2. Vectors

- vector notation
- vector operations
- linear, affine functions
- norm, distance, angle
- standard deviation, correlation
- complexity

Vector

a *vector* is a collection of elements written as

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{or} \quad a = (a_1, a_2, \ldots, a_n)$$

- a_i is the ith *element* (*entry, coefficient, component*) of vector a
- i is the *index* of the ith element a_i
- number of elements n is the *size* (*length, dimension*) of the vector
- a vector of size n is called an n*-vector*
- example of a 4-vector:

$$a = \begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ 7.2 \end{bmatrix} = (-1.1, 0.0, 3.6, 7.2), \qquad a_3 = 3.6$$

Notes and conventions

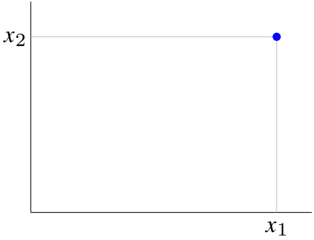
- $\mathbb{R}^n$ is the set of n-vectors with real entries
- $a \in \mathbb{R}^n$ means a is n-vector with real entries
- two n-vectors a and b are equal, denoted as $a = b$, if $a_i = b_i$ for all i
- warning: a_i can refer to an ith vector in a collection of vectors
  - in this case, we use $(a_i)_j$ to denote the jth entry of vector a_i
  - example: if $a_2 = (-1, 2, -5)$, then $(a_2)_3 = -5$

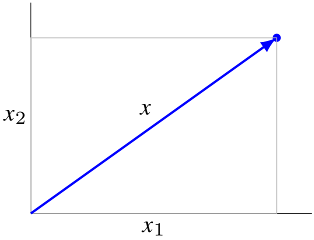
Conventions

- parentheses are also used instead of rectangular brackets to represent a vector
- other notations exist to distinguish vectors from numbers (*e.g.*, $\boldsymbol{a}, \vec{a}, \mathbf{a}$)
- conventions vary; be prepared to distinguish scalars from vectors

Geometric interpretation: location and displacement

- location (position): coordinates of a point in 2-D (plane) or 3-D space
- displacement: vector represents the change in position from one point to another (shown as an arrow in plane or 3-D space)





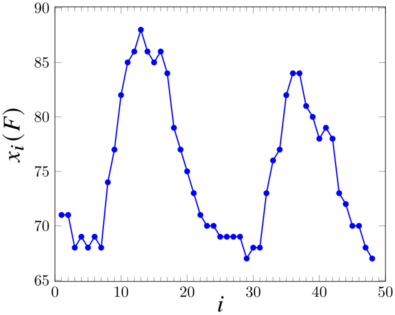
- other quantities that have direction and magnitude (velocity, force vector, ...)

Examples of vectors

Time series or signal

elements of n-vector are values of some quantity at n different times

- hourly temperature over a period of n hours



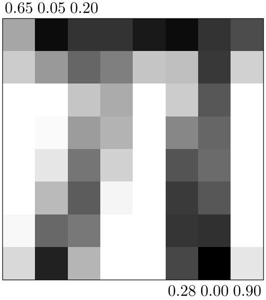
- audio signal: entries give the value of acoustic pressure at equally spaced times

Examples of vectors

Color: 3-vector can represent a color, with RGB intensity values

Monochrome (black and white) image

grayscale values of $M \times N$ pixels stored as MN-vector (row-wise or column-wise)



$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{62} \\ x_{63} \\ x_{64} \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.05 \\ 0.20 \\ \vdots \\ 0.28 \\ 0.00 \\ 0.90 \end{bmatrix}$$

Color image: $3MN$-vectors with R, G, B values of the MN pixels

Video: vector of size KMN represents K monochrome images of $M \times N$ pixels

Examples of vectors

Quantities

- elements of n-vector represent quantities of n resources or products
- sign indicates whether quantity is held or owed, produced or consumed, ...
- example: *bill of materials* is the list of resources (items) required to build a product represented as an n-vector whose entries give the amounts resources required

Portfolio vector

- n-vector s can represent stock portfolio (*e.g.*, investment in n assets)
  - assets can be stocks, bonds, cash, commodities (*e.g.*, gold), real estate ...
- s_i is the number of shares of asset i held (or invested in asset i)
- elements can be the no. of shares, dollar values, fractions of total dollar amount
- shares you owe another party (short positions) are represented by negative values

Examples of vectors

Daily return

- daily fractional return of a stock for a period of n trading days
- example: return time series vector $(-0.022, +0.014, +0.004)$ means stock price
  - went down 2.2% on the first day
  - then up 1.4% the next day
  - and up again 0.4% on the third day

Cash flow

- cash flow: payments into and out of an entity over n periods
- example: vector $(1000, -10, -10, -10, -1010)$ represents
  - a one year loan of 1000
  - with 1% interest only payments made each period (*e.g.*, quarter)
  - and the principal and last interest payment at the end

Examples of vectors

Word count vectors

- vector represents a document
- size of vector is the number of words in a dictionary
- word *count vector:* entry i is the number of times word i occurs in document
- word *histogram:* entry i is frequency of word i in document (in percentage)

Example: *word count vectors are used in computer-based document analysis; each entry of the word count vector represents the number of times the associated dictionary word appears in the document*

$$\begin{array}{l} \text{word} \\ \text{in} \\ \text{number} \\ \text{horse} \\ \text{document} \end{array} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Examples of vectors

Feature vector

- collects together n different quantities that relate to a single object
- entries are called the *features* or *attributes*

Examples

- age, height, weight, blood pressure, gender, etc., of patients
- square footage, number of bedrooms, list price, etc., of houses in an inventory

Notes

- vector elements can represent very different quantities, in different units
- can contain categorical features (*e.g.*, 1/0 for house/condo)
- ordering has no particular meaning

Row vector and transpose

an *row* vector b of size n with entries $b_1, \ldots, b_n$ has the form:

$$b = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}$$

- all vectors are column vectors unless otherwise stated
- other notation exists, *e.g.*, $b = [b_1, b_2, \ldots, b_n]$ (we will not use)

Transpose: the *transpose* of an n-column vector a is the row vector a^T:

$$a^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^T = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$$

- $(\cdot)^T$ is transpose operation
- $(a^T)^T = a$ (transpose of row vector is a column vector)

Block vectors, subvectors

Stacking

- vectors can be *stacked* (*concatenated*) to create larger vectors
- stacking vectors b, c, d of size m, n, p gives an $(m+n+p)$-vector

$$a = \begin{bmatrix} b \\ c \\ d \end{bmatrix} = (b, c, d) = \big(b_1, \ldots, b_m, c_1, \ldots, c_n, d_1, \ldots, d_p\big)$$

- b, c, and d are called *subvectors* or *slices* of a
- example: if $a = 1$, $b = (2, -1)$, $c = (4, 2, 7)$, then $(a, b, c) = (1, 2, -1, 4, 2, 7)$

Subvectors slicing

- colon (:) notation is used to define subvectors (slices) of a vector
- for vector a, we define $a_{r:s} = (a_r, \ldots, a_s)$
- example: if $a = (1, -1, 2, 0, 3)$, then $a_{2:4} = (-1, 2, 0)$

Special vectors

Zero vector and ones vector

$$0 = (0, 0, \ldots, 0), \quad \mathbf{1} = (1, 1, \ldots, 1)$$

size follows from context (if not, we add a subscript and write $0_n, \mathbf{1}_n$)

Unit vectors

- there are n *unit vectors* of size n, denoted by $e_1, e_2, \ldots, e_n$:

$$(e_i)_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

- the ith unit vector is zero except its ith element which is 1
- example: for $n = 3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- the size of e_i follows from context (or should be specified explicitly)

Sparsity

- a vector is *sparse* if many of its entries are 0
- can be stored and manipulated efficiently on a computer
- $\mathbf{nnz}(x)$ is number of entries that are nonzero
- examples:
  - $x = 0$ with $\mathbf{nnz}(x) = 0$
  - $x = e_i$ (unit vectors), $\mathbf{nnz}(x) = 1$
  - $x = (0, 0, 1, 0, 0, 0, -2, 0, 5, 0, 0)$, $\mathbf{nnz}(x) = 3$
- sparse vectors arise in many applications

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Addition and subtraction

for n-vectors a and b,

$$a + b = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}, \quad a - b = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}$$

Example

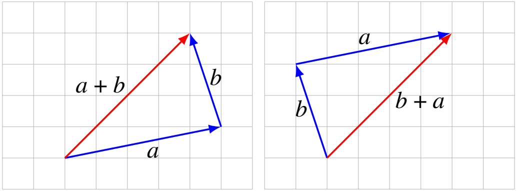
$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

Properties: for vectors a, b of equal size

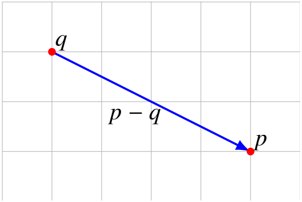
- commutative: $a + b = b + a$
- associative: $a + (b + c) = (a + b) + c$

Geometric interpretation: displacements addition

- if a and b are displacements, $a + b$ is the net displacement



- displacement from point q to point p is $p - q$



Scalar-vector multiplication

for scalar β and n-vector a,

$$\beta \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \beta a_1 \\ \beta a_2 \\ \vdots \\ \beta a_n \end{bmatrix}$$

example:

$$(-2) \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}$$

Properties: for vectors a, b of equal size, scalars β, γ

- commutative: $\beta a = a\beta$
- associative: $(\beta\gamma)a = \beta(\gamma a)$, we write as $\beta\gamma a$
- distributive with scalar addition: $(\beta + \gamma)a = \beta a + \gamma a$
- distributive with vector addition: $\beta(a + b) = \beta a + \beta b$

Linear combination

a *linear combination* of vectors $a_1, \ldots, a_m$ is a sum of scalar-vector products:

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_m a_m$$

- scalars $\beta_1, \ldots, \beta_m$ are the *coefficients* of the linear combination
- example: any n-vector b can be written as

$$b = b_1 e_1 + \cdots + b_n e_n$$

Special linear combinations

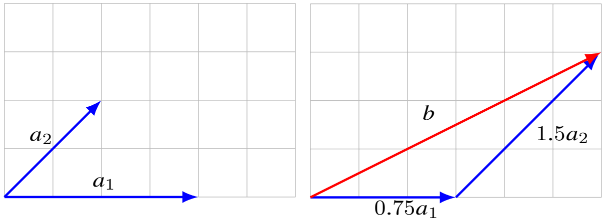
- *affine combination*: when $\beta_1 + \cdots + \beta_m = 1$
- *convex combination* (*weighted average*): when $\beta_1 + \cdots + \beta_m = 1$ and $\beta_i \geq 0$

Example: combination of displacements

- vector a represents a displacement
- for $\beta > 0$, βa is displacement in same direction of a, with magnitude scaled by β
- for $\beta < 0$, βa is displac. in the opposite direction of a, with mag. scaled by $|\beta|$

Example

$$b = 0.75 a_1 + 1.5 a_2$$

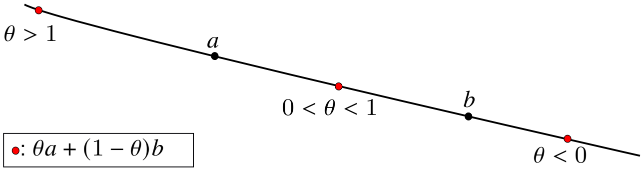


Line segment

any point on the line passing through distinct a and b can be written as

$$c = \theta a + (1 - \theta) b$$

- θ is a scalar
- an affine combination
- for $0 \leq \theta \leq 1$, point c lie on the segment between a and b



Addition and multiplication examples

Word count

- a and b are word count vectors (using the same dictionary) for two documents
- $a + b$ is the word count vector of the document combining the original two
- $a - b$ how many more times each word appears in 1st document compared to 2nd

Audio mixing

- $a_1, \dots, a_m$ are vectors representing audio signals over the same period of time
- βa_i is the same audio signal, but changed in volume (loudness) by the factor $|\beta_i|$
- linear combination $\beta_1 a_1 + \cdots + \beta_m a_m$ is a mixture of the audio tracks

Portfolio trading

- s is n-vector giving no. of shares of n assets in a portfolio
- b is n-vector giving no. of shares of assets that we buy ($b_i > 0$) or sell ($b_i < 0$)
- after trading, our portfolio is $s + b$, which is called the *trade vector* or *trade list*

Inner product

the *inner product* (or *dot product*) of two n-vectors a, b is

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- a scalar
- example:

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = (-1)(1) + (2)(0) + (2)(-3) = -7$$

- other notation exists: $\langle a, b \rangle$, $\langle a \mid b \rangle$, $a \cdot b$

Properties of inner product

for vectors a, b, c of equal size, scalar γ

- nonnegativity: $a^T a \geq 0$, and $a^T a = 0$ if and only if $a = 0$.
- commutative: $a^T b = b^T a$
- associative with scalar multiplication: $(\gamma a)^T b = \gamma(a^T b)$
- distributive with vector addition: $(a + b)^T c = a^T c + b^T c$

Useful combination: for vectors a, b, c, d

$$(a + b)^T (c + d) = a^T c + a^T d + b^T c + b^T d$$

Block vectors: if vectors a, b are block vectors, and corresponding blocks $a_i, b_i \in \mathbb{R}^{n_i}$ have the same sizes (they conform),

$$a^T b = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}^T \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} = a_1^T b_1 + \cdots + a_k^T b_k$$

Simple examples

Inner product with unit vector

$$e_i^T a = a_i$$

Differencing

$$(e_i - e_j)^T a = a_i - a_j$$

Sum and average

$$\mathbf{1}^T a = a_1 + a_2 + \cdots + a_n$$

$$\operatorname{avg}(a) = \frac{a_1 + a_2 + \cdots + a_n}{n} = \left(\frac{1}{n}\mathbf{1}\right)^T a$$

Inner product examples

Weights, features, scores

- vectors of features f and weights w
- $w^T f = w_1 f_1 + w_2 f_2 + \cdots + w_n f_n$ is the total score
- example: features are associated with a loan applicant (*e.g.*, age, income, . . .)
  - we can interpret $s = w^T f$ as a credit score
  - we can interpret w_i as the weight given to feature i in forming the score

Price quantity (cost)

- vectors of prices p and quantities q of n goods
- $p^T q = p_1 q_1 + p_2 q_2 + \cdots + p_n q_n$ is the total cost

Speed time

- vehicle travels over n segments with constant speed in each segment
- n-vector s gives the speed in the segments
- n-vector t gives the times taken to traverse the segments
- $s^T t$ is the total distance traveled

Inner product examples

Polynomial evaluation

- n-vector c represents the coefficients of a polynomial p of degree $n-1$ or less:

$$p(x) = c_1 + c_2 x + \cdots + c_{n-1} x^{n-2} + c_n x^{n-1}$$

- let $z = (1, t, t^2, \ldots, t^{n-1})$ be the n-vector of powers of a scalar t
- $c^T z = p(t)$ is the value of the polynomial p at the point t

Discounted total

- cash flow vector c where c_i is value at period i
- r is interest rate and $d = (1, 1/(1+r), \ldots, 1/(1+r)^{n-1})$
- $d^T c = c_1 + c_2/(1+r) + \ldots, c_n/(1+r)^{n-1}$ is the discounted total of cash flow
  - money received in the future is worth less than money received today
- called *net present value* (NPV) with interest rate r

Inner product examples

Portfolio value

- s is an n-vector of holdings in shares of a portfolio of n assets
- p is an n-vector for the prices of the assets
- $p^T s$ is the total (or net) value of the portfolio

Portfolio return

- portfolio vector x with x_i representing dollar value of asset i
- r_i is rate (fraction) of return of asset i over the investment period:

$$p_i^{\text{final}} = (1 + r_i) p_i^{\text{init}}, \qquad r_i = \frac{p_i^{\text{final}} - p_i^{\text{init}}}{p_i^{\text{init}}}$$

  p_i^{init} and p_i^{final} are the prices of asset i at the beginning and end of the period

- $r^T x = r_1 x_1 + \cdots + r_n x_n$ is total return in dollars over the period
- if w is the fractional (dollar) holdings of our portfolio, then $r^T w$ is rate of return
  - example: if $r^T w = 0.09$, then our portfolio return is 9%; if we had invested 10000 initially, we would have earned $\$900$

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- vector operations
- **linear, affine functions**
- norm, distance, angle
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Linear functions

- $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ means f is a *function* mapping n-vectors to numbers
- example: $f(x) = x_1 + x_2 - x_4^2$ ($f : \mathbb{R}^4 \longrightarrow \mathbb{R}$)

Linear functions: f is *linear* if it satisfies the superposition property

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all scalars α, β, and all n-vectors x, y

Extension: if f is linear, then

$$f\left(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m\right) = \alpha_1 f\left(u_1\right) + \alpha_2 f\left(u_2\right) + \cdots + \alpha_m f\left(u_m\right)$$

for all n-vectors $u_1, \ldots, u_m$ and all scalars $\alpha_1, \ldots, \alpha_m$

Inner product function

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

the inner product function is linear:

$$\begin{aligned} f(\alpha x + \beta y) &= a^T(\alpha x + \beta y) \\ &= a^T(\alpha x) + a^T(\beta y) \\ &= \alpha(a^T x) + \beta(a^T y) \\ &= \alpha f(x) + \beta f(y) \end{aligned}$$

All linear functions are inner products

- if $f : \mathbb{R}^n \to \mathbb{R}$ is linear, then $f(x) = a^T x$ for some (unique) a
- this follows from

$$\begin{aligned} f(x) &= f\left(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n\right) \\ &= x_1 f\left(e_1\right) + x_2 f\left(e_2\right) + \cdots + x_n f\left(e_n\right) = a^T x \end{aligned}$$

  with $a = (f(e_1), \ldots, f(e_n))$

Example

- mean or average value of an n-vector is linear

$$f(x) = \mathrm{avg}(x) = (x_1 + x_2 + \cdots + x_n)\,/n = a^T x$$

  where $a = (1/n, \ldots, 1/n) = (1/n)\mathbf{1}$ (sometimes denoted $\bar{x}$ or μ_x)

- maximum element func. $f(x) = \max\{x_1, \ldots, x_n\}$, is not linear (unless $n = 1$)
  - we can show this by a counterexample for $n = 2$
  - take $x = (1, -1), y = (-1, 1), \alpha = 1/2, \beta = 1/2$
  - then

$$f(\alpha x + \beta y) = 0 \neq \alpha f(x) + \beta f(y) = 1$$

Affine functions

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *affine* if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all n-vectors and scalars $\alpha + \beta = 1$

- extension: if f is affine, then

$$f(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m) = \alpha_1 f(u_1) + \alpha_2 f(u_2) + \cdots + \alpha_m f(u_m)$$

  for all n-vectors $u_1, \ldots, u_m$ and all scalars $\alpha_1, \ldots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = 1$

- every affine function f can be expressed as $f(x) = a^T x + b$ with

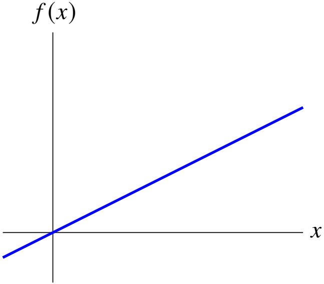
$$\begin{aligned} a &= \big(f(e_1) - f(0), f(e_2) - f(0), \ldots, f(e_n) - f(0)\big) \\ b &= f(0) \end{aligned}$$

- an affine function is a linear function plus a constant

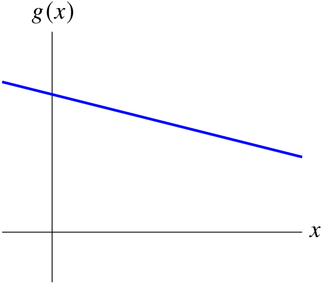
- often affine functions are called linear (which is mathematically not true)

Linear versus affine functions

f is linear



g is affine



linear functions pass through zero since $f(0) = 0$

First-order Taylor (affine) approximation

first-order *Taylor approximation* of $f : \mathbb{R}^n \to \mathbb{R}$, near point z:

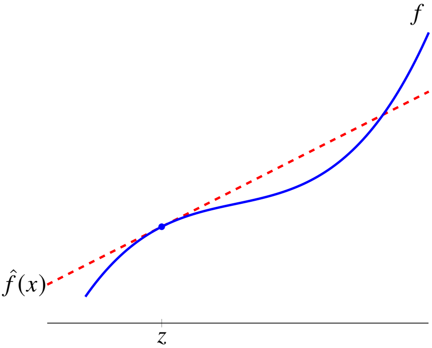
$$\begin{aligned}\hat{f}(x) &= f(z) + \frac{\partial f}{\partial x_1}(z)\,(x_1 - z_1) + \cdots + \frac{\partial f}{\partial x_n}(z)\,(x_n - z_n) \\ &= f(z) + \nabla f(z)^T (x - z)\end{aligned}$$

- n-vector $\nabla f(z)$ is the *gradient* of f at z,

$$\nabla f(z) = \left(\frac{\partial f}{\partial x_1}(z), \ldots, \frac{\partial f}{\partial x_n}(z)\right)$$

- $\hat{f}(x)$ is very close to $f(x)$ when x_i are all near z_i
- sometimes written $\hat{f}(x; z)$, to indicate that z where the approximation appear
- $\hat{f}$ is an affine function of x
- often called *linear approximation* of f near z, even though it is in general affine

Example with one variable



$$\hat{f}(x) = f(z) + f'(z)(x - z)$$

Example with two variables

$$f(x_1, x_2) = x_1 - 3x_2 + e^{2x_1+x_2-1}$$

- gradient:

$$\nabla f(x) = \left[\begin{array}{c} 1 + 2e^{2x_1+x_2-1} \\ -3 + e^{2x_1+x_2-1} \end{array}\right]$$

- Taylor approximation around $z = 0$:

$$\begin{aligned} \hat{f}(x) &= f(0) + \nabla f(0)^T(x - 0) \\ &= e^{-1} + (1 + 2e^{-1})x_1 + (-3 + e^{-1})x_2 \end{aligned}$$

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Euclidean norm

Euclidean norm of vector $a \in \mathbb{R}^n$:

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{a^T a}$$

- reduces to absolute value $|a|$ when $n = 1$
- measures the magnitude of a
- examples

$$\left\|\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right\| = \sqrt{9} = 3, \quad \left\|\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\| = 1$$

Properties

Positive definiteness

$$\|a\| \geq 0 \quad \text{for all } a, \quad \|a\| = 0 \quad \text{only if } a = 0$$

Homogeneity

$$\|\beta a\| = |\beta|\|a\| \quad \text{for all vectors } a \text{ and scalars } \beta$$

Triangle inequality

$$\|a + b\| \leq \|a\| + \|b\| \text{ for all vectors } a \text{ and } b \text{ of equal length}$$

- any real function that satisfies these properties is called a (general) *norm*
- Euclidean norm is often written as $\|a\|_2$ to distinguish from other norms
  - also called 2-norm or ℓ_2-norm
- examples of other norms are the one-norm and infinity-norm:

$$\|a\|_1 = |a_1| + |a_2| + \cdots + |a_n|$$
$$\|a\|_\infty = \max\{|a_1|, |a_2|, \ldots, |a_n|\}$$

Norm of block vector and norm of sum

Norm of block vector: for vectors a, b, c,

$$\left\| \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\| = \sqrt{\|a\|^2 + \|b\|^2 + \|c\|^2} = \|(\|a\|, \|b\|, \|c\|)\|$$

Norm of sum: for vectors a, b,

$$\|a + b\| = \sqrt{\|a\|^2 + 2a^T b + \|b\|^2}$$

Cauchy-Schwarz inequality

$$|a^Tb| \leq \|a\|\|b\| \quad \text{for all } a, b \in \mathbb{R}^n$$

moreover, equality $|a^Tb| = \|a\|\|b\|$ holds if:

- $a = 0$ or $b = 0$; in this case $a^Tb = 0 = \|a\|\|b\|$
- $b = \gamma a$ for some $\gamma > 0$; in this case

$$0 < a^Tb = \gamma\|a\|^2 = \|a\|\|b\|$$

- $b = -\gamma a$ for some $\gamma > 0$; in this case

$$0 > a^Tb = -\gamma\|a\|^2 = -\|a\|\|b\|$$

Proof of Cauchy-Schwarz inequality

1. trivial if $a = 0$ or $b = 0$

2. assume $\|a\| = \|b\| = 1$; we show that $-1 \le a^T b \le 1$

$$\begin{aligned} 0 &\le \|a - b\|^2 \\ &= (a-b)^T(a-b) \\ &= \|a\|^2 - 2a^T b + \|b\|^2 \\ &= 2(1 - a^T b) \end{aligned}$$

with equality only if $a = b$

$$\begin{aligned} 0 &\le \|a + b\|^2 \\ &= (a+b)^T(a+b) \\ &= \|a\|^2 + 2a^T b + \|b\|^2 \\ &= 2(1 + a^T b) \end{aligned}$$

with equality only if $a = -b$

3. for general nonzero a, b, apply case 2 to the unit-norm vectors

$$\frac{1}{\|a\|}a, \quad \frac{1}{\|b\|}b$$

Triangle inequality from Cauchy-Schwarz inequality

for vectors a, b of equal size

$$\begin{aligned}
\|a+b\|^2 &= (a+b)^T(a+b) \\
&= a^Ta + b^Ta + a^Tb + b^Tb \\
&= \|a\|^2 + 2a^Tb + \|b\|^2 \\
&\leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 \\
&= (\|a\| + \|b\|)^2
\end{aligned}$$

- taking square roots gives the triangle inequality
- triangle inequality is an equality if and only if $a^Tb = \|a\|\|b\|$
- also note from line 3 that $\|a+b\|^2 = \|a\|^2 + \|b\|^2$ if $a^Tb = 0$

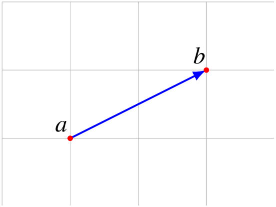
Euclidean distance

Euclidean distance between two vectors a and b,

$$\mathrm{dist}(a, b) = \|a - b\|$$

- agrees with ordinary distance for $n = 1, 2, 3$

2-D illustration

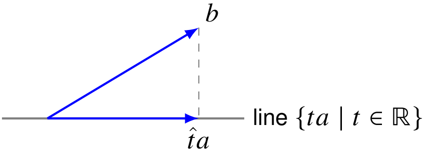


- when the distance between two vectors is small, we say they are 'close' or 'nearby', and when the distance is large, we say they are 'far'

Projection onto a vector

given two vectors $a, b \in \mathbb{R}^n$, with $a \neq 0$, the vector multiple ta closest to b has

$$\hat{t} = \frac{a^T b}{a^T a} = \frac{a^T b}{\|a\|^2}$$



Proof: squared distance between ta and b is

$$\|ta - b\|^2 = (ta - b)^T (ta - b) = t^2 a^T a - 2t a^T b + b^T b$$

derivative w.r.t. t is zero for

$$\hat{t} = \frac{a^T b}{a^T a} = \frac{a^T b}{\|a\|^2}$$

Geometric interpretation: $b - \hat{t}a \perp a$:

$$(b - \hat{t}a)^T a = 0 \Longrightarrow \hat{t} = \frac{a^T b}{\|a\|^2}$$

Feature distance and nearest neighbors

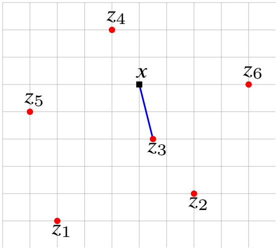
Feature distance

- let x and y be feature vectors for two entities
- $\|x - y\|$ is the *feature distance*; gives a measure of how different the objects are
  - example: features associated with patients in a hospital (weight, age, results of tests)
  - feature vector distance gives similarity between one patient case and another one

Nearest neighbor

- $z_1, \ldots, z_m$ is a list of vectors
- z_j is the nearest neighbor of x if

$$\|x - z_j\| \leq \|x - z_i\|, \quad i = 1, \ldots, m$$



Document dissimilarity

- if x_i represent histogram of word occurrence in document i
- $\|x_i - x_j\|$ measures the dissimilarity between documents

Example

- 5 Wikipedia articles: 'Veterans Day', 'Memorial Day', 'Academy Awards', 'Golden Globe Awards', 'Super Bowl'
- word count histograms, dictionary of 4423 words
- pairwise distances shown below

	Veterans Day	Memorial Day	Academy Awards	Golden Globe Awards	Super Bowl
Veterans Day	0	0.095	0.130	0.153	0.170
Memorial Day	0.095	0	0.122	0.147	0.164
Academy A.	0.130	0.122	0	0.108	0.164
Golden Globe A.	0.153	0.147	0.108	0	0.181
Super Bowl	0.170	0.164	0.164	0.181	0

Units for heterogeneous vector entries

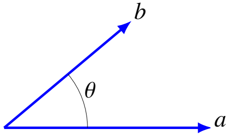
$$\|a - b\|^2 = (a_1 - b_1)^2 + \cdots + (a_n - b_n)^2$$

- suppose entries of vectors a_i, b_i represent different types of quantities
- choice of units for each entry affects the distance between a and b
- general rule: choose units so typical vector entries have similar ranges of values

Angle between vectors

the *angle* between nonzero real vectors a, b is defined as

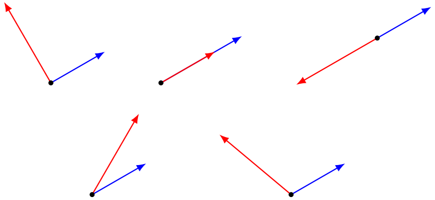
$$\theta = \angle(a, b) = \arccos\left(\frac{a^T b}{\|a\|\|b\|}\right)$$



- this is the unique value of $\theta \in [0, \pi]$ that satisfies $a^T b = \|a\|\|b\| \cos\theta$
- coincides with ordinary angle between vectors in 2-D and 3-D
- symmetric: $\angle(a, b) = \angle(b, a)$
- unaffected by positive scaling: $\angle(\alpha a, \beta b) = \angle(a, b)$ for +ve α, β

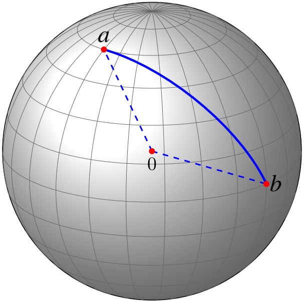
Classification of angles

$\theta = 0$	$a^Tb = \|a\|\|b\|$	aligned/parallel
$0 \leq \theta < \pi/2$	$a^Tb > 0$	acute angle
$\theta = \pi/2$	$a^Tb = 0$	orthogonal $(a \perp b)$
$\pi/2 < \theta \leq \pi$	$a^Tb < 0$	obtuse angle
$\theta = \pi$	$a^Tb = -\|a\|\|b\|$	anti-aligned/opposed



Example: spherical distance

if a, b are on sphere of radius R, distance along the sphere is $R\angle(a, b)$



Document dissimilarity by angles

- if n-vectors x_i are word counts for documents, their angle $\angle(x_i, x_j)$ can be used as a measure of document dissimilarity
- example: pairwise angles (in degrees) for 5 Wikipedia pages shown below

	Veterans Day	Memorial Day	Academy Awards	Golden globe Awards	Super Bowl
Veterans Day	0	60.6	85.7	87.0	87.7
Memorial Day	60.6	0	85.6	87.5	87.5
Academy A.	85.7	85.6	0	58.7	86.1
Golden Globe A.	87.0	87.5	58.7	0	86.0
Super Bowl	87.7	87.5	86.1	86.0	0

Norm of sum via angles

for vectors a and b we have

$$\begin{aligned}\|a+b\|^2 &= \|a\|^2 + 2a^T b + \|b\|^2 \\ &= \|a\|^2 + 2\|a\|\|b\|\cos\theta + \|b\|^2\end{aligned}$$

- if a and b are aligned $(\theta = 0)$, then $\|a+b\| = \|a\| + \|b\|$
- if a and b are orthogonal $(\theta = 90^\circ)$, then

$$\|a+b\|^2 = \|a\|^2 + \|b\|^2$$

  and $\|a+b\| = \sqrt{\|a\|^2 + \|b\|^2}$ (called the Pythagorean theorem)

Outline

- vector notation
- vector operations
- linear, affine functions
- norm, distance, angle
- **standard deviation, correlation**
- complexity

RMS value

the *root-mean-square* value of $a \in \mathbb{R}^n$ is the root of the average squared entry

$$\operatorname{rms}(x) = \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} = \frac{\|a\|}{\sqrt{n}}$$

- it is root of *mean-square value*: $\operatorname{ms} = (a_1^2 + \cdots + a_n^2)/n$
- RMS value useful for comparing sizes of vectors of different lengths
- $\operatorname{rms}(a)$ gives 'typical' value of $|a_i|$
- *e.g.*, $\operatorname{rms}(\alpha \mathbf{1}) = |\alpha|$ (independent of n)
- $\operatorname{rms}(a - b)$ is called the RMS *deviation* between a and b

Standard deviation

the *standard deviation* of $a \in \mathbb{R}^n$ is

$$\mathrm{std}(a) = \mathrm{rms}(a - \mathrm{avg}(a)\mathbf{1}) = \left\| a - ((\mathbf{1}^T a)/n)\mathbf{1} \right\| / \sqrt{n}$$

- std is RMS deviation from the average
- std gives us the 'typical' amount a vector entries deviate from their average (mean)
- $\tilde{a} = a - \mathrm{avg}(a)\mathbf{1}$ is called *de-meaned* vector (since $\mathrm{avg}(\tilde{a}) = 0$)
- other notation: μ and σ are often used for mean and standard deviation

Standard deviation formula

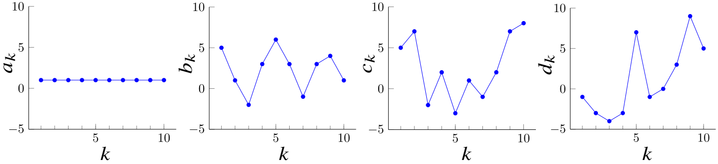
$$\mathrm{rms}(a)^2 = \mathrm{avg}(a)^2 + \mathrm{std}(a)^2$$

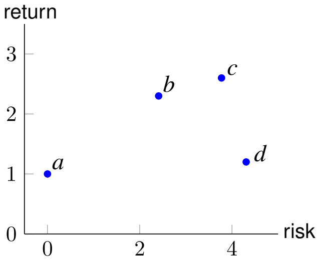
Proof

$$\begin{aligned}
\mathrm{std}(a)^2 &= \frac{\|a - \mathrm{avg}(a)\mathbf{1}\|^2}{n} \\
&= \frac{1}{n}\left(a - \frac{\mathbf{1}^T a}{n}\mathbf{1}\right)^T \left(a - \frac{\mathbf{1}^T a}{n}\mathbf{1}\right) \\
&= \frac{1}{n}\left(a^T a - \frac{(\mathbf{1}^T a)^2}{n} - \frac{(\mathbf{1}^T a)^2}{n} + \left(\frac{\mathbf{1}^T a}{n}\right)^2 n\right) \\
&= \frac{1}{n}\left(a^T a - \frac{(\mathbf{1}^T a)^2}{n}\right) \\
&= \mathrm{rms}(a)^2 - \mathrm{avg}(a)^2
\end{aligned}$$

Mean return and risk of investment

- vectors represent time series of returns on an investment (as a percentage)
- average value is (mean) return of the investment
- standard deviation measures variation around the mean (called risk)





Correlation coefficient

correlation coefficient (between a and b)

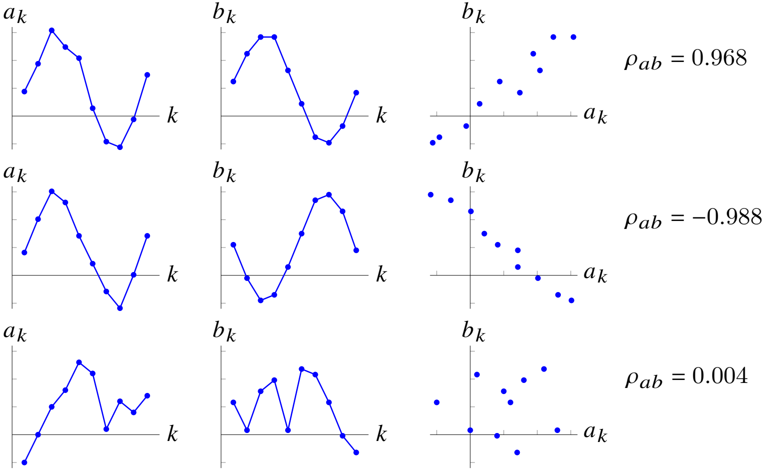
$$\rho = \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \|\tilde{b}\|}$$

where vectors $\tilde{a}$ and $\tilde{b}$ are de-meaned vectors ($\tilde{a} \neq 0, \tilde{b} \neq 0$):

$$\tilde{a} = a - \mathrm{avg}(a)\mathbf{1}, \quad \tilde{b} = b - \mathrm{avg}(b)\mathbf{1}$$

- $\rho = \cos \angle(\tilde{a}, \tilde{b})$ hence $-1 \leq \rho \leq 1$
- $\rho = 0$, a and b are uncorrelated
- $\rho > 0.8$ (or so), a and b are highly correlated
- $\rho < -0.8$ (or so), a and b are highly anti-correlated
- highly correlated "means" many a_i, b_i are both above (below) their means

Example



a_k
k
b_k
k
b_k
a_k
$\rho_{ab} = 0.968$
a_k
k
b_k
k
b_k
a_k
$\rho_{ab} = -0.988$
a_k
k
b_k
k
b_k
a_k
$\rho_{ab} = 0.004$

Examples

highly correlated vectors:

- rainfall time series at nearby locations
- daily returns of similar companies in same industry
- word count vectors of closely related documents (*e.g.*, same author, topic, ...)
- sales of shoes and socks (at different locations or periods)

approximately uncorrelated vectors

- unrelated vectors
- audio signals (even different tracks in multi-track recording)

(somewhat) negatively correlated vectors

- daily temperatures in Palo Alto and Melbourne

Properties and standardization

Properties of standard deviation

- *adding a constant:* $\mathrm{std}(a + \beta\mathbf{1}) = \mathrm{std}(a)$ for vector a and number β
- *multiplying by a scalar:* $\mathrm{std}(\beta a) = |\beta|\,\mathrm{std}(a)$ for vector a and number β
- *sum:* $\mathrm{std}(a + b) = \sqrt{\mathrm{std}(a)^2 + 2\rho\,\mathrm{std}(a)\,\mathrm{std}(b) + \mathrm{std}(b)^2}$ for vectors a, b

Standardization

- de-meaned vector of a in standard units is

$$z = \frac{1}{\mathrm{std}(a)}(a - \mathrm{avg}(a)\mathbf{1})$$

- z is called *standardized* or z*-scored* version of a ($\mathrm{avg}(z) = 0$ and $\mathrm{std}(z) = 1$)
- $z_4 = 1.4$ means a_4 is 1.4 standard deviations above the mean of a

Example: hedging investments

- a and b are time series of returns for two assets with the same return (average) μ, risk (standard deviation) σ, and correlation coefficient ρ
- $c = (a + b)/2$ is time series of returns for an investment with 50% in each asset
- this blended investment has the same return as the original assets, since

$$\operatorname{avg}(c) = \operatorname{avg}((a + b)/2) = (\operatorname{avg}(a) + \operatorname{avg}(b))/2 = \mu$$

- the risk (standard deviation) of this blended investment is

$$\operatorname{std}(c) = \sqrt{2\sigma^2 + 2\rho\sigma^2}/2 = \sigma\sqrt{(1 + \rho)/2}$$

- risk of the blended investment is never more than the risk of the original assets, and is smaller when the correlation of the original asset returns is smaller
- investing in two uncorrelated or -ve correlated assets is called *hedging*

Outline

- vector notation
- vector operations
- linear, affine functions
- norm, distance, angle
- standard deviation, correlation
- **complexity**

Floating point operation (FLOP)

Computer representation of numbers

- computers store (real) numbers in *floating-point format*
- number represented as 64 bits (0s and 1s), or 8 bytes (group of bits)
- each of 2^{64} sequences of bits corresponds to a specific number

Floating point operations

- 1 flop = one basic arithmetic operation ($+, -, *, /, \sqrt{}, \ldots$) in $\mathbb{R}$ (or complex $\mathbb{C}$)
- speed with which a computer can carry out flops is typically in 1-10 Gflop/s
- *complexity* of an operation is the number of flops required to carry it out
- flop is the unit of complexity when comparing algorithms; run time of the algorithm:

$$\text{run time} \approx \frac{\text{number of operations (flops)}}{\text{computer speed (flops per second)}}$$

this is a very crude and simplified model of complexity of algorithms

Dominant terms

- typically, complexity is highly simplified, dropping small or negligible terms
- dominant term: the highest-order term in the flop count

$$\frac{1}{3}n^3 + 100n^2 + 10n + 5 \approx \frac{1}{3}n^3$$

- order: the power in the dominant term

$$\frac{1}{3}n^3 + 10n^2 + 100 = \text{order } n^3 = \mathcal{O}(n^3)$$

- order is useful in understanding how the time to execute the computation will scale when the size of the operands changes

Complexity of vector operations

for vectors of size n

- $x + y$ needs n additions, so n flops
- scalar multiplication: n flops
- inner product: $2n - 1 \approx 2n$ flops
  - we simplify this to $2n$ (or even n) flops
- these operations are all order n

Sparse vectors: when x and/or y is sparse

- αx requires $\mathbf{nnz}(x)$ flops
- $x + y$ requires $\min\{\mathbf{nnz}(x), \mathbf{nnz}(y)\}$ flops
- if sparsity pattern do not overlap, $x + y$ requires zero flops
- $x^T y$ requires no more than $2\min\{\mathbf{nnz}(x), \mathbf{nnz}(y)\}$ flops

Complexity of norms

for n-vectors

- $\|x\|$ requires $2n$ flops
  - n multiplications (to square each entry)
  - $n-1$ additions (to add the squares)
  - one squareroot
- RMS value costs $2n$ (ignore two flops from division of $\sqrt{n}$)
- distance between two vectors costs $3n$ flops
- angle between them costs $6n$ flops
- de-meaning an n-vector requires $2n$ flops
  - n for forming the average
  - n flops for subtracting the average from each entry
- standard deviation costs $4n$ flops
  - $2n$ for computing the de-meaned vector
  - $2n$ for computing its RMS value
- correlation coefficient costs $10n$ flops to compute

References and further readings

- S. Boyd and L. Vandenberghe. *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares.* Cambridge University Press, 2018.

- L. Vandenberghe, *EE133A Lecture Notes,* University of California, Los Angeles.