The Reidemeister spectra of low dimensional almost-crystallographic groups

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Abstract

We determine which non-crystallographic, almost-crystallographic groups of dimension 4 have the R_∞-property. We then calculate the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and the 4-dimensional almost-Bieberbach groups.

1 Introduction

Let G be any group and $\varphi : G \to G$ an endomorphism of this group. Define an equivalence relation $\sim_\varphi$ on G given by

$$\forall g, g' \in G : g \sim_\varphi g' \iff \exists h \in G : g = hg'\varphi(h)^{-1}.$$

An equivalence class $[g]_\varphi$ is called a *Reidemeister class* of φ or *φ-twisted conjugacy class.* The *Reidemeister number* $R(\varphi)$ is the number of Reidemeister classes of φ and is therefore always a positive integer or infinity. The *Reidemeister spectrum* of a group G is the set of all Reidemeister numbers when considering all possible automorphisms of that group:

$$\mathrm{Spec}_R(G) := \{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\}.$$

If $\mathrm{Spec}_R(G) = \{\infty\}$ we say that G has the *R_∞-property.*

Reidemeister numbers originate in Nielsen fixed point theory, where they are defined as the number of fixed point classes of a self-map of a topological space [Jiang 1983], although they also yield applications in algebraic geometry and representation theory [Fel'shtyn and Troitsky 2015].

It turns out that many (infinite) groups admit the R_∞-property. This is also the case for most almost-crystallographic groups, e.g. in [Dekimpe and Penninckx 2011] it was shown that 207 of the 219 3-dimensional crystallographic groups and 15 of the 17 families of 3-dimensional (non-crystallographic) almost-crystallographic groups all have the R_∞-property. Furthermore, in [Dekimpe et al. 2019a] it was shown that 4692 of the 4783 4-dimensional crystallographic groups admit the R_∞ property. Moreover, the Reidemeister spectra of all crystallographic groups of dimensions 1, 2 and 3 were calculated, as well as the spectra of

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the 4-dimensional Bieberbach groups. In this paper we extend these results by studying the 4-dimensional almost-crystallographic groups.

This paper is structured as follows. In the next two sections, we provide the necessary preliminaries on Reidemeister numbers and almost-crystallographic groups. In section 4 we determine which almost-crystallographic groups of dimension 4 possess the R_∞-property. Sections 5 and 6 are devoted to calculating the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and the 4-dimensional almost-Bieberbach groups respectively. The final section summarises the obtained results.

2 Reidemeister numbers and spectra

In this section we introduce basic notions concerning the Reidemeister number. For a general reference on Reidemeister numbers and their connection to fixed point theory, we refer the reader to [Jiang 1983].

The definitions of the Reidemeister number and Reidemeister spectrum were given in the introduction. However, nothing was said on how we actually determine whether a group has the R_∞-property, and if not, how we calculate its Reidemeister spectrum. The following lemma is an essential tool for the former.

Lemma 2.1 (see [Fel'shtyn and Troitsky 2015, Section 2.2], [Gonçalves and Wong 2009, Lemma 1.1])**.** *Let N be a normal subgroup of a group G and $\varphi \in \operatorname{Aut}(G)$ with $\varphi(N) = N$. We denote the restriction of φ to N by $\varphi|_N$, and the induced automorphism on the quotient G/N by φ'. We then get the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \varphi|_N} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \\
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 1
\end{array}$$

We obtain the following properties:

(1) $R(\varphi) \geq R(\varphi')$,

(2) if $R(\varphi') < \infty$, $R(\varphi|_N) = \infty$ and $|\operatorname{Fix}(\varphi')| < \infty$, then $R(\varphi) = \infty$.

A direct consequence for characteristic subgroups is the following:

Corollary 2.2. *Let N be a characteristic subgroup of G. If either*

(1) the quotient G/N has the R_∞-property, or

(2) N has finite index in G and has the R_∞-property,

then G has the R_∞-property as well.

3 Almost-crystallographic groups

Let G be a connected, simply connected, nilpotent Lie group with automorphism group $\operatorname{Aut}(G)$. The affine group $\operatorname{Aff}(G)$ is the semi-direct product $\operatorname{Aff}(G) = G \rtimes \operatorname{Aut}(G)$, where multiplication is defined by $(d_1, D_1)(d_2, D_2) = (d_1 D_1(d_2), D_1 D_2)$. If C is a maximal compact subgroup of $\operatorname{Aut}(G)$, then $G \rtimes C$ is a subgroup of $\operatorname{Aff}(G)$. A cocompact discrete subgroup Γ of $G \rtimes C$ is called an *almost-crystallographic group* modelled on the Lie group G. The dimension of Γ is defined as the dimension of G.

If Γ is torsion-free, then it is called an *almost-Bieberbach group.* If $G = \mathbb{R}^n$, then it is called a *crystallographic group*, or a *Bieberbach group* if it also torsion-free.

Crystallographic groups were historically studied first, and are well understood by the three Bieberbach theorems. These theorems have since been generalised to almost-crystallographic groups, which we will briefly discuss below. We refer to [Szczepański 2012] and [Dekimpe 1996] for more information on the original and generalised theorems respectively.

The generalised first Bieberbach theorem says that if $\Gamma \subseteq \mathrm{Aff}(G)$ is an n-dimensional almost-crystallographic group, then its *translation subgroup* $N := \Gamma \cap G$ is a uniform lattice of G and is of finite index in Γ. Moreover, N is the unique maximal nilpotent normal subgroup of Γ, and is therefore characteristic in Γ. The quotient group $F := \Gamma/N$ is a finite group called the *holonomy group* of Γ. In fact $F = \{A \in \mathrm{Aut}(G) \mid \exists a \in G : (a, A) \in \Gamma\}$. If Γ is crystallographic ($G = \mathbb{R}^n$), we may assume that $N = \mathbb{Z}^n$ and F is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$.

The generalised second Bieberbach theorem tells us more about automorphisms of almost-crystallographic groups.

Theorem 3.1 (generalised second Bieberbach theorem)**.** *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of an almost-crystallographic group $\Gamma \subseteq \mathrm{Aff}(G)$ with holonomy group F. Then there exists a $(d, D) \in \mathrm{Aff}(G)$ such that $\varphi(\gamma) = (d, D) \circ \gamma \circ (d, D)^{-1}$ for all $\gamma \in \Gamma$. To shorten notation, we will write $\varphi = \xi_{(d,D)}$.*

An automorphism $\Phi : G \to G$ of a Lie group G induces an automorphism $\Phi_* : \mathfrak{g} \to \mathfrak{g}$ of the associated Lie algebra $\mathfrak{g}$. We will henceforth always denote an induced automorphisms on a Lie algebra with a star ($*$) subscript, for example A_* is the Lie algebra automorphism induced by some $A \in F$ where $F \subseteq \mathrm{Aut}(G)$ is the holonomy group of an almost-crystallographic group. In particular, an automorphism $\varphi = \xi_{(d,D)}$ of an almost-crystallographic group has an associated matrix D_*.

The generalised third Bieberbach theorem is less straightforward to generalise. Unlike for crystallographic groups, it is not true that there are only finitely many n-dimensional almost-crystallographic groups for a given dimension n. However, we can state that for a given finitely generated torsion-free nilpotent group N, there are (up to isomorphism) only finitely many almost-crystallographic groups Γ such that the translation subgroup of Γ is isomorphic to N.

In [Dekimpe 1996, Section 2.5], this generalisation is proved using the concept of an *isolator*, which shall prove useful to us as well.

Definition 3.2. Let G be a group with subgroup H. The isolator of H in G is defined as

$$\sqrt[G]{H} := \{g \in G \mid g^k \in H \text{ for some } k \geq 1\}.$$

Although much can be said about isolators, for the purposes of this paper we only care about a very specific result.

Lemma 3.3 (see [Dekimpe 1996, Lemma 2.4.2])**.** *Let Γ be an almost-crystallographic group with translation subgroup N of nilpotency class c. Then the isolator $\sqrt[N]{\gamma_c(N)} \leq Z(N)$ is a characteristic subgroup of Γ. Moreover, the quotient group $\Gamma/\sqrt[N]{\gamma_c(N)}$ is an almost-crystallographic group whose translation subgroup $N/\sqrt[N]{\gamma_c(N)}$ has nilpotency class $c-1$. If $c = 2$, then this quotient is a crystallographic group.*

We will now give the most important results for Reidemeister theory applied to almost-crystallographic groups. A first result allows us to easily determine whether an almost-crystallographic group admits the R_∞-property or not.

Theorem 3.4 (see [Dekimpe and Penninckx 2011, Corollary 3.10])**.** *Let* Γ *be an* n*-dimensional almost-crystallographic group with holonomy group* $F \subseteq \mathrm{Aut}(G)$ *and* $\varphi = \xi_{(d,D)} \in \mathrm{Aut}(\Gamma)$ *(where we use the notation of theorem 3.1). Then*

$$\begin{aligned} &R(\varphi) = \infty \\ &\iff \exists A \in F \text{ such that } \det(\mathbb{1}_n - A_* D_*) = 0 \\ &\iff \exists A \in F \text{ such that } A_* D_* \text{ has eigenvalue } 1. \end{aligned}$$

The second result only holds for almost-Bieberbach groups, and allows for an easy computation of the Reidemeister number of an automorphism.

Theorem 3.5 (averaging formula, see [Ha et al. 2012, Theorem 6.11] and [Lee and Lee 2009, Theorem 4.3])**.** *Let* Γ *be an* n*-dimensional almost-Bieberbach group with holonomy group* $F \subseteq \mathrm{Aut}(G)$*, and* $\varphi = \xi_{(d,D)} \in \mathrm{Aut}(\Gamma)$ *with* $R(\varphi) < \infty$*. Then*

$$R(\varphi) = \frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_n - A_* D_*)|.$$

In general, this formula does not hold for automorphisms of almost-crystallographic groups, examples can be found in [Dekimpe et al. 2019a] and later in this paper. Therefore, the calculation of the Reidemeister spectra usually requires a deeper understanding of how the Reidemeister classes are formed in a specific group.

4 The R_∞-property for 4-dimensional almost-crystallographic groups

Every almost-crystallographic group of dimension 1 or 2 is crystallographic. In [Dekimpe and Penninckx 2011] it was determined which 3-dimensional almost-crystallographic groups admit the R_∞-property. We extend these results to dimension 4. In this case the translation subgroup N is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class at most 3. Nilpotency class 1 is of course the crystallographic case, which was done in [Dekimpe et al. 2019a].

4.1 Nilpotency class 2

Let Γ be an almost-crystallographic group whose translation subgroup N is a nilpotent group of rank 4 and nilpotency class 2. In [Dekimpe 1996] it was shown that N can be given the following presentation:

$$\left\langle e_1, e_2, e_3, e_4 \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & [e_3, e_2] = e_1^{l_1} \\ {[e_3, e_1]} = 1 & [e_4, e_2] = e_1^{l_2} \\ {[e_4, e_1]} = 1 & [e_4, e_3] = e_1^{l_3} \end{array} \right\rangle.$$

Moreover, let G be the Lie group that Γ is modelled on. By [Dekimpe 1995, Theorem 4.1], there exists a faithful affine representation $\lambda : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aff}(\mathbb{R}^4)$ such that its restriction to Γ is again a faithful affine representation. In particular,

$$\lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda(e_2) = \begin{pmatrix} 1 & 0 & -\frac{l_1}{2} & -\frac{l_2}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\lambda(e_3) = \begin{pmatrix} 1 & \frac{l_1}{2} & 0 & -\frac{l_3}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda(e_4) = \begin{pmatrix} 1 & \frac{l_2}{2} & \frac{l_3}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the values of l_1, l_2 and l_3 are determined by the relations $[e_3, e_2] = e_1^{l_1}$, $[e_4, e_2] = e_1^{l_2}$ and $[e_4, e_3] = e_1^{l_3}$.

Lemma 3.3 tells us that the subgroup $\langle e_1 \rangle = \sqrt[N]{\gamma_2(N)}$ is characteristic and the quotient $\Gamma' := \Gamma/\langle e_1 \rangle$ is a 3-dimensional crystallographic group. Using corollary 2.2, we know that if Γ' has the R_∞-property, then so does Γ. In [Dekimpe 1996; Dekimpe and Eick 2002] the almost-crystallographic groups were classified into families based on which crystallographic group Γ' is. Since only twelve 3-dimensional crystallographic groups do not have the R_∞-property, we need only consider the corresponding twelve families of 4-dimensional almost-crystallographic groups.

Each of these families can be split in smaller subfamilies, determined by the action of F on $\sqrt[N]{\gamma_2(N)}$: every $A \in F$ acts on e_1 by ${}^A e_1 = e_1^{\epsilon_A}$ with $\epsilon_A \in \{-1, 1\}$. The following proposition quickly deals with the subfamilies where F does not act trivially on $\sqrt[N]{\gamma_2(N)}$.

Proposition 4.1. *Let Γ be an almost-crystallographic group with translation subgroup N of rank 4 and nilpotency class 2, and holonomy group F. If F acts non-trivially on $\sqrt[N]{\gamma_2(N)}$, then Γ has the R_∞-property.*

Proof. Let $A \in F$ arbitrary and $\varphi = \xi_{(d,D)} \in \operatorname{Aut}(\Gamma)$. Since A acts on $\langle e_1 \rangle = \sqrt[N]{\gamma_2(N)}$ by ${}^A e_1 = e_1^{\epsilon_A}$ with $\epsilon_A \in \{-1, 1\}$ and $\varphi(e_1) = e_1^{\nu}$ with $\nu \in \{-1, 1\}$, A_* and D_* must have the following forms:

$$A_* = \begin{pmatrix} \epsilon_A & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad D_* = \begin{pmatrix} \nu & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Thus, $\mathbb{1}_4 - A_* D_*$ is of the form

$$\mathbb{1}_4 - A_* D_* = \begin{pmatrix} 1 - \nu\epsilon_A & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Now let us look at specific $A \in F$. First, let A be the neutral element of F, which necessarily acts trivially on e_1. The above matrix then has upper left entry $1-\nu$, hence $\det(\mathbb{1}_4 - D_*) \neq 0$ if and only if $\nu = -1$.

Second, let A be an element of F for which $\epsilon_A = -1$. Such element exists since we assumed F acts non-trivially on $\sqrt[N]{\gamma_2(N)}$. Then the matrix $\mathbb{1}_4 - A_* D_*$ has upper left entry $1 + \nu$, and $\det(\mathbb{1}_4 - A_* D_*) \neq 0$ if and only if $\nu = 1$.

Family	δ
1,2	$\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}$
3,4	$\begin{pmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}$
5	$\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}$
143	$\begin{pmatrix} 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{pmatrix}$
146	$\begin{pmatrix} 1 & -\frac{k_1}{2}+k_2+2k_3 & -k_2+k_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Table 1: Conjugacy matrices between representations

As ν cannot be -1 and 1 at the same time, we always have some $A \in F$ for which $\det(\mathbb{1}_4 - A_* D_*) = 0$, and by theorem 3.4 this means that $R(\varphi) = \infty$. Since this holds for any automorphism, Γ has the R_∞-property. $\square$

From the proof of the theorem above, we can also conclude the following:

Proposition 4.2. *Let Γ be an almost-crystallographic group with translation subgroup N of rank 4 and nilpotency class 2, and let e_1 be a generator of $\sqrt[N]{\gamma_2(N)}$. If $\varphi \in \operatorname{Aut}(\Gamma)$ has finite Reidemeister number, then $\varphi(e_1) = e_1^{-1}$.*

We will number the twelve families under consideration according to the crystallographic group $\Gamma / \sqrt[N]{\gamma_2(N)}$, using the classification in the International Tables in Crystallography [Aroyo 2016]: they are families 1-5, 16, 19, 22-24, 143 and 146. When we write $\Gamma_{n/m}$, we mean the n-dimensional crystallographic group with IT-number m.

Using the techniques in [Dekimpe 1996, Section 5.4], we find that for an almost-crystallographic group belonging to one of the families 16, 19 or 22-24, F acting trivially on $\sqrt[N]{\gamma_2(N)}$ implies that the group is actually crystallographic. Therefore we may omit these families and we are left with only 7 families to study.

Note that the presentations given in this paper may vary from those in [Dekimpe 1996; Dekimpe and Eick 2002]. Let Γ and λ denote a group and its faithful representation as given in this paper, and let Γ' and μ be the corresponding group and representation as given by [Dekimpe 1996] or [Dekimpe and Eick 2002]. Table 1 contains a matrix δ such that

$$\lambda(\Gamma) = \delta \mu(\Gamma') \delta^{-1},$$

hence $\lambda(\Gamma)$ and $\mu(\Gamma')$ are conjugate subgroups of $\operatorname{Aff}(\mathbb{R}^4)$ and therefore Γ and Γ' are isomorphic.

Family 1. This family consists of the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 4. It was shown in [Dekimpe et al. 2019b, Section 3.2] that these groups do not have the R_∞-property.

Family 2. Every group in this family has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_1^{k_4} e_2^{-1} \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_1^{k_5} e_3^{-1} \alpha \\ [e_3, e_2] = e_1^{k_1} & \alpha e_4 = e_1^{k_6} e_4^{-1} \alpha \\ [e_4, e_2] = e_1^{k_2} & \alpha^2 = e_1^{k_7} \\ [e_4, e_3] = e_1^{k_3} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_4 & k_5 & k_6 & \frac{k_7}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Set $k := \gcd(k_1, k_2, k_3)$ and $g := e_2^{k_3/k} e_3^{-k_2/k} e_4^{k_1/k}$, then the centre $Z(N)$ of the translation subgroup is generated by e_1 and g. Let $\varphi : \Gamma \to \Gamma$ be any automorphism. Since $\langle e_1 \rangle$ and $Z(N)$ are both characteristic in Γ, we have that $\varphi(g) = g^\epsilon e_1^m$ for some $\epsilon \in \{-1, 1\}$ and $m \in \mathbb{Z}$. Consider the induced automorphism $\varphi' = \xi_{(d',D')}$ on $\Gamma / \langle e_1 \rangle \cong \Gamma_{3/2}$. Then

$$\varphi'(g\langle e_1 \rangle) = D'(g\langle e_1 \rangle) = \varphi(g)\langle e_1 \rangle = g^\epsilon \langle e_1 \rangle.$$

Depending on the value of ϵ, D'_* has either eigenvalue 1, in which case $\det(\mathbb{1}_3 - D'_*) = 0$, or eigenvalue -1, in which case $\det(\mathbb{1}_3 + D'_*) = 0$. Since the holonomy group of $\Gamma_{3/2}$ is $\{\mathbb{1}_3, -\mathbb{1}_3\}$, we obtain by theorem 3.4 that $R(\varphi') = \infty$ and by lemma 2.1 that therefore $R(\varphi) = \infty$. Since this holds for an arbitrary automorphism, Γ has the R_∞-property.

Families 3, 4 and 5. Every group in one of these families has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2 \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_1^{k_2} e_2^{-\nu} e_3^{-1} \alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_1^{k_3} e_4^{-1} \alpha \\ [e_4, e_2] = 1 & \alpha^2 = e_1^{k_4} e_2^{\mu} \\ [e_4, e_3] = e_1^{k_1} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & k_2 & k_3 & \frac{k_4}{2} \\ 0 & 1 & -\nu & 0 & \frac{\mu}{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Family 3 is given by $\mu, \nu = 0$, family 4 by $\mu = 1, \nu = 0$ and family 5 by $\mu = 0, \nu = 1$. Define an automorphism $\varphi = \xi_{(d,D)}$ by

$$\begin{aligned} \varphi(e_1) &= e_1^{-1}, \\ \varphi(e_2) &= e_2^{-1}, \\ \varphi(e_3) &= e_1^{k_1 - k_2 - k_3} e_2^{\nu} e_3 e_4^2, \\ \varphi(e_4) &= e_1^{3k_1 - k_2 - 2k_3} e_2^{\nu} e_3^2 e_4^3, \\ \varphi(\alpha) &= e_1^{-k_4} e_2^{-\mu} \alpha, \end{aligned}$$

then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{pmatrix}.$$

We can apply theorem 3.4 to show that $R(\varphi) < \infty$ and hence Γ does not have the R_∞-property.

Families 143 and 146. Every group in one of these families has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2 \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_1^{k_2} e_4 \alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_1^{k_3} e_2^{\mu} e_3^{-1} e_4^{-1} \alpha \\ [e_4, e_2] = 1 & \alpha^3 = e_1^{k_4} \\ [e_4, e_3] = e_1^{k_1} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & k_2 & -\frac{k_1}{2} + k_3 & \frac{k_4}{3} \\ 0 & 1 & 0 & \mu & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Family 143 is given by $\mu = 0$ and family 146 by $\mu = 1$. Using an argument identical to the proof of [Dekimpe and Penninckx 2011, Theorem 4.4, family 13], we may conclude that all groups in these families have the R_∞-property.

4.2 Nilpotency class 3

By an argument analogous to [Gonçalves and Wong 2009, Example 5.2], a finitely-generated, torsion-free, nilpotent group of nilpotency class 3 and rank 4 has the R_∞-property. Applying corollary 2.2 then proves that every 4-dimensional almost-crystallographic group with translation subgroup of nilpotency class 3 has the R_∞-property.

5 The Reidemeister spectra of the 3-dimensional almost-crystallographic groups

Let Γ be an almost-crystallographic group whose translation subgroup N is a nilpotent group of rank 3 and nilpotency class 2. Such N can be given the following presentation:

$$\left\langle e_1, e_2, e_3 \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & [e_3, e_2] = e_1^{l_1} \\ [e_3, e_1] = 1 & \end{array} \right\rangle,$$

with $l_1 > 0$. Moreover, let G be the Lie group that Γ is modelled on. By [Dekimpe 1995, Theorem 4.1], there exists a faithful affine representation $\lambda : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aff}(\mathbb{R}^3)$ such that its restriction to Γ is again a faithful affine representation. In particular,

$$\lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \lambda(e_2) = \begin{pmatrix} 1 & 0 & -\frac{l_1}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \lambda(e_3) = \begin{pmatrix} 1 & \frac{l_1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the value of l_1 is determined by the relation $[e_3, e_2] = e_1^{l_1}$. Like in section 4.1, we have that the subgroup $\langle e_1 \rangle = \sqrt[N]{\gamma_2(N)}$ is characteristic in Γ, and an automorphism φ must satisfy $\varphi(e_1) = e_1^{-1}$ to have finite Reidemeister number.

As mentioned before, in [Dekimpe and Penninckx 2011, Theorem 4.4] it was shown that there are only 2 families of almost-crystallographic groups that do not admit the R_∞-property. We again number these families according to the IT-number of the quotient $\Gamma/\sqrt[N]{\gamma_2(N)}$.

Family 1. The groups in this family are exactly the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 3. In [Roman'kov 2011, Section 3] it was shown that these groups have Reidemeister spectrum $2\mathbb{N} \cup \{\infty\}$. This was shown specifically for the case $k_1 = 1$, but the argument holds for any $k_1 > 0$.

Family 2. Every group in this family has a presentation of the form

$$\left\langle e_1, e_2, e_3, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_1^{k_2} e_2^{-1} \alpha \\ [e_3, e_2] = e_1^{k_1} & \alpha e_3 = e_1^{k_3} e_3^{-1} \alpha \\ & \alpha^2 = e_1^{k_4} \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_2 & k_3 & \frac{k_4}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let φ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation λ, this automorphism will correspond to a matrix $\delta \in \mathrm{Aff}(\mathbb{R}^4)$ such that

$$\lambda(\varphi(\gamma)) = \delta \lambda(\gamma) \delta^{-1}.$$

for all $\gamma \in \Gamma$. Since we assumed that $R(\varphi) < \infty$, we have that $\varphi(e_1) = e_1^{-1}$. Moreover, φ induces an automorphism φ' on $\Gamma' := \Gamma/\langle e_1 \rangle$. Thus, δ must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & 0 \\ 0 & m_1 & m_3 & d_1/2 \\ 0 & m_2 & m_4 & d_2/2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the constants m_i, d_j are integers, $m_1 m_4 - m_2 m_3 = -1$ and $n_1, n_2 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of n_1, n_2 and l_1, l_2, l_3 such that

$$\begin{aligned} \delta \lambda(e_2) \delta^{-1} &= \lambda(e_1)^{l_1} \lambda(e_2)^{m_1} \lambda(e_3)^{m_2}, \\ \delta \lambda(e_3) \delta^{-1} &= \lambda(e_1)^{l_2} \lambda(e_2)^{m_3} \lambda(e_3)^{m_4}, \\ \delta \lambda(\alpha) \delta^{-1} &= \lambda(e_1)^{l_3} \lambda(e_2)^{d_1} \lambda(e_3)^{d_2} \lambda(\alpha). \end{aligned}$$

From the obtained values of l_1, l_2 and l_3, we get

$$\begin{aligned} \varphi(e_1) &= e_1^{-1}, \\ \varphi(e_2) &= e_1^{\frac{k_1}{2}(m_1 m_2 + m_1 d_2 - m_2 d_1) - \frac{k_2}{2}(m_1 + 1) - \frac{k_3}{2} m_2} e_2^{m_1} e_3^{m_2}, \\ \varphi(e_3) &= e_1^{\frac{k_1}{2}(m_3 m_4 + m_3 d_2 - m_4 d_1) - \frac{k_2}{2} m_3 - \frac{k_3}{2}(m_4 + 1)} e_2^{m_3} e_3^{m_4}, \\ \varphi(\alpha) &= e_1^{\frac{k_1}{2} d_1 d_2 - \frac{k_2}{2} d_1 - \frac{k_3}{2} d_2 - k_4} e_2^{d_1} e_3^{d_2} \alpha, \end{aligned}$$

where all exponents must be integers. This places four conditions on the m_i and d_j:

(a) $k_1(m_1m_2 + m_1d_2 - m_2d_1) - k_2(m_1 + 1) - k_3m_2 \equiv 0 \mod 2$,

(b) $k_1(m_3m_4 + m_3d_2 - m_4d_1) - k_2m_3 - k_3(m_4 + 1) \equiv 0 \mod 2$,

(c) $k_1d_1d_2 - k_2d_1 - k_3d_2 \equiv 0 \mod 2$,

(d) $m_1m_4 - m_2m_3 = -1$.

For ease of notation, let us set

$$M := \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad d := \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{Z}^2.$$

We will determine $R(\varphi)$ in a very similar way to the proof of [Dekimpe et al. 2019a, Proposition 5.11]. Let $[x]_\varphi$ be a Reidemeister class of Γ, then for any $k \in \mathbb{Z}$,

$$x = (e_1^{-k})xe_1^{2k}\varphi(e_1^{-k})^{-1},$$

therefore $x \sim_\varphi xe_1^{2k}$ for all $k \in \mathbb{Z}$. Consider the quotient group $\Gamma' = \Gamma/\langle e_1\rangle$ and let $\varphi' = \xi_{(d/2,M)}$ be the induced automorphism on this quotient. Since we assumed that $R(\varphi) < \infty$, we have that $R(\varphi') < \infty$ as well. [Dekimpe et al. 2019a, Proposition 5.10] tells us that $R(\varphi') = |\operatorname{tr}(M)| + O(\mathbb{1}_2 - M, d)$ with

$$O(A, a) := \#\left\{\bar{x} \in \mathbb{Z}_2^2 \mid \bar{A}\bar{x} = \bar{a}\right\},$$

where the bar-notation denotes the element-wise projection to $\mathbb{Z}_2$. A Reidemeister class $[x\langle e_1\rangle]_{\varphi'}$ of Γ' will lift to at most 2 Reidemeister classes of Γ: $[x]_\varphi$ and $[xe_1]_\varphi$; so the number of lifts is either 2 (when $x \not\sim_\varphi xe_1$) or 1 (when $x \sim_\varphi xe_1$). The latter happens if and only if

$$\exists z \in \Gamma : xe_1 = zx\varphi(z)^{-1}. \tag{1}$$

Projecting this to the quotient Γ', we have

$$\exists z \in \Gamma : x\langle e_1\rangle = zx\varphi(z)^{-1}\langle e_1\rangle. \tag{2}$$

Since e_1 is central in Γ and x appears exactly once on each side of the equality sign in (1), the e_1-component of x does not matter. Set $x = e_2^{x_2}e_3^{x_3}\alpha^{\epsilon_x}$ and $z = e_1^{z_1}e_2^{z_2}e_3^{z_3}\alpha^{\epsilon_z}$. Let us first assume that $\epsilon_z = 0$, then (2) is equivalent to

$$\exists z_2, z_3 \in \mathbb{Z} : (\mathbb{1}_2 - AM)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = 0,$$

with A the holonomy part of $x\langle e_1\rangle$. As $R(\varphi') < \infty$, we must have $z_2 = z_3 = 0$. But then $z = e_1^{z_1}$, and (1) then becomes $xe_1 = xe_1^{2z_1}$. As z_1 is an integer, this is impossible. So, let us assume that $\epsilon_z = 1$. Writing out (1) component-wise, we find that this condition is equivalent to the following:

There exist $z_1, z_2, z_3 \in \mathbb{Z}$ such that:

(i) $2\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = (\mathbb{1}_2 - (-1)^{\epsilon_x}M)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} - (-1)^{\epsilon_x}d$,

(ii) $k_1z_2z_3 - k_2z_2 - k_3z_3 - k_4 + 1 = 2z_1$.

Condition (i) is independent of the e_1-components, and hence can be interpreted in terms of the quotient group Γ'. In the proof of [Dekimpe et al. 2019a, Proposition 5.11] it was shown that, for a fixed value of ϵ_x, the number of Reidemeister classes $[x\langle e_1\rangle]_{\varphi'}$ for which a pair (z_2, z_3) satisfying (i) exists is exactly $O(\mathbb{1}_2 - M, d)$, i.e. the number of solutions $(\bar{z}_2, \bar{z}_3) \in \mathbb{Z}_2^2$ of the linear system of equations

$$\text{(i')} \quad \left(\overline{\mathbb{1}_2 - M}\right)\begin{pmatrix}\bar{z}_2\\ \bar{z}_3\end{pmatrix} = \bar{d}.$$

Note that the above equation is exactly condition (i) taken modulo 2.

Since ϵ_x can take two values (1 and -1), there are in total $2O(\mathbb{1}_2 - M, d)$ Reidemeister classes $[x\langle e_1\rangle]_{\varphi'}$ satisfying condition (i). On the other hand, there are $|\operatorname{tr}(M)| - O(\mathbb{1}_2 - M, d)$ Reidemeister classes of Γ' for which condition (i) does not hold (see [Dekimpe et al. 2019a, Section 5]).

Recall that the variable z_1 appears only in condition (ii). If we have a Reidemeister class $[x\langle e_1\rangle]_{\varphi'}$ and a pair (z_2, z_3) for which (i) holds, then we can find a $z_1 \in \mathbb{Z}$ to make condition (ii) hold if and only if

$$\text{(ii')} \quad \bar{k}_1\bar{z}_2\bar{z}_3 - \bar{k}_2\bar{z}_2 - \bar{k}_3\bar{z}_3 - \bar{k}_4 + \bar{1} = \bar{0},$$

which is exactly condition (ii) taken modulo 2.

We partition the solutions of (i') into those that do not satisfy condition (ii') and those that do. Let S be the number of the former and T the number of the latter, then $S + T = O(\mathbb{1}_2 - M, d)$. Of the $2O(\mathbb{1}_2 - M, d)$ Reidemeister classes $[x\langle e_1\rangle]_{\varphi'}$ satisfying condition (i), $2S$ lift to two distinct Reidemeister classes $[x]_\varphi$ and $[xe_1]_\varphi$, and $2T$ lift to a single Reidemeister class $[x]_\varphi$. All together, we have

$$\begin{aligned} R(\varphi) &= 2(|\operatorname{tr}(M)| - S - T) + 2(2S) + 2T\\ &= 2(|\operatorname{tr}(M)| + S). \end{aligned}$$

In particular, we get that $R(\varphi) \in 2\mathbb{N}$. Taking the parity of $\operatorname{tr}(M)$ into account, we can further determine the possible Reidemeister numbers:

$$R(\varphi) \in \begin{cases} 4\mathbb{N} + 2S & \text{if } \operatorname{tr}(M) \equiv 0 \pmod 2,\\ 4\mathbb{N} + 2S - 2 & \text{if } \operatorname{tr}(M) \equiv 1 \pmod 2, \end{cases}$$

where

$$S \le O(\mathbb{1}_2 - M, d) \le \begin{cases} 4 & \text{if } \operatorname{tr}(M) \equiv 0 \pmod 2,\\ 1 & \text{if } \operatorname{tr}(M) \equiv 1 \pmod 2. \end{cases}$$

There is one special case, however. If $M \equiv \mathbb{1}_2 \mod 2$ all entries of $\mathbb{1}_2 - M$ will be multiples of 2; so $|\det(\mathbb{1}_2 - M)| = |\operatorname{tr}(M)| \in 4\mathbb{N}$ and therefore $R(\varphi) \in 8\mathbb{N} + 2S$.

For a fixed group Γ in this family (i.e. a fixed 4-tuple of parameters (k_1, k_2, k_3, k_4)), an automorphism $\varphi \in \operatorname{Aut}(\Gamma)$ is uniquely determined by the matrix $M \in \operatorname{GL}_2(\mathbb{Z})$ and the vector $d \in \mathbb{Z}^2$. Our goal is to find out, for each group in the family (or equivalently, for each tuple (k_1, k_2, k_3, k_4)), which M and d satisfy conditions (a) - (d) and thus produce an automorphism.

Conditions (a) - (c) are actually conditions over $\mathbb{Z}_2$, and none of the parameters k_i appear in condition (d). Therefore, only the parity of the k_i will play a role, so we need to check 16 cases, each corresponding to an element of $\mathbb{Z}_2^4$. Furthermore, a group with parameters (k_1, k_2, k_3, k_4) is isomorphic to the group with parameters $(-k_1, k_3, k_2, k_4)$, which allows us to omit the cases $(0,1,0,0)$, $(0,1,0,1)$, $(1,1,0,0)$ and $(1,1,0,1)$, leaving only 12 cases.

Rather than trying to find all couples (M, d) (of which there are likely to be infinitely many), we can start by finding all couples $(\bar{M}, \bar{d}) \in \mathrm{GL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2$ satisfying conditions (a)-(c).

The function MakeList defined in algorithm 1 does exactly this. Moreover, it assigns to every couple a set R, which is the set of possible Reidemeister numbers the corresponding automorphisms can have. The results can be found in tables 2 to 13. The Reidemeister spectrum of a group is a subset of (or the entirety of) the union of all these sets R.

Next, for each quadruplet of parameters, we tried to find a family of automorphisms whose Reidemeister numbers produce the union of these sets R. We succeeded in this for every choice of parameters, hence the Reidemeister spectrum always equals the union of the R. These automorphisms and their Reidemeister spectra, for all (k_1, k_2, k_3, k_4), can be found in table 14. For the sake of brevity, we omitted ∞ from the spectra in this table.

We may thus conclude that, depending on the parity of the parameters k_1, k_2, k_3 and k_4, the Reidemeister spectrum is $2\mathbb{N} \cup \{\infty\}$, $4\mathbb{N} \cup \{\infty\}$, $(4\mathbb{N} - 2) \cup \{\infty\}$ or $(2\mathbb{N} + 2) \cup \{\infty\}$. Note that all almost-Bieberbach groups have parameters with parities $(0, 0, 0, 1)$ and therefore have spectrum $2\mathbb{N} \cup \{\infty\}$.

Algorithm 1 MakeList function

function MakeList(k_1, k_2, k_3, k_4)
    AutList $:= \varnothing$
    for $\bar{M} \in \mathrm{GL}_2(\mathbb{Z}_2)$, $\bar{d} \in \mathbb{Z}_2^2$ **do**
        if conditions (1), (2), (3) are met **then**
            $S := 0$
            for $\bar{z} \in \mathbb{Z}_2^2$ **do**
                if $\bar{z}$ satisfies (i') but not (ii') **then**
                    $S := S + 1$
                end if
            end for
            if $\mathrm{tr}(M) \equiv 0 \mod 2$ **then**
                if $M \equiv \mathbb{1}_2 \mod 2$ **then**
                    $R := 8\mathbb{N} + 2S$
                else
                    $R := 4\mathbb{N} + 2S$
                end if
            else
                $R := 4\mathbb{N} + 2S - 2$
            end if
            AutList := AutList $\cup \left\{(\bar{M}, \bar{d}, R)\right\}$
        end if
    end for
    return AutList
end function

6 Spectra of 4D almost-Bieberbach groups

We already determined in section 4 which families of four-dimensional almost-crystallographic groups do not have the R_∞-property. In [Dekimpe 1996] it is determined which groups among these families are almost-Bieberbach groups. We use the presentations from section 4.

Family 1. Every group in this family is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class 2. In [Dekimpe et al. 2019b, Section 3.2] it was shown that the Reidemeister spectrum of such group is always $4\mathbb{N} \cup \{\infty\}$.

Family 3. The almost-Bieberbach groups in this family are those with $(k_1, k_2, k_3, k_4) = (2k, 0, 0, 1)$ for some $k \in \mathbb{N}$. An automorphism $\varphi = \xi_{(d,D)}$ with $R(\varphi) < \infty$ must be of the form

$$
\begin{aligned}
\varphi(e_1) &= e_1^{-1},\\
\varphi(e_2) &= e_1^{l} e_2^{-1},\\
\varphi(e_3) &= e_1^{k(m_1m_2+m_1d_2-m_2d_1)} e_3^{m_1} e_4^{m_2},\\
\varphi(e_4) &= e_1^{k(m_3m_4+m_3d_2-m_4d_1)} e_3^{m_3} e_4^{m_4},\\
\varphi(\alpha) &= e_1^{kd_1d_2-1} e_3^{d_1} e_4^{d_2} \alpha,
\end{aligned}
$$

with m_1, m_2, m_3, m_4, d_1, d_2, $l \in \mathbb{Z}$ and $m_1m_4 - m_2m_3 = -1$. Then D_* is of the form

$$
D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & m_1 & m_3 \\ 0 & 0 & m_2 & m_4 \end{pmatrix}.
$$

Using theorem 3.5, we find that $R(\varphi) = 4|m_1 + m_4| \in 4\mathbb{N}$. Now, take the automorphism φ_m given by

$$
\begin{aligned}
\varphi_m(e_1) &= e_1^{-1}, & \varphi_m(e_4) &= e_1^{km} e_3 e_4^{m},\\
\varphi_m(e_2) &= e_2^{-1}, & \varphi_m(\alpha) &= e_1^{-1}\alpha,\\
\varphi_m(e_3) &= e_4,
\end{aligned}
$$

with $m \in \mathbb{N}$. Then $R(\varphi_m) = 4m$ and hence $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$.

Family 4. The almost-Bieberbach groups in this family are those where either $(k_1, k_2, k_3, k_4) = (k, 0, 0, 0)$ with $k \in \mathbb{N}$ or $(k_1, k_2, k_3, k_4) = (2k, 1, 0, 0)$ with $k \in \mathbb{N}$. In the former case, such almost-Bieberbach group can be seen as an internal semidirect product $H_k \rtimes \mathbb{Z}$, where $H_k = \langle e_1, e_3, e_4 \rangle$ and $\mathbb{Z} = \langle \alpha \rangle$. Similarly, in the latter case, a group is an internal semidirect product $H_{2k} \rtimes \mathbb{Z}$.

Both of these semidirect products were studied in [Dekimpe et al. 2019b, Proposition 5.23], their Reidemeister spectra are respectively $4\mathbb{N} \cup \{\infty\}$ and $8\mathbb{N} \cup \{\infty\}$.

Family 5. The almost-Bieberbach groups in this family are those where $(k_1, k_2, k_3, k_4) = (k, 0, 0, 1)$ with $k \in \mathbb{N}$. An automorphism $\varphi = \xi_{(d,D)}$ with $R(\varphi) < \infty$ must be of the form

$$
\begin{aligned}
\varphi(e_1) &= e_1^{-1},\\
\varphi(e_2) &= e_2^{-1} e_1^{k(2m_1m_2+2m_1d_2-2m_2d_1-m_2-d_2)-2l},\\
\varphi(e_3) &= e_2^{m_1} e_3^{-1+2m_1} e_4^{m_2} e_1^{l},\\
\varphi(e_4) &= e_2^{m_3} e_3^{2m_3} e_4^{1+2m_4} e_1^{k(2m_3m_4+m_3d_2+m_3-2m_4d_1-d_1)},\\
\varphi(\alpha) &= e_2^{d_1} e_3^{2d_1} e_4^{d_2} e_1^{kd_1d_2-1}\alpha,
\end{aligned}
$$

with m_1, m_2, m_3, m_4, d_1, d_2, $l \in \mathbb{Z}$ and $m_1 - m_4 + 2m_1m_4 - m_2m_3 = 0$. Then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & -1+2m_1 & 2m_3 \\ 0 & 0 & m_2 & 1+2m_4 \end{pmatrix}.$$

Using theorem 3.5, we find that $R(\varphi) = 8|m_1 + m_4| \in 8\mathbb{N} \cup \{\infty\}$. Now, take the automorphism φ_m given by

$$\begin{aligned} \varphi_m(e_1) &= e_1^{-1}, & \varphi_m(e_4) &= e_1^{km} e_2^{m} e_3^{2m} e_4, \\ \varphi_m(e_2) &= e_1^{k(2m-1)} e_2^{-1}, & \varphi_m(\alpha) &= e_1^{-1}\alpha, \\ \varphi_m(e_3) &= e_2^{m} e_3^{2m-1} e_4, \end{aligned}$$

with $m \in \mathbb{N}$. Then $R(\varphi_m) = 8m$ and hence $\mathrm{Spec}_R(\Gamma) = 8\mathbb{N} \cup \{\infty\}$.

7 Conclusion

We have determined which (non-crystallographic) almost-crystallographic groups of dimension 4 admit the R_∞ property, and calculated the Reidemeister spectra of the non-crystallographic 3-dimensional almost-crystallographic groups, as well as the spectra of the non-crystallographic 4-dimensional almost-Bieberbach groups. Together with the results of [Dekimpe et al. 2019a], this completes the calculation of the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and of the 4-dimensional almost-Bieberbach groups.

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$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+8$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$

Table 2: MakeList$(0, 0, 0, 0)$

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$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}-2$

Table 3: $\textsc{MakeList}(0,0,0,1)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$

Table 4: $\textsc{MakeList}(0,0,1,0)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$8\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$

Table 5: $\textsc{MakeList}(0,0,1,1)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$8\mathbb{N}$

Table 6: $\textsc{MakeList}(0,1,1,0)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$8\mathbb{N}$

Table 7: MakeList$(0,1,1,1)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+6$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}-2$

Table 8: MakeList$(1,0,0,0)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$

Table 9: MakeList$(1,0,0,1)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}-2$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+6$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}-2$

Table 10: MakeList$(1,0,1,0)$

$\bar{M}$	$\bar{d}$	R
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$8\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	$4\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$4\mathbb{N}$

Table 11: MakeList$(1,0,1,1)$

$\bar{M}$	$\bar{d}$	R
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}+2$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$8\mathbb{N}+2$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}$

Table 12: MakeList$(1, 1, 1, 0)$

$\bar{M}$	$\bar{d}$	R
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}+2$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}-2$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$8\mathbb{N}+6$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4\mathbb{N}-2$

Table 13: MakeList$(1, 1, 1, 1)$

(k_1, k_2, k_3, k_4)	M	d	$R(\varphi)$	$\mathrm{Spec}_R(\Gamma)$
$(0, 0, 0, 0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$4m$	$4\mathbb{N}$
$(0, 0, 0, 1)$	$\begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N}$
$(0, 0, 1, 0)$	$\begin{pmatrix} 1 & 1 \\ 2m & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$4m$	$4\mathbb{N}$
$(0, 0, 1, 1)$	$\begin{pmatrix} 1 & 1 \\ 2m & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4m$	$4\mathbb{N}$
$(0, 1, 1, 0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$4m$	$4\mathbb{N}$
$(0, 1, 1, 1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4m$	$4\mathbb{N}$
$(1, 0, 0, 0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$4m-2$	$4\mathbb{N}-2$
$(1, 0, 0, 1)$	$\begin{pmatrix} 1 & 1 \\ m & m-1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(1, 0, 1, 0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$4m-2$	$4\mathbb{N}-2$
$(1, 0, 1, 1)$	$\begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(1, 1, 1, 0)$	$\begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(1, 1, 1, 1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$4m-2$	$4\mathbb{N}-2$

Table 14: Automorphisms and Reidemeister spectra and for all (k_1, k_2, k_3, k_4)