9. Convex optimization problems

- convex sets
- convex functions
- operations preserving convexity
- basic properties
- convex problems

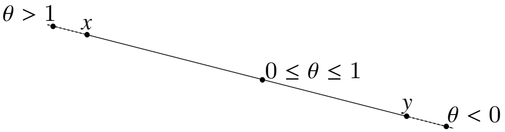
Line segment

Line through non-equal points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ has the form

$$\{\theta x + (1-\theta) y \mid \theta \in \mathbb{R}\}$$

Line segment between x and y:

$$\{\theta x + (1-\theta) y \mid \theta \in [0,1]\}$$

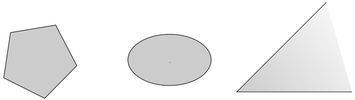


Convex sets

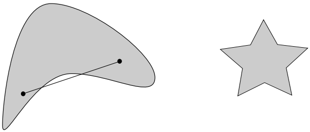
a set $C \subseteq \mathbb{R}^n$ is *convex* if for any $x, y \in C$, we have

$$\theta x + (1-\theta)y \in C \quad \text{for any} \quad \theta \in [0,1]$$

i.e., a convex set contain the line segment between any two points in the set



convex sets



nonconvex sets

a point on line segment between x and y is called a *convex combination* of x and y

Affine sets

a set $C \subseteq \mathbb{R}^n$ is *affine* if for any $x, y \in C$ and $\theta \in \mathbb{R}$, we have

$$\theta x + (1 - \theta) y \in C$$

- a set that contains the line through any two distinct points in the set
- a convex set since it holds for any θ, so it holds also for $\theta \in [0, 1]$
- a point $\theta x + (1 - \theta) y$ is called an *affine combination* of x, y

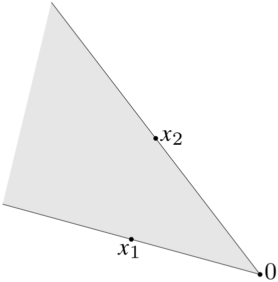
Examples

- solution set of linear equations $\{x \mid Ax = b\}$ is affine
- every affine set can be expressed as solution set of linear equations
- the empty set, any single point (singleton), and $\mathbb{R}^n$ are affine, hence convex
- a line $\mathcal{L} = \{x_0 + tv \mid t \in \mathbb{R}\}$ with $x_0, v \in \mathbb{R}^n$ and $v \neq 0$ is affine and convex

Convex cones and rays

Convex cone: $C \subseteq \mathbb{R}^n$ is a *convex cone* if for every $x, y \in C$,

$$\theta_1 x + \theta_2 y \in C \quad \text{for all} \quad \theta_1, \theta_2 \geq 0$$



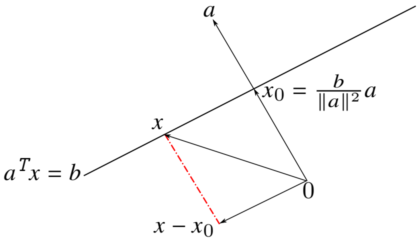
- a point $\theta_1 x + \theta_2 y$ with $\theta_1, \theta_2 \geq 0$ is called a *conic (nonnegative) combination*
- an example of a convex cone is the *norm cone*: $\{(x,t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$
- called *second-order cone* for Euclidean norm, *i.e.*,

$$\{(x,t) \mid \|x\|_2 \leq t\} = \{(x,t) \mid \|x\|_2^2 \leq t^2, t \geq 0\}$$

Rays: $\{x_0 + tv \mid t \geq 0\}$ with $v \neq 0$, is convex (not affine); it is a convex cone if $x_0 = 0$

Hyperplane

a *hyperplane* $\mathcal{H} = \{x \in \mathbb{R}^n \mid a^T x = b\}$ with $a \neq 0$ is affine and convex



- a is called the *normal vector*

- for any $x_0 \in \mathcal{H}$ (*e.g.*, $x_0 = (b/\|a\|^2)a$), $x \in \mathcal{H}$ if and only if $x - x_0 \perp a$:

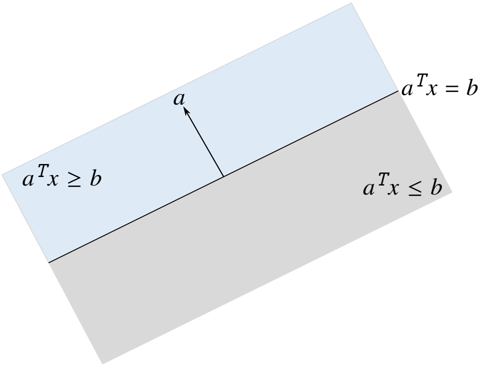
$$a^T x = b = a^T x_0 \Longrightarrow a^T (x - x_0) = 0$$

Halfspaces

the hyperplane $\{x \in \mathbb{R}^n \mid a^T x = b\}$ divides $\mathbb{R}^n$ in two *halfspaces*

$$\mathcal{H}^- = \{x \in \mathbb{R}^n \mid a^T x \leq b\} \quad \text{and} \quad \mathcal{H}^+ = \{x \in \mathbb{R}^n \mid a^T x \geq b\}$$

a halfspace is convex



Balls and ellipsoids

Balls: for $x_c \in \mathbb{R}^n$, $r > 0$, and $\|\cdot\|$ an arbitrary norm, the open and closed balls

$$\mathcal{B}(x_c, r) = \{x \mid \|x - x_c\| < r\} = \{x_c + ru \mid \|u\| < 1\}$$

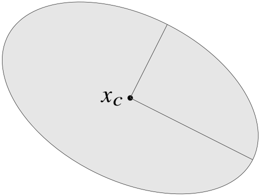
$$\mathcal{B}[x_c, r] = \{x \mid \|x - x_c\| \le r\} = \{x_c + ru \mid \|u\| \le 1\}$$

are convex

Ellipsoids: an ellipsoid

$$\mathcal{E} = \{x \mid x^T Q x + r^T x + c \le 0\}$$

is convex with $Q \in \mathbb{S}^n_{++}$ positive definite, $r \in \mathbb{R}^n$, and $c \in \mathbb{R}$



also written as $\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$ with $P \in \mathbb{S}^n_{++}$ and center $x_c \in \mathbb{R}^n$

Linear matrix inequality

a *linear matrix inequality* (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \preceq 0$$

- $x \in \mathbb{R}^n$, $F_0, \ldots, F_n$ are $m \times m$ symmetric matrices
- the solution set of a linear matrix inequality, $\{x \mid F(x) \preceq 0\}$, is convex

Example any solution $w(t)$ to the linear differential equation

$$\dot{w}(t) = Aw(t), \quad A \in \mathbb{R}^{n \times n}$$

converges to the origin iff there exists a real symmetric matrix X satisfying:

$$AX + XA^T \prec 0, \quad X \succ 0 \tag{9.1}$$

let us express the variable vector $x \in \mathbb{R}^m$ as:

$$X = x_1 X_1 + x_2 X_2 + \cdots + x_m X_m$$

with $X_i \ (i = 1, 2, \ldots, m)$ basis for subspace spanned by $n \times n$ symmetric matrices (with $m = n(n+1)/2$); for instance, when $n = 2$, we have $m = 3$ and:

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

given this representation, the inequality in (9.1) can be recast as:

$$F(x) \triangleq \begin{bmatrix} -X & 0 \\ 0 & AX + XA^T \end{bmatrix} \prec 0,$$

which can then be expressed as LMI with $F_0 = 0$ and

$$F_i = \begin{bmatrix} -X_i & 0 \\ 0 & AX_i + X_i A^T \end{bmatrix}, \quad (i = 1, \ldots, m)$$

Methods for establishing convexity of a set

1. apply definition; recommended only for very simple sets

2. use convex functions (explained later)

3. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity

   – intersection

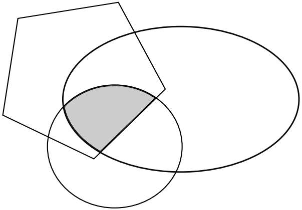
   – affine mapping

   – perspective mapping

   – linear-fractional mapping

Intersection, scaling, summation

Intersection: the intersection of any collection of convex sets is convex



Scaling: if C is a convex set and β is a real number, then the set

$$\beta C = \{\beta y \mid y \in C\} \quad \text{is also convex}$$

Summation: if C_1 and C_2 are convex sets, then the set

$$C_1 + C_2 = \{x_1 + x_2 \mid x_1 \in C_1, x_2 \in C_2\} \quad \text{is convex}$$

Affine transformation

let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the affine function

$$f(x) = Ax + b$$

- the image of a convex set $C \subseteq \mathbb{R}^n$ under f is convex

$$C \subseteq \mathbb{R}^n \quad \text{convex} \quad \Longrightarrow \quad f(C) = \{Ax + b \mid x \in C\} \quad \text{is convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^n \quad \text{convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbb{R}^n \mid Ax + b \in C\} \quad \text{is convex}$$

Examples

- the image of norm ball under affine transformation

$$\{Ax + b \mid \|x\| \leq 1\}$$

  – for example, an ellipsoid

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\} = \{P^{1/2}u + x_c \mid \|u\|_2 \leq 1\}$$

  is the image of the unit Euclidean ball $\{u \mid \|u\|_2 \leq 1\}$ via $f(u) = P^{1/2}u + x_c$

- the inverse image of norm ball under affine transformation

$$\{x \mid \|Ax + b\| \leq 1\}$$

- hyperbolic cone

$$\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\} \quad \text{with } P \in \mathbb{S}^n_+$$

  – inverse image of 2nd cone $\{(z, t) \mid z^T z \leq t^2, t \geq 0\}$ under $f(x) = (P^{1/2}x, c^T x)$

- solution set of linear matrix inequality

$$\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\} \quad \text{with } A_i, B \in \mathbb{S}^p$$

Perspective and linear-fractional function

Perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$:

$$P(x,t) = x/t, \quad \mathrm{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

Linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \quad \mathrm{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

Outline

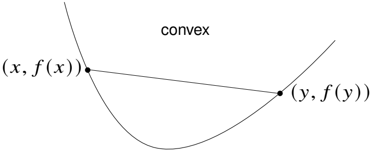
- convex sets
- **convex functions**
- operations preserving convexity
- basic properties
- convex problems

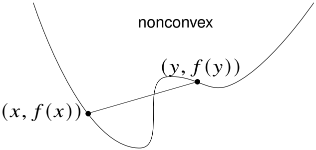
Definition

$f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if $\mathrm{dom}\, f$ is a convex set and

$$f\big(\theta x + (1-\theta)y\big) \le \theta f(x) + (1-\theta) f(y) \qquad (9.2)$$

for all $x, y \in \mathrm{dom}\, f$, $0 \le \theta \le 1$





- f is *strictly convex* if strict inequality holds in (9.2)
- f is *concave* (*strictly concave*) if $-f$ is convex (strictly convex)
- f is convex over convex set $\mathcal{X} \subseteq \mathbb{R}^n$ if (9.2) holds for all $x, y \in \mathcal{X}$

Examples

- *affine functions:* $f(x) = a^T x + b$ with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, is convex and concave:

$$\begin{aligned} f(\theta x + (1-\theta)y) &= a^T((\theta x + (1-\theta)y)) + b \\ &= \theta(a^T x + b) + (1-\theta)(a^T y + b) \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned}$$

- *norm functions:* any norm $\|\cdot\|$ is convex:

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \|\theta x + (1-\theta)y\| \\ &\le \|\theta x\| + \|(1-\theta)y\| = \theta f(x) + (1-\theta) f(y) \end{aligned}$$

  where the inequality follows from the triangle inequality

- $f(x) = x^T Q x$ with $Q \in \mathbb{S}^n$ and convex $\mathrm{dom}\, f$ is convex if

$$(x-y)^T Q (x-y) \ge 0 \quad \text{for all} \quad x, y \in \mathrm{dom}\, f$$

- the function

$$f(x_1, x_2) = x_1 x_2 \quad \text{with} \quad \operatorname{dom} f = \{x \mid x_1, x_2 \geq 0\}$$

is nonconvex since for $x = (1, 2)$, $y = (2, 1)$, $\theta = 0.5$, we have

$$f(0.5x + 0.5y) = \frac{9}{4} \not\leq 0.5f(x) + 0.5f(y) = 2,$$

which violates the definition of convexity

- the function

$$f(x) = x \quad \text{over} \quad \operatorname{dom} f = \{x \mid x \neq 1\}$$

is not convex even though it is linear; this is because its domain is nonconvex

Extended-value extension

extended-value extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ of f:

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \operatorname{dom} f \\ \infty & x \notin \operatorname{dom} f \end{cases}$$

often simplifies notation; for example, the condition

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \leq \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbb{R} \cup \{\infty\}$), means the same as the two conditions

- $\operatorname{dom} f$ is convex
- for $x, y \in \operatorname{dom} f$,

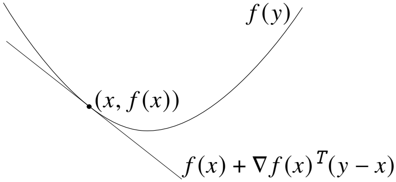
$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

First-order convexity condition

suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable (with open domain)

f is convex if and only if its domain is convex and for any $x, y \in \mathrm{dom}\, f$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$



- f is strictly convex if strict inequality holds
- first order Taylor approximation of convex f is a global underestimator
- if $\nabla f(x) = 0$, then $f(x) \leq f(y)$ for all $y \in \mathrm{dom}\, f$ so x is a global minimizer of f

Second-order convexity condition

suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable (with open domain)

f is convex if and only if its domain is convex and

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \operatorname{dom} f \tag{9.3}$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \operatorname{dom} f$, then f is strictly convex
- converse is not true (*e.g.*, $f(x) = x^4$ is strictly convex but $f''(x) = 0$ at $x = 0$

Convexity of domain

- $\operatorname{dom} f$ must be convex to use the first or second order convexity characterization
- for example, the function

$$f(x) = 1/x^2 \quad \text{with} \quad \operatorname{dom} f = \{x \in \mathbb{R} \mid x \neq 0\}$$

  satisfies $f''(x) = 6/x^4 > 0$ for all $x \in \operatorname{dom} f$, but is not a convex function

Examples

the following can be shown using the definition or the second order condition

Convex

- *exponential:* $e^{\alpha x}$ is convex for any $\alpha \in \mathbb{R}$
- *powers:* x^{α} is convex on $\mathbb{R}_{++}$ when $\alpha \geq 1$ or $\alpha \leq 0$
- *powers of absolute value:* $|x|^p$ is convex on $\mathbb{R}$ for $p \geq 1$
- *negative entropy:* $x \log x$ is convex on $\mathbb{R}_{++}$

Concave

- *powers:* x^{α} on $\mathbb{R}_{++}$ is concave for $0 \leq \alpha \leq 1$
- *logarithm:* $\log x$ is concave on $\mathbb{R}_{++}$

Example: quadratic functions

$$f(x) = x^T Q x + r^T x + c \quad \text{with } Q = Q^T$$

is convex if and only if $Q \succeq 0$

- $f(x) = 4x_1^2 + 2x_2^2 + 3x_1x_2 + 4x_1 + 5x_2$ is convex since its Hessian

$$\nabla^2 f(x) = \begin{bmatrix} 8 & 3 \\ 3 & 4 \end{bmatrix}$$

  is positive definite

- $f(x) = 4x_1^2 - 2x_2^2 + 3x_1x_2 + 4x_1 + 5x_2$ is nonconvex since its Hessian

$$\nabla^2 f(x) = \begin{bmatrix} 8 & 3 \\ 3 & -4 \end{bmatrix}$$

  is indefinite

Example: quadratic over linear

the function

$$f(x, t) = x^2/t \quad \text{with} \quad \operatorname{dom} f = \{(x, t) \mid t > 0\}$$

is convex

this is because the Hessian

$$\begin{aligned} \nabla^2 f(x) &= 2 \begin{bmatrix} 1/t & -x/t^2 \\ -x/t^2 & x^2/t^3 \end{bmatrix} \\ &= \frac{2}{t^3} \begin{bmatrix} t \\ -x \end{bmatrix} \begin{bmatrix} t & -x \end{bmatrix} \succeq 0 \end{aligned}$$

over its domain ($t > 0$)

Example: log-sum-exp function

the softmax of log-sum-exp function $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex over $\mathbb{R}^n$

- the partial derivatives of f are:

$$\frac{\partial f}{\partial x_i} = \frac{e^{x_i}}{\sum_{k=1}^n e^{x_k}}$$

the second partial derivatives are

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} \frac{e^{x_i}}{\sum_{k=1}^n e^{x_k}} - \frac{e^{x_i} e^{x_i}}{(\sum_{k=1}^n e^{x_k})^2} & \text{if } i = j \\ -\frac{e^{x_i} e^{x_j}}{(\sum_{k=1}^n e^{x_k})^2} & \text{if } i \neq j \end{cases}$$

- thus, we can express the Hessian as

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^T w} \mathrm{diag}(w) - \frac{1}{(\mathbf{1}^T w)^2} w w^T, \quad w = (e^{x_1}, \ldots, e^{x_n})$$

- for any $v \in \mathbb{R}^n$, we have

$$v^T \nabla^2 f(x) v = \frac{(\sum_i w_i v_i^2)(\sum_i w_i) - (v^T w)^2}{(\sum_i w_i)^2} \geq 0$$

- follows by applying Cauchy-Schwarz on the vectors a and b with entries

$$a_i = \sqrt{w_i} v_i, \quad b_i = \sqrt{w_i}, \quad i = 1, \ldots, n$$

i.e.,

$$(v^T w)^2 = (a^T b)^2 \leq \|a\|^2 \|b\|^2 = \left(\sum_{i=1}^n w_i v_i^2\right) \left(\sum_{i=1}^n w_i\right)$$

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Operations that preserves convexity

Weighted nonnegative sum

$$f = w_1 f_1 + \cdots + w_k f_k$$

- f convex if f_i are convex and $w_i \geq 0$
- a nonnegative weighted sum of concave functions is concave
- +ve weighted sum of strictly convex (concave) f_i is strictly convex (concave)

Integral: if $f(x, \alpha)$ is convex in x for each $\alpha \in \mathcal{A}$, then $\int_{\alpha \in \mathcal{A}} f(x, \alpha) d\alpha$ is convex

Composition with affine function: for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, let

$$f(x) = g(Ax + b), \quad \text{with} \quad \mathrm{dom}\, f = \{x \mid Ax + b \in \mathrm{dom}\, g\}$$

f is convex (concave) if g is convex (concave)

Example

- *negative entropy function*

$$f(x) = \sum_{i=1}^{n} x_i \log x_i, \quad \mathrm{dom}\, f = \mathbb{R}^n_{++} = \{x \mid x_i > 0\}$$

  f is convex since it is the sum of convex functions $x_i \log x_i$

- logarithmic barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \quad \mathrm{dom}\, f = \{x \mid a_i^T x < b_i,\ i = 1, \ldots, m\}$$

  is convex since it is a sum of convex functions

- for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$

$$f(x) = e^{a^T x + b}$$

  is convex over $\mathbb{R}^n$ since $f(x) = g(a^T x + b)$ where $g(t) = e^t$ is a convex function

- the function

$$f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2 + 2x_1 - 3x_2 + e^{x_1}$$

is convex it is the sum of two convex functions $f = f_1 + f_2$ with

$$f_1(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2 + 2x_1 - 3x_2, \quad f_2(x_1, x_2) = e^{x_1}$$

  - f_1 is convex since $\nabla^2 f(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ is positive semidefinite
  - f_2 is also convex since $g(t) = e^t$ is convex and $f_2(x_1, x_2) = g(x_2)$

- the function

$$f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$$

is convex over $\mathbb{R}^3$; it is the sum of three convex functions: $e^{x_1 - x_2 + x_3}$, e^{2x_2}, x_1

Example: generalized quadratic-over-linear

let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ ($c \neq 0$), and $d \in \mathbb{R}$, then the function

$$f(x) = \frac{\|Ax+b\|^2}{c^T x + d}$$

is convex over $\mathrm{dom}\, f = \{x \mid c^T x + d > 0\}$

- we can write f as

$$f(x) = g(Ax+b, c^T x + d), \qquad g(y,t) = \frac{\|y\|^2}{t} = \sum_{i=1}^m \frac{y_i^2}{t}$$

  with $\mathrm{dom}\, f = \{(y,t) \mid y \in \mathbb{R}^m,\ t > 0\}$

- g is sum of convex functions $g_i(y,t) = \frac{y_i^2}{t}$ over $\{(y_i,t) \mid y_i \in \mathbb{R},\ t > 0\}$
- thus f is convex (composition of convex function with an affine mapping)

Pointwise maximum

the max of convex functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$

$$f(x) = \max\{f_1(x), \ldots, f_k(x)\}$$

is convex

Examples

- piece-wise linear function $f(x) = \max_{i=1,\ldots,k}\{a_i^T x + b_i\}$ is convex
- sum of k largest values

  $$f_k(x) = x_{[1]} + \cdots + x_{[k]} \quad (x_{[i]} \text{ is } i\text{th largest component of } x)$$

  is convex since it is a maximum of linear functions

  $$f_k(x) = \max\{x_{i_1} + \cdots + x_{i_k} \mid i_1, \ldots, i_k \in \{1, 2, \ldots, n\} \text{ are different}\}$$

Pointwise supremum

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then $g(x) = \sup_{y \in \mathcal{A}} f(x, y)$ is convex

Examples

- the distance to farthest point in a set C:

$$\sup_{y \in C} \|x - y\|$$

  is convex

- the maximum eigenvalue of symmetric matrix $X \in \mathbb{S}$:

$$\lambda_{\max}(X) = \sup_{\|y\|=1} y^T X y$$

  is convex

Partial minimization

if $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex (provided that $g(x) > \infty$ for some x)

Example: for a convex set $C \subset \mathbb{R}^n$, the *distance function*

$$d(x, C) = \min_{y} \{\|x - y\| \mid y \in C\}$$

is convex because $f(x, y) = \|x - y\|$ is convex in both (x, y)

Summary of minimization/maximization rules

if we include the counterparts for concave functions, there are four rules

Maximization

$$g(x) = \sup_{y \in C} f(x, y)$$

- g is convex if f is convex in x for fixed y; C can be any set
- g is concave if f is jointly concave in (x, y) and C is a convex set

Minimization

$$g(x) = \inf_{y \in C} f(x, y)$$

- g is convex if f is jointly convex in (x, y) and C is a convex set
- g is concave if f is concave in x for fixed y; C can be any set

Composition with scalar functions

composition of $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$:

$$f(x) = g(h(x)), \quad \mathrm{dom}\, f = \{x \in \mathrm{dom}\, h \mid h(x) \in \mathrm{dom}\, g\}$$

f is convex if g is convex and one of the following three cases holds

- h is convex, and $\tilde{g}$ is nondecreasing
- h is concave, and $\tilde{g}$ is nonincreasing
- g is affine

f is concave if g is concave and one of the following three cases holds

- h is concave, and $\tilde{g}$ is nondecreasing
- h is convex, and $\tilde{g}$ is nonincreasing
- g is affine

Proof

$$
\begin{aligned}
f(\theta x + (1-\theta)y) &= g(h(\theta x + (1-\theta)y)) \\
&\leq g(\theta h(x) + (1-\theta)h(y)) \\
&\leq \theta g(h(x)) + (1-\theta)g(h(y)) \\
&= \theta f(x) + (1-\theta)f(x)
\end{aligned}
$$

- the first inequality arises from convexity of h and the nondecreasing nature of g
- the second inequality is a result of the convexity of g

Examples

- $f(x) = \exp(\|x\|^2)$ is convex since $f(x) = g(h(x))$ where
  - $h(x) = \|x\|^2$ is a convex function
  - $g(t) = e^t$ is a nondecreasing convex function

  more generally, $\exp h(x)$ is convex if h is convex

- $f(x) = (1 + \|x\|^2)^2$ is a convex function since $f(x) = g(h(x))$ where
  - $h(x) = 1 + \|x\|^2$ is convex
  - $g(t) = t^2$ is convex and nondecreasing over h (*i.e.*, the interval $[1, \infty)$)

- $h(x)^p$ is convex for $p \geq 1$ if h is convex and nonnegative

- $-\log(-h(x))$ is convex if h is convex and negative

- $1/h(x)$ is convex if h is concave and positive

- $\log h(x)$ is concave if h is concave and positive

Vector functions composition

composition of $h : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}$:

$$f(x) = g(h(x)) = g(h_1(x), \ldots, h_k(x))$$

f is convex if g is convex and for each i, one of the following holds

- h_i is convex and $\tilde{g}$ nondecreasing in its ith argument
- h_i is concave and $\tilde{g}$ nonincreasing in its ith argument
- h_i is affine

Examples

- $f(x) = \log \sum_{i=1}^{k} e^{h_i(x)}$ is convex when h_i are convex
  - $f(x) = g(h(x))$, $g(z) = \log \sum_{i=1}^{k} e^{z_i}$ is convex and nondecreasing in each argument
- $\left(\sum_{i=1}^{k} h_i(x)^p\right)^{\frac{1}{p}}$ is convex for $p \geq 1$ and $h_1, \ldots, h_k$ convex and nonnegative
  - $g : \mathbb{R}^k \to \mathbb{R}$
$$g(z) = \left(\sum_{i=1}^{k} \max\{z_i, 0\}^p\right)^{\frac{1}{p}}$$
  - $g(h(x))$ is convex since g is both convex and nondecreasing in its arguments
  - for nonnegative values of z, $g(z)$ simplifies to
$$\left(\sum_{i=1}^{k} z_i^p\right)^{\frac{1}{p}}$$
  - we conclude that $\left(\sum_{i=1}^{k} h_i(x)^p\right)^{\frac{1}{p}}$ is convex
- $f(x) = \sum_{i=1}^{k} \log h_i(x)$ is concave if h_i are concave and positive

Examples

- $f(x) = p(x)^2/q(x)$ is convex if
  - p is nonnegative and convex
  - q is positive and concave

- the function

$$f(x, y) = \frac{(x-y)^2}{1 - \max(x, y)}, \quad x < 1, \quad y < 1$$

  is convex

  - x, y, and 1 are affine
  - $\max(x, y)$ is convex; $x - y$ is affine
  - $1 - \max(x, y)$ is concave
  - function u^2/v is convex, monotone decreasing in v for $v > 0$
  - f is compos. of $g(u, v) = \frac{u^2}{v}$ with $u = x - y, v = 1 - \max(x, y)$, hence convex

Perspective function

the *perspective* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$g(x, t) = tf(x/t), \quad \mathrm{dom}\, g = \{(x, t) \mid x/t \in \mathrm{dom}\, f, t > 0\}$$

g is convex if f is convex

Examples

- $f(x) = x^T x$ is convex, so $g(x, t) = x^T x/t$ is convex for $t > 0$
- $f(x) = -\log x$ is convex, so the relative entropy

$$g(x, t) = t \log t - t \log x$$

  is convex on $\mathbb{R}^2_{++}$

- if f is convex, then

$$g(x) = (c^T x + d) f\big((Ax + b)/(c^T x + d)\big)$$

  is convex on $\{x \mid c^T x + d > 0, (Ax + b)/(c^T x + d) \in \mathrm{dom}\, f\}$

Outline

- convex sets
- convex functions
- operations preserving convexity
- **basic properties**
- convex problems

Restriction of a convex function to a line

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$$g(t) = f(x + tv), \quad \mathrm{dom}\, g = \{t \mid x + tv \in \mathrm{dom}\, f\}$$

is convex in t for any $x \in \mathrm{dom}\, f$ and $v \in \mathbb{R}^n$

- f convex if it remains convex when restricted to any line intersecting its domain
- allows us to check convexity of f by checking convexity of one variable functions

Example: log-determinant function

$f : \mathbb{S}^n \to \mathbb{R}$ with $f(X) = \log\det X$ is concave over $\mathrm{dom}\, f = \mathbb{S}^n_{++}$

Proof

- let $X_0 = X_0^{1/2} X_0^{1/2} \in \mathbb{S}^n_{++}$, $V \in \mathbb{R}^{n\times n}$ be symmetric, then

$$\begin{aligned} g(t) = \log\det(X_0 + tV) &= \log\det(X_0^{1/2} X_0^{1/2} + tV) \\ &= \log\det X_0 + \log\det(I + tX_0^{-1/2} V X_0^{-1/2}) \\ &= \log\det X_0 + \log\prod_i (1 + t\lambda_i) \\ &= \log\det X_0 + \sum_{i=1}^{n} \log(1 + t\lambda_i) \end{aligned}$$

  where λ_i, are the eigenvalues of $X_0^{-1/2} V X_0^{-1/2}$

- 2nd term is sum of concave functions; hence $g(t)$ is concave and f is concave

Sublevel sets and convexity

the sublevel set of $f : \mathbb{R}^n \to \mathbb{R}$ at level γ is defined as

$$\mathcal{S}_\gamma = \{x \in \operatorname{dom} f \mid f(x) \le \gamma\}$$

- for a convex function f, the sublevel set $\mathcal{S}_\gamma$ is also convex:

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y) \le \gamma, \quad \text{for all } x, y \in \mathcal{S}_\gamma$$

- useful to show convexity of a set
- a function can have all its sublevel sets convex, but not be a convex
  - for example, $f(x) = -e^x$ is not convex on $\mathbb{R}$ but all its sublevel sets are convex
  - another example is $f(x) = \log(x)$, which is concave; with convex sublevel sets $(0, e^\gamma]$

Example

let $P \succeq 0$ is an $n \times n$ matrix, then the set:

$$C = \left\{ x \mid (x^T P x + 1)^2 + \log\big(\sum_{i=1}^{n} e^{x_i}\big) \leq 3 \right\}$$

is convex since it is the level set of a convex function

$$f(x) = (x^T P x + 1)^2 + \log\big(\sum_{i=1}^{n} e^{x_i}\big)$$

- the log-sum-exp function, previously established as convex
- $(x^T P x + 1)^2$ is convex since it is equal $g(x^T P x)$ with $g(t) = (t+1)^2$
  - g is nondecreasing convex function (defined on $\mathbb{R}_+$)
  - $x^T P x$ convex quadratic function
  - convexity follows from composition rule
- f is convex, being the sum of two convex functions

Epigraph

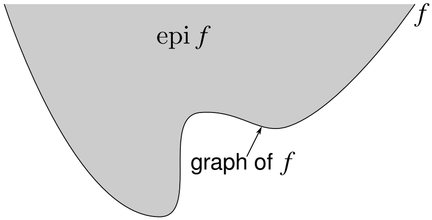
the *graph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$\{(x, f(x)) \mid x \in \operatorname{dom} f\} \subset \mathbb{R}^{n+1}$$

the *epigraph* of $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\operatorname{epi}(f) = \{(x, s) \mid x \in \operatorname{dom} f,\ f(x) \le s\} \subset \mathbb{R}^{n+1}$$

- the epigraph encompasses the points situated on or above the graph of f



- a function is convex if and only if its epigraph is a convex set

Example

consider the function $f : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$, represented by

$$f(x, Y) = x^T Y^{-1} x, \quad Y \in \mathbb{S}^n_{++}$$

- we can determine the convexity of f is by examining its epigraph:

$$\begin{aligned} \mathrm{epi}\, f &= \{(x, Y, t) \mid Y \succ 0,\ x^T Y^{-1} x \le t\} \\ &= \left\{(x, Y, t) \mid \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0,\ Y \succ 0 \right\} \end{aligned}$$

  last line follows from Schur complement criteria for positive semidefiniteness

- the latter condition is an LMI in the variables (x, Y, t)
- hence the epigraph of f is convex, and consequently f is convex

Outline

- convex sets
- convex functions
- operations preserving convexity
- basic properties
- **convex problems**

Definition

Convex optimization problem in standard form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_j(x) = 0, \quad j = 1, \ldots, p \end{array}$$

- f and g_i are convex
- $h_j(x)$ are affine, *i.e.*, $h_j(x) = a_j^T x - b_j$ for some $a_j \in \mathbb{R}^n$ and $b_j \in \mathbb{R}$
- the feasible set is convex since it is the intersection of convex sets

Concave problems

- maximization with concave objective and convex constraints
- a concave problem is also referred to as a convex problem

Examples

- the problem

$$\begin{array}{ll} \text{minimize} & -2x_1 + x_2 \\ \text{subject to} & x_1^2 + x_2^2 \leq 4 \end{array}$$

  is convex

- the problem

$$\begin{array}{ll} \text{minimize} & -2x_1 + x_2 \\ \text{subject to} & x_1^2 + x_2^2 = 4 \end{array}$$

  is nonconvex since the equality constraint function $h(x) = x_1^2 + x_2^2 - 4$ is not affine

Example

$$\begin{array}{ll}\text{minimize} & f(x) = x_1^2 + x_2^2 \\ \text{subject to} & g_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0\end{array}$$

- problem has convex objective f
- the feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- for our definition, this is not a convex problem (g_1 not convex and h_1 not affine)
- problem is equivalent (but not identical) to the convex problem:

$$\begin{array}{ll}\text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0\end{array}$$

Example

- an investor wants to invest a total value of at most d into n possible investments
- let x_i is investment deposit for investment i
- in economy it is frequently assumed that $f_i(x_i)$ have forms:

$$f_i(x_i) = \alpha_i(1 - e^{-\beta_i x_i}), \quad f_i(x_i) = \alpha_i \log(1 + \beta_i x_i), \quad f_i(x_i) = \frac{\alpha_i x_i}{x_i + \beta_i}$$

  with $\alpha_i, \beta_i > 0$; the above functions are concave

- formulation: determine the investment deposits that maximize expected profit

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & \sum_{i=1}^{n} x_i \leq d \\ & x_i \geq 0, \quad i = 1, \ldots, n \end{array}$$

  this is a convex problem (we can transform max into min)

Convexity of feasible and optimal set

- feasible set is convex since it is the intersection of convex sets:

$$\mathrm{dom}\, f, \text{ sublevel sets } \{x \mid g_i(x) \le 0\}, \text{ and affine sets } \{x \mid a_j^T x = b_j\}$$

- optimal set is convex: any convex combination of optimal x_1, x_2 is feasible and

$$f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2) = p^\star$$

so $f(\theta x_1 + (1-\theta)x_2) = p^\star$, *i.e.*, any convex combination is optimal

Local minimizers are global minimizers

any locally optimal point of a convex problem is (globally) optimal

Proof

- if x° is a local minimizer, then $f(x^\circ) \le f(z)$ for all feasible z with $\|z - x^\circ\| \le R$
- assume $f(y) < f(x^\circ)$ for some feasible y so that x° is not a global minimizer
- since $f(y) < f(x^\circ)$, we have $\|y - x^\circ\| > R$
- let $z = \theta y + (1-\theta)x^\circ$, from convexity definition, we have

$$f(z) = f(\theta y + (1-\theta)x^\circ) \le \theta f(y) + (1-\theta) f(x^\circ) < f(x^\circ)$$

- for $\theta = R/2\|y - x^\circ\|$, we have $\|z - x^\circ\| = R/2 < R$
- this implies that there is z close to x° such that $f(z) < f(x^\circ)$ (contradiction)
- hence, there is no feasible y such that $f(y) < f(x^\circ)$, *i.e.*, x° is a global minimizer

First-order optimality condition

- suppose $f : \mathcal{X} \to \mathbb{R}$ is convex over a convex set $\mathcal{X} \subset \mathbb{R}^n$
- the point $x^\star$ is optimal if and only if

$$\nabla f(x^\star)^T(y - x^\star) \geq 0, \quad \forall\, y \in \mathcal{X} \tag{9.4}$$

(the above condition is difficult to verify in practice)

Unconstrained case: for $\mathcal{X} = \mathbb{R}^n$, the above condition reduces to

$$\nabla f(x^\star) = 0$$

to see this suppose that $x \in \operatorname{dom} f$ is optimal and let $y = x - t\nabla f(x)$, which is in the domain of f for sufficiently small t (since domain is open by definition); note that

$$\nabla f(x)^T(y - x) = -t\|\nabla f(x)\|^2 \geq 0 \implies \nabla f(x) = 0$$

Examples

- $f(x) = x\log x$ with $\mathrm{dom}\, f = \mathbb{R}_{++}$; setting the derivative to zero

  $$f'(x) = \log x + 1 = 0 \Longrightarrow x = 1/e$$

  g the second derivative is

  $$f''(x) = 1/x > 0 \quad \text{for all } x \in \mathrm{dom}\, f$$

  hence, the function is convex and $x = 1/e$ is a global minimizer

- *minimization over the nonnegative orthant*

  $$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \succeq 0 \end{array}$$

  using the optimality condition:

  $$x \succeq 0, \quad \nabla f(x)^T(y - x) \geq 0 \text{ for all } y \succeq 0$$

  equivalent to

  $$x \succeq 0, \quad \nabla f(x) \succeq 0, \quad x_i \nabla f(x)_i = 0, \quad i = 1, \ldots, n$$

Sufficiency of KKT conditions

for cvx problems, if there exists $x^\star \in \mathcal{D}$, $\mu^\star \in \mathbb{R}^m$, $\lambda^\star \in \mathbb{R}^p$ satisfying

$$\begin{aligned}
\nabla f(x^\star) + \sum_{i=1}^m \mu_i^\star \nabla g_i(x^\star) + \sum_{j=1}^p \lambda_j^\star \nabla h_j(x^\star) &= 0 \\
g_i(x^\star) &\leq 0, \quad i = 1, \ldots, m \\
Ax^\star &= b \\
\mu_i^\star &\geq 0, \quad i = 1, \ldots, m \\
g_i(x^\star)\mu_i^\star &= 0, \quad i = 1, \ldots, m
\end{aligned}$$

then, $x^\star$ is a global minimizer

- there may be optimal points that do not satisfy KKT conditions
- when we discuss duality, we will provide conditions such that the KKT conditions are both necessary and sufficient

Proof

- let x be a feasible solution; note that the function

$$J(x) = L(x, \mu^\star, \lambda^\star) = f(x) + \sum_{i=1}^{m} \mu_i^\star g_i(x) + \sum_{i=1}^{p} \lambda_j^\star h_j(x)$$

  is convex since it is the sum of convex functions

- since $\nabla J(x^\star) = 0$, $x^\star$ is a minimizer of J over $\mathbb{R}^n$; thus,

$$\begin{aligned} f(x^\star) &\overset{\text{kkt}}{=} f(x^\star) + \sum_{i=1}^{m} \mu_i^\star g_i(x^\star) + \sum_{i=1}^{p} \lambda_j^\star h_j(x^\star) \\ &= J(x^\star) \\ &\leq J(x) \\ &= f(x) + \sum_{i=1}^{m} \mu_i^\star g_i(x) + \sum_{i=1}^{p} \lambda_j^\star h_j(x) \\ &\leq f(x) \end{aligned}$$

- hence, $x^\star$ is optimal

Example

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\ \text{subject to} & x_1 + x_2 + x_3 = 3 \end{array}$$

the above problem is convex with an equality constraint; the Lagrangian is

$$L(x, \lambda) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda(x_1 + x_2 + x_3 - 3)$$

the KKT conditions are

$$\begin{aligned} x_1 + \lambda &= 0 \\ x_2 + \lambda &= 0 \\ x_3 + \lambda &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

the unique optimal solution is $x = (1, 1, 1)$ and $\lambda = -1$

Example

$$\begin{array}{ll} \text{minimize} & x_1^2 - x_2 \\ \text{subject to} & x_2^2 \leq 0 \end{array}$$

it is easy to see that the solution is $x^\star = (0, 0)$; for this the Lagrangian is

$$L(x, \mu) = \tfrac{1}{2}x_1^2 - x_2 + \mu x_2^2$$

the KKT conditions are

$$\begin{aligned} 2x_1 &= 0 \\ -1 + 2\mu x_2 &= 0 \\ \mu x_2^2 &= 0 \\ x_2^2 &\leq 0 \\ \mu &\geq 0 \end{aligned}$$

the above nonlinear system of equations is infeasible

References and further readings

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