Introduction

- course introduction
- optimization examples

Mathematical optimization

(mathematical) Optimization problem

$$
\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & g_i(x) \leq 0, \; i = 1, \ldots, m \quad \text{(inequality constraints)} \\
 & h_j(x) = 0, \; j = 1, \ldots, p \quad \text{(equality constraints)}
\end{array}
$$

- $x = (x_1, \ldots, x_n)$ is the *optimization variable*
- $f(\cdot)$ is the *objective function* or *cost function* to be minimized
- $g_i(\cdot)$ are *inequality constraints functions*
- $h_j(\cdot)$ are *equality constraints functions*
- maximization problems are the same as minimizing the negative of the function

Optimal point or solution: a point $x^\star$ is an *optimal point* or *solution* if it attains the smallest (largest) objective value among all points that satisfy the constraints

Applications

Applications

- allocate portfolio investments to maximize returns and minimize risk
- design efficient electrical networks
- create lightweight, structurally sound aircraft and aerospace structures
- optimize fuel-efficient trajectories for space vehicles
- design cost-effective structures like frames and dams, ensuring safety
- improve personalized recommendations by factoring user-item interactions
- develop machine learning models for:
  - object classification (*e.g.*, identifying animals in images)
  - prediction (*e.g.*, estimating house prices based on features like location and size)

Modeling: the process of identifying the objective, constraints, variables of a problem

Optimal decision making

- the variable, x, represent some *action* such as:
  - trades in a portfolio
  - adjustments to airplane control surfaces
  - task scheduling or assignment
  - resource allocation decisions
  - transmitted signal...
- constraint functions limit the action or impose conditions on outcome:
  - physical or technical limits
  - resource budgets
  - design requirements that need be satisfied...
- objective represents some criteria, we want to minimize:
  - total cost
  - deviation from desired outcome (error)
  - consumption of fuel
  - risk...

Linear and nonlinear optimization

an optimization problem is called *linear program* if it has the form

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} g_{ij} x_j = h_i, \quad i = 1, \ldots, p \end{array}$$

- $\{c_i, a_{ij}, g_{ij}, h_i, b_i\}$ are given coefficients
- the objective and constraint functions are “linear”

Nonlinear program: an optimization problem that is not a linear program

Other optimization classes

- *unconstrained optimization*: no constraints, *i.e.*, $h_j(x) = g_i(x) = 0$
- *discrete optimization*: variables take only discrete or integer values
- *integer linear program*: a discrete optimization with linear objective and constraints
- *mixed integer optimization*: variables can be both integer and continuous

Note: this course focuses solely on optimization with continuous variables

Examples

- an instance of an unconstrained nonlinear optimization

$$\text{minimize} \quad (x_1 + x_2 - 1)^2 + (x_1 - x_2 + 1)^2 + (2x_1 + 5x_2 - 10)^2$$

- an instance of a constrained nonlinear optimization is given by

$$\begin{array}{ll} \text{minimize} & x_1^3 + x_2 x_1 + e^{x_1} \\ \text{subject to} & x_1^2 + x_2^2 = 1 \\ & x_1 \geq 0 \end{array}$$

- an example of a linear program is:

$$\begin{array}{ll} \text{minimize} & x_1 - 2x_2 + x_3 \\ \text{subject to} & x_1 + x_2 \leq 5 \\ & x_1 + x_2 \geq -1 \\ & x_1 + x_2 + x_3 = 1 \end{array}$$

Solving optimization problems

- various methods exist to solve optimization problems
- chosen method depend on several factors (*e.g.*, problem class and structure)
- solutions guide decision-makers, who oversee, validate, and adjust the approach or problem as required

Can you solve it exactly?

- very difficult to solve with guarantees of global optimality
- but you can try to solve it approximately, and it often doesn't matter
- the exception: **convex optimization**
  - includes linear programming (LP), quadratic programming (QP), many others
  - we can solve these problems reliably and efficiently
  - come up in many applications across many fields

Course topics

General course topics

- unconstrained and constrained optimization: optimality conditions
- convex optimization and duality
- solution methods: unconstrained and constrained
- modeling and applications in optimization

Prerequisites

- good knowledge of linear algebra and calculus (we will review the essential topics)
- MATLAB programming: prior experience not mandatory, but self-study is expected

Course objectives

- understand the mathematical theory of nonlinear and convex optimization and their practical applications
- learn and implement fundamental and some advanced optimization methods
- develop skills to identify optimization problems and select suitable solution method
- gain optimization knowledge for research and real-world applications

Course information

Course materials: all course material will be posted on Moodle

Grading

- (bi)weekly homework (20%)
- midterm exam (30%)
- final exam and/or project (50%)

(these weights are approximate; we reserve the right to change them later)

refer to the syllabus on the Moodle course website for more information, such as course references, office hours, class policy, exam dates, etc.

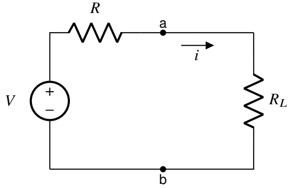
AI tools policy

- unauthorized use of AI tools, like ChatGPT, is treated as plagiarism
- AI is an aid, not a substitute for genuine understanding; reliance solely on AI without understanding can result in penalties
- suspected misuse of AI may lead to oral exams or alternative assessments

Outline

- course introduction
- **optimization examples**

Maximum power transfer



- voltage source: V (in volts)
- line resistor: R (given value)
- objective: determine R_L to maximize power to it

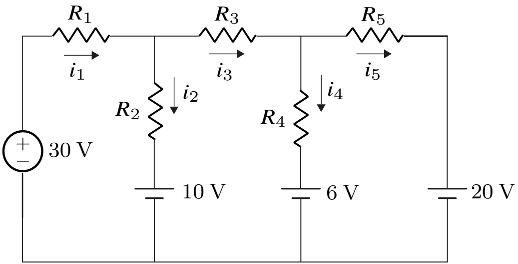
power delivered to R_L is $p(R_L) = i^2 R_L$ and $i = V/(R + R_L)$; hence, we can formulate the problem as

$$\text{maximize} \quad \frac{V^2 x}{(R+x)^2}$$

with variable $x = R_L$; this is an unconstrained nonlinear program

Battery charging

electric circuit is designed to use a $30\ \mathrm{V}$ source to charge $10\ \mathrm{V}$, $6\ \mathrm{V}$, $20\ \mathrm{V}$ batteries



- currents i_1, i_2, i_3, i_4, i_5 limited to a maximum of $4\ \mathrm{A}$, $3\ \mathrm{A}$, $3\ \mathrm{A}$, $2\ \mathrm{A}$, $2\ \mathrm{A}$
- batteries must not be discharged; *i.e.*, currents i_k must be nonnegative
- goal: find $i_1, i_2, \ldots, i_5$ that maximizes total power transferred to the batteries

using circuit analysis, the problem can be modeled as the linear program:

$$
\begin{array}{ll}
\text{maximize} & 10i_2 + 6i_4 + 20i_5 \\
\text{subject to} & i_1 = i_2 + i_3 \\
& i_3 = i_4 + i_5 \\
& i_1 \leq 4 \\
& i_2 \leq 3 \\
& i_3 \leq 3 \\
& i_4 \leq 2 \\
& i_5 \leq 2 \\
& i_1, i_2, i_3, i_4, i_5 \geq 0
\end{array}
$$

once the currents are found, we can find the resistors $R_1, \ldots, R_5$ that draw such currents using Ohm's and Kirchhoff's laws

Concrete mixture

property	concrete type 1	concrete type 2
cost	\$5/kg	\$1/kg
cement	30%	10%
gravel	40%	20%
sand	30%	70%

find mixture with at least: 5 kg cement, 3 kg gravel, 4 kg sand, while minimizing cost

Problem formulation:

$$
\begin{array}{ll}
\text{minimize} & 5x_1 + x_2 \\
\text{subject to} & 0.3x_1 + 0.1x_2 \geq 5 \\
 & 0.4x_1 + 0.2x_2 \geq 3 \\
 & 0.3x_1 + 0.7x_2 \geq 4 \\
 & x_1 \geq 0, x_2 \geq 0
\end{array}
$$

variables x_i represent the weight of concrete i that we want to buy

Knapsack problem

Description

- n items, each with a weight and value
- select items to maximize total value while keeping total weight within a set limit

Investment example

- goal: invest among n opportunities
- budget: at most d dollars
- ith investment:
  - cost: c_i dollars
  - expected profit: p_i
  - available units: b_i

how many items of each type should be bought to maximize the expected profit?

problem can be formulated as

$$
\begin{array}{ll}
\text{maximize} & \sum_{i=1}^{n} p_i x_i \\
\text{subject to} & \sum_{i=1}^{n} c_i x_i \leq d, \text{ (total cost} \leq \text{budget)}, \\
& x_i \in \{0, 1, 2, \ldots, b_i\}, \quad i = 1, \ldots, n
\end{array}
$$

an integer linear program since the objective and constraints are “linear” and the variables are integer

Facility placement

given locations of some facilities $(a_1, b_1), \ldots, (a_m, b_m)$ in 2D space

- $x = (x_1, x_2)$ is location of distribution center that we want to find
- goal: find x to minimize total daily distance between facilities and center
- distance to facility (a_i, b_i):

$$d_i = \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2}$$

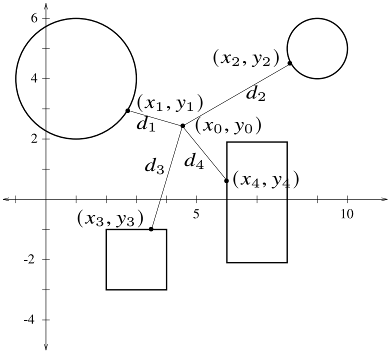
the problem can be formulated as

$$\text{minimize} \quad \sum_{i=1}^{m} w_i \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2}$$

- w_i is weight for distance d_i (*e.g.*,, higher for high-traffic areas)
- this problem is known as the *Fermat-Weber problem*

Electrical wires connections

four buildings are to be connected by electrical wires



- central joining point is (x_0, y_0)
- each building i connects at position (x_i, y_i) with wire length d_i
- goal: find the positions (x_i, y_i) that minimize the total length of wires used

- building 1 (circular): center $(1, 4)$, radius 2
- building 2 (circular): center $(9, 5)$, radius 1
- building 3 (square): center $(3, -2)$, side length 2
- building 4 (rectangle): center $(7, 0)$, height 4, width 2

Problem formulation:

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{4} \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} \\
\text{subject to} & (x_1 - 1)^2 + (y_1 - 4)^2 \leq 4 \\
& (x_2 - 9)^2 + (y_2 - 5)^2 \leq 1 \\
& 2 \leq x_3 \leq 4 \\
& -3 \leq y_3 \leq -1 \\
& 6 \leq x_4 \leq 8 \\
& -2 \leq y_4 \leq 2
\end{array}
$$

with variables (x_i, y_i) ($i = 0, 1, \ldots, 4$)

References and further readings

- S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004. (`web.stanford.edu/~boyd/cvxbook`)

- G. C. Calafiore and L. El Ghaoui. *Optimization Models.* Cambridge University Press, 2014.

- I. Griva and S. G. Nash and A. Sofer. *Linear and Nonlinear Optimization.* SIAM, 2009.