

## 7. Least squares

- linear least squares
- regularized least squares
- nonlinear least squares
- Gauss-Newton method
- Levenberg-Marquardt method

# Linear least squares

## Inconsistent linear equations

- let  $A$  be  $m \times n$  and consider  $Ax = b$  where  $b$  is an  $m$ -vector
- in most applications,  $m > n$  and there is no  $x$  satisfying  $Ax = b$  ( $b \notin \text{range}(A)$ )
- it is desirable to find an  $x$  such that  $Ax \approx b$

(linear) **Least squares:** choose  $x$  that solves:

$$\text{minimize} \quad \|Ax - b\|^2 = \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij}x_j - b_i \right)^2$$

- $r = Ax - b$  is called the *residual*
- $A$  and  $b$  are called the *data* for the problem
- also called *regression* (in data-fitting context)

## Column and row interpretations

let  $a_i$  denote the  $i$ th column of  $A$  and  $\hat{a}_j^T$  denote the  $j$ th row of  $A$ :

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} \hat{a}_1^T \\ \vdots \\ \hat{a}_m^T \end{bmatrix}$$

### Row interpretation

$$\text{minimize } \|Ax - b\|^2 = (\hat{a}_1^T x - b_1)^2 + \cdots + (\hat{a}_m^T x - b_m)^2$$

minimize the sum of squares of the residuals  $r_i = \hat{a}_i^T x - b_i$

### Column interpretation

$$\text{minimize } \|Ax - b\|^2 = \|(x_1 a_1 + \cdots + x_n a_n) - b\|^2$$

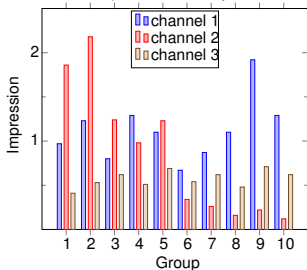
find the coefficients of the linear combination of the columns that is closest to  $b$

## Example: advertising purchases

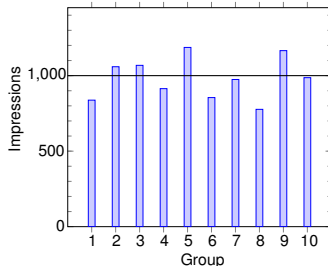
- $m$  demographics groups (audiences),  $n$  advertising channels
- $v_i^{\text{des}}$  is target number of views or impressions for group  $i$
- $R_{ij}$  is # views in group  $i$  per dollar spent on ads in channel  $j$
- $s_j$  is amount of advertising purchased in channel  $j$
- $(Rs)_i$  is total number of views in group  $i$
- least squares problem: minimize  $\|Rs - v^{\text{des}}\|^2$  (ignoring  $s \geq 0$  and budget)

**Example:**  $m = 10$ ,  $n = 3$ ,  $v^{\text{des}} = 10^3 \times \mathbf{1}$ ,  $\hat{s} = (62, 100, 1443)$

columns  $R$  (1000 views per dollar)



$v^{\text{des}}$  and achieved views  $R\hat{s}$



## Solution

a solution must satisfy the *normal equations*:

$$A^T A x^\star = A^T b$$

if the columns of  $A$  are linearly independent, then the solution is unique:

$$x^\star = (A^T A)^{-1} A^T b = A^\dagger b$$

- $A^\dagger = (A^T A)^{-1} A^T$  is the psuedo-inverse of  $A$ , which is also a left inverse
- solution is sometimes called the *least squares approximate solution* of  $Ax = b$ 
  - $x^\star = A^\dagger b$  solves the linear equation  $Ax = b$  if it has a solution ( $b \in \text{range}(A)$ )
  - if  $Ax = b$  does not have a solution, then  $Ax^\star \neq b$
- any  $x$  satisfying the above is a global minimizer since  $\nabla^2 f(x) = 2A^T A \geq 0$

### MATLAB command

```
>> A=[] % define the matrix A
>> b=[] % define the vector b
>> x=A\b % solution
```

## Example

we are given two different types of concrete:

first type	second type
30% cement	10% cement
40% gravel	20% gravel
30% sand	70% sand (all percentages of weight)

how many pounds of each type to mix so that you get a mixture close to

- 5 pounds of cement
- 3 pounds of gravel
- 4 pounds of sand?

- letting  $x_1$  and  $x_2$  to be the amounts of concrete of the first and second types
- the above problem can be formulated as the least squares problem:

$$\text{minimize} \quad \left\| \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} \right\|^2 = \|Ax - b\|^2$$

where  $x = (x_1, x_2)$

- since the columns of  $A$  are linearly independent, the solution is

$$x^\star = (A^T A)^{-1} A^T b = \begin{bmatrix} 10.6 \\ 0.961 \end{bmatrix}$$

## Optimality proof using algebra

$$\begin{aligned}\|Ax - b\|^2 &= \|(Ax - Ax^\star) + (Ax^\star - b)\|^2 \\ &= \|A(x - x^\star)\|^2 + \|Ax^\star - b\|^2 \\ &\quad + 2(Ax - Ax^\star)^T(Ax^\star - b)\end{aligned}$$

using  $A^T Ax^\star = A^T b$ , the cross product term is zero; hence

$$\|Ax - b\|^2 = \|A(x - x^\star)\|^2 + \|Ax^\star - b\|^2$$

- since  $\|A(x - x^\star)\|^2 \geq 0$ , we have  $\|Ax - b\|^2 \geq \|Ax^\star - b\|^2$
- if  $A$  has linearly independent columns, then

$$\|Ax - b\|^2 > \|Ax^\star - b\|^2 \quad (\text{unique solution})$$

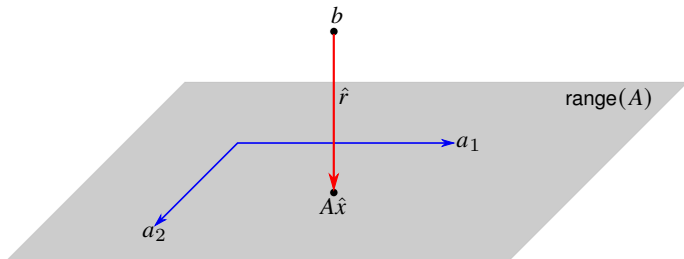
this is because  $\|A(x - x^\star)\|^2 = 0 \Rightarrow A(x - x^\star) = 0 \Rightarrow x = x^\star$



## Geometric interpretation

- $Ax^\star$  is the vector in  $\text{range}(A) = \text{span}(a_1, \dots, a_n)$  closest to  $b$
- the optimal residual  $r^\star = Ax^\star - b$  is orthogonal to  $\text{range}(A)$ :  $r^\star \perp Aw$  for any  $w$

$$(Aw)^T r^\star = (Aw)^T (Ax^\star - b) = w^T A^T (Ax^\star - b) = w^T 0 = 0,$$



- $Ax^\star = AA^\dagger b$  is projection on  $\text{range}(A)$

## Data fitting

**Setup:** a scalar  $y$  and an  $n$ -vector  $z$  are related by model,  $g : \mathbb{R}^p \rightarrow \mathbb{R}$ ,

$$y \approx g(x)$$

- $z$  is the *independent variable* or *feature vector*
- $y$  is the *outcome* or *response variable*
- we don't know  $g$ , which gives the 'true' relationship between  $x$  and  $y$

**Data:** we are given some data (*observations, samples, or measurements*)

$$z^{(1)}, \dots, z^{(m)}, \quad y^{(1)}, \dots, y^{(m)}$$

- $(z^{(i)}, y^{(i)})$  is  *$i$ th data pair*
- $z_j^{(i)}$  is the  $j$ th component of  $i$ th data point  $x^{(i)}$

**Model fitting:** find  $g$  or an approximation of it based on observations by minimizing

$$\frac{1}{m} \sum_{i=1}^m (g(z^{(i)}) - y^{(i)})^2 \tag{7.1}$$

# Linear in parameters model fitting

## Linear in parameters model

$$g(z) = x_1 g_1(z) + x_2 g_2(z) + \cdots + x_n g_n(z)$$

- $g_i(z)$  are *basis functions* or *feature mappings* that we choose
- $x_i$  are *model parameters*
- we want to estimate  $x$  such that the cost (7.1) is minimized

**Least squares formulation:** minimize  $\|Ax - b\|^2$  where

$$A = \begin{bmatrix} g_1(z^{(1)}) & g_2(z^{(1)}) & \cdots & g_n(z^{(1)}) \\ g_1(z^{(2)}) & g_2(z^{(2)}) & \cdots & g_n(z^{(2)}) \\ \vdots & \vdots & \cdots & \vdots \\ g_1(z^{(m)}) & g_2(z^{(m)}) & \cdots & g_n(z^{(m)}) \end{bmatrix}, \quad b = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

## Polynomial fit

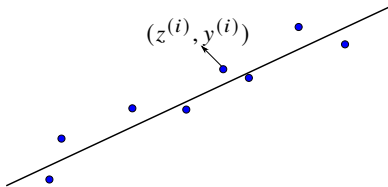
model is a polynomial of degree at most  $n - 1$ :

$$g(z) = x_1 + x_2 z + \cdots + x_n z^{n-1}$$

- $f_i(x) = z^{i-1}$ ,  $i = 1, \dots, n$ ; here  $z^i$  means scalar  $z$  to  $i$ th power
- $z^{(i)}$  is  $i$ th data point
- $A$  is Vandermonde matrix

$$A = \begin{bmatrix} 1 & z^{(1)} & \cdots & (z^{(1)})^{p-1} \\ 1 & z^{(2)} & \cdots & (z^{(2)})^{p-1} \\ \vdots & \vdots & & \vdots \\ 1 & z^{(m)} & \cdots & (z^{(m)})^{p-1} \end{bmatrix}$$

## Line fitting



find a straight line that best fits the data  $(z^{(i)}, y^{(i)})$ :

$$x_1 + x_2 z \approx y$$

- $x_1$  is the displacement
- $x_2$  is the slope of the line
- $g(z) = x_1 + x_2 z$ ,  $g_1(z) = 1$ ,  $g_2(z) = z$

$$A = \begin{bmatrix} 1 & z^{(1)} \\ 1 & z^{(2)} \\ \vdots & \vdots \\ 1 & z^{(m)} \end{bmatrix}, \quad b = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

## Example

fit a straight line  $y^{(i)} \approx x_1 + x_2 z^{(i)}$  to the data:

$$(z^{(1)}, y^{(1)}) = (2, 3), \quad (z^{(2)}, y^{(2)}) = (3, 4), \quad (z^{(3)}, y^{(3)}) = (4, 15)$$

- we can minimize

$$\begin{aligned} \sum_{i=1}^3 (x_1 + x_2 z^{(i)} - y^{(i)})^2 &= (x_1 + 2x_2 - 3)^2 + (x_1 + 3x_2 - 4)^2 + (x_1 + 4x_2 - 15)^2 \\ &= \|Ax - b\|^2 \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \\ 15 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- the solution is

$$x^\star = \begin{bmatrix} x_1^\star \\ x_2^\star \end{bmatrix} = (A^T A)^{-1} A^T b = \begin{bmatrix} -32/3 \\ 6 \end{bmatrix}$$

the equation of the line is  $g(z) = 6z - 32/3$

# Regression

consider the *regression model*:

$$y \approx g(z) = z^T \beta + v$$

- least squares regression: choose the model parameters  $v, \beta$  that minimize

$$\frac{1}{m} \sum_{i=1}^m (v + (z^{(i)})^T \beta - y^{(i)})^2 = \|Ax - y\|^2$$

with

$$A = \begin{bmatrix} 1 & (z^{(1)})^T \\ \vdots & \vdots \\ 1 & (z^{(m)})^T \end{bmatrix}, \quad x = \begin{bmatrix} v \\ \beta \end{bmatrix}, \quad y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

- same as data fitting with basis functions

$$g_1(z) = 1, \quad g_i(z) = z_{i-1}, \quad i = 2, \dots, n+1$$

## Linear estimation

we have  $m$  measurements  $y_1, \dots, y_m$  of some time-varying linear system:

$$y_t = a_t^T x + v_t, \quad t = 1, \dots, m$$

- $a_t^T$  are known or measured linear system parameters
- $v_t$  is an unknown small measurement noise
- the estimation problem is to find a good  $x$  such that  $y_t - a_t^T x$  is minimized for all  $t$
- we can formulate this as a least square problem with

$$A = \begin{bmatrix} h_1^T \\ \vdots \\ h_m^T \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$



## Example

- $n$  measurements of voltage across resistor with 1-ampere current

$$V_i = R + n_i \quad i = 1, \dots, n$$

- we wish to find  $R$  that best fits our measurements

this problem can be formulated as

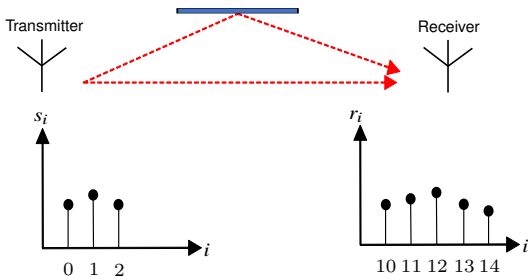
$$\text{minimize} \quad \left\| \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} R - \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \right\|^2$$

least squares problem with  $A = \mathbf{1}$  and  $b = (V_1, \dots, V_n)$ ; hence solution is

$$R^* = (A^T A)^{-1} A^T b = \frac{1}{n} \sum_{i=1}^n V_i = \text{avg}(b)$$

## Example

- a wireless transmitter sends three signals  $s_0, s_1$ , and  $s_2$  at times  $t = 0, 1, 2$
- the transmitted signal takes two paths to the receiver:
  - I. direct path, with delay 10 and attenuation factor  $\alpha_1$
  - II. indirect (reflected) path, with delay 12 and attenuation factor  $\alpha_2$
- the received signal is measured from  $t = 10$  to  $t = 14$ , which is the sum of the signals from these two paths plus some unknown noise



find the channel attenuation factors  $\alpha_1$  and  $\alpha_2$  that “best” fits the signals:

$$s = (s_0, s_1, s_2) = (1, 2, 1) \\ (r_{10}, r_{11}, r_{12}, r_{13}, r_{14}) = (4, 7, 8, 6, 3)$$

we can formulate this as a least squares problem with

$$A = \begin{bmatrix} s_0 & 0 \\ s_1 & 0 \\ s_2 & s_0 \\ 0 & s_1 \\ 0 & s_2 \end{bmatrix}, \quad b = \begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_{13} \\ r_{14} \end{bmatrix}, \quad x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

the least squares solution is

$$\begin{aligned} x^\star &= (A^T A)^{-1} A^T b \\ &= \begin{bmatrix} \|s\|^2 & s_0 s_2 \\ s_0 s_2 & \|s\|^2 \end{bmatrix}^{-1} \begin{bmatrix} s_0 r_{10} + s_1 r_{11} + s_0 r_{12} \\ s_0 r_{12} + s_1 r_{13} + s_0 r_{14} \end{bmatrix} \\ &= \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 + 14 + 8 \\ 8 + 12 + 3 \end{bmatrix} = \begin{bmatrix} \frac{133}{35} \\ \frac{112}{35} \end{bmatrix} \end{aligned}$$

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## Regularized least squares

$$\text{minimize} \quad \|Ax - b\|^2 + \delta \|Rx\|^2$$

- $R \in \mathbb{R}^{p \times n}$  is the *regularization matrix*;  $\delta$  is the *regularization parameter*
- large  $\delta$  gives more emphasis on making the term  $\delta \|Rx\|^2$  small

### Why regularization?

- useful when  $A$  has linearly dependent columns
- utilize some prior information about  $x$

### Solution

$$(A^T A + \delta R^T R)x = A^T b$$

if  $A^T A + \delta R^T R$  is invertible, then

$$x^\star = (A^T A + \delta R^T R)^{-1} A^T b$$

## Example: signal de-noising

- $x = (x_1, x_2, \dots, x_n)$  represent some signal (e.g., audio signals)
- $x_i$  represents the value of the signal sampled at time  $i$
- the signal can be measured with some additive noise

$$y = x + v$$

where  $v$  is some noise

- the signal does not vary too much  $|x_{i+1} - x_i| \ll 1$
- given  $y$ , we want to find a “good” estimate of  $x$

### Naive solution

- directly set  $x = y$
- this can result in a bad estimate if some noise components  $v_i$  are large

## Least squares formulation

$$\text{minimize} \quad \|x - y\|^2 + \delta \|Rx\|^2$$

- $\delta$  is a smoothing regularization parameter
- $R$  is an  $(n - 1) \times n$  smoothing matrix:

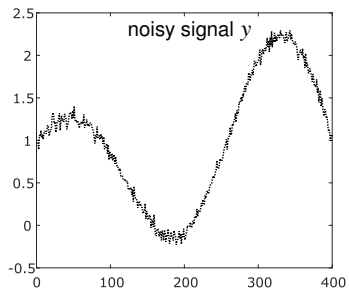
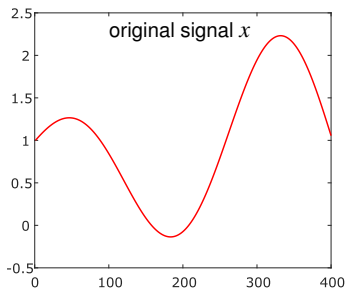
$$\|Rx\|^2 = \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$$

the matrix  $R$  is the difference matrix

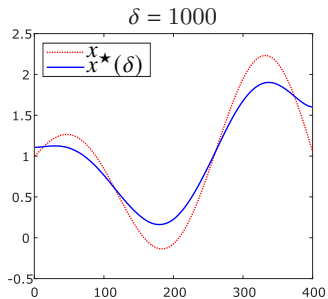
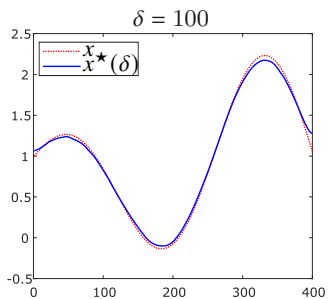
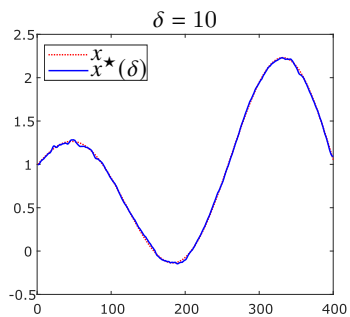
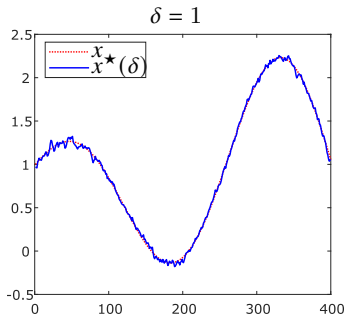
$$R = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

- the optimal solution is given by

$$x^*(\delta) = (I + \delta R^T R)^{-1} y$$







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## Nonlinear least squares

$$\text{minimize} \quad \|r(x)\|^2 = r_1(x)^2 + \cdots + r_m(x)^2$$

- $r : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is nonlinear function with components  $r_i : \mathbb{R}^n \rightarrow \mathbb{R}$
- when  $r(x) = Ax - b$ , we recover the linear least squares problem
- nonlinear least squares are hard to solve
- solution solves/approximate the solution to a set of  $m$  *nonlinear* equations:

$$r_i(x) = 0, \quad i = 1, \dots, m$$

## Location from distance of measurements

- locate some object with unknown location  $x \in \mathbb{R}^n$  ( $n = 2$  or  $n = 3$ )
- given noisy measurements of distance to from  $x$  to some known locations  $y_i$ :

$$\gamma_i = \|x - y_i\| + v_i, \quad i = 1, \dots, m$$

where  $v_i$  is some small measurement noise

- we can estimate the position of  $x$  by solving

$$\text{minimize} \quad \sum_{i=1}^m (\|x - y_i\| - \gamma_i)^2$$

this is a nonlinear least squares problem with  $r_i(x) = \|x - y_i\| - \gamma_i$

# Nonlinear data-fitting

## Model fitting problem

- given  $m$  data points/measurements  $(z^{(i)}, y^{(i)}), i = 1, \dots, m, z_i \in \mathbb{R}^n, y_i \in \mathbb{R}$
- these points are approximately related by the equation

$$g(z; x) \approx y$$

where  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x$  are unknown parameters

## Nonlinear least squares formulation

$$\text{minimize} \quad \sum_{i=1}^m (g(z^{(i)}; x) - y^{(i)})^2$$

if  $g$  is linear in parameters  $x_i$ , then we get a linear least squares

## Example

- given  $m$  measurements,  $y^{(1)}, \dots, y^{(m)}$ , at  $m$  times,  $t^{(1)}, \dots, t^{(m)}$  of:

$$y^{(i)} = \beta \sin(\omega t^{(i)} + \phi) + n(t^{(i)})$$

where  $n(t^{(i)})$  is a random noise

- find the parameters  $\beta$ ,  $\omega$  and  $\phi$  that gives some optimal fit to these measurements

### Nonlinear least squares formulation

$$\text{minimize} \quad \sum_{i=1}^m r_i(x)^2 = \sum_{i=1}^m (y^{(i)} - \beta \sin(\omega t^{(i)} + \phi))^2$$

with variable  $x = (\beta, \omega, \phi)$  and  $r_i(x) = y^{(i)} - \beta \sin(\omega t^{(i)} + \phi)$

# Classification

## Classification

- $m$  training data points  $(z^{(1)}, y^{(1)}), \dots, (z^{(m)}, y^{(m)})$
- outcome  $y_i$  takes on finite number of values called *labels* or *categories*
  - TRUE or FALSE
  - SPAM or NOT SPAM
  - DOG, HORSE, or MOUSE
- data fitting  $g(z^{(i)}) \approx y^{(i)}$  is called *classification*
- model used to determine which class the a new data point  $z$  belongs to

## Boolean (2-way) classification

- two possible outcomes only encoded as  $y \in \{+1, -1\}$
- Boolean classifier has form  $y = g(z), g : \mathbb{R}^n \rightarrow \{-1, +1\}$

## Least squares Boolean classifier

- we are given the data points  $(z^{(i)}, y^{(i)}), i = 1, \dots, m$
- determine basis functions  $g_1, \dots, g_n$  for linear-in-parameter model

$$\tilde{g}(z) = x_1 g_1(z) + x_2 g_2(z) + \dots + x_n g_n(z)$$

- use least squares data-fitting to find parameters  $x_1, \dots, x_n$
- take the sign of  $\tilde{g}(z)$  to get the *Boolean classifier*:

$$g(z) = \text{sign}(\tilde{g}(z)) = \begin{cases} +1 & \text{if } \tilde{g}(z) \geq 0 \\ -1 & \text{if } \tilde{g}(z) < 0 \end{cases}$$

- often used with regression model  $\tilde{g}(z) = z^T \beta + v$



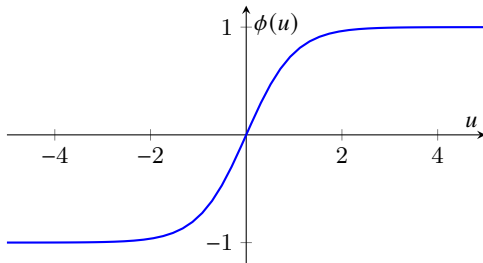
## Nonlinear least squares classification

$$\text{minimize } \sum_{i=1}^m (\phi(x_1 g_1(z_i) + x_2 g_2(z_i) + \cdots + x_n g_n(z_i)) - y_i)^2$$

where  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is the sigmoidal function:

$$\phi(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}},$$

which is a differentiable approximation of  $\text{sign}(u)$



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## Linear least square approximation at each iteration

$$\text{minimize } f(x) = \sum_{i=1}^m r_i(x)^2$$

- $x^{(k)}$  is estimate of a solution at time  $k$
- $\hat{r}(x; x^{(k)})$  is first order Taylor approximation of  $r$  around  $x^{(k)}$ :

$$\hat{r}(x; x^{(k)}) = r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})$$

this is a good approximation if  $x$  near  $x^{(k)}$  ( $\|x - x^{(k)}\|$  is small)

- Gauss-Newton method produces new estimate  $x^{(k+1)}$  that solves the problem

$$\text{minimize } \|\hat{r}(x; x^{(k)})\|^2 = \|r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})\|^2$$

- the above problem is a linear least squares problem with

$$A = Dr(x^{(k)}), \quad b = Dr(x^{(k)})x^{(k)} - r(x^{(k)})$$

## Gauss-Newton method

setting  $x^{(k+1)}$  to be the solution of the previous problem, we have

$$\begin{aligned}x^{(k+1)} &= (A^T A)^{-1} A^T b \\&= (Dr(x^{(k)})^T Dr(x^{(k)}))^{-1} Dr(x^{(k)})^T (Dr(x^{(k)})x^{(k)} - r(x^{(k)})) \\&= x^{(k)} - (Dr(x^{(k)})^T Dr(x^{(k)}))^{-1} Dr(x^{(k)})^T r(x^{(k)})\end{aligned}$$

- assumes that  $A = Dr(x^{(k)})$  has linearly independent columns
- if converged (i.e.,  $x^{(k+1)} = x^{(k)}$ ) then

$$Dr(x^{(k)})^T r(x^{(k)}) = 0$$

hence  $x^{(k)}$  satisfies the optimality condition since  $\nabla f(x) = 2Dr(x)^T r(x)$

## Gauss-Newton algorithm

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**given** a starting point  $x^{(0)}$  and solution tolerance  $\epsilon$

**repeat for**  $k \geq 0$ :

1. evaluate  $Dr(x^{(k)}) = (\nabla r_1(x^{(k)})^T, \dots, \nabla r_m(x^{(k)})^T)$
2. **if**  $\|2Dr(x^{(k)})^T r(x^{(k)})\|^2 \leq \epsilon$  or  $\|r(x^{(k)})\| < \epsilon$ , stop and output  $x^{(k)}$
3. set  $x^{(k+1)}$  to be the minimizer of  $\|Dr(x^{(k)})x - (Dr(x^{(k)})x^{(k)} - r(x^{(k)}))\|^2$ :

$$x^{(k+1)} = x^{(k)} - (Dr(x^{(k)})^T Dr(x^{(k)}))^{-1} Dr(x^{(k)})^T r(x^{(k)})$$

---

- if  $x^{(k+1)} = x^{(k)}$ , then  $x^{(k)}$  satisfies the optimality condition
- this does not mean that  $x^{(k)}$  is a good solution
- it is common to run the algorithm from different starting points and choose the best solution of these multiple runs

## Issues with Gauss-Newton method

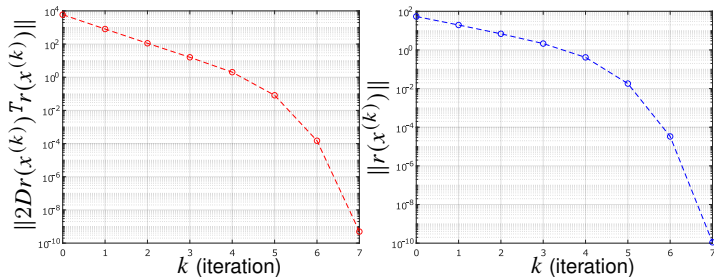
- approximation  $\|r(x)\|^2 \approx \|\hat{r}(x; x^{(k)})\|^2$  holds when  $x$  near  $x^{(k)}$
- when  $x^{(k+1)}$  is not near  $x^{(k)}$ , the affine approximation will not be accurate
- so the algorithm may fail or diverge ( $\|r(x^{(k+1)})\| > \|r(x^{(k)})\|$ )
- a second major issue is that columns of the matrix  $Dr(x^{(k)})$  may not always be linearly independent; in this case, the next iterate is not defined

## Example

$$r(x) = e^x - e^{-x} - 1$$

since  $r'(x) = e^x + e^{-x}$ , the Gauss-Newton iteration is

$$x^{(k+1)} = x^{(k)} - \frac{e^{x^{(k)}} - e^{-x^{(k)}} - 1}{e^{x^{(k)}} + e^{-x^{(k)}}}$$



evolution of error with  $x^{(0)} = 5$ ; the algorithm quickly converges to  $x^\star = 0.4812$

## Example

$$r_i(x) = \sqrt{(x_1 - p_i)^2 + (x_2 - q_i)^2} - \gamma_i, \quad i = 1, \dots, 5$$

where  $p_i, q_i, \gamma_i$  are given

the gradient of  $r_i$  is

$$\nabla r_i(x) = \begin{bmatrix} \frac{x_1 - p_i}{\sqrt{(x_1 - p_i)^2 + (x_2 - q_i)^2}} \\ \frac{x_2 - q_i}{\sqrt{(x_1 - p_i)^2 + (x_2 - q_i)^2}} \end{bmatrix}$$

thus, the Jacobian of  $r$  is

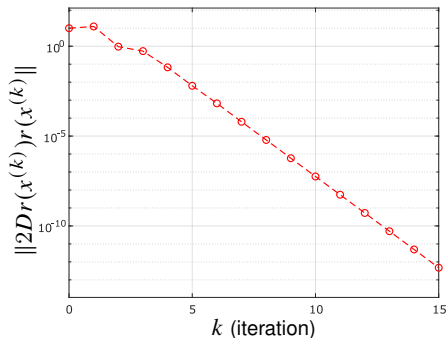
$$Dr(x) = \begin{bmatrix} \frac{x_1 - p_1}{\sqrt{(x_1 - p_1)^2 + (x_2 - q_1)^2}} & \frac{x_2 - q_1}{\sqrt{(x_1 - p_1)^2 + (x_2 - q_1)^2}} \\ \frac{x_1 - p_2}{\sqrt{(x_1 - p_2)^2 + (x_2 - q_2)^2}} & \frac{x_2 - q_2}{\sqrt{(x_1 - p_2)^2 + (x_2 - q_2)^2}} \\ \frac{x_1 - p_3}{\sqrt{(x_1 - p_3)^2 + (x_2 - q_3)^2}} & \frac{x_2 - q_3}{\sqrt{(x_1 - p_3)^2 + (x_2 - q_3)^2}} \\ \frac{x_1 - p_4}{\sqrt{(x_1 - p_4)^2 + (x_2 - q_4)^2}} & \frac{x_2 - q_4}{\sqrt{(x_1 - p_4)^2 + (x_2 - q_4)^2}} \\ \frac{x_1 - p_5}{\sqrt{(x_1 - p_5)^2 + (x_2 - q_5)^2}} & \frac{x_2 - q_5}{\sqrt{(x_1 - p_5)^2 + (x_2 - q_5)^2}} \end{bmatrix}$$

where we assume  $(x_1, x_2) \neq (p_i, q_i)$



results with data

$$p = \begin{bmatrix} 8 \\ 2.0 \\ 1.5 \\ 1.5 \\ 2.5 \end{bmatrix}, \quad q = \begin{bmatrix} 5 \\ 1.7 \\ 1.5 \\ 2.0 \\ 1.5 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 1.87 \\ 1.24 \\ 0.53 \\ 1.29 \\ 1.49 \end{bmatrix}$$



evolution of error with  $x^{(0)} = (1, 3)$ ; algorithm converges to solution  $x^\star = (1.1833, 0.8275)$

## Relation to Newton method for nonlinear equations

- Gauss-Newton update

$$x^{(k+1)} = x^{(k)} - \left( Dr(x^{(k)})^T Dr(x^{(k)}) \right)^{-1} Dr(x^{(k)})^T r(x^{(k)})$$

- if  $m = n$ , then  $Dr(x)$  is square and this is the Newton update

$$x^{(k+1)} = x^{(k)} - Dr(x^{(k)})^{-1} r(x^{(k)})$$

## Relation to Newton method for unconstrained minimization

$$f(x) = \|r(x)\|^2 = \sum_{i=1}^m r_i(x)^2$$

- gradient:

$$\nabla f(x) = 2 \sum_{i=1}^m r_i(x) \nabla r_i(x) = 2Dr(x)^T r(x)$$

- second derivatives:

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(x) = 2 \sum_{i=1}^m \left( \frac{\partial r_i}{\partial x_j}(x) \frac{\partial r_i}{\partial x_k}(x) + r_i(x) \frac{\partial^2 r_i}{\partial x_j \partial x_k}(x) \right)$$

- Hessian

$$\nabla^2 f(x) = 2Dr(x)^T Dr(x) + 2 \sum_{i=1}^m r_i(x) \nabla^2 r_i(x)$$

## Newton and Gauss-Newton steps

- (Undamped) Newton step at  $x = x^{(k)}$ :

$$\begin{aligned}v_{\text{nt}} &= -\nabla^2 f(x)^{-1} \nabla f(x) \\&= -\left(Dr(x)^T Dr(x) + \sum_{i=1}^m r_i(x) \nabla^2 r_i(x)\right)^{-1} Dr(x)^T r(x)\end{aligned}$$

- Gauss-Newton step at  $x = x^{(k)}$ :

$$v_{\text{gn}} = -\left(Dr(x)^T Dr(x)\right)^{-1} Dr(x)^T r(x)$$

can be written as  $v_{\text{gn}} = -H_{\text{gn}}^{-1} \nabla f(x)$  where  $H_{\text{gn}} = Dr(x)^T Dr(x)$

- $H_{\text{gn}}$  is the Hessian without the term  $\sum_i r_i(x) \nabla^2 r_i(x)$

# Comparison

## Newton step

- requires second derivatives of  $f$
- not always a descent direction ( $\nabla^2 f(x)$  is not necessarily positive definite)
- fast convergence near local minimum

## Gauss-Newton step

- Gauss-Newton iteration is cheaper (does not require second derivatives)
- a descent direction (if columns of  $Dr(x)$  are linearly independent):

$$\nabla f(x)^T v_{\text{gn}} = -2v_{\text{gn}}^T Dr(x)^T Dr(x) v_{\text{gn}} < 0 \quad \text{if } v_{\text{gn}} \neq 0$$

- local convergence to  $x^\star$  is similar to Newton method if

$$\sum_{i=1}^m r_i(x^\star) \nabla^2 r_i(x^\star)$$

is small (for each  $i$ ,  $r_i(x^\star)$  is small or  $r_i$  is nearly affine around  $x^\star$ )

## Outline

- linear least squares
- regularized least squares
- nonlinear least squares
- Gauss-Newton method
- **Levenberg-Marquardt method**

## Regularized approximate problem

ensure  $x$  is close to  $x^{(k)}$  by regularization

$$\text{minimize} \quad \|r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})\|^2 + \delta_k \|x - x^{(k)}\|^2$$

- regularization parameter  $\delta_k$  controls how close  $x^{(k+1)}$  is to  $x^{(k)}$
- regularization fixes invertibility issue of Gauss-Newton
- the above problem can be rewritten as

$$\text{minimize} \quad \left\| \begin{bmatrix} Dr(x^{(k)}) \\ \sqrt{\delta_k} I \end{bmatrix} x - \begin{bmatrix} Dr(x^{(k)})x^{(k)} - r(x^{(k)}) \\ \sqrt{\delta_k} x^{(k)} \end{bmatrix} \right\|^2$$

this is just a least squares problem with cost  $\|Ax - b\|^2$  where

$$A = \begin{bmatrix} Dr(x^{(k)}) \\ \sqrt{\delta_k} I \end{bmatrix}, \quad b = \begin{bmatrix} Dr(x^{(k)})x^{(k)} - r(x^{(k)}) \\ \sqrt{\delta_k} x^{(k)} \end{bmatrix}$$

## Levenberg-Marquardt update

the solution is

$$x^{(k+1)} = x^{(k)} - (Dr(x^{(k)})^T Dr(x^{(k)}) + \delta_k I)^{-1} Dr(x^{(k)})^T r(x^{(k)})$$

### Updating $\delta$

- if  $\delta_k$  is very small, then  $x^{(k+1)}$  can be far from  $x^{(k)}$ , and the method may fail
- if  $\delta_k$  is large enough, then  $x^{(k+1)}$  becomes close to  $x^{(k)}$  and the affine approximation will be accurate enough
- a simple way to update  $\delta_k$  is to check whether

$$\|r(x^{(k+1)})\|^2 < \|r(x^{(k)})\|^2$$

if so, then we can decrease  $\delta_{k+1}$ ; otherwise, we increase  $\delta_{k+1}$



# Levenberg-Marquardt algorithm

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**given** a starting point  $x^{(0)}$ , solution tolerance  $\epsilon$ , and  $\delta^{(0)} > 0$

**repeat for**  $k \geq 0$

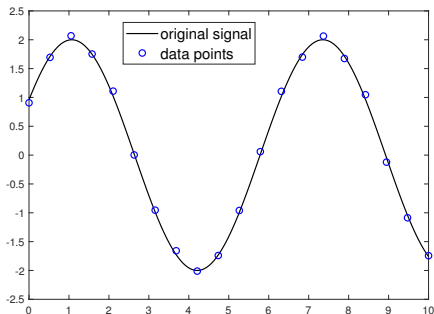
1. evaluate  $Dr(x^{(k)}) = (\nabla r_1(x^{(k)})^T, \dots, \nabla r_m(x^{(k)})^T)$ .
2. **if**  $\|2Dr(x^{(k)})^T r(x^{(k)})\|^2 \leq \epsilon$  or  $\|r(x^{(k)})\| < \epsilon$ , stop and output  $x^{(k)}$
3. set  $x^{(k+1)}$  to be the minimizer of

$$\|r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})\|^2 + \delta_k \|x - x^{(k)}\|^2$$

4. **if**  $\|r(x^{(k+1)})\|^2 < \|r(x^{(k)})\|^2$ , then decrease  $\delta_{k+1}$  (e.g.,  $\delta_{k+1} = 0.8\delta_k$ );  
otherwise, increase  $\delta_{k+1}$  (e.g.,  $\delta_{k+1} = 2\delta_k$ ) and set  $x^{(k+1)} = x^{(k)}$
-

## Example

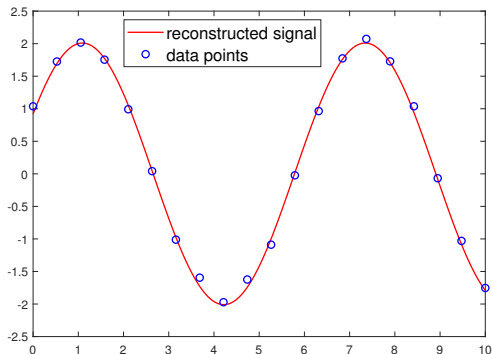
- data-fitting problem with  $r_i(\beta, \omega, \phi) = y^{(i)} - \beta \sin(\omega t^{(i)} + \phi)$
- find  $(\beta, \omega, \phi)$  given  $m = 20$  data points



- for this problem, we have

$$\nabla r_i(\beta, \omega, \phi) = \begin{bmatrix} -\sin(\omega t^{(i)} + \phi) \\ -\beta t^{(i)} \cos(\omega t^{(i)} + \phi) \\ -\beta \cos(\omega t^{(i)} + \phi) \end{bmatrix}$$

- applying Levenberg-Marquardt algorithm results in



## References and further readings

- S. Boyd and L. Vandenberghe. *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares*. Cambridge University Press, 2018. (chapters 12, 13, 14, 15, 18)
- L. Vandenberghe, *EE133A Lecture Notes*, UCLA.
- E. K.P. Chong, Wu-S. Lu, and S. H. Zak. *An Introduction to Optimization: With Applications to Machine Learning*. John Wiley & Sons, 2023. (chapter 12)
- A. Beck. *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with Python and MATLAB*. SIAM, 2023. (chapter 3)