. In [4], for a C^∞-hermitian invertible sheaf $\overline{L}$ on X, we introduce the arithmetic volume $\widehat{\mathrm{vol}}(\overline{L})$ defined by

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{n \to \infty} \frac{\log \#\{s \in H^0(X, nL) \mid \|s\|_{\sup} \leq 1\}}{n^d/d!}.$$

By Chen's recent work [2], "lim sup" in the above equation can be replaced by "lim". Moreover, in [4], we construct a positively homogeneous function $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$ of degree d such that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{\mathrm{Pic}}(X) & \xrightarrow{\widehat{\mathrm{vol}}} & \mathbb{R} \\ \downarrow & \nearrow_{\widehat{\mathrm{vol}}} & \\ \widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & & \end{array}$$

The most important result of [4] is the continuity of $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$, which has a lot of applications as treated in [4]. In this paper, we would like to consider its continuous extension over $\mathbb{R}$, which is not obvious because a continuous and positively homogeneous

function on a vector space over $\mathbb{Q}$ does not necessarily have a continuous extension over $\mathbb{R}$ (cf. Example 4.1).

Let $C^0(X)$ be the set of real valued continuous functions f on $X(\mathbb{C})$ with $F_\infty^*(f) = f$, where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map on $X(\mathbb{C})$. We denote the group of isomorphism classes of continuous hermitian invertible sheaves on X by $\widehat{\mathrm{Pic}}_{C^0}(X)$. For details, see Conventions and terminology 15. Here we consider four natural homomorphisms:

$$\begin{cases} \overline{\mathcal{O}} : C^0(X) \to \widehat{\mathrm{Pic}}_{C^0}(X) \quad (f \mapsto (\mathcal{O}_X, \exp(-f)|\cdot|_{can})), \\ \zeta : \widehat{\mathrm{Pic}}_{C^0}(X) \to \mathrm{Pic}(X) \quad ((L, |\cdot|) \mapsto L), \\ \mu : C^0(X) \otimes_{\mathbb{Z}} \mathbb{R} \to C^0(X) \quad (f \otimes x \mapsto xf), \\ \overline{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}} : C^0(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \quad (f \otimes x \mapsto \overline{\mathcal{O}}(f) \otimes x). \end{cases}$$

If we define $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ to be

$$\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} := \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R}/(\overline{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})(\mathrm{Ker}(\mu)),$$

then the above homomorphisms yield a commutative diagram

$$\begin{array}{ccccccc} C^0(X) & \xrightarrow{\overline{\mathcal{O}}} & \widehat{\mathrm{Pic}}_{C^0}(X) & \xrightarrow{\zeta} & \mathrm{Pic}(X) & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ C^0(X) & \xrightarrow{\overline{\mathcal{O}}} & \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} & \xrightarrow{\zeta} & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & 0 \end{array}$$

with exact horizontal sequences. The purpose of this paper is to prove the following theorem:

Theorem A. *There is a unique positively homogeneous function* $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ *of degree* d *with the following properties* (*cf. Theorem* 4.4):

(1) (*cf. Proposition* 4.6) *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in a finite dimensional real vector subspace* V *of* $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ *and* $\{f_n\}_{n=1}^{\infty}$ *a sequence in* $C^0(X)$ *such that* $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in the usual topology of* V *and* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to* f. *Then*

$$\lim_{n \to \infty} \widehat{\mathrm{vol}}\left(x_n + \overline{\mathcal{O}}(f_n)\right) = \widehat{\mathrm{vol}}\left(x + \overline{\mathcal{O}}(f)\right).$$

(2) (*cf. Theorem* 5.1) *Let* $\{\overline{A}_n\}_{n=1}^{\infty}$ *be a sequence in a finitely generated* $\mathbb{Z}$-*submodule* M *of* $\mathrm{Pic}_{C^0}(X)$ *and* $\{f_n\}_{n=1}^{\infty}$ *a sequence in* $C^0(X)$ *such that* $\{\overline{A}_n \otimes 1/n\}_{n=1}^{\infty}$ *converges to* $\overline{A}$ *in* $M \otimes \mathbb{R}$ *in the usual topology and* $\{f_n/n\}_{n=1}^{\infty}$ *converges uniformly to* f. *Then*

$$\lim_{n \to \infty} \frac{\hat{h}^0(\overline{L}_n + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \widehat{\mathrm{vol}}(\pi(\overline{A}) + \overline{\mathcal{O}}(f)),$$

where π *is the canonical homomorphism* $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ *and* $\hat{h}^0$ *means the logarithm of the number of small sections* (*for details, see Conventions and terminology* 16).

The composition $\widehat{\mathrm{vol}} \cdot \pi : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\pi} \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \xrightarrow{\widehat{\mathrm{vol}}} \mathbb{R}$ gives an affirmative answer to a continuous extension over $\mathbb{R}$ of the arithmetic volume function on $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The last result is a generalization of Chen's theorem [2]. This also gives an interpretation of the value of $\widehat{\mathrm{vol}}$ in terms of $\hat{h}^0$. Namely, if we want to evaluate

$\widehat{\mathrm{vol}}(\pi(\overline{L}_1 \otimes a_1 + \cdots + \overline{L}_r \otimes a_r))$ for $\overline{L}_1, \ldots, \overline{L}_r \in \widehat{\mathrm{Pic}}_{C^0}(X)$ and $a_1, \ldots, a_r \in \mathbb{R}$, then (2) of Theorem A says

$$\widehat{\mathrm{vol}}(\pi(\overline{L}_1 \otimes a_1 + \cdots + \overline{L}_r \otimes a_r)) = \lim_{n \to \infty} \frac{\hat{h}^0([na_1]\overline{L}_1 + \cdots + [na_r]\overline{L}_r)}{n^d/d!}.$$

The most important tool to establish the continuity of $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X) \otimes \mathbb{Q} \to \mathbb{R}$ was the fundamental estimation theorem [4, Theorem 3.1]. Unfortunately, it is insufficient to prove a continuous extension of the arithmetic volume function over $\mathbb{R}$. Actually, we need the following generalization of the fundamental estimation theorem as a multi-indexed version:

Theorem B. *We assume that X is generically smooth. Let $\overline{L}_1, \ldots, \overline{L}_r, \overline{A}$ be C^∞-hermitian invertible sheaves on X. Then there are positive constants a_0, C and D depending only on X and $\overline{L}_1, \ldots, \overline{L}_r, \overline{A}$ such that*

$$\hat{h}^0\left(a_1\overline{L}_1 + \cdots + a_r\overline{L}_r + (b-c)\overline{A}\right)$$
$$\leq \hat{h}^0\left(a_1\overline{L}_1 + \cdots + a_r\overline{L}_r - c\overline{A}\right)$$
$$+ Cb\left(|a_1| + \cdots + |a_r|\right)^{d-1}$$
$$+ D\left(|a_1| + \cdots + |a_r|\right)^{d-1} \log\left(|a_1| + \cdots + |a_r|\right)$$

for all $a_1, \ldots, a_r, b, c \in \mathbb{Z}$ with

$$|a_1| + \cdots + |a_r| \geq b \geq c \geq 0 \quad and \quad |a_1| + \cdots + |a_r| \geq a_0.$$

Using the above estimate, we can show the uniform continuity of $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes \mathbb{Q} \to \mathbb{R}$ in the following sense; if $\overline{L}_1, \ldots, \overline{L}_r$ are continuous hermitian invertible sheaves on X and $f : \mathbb{Q}^r \to \mathbb{R}$ is a function given by

$$f(x_1, \ldots, x_r) = \widehat{\mathrm{vol}}(x_1\overline{L}_1 + \cdots + x_r\overline{L}_r),$$

then f is uniformly continuous on any bounded set of $\mathbb{Q}^r$. This fact gives us a continuous extension of $\widehat{\mathrm{vol}}$ over $\mathbb{R}$.

This paper is organized as follows: In Section 1, we give the proof of the multi-indexed version of the fundamental estimation. In Section 2, elementary properties of the arithmetic volume function for continuous hermitian $\mathbb{Q}$-invertible sheaves are treated. In Section 3, we consider the uniform continuity of the arithmetic volume function over $\mathbb{Q}$. In Section 4, we establish a continuous extension of the arithmetic volume function over $\mathbb{R}$. In Section 5, we prove that the arithmetic volume function over $\mathbb{R}$ can realized as the limit of $\hat{h}^0$ of continuous hermitian invertible sheaves.

Conventions and terminology. We use the same conventions and terminology as in [4]. Besides them, we fix the following conventions and terminology for this paper.

10. Let S be a set and r a positive integer. For $\boldsymbol{x} = (x_1, \ldots, x_r) \in S^r$, the i-th entry x_i of $\boldsymbol{x}$ is denoted by $\boldsymbol{x}(i)$. We assume that S has an order $\leq$. Then, for $\boldsymbol{x}, \boldsymbol{x}' \in S^r$, $\boldsymbol{x} \leq \boldsymbol{x}'$ means that $\boldsymbol{x}(i) \leq \boldsymbol{x}'(i)$ for all $i = 1, \ldots, r$.

11. Let p be a real number with $p \geq 1$. For $\boldsymbol{x} \in \mathbb{C}^r$, we set

$$|\boldsymbol{x}|_p = \left(|\boldsymbol{x}(1)|^p + \cdots + |\boldsymbol{x}(r)|^p \right)^{1/p} .$$

In particular, $|\boldsymbol{x}|_1 = |\boldsymbol{x}(1)| + \cdots + |\boldsymbol{x}(r)|$ and $|\boldsymbol{x}|_2 = \sqrt{|\boldsymbol{x}(1)|^2 + \cdots + |\boldsymbol{x}(r)|^2}$ (cf. [4, Conventions and terminology 2]). Moreover, we set $|\boldsymbol{x}|_\infty = \max\{|\boldsymbol{x}(1)|, \ldots, |\boldsymbol{x}(r)|\}$. Note that $\lim_{p \to \infty} |\boldsymbol{x}|_p = |\boldsymbol{x}|_\infty$.

12. Let N be a $\mathbb{Z}$-module and K a field. Then $N \otimes_{\mathbb{Z}} K$ is a K-vector space in the natural way. We denote the K-scalar product of $N \otimes_{\mathbb{Z}} K$ by $\cdot$, that is,

$$a \cdot (x_1 \otimes a_1 + \cdots + x_r \otimes a_r) = x_1 \otimes aa_1 + \cdots + x_r \otimes aa_r,$$

where $x_1, \ldots, x_r \in N$ and $a, a_1, \ldots, a_r \in K$. Note that the kernel of the natural homomorphism $N \to N \otimes_{\mathbb{Z}} \mathbb{Q}$ is the subgroup consisting of torsion elements of N.

13. Let M be a module over a ring R. Let $\boldsymbol{a} \in R^r$ and $\boldsymbol{L} \in M^r$. For simplicity, we denote $\boldsymbol{a}(1) \cdot \boldsymbol{L}(1) + \cdots + \boldsymbol{a}(r) \cdot \boldsymbol{L}(r)$ by $\boldsymbol{a} \cdot \boldsymbol{L}$.

14. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let V be a vector space over $\mathbb{K}$ and $f : V \to \mathbb{R}$ a function. Let d be a non-negative real number. We say f is a *positively homogeneous function* of degree d if $f(\lambda x) = \lambda^d f(x)$ for all $\lambda \in \mathbb{K}_{\geq 0}$ and $x \in V$. Moreover, f is said to be *weakly continuous* if, for any finite dimensional vector subspace W of V, $f|_W : W \to \mathbb{R}$ is continuous in the usual topology.

15. Let X be a d-dimensional projective arithmetic variety. Let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation map on $X(\mathbb{C})$. The set of real valued continuous (resp. C^∞-) functions f on $X(\mathbb{C})$ with $F_\infty^*(f) = f$ is denoted by $C^0(X)$ (resp. $C^\infty(X)$). A pair $\overline{L} = (L, |\cdot|)$ of an invertible sheaf L on X and a continuous hermitian metric $|\cdot|$ of L is called a *continuous hermitian invertible sheaf* on X if the hermitian metric is invariant under F_∞. Moreover, if the metric $|\cdot|$ is C^∞, then $\overline{L}$ is called a C^∞-*hermitian invertible sheaf* on X. An element of $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called a *continuous hermitian $\mathbb{Q}$-invertible sheaf* (resp. C^∞-*hermitian $\mathbb{Q}$-invertible sheaf*) on X.

16. Let $(L, \|\cdot\|)$ be a normed $\mathbb{Z}$-module, that is, M is finitely generated $\mathbb{Z}$-module and $\|\cdot\|$ is a norm of $M \otimes_{\mathbb{Z}} \mathbb{R}$. According to [4], $\hat{h}^0(M, \|\cdot\|)$ is defined by

$$\hat{h}^0(M, \|\cdot\|) := \log \#\{x \in M \mid \|x \otimes 1\| \leq 1\}.$$

Let $\overline{L}$ be a continuous hermitian invertible sheaf on a projective arithmetic variety. For simplicity, $\hat{h}^0(H^0(L), \|\cdot\|_{\sup})$ is often denoted by $\hat{h}^0(\overline{L})$.

17. Let M be a compact complex manifold and Φ a volume form on M. Let $\overline{A} = (A, |\cdot|_A)$ and $\overline{B} = (B, |\cdot|_B)$ be C^∞-hermitian invertible sheaves on M. Let t be a section of $H^0(M, B)$ such that t is non-zero on each connected component of M. The subnorm induced by an injective homomorphism $H^0(M, A - B) \xrightarrow{\otimes t} H^0(M, A)$ and the natural L^2-norm of $H^0(M, A)$ given by Φ and $|\cdot|_A$ is denoted by $\|\cdot\|_{L^2, t, \mathrm{sub}}^{\overline{A}, A-B}$, that is,

$$\|s\|_{L^2, t, \mathrm{sub}}^{\overline{A}, A-B} = \sqrt{\int_M |s \otimes t|_A^2 \Phi}$$

for $s \in H^0(M, A - B)$. For simplicity, $\|\cdot\|_{L^2, t, \mathrm{sub}}^{\overline{A}, A-B}$ is often denoted by $\|\cdot\|_{L^2, t, \mathrm{sub}}^{\overline{A}}$.

1. A MULTI-INDEXED VERSION OF THE FUNDAMENTAL ESTIMATION

1.1. Let X be a d-dimensional generically smooth projective arithmetic variety. Let $\overline{A}$ be a C^∞-hermitian invertible sheaf on X and $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_r)$ a finite sequence of C^∞-hermitian invertible sheaves on X. Let $\boldsymbol{L} = (L_1, \ldots, L_r)$ be the sequence of invertible sheaves obtained by forgetting metrics of $\overline{\boldsymbol{L}}$. The following theorem is a generalization of [4, Theorem 3.1].

Theorem 1.1.1. *There are positive constants a_0, C and D depending only on X, $\overline{\boldsymbol{L}}$ and $\overline{A}$ such that*

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A), \|\cdot\|_{\sup}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) \leq \hat{h}^0\left(H^0(\boldsymbol{a}\cdot\boldsymbol{L} - cA), \|\cdot\|_{\sup}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}}\right)$$
$$+ Cb|\boldsymbol{a}|_1^{d-1} + D|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq a_0$, where

$$\begin{cases} |\boldsymbol{a}|_1 = |a_1| + \cdots + |a_r|, \\ \boldsymbol{a} \cdot \boldsymbol{L} = a_1 L_1 + \cdots + a_r L_r, \\ \boldsymbol{a} \cdot \overline{\boldsymbol{L}} = a_1\overline{L}_1 + \cdots + a_r\overline{L}_r \end{cases}$$

for $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ (cf. Conventions and terminology 11 and 13).

The proof of Theorem 1.1.1 is almost same as one of [4, Theorem 3.1]. For reader's convenience, we will give its proof in the remaining of this section. We use the same notation as in [4]. Let us begin with distorsion functions.

1.2. Distorsion function. Let M be an n-equidimensional projective complex manifold. First let us recall distorsion functions. Let Φ be a volume form of M and $\overline{H} = (H, h)$ a C^∞-hermitian invertible sheaf on M. For $s, s' \in H^0(M, H)$, we set

$$\langle s, s' \rangle_{\overline{H}, \Phi} = \int_M h(s, s')\Phi.$$

Let $s_1, \ldots, s_N$ be an orthonormal basis of $H^0(X, H)$ with respect to $\langle\ ,\ \rangle_{\overline{H}, \Phi}$. Then it is easy to see that, for all $x \in M$, the quantity $\sum_{i=1}^N h(s_i, s_i)(x)$ does not depend on the choice of the orthonormal basis $s_1, \ldots, s_N$, so that we define

$$\mathrm{dist}(\overline{H}, \Phi)(x) = \sum_{i=1}^N h(s_i, s_i)(x).$$

The function $\mathrm{dist}(\overline{H}, \Phi)$ is called the *distorsion function of $\overline{H}$ with respect to* Φ. For a positive number λ, it is easy to check that $\mathrm{dist}(\overline{H}, \lambda\Phi) = \lambda^{-1}\mathrm{dist}(\overline{H}, \Phi)$. Moreover, if $M_1, \ldots, M_l$ are connected components of M, then

$$\mathrm{dist}(\overline{H}, \Phi) = \mathrm{dist}\left(\overline{H}\big|_{X_1}, \Phi\big|_{X_1}\right) + \cdots + \mathrm{dist}\left(\overline{H}\big|_{X_l}, \Phi\big|_{X_l}\right).$$

Let $\overline{A}$ be a positive C^∞-hermitian invertible sheaf on M and $\overline{\boldsymbol{B}} = (\overline{B}_1, \ldots, \overline{B}_l)$ a finite sequence of positive C^∞-hermitian invertible sheaves on M. Then we have the following:

Theorem 1.2.1. *For any real number ϵ with $0 < \epsilon < 1$, there is a positive constant $a(\epsilon)$ such that*

$$\mathrm{dist}(a\overline{A} - \boldsymbol{b} \cdot \overline{\boldsymbol{B}}, c_1(\overline{A})^n) \leq \frac{(1+\epsilon)^{l+1}}{(1-\epsilon)^l}h^0(aA)$$

for all $a \in \mathbb{Z}$ and $\boldsymbol{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ with $a \geq a(\epsilon)$, $b_1 \geq a(\epsilon), \ldots, b_l \geq a(\epsilon)$, where $\boldsymbol{b} \cdot \overline{\boldsymbol{B}} = b_1\overline{B}_1 + \cdots + b_l\overline{B}_l$.

Proof. Clearly we may assume that M is connected. In the following, the hermitian metrics of $a\overline{A}$, $b_i\overline{B}_i$ and $\boldsymbol{b} \cdot \overline{\boldsymbol{B}}$ are denoted by $h_{a\overline{A}}$, $h_{b_i\overline{B}_i}$ and $h_{\boldsymbol{b} \cdot \overline{\boldsymbol{B}}}$ respectively. By Bouche-Tian's theorem ([1], [5]), there is a positive constant $a(\epsilon)$ such that

$$\begin{cases} h^0(aA)\,(1-\epsilon) \le \operatorname{dist}(a\overline{A}, \Phi(\overline{A}))(z) \le h^0(aA)\,(1+\epsilon) \\ h^0(b_iB_i)\,(1-\epsilon) \le \operatorname{dist}(b_i\overline{B}_i, \Phi(\overline{B}_i))(z) \le h^0(b_iB_i)\,(1+\epsilon) \quad (i=1,\ldots,l) \end{cases}$$

hold for all $z \in M$ and all $a \ge a(\epsilon), b_1 \ge a(\epsilon), \ldots, b_l \ge a(\epsilon)$, where

$$\Phi(\overline{A}) = \frac{c_1(\overline{A})^n}{\int_M c_1(\overline{A})^n} \quad \text{and} \quad \Phi(\overline{B}_i) = \frac{c_1(\overline{B}_i)^n}{\int_M c_1(\overline{B}_i)^n}$$

for $i = 1, \ldots, l$.

Fix an arbitrary point x of M. Let $\boldsymbol{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ with $b_1 \ge a(\epsilon), \ldots, b_l \ge a(\epsilon)$. We consider an orthonormal basis of $H^0(b_iB_i)$ with respect to the L^2-norm $\langle \, , \, \rangle_{b_i\overline{B}_i, \Phi(\overline{B}_i)}$ arising from $h_{b_i\overline{B}_i}$ and $\Phi(\overline{B}_i)$ such that the only one element of the basis has non-zero value at x. Let s_{b_i} be a such element of the basis. Then we have

$$h_{b_i\overline{B}_i}(s_{b_i}, s_{b_i})(x) = \operatorname{dist}(b_i\overline{B}_i, \Phi(\overline{B}_i))(x) \ge (1-\epsilon)h^0(b_iB_i).$$

On the other hand, since

$$\|s_{b_i}\|_{\sup}^2 \le \sup_{z \in M} \operatorname{dist}(b_i\overline{B}_i, \Phi(\overline{B}_i))(z) \le (1+\epsilon)h^0(b_iB_i),$$

we obtain

$$\frac{h_{b_i\overline{B}_i}(s_{b_i}, s_{b_i})(x)}{\|s_{b_i}\|_{\sup}^2} \ge \frac{1-\epsilon}{1+\epsilon}.$$

If we set $s_{\boldsymbol{b}} = s_{b_1} \otimes \cdots \otimes s_{b_l}$, then

$$\frac{h_{\boldsymbol{b} \cdot \overline{\boldsymbol{B}}}(s_{\boldsymbol{b}}, s_{\boldsymbol{b}})(x)}{\|s_{\boldsymbol{b}}\|_{\sup}^2} \ge \frac{h_{b_1\overline{B}_1}(s_{b_1}, s_{b_1})(x) \cdots h_{b_l\overline{B}_l}(s_{b_l}, s_{b_l})(x)}{\|s_{b_1}\|_{\sup}^2 \cdots \|s_{b_l}\|_{\sup}^2} \ge \left(\frac{1-\epsilon}{1+\epsilon}\right)^l.$$

For $a \ge a(\epsilon)$, let $t_1, \ldots, t_r$ be an orthonormal basis of $H^0(aA - \boldsymbol{b} \cdot \overline{\boldsymbol{B}})$ with respect to $\langle \, , \, \rangle_{a\overline{A}-\boldsymbol{b} \cdot \overline{\boldsymbol{B}}, \Phi(\overline{A})}$ such that $s_{\boldsymbol{b}} \otimes t_1, \ldots s_{\boldsymbol{b}} \otimes t_r$ are orthogonal with respect to $\langle \, , \, \rangle_{a\overline{A}, \Phi(\overline{A})}$ as elements of $H^0(aA)$. Then, since

$$\left\{ s_{\boldsymbol{b}} \otimes t_i / \|s_{\boldsymbol{b}} \otimes t_i\|_{a\overline{A}, \Phi(\overline{A})} \right\}_{i=1,\ldots,r}$$

form a part of an orthonormal basis of $H^0(aA)$,

$$\sum_{i=1}^r \frac{h_{a\overline{A}}(s_{\boldsymbol{b}} \otimes t_i, s_{\boldsymbol{b}} \otimes t_i)(x)}{\|s_{\boldsymbol{b}} \otimes t_i\|_{a\overline{A}, \Phi(\overline{A})}^2} \le \operatorname{dist}(aA, \Phi(\overline{A}))(x) \le (1+\epsilon)h^0(aA).$$

Note that $\|s_{\boldsymbol{b}} \otimes t_i\|_{a\overline{A}, \Phi(\overline{A})}^2 \le \|s_{\boldsymbol{b}}\|_{\sup}^2$. Moreover, since $\lambda = \int_M c_1(\overline{A})^n \ge 1$,

$$\operatorname{dist}(a\overline{A} - \boldsymbol{b} \cdot \overline{\boldsymbol{B}}, c_1(\overline{A})^n) = \lambda^{-1}\operatorname{dist}(a\overline{A} - \boldsymbol{b} \cdot \overline{\boldsymbol{B}}, \Phi(\overline{A})) \le \operatorname{dist}(a\overline{A} - \boldsymbol{b} \cdot \overline{\boldsymbol{B}}, \Phi(\overline{A})).$$

Therefore,

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^l \operatorname{dist}(a\overline{A} - \boldsymbol{b} \cdot \overline{\boldsymbol{B}}, c_1(\overline{A})^n)(x)$$

$$\leq \frac{h_{\boldsymbol{b}\cdot\overline{\boldsymbol{B}}}(s_{\boldsymbol{b}}, s_{\boldsymbol{b}})(x)}{\|s_{\boldsymbol{b}}\|_{\sup}^2} \sum_{i=1}^r h_{a\overline{A}-\boldsymbol{b}\cdot\overline{\boldsymbol{B}}}(t_i, t_i)(x)$$

$$\leq \sum_{i=1}^r \frac{h_{\boldsymbol{b}\cdot\overline{\boldsymbol{B}}}(s_{\boldsymbol{b}}, s_{\boldsymbol{b}})(x)}{\|s_{\boldsymbol{b}} \otimes t_i\|_{a\overline{A},\Phi(\overline{A})}^2} h_{a\overline{A}-\boldsymbol{b}\cdot\overline{\boldsymbol{B}}}(t_i, t_i)(x)$$

$$= \sum_{i=1}^r \frac{h_{a\overline{A}}(s_{\boldsymbol{b}} \otimes t_i, s_{\boldsymbol{b}} \otimes t_i)(x)}{\|s_{\boldsymbol{b}} \otimes t_i\|_{a\overline{A},\Phi(\overline{A})}^2} \leq (1+\epsilon)h^0(aA).$$

Thus the theorem follows. $\qquad\qquad\square$

Here we recall several notations: Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional vector space over $\mathbb{K}$. A map $\langle\ ,\ \rangle : V \times V \to \mathbb{K}$ is called a $\mathbb{K}$-*inner product* if the following conditions (1) $\sim$ (4) are satisfied: (1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ ($\forall x, y \in V$), (2) $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$ and $\langle ax, y \rangle = a\langle x, y \rangle$ ($\forall x, x', y \in V, \forall a \in \mathbb{K}$), (3) $\langle x, x \rangle \geq 0$ ($\forall x \in V$), (4) $\langle x, x \rangle = 0 \iff x = 0$. Let $(V_1, \langle\ ,\ \rangle_1)$ and $(V_2, \langle\ ,\ \rangle_2)$ be finite dimensional vector spaces over $\mathbb{K}$ with $\mathbb{K}$-inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, and let $\phi : V_1 \to V_2$ be an isomorphism over $\mathbb{K}$. For a basis $\{x_1, \ldots, x_n\}$ of V_1, we consider a quantity

$$-\frac{1}{2} \log\left(\frac{\det(\langle x_i, x_j \rangle_1)}{\det(\langle \phi(x_i), \phi(x_j) \rangle_2)}\right),$$

which does not depend on the choice of the basis $\{x_1, \ldots, x_n\}$ of V_1. It is called the *volume difference* of $(V_1, \langle\ ,\ \rangle_1) \xrightarrow{\phi} (V_2, \langle\ ,\ \rangle_2)$ and is denoted by

$$\gamma((V_1, \langle\ ,\ \rangle_1) \xrightarrow{\phi} (V_2, \langle\ ,\ \rangle_2)).$$

It is easy to check that

$$[\mathbb{K} : \mathbb{R}]\gamma((V_1, \langle\ ,\ \rangle_1) \xrightarrow{\phi} (V_2, \langle\ ,\ \rangle_2)) = \log\left(\frac{\operatorname{vol}\{x \in V_1 \mid \langle x, x \rangle_1 \leq 1\}}{\operatorname{vol}\{x \in V_1 \mid \langle \phi(x), \phi(x) \rangle_2 \leq 1\}}\right),$$

where vol is a Haar measure of V_1. Thus if $(M, \|\cdot\|_1)$ and $(M, \|\cdot\|_2)$ are normed $\mathbb{Z}$-modules and $\|\cdot\|_1$ and $\|\cdot\|_2$ are L^2-norms, then

$$\hat{\chi}(M, \|\cdot\|_1) - \hat{\chi}(M, \|\cdot\|_2) = \gamma((M \otimes_{\mathbb{Z}} \mathbb{R}, \|\cdot\|_1) \xrightarrow{\operatorname{id}} (M \otimes_{\mathbb{Z}} \mathbb{R}, \|\cdot\|_2))$$

Let us consider a corollary of Theorem 1.2.1. Let $\overline{L}_1, \ldots, \overline{L}_r, \overline{A}$ be C^∞-hermitian invertible sheaves on M such that $\overline{A}$ and $\overline{L}_i + \overline{A}$ are positive for $i = 1, \ldots, r$. We set

$$\boldsymbol{L} = (L_1, \ldots, L_r), \quad \overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_r), \quad \Phi = c_1(\overline{L}_1 + \cdots + \overline{L}_r + r\overline{A})^n.$$

Let $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}_{\geq 0}$. Let s be a section of $H^0(bA)$ such that $\|s\|_{\sup} \leq 1$ and s is non-zero on each connected component of M. Let

$$\langle\ ,\ \rangle_{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}} \quad \text{and} \quad \langle\ ,\ \rangle_{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}$$

be the natural $\mathbb{C}$-inner products of $H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA)$ and $H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)$ with respect to Φ. Here we consider a submetric $\langle\ ,\ \rangle_{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A},s,\mathrm{sub}}$ of $H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA)$ induced by s and the metric $\langle\ ,\ \rangle_{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}$, that is,

$$\langle t, t' \rangle_{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A},s,\mathrm{sub}} = \langle st, st' \rangle_{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}$$

for $t, t' \in H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA)$. Then we have the following corollary.

Corollary 1.2.2. *Let $\gamma(\boldsymbol{a}, c, s)$ be the volume difference of*

$$\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \langle, \rangle_{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}\right) \xrightarrow{\mathrm{id}} \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \langle, \rangle_{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}, s, \mathrm{sub}}\right).$$

For any real number ϵ with $0 < \epsilon < 1$, there is positive constant $a(\epsilon)$ such that

$$\gamma(\boldsymbol{a}, c, s) \geq \frac{(1 + \epsilon)^{r+2}}{(1 - \epsilon)^{r+1}} h^0(2|\boldsymbol{a}|_1(L_1 + \cdots + L_r + rA)) \left(\int_M \log(|s|)\Phi\right)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $c \in \mathbb{Z}_{\geq 0}$ with $|\boldsymbol{a}|_1 \geq a(\epsilon)$.

Proof. Note that, for all $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r$ and $c \in \mathbb{Z}_{\geq 0}$,

$$\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A} = 2|\boldsymbol{a}|_1(\overline{L}_1 + \cdots + \overline{L}_r + r\overline{A})$$
$$- (2|\boldsymbol{a}|_1 - a_1)(\overline{L}_1 + \overline{A}) - \cdots - (2|\boldsymbol{a}|_1 - a_r)(\overline{L}_r + \overline{A}) - (c + |\boldsymbol{a}|_1)\overline{A}$$

and that $2|\boldsymbol{a}|_1 - a_i \geq |\boldsymbol{a}|_1$ for all i. Therefore, by Theorem 1.2.1, there is a positive constant $a(\epsilon)$ such that

$$\mathrm{dist}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}, \Phi) \leq \frac{(1 + \epsilon)^{r+2}}{(1 - \epsilon)^{r+1}} h^0(2|\boldsymbol{a}|_1(L_1 + \cdots + L_r + rA))$$

for all $x \in M$ and all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $c \in \mathbb{Z}_{\geq 0}$ with $|\boldsymbol{a}|_1 \geq a(\epsilon)$. Let $t_1, \ldots, t_l$ be an orthonormal basis of $H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA)$ with respect to $\langle , \rangle_{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}$ such that $st_1, \ldots, st_l$ are orthogonal with respect to $\langle , \rangle_{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$. Then

$$\gamma(\boldsymbol{a}, c, s) = -\frac{1}{2} \log \left(\frac{\det(\langle t_i, t_j \rangle_{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}})}{\det(\langle st_i, st_j \rangle_{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}})}\right) = \frac{1}{2} \sum_{i=1}^{l} \log \int_M |s|^2 |t_i|^2 \Phi.$$

Thus, using Jensen's inequality, for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $c \in \mathbb{Z}_{\geq 0}$ with $|\boldsymbol{a}|_1 \geq a(\epsilon)$,

$$\gamma(\boldsymbol{a}, c, s) \geq \frac{1}{2} \sum_{i=1}^{l} \int_M \log(|s|^2)|t_i|^2 \Phi$$
$$= \int_M \log(|s|) \mathrm{dist}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}, \Phi)\Phi$$
$$\geq \frac{(1 + \epsilon)^{r+2}}{(1 - \epsilon)^{r+1}} h^0(2|\boldsymbol{a}|_1(L_1 + \cdots + L_r + rA)) \left(\int_M \log(|s|)\Phi\right).$$

$\square$

1.3. The proof of Theorem 1.1.1. In this subsection, let us give the proof of Theorem 1.1.1. Here we consider variants of Theorem 1.1.1, that is, the restricted version of Theorem 1.1.1 and the L^2-version of Theorem 1.3.1.

Theorem 1.3.1. *In the situation of Theorem 1.1.1, there are positive constants a_0, C and D depending only on X, $\overline{\boldsymbol{L}}$ and $\overline{A}$ such that*

$$\hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A), \| \cdot \|_{\mathrm{sup}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}\right) \leq \hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{\mathrm{sup}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}\right)$$
$$+ Cb|\boldsymbol{a}|_1^{d-1} + D|\boldsymbol{a}|_1^{d-1} \log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq a_0$.

Theorem 1.3.2. *In the situation of Theorem* 1.1.1, *we fix a volume form of $X(\mathbb{C})$ to give L^2-norms $\| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$ and $\| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}$. Then there are positive constants a_0', C' and D' depending only on X, $\overline{\boldsymbol{L}}$, $\overline{A}$ and the volume form of $X(\mathbb{C})$ such that*

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A), \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}\right) \leq \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}\right)$$
$$+ C'b|\boldsymbol{a}|_1^{d-1} + D'|\boldsymbol{a}|_1^{d-1} \log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq a_0'$.

First of all, let us see

$$\begin{cases} \text{Theorem 1.3.1} \Longrightarrow \text{Theorem 1.1.1,} \\ \text{Theorem 1.3.2} \Longrightarrow \text{Theorem 1.3.1,} \end{cases}$$

so that it is sufficient to show Theorem 1.3.2.

Theorem 1.3.1 $\Longrightarrow$ Theorem 1.1.1. For $\boldsymbol{\epsilon} \in \{\pm 1\}^r$ and $\boldsymbol{a} \in \mathbb{Z}^r$, we set

$$\overline{\boldsymbol{L}}(\boldsymbol{\epsilon}) = (\boldsymbol{\epsilon}(1)\overline{L}_1, \ldots, \boldsymbol{\epsilon}(r)\overline{L}_r) \quad \text{and} \quad \boldsymbol{a}(\boldsymbol{\epsilon}) = (\boldsymbol{\epsilon}(1)\boldsymbol{a}(1), \ldots, \boldsymbol{\epsilon}(r)\boldsymbol{a}(r)).$$

By Theorem 1.3.1, for each $\boldsymbol{\epsilon} \in \{\pm 1\}^r$, there are positive constants $a_0(\boldsymbol{\epsilon})$, $C(\boldsymbol{\epsilon})$ and $D(\boldsymbol{\epsilon})$ depending only on X, $\overline{\boldsymbol{L}}(\boldsymbol{\epsilon})$ and $\overline{A}$ such that

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L}(\boldsymbol{\epsilon}) + (b-c)A), \| \cdot \|_{\sup}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}}(\boldsymbol{\epsilon}) + (b-c)\overline{A}}\right)$$
$$\leq \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L}(\boldsymbol{\epsilon}) - cA), \| \cdot \|_{\sup}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}}(\boldsymbol{\epsilon}) - c\overline{A}}\right)$$
$$+ C(\boldsymbol{\epsilon})b|\boldsymbol{a}|_1^{d-1} + D(\boldsymbol{\epsilon})|\boldsymbol{a}|_1^{d-1} \log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq a_0(\boldsymbol{\epsilon})$. Note that, for any $\boldsymbol{a} \in \mathbb{Z}^r$, there is $\boldsymbol{\epsilon} \in \{\pm 1\}^r$ with $\boldsymbol{a}(\boldsymbol{\epsilon}) \in \mathbb{Z}_{\geq 0}^r$, and that $\boldsymbol{a}(\boldsymbol{\epsilon}) \cdot \overline{\boldsymbol{L}}(\boldsymbol{\epsilon}) = \boldsymbol{a} \cdot \overline{\boldsymbol{L}}$ and $|\boldsymbol{a}(\boldsymbol{\epsilon})|_1 = |\boldsymbol{a}|_1$ for $\boldsymbol{\epsilon} \in \{\pm 1\}^r$ and $\boldsymbol{a} \in \mathbb{Z}^r$. Thus, if we set

$$a_0 = \max_{\boldsymbol{\epsilon} \in \{\pm 1\}^r} \{a_0(\boldsymbol{\epsilon})\}, \quad C = \max_{\boldsymbol{\epsilon} \in \{\pm 1\}^r} \{C(\boldsymbol{\epsilon})\} \quad \text{and} \quad D = \max_{\boldsymbol{\epsilon} \in \{\pm 1\}^r} \{D(\boldsymbol{\epsilon})\},$$

then Theorem 1.1.1 follows. $\qquad\square$

Theorem 1.3.2 $\Longrightarrow$ Theorem 1.3.1. Since $\| \cdot \|_{\sup}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \geq \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$, we have

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A), \| \cdot \|_{\sup}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}\right)$$
$$\leq \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A), \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}\right).$$

Moreover, applying Gromov's inequality (cf. [4, Corollary 1.1.2]) to $\overline{L}_{1\mathbb{C}}, \ldots, \overline{L}_{r\mathbb{C}}, -\overline{A}_{\mathbb{C}}$, there is a constant $D \geq 1$ such that

$$\| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}} \geq D^{-1}(|\boldsymbol{a}|_1 + c + 1)^{-(d-1)} \| \cdot \|_{\sup}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $c \in \mathbb{Z}_{\geq 0}$. Therefore, since $|\boldsymbol{a}|_1 \geq c$, by using [4, Proposition 2,1], we obtain

$$\hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}} \right)$$

$$\leq \hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), D^{-1}(|\boldsymbol{a}|_1 + c + 1)^{-(d-1)}\|\cdot\|_{\sup}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}} \right)$$

$$\leq \hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \|\cdot\|_{\sup}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}} \right)$$

$$+ \log(D(2|\boldsymbol{a}|_1 + 1)^{d-1})C_1|\boldsymbol{a}|_1^{d-1} + C_2|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1),$$

where C_1 and C_2 are positive constants with the following properties:

$$(1.3.2.1) \quad \begin{cases} \operatorname{rk} H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A)) \leq C_1|\boldsymbol{a}|_1^{d-1} \quad (|\boldsymbol{a}|_1 \geq 1), \\ \log(18)\operatorname{rk} H^0(a_1(L + A) + \cdots + a_r(L_r + A)) \\ \quad + 2\log\left((\operatorname{rk} H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A)))!\right) \\ \qquad\qquad\qquad\qquad \leq C_2|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1) \quad (|\boldsymbol{a}|_1 \geq 2). \end{cases}$$

Thus we get our assertion. $\qquad\square$

The proof of Theorem 1.3.2. First let us see the following claim.

Claim 1.3.2.2. *Let $\overline{A}'$ be another C^∞-hermitian invertible sheaf on X with $\overline{A} \leq \overline{A}'$. If the theorem holds for $\overline{L}$ and $\overline{A}'$, then it also holds for $\overline{L}$ and $\overline{A}$.*

Proof. This is obvious because $\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b - c)\overline{A} \leq \boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b - c)\overline{A}'$ and $\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}' \leq \boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}$. $\qquad\square$

By the above claim, we may assume the following:

(1) A is very ample on X.
(2) $\overline{A}$ and $\overline{L}_i + \overline{A}$ $(i = 1, \ldots, r)$ are positive on $X(\mathbb{C})$.

Moreover, we fix positive constants C_1 and C_2 as in (1.3.2.1).

Claim 1.3.2.3. *If the theorem holds for a volume form on $X(\mathbb{C})$, then so does for any volume form.*

Proof. We assume that the theorem holds for a volume form Φ on $X(\mathbb{C})$. Let Φ' be another volume form on $X(\mathbb{C})$. Then there are constants $0 < \sigma_0 < 1$ and $\sigma_1 > 1$ with

$$\sigma_0^2\Phi' \leq \Phi \leq \sigma_1^2\Phi'.$$

Thus

$$\|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A},\Phi} \leq \sigma_1\|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A},\Phi'} \quad \text{and} \quad \sigma_0\|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A},\Phi'} \leq \|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A},\Phi}.$$

Therefore we have

$$\hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A), \sigma_1\|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A},\Phi'} \right)$$

$$\leq \hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A), \|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A},\Phi} \right)$$

and

$$\hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A},\Phi} \right) \leq \hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \sigma_0\|\cdot\|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A},\Phi'} \right).$$

Note that

$$\begin{cases} \operatorname{rk} H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A) \le \operatorname{rk} H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A)) \\ \operatorname{rk} H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA) \le \operatorname{rk} H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A)). \end{cases}$$

Then, by [4, Proposition 2.1],

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A), \|\cdot\|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}, \Phi'}\right)$$

$$\le \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b-c)A), \sigma_1\|\cdot\|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}, \Phi'}\right)$$
$$+ \log(\sigma_1)C_1|\boldsymbol{a}|_1^{d-1} + C_2 \log(|\boldsymbol{a}|_1)|\boldsymbol{a}|_1^{d-1}$$

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \sigma_0\|\cdot\|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}, \Phi'}\right) \le \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \|\cdot\|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}, \Phi'}\right)$$
$$+ \log(\sigma_0^{-1})C_1|\boldsymbol{a}|_1^{d-1} + C_2 \log(|\boldsymbol{a}|_1)|\boldsymbol{a}|_1^{d-1}.$$

Thus we get the claim. □

Therefore, we may assume that a volume form Φ on $X(\mathbb{C})$ is given by

$$\Phi = c_1(\overline{L}_1 + \cdots + \overline{L}_r + r\overline{A})^{d-1}.$$

Here we fix a notation: For a real number λ, we set

$$\overline{A}^\lambda = \overline{A} + (\mathcal{O}_X, \exp(-\lambda)|\cdot|_{can}).$$

Let us see the following claim.

Claim 1.3.2.4. *We may assume the following:*

(a) *There is a non-zero small section s of A such that $\operatorname{div}(s)$ is smooth over $\mathbb{Q}$. (In the case where $d = 1$, $\operatorname{div}(s)$ is empty on $X_{\mathbb{Q}}$.)*

(b) *We can find a positive integer n with the following property: For each $i = 1, \ldots, r$, there is a non-zero small section t_i of $nA - L_i$ such that t_i is non-zero on every irreducible component of $\operatorname{div}(s)$.*

Proof. Since A is very ample, there is a non-zero section s of A such that $\operatorname{div}(s)$ is smooth over $\mathbb{Q}$. Moreover, by [4, Lemma 3.2], we can find a positive integer n with the following property: For each $i = 1, \ldots, r$, there is a non-zero section t_i of $nA - L_i$ such that t_i is non-zero on every irreducible component of $\operatorname{div}(s)$. Let λ be a non-negative real number with $\exp(-\lambda)\|s\|_{\sup} \le 1$ and $\exp(-n\lambda)\|t_i\|_{\sup} \le 1$ for $i = 1, \ldots, r$. Then s and t_i are small sections of $\overline{A}^\lambda$ and $n\overline{A}^\lambda - \overline{L}_i$ respectively. On the other hand, $\overline{A} \le \overline{A}^\lambda$, and $\overline{A}^\lambda$ satisfy the conditions (1) and (2). Moreover,

$$c_1(\overline{L}_1 + \cdots + \overline{L}_r + r\overline{A}) = c_1(\overline{L}_1 + \cdots + \overline{L}_r + r\overline{A}^\lambda).$$

Thus, by Claim 1.3.2.2, we get our claim. □

For a coherent sheaf $\mathcal{F}$ on X and a closed subscheme Z of X, the image of the natural homomorphism

$$H^i(X, \mathcal{F}) \to H^i(Z, \mathcal{F}|_Z)$$

is denoted by $I^i(Z, \mathcal{F}|_Z)$ or $I^i(\mathcal{F}|_Z)$ for simplicity.

Let us start the proof of Theorem 1.3.2. If $b = 0$, then $c = 0$. In this case, the assertion of the theorem is obvious, so that we may assume that $b \ge 1$. As in Claim 1.3.2.4, let s be a non-zero small section of A such that $Y := \operatorname{div}(s)$ is smooth over $\mathbb{Q}$.

Here we fix constants C_3 and C_4 as follows:

$$\begin{cases} \mathrm{rk}\, H^0(Y, a_1(L_1 + A) + \cdots + a_r(L_r + A)|_Y) \leq C_3|\boldsymbol{a}|_1^{d-2} \qquad (|\boldsymbol{a}|_1 \geq 1) \\ \log(18)\, \mathrm{rk}\, H^0(Y, a_1(L_1 + A) + \cdots + a_r(L_r + A)|_Y) \\ \qquad + 2\log\left((\mathrm{rk}\, H^0(Y, a_1(L_1 + A) + \cdots + a_r(L_r + A)|_Y))!\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \leq C_4|\boldsymbol{a}|_1^{d-2}\log(|\boldsymbol{a}|_1) \qquad (|\boldsymbol{a}|_1 \geq 2) \end{cases}$$

In the case where $d = 1$, $\mathrm{rk}\, H^0(Y, a_1(L_1 + A) + \cdots + a_r(L_r + A)|_Y) = 0$.

Let $\| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}$ be the quotient norm of $I^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{bY})$ induced by the surjective homomorphism $H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A) \to I^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{bY})$ and L^2-norm $\| \cdot \|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}$ of $H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)$. An exact sequence

$$0 \to H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA) \xrightarrow{s^b} H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A) \to I^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{bY}) \to 0$$

gives rise to an exact sequence of normed $\mathbb{Z}$-modules:

$$0 \to \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2,s^b,\mathrm{sub}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) \to \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A), \| \cdot \|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right)$$
$$\to \left(I^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{bY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) \to 0.$$

Since $\mathrm{rk}\, H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA) \leq \mathrm{rk}\, H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A))$, by [4, (4) of Proposition 2.1], the above exact sequence yields

(1.3.2.4)
$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A), \| \cdot \|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) \leq \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2,s^b,\mathrm{sub}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right)$$
$$+ \hat{h}^0\left(I^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{bY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) + C_2|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}_{\geq 0}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq 2$.

Here let us consider two lemmas.

Lemma 1.3.3. *There are constants a_0 and C_5 depending only on $\overline{L}$ and $\overline{A}$ such that*

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2,s^b,\mathrm{sub}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) \leq \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}}\right)$$
$$+ C_5 b|\boldsymbol{a}|_1^{d-1} + C_2|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq a_0$.

Lemma 1.3.4. *There are constants C_6 and C_7 depending only on $\overline{L}$ and $\overline{A}$ such that*

$$\hat{h}^0\left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{bY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right) \leq C_6 b|\boldsymbol{a}|_1^{d-1} + (C_4 + C_7)|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq 2$.

We will prove these lemmas in the next subsection. Assuming them, we proceed with the proof of our theorem. Gathering (1.3.2.4), Lemma 1.3.3 and Lemma 1.3.4, if we put $C = C_5 + C_6$ and $D = 2C_2 + C_4 + C_7$, then

$$\hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A), \| \cdot \|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}+(b-c)\overline{A}}\right)$$
$$\leq \hat{h}^0\left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2}^{\boldsymbol{a}\cdot\overline{\boldsymbol{L}}-c\overline{A}}\right) + Cb|\boldsymbol{a}|_1^{d-1} + D|\boldsymbol{a}|_1^{d-1}\log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}|_1 \geq a_0$. This proves Theorem 1.3.2. $\qquad\square$

1.4. The proofs of Lemma 1.3.3 and Lemma 1.3.4. In this subsection, we consider the proofs of Lemma 1.3.3 and Lemma 1.3.4. We keep every notation as in the previous subsection.

Proof of Lemma 1.3.3. Note that $\| \cdot \|_{L^2, s^b, \mathrm{sub}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \leq \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}}$. Thus, by [4, (2) of Proposition 2.1],

$$\hat{h}^0 \left(H^0(aL - cA), \| \cdot \|_{L^2}^{a\overline{L} - c\overline{A}} \right) - \hat{h}^0 \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2, s^b, \mathrm{sub}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$
$$+ C_2 |\boldsymbol{a}|_1^{d-1} \log(|\boldsymbol{a}|_1)$$
$$\geq \hat{\chi} \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - c\overline{A}} \right) - \hat{\chi} \left(H^0(\boldsymbol{a} \cdot \boldsymbol{L} - cA), \| \cdot \|_{L^2, s^b, \mathrm{sub}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right).$$

Hence, by Corollary 1.2.2, we obtain Lemma 1.3.3. $\qquad\square$

Proof of Lemma 1.3.4. This is very complicated. Let k be an integer with $0 \leq k < b$. Let $\| \cdot \|_{L^2, s^k, \mathrm{sub}, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$ be the quotient norm induced by the surjective homomorphism

$$H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \to I^0(Y, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_Y)$$

and the norm $\| \cdot \|_{L^2, s^k, \mathrm{sub}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$ of $H^0(aL + (b - c - k)A)$. Let Y' be the horizontal part of Y, that is, the Zariski closure of $Y \cap X_{\mathbb{Q}}$ in X. In the case where $d = 1$, $Y' = \emptyset$. Since the kernel of

$$I^0(Y, aL + (b - c - k)A|_Y) \to I^0(Y', aL + (b - c - k)A|_{Y'})$$

is a torsion group, $I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'})$ has the same norm $\| \cdot \|_{L^2, s^k, \mathrm{sub}, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$. Then we have the following.

Claim 1.4.1. *There are constants C_6' and C_7' depending only on $\overline{L}$ and $\overline{A}$ such that*

$$\hat{h}^0 \left(I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}), \| \cdot \|_{L^2, s^k, \mathrm{sub}, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right) \leq C_6' |\boldsymbol{a}|_1^{d-1} + C_7' |\boldsymbol{a}|_1^{d-2} \log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c, k \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$, $|\boldsymbol{a}|_1 \geq 2$ and $0 \leq k < b$.

Proof. If $d = 1$, then $I^0(Y', aL + (b - c - k)A|_{Y'}) = 0$. Thus our assertion is obvious, so that we assume $d \geq 2$.

Let U be an open set of $X(\mathbb{C})$ such that the closure of U does not intersect with $Y'(\mathbb{C})$ and that U is not empty on each connected component of $X(\mathbb{C})$. Applying [4, Lemma 1.4] to $L_{1\mathbb{C}}, \ldots, L_{r\mathbb{C}}, A_{\mathbb{C}}$ and $L_{1\mathbb{C}}, \ldots, L_{r\mathbb{C}}, -A_{\mathbb{C}}$, we can find constants $D_1 \geq 1$ and $D_1' \geq 1$ such that, for all $\boldsymbol{l} \in \mathbb{Z}_{\geq 0}^r$, $m \in \mathbb{Z}$ and $u \in H^0(X(\mathbb{C}), \boldsymbol{l} \cdot \boldsymbol{L} + mA)$,

$$D_1' D_1^{|\boldsymbol{l}|_1 + |m|} \int_U |u|^2 \Phi \geq \int_{X(\mathbb{C})} |u|^2 \Phi.$$

Since $0 < \inf_{x \in U}\{|s|(x)\} \leq 1$, if we set $D_2 = 1/\inf_{x \in U}\{|s|(x)\}$ then $D_2 \geq 1$. Thus, if we set $D_3 = \max\{D_2, D_1\}$, then, for $u \in H^0(X, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A)$,

$$\int_{X(\mathbb{C})} |s^k \otimes u|^2 \Phi \geq \int_U |s^k \otimes u|^2 \Phi \geq D_2^{-2k} \int_U |u|^2 \Phi$$

$$\geq D_2^{-2k} D_1'^{-1} D_1^{-(|\boldsymbol{a}|_1 + |b-c-k|)} \int_{X(\mathbb{C})} |u|^2 \Phi$$

$$\geq D_1'^{-1} D_3^{-4|\boldsymbol{a}|_1} \int_{X(\mathbb{C})} |u|^2 \Phi.$$

The above inequality means that

$$\| \cdot \|_{L^2, s^k, \mathrm{sub}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \geq D_1'^{-1/2} D_3^{-2|\boldsymbol{a}|_1} \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}}.$$

Therefore, we have

$$\| \cdot \|_{L^2, s^k, \mathrm{sub}, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \geq D_1'^{-1/2} D_3^{-2|\boldsymbol{a}|_1} \| \cdot \|_{L^2, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}},$$

where $\| \cdot \|_{L^2, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}}$ is the quotient norm of $I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'})$ induced by the surjective homomorphism

$$H^0(X, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \to I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}).$$

Note that $e^x \geq x + 1$ ($x \geq 0$). Applying [4, Corollary 1.1.3] to $L_{1\mathbb{C}}, \ldots, L_{r\mathbb{C}}, A_{\mathbb{C}}$ and $L_{1\mathbb{C}}, \ldots, L_{r\mathbb{C}}, -A_{\mathbb{C}}$, we can find constants $D_4, D_4' \geq 1$ such that

$$\| \cdot \|_{L^2, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}} \geq D_4'^{-1/2} D_4^{-(|\boldsymbol{a}|_1 + |b-c-k|)/2} \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{Y'}}$$

$$\geq D_4'^{-1/2} D_4^{-|\boldsymbol{a}|_1} \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{Y'}}$$

holds on $I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'})$. Here a volume form of Y is given by the C^∞-hermitian invertible sheaf $(\overline{L}_1 + \cdots + \overline{L}_r + r\overline{A})|_{Y'}$. Therefore, if we set $D_5 = \max\{D_3, D_4\}$ and $D_5' = \max\{D_1', D_4'\}$, then

$$\| \cdot \|_{L^2, s^k, \mathrm{sub}, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \geq D_5'^{-1} D_5^{-3|\boldsymbol{a}|_1} \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{Y'}}$$

holds on $I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'})$. Note that $\mathrm{rk}\, H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}) \leq \mathrm{rk}\, H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A)|_{Y'})$. Thus, by [4, (3) of Proposition 2.1],

$$\hat{h}^0\left(I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}), \| \cdot \|_{L^2, s^k, \mathrm{sub}, \mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}\right)$$

$$\leq \hat{h}^0\left(I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}), \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{Y'}}\right)$$

$$+ \log(D_5' D_5^{3|\boldsymbol{a}|_1}) C_3 |\boldsymbol{a}|_1^{d-2} + C_4 |\boldsymbol{a}|_1^{d-2} \log(|\boldsymbol{a}|_1)$$

$$\leq \hat{h}^0\left(H^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}), \| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{Y'}}\right)$$

$$+ \log(D_5' D_5^{3|\boldsymbol{a}|_1}) C_3 |\boldsymbol{a}|_1^{d-2} + C_4 |\boldsymbol{a}|_1^{d-2} \log(|\boldsymbol{a}|_1).$$

Let $\widetilde{Y'}$ be the normalization of Y'. Let $t_1, \ldots, t_r$ be small sections as in Claim 1.3.2.4. Then t_i gives rise to an inequality $\overline{L}_i|_{\widetilde{Y'}} \leq n\overline{A}|_{\widetilde{Y'}}$. Therefore,

$$\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b - c - k)\overline{A}|_{\widetilde{Y'}} \leq (n|\boldsymbol{a}|_1 + b - c - k)\overline{A}|_{\widetilde{Y'}}.$$

Thus we have

$$\hat{h}^0\left(H^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}), \|\cdot\|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{Y'}}\right)$$

$$\leq \hat{h}^0\left(H^0(\widetilde{Y'}, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{\widetilde{Y'}}), \|\cdot\|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c-k)\overline{A}|_{\widetilde{Y'}}}\right)$$

$$\leq \hat{h}^0\left(H^0(\widetilde{Y'}, (n|\boldsymbol{a}|_1 + b - c - k)A|_{\widetilde{Y'}}), \|\cdot\|_{L^2}^{(n|\boldsymbol{a}|_1 + b - c - k)\overline{A}|_{\widetilde{Y'}}}\right).$$

Moreover, by [4, Lemma 3.3], there is a constant D_6 such that

$$\hat{h}^0\left(H^0(\widetilde{Y'}, mA|_{\widetilde{Y'}}), \|\cdot\|_{L^2}^{m\overline{A}|_{\widetilde{Y'}}}\right) \leq D_6 m^{d-1}$$

for all $m \geq 1$. Thus we get the claim. $\qquad \square$

We would like to extend Claim 1.4.1 to Y. First of all, we consider the following claim.

Claim 1.4.2. *Let I and I' be the defining ideals of Y and Y' respectively. Then there is a constant C_6'' such that*

$$\log \#H^0(X, (\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \otimes I'/I) \leq C_6'' |\boldsymbol{a}|_1^{d-1}$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c, k \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$, $|\boldsymbol{a}|_1 \geq 2$ and $0 \leq k < b$.

Proof. Since A is ample, there is a positive integer m_0 such that $H^j(X, mA \otimes I'/I) = 0$ for all $m \geq m_0$ and $j > 0$. We set $\mathrm{Ass}_{\mathcal{O}_X}(I'/I) = \{x_1, \ldots, x_r\}$. Then, by [4, Lemma 3.2], there is n_0 independent from a, b, c, k with the following properties:

(i) For each $i = 1, \ldots, r$, there is a non-zero section l_i of $H^0(X, n_0 A - L_i)$ such that $l_i(x_j) \neq 0$ for all j.
(ii) $2(n_0 - 1) \geq m_0$.

By (i), we have an injective homomorphism

$$(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \otimes I'/I \xrightarrow{\otimes l_1^{\otimes \boldsymbol{a}(1)} \otimes \cdots \otimes l_r^{\otimes \boldsymbol{a}(r)}} (n_0|\boldsymbol{a}|_1 + b - c - k)A \otimes I'/I.$$

Note that

$$n_0|\boldsymbol{a}|_1 + b - c - k \geq (n_0 - 1)|\boldsymbol{a}|_1 + |\boldsymbol{a}|_1 + b - c - k \geq 2(n_0 - 1) \geq m_0.$$

Then we have

$$\log \#H^0(X, (\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \otimes I'/I)$$

$$\leq \log \#H^0(X, (n_0|\boldsymbol{a}|_1 + b - c - k)A \otimes I'/I)$$

$$= \sum_{j \geq 0} (-1)^j \log \#H^j(X, (n_0|\boldsymbol{a}|_1 + b - c - k)A \otimes I'/I).$$

Since $\mathrm{Supp}(I'/I)$ is contained in fibers of $X \to \mathrm{Spec}(\mathbb{Z})$, by using Snapper's theorem, we can find a polynomial $P(X) \in \mathbb{R}[X]$ of degree $\leq d - 1$ such that

$$\sum_{j \geq 0} (-1)^j \log \#H^j(X, (n_0|\boldsymbol{a}|_1 + b - c - k)A \otimes I'/I) = P(n_0|\boldsymbol{a}|_1 + b - c - k).$$

Thus we get the claim. $\qquad \square$

Claim 1.4.3. *There are constants C_6 and C_7 depending only on $\overline{L}$ and $\overline{A}$ such that*

$$\hat{h}^0 \left(I^0(Y, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_Y), \| \cdot \|^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}_{L^2, s^k, \mathrm{sub, quot}} \right) \leq C_6 |\boldsymbol{a}|_1^{d-1} + C_7 |\boldsymbol{a}|_1^{d-2} \log(|\boldsymbol{a}|_1)$$

for all $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, c, k \in \mathbb{Z}$ with $|\boldsymbol{a}|_1 \geq b \geq c \geq 0$, $|\boldsymbol{a}|_1 \geq 2$ and $0 \leq k < b$.

Proof. Since the kernel of the natural surjective homomorphism

$$I^0(Y, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_Y) \to I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'})$$

is contained in the torsion group $H^0(X, (\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \otimes I'/I)$, we have

$$\hat{h}^0 \left(I^0(Y, \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_Y), \| \cdot \|^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}_{L^2, s^k, \mathrm{sub, quot}} \right)$$

$$\leq \hat{h}^0 \left(I^0(Y', \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_{Y'}), \| \cdot \|^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}_{L^2, s^k, \mathrm{sub, quot}} \right)$$

$$+ \log \# H^0(X, (\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) \otimes I'/I).$$

Thus our claim follows from Claim 1.4.1 and Claim 1.4.2. $\qquad\qquad\square$

Let us start the proof of Lemma 1.3.4. A commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & -(k+1)A & \xrightarrow{s^{k+1}} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{(k+1)Y} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle s} & & \| & & \downarrow & & \\
0 & \longrightarrow & -kA & \xrightarrow{s^k} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{kY} & \longrightarrow & 0 \\
& & \downarrow & & & & & & \\
& & -kA|_Y & & & & & &
\end{array}
$$

yields an injective homomorphism $\alpha_k : -kA|_Y \to \mathcal{O}_{(k+1)Y}$ together with a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & -kA & \xrightarrow{s^k} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{kY} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & -kA|_Y & \xrightarrow{\alpha_k} & \mathcal{O}_{(k+1)Y} & \longrightarrow & \mathcal{O}_{kY} & \longrightarrow & 0,
\end{array}
$$

where two horizontal sequence are exact. Thus, tensoring with $\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A$, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
0 \to & \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A & \xrightarrow{s^k} & \boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A & \to \boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{kY} \to 0 \\
& \downarrow & & \downarrow & \| \\
0 \to \boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_Y & \xrightarrow{\alpha_k} & \boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{(k+1)Y} & \to \boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A|_{kY} \to 0.
\end{array}
$$

Therefore, we get an exact sequence

$$0 \to I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A)|_Y) \to I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{(k+1)Y})$$

$$\to I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{kY}) \to 0.$$

Note that in a commutative diagram

$$\begin{array}{ccc} H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A) & \xrightarrow{s^k} & H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A) \\ \downarrow & & \downarrow \\ I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A)|_Y) & \xrightarrow{\alpha_k} & I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{(k+1)Y}), \end{array}$$

two vertical homomorphisms have the same kernel. Thus, by [4, Lemma 3.4],

$$0 \to \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A)|_Y), \| \cdot \|_{L^2,s^k,\mathrm{sub,quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$\to \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{(k+1)Y}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$\to \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{kY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right) \to 0$$

is a normed exact sequence, where, for each i, the norm of

$$\left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{iY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

is the quotient norm induced by the surjective homomorphism $H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A) \to I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{iY})$ and L^2-norm $\| \cdot \|_{L^2}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}}$ of $H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)$. Note that

$$\mathrm{rk}\, H^0(\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A|_Y) \le \mathrm{rk}\, H^0(a_1(L_1 + A) + \cdots + a_r(L_r + A)|_Y).$$

Thus, by using [4, (4) of Proposition 2.1],

$$\hat{h}^0 \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{(k+1)Y}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$- \hat{h}^0 \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{kY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$\le \hat{h}^0 \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A)|_Y), \| \cdot \|_{L^2,s^k,\mathrm{sub,quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right) + C_4 |\boldsymbol{a}|_1^{d-2} \log(|\boldsymbol{a}|_1).$$

Therefore, taking $\sum_{k=1}^{b-1}$, the above inequalities imply

$$\hat{h}^0 \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_{bY}), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$- \hat{h}^0 \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c)A)|_Y), \| \cdot \|_{L^2,\mathrm{quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$\le \sum_{k=1}^{b-1} \hat{h}^0 \left(I^0((\boldsymbol{a} \cdot \boldsymbol{L} + (b - c - k)A)|_Y), \| \cdot \|_{L^2,s^k,\mathrm{sub,quot}}^{\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + (b-c)\overline{A}} \right)$$

$$+ (b - 1)C_4 |\boldsymbol{a}|_1^{d-2} \log(|\boldsymbol{a}|_1).$$

Hence, by using Claim 1.4.3, we have Lemma 1.3.4. $\qquad\square$

2. ARITHMETIC VOLUME FUNCTION

Let X be a d-dimensional projective arithmetic variety. Let $\overline{L}$ be a continuous hermitian invertible sheaf on X (cf. Conventions and terminology 15). As mentioned in Conventions and terminology 16, $\hat{h}^0(\overline{L})$ is given by

$$\hat{h}^0(\overline{L}) = \log \# \left\{ s \in H^0(X, L) \mid \|s\|_{\sup} \le 1 \right\}.$$

In the same way as in [4], the arithmetic volume $\widehat{\mathrm{vol}}(\overline{L})$ of $\overline{L}$ is defined by

$$\widehat{\mathrm{vol}}(\overline{L}) = \limsup_{m \to \infty} \frac{\hat{h}^0(m\overline{L})}{m^d/d!}.$$

By virtue of Chen's theorem [2], $\widehat{\mathrm{vol}}(\overline{L})$ is actually given by

$$\widehat{\mathrm{vol}}(\overline{L}) = \lim_{m \to \infty} \frac{\hat{h}^0(m\overline{L})}{m^d/d!}.$$

Fix a volume form Φ on $X(\mathbb{C})$. For a real number $p \geq 1$, let $\| \cdot \|_{L^p}$ be the L^p-norm of $H^0(X(\mathbb{C}), L_{\mathbb{C}})$ induced by the hermitian metric $| \cdot |$ of $\overline{L}$ and Φ, that is,

$$\|s\|_{L^p} := \left(\int_{X(\mathbb{C})} |s|^p \Phi \right)^{1/p} \qquad (s \in H^0(X(\mathbb{C}), L_{\mathbb{C}})).$$

Similarly as above, we can define a volume with respect to the L^p-norm to be

$$\widehat{\mathrm{vol}}_{L^p}(\overline{L}) = \limsup_{m \to \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \|s\|_{L^p} \leq 1\}}{m^d/d!}.$$

Then, using Gromov's inequality [4, Corollary 1.1.2] and Chen's result [2], it is easy to see that $\widehat{\mathrm{vol}}_{L^p}(\overline{L}) = \widehat{\mathrm{vol}}(\overline{L})$ and

$$\widehat{\mathrm{vol}}_{L^p}(\overline{L}) = \lim_{m \to \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \|s\|_{L^p} \leq 1\}}{m^d/d!}.$$

For the definitions of $C^0(X)$ and $\widehat{\mathrm{Pic}}_{C^0}(X)$, see Conventions and terminology 15. Let $\overline{\mathcal{O}} : C^0(X) \to \widehat{\mathrm{Pic}}_{C^0}(X)$ be a homomorphism defined by

$$f \mapsto \overline{\mathcal{O}}(f) := (\mathcal{O}_X, \exp(-f)| \cdot |_{can}).$$

Let $\zeta : \widehat{\mathrm{Pic}}_{C^0}(X) \to \mathrm{Pic}(X)$ be a natural homomorphism given by forgetting the equipped hermitian metric. Then the following sequence

$$(2.1) \qquad C^0(X) \xrightarrow{\overline{\mathcal{O}}} \widehat{\mathrm{Pic}}_{C^0}(X) \xrightarrow{\zeta} \mathrm{Pic}(X) \longrightarrow 0$$

is exact. It is easy to see that

$$(2.2) \qquad \mathrm{Ker}(\overline{\mathcal{O}}) = \left\{ \log |u| \mid u \in H^0(X, \mathcal{O}_X^\times) \right\}.$$

Note that the natural homomorphism $C^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to C^0(X)$ $(f \otimes a \mapsto af)$ is an isomorphism. Thus (2.1) yields an exact sequence

$$(2.3) \qquad C^0(X) \xrightarrow{\overline{\mathcal{O}}} \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\zeta \otimes \mathrm{id}_{\mathbb{Q}}} \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0$$

together with the following commutative diagram:

$$
\begin{array}{ccccccc}
C^0(X) & \xrightarrow{\overline{\mathcal{O}}} & \widehat{\mathrm{Pic}}_{C^0}(X) & \xrightarrow{\zeta} & \mathrm{Pic}(X) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
C^0(X) & \xrightarrow{\overline{\mathcal{O}}} & \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\zeta \otimes \mathrm{id}_{\mathbb{Q}}} & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & 0.
\end{array}
$$

Let us begin with the following lemma.

Lemma 2.4. *Let $\overline{L}$ be a continuous hermitian $\mathbb{Q}$-invertible sheaf on X (cf. Conventions and terminology 15). For any positive real number ϵ, there is $\phi \in C^0(X)$ such that $\|\phi\|_{\sup} \leq \epsilon$ and $\overline{L} + \overline{\mathcal{O}}(\phi)$ is C^∞.*

Proof. Let us choose a positive integer n and a continuous hermitian invertible sheaf $\overline{M}$ such that $n\overline{L} = \overline{M} \otimes 1$ in $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $|\cdot|$ be the metric of $\overline{M}$ and fix a C^∞-metric $|\cdot|_0$ of M. Then there is $f \in C^0(X)$ such that $|\cdot| = \exp(-f)|\cdot|_0$. By Stone-Weierstrass theorem, there is $g \in C^\infty(X)$ such that $\|f - g\|_{\sup} \le n\epsilon$. Note that $\exp(-(f-g))|\cdot| = \exp(g)|\cdot|_0$. Thus $\overline{M} + \overline{\mathcal{O}}(f-g)$ is a C^∞-hermitian invertible sheaf. Therefore, if we set $\phi = (f-g)/n$, then $\|\phi\|_{\sup} \le \epsilon$ and

$$n(\overline{L} + \overline{\mathcal{O}}(\phi)) = \overline{M} \otimes 1 + \overline{\mathcal{O}}(f-g) = (\overline{M} + \overline{\mathcal{O}}(f-g)) \otimes 1.$$

Hence the lemma follows. □

Proposition 2.5. *For a continuous hermitian invertible sheaf $\overline{L}$ on X, we have the following:*

(1) $|\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(f)) - \widehat{\mathrm{vol}}(\overline{L})| \le d\|f\|_{\sup} \mathrm{vol}(L_{\mathbb{Q}})$ *for $f \in C^0(X)$.*

(2) $\widehat{\mathrm{vol}}(n\overline{L}) = n^d \widehat{\mathrm{vol}}(\overline{L})$ *for a non-negative integer n.*

(3) *If $\nu : X' \to X$ is a birational morphism of projective arithmetic varieties, then $\widehat{\mathrm{vol}}(\nu^*(\overline{L})) = \widehat{\mathrm{vol}}(\overline{L})$ for any $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X)$.*

Proof. (1) We set $\lambda = \|f\|_{\sup}$. Then $-\lambda \le f \le \lambda$. Thus it is easy to see that

$$\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(-\lambda)) \le \widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(f)) \le \widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\lambda)).$$

Therefore, it is a consequence of [4, Proposition 2.1, (3)] in the same way as in [4, Proposition 4.2].

(2) Fix a positive integer n. By Lemma 2.4, for any positive number ϵ, there is $\phi \in C^0(X)$ such that $\|\phi\|_{\sup} \le \epsilon$ and $\overline{L} + \overline{\mathcal{O}}(\phi)$ is C^∞. Hence, by [4, Propostion 4.8], $\widehat{\mathrm{vol}}(n(\overline{L} + \overline{\mathcal{O}}(\phi))) = n^d \widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi))$. Therefore, using (1), we have

$$|\widehat{\mathrm{vol}}(n\overline{L}) - n^d\widehat{\mathrm{vol}}(\overline{L})| \le |\widehat{\mathrm{vol}}(n\overline{L}) - \widehat{\mathrm{vol}}(n(\overline{L} + \overline{\mathcal{O}}(\phi)))| + |\widehat{\mathrm{vol}}(n(\overline{L} + \overline{\mathcal{O}}(\phi))) - n^d\widehat{\mathrm{vol}}(\overline{L})|$$
$$\le d\|n\phi\|_{\sup} \mathrm{vol}(nL_{\mathbb{Q}}) + n^d|\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi)) - \widehat{\mathrm{vol}}(\overline{L})|$$
$$\le d(n\|\phi\|_{\sup})(n^{d-1}\mathrm{vol}(L_{\mathbb{Q}})) + n^d(d\|\phi\|_{\sup}\mathrm{vol}(L_{\mathbb{Q}}))$$
$$= 2n^d d\,\mathrm{vol}(L_{\mathbb{Q}})\|\phi\|_{\sup} \le (2n^d d\,\mathrm{vol}(L_{\mathbb{Q}}))\epsilon.$$

Here ϵ is arbitrary. The above estimation implies (2).

(3) By [4, Theorem 4.3], (3) holds if $\overline{L}$ is C^∞. For a positive real number ϵ, by Lemma 2.4, we can find $\phi \in C^0(X)$ such that $\overline{L} + \overline{\mathcal{O}}(\phi)$ is C^∞ and $\|\phi\|_{\sup} \le \epsilon$. Then, by (1),

$$\begin{cases} |\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi)) - \widehat{\mathrm{vol}}(\overline{L})| \le d\epsilon\,\mathrm{vol}(L_{\mathbb{Q}}), \\ |\widehat{\mathrm{vol}}(\nu^*(\overline{L} + \overline{\mathcal{O}}(\phi))) - \widehat{\mathrm{vol}}(\nu^*(\overline{L}))| \le d\|\nu^*(\phi)\|_{\sup}\mathrm{vol}(\nu^*(L_{\mathbb{Q}})) = d\epsilon\,\mathrm{vol}(L_{\mathbb{Q}}). \end{cases}$$

Thus,

$$|\widehat{\mathrm{vol}}(\nu^*(\overline{L})) - \widehat{\mathrm{vol}}(\overline{L})| \le |\widehat{\mathrm{vol}}(\nu^*(\overline{L})) - \widehat{\mathrm{vol}}(\nu^*(\overline{L} + \overline{\mathcal{O}}(\phi)))|$$
$$+ |\widehat{\mathrm{vol}}(\nu^*(\overline{L} + \overline{\mathcal{O}}(\phi))) - \widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi))| + |\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi)) - \widehat{\mathrm{vol}}(\overline{L})|$$
$$\le \epsilon(2d\,\mathrm{vol}(L_{\mathbb{Q}})).$$

Therefore we get (3). □

By virtue of (2) of the above proposition, if $\overline{L} \otimes \alpha = \overline{M} \otimes \beta$ in $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $\overline{L}, \overline{M} \in \widehat{\mathrm{Pic}}_{C^0}(X)$ and $\alpha, \beta \in \mathbb{Q}_{\geq 0}$, then $\alpha^d \widehat{\mathrm{vol}}(\overline{L}) = \beta^d \widehat{\mathrm{vol}}(\overline{M})$. Indeed, we choose a positive integer m such that $m\alpha, m\beta \in \mathbb{Z}$. Then

$$m\alpha \overline{L} \otimes 1 = m(\overline{L} \otimes \alpha) = m(\overline{M} \otimes \beta) = m\beta \overline{M} \otimes 1.$$

Thus there is a positive integer n with $nm\alpha \overline{L} = nm\beta \overline{M}$, which implies $(nm\alpha)^d \widehat{\mathrm{vol}}(\overline{L}) = (nm\beta)^d \widehat{\mathrm{vol}}(\overline{M})$ by (2) of the above proposition. Hence $\alpha^d \widehat{\mathrm{vol}}(\overline{L}) = \beta^d \widehat{\mathrm{vol}}(\overline{M})$. Therefore, we can define $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$ to be $\widehat{\mathrm{vol}}(\overline{L} \otimes \alpha) = \alpha^d \widehat{\mathrm{vol}}(\overline{L})$, where $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X)$ and $\alpha \in \mathbb{Q}_{\geq 0}$. By the definition of $\widehat{\mathrm{vol}}$, a diagram

$$\begin{array}{ccc} \widehat{\mathrm{Pic}}_{C^0}(X) & \xrightarrow{\widehat{\mathrm{vol}}} & \mathbb{R} \\ \downarrow & \nearrow_{\widehat{\mathrm{vol}}} & \\ \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & & \end{array}$$

is commutative. The following proposition is an immediate consequence of the above proposition.

Proposition 2.6. *For $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, the following hold:*
  (1) $|\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(f)) - \widehat{\mathrm{vol}}(\overline{L})| \leq d\|f\|_{\sup} \mathrm{vol}(L_{\mathbb{Q}})$ *for $f \in C^0(X)$.*
  (2) $\widehat{\mathrm{vol}}(a\overline{L}) = a^d \widehat{\mathrm{vol}}(\overline{L})$ *for a non-negative rational number a.*
  (3) *If $\nu : X' \to X$ is a birational morphism of projective arithmetic varieties, then $\widehat{\mathrm{vol}}(\nu^*(\overline{L})) = \widehat{\mathrm{vol}}(\overline{L})$ for any $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Let $\overline{L}$ be a continuous hermitian invertible sheaf on X. We say $\overline{L}$ is *effective* if there is a non-zero section $s \in H^0(X, L)$ with $\|s\|_{\sup} \leq 1$. For $\overline{L}_1, \overline{L}_2 \in \widehat{\mathrm{Pic}}_{C^0}(X)$, if $\overline{L}_1 - \overline{L}_2$ is effective, then it is denoted by $\overline{L}_1 \geq \overline{L}_2$ or $\overline{L}_2 \leq \overline{L}_1$. Moreover, for $\overline{M} \in \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we say $\overline{M}$ is $\mathbb{Q}$-*effective* if there are a positive integer n and $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X)$ such that $\overline{L}$ is effective and $n\overline{M} = \overline{L} \otimes 1$ in $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For $\overline{M}_1, \overline{M}_2 \in \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, if $\overline{M}_1 - \overline{M}_2$ is $\mathbb{Q}$-effective, then it is denoted by $\overline{M}_1 \geq_{\mathbb{Q}} \overline{M}_2$ or $\overline{M}_2 \leq_{\mathbb{Q}} \overline{M}_1$.

Proposition 2.7.   (1) *If $\overline{L}_1 \geq \overline{L}_2$ for $\overline{L}_1, \overline{L}_2 \in \widehat{\mathrm{Pic}}_{C^0}(X)$, then $\hat{h}^0(\overline{L}_1) \geq \hat{h}^0(\overline{L}_2)$ and $\widehat{\mathrm{vol}}(\overline{L}_1) \geq \widehat{\mathrm{vol}}(\overline{L}_2)$.*
  (2) *If $\overline{M}_1 \geq_{\mathbb{Q}} \overline{M}_2$ for $\overline{M}_1, \overline{M}_2 \in \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, then $\widehat{\mathrm{vol}}(\overline{M}_1) \geq \widehat{\mathrm{vol}}(\overline{M}_2)$.*

Proof. (1) is easily checked. Let us consider (2). Since $\overline{M}_1 \geq_{\mathbb{Q}} \overline{M}_2$, there are a positive integer n and $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X)$ such that $\overline{L}$ is effective and $n(\overline{M}_1 - \overline{M}_2) = \overline{L} \otimes 1$. Moreover, we can find a positive integer m and $\overline{L}_1, \overline{L}_2 \in \widehat{\mathrm{Pic}}_{C^0}(X)$ such that $m\overline{M}_1 = \overline{L}_1 \otimes 1$ and $m\overline{M}_2 = \overline{L}_2 \otimes 1$. Then

$$m\overline{L} \otimes 1 = mn(\overline{M}_1 - \overline{M}_2) = n(\overline{L}_1 - \overline{L}_2) \otimes 1.$$

Thus there is a positive integer l with $lm\overline{L} = ln(\overline{L}_1 - \overline{L}_2)$, which implies that $\widehat{\mathrm{vol}}(ln\overline{L}_1) \geq \widehat{\mathrm{vol}}(ln\overline{L}_2)$. Therefore,

$$\widehat{\mathrm{vol}}(lnm\overline{M}_1) = \widehat{\mathrm{vol}}(ln\overline{L}_1 \otimes 1) = \widehat{\mathrm{vol}}(ln\overline{L}_1)$$
$$\geq \widehat{\mathrm{vol}}(ln\overline{L}_2) = \widehat{\mathrm{vol}}(ln\overline{L}_2 \otimes 1) = \widehat{\mathrm{vol}}(lnm\overline{M}_2).$$

Hence, using the homogeneity of $\widehat{\mathrm{vol}}$, we have $\widehat{\mathrm{vol}}(\overline{M}_1) \geq \widehat{\mathrm{vol}}(\overline{M}_2)$. $\square$

3. UNIFORM CONTINUITY OF THE ARITHMETIC VOLUME FUNCTION

Let X be a d-dimensional projective arithmetic variety. The purpose of this section is to prove the uniform continuity of the arithmetic volume function $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$ in the following sense; $\widehat{\mathrm{vol}}$ is uniformly continuous on any bounded set in any finite dimensional vector subspace of $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (cf. Theorem 3.4).

Let us begin with the following lemma.

Lemma 3.1. *Let V be a projective variety over a field and let $\boldsymbol{L} = (L_1, \ldots, L_r)$ be a finite sequence of $\mathbb{Q}$-invertible sheaves on V. Then there is a constant C depending only on V and $\boldsymbol{L}$ such that $\mathrm{vol}(\boldsymbol{a} \cdot \boldsymbol{L}) \leq C|\boldsymbol{a}|_1^{\dim V}$ for all $\boldsymbol{a} \in \mathbb{R}^r$.*

Proof. We set $f(\boldsymbol{a}) = \mathrm{vol}(\boldsymbol{a} \cdot \boldsymbol{L})$ for $\boldsymbol{a} \in \mathbb{R}^r$. It is well known that f is a continuous and homogeneous function of degree $\dim V$ on $\mathbb{R}^r$ (cf. [3]). We set $K = \{\boldsymbol{x} \in \mathbb{R}^r \mid |\boldsymbol{x}|_1 = 1\}$. Since K is compact, if we set $C = \sup_{\boldsymbol{x} \in K} f(\boldsymbol{x})$, then, for $\boldsymbol{y} \in \mathbb{R}^r \setminus \{0\}$, $f(\boldsymbol{y}/|\boldsymbol{y}|_1) \leq C$. Thus, since f is homogeneous of degree $\dim V$, we have $f(\boldsymbol{y}) \leq C|\boldsymbol{y}|_1^{\dim V}$. $\square$

Next we consider the strong estimate of $\widehat{\mathrm{vol}}$ in $\widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 3.2. *Let $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_r)$ and $\overline{\boldsymbol{A}} = (\overline{A}_1, \ldots, \overline{A}_{r'})$ be finite sequences of C^∞-hermitian $\mathbb{Q}$-invertible sheaves on X. Then there is a positive constant C depending only on X, $\overline{\boldsymbol{L}}$ and $\overline{\boldsymbol{A}}$ such that*

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})| \leq C|(\boldsymbol{a}, \boldsymbol{\delta})|_1^{d-1}|\boldsymbol{\delta}|_1$$

for all $\boldsymbol{a} \in \mathbb{Q}^r$ and $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$.

Proof. First let us see the following claim:

Claim 3.2.1. *Let $\overline{A}$ be a $\mathbb{Q}$-effective C^∞-hermitian $\mathbb{Q}$-invertible sheaf on X. Then there is a positive constant C_1 depending only on X, $\overline{\boldsymbol{L}}$ and $\overline{A}$ such that*

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \delta \cdot \overline{A}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})| \leq C_1|(\boldsymbol{a}, \delta)|_1^{d-1}|\delta|$$

for all $\boldsymbol{a} \in \mathbb{Q}^r$ and $\delta \in \mathbb{Q}$.

Proof. Let $\nu : X' \to X$ be a generic resolution of singularities of X. Then, by [4, Theorem 4.3] or Proposition 2.6,

$$\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}) = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \nu^*(\overline{\boldsymbol{L}})), \quad \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \delta \cdot \overline{A}) = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \nu^*(\overline{\boldsymbol{L}}) + \delta \cdot \nu^*(\overline{A})).$$

Thus we may assume that X is generically smooth. Moreover, since

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot n\overline{\boldsymbol{L}} + \delta n\overline{A}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot n\overline{\boldsymbol{L}})| = n^d|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \delta\overline{A}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})|$$

for a positive integer n, we may assume that $\overline{L}_1, \ldots, \overline{L}_r, \overline{A} \in \widehat{\mathrm{Pic}}(X)$ and $\overline{A}$ is effective.

We set $\overline{\boldsymbol{L}}' = (\overline{L}_1, \ldots, \overline{L}_r, 0)$. By Theorem 1.1.1, there are a positive constants a_0', C_1' and D_1' depending only on X, $\overline{L}_1, \ldots, \overline{L}_r$ and $\overline{A}$ such that

$$\hat{h}^0\left(\boldsymbol{a}' \cdot \overline{\boldsymbol{L}}' + (b - c)\overline{A}\right) \leq \hat{h}^0\left(\boldsymbol{a}' \cdot \overline{\boldsymbol{L}}' - c\overline{A}\right) + C_1'b|\boldsymbol{a}'|_1^{d-1} + D_1'|\boldsymbol{a}'|_1^{d-1}\log(|\boldsymbol{a}'|_1)$$

for all $\boldsymbol{a}' \in \mathbb{Z}^{r+1}$ and $b, c \in \mathbb{Z}$ with $|\boldsymbol{a}'|_1 \geq b \geq c \geq 0$ and $|\boldsymbol{a}'|_1 \geq a_0$.

If $\delta = 0$, then the assertion of the claim is obvious, so that we assume that $\delta \neq 0$. Let m_0 be a positive integer such that $m_0/|\delta| \in \mathbb{Z}$ and $(m_0/|\delta|)\boldsymbol{a} \in \mathbb{Z}^r$.

Applying the above estimate to the case where $\boldsymbol{a}' = m(m_0/|\delta|)(\boldsymbol{a}, \delta)$, $b = mm_0$ and $c = 0$ ($m \gg 0$), we have

$$\widehat{\mathrm{vol}}((m_0/|\delta|)\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + m_0\overline{A}) \leq \widehat{\mathrm{vol}}((m_0/|\delta|)\boldsymbol{a} \cdot \overline{\boldsymbol{L}}) + d!C_1'm_0^d(1/|\delta|)^{d-1}|(\boldsymbol{a}, \delta)|_1^{d-1}$$

becuase $m(m_0/|\delta|)(\boldsymbol{a},\delta) \cdot \overline{\boldsymbol{L}}' = m(m_0/|\delta|)\boldsymbol{a} \cdot \overline{\boldsymbol{L}}$. Thus, using the homogeneity of $\widehat{\mathrm{vol}}$, we obtain

$$0 \leq \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + |\delta|\overline{A}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}) \leq d! C_1' |\delta| |(\boldsymbol{a},\delta)|_1^{d-1}.$$

Next, applying the above estimate to the case where $\boldsymbol{a}' = m(m_0/|\delta|)(\boldsymbol{a},\delta)$, $b = mm_0$ and $c = mm_0$ ($m \gg 0$), we have

$$0 \leq \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - |\delta|\overline{A}) \leq d! C_1' |\delta| |(\boldsymbol{a},\delta)|_1^{d-1}.$$

Thus the claim follows. $\qquad\square$

Next we consider a general case. We can find C^∞-hermitian $\mathbb{Q}$-invertible sheaves $\overline{A}_1', \overline{A}_1'', \ldots, \overline{A}_{r'}', \overline{A}_{r'}''$ such that $\overline{A}_i = \overline{A}_i' - \overline{A}_i''$, $\overline{A}_i' \geq_{\mathbb{Q}} 0$ and $\overline{A}_i'' \geq_{\mathbb{Q}} 0$ for all $i = 1, \ldots, r'$. Then, since $|(\boldsymbol{a},\boldsymbol{\delta},-\boldsymbol{\delta})|_1 \leq 2|(\boldsymbol{a},\boldsymbol{\delta})|_1$, $|(\boldsymbol{\delta},-\boldsymbol{\delta})|_1 = 2|\boldsymbol{\delta}|_1$ and

$$\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} = \boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}' + (-\boldsymbol{\delta}) \cdot \overline{\boldsymbol{A}}'',$$

we may assume that $\overline{A}_i \geq_{\mathbb{Q}} 0$ for all i. We set $\overline{B} = \overline{A}_1 + \cdots + \overline{A}_{r'}$. Then we have $-|\boldsymbol{\delta}|_1 \overline{B} \leq \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} \leq |\boldsymbol{\delta}|_1 \overline{B}$, which implies that

$$\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} - |\boldsymbol{\delta}|_1 \overline{B}) \leq \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}) \leq \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + |\boldsymbol{\delta}|_1 \overline{B}).$$

Thus the theorem follows from the previous claim. $\qquad\square$

Corollary 3.3. *Let $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_r)$ and $\overline{\boldsymbol{A}} = (\overline{A}_1, \ldots, \overline{A}_{r'})$ be finite sequences of C^∞-hermitian $\mathbb{Q}$-invertible sheaves on X. Then there are positive constants C and C' depending only on X, $\overline{\boldsymbol{L}}$ and $\overline{\boldsymbol{A}}$ such that*

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})| \leq C |(\boldsymbol{a},\boldsymbol{\delta})|_1^{d-1} |\boldsymbol{\delta}|_1 + C' |(\boldsymbol{a},\boldsymbol{\delta})|_1^{d-1} \|g\|_{\sup}$$

for all $\boldsymbol{a} \in \mathbb{Q}^r$, $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$ and $g \in C^0(X)$.

Proof. By (1) of Proposition 2.6 and Lemma 3.1, there is a positive constant C' depending only on d, $\boldsymbol{L}_{\mathbb{Q}}$ and $\boldsymbol{A}_{\mathbb{Q}}$ such that

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}})| \leq C' \|g\|_{\sup} |(\boldsymbol{a},\boldsymbol{\delta})|_1^{d-1}$$

for all $\boldsymbol{a} \in \mathbb{Q}^r$, $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$ and $g \in C^0(X)$. Therefore, the corollary follows from Theorem 3.2. $\qquad\square$

Theorem 3.4. *Let $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_r)$ and $\overline{\boldsymbol{A}} = (\overline{A}_1, \ldots, \overline{A}_{r'})$ be finite sequences of continuous hermitian $\mathbb{Q}$-invertible sheaves on X. Let B be a bounded set in $\mathbb{Q}^r$. Then, for any positive real number ϵ, there are positive real numbers δ and δ' such that*

$$\left| \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}) \right| \leq \epsilon$$

for all $\boldsymbol{a} \in B$, $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$ and $g \in C^0(X)$ with $|\boldsymbol{\delta}|_1 \leq \delta$ and $\|g\|_{\sup} \leq \delta'$. In particular, if we set $f(\boldsymbol{x}) = \widehat{\mathrm{vol}}(\boldsymbol{x} \cdot \overline{\boldsymbol{L}})$ for $\boldsymbol{x} \in \mathbb{Q}^r$, then f is uniformly continuous on B.

Proof. By Lemma 3.1, there is a constant C_1 such that

$$\begin{cases} \mathrm{vol}((\boldsymbol{a} \cdot \boldsymbol{L})_{\mathbb{Q}}) \leq C_1 |\boldsymbol{a}|_1^{d-1}, \\ \mathrm{vol}((\boldsymbol{a} \cdot \boldsymbol{L} + \boldsymbol{\delta} \cdot \boldsymbol{A})_{\mathbb{Q}}) \leq C_1 |(\boldsymbol{a},\boldsymbol{\delta})|_1^{d-1} \end{cases}$$

for all $\boldsymbol{a} \in \mathbb{Q}^r$ and $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$. We set $M = \sup\{|\boldsymbol{a}|_1 \mid \boldsymbol{a} \in B\}$. By Lemma 2.4, we can find $\phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_{r'} \in C^0(X)$ such that

$$\overline{\boldsymbol{L}}^\phi = (\overline{L}_1 + \overline{\mathcal{O}}(\phi_1), \ldots, \overline{L}_r + \overline{\mathcal{O}}(\phi_r)) \quad \text{and} \quad \overline{\boldsymbol{A}}^\psi = (\overline{A}_1 + \overline{\mathcal{O}}(\psi_1), \ldots, \overline{A}_{r'} + \overline{\mathcal{O}}(\psi_{r'}))$$

are C^∞ and that

$$\|\phi_i\|_{\sup} \le \frac{\epsilon}{3C_1 d(M+1)^d} \quad \text{and} \quad \|\psi_j\|_{\sup} \le \frac{\epsilon}{3C_1 d(M+1)^d}$$

for all i and j. Since

$$\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} = \boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(\boldsymbol{a} \cdot \boldsymbol{\phi}) \quad \text{and} \quad \boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}^{\boldsymbol{\psi}} = \boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(\boldsymbol{a} \cdot \boldsymbol{\phi} + \boldsymbol{\delta} \cdot \boldsymbol{\psi}),$$

by (1) of Proposition 2.6, we have

$$\begin{cases} |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})| \le dC_1 \|\boldsymbol{a} \cdot \boldsymbol{\phi}\|_{\sup} |\boldsymbol{a}|_1^{d-1} \\ |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}^{\boldsymbol{\psi}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g))| \\ \qquad\qquad \le dC_1 \|\boldsymbol{a} \cdot \boldsymbol{\phi} + \boldsymbol{\delta} \cdot \boldsymbol{\psi}\|_{\sup} |(\boldsymbol{a}, \boldsymbol{\delta})|_1^{d-1} \end{cases}$$

Note that

$$dC_1 \|\boldsymbol{a} \cdot \boldsymbol{\phi}\|_{\sup} |\boldsymbol{a}|_1^{d-1} \le dC_1 |\boldsymbol{a}|_1^d \frac{\epsilon}{3C_1 d(M+1)^d} \le \epsilon/3$$

and

$$dC_1 \|\boldsymbol{a} \cdot \boldsymbol{\phi} + \boldsymbol{\delta} \cdot \boldsymbol{\psi}\|_{\sup} |(\boldsymbol{a}, \boldsymbol{\delta})|_1^{d-1} \le dC_1 |(\boldsymbol{a}, \boldsymbol{\delta})|_1^d \frac{\epsilon}{3C_1 d(M+1)^d} \le \epsilon/3$$

for all $\boldsymbol{a} \in B$ and $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$ with $|\boldsymbol{\delta}|_1 \le 1$. Thus we get

$$\begin{cases} |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})| \le \epsilon/3, \\ |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}^{\boldsymbol{\psi}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g))| \le \epsilon/3 \end{cases}$$

for all $\boldsymbol{a} \in B$, $g \in C^0(X)$ and $\boldsymbol{\delta} \in \mathbb{Q}^{r'}$ with $|\boldsymbol{\delta}|_1 \le 1$.

On the other hand, by Corollary 3.3, we can find positive real constants δ and δ' depending only on $B, \overline{\boldsymbol{L}}^{\boldsymbol{\phi}}, \overline{\boldsymbol{A}}^{\boldsymbol{\psi}}$ and X such that

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}^{\boldsymbol{\psi}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}})| \le \epsilon/3$$

for $\boldsymbol{a} \in B$, $|\boldsymbol{\delta}|_1 \le \delta$ and $\|g\|_{\sup} \le \delta'$. Therefore, our theorem follows because

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})|$$
$$\le |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}^{\boldsymbol{\psi}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}} + \overline{\mathcal{O}}(g))|$$
$$+ |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}^{\boldsymbol{\psi}} + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}})|$$
$$+ |\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}}^{\boldsymbol{\phi}}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})|.$$

$\square$

4. Continuous extension of the arithmetic volume function over $\mathbb{R}$

Let V be a vector space over $\mathbb{Q}$ and $f : V \to \mathbb{R}$ a weakly continuous function in the sense of Conventions and terminology 14. We say f has a *weakly continuous extension over* $\mathbb{R}$ if there is a weakly continuous function $\tilde{f} : V \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ with $\tilde{f}\Big|_V = f$. Note that if there is a weakly continuous extension $\tilde{f}$ of f, then $\tilde{f}$ is uniquely determined. The following example shows that a weakly continuous function over $\mathbb{Q}$ does not necessarily have a weakly continuous extension over $\mathbb{R}$.

Example 4.1. Let a be a positive irrational number, and let $f : \mathbb{Q}^2 \to \mathbb{R}$ be a function given by

$$f(x, y) = \begin{cases} \max\{|x|, a|y|\} & \text{if } x + ay > 0, \\ 0 & \text{if } x + ay \le 0. \end{cases}$$

Then it is easy to see the following:

(1) f is positively homogeneous of degree 1.
(2) f is continuous on $\mathbb{Q}^2$.
(3) f is monotonically increasing, that is, $f(x, y) \le f(x', y')$ for all $(x, y), (x', y') \in \mathbb{Q}^2$ with $x \le x'$ and $y \le y'$.
(4) f has no continuous extension over $\mathbb{R}$.
(5) f is not uniformly continuous on $\{(x, y) \in \mathbb{Q}^2 \mid \max\{|x|, a|y|\} \le 1\}$.

The following lemma gives a condition to guarantee a weakly continuous extension over $\mathbb{R}$.

Lemma 4.2. *Let $f : V \to \mathbb{R}$ be a weakly continuous function on a vector space V over $\mathbb{Q}$. Then the following are equivalent:*

(1) *f has a weakly continuous extension over $\mathbb{R}$.*
(2) *f is uniformly continuous on any bounded set B in any finite dimensional vector subspace of V.*

Moreover, if f is positively homogenous, then the weakly continuous extension of f is also positively homogeneous.

Proof. "(1) $\Longrightarrow$ (2)" is obvious by Heine's theorem.

Let us consider "(2) $\Longrightarrow$ (1)". For a vector subspace W of V, we denote $f|_W$ by f_W.

(a) We assume that W is finite dimensional. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in W. Then there is a bounded set B in W with $a_n \in B$ for all n. Thus, $\{f(a_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence because $f|_B$ is uniformly continuous. Hence, by using the well-known way, there is a continuous function $\tilde{f}_W : W \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ with $\tilde{f}_W\big|_W = f_W$. Namely, if $x \in W \otimes_{\mathbb{Q}} \mathbb{R}$ and $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in W with $x = \lim\limits_{n \to \infty} a_n$, then $\tilde{f}_W(x) = \lim\limits_{n \to \infty} f_W(a_n)$.

(b) Let $W \subseteq W'$ be finite dimensional vector subspaces of V. Then

$$\tilde{f}_{W'}\big|_{W \otimes_{\mathbb{Q}} \mathbb{R}} = \tilde{f}_W$$

because a Cauchy sequence in W is a Cauchy sequence in W'.

Let $x \in V \otimes_{\mathbb{Q}} \mathbb{R}$. Then there is a finite dimensional vector space W of V with $x \in W \otimes_{\mathbb{Q}} \mathbb{R}$. The above (b) shows that $\tilde{f}_W(x)$ does not depend on the choice of W, so that $\tilde{f}(x)$ is defined by $\tilde{f}_W(x)$.

We need to show that $\tilde{f} : V \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ is weakly continuous. Let T be a finite dimensional vector subspace of $V \otimes_{\mathbb{Q}} \mathbb{R}$. Then there is a finite dimensional vector subspace W of V with $T \subseteq W \otimes_{\mathbb{Q}} \mathbb{R}$. Note that $\tilde{f}\big|_{W \otimes_{\mathbb{Q}} \mathbb{R}} = \tilde{f}_W$. Thus $\tilde{f}\big|_T$ is continuous.

The last assertion is obvious by our construction. $\qquad\square$

Let X be a d-dimensional projective arithmetic variety. The exact sequence (2.3) gives rise to the following exact sequence:

$$C^0(X) \otimes_{\mathbb{Q}} \mathbb{R} \to \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to 0.$$

Let us consider the natural homomorphisms $\mu : C^0(X) \otimes_{\mathbb{Q}} \mathbb{R} \to C^0(X)$ given by $\mu(f \otimes x) = xf$. Let N be the image of $\mathrm{Ker}(\mu)$ via $C^0(X) \otimes_{\mathbb{Q}} \mathbb{R} \to \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We set

$$\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} = (\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R})/N$$

and the canonical homomorphism $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ is denoted by π. Then the above exact sequence yields the following commutative diagram:

(4.3.0)

$$\begin{array}{ccccccc}
C^0(X) & \longrightarrow & \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
C^0(X) & \longrightarrow & \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} & \longrightarrow & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & 0,
\end{array}$$

where each horizontal sequence is exact.

The following theorem is one of the main theorem of this paper.

Theorem 4.4.   (1) *There is a unique weakly continuous and positively homogeneous function* $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ *of degree d, which is a continuous extension of* $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$ *over* $\mathbb{R}$.

   (2) *The above arithmetic volume function* $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ *descend to* $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$, *that is, there is a weakly continuous and positively homogeneous function*

$$\widehat{\mathrm{vol}}' : \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$$

of degree d such that the following diagram is commutative:

By abuse of notation, $\widehat{\mathrm{vol}}'$ *is also denoted by* $\widehat{\mathrm{vol}}$.

Proof. The first assertion follows from Theorem 3.4 and Lemma 4.2. For the second assertion, let us consider the following claim:

Claim 4.4.1. *Every element of N can be written by a form*

$$\overline{\mathcal{O}}(f_1) \otimes x_1 + \cdots + \overline{\mathcal{O}}(f_r) \otimes x_r$$

for $f_1, \ldots, f_r \in C^0(X)$ and $x_1, \ldots, x_r \in \mathbb{R}$ with $x_1 f_1 + \cdots + x_r f_r = 0$.

Proof. For $\omega = f_1 \otimes x_1 + \cdots + f_r \otimes x_r \in C^0(X) \otimes_{\mathbb{Q}} \mathbb{R}$, $\omega \in \mathrm{Ker}(\mu)$ if and only if $x_1 f_1 + \cdots + x_r f_r = 0$, which proves the claim. $\square$

Let $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X) \otimes \mathbb{R}$, $f_1, \ldots, f_r \in C^0(X)$ and $x_1, \ldots, x_r \in \mathbb{R}$ with $x_1 f_1 + \cdots + x_r f_r = 0$. It is sufficient to show that

$$\widehat{\mathrm{vol}}\left(\overline{L} + \overline{\mathcal{O}}(f_1) \otimes x_1 + \cdots + \overline{\mathcal{O}}(f_r) \otimes x_r\right) = \widehat{\mathrm{vol}}(\overline{L}).$$

Let us choose a sequence $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ in $\mathbb{Q}^r$ with $\lim_{n\to\infty} \boldsymbol{x}_n = (x_1, \ldots, x_r)$. Then, by Theorem 4.4,

$$\lim_{n\to\infty} \widehat{\mathrm{vol}}\left(\overline{L} + \overline{\mathcal{O}}(f_1) \otimes \boldsymbol{x}_n(1) + \cdots + \overline{\mathcal{O}}(f_r) \otimes \boldsymbol{x}_n(r)\right)$$
$$= \widehat{\mathrm{vol}}\left(\overline{L} + \overline{\mathcal{O}}(f_1) \otimes x_1 + \cdots + \overline{\mathcal{O}}(f_r) \otimes x_r\right).$$

Since $\boldsymbol{x}_n \in \mathbb{Q}^r$, if we set $\phi_n = \boldsymbol{x}_n(1)f_1 + \cdots + \boldsymbol{x}_n(r)f_r$, then

$$\overline{\mathcal{O}}(f_1) \otimes \boldsymbol{x}_n(1) + \cdots + \overline{\mathcal{O}}(f_r) \otimes \boldsymbol{x}_n(r) = \overline{\mathcal{O}}(\phi_n).$$

Note that

$$\|\phi_n\|_{\sup} = \|(\boldsymbol{x}_n(1) - x_1)f_1 + \cdots + (\boldsymbol{x}_n(r) - x_r)f_r\|_{\sup}$$
$$\leq |\boldsymbol{x}_n(1) - x_1|\|f_1\|_{\sup} + \cdots + |\boldsymbol{x}_n(r) - x_r|\|f_r\|_{\sup}.$$

Thus $\lim_{n\to\infty} \|\phi_n\|_{\sup} = 0$. On the other hand, by (1) of Proposition 4.6,

$$|\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi_n)) - \widehat{\mathrm{vol}}(\overline{L})| \leq d\|\phi_n\|_{\sup}\mathrm{vol}(L_{\mathbb{Q}}).$$

Therefore,

$$\lim_{n\to\infty} \widehat{\mathrm{vol}}\left(\overline{L} + \overline{\mathcal{O}}(f_1) \otimes \boldsymbol{x}_n(1) + \cdots + \overline{\mathcal{O}}(f_r) \otimes \boldsymbol{x}_n(r)\right)$$
$$= \lim_{n\to\infty} \widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(\phi_n)) = \widehat{\mathrm{vol}}(\overline{L}).$$

Thus we get (2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 4.5. Let $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ and $X = \mathrm{Spec}(\mathcal{O}_K)$. Note that $\{\sigma : K \hookrightarrow \mathbb{C}\} = \{\sigma_1, \sigma_2\}$, where $\sigma_1(\sqrt{2}) = \sqrt{2}$ and $\sigma_2(\sqrt{2}) = -\sqrt{2}$. Then $C^0(X) = \mathbb{R}^2$ in the natural way. Moreover, the class number of K is 1 and the fundamental unit of K is $\sqrt{2}+1$. Thus, if we set $\omega = \left(\log(\sqrt{2}+1), \log(\sqrt{2}-1)\right)$, then we have an exact sequence

$$0 \to \mathbb{Z}\omega \to \mathbb{R}^2 \xrightarrow{\overline{\mathcal{O}}} \widehat{\mathrm{Pic}}_{C^0}(X) \to 0,$$

which yields the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Q}\omega & \longrightarrow & \mathbb{R}^2 & \xrightarrow{\overline{\mathcal{O}}} & \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R}\omega & \longrightarrow & \mathbb{R}^2 & \xrightarrow{\overline{\mathcal{O}}} & \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} & \longrightarrow & 0,
\end{array}
$$

where each horizontal sequence is exact. In particular, the canonical homomorphism $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ is not injective. Moreover, it is easy to see that

$$\widehat{\mathrm{vol}}(\overline{\mathcal{O}}(\lambda_1, \lambda_2)) = \begin{cases} \lambda_1 + \lambda_2 & \text{if } \lambda_1 + \lambda_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The arithmetic volume function $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ has the following properties:

Proposition 4.6.   (1) *For all $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ and $f \in C^0(X)$, we have*

$$\left|\widehat{\mathrm{vol}}(\overline{L} + \overline{\mathcal{O}}(f)) - \widehat{\mathrm{vol}}(\overline{L})\right| \leq d\|f\|_{\sup}\mathrm{vol}(L_{\mathbb{Q}}).$$

(2) *Let $\nu : X' \to X$ be a morphism of projective arithmetic varieties. Then there is a unique homomorphism $\nu^* : \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \to \widehat{\mathrm{Pic}}_{C^0}(X')_{\mathbb{R}}$ such that the following diagram is commutative:*

$$
\begin{array}{ccc}
\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\nu^*\otimes\mathrm{id}} & \widehat{\mathrm{Pic}}_{C^0}(X') \otimes_{\mathbb{Z}} \mathbb{R} \\
\pi \downarrow & & \pi \downarrow \\
\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} & \xrightarrow{\nu^*} & \widehat{\mathrm{Pic}}_{C^0}(X')_{\mathbb{R}}.
\end{array}
$$

Moreover, if ν is birational, then $\widehat{\mathrm{vol}}(\nu^*(\overline{L})) = \widehat{\mathrm{vol}}(\overline{L})$ for $\overline{L} \in \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$.

(3) *Let* $\overline{L}_1, \ldots, \overline{L}_r, \overline{A}_1, \ldots, \overline{A}_{r'}$ *be* C^∞*-hermitian* $\mathbb{Q}$*-invertible sheaves on* X*. If we set* $\overline{\boldsymbol{L}} = (\pi(\overline{L}_1), \ldots, \pi(\overline{L}_r))$ *and* $\overline{\boldsymbol{A}} = (\pi(\overline{A}_1), \ldots, \pi(\overline{A}_{r'}))$*, then there is a positive constant* C *depending only on* X *and* $\overline{L}_1, \ldots, \overline{L}_r, \overline{A}_1, \ldots, \overline{A}_{r'}$ *such that*

$$|\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \boldsymbol{\delta} \cdot \overline{\boldsymbol{A}}) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})| \le C|(\boldsymbol{a}, \boldsymbol{\delta})|_1^{d-1}|\boldsymbol{\delta}|_1$$

for all $\boldsymbol{a} \in \mathbb{R}^r$ *and* $\boldsymbol{\delta} \in \mathbb{R}^{r'}$*.*

(4) *Let* V *be a finite dimensional vector subspace of* $\widehat{\mathrm{Pic}}_{C^0}(X)_\mathbb{R}$ *and* $\| \cdot \|$ *a norm of* V*. Let* K *be a compact set in* V*. For any positive real number* ϵ*, there are positive real number* δ *and* δ' *such that*

$$|\widehat{\mathrm{vol}}(x + a + \overline{\mathcal{O}}(g)) - \widehat{\mathrm{vol}}(x)| \le \epsilon$$

for all $x \in K$*,* $a \in V$ *and* $g \in C^0(X)$ *with* $\|a\| \le \delta$ *and* $\|g\|_{\sup} \le \delta'$*.*

(5) *Let* $\{x_n\}_{n=1}^\infty$ *be a sequence in a finite dimensional vector subspace of* $\widehat{\mathrm{Pic}}_{C^0}(X)_\mathbb{R}$ *and* $\{f_n\}_{n=1}^\infty$ *a sequence in* $C^0(X)$ *such that* $\{x_n\}_{n=1}^\infty$ *converges to* x *in the usual topology and* $\{f_n\}_{n=1}^\infty$ *converges uniformly to* f*. Then*

$$\lim_{n \to \infty} \widehat{\mathrm{vol}}\left(x_n + \overline{\mathcal{O}}(f_n)\right) = \widehat{\mathrm{vol}}\left(x + \overline{\mathcal{O}}(f)\right).$$

(6) *Let* $\overline{L}_1, \ldots, \overline{L}_r$ *be* $\mathbb{Q}$*-effective continuous hermitian* $\mathbb{Q}$*-invertible sheaves on* X*. For* $(a_1, \ldots, a_r), (a_1', \ldots, a_r') \in \mathbb{R}^r$ *and* $h, h' \in C^0(X)$*, if* $a_i \le a_i'$ $(\forall i)$ *and* $h \le h'$*, then*

$$\widehat{\mathrm{vol}}(a_1\pi(\overline{L}_1) + \cdots a_r\pi(\overline{L}_r) + \overline{\mathcal{O}}(h)) \le \widehat{\mathrm{vol}}(a_1'\pi(\overline{L}_1) + \cdots a_r'\pi(\overline{L}_r) + \overline{\mathcal{O}}(h')).$$

Proof. (1) It follows from (1) of Proposition 2.6 and Theorem 4.4.

(2) Let $f_1, \ldots, f_r \in C^0(X)$ and $x_1, \ldots, x_r \in \mathbb{R}$ with $x_1 f_1 + \cdots + x_r f_r = 0$. Then

$$(\nu^* \otimes \mathrm{id})\left(\sum \overline{\mathcal{O}}(f_i) \otimes x_i\right) = \sum \overline{\mathcal{O}}(\nu^*(f_i)) \otimes x_i$$

and

$$x_1\nu^*(f_1) + \cdots + x_r\nu^*(f_r) = \nu^*(x_1 f_1 + \cdots + x_r f_r) = 0.$$

This observation shows the first assertion. The second assertion is a consequence of (3) of Proposition 2.6 and Theorem 4.4.

(3) It is implied by Theorem 3.2 and Theorem 4.4.

(4) We can find a finite dimensional vector subspace W of $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_\mathbb{Z} \mathbb{Q}$ such that $V \subseteq \pi(W \otimes_\mathbb{Q} \mathbb{R})$. Thus it follows from Theorem 3.4 and Theorem 4.4.

(5) Let V be a vector space generated by $\{x_n\}_{n=1}^\infty$ and $\overline{\mathcal{O}}(f)$. Note that V is finite dimensional and $x_n + \overline{\mathcal{O}}(f_n) = x + \overline{\mathcal{O}}(f) + (x_n - x) + \overline{\mathcal{O}}(f_n - f)$. Thus, applying (4) to V, we can see (5)

(6) This can be proved by (2) of Proposition 2.7 and Theorem 4.4. $\qquad \square$

5. APPROXIMATION OF THE ARITHMETIC VOLUME FUNCTION

Let X be a d-dimensional projective arithmetic variety. The purpose of this section is to prove the following theorem, which gives an approximation of the arithmetic volume function in term of $\hat{h}^0$.

Theorem 5.1. *Let* M *be a finitely generated* $\mathbb{Z}$*-submodule of* $\mathrm{Pic}_{C^0}(X)$*. Let* $\{\overline{A}_n\}_{n=1}^\infty$ *be a sequence in* M *and* $\{f_n\}_{n=1}^\infty$ *a sequence in* $C^0(X)$ *such that* $\{\overline{A}_n \otimes 1/n\}_{n=1}^\infty$ *converges to* $\overline{A}$ *in* $M \otimes \mathbb{R}$ *in the usual topology and* $\{f_n/n\}_{n=1}^\infty$ *converges uniformly to* f*. Then*

$$\lim_{n \to \infty} \frac{\hat{h}^0(\overline{L}_n + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \widehat{\mathrm{vol}}(\overline{A} + \overline{\mathcal{O}}(f)).$$

Proof. Let $\overline{L}_1, \ldots, \overline{L}_l$ be a generator of M such that $\overline{L}_1, \ldots, \overline{L}_r$ ($r \leq l$) gives rise to a free basis of M/M_{tor} and that $\overline{L}_{r+1}, \ldots, \overline{L}_l$ are torsion elements of M. Here we set $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_l)$. Let N be a positive integer such that $N\overline{L}_{r+1} = \cdots = N\overline{L}_l = 0$. Then we can find $\boldsymbol{a}_n \in \mathbb{Z}^l$ such that $\overline{A}_n = \boldsymbol{a}_n \cdot \overline{\boldsymbol{L}}$ and $0 \leq \boldsymbol{a}_n(i) \leq N$ for all $i = r+1, \ldots, l$. By our assumption, for $1 \leq i \leq r$, $\{\boldsymbol{a}_n(i)/n\}_{n=1}^\infty$ converges to $a_i \in \mathbb{R}$. Thus, if we set $\boldsymbol{a} = (a_1, \ldots, a_r, 0, \ldots, 0)$, then $\lim_{n \to \infty} \boldsymbol{a}_n/n = \boldsymbol{a}$. Therefore, it is sufficient to show the following theorem. $\qquad\square$

Theorem 5.2. *Let $\{\boldsymbol{a}_n\}_{n=1}^\infty$ be a sequence in $\mathbb{Z}^l$ and $\{f_n\}_{n=1}^\infty$ a sequence in $C^0(X)$ such that*

$$\boldsymbol{a} = \lim_{n \to \infty} \boldsymbol{a}_n/n \in \mathbb{R}^l \quad \text{and} \quad \lim_{n \to \infty} \|(f_n/n) - f\|_{\sup} = 0$$

for some $f \in C^0(X)$. Then, for a finite sequence $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_l)$ in $\widehat{\mathrm{Pic}}_{C^0}(X)$,

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

Proof. First of all, let us see the following claim:

Claim 5.2.1. *We may assume that $\overline{\boldsymbol{L}}$ is effective, that is, $\overline{L}_i$ is effective for every i.*

Proof. We can find $\overline{L}_i' \geq 0$ and $\overline{L}_i'' \geq 0$ with $\overline{L}_i = \overline{L}_i' - \overline{L}_i''$. We set $\boldsymbol{a}'$, $\boldsymbol{a}_n'$ and $\overline{\boldsymbol{L}}'$ as follows:

$$\begin{cases} \boldsymbol{a}' = (\boldsymbol{a}(1), \ldots, \boldsymbol{a}(l), -\boldsymbol{a}(1), \ldots, -\boldsymbol{a}(l)), \\ \boldsymbol{a}_n' = (\boldsymbol{a}_n(1), \ldots, \boldsymbol{a}_n(l), -\boldsymbol{a}_n(1), \ldots, -\boldsymbol{a}_n(l)), \\ \overline{\boldsymbol{L}}' = (\overline{L}_1', \ldots, \overline{L}_l', \overline{L}_1'', \ldots, \overline{L}_l''). \end{cases}$$

Then $\boldsymbol{a} \cdot \overline{\boldsymbol{L}} = \boldsymbol{a}' \cdot \overline{\boldsymbol{L}}'$, $\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} = \boldsymbol{a}_n' \cdot \overline{\boldsymbol{L}}'$ and $\lim_{n \to \infty} \boldsymbol{a}_n'/n = \boldsymbol{a}'$. Thus the claim follows. $\qquad\square$

Under the assumption that $\overline{\boldsymbol{L}}$ is effective, we will prove this theorem in the following steps:

Step 1. If X is generically smooth, f is C^∞ and $\overline{\boldsymbol{L}} = (\overline{L}_1, \ldots, \overline{L}_l)$ is a finite sequence of C^∞-hermitian invertible sheaves on X, then the assertion of Theorem 5.2 holds.

Step 2. If X is generically smooth, then the assertion of Theorem 5.2 holds.

Step 3. If X is normal, then the assertion of Theorem 5.2 holds.

Step 4. In general, Theorem 5.2 holds.

Step 1: Let us begin with the following claim:

Claim 5.2.2. *Let $\overline{L}$ and $\overline{A}$ be C^∞-hermitian invertible sheaves on X. Then there are positive constants C, D and n_1 depending only on $\overline{L}$, $\overline{A}$ and X such that*

$$\begin{cases} \hat{h}^0\left(n\overline{L} + \lceil n\epsilon \rceil \overline{A}\right) \leq \hat{h}^0\left(n\overline{L}\right) + C\lceil n\epsilon \rceil n^{d-1} + Dn^{d-1}\log(n) \\ \hat{h}^0\left(n\overline{L}\right) \leq \hat{h}^0\left(n\overline{L} - \lceil n\epsilon \rceil \overline{A}\right) + C\lceil n\epsilon \rceil n^{d-1} + Dn^{d-1}\log(n) \end{cases}$$

for all $n \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$ with $n \geq n_1$ and $0 \leq \epsilon \leq 1/2$.

Proof. Note that if $n \geq 2$ and $0 \leq \epsilon \leq 1/2$, then $n \geq (n/2) + 1 \geq \lceil n/2 \rceil \geq \lceil n\epsilon \rceil$. Thus the claim follows from Theorem 1.1.1 or [4, Theorem 3.1]. $\qquad\square$

First we consider the case where $\boldsymbol{a} \in \mathbb{Z}^l$. In this case, by [2],

$$\lim_{n \to \infty} \frac{\hat{h}^0(n(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)))}{n^d/d!} = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

For any $0 < \epsilon < 1/2$, there is a positive integer n_0 such that

$$|\boldsymbol{a}_n - n\boldsymbol{a}|_1 \leq n\epsilon \quad \text{and} \quad \|f_n - nf\|_{\sup} \leq n\epsilon$$

for all $n \geq n_0$. Thus if we set $\mathbf{1} = (1, \ldots, 1)$, then

$$n\boldsymbol{a} - \lceil n\epsilon \rceil \mathbf{1} \leq \boldsymbol{a}_n \leq n\boldsymbol{a} + \lceil n\epsilon \rceil \mathbf{1} \quad \text{and} \quad nf - \lceil n\epsilon \rceil \leq f_n \leq nf + \lceil n\epsilon \rceil,$$

where the first inequality means that $n\boldsymbol{a}(i) - \lceil n\epsilon \rceil \leq \boldsymbol{a}_n(i) \leq n\boldsymbol{a}(i) + \lceil n\epsilon \rceil$ for all i. Therefore,

$$\hat{h}^0\left(n(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) - \lceil n\epsilon \rceil(\mathbf{1} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(1))\right) \leq \hat{h}^0\left(\boldsymbol{a}_n \cdot \boldsymbol{L} + \overline{\mathcal{O}}(f_n)\right)$$
$$\leq \hat{h}^0\left(n(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) + \lceil n\epsilon \rceil(\mathbf{1} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(1))\right).$$

Thus, by Claim 5.2.2, there are constant C and D depending only on $\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)$ and $\mathbf{1} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(1)$ such that

$$\hat{h}^0(n(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f))) - C\lceil n\epsilon \rceil n^{d-1} - Dn^{d-1}\log(n) \leq \hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n)) \leq$$
$$\hat{h}^0(n(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f))) + C\lceil n\epsilon \rceil n^{d-1} + Dn^{d-1}\log(n).$$

for all $n \gg 1$. Thus, taking $n \to \infty$, we obtain the following:

$$\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) - Cd!\epsilon \leq \liminf_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$
$$\leq \limsup_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} \leq \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) + Cd!\epsilon.$$

Here ϵ is arbitrary. Thus

$$\liminf_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \limsup_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}})$$

which shows the case where $\boldsymbol{a} \in \mathbb{Z}^l$.

Next we consider the case where $\boldsymbol{a} \in \mathbb{Q}^l$. Let N be a positive integer with $N \cdot \boldsymbol{a} \in \mathbb{Z}^l$. Since $\lim_{n \to \infty} \boldsymbol{a}_{Nn+k}/n = N\boldsymbol{a}$ and $\lim_{n \to \infty} \|f_{Nn+k}/n - Nf\|_{\sup} = 0$ for $0 \leq k < N$, by using the previous case, we have

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_{Nn+k} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_{Nn+k}))}{n^d/d!} = \widehat{\mathrm{vol}}(N\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(Nf)) = N^d\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

On the other hand,

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_{Nn+k} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_{Nn+k}))}{n^d/d!} = \lim_{n \to \infty} \frac{(Nn+k)^d}{n^d} \frac{\hat{h}^0(\boldsymbol{a}_{Nn+k} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_{Nn+k}))}{(Nn+k)^d/d!}$$
$$= N^d \lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_{Nn+k} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_{Nn+k}))}{(Nn+k)^d/d!}.$$

Thus we get

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_{Nn+k} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_{Nn+k}))}{(Nn+k)^d/d!} = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f))$$

for all k with $0 \leq k < N$, which proves the case where $\boldsymbol{a} \in \mathbb{Q}^l$.

Finally we consider a general case. For $\epsilon > 0$, let us choose $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_l), \boldsymbol{\delta}' = (\delta_1', \ldots, \delta_l') \in \mathbb{R}_{\geq 0}^l$ such that $\boldsymbol{a} + \boldsymbol{\delta}, \boldsymbol{a} - \boldsymbol{\delta}' \in \mathbb{Q}^l$ and $|\boldsymbol{\delta}|_1, |\boldsymbol{\delta}'|_1 \leq \epsilon$. If we set

$$\boldsymbol{b}_n = \boldsymbol{a}_n + ([n\delta_1], \ldots, [n\delta_l]) \quad \text{and} \quad \boldsymbol{b}_n' = \boldsymbol{a}_n - ([n\delta_1'], \ldots, [n\delta_l']),$$

then $\lim\limits_{n\to\infty} \boldsymbol{b}_n/n = \boldsymbol{a} + \boldsymbol{\delta}$ and $\lim\limits_{n\to\infty} \boldsymbol{b}'_n/n = \boldsymbol{a} - \boldsymbol{\delta}'$. Thus, using the previous case, we have

$$\widehat{\mathrm{vol}}((\boldsymbol{a} - \boldsymbol{\delta}') \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) = \liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{b}'_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$

$$\le \liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} \le \limsup_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$

$$\le \limsup_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{b}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \widehat{\mathrm{vol}}((\boldsymbol{a} + \boldsymbol{\delta}) \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

By (6) of Proposition 4.6,

$$\widehat{\mathrm{vol}}((\boldsymbol{a} - \epsilon\boldsymbol{1}) \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) \le \widehat{\mathrm{vol}}((\boldsymbol{a} - \boldsymbol{\delta}') \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f))$$

and

$$\widehat{\mathrm{vol}}((\boldsymbol{a} + \boldsymbol{\delta}) \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) \le \widehat{\mathrm{vol}}((\boldsymbol{a} + \epsilon\boldsymbol{1}) \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

Therefore,

$$\widehat{\mathrm{vol}}((\boldsymbol{a} - \epsilon\boldsymbol{1}) \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) \le \liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$

$$\le \limsup_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} \le \widehat{\mathrm{vol}}((\boldsymbol{a} + \epsilon\boldsymbol{1}) \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

Thus, taking $\epsilon \to 0$ and using the continuity of the volume function, we have

$$\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) = \liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \limsup_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}.$$

Hence we get Step 1.

Step 2: It is sufficient to show the following inequality:

$$(5.2.3) \quad \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) - 2d\epsilon(|\boldsymbol{a}|_1 + 1)\,\mathrm{vol}((\boldsymbol{a} + \boldsymbol{1}) \cdot \boldsymbol{L}_{\mathbb{Q}})$$

$$\le \liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} \le \limsup_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$

$$\le \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) + 2d\epsilon(|\boldsymbol{a}|_1 + 1)\,\mathrm{vol}((\boldsymbol{a} + \boldsymbol{1}) \cdot \boldsymbol{L}_{\mathbb{Q}})$$

for any positive real number ϵ. By Lemma 2.4, there are $g_1, \ldots, g_l, h \in C^0(X)$ such that $\|g_i\|_{\sup} \le \epsilon$ $(i = 1, \ldots, l)$, $\|h\|_{\sup} \le \epsilon$, $f + h$ is C^∞ and that

$$\overline{\boldsymbol{L}}^{\boldsymbol{g}} = (\overline{L}_1 + \overline{\mathcal{O}}(g_1), \ldots, \overline{L}_l + \overline{\mathcal{O}}(g_l))$$

is C^∞. Then it is easy to see that

$$\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n) + \overline{\mathcal{O}}(-\epsilon(|\boldsymbol{a}_n|_1 + n)) \le \boldsymbol{a}_n \cdot \overline{\boldsymbol{L}}^{\boldsymbol{g}} + \overline{\mathcal{O}}(f_n + nh)$$

$$\le \boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n) + \overline{\mathcal{O}}(\epsilon(|\boldsymbol{a}_n|_1 + n)),$$

which implies that

$$\hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n) + \overline{\mathcal{O}}(-\epsilon(|\boldsymbol{a}_n|_1 + n))\right) \le \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}}^{\boldsymbol{g}} + \overline{\mathcal{O}}(f_n + nh)\right)$$

$$\le \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n) + \overline{\mathcal{O}}(\epsilon(|\boldsymbol{a}_n|_1 + n))\right).$$

For each i, we choose an integer b_i with $\boldsymbol{a}(i) < b_i \le \boldsymbol{a}(i) + 1$. Then there is a positive integer n_0 such that $\boldsymbol{a}_n(i) \le nb_i$ for all $n \ge n_0$ and i. Thus, if we set $\boldsymbol{b} = (b_1, \ldots, b_l)$,

then $\boldsymbol{a}_n \leq n\boldsymbol{b}$ for all $n \geq n_0$ and $\boldsymbol{b} \leq \boldsymbol{a} + \mathbf{1}$. Thus $h^0(\boldsymbol{a}_n \cdot \boldsymbol{L}_{\mathbb{Q}}) \leq h^0(n\boldsymbol{b} \cdot \boldsymbol{L}_{\mathbb{Q}})$ for $n \geq n_0$. Hence, by using [4, (3) of Proposition 2.1], if we set

$$\beta(n) = \epsilon(|\boldsymbol{a}_n|_1 + n)h^0(n\boldsymbol{b} \cdot \boldsymbol{L}_{\mathbb{Q}}) + C_1 n^{d-1} \log(n)$$

for some positive constant C_1, then

$$\begin{cases} \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n) + \overline{\mathcal{O}}(\epsilon(|\boldsymbol{a}_n|_1 + n))\right) \leq \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n)\right) + \beta(n), \\ \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n) + \overline{\mathcal{O}}(-\epsilon(|\boldsymbol{a}_n|_1 + n))\right) \geq \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n)\right) - \beta(n) \end{cases}$$

for $n \gg 1$. Thus,

$$-\beta(n) \leq \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n)\right) - \hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}^{\boldsymbol{g}}} + \overline{\mathcal{O}}(f_n + nh)\right) \leq \beta(n)$$

for $n \gg 1$. Therefore, since

$$\begin{cases} \lim_{n \to \infty} \dfrac{\hat{h}^0\left(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}^{\boldsymbol{g}}} + \overline{\mathcal{O}}(f_n + nh)\right)}{n^d/d!} = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}^{\boldsymbol{g}}} + \overline{\mathcal{O}}(f + h)) \quad (\because \text{Step 1}), \\ \lim_{n \to \infty} \dfrac{\beta(n)}{n^d/d!} = d\epsilon(|\boldsymbol{a}|_1 + 1) \, \mathrm{vol}(\boldsymbol{b} \cdot \boldsymbol{L}_{\mathbb{Q}}), \\ \mathrm{vol}(\boldsymbol{b} \cdot \boldsymbol{L}_{\mathbb{Q}}) \leq \mathrm{vol}((\boldsymbol{a} + \mathbf{1}) \cdot \boldsymbol{L}_{\mathbb{Q}}), \end{cases}$$

we have

$$\widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}^{\boldsymbol{g}}} + \overline{\mathcal{O}}(f + h)) - d\epsilon(|\boldsymbol{a}|_1 + 1) \, \mathrm{vol}((\boldsymbol{a} + \mathbf{1}) \cdot \boldsymbol{L}_{\mathbb{Q}})$$

$$\leq \liminf_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} \leq \limsup_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$

$$\leq \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}^{\boldsymbol{g}}} + \overline{\mathcal{O}}(f + h)) + d\epsilon(|\boldsymbol{a}|_1 + 1) \, \mathrm{vol}((\boldsymbol{a} + \mathbf{1}) \cdot \boldsymbol{L}_{\mathbb{Q}}).$$

On the other hand, by (1) of Proposition 4.6,

$$\left| \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}^{\boldsymbol{g}}} + \overline{\mathcal{O}}(f + h)) - \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)) \right|$$

$$= \left| \widehat{\mathrm{vol}}((\boldsymbol{a}, 1) \cdot (\overline{\boldsymbol{L}}, \overline{\mathcal{O}}(f)) + \overline{\mathcal{O}}(\boldsymbol{a} \cdot \boldsymbol{g} + h)) - \widehat{\mathrm{vol}}((\boldsymbol{a}, 1) \cdot (\overline{\boldsymbol{L}}, \overline{\mathcal{O}}(f))) \right|$$

$$\leq d\epsilon(|\boldsymbol{a}|_1 + 1) \, \mathrm{vol}((\boldsymbol{a}, 1) \cdot (\boldsymbol{L}_{\mathbb{Q}}, 0)) \leq d\epsilon(|\boldsymbol{a}|_1 + 1) \, \mathrm{vol}((\boldsymbol{a} + \mathbf{1}) \cdot \boldsymbol{L}_{\mathbb{Q}}).$$

Hence (5.2.3) follows.

Step 3: Let $\nu : X' \to X$ be a generic resolution of singularities of X such that X' is normal. Then, since $\nu_* \mathcal{O}_{X'} = \mathcal{O}_X$, we have

$$H^0(X, \boldsymbol{a}_n \cdot \boldsymbol{L}) = H^0(X', \boldsymbol{a}_n \cdot \nu^*(\boldsymbol{L})).$$

Thus $\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n)) = \hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))$. Therefore, by using Step 2 and (2) of Proposition 4.6,

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}$$

$$= \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f))) = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

Step 4: Let $\nu : X' \to X$ be the normalization of X. It is sufficient to see that

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}$$

because, by using (2) of Proposition 4.6 and Step 3, the above equation implies that

$$\lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} = \lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}$$
$$= \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f))) = \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f)).$$

Since $H^0(X, \boldsymbol{a}_n \cdot \boldsymbol{L}) \subseteq H^0(X', \boldsymbol{a}_n \cdot \nu^*(\boldsymbol{L}))$, we have

$$\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n)) \le \hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n))).$$

Thus

$$\liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!} \le \limsup_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}$$
$$\le \lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}.$$

Therefore, we need to show

$$\lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!} \le \liminf_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}.$$

The proof of the above inequality is similar to one of [4, Theorem 4.3]. Let $\mathcal{I}_{X'/X}$ be the conductor ideal sheaf of $X' \to X$. Let H be an ample invertible sheaf on X' with $H^0(X', H \otimes \mathcal{I}_{X'/X}) \ne 0$. Let s be a non-zero element of $H^0(X', H \otimes \mathcal{I}_{X'/X})$. Let us choose a C^∞-hermitian norm $|\cdot|$ of H with $\|s\|_{\sup} \le 1$. We set $\overline{H} = (H, |\cdot|)$.

Claim 5.2.4. $\displaystyle\lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!} = \lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) - \overline{H} + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}.$

Proof. We set

$$\begin{cases} \overline{\boldsymbol{L}}' = (\nu^*(\overline{L}_1), \dots, \nu^*(\overline{L}_l), \overline{H}), \\ \boldsymbol{a}_n' = (\boldsymbol{a}_n(1), \dots, \boldsymbol{a}_n(l), -1), \\ \boldsymbol{a}' = (\boldsymbol{a}(1), \dots, \boldsymbol{a}(l), 0). \end{cases}$$

Then $\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) - \overline{H} = \boldsymbol{a}_n' \cdot \overline{\boldsymbol{L}}'$ and $\boldsymbol{a}' = \lim_{n\to\infty} \boldsymbol{a}_n'/n$. By Step 3,

$$\lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) - \overline{H} + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!} = \lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n' \cdot \overline{\boldsymbol{L}}' + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}$$
$$= \widehat{\mathrm{vol}}(\boldsymbol{a}' \cdot \overline{\boldsymbol{L}}' + \overline{\mathcal{O}}(\nu^*(f)))$$
$$= \widehat{\mathrm{vol}}(\boldsymbol{a} \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f)))$$
$$= \lim_{n\to\infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}.$$

$\square$

In the same way as in the proof of [4, Theorem 4.3], we can see

$$\mathrm{Image}\left(H^0(X', \boldsymbol{a}_n \cdot \nu^*(\boldsymbol{L}) - H) \xrightarrow{s} H^0(X', \boldsymbol{a}_n \cdot \nu^*(\boldsymbol{L}))\right) \subseteq H^0(X, \boldsymbol{a}_n \cdot \boldsymbol{L}).$$

Thus $\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) - \overline{H} + \overline{\mathcal{O}}(\nu^*(f_n))) \leq \hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))$. Therefore, using the above claim,

$$\lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!} = \lim_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \nu^*(\overline{\boldsymbol{L}}) - \overline{H} + \overline{\mathcal{O}}(\nu^*(f_n)))}{n^d/d!}$$
$$\leq \liminf_{n \to \infty} \frac{\hat{h}^0(\boldsymbol{a}_n \cdot \overline{\boldsymbol{L}} + \overline{\mathcal{O}}(f_n))}{n^d/d!}.$$

$\square$

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

E-mail address: moriwaki@math.kyoto-u.ac.jp