

## 12. Algorithms for constrained optimization

- penalty method
- augmented Lagrangian method
- ADMM
- distributed optimization via ADMM

## Penalized formulation

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

### Penalized formulation

$$\text{minimize} \quad f(x) + \rho P(h(x))$$

- $h(x) = (h_1(x), \dots, h_p(x))$
- $P : \mathbb{R}^p \rightarrow \mathbb{R}$  is the *penalty function*
- $\rho \in \mathbb{R}$  is the *penalty parameter*
- $\rho P(x)$  penalize constraints violation, *i.e.*, has large values for infeasible points

# Penalty function

**Penalty function:** the penalty function  $P$  satisfies the following conditions:

1.  $P$  is continuous
2.  $P(h(x)) \geq 0$  for all  $x \in \mathbb{R}^n$
3.  $P(h(x)) = 0$  if and only if  $x$  is feasible ( $h(x) = 0$ )

**Example: quadratic penalty function**

$$P(h(x)) = \|h(x)\|^2 = \sum_{i=1}^p (h_i(x))^2$$

## Quadratic penalty formulation

$$\text{minimize } f(x) + \rho \|h(x)\|^2$$

- a solution of the above problem might not be feasible
- for large  $\rho$  we expect to have small values  $(h_i(x))^2$   
i.e., an approximate solution to the original problem
- solving the above for an increasing sequence of  $\rho$  is called the *penalty method*

## Quadratic penalty method

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**given** a starting point  $x^{(0)}$ ,  $\rho_0$ , and a solution tolerance  $\epsilon > 0$

**repeat for**  $k = 0, 1, \dots$

1. set  $x^{(k+1)}$  to be the (approximate) minimizer of

$$\text{minimize } f(x) + \rho_k \|h(x)\|^2$$

using an unconstrained optimization method with initial point  $x^{(k)}$

2. update  $\rho_{k+1} = 2\rho_k$
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- terminate if  $\|h(x)\|^2$  is small enough
- simple and easy to implement
- but has a major issue:
  - $\rho_k$  rapidly increases with iterations
  - solving penalty problem can be very slow or simply fail

## Connection to optimality condition

recall the Lagrange optimality conditions:

$$\nabla f(x^\star) + Dh(x^\star)^T \lambda^\star = 0, \quad h(x^\star) = 0$$

- $x^{(k+1)}$  satisfies optimality condition for the unconstrained penalized problem:

$$\nabla f(x^{(k+1)}) + 2\rho_k Dh(x^{(k+1)})^T h(x^{(k+1)}) = 0$$

- letting  $\lambda^{(k+1)} = 2\rho_k h(x^{(k+1)})$ , then

$$\nabla f(x^{(k+1)}) + Dh(x^{(k+1)})^T \lambda^{(k+1)} = 0$$

- so  $x^{(k+1)}$  and  $\lambda^{(k+1)}$  satisfy first equation in the Lagrange optimality condition
- feasibility  $h(x^{(k+1)}) = 0$  is approximately satisfied for  $\rho_k$  large
  - feasibility holds in the limit only  $\rho_k \rightarrow \infty$

## Inequality constraints

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_j(x) = 0, \quad j = 1, \dots, p\end{array}$$

can be handled using the penalized formulation

$$\text{minimize} \quad f(x) + \rho \|h(x)\|^2 + \rho \|g^+(x)\|^2$$

- $g^+(x) = (g_1^+(x), \dots, g_m^+(x))$  and

$$g_i^+(x) = \max\{0, g_i(x)\} = \begin{cases} 0 & \text{if } g_i(x) \leq 0 \\ g_i(x) & \text{if } g_i(x) > 0 \end{cases}$$

- there are other choices of penalty functions
- we just consider the simple quadratic penalization function

# Outline

- penalty method
- **augmented Lagrangian method**
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## Constrained problem

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & h(x) = 0\end{array}$$

- $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $h : \mathbb{R}^p \rightarrow \mathbb{R}$
- Lagrangian:  $L(x, \lambda) = f(x) + \lambda^T h(x)$  where  $\lambda \in \mathbb{R}^p$
- problem is equivalent to penalized formulation

$$\begin{array}{ll}\text{minimize} & f(x) + (\rho/2)\|h(x)\|^2 \\ \text{subject to} & h(x) = 0\end{array}$$

where  $\rho$  is a penalty parameter

## Augmented Lagrangian

the **augmented Lagrangian** (AL) is

$$\begin{aligned}L_{\rho}(x, \lambda) &= L(x, \lambda) + (\rho/2)\|h(x)\|^2 \\ &= f(x) + \lambda^T h(x) + (\rho/2)\|h(x)\|^2\end{aligned}$$

- augmented Lagrangian is the Lagrangian of the penalized problem
  - this is the Lagrangian  $L(x, \lambda)$  augmented with a quadratic penalty
- if  $x^{\star}$  is a solution of original (or penalized) problem and a regular point, then

$$\nabla_x L_{\rho}(x^{\star}, \lambda^{\star}) = 0 \quad \text{for some } \lambda^{\star}$$

- AL method minimizes  $L_{\rho}(x, \lambda)$  for a sequence of values of  $\lambda$  and  $\rho$

## AL and Lagrange multiplier update

- minimizer  $\tilde{x}$  of augmented Lagrangian  $L_\rho(x, \lambda)$  satisfies

$$\nabla f(\tilde{x}) + Dh(\tilde{x})^T(\rho h(\tilde{x}) + \lambda) = 0$$

- if we define  $\tilde{\lambda} = \lambda + \rho h(\tilde{x})$  this can be written as

$$\nabla f(\tilde{x}) + Dh(\tilde{x})^T \tilde{\lambda} = 0$$

- this is the first equation in the optimality conditions

$$\nabla f(x) + Dh(x)^T \lambda = 0, \quad h(x) = 0$$

- shows that if  $h(\tilde{x}) = 0$ , then  $\tilde{x}$  satisfies optimality conditions
- if  $h(\tilde{x})$  is not small, suggests  $\tilde{\lambda}$  is a good update for  $\lambda$
- we hope for large  $\rho$ , minimizer of  $L_\rho(x, \lambda)$  is feasible

## Augmented Lagrangian algorithm

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**given**  $x^{(0)}$ ,  $\lambda^{(0)}$ ,  $\rho^{(0)}$ , and a solution tolerance  $\epsilon > 0$

**repeat for**  $k = 0, 1, \dots$

1. set  $x^{(k+1)}$  to be an (approximate) solution to

$$\text{minimize } f(x) + (\lambda^{(k)})^T h(x) + (\rho_k/2) \|h(x)\|^2$$

using any unconstrained optimization method with initial point  $x^{(k)}$

2. update  $\lambda^{(k)}$ :

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho_k h(x^{(k+1)})$$

3. set  $\rho_k$  as constant or

$$\begin{cases} \rho_k & \text{if } \|h(x^{(k+1)})\| < 0.25 \|h(x^{(k)})\| \\ 2\rho_k & \text{otherwise} \end{cases}$$

- 
- $\rho$  is increased only when needed, more slowly than in penalty method
  - continues until  $h(x^{(k)})$  and/or  $\nabla L(x^{(k)}, \lambda^{(k)})$  are sufficiently small

## Example

consider applying the augmented Lagrangian method to the problem:

$$\begin{array}{ll}\text{minimize} & e^{3x_1} + e^{-4x_2} \\ \text{subject to} & x_1^2 + x_2^2 = 1\end{array}$$

with  $x^{(0)} = (-1, 1)$  and  $\lambda^{(0)} = -1$ , we set a constant penalty parameter  $\rho_k = 10$

the augmented Lagrangian function is

$$L_\rho(x, \lambda) = e^{3x_1} + e^{-4x_2} + \lambda (x_1^2 + x_2^2 - 1) + (\rho/2) (x_1^2 + x_2^2 - 1)^2$$

for the inner minimization problems, we employ Newton's method:

$$\hat{x} \leftarrow \hat{x} + \nabla^2 L_\rho(\hat{x}, \lambda^{(k)})^{-1} \nabla L_\rho(\hat{x}, \lambda^{(k)})$$

the gradient and Hessian are:

$$\nabla L_{\rho}(x, \lambda) = \begin{bmatrix} 3e^{3x_1} + 2\lambda x_1 + 2\rho x_1(x_1^2 + x_2^2 - 1) \\ -4e^{-4x_2} + 2\lambda x_2 + 2\rho x_2(x_1^2 + x_2^2 - 1) \end{bmatrix}$$

and

$$\nabla^2 L_{\rho}(x, \lambda) = \begin{bmatrix} 9e^{3x_1} + 2\lambda + 2\rho(x_1^2 + x_2^2 - 1) + 4\rho x_1^2 & 4\rho x_1 x_2 \\ 4\rho x_1 x_2 & 16e^{-4x_2} + 2\lambda + 2\rho(x_1^2 + x_2^2 - 1) + 4\rho x_2^2 \end{bmatrix}$$

iteration starts from  $\hat{x} = x^{(k)}$  and continues until  $\|\nabla L_{\rho}(\hat{x}, \lambda^{(k)})\| < 10^{-4}$

the value  $x^{(k+1)}$  is then set to  $\hat{x}$  and the Lagrange multiplier is subsequently updated:

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho((x_1^{(k+1)})^2 + (x_2^{(k+1)})^2 - 1)$$

## MATLAB code implementation

```
rho=10;
x=[-1;1];
lam=-1;
%% AL gradient and Hessian
g=@(x,lam)[3*exp(3*x(1))+2*lam*x(1)+2*rho*x(1)*(x(1)^2+x(2)^2-1);
-4*exp(-4*x(2))+2*lam*x(2)+2*rho*x(2)*(x(1)^2+x(2)^2-1)];
hess=@(x,lam)[9*exp(3*x(1))+2*lam+2*rho*(x(1)^2+x(2)^2-1)+4*rho*x(1)^2 4*rho*x(1)*x(2);
4*rho*x(1)*x(2) 16*exp(-4*x(2))+2*lam+2*rho*(x(1)^2+x(2)^2-1)+4*rho*x(2)^2];
%% AL method
for i=1:100
% Newton inner minimization
while (norm(g(x,lam)) >= 1e-4)
d = -hess(x,lam)\g(x,lam);
x = x+d;
end
% Lagrange update
lam=lam+rho*(x(1)^2+x(2)^2-1);
end
```

running for 100 iterations, we get  $x^* = (-0.7483, 0.6633)$  and  $\lambda^* = 0.2123$

## AL for constrained nonlinear least squares

$$\begin{array}{ll}\text{minimize} & \|r(x)\|^2 \\ \text{subject to} & h(x) = 0\end{array}$$

$$r(x) = (r_1(x), \dots, r_m(x)), h(x) = (h_1(x), \dots, h_p(x))$$

### Augmented Lagrangian

$$\begin{aligned}L_\rho(x, \lambda) &= \|r(x)\|^2 + h(x)^T \lambda + (\rho/2) \|h(x)\|^2 \\ &= \|r(x)\|^2 + (\rho/2) \|h(x) + \frac{1}{\rho} \lambda\|^2 - \frac{1}{2\rho} \|\lambda\|^2 \\ &= \left\| \begin{bmatrix} r(x) \\ \sqrt{\rho/2} h(x) + \lambda/(\sqrt{2\rho}) \end{bmatrix} \right\|^2 - \frac{1}{2\rho} \|\lambda\|^2\end{aligned}$$

can be minimized over  $x$  (for fixed  $\rho, \lambda$ ) by Levenberg-Marquardt method:

$$\text{minimize} \quad \left\| \begin{bmatrix} r(x) \\ \sqrt{\rho/2} h(x) + \lambda/(\sqrt{2\rho}) \end{bmatrix} \right\|^2$$



## AL for constrained nonlinear least squares

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**given:**  $\lambda^{(0)} = 0$ ,  $\rho_0 = 1$ , and  $x^{(0)}$

**repeat** for  $k = 0, 1 \dots$

1. set  $x^{(k+1)}$  to be the (approximate) solution of

$$\text{minimize} \quad \left\| \begin{bmatrix} r(x) \\ \sqrt{\rho_k/2}h(x) + \lambda^{(k)}/(\sqrt{2\rho_k}) \end{bmatrix} \right\|^2$$

using Levenberg-Marquardt algorithm, starting from initial point  $x^{(k)}$

2. multiplier update:

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho_k h(x^{(k+1)})$$

3. penalty parameter update:

$$\rho_{k+1} = \begin{cases} \rho_k & \text{if } \|h(x^{(k+1)})\| < 0.25\|h(x^{(k)})\| \\ \rho_{k+1} = 2\rho_k & \text{otherwise} \end{cases}$$

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## ADMM problem form

the alternating direction method of multiplier (ADMM) solves problem of form:

$$\begin{array}{ll}\text{minimize} & f(x) + g(z) \\ \text{subject to} & Ax + Bz = c\end{array}$$

- variables  $x \in \mathbb{R}^n$  and  $z \in \mathbb{R}^m$
- $A \in \mathbb{R}^{p \times n}$ ,  $B \in \mathbb{R}^{p \times m}$ , and  $c \in \mathbb{R}^p$
- augmented Lagrangian

$$L_\rho(x, z, \lambda) = f(x) + g(z) + \lambda^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|^2$$

## ADMM update

$$x^{(k+1)} = \underset{x}{\operatorname{argmin}} L_{\rho}(x, z^{(k)}, \lambda^{(k)})$$

$$z^{(k+1)} = \underset{z}{\operatorname{argmin}} L_{\rho}(x^{(k+1)}, z, \lambda^{(k)})$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho(Ax^{(k+1)} + Bz^{(k+1)} - c)$$

- $\rho > 0$  is the ADMM penalty parameter
- $x$  and  $z$  are updated in an alternating or sequential fashion
- this is different from AL method where  $x$  and  $z$  are minimized jointly

$$(x, z) = \underset{x, z}{\operatorname{argmin}} L_{\rho}(x, z, \lambda^{(k)})$$

- separating the minimization over  $x$  and  $z$  allows to decompose large problems into smaller ones when  $f$  or  $g$  are separable

## ADMM scaled form

define the residual  $r = Ax + Bz - c$  and  $u = (1/\rho)\lambda$ , then

$$\begin{aligned}\lambda^T r + (\rho/2)\|r\|^2 &= (\rho/2)\|r + (1/\rho)\lambda\|^2 - (1/2\rho)\|\lambda\|^2 \\ &= (\rho/2)\|r + u\|^2 - (\rho/2)\|u\|^2\end{aligned}$$

## ADMM scaled form

$$\begin{aligned}x^{(k+1)} &= \underset{x}{\operatorname{argmin}} \left( f(x) + (\rho/2)\|Ax + Bz^{(k)} - c + u^{(k)}\|^2 \right) \\ z^{(k+1)} &= \underset{z}{\operatorname{argmin}} \left( g(z) + (\rho/2)\|Ax^{(k+1)} + Bz - c + u^{(k)}\|^2 \right) \\ u^{(k+1)} &= u^{(k)} + Ax^{(k+1)} + Bz^{(k+1)} - c\end{aligned}$$

## Example: quadratic programs

$$\begin{array}{ll}\text{minimize} & (1/2)x^T Q x + r^T x \\ \text{subject to} & Cx = d \\ & x \geq 0\end{array}$$

- $Q$  is positive semidefinite (reduces to an LP when  $Q = 0$ )
- we can express this problem in the ADMM form:

$$\begin{array}{ll}\text{minimize} & f(x) + g(z) \\ \text{subject to} & x - z = 0\end{array}$$

where

$$f(x) = (1/2)x^T Q x + r^T x, \quad \text{dom } f = \{x \mid Cx = d\}$$

is the original objective with restricted domain

- $g$  is the indicator function of the nonnegative orthant  $\mathbb{R}_+^n$

the scaled form of ADMM consists of the iterations

$$\begin{aligned}x^{(k+1)} &= \underset{x}{\operatorname{argmin}} \left( f(x) + (\rho/2) \|x - z^{(k)} + u^{(k)}\|^2 \right) \\z^{(k+1)} &= (x^{(k+1)} + u^{(k)})_+ \\u^{(k+1)} &= u^{(k)} + x^{(k+1)} - z^{(k+1)}\end{aligned}$$

the  $x$ -update is a constrained least squares problem with optimality conditions

$$\begin{bmatrix} Q + \rho I & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ v \end{bmatrix} + \begin{bmatrix} r - \rho(z^{(k)} - u^{(k)}) \\ -d \end{bmatrix} = 0$$

## Norm-one regularized least squares

the **lasso** problem is the  $\ell_1$  regularized least squares

$$\text{minimize} \quad (1/2)\|Ax - b\|^2 + \eta\|x\|_1$$

- $\eta > 0$  is a scalar regularization parameter
- in ADMM form, the lasso problem can be written as

$$\begin{array}{ll}\text{minimize} & f(x) + g(z) \\ \text{subject to} & x - z = 0\end{array}$$

where  $f(x) = (1/2)\|Ax - b\|^2$  and  $g(z) = \eta\|z\|_1$



the ADMM iteration is

$$x^{(k+1)} = (A^T A + \rho I)^{-1} (A^T b + \rho(z^{(k)} - u^{(k)}))$$

$$z^{(k+1)} = S_{\eta/\rho}(x^{(k+1)} + u^{(k)})$$

$$u^{(k+1)} = u^{(k)} + x^{(k+1)} - z^{(k+1)}$$

where  $S$  is the soft thresholding operator defined element-wise as

$$\begin{aligned} S_{\kappa}(a) &= \begin{cases} a - \kappa & a > \kappa \\ 0 & |a| \leq \kappa \\ a + \kappa & a < -\kappa \end{cases} \\ &= (a - \kappa)_+ - (-a - \kappa)_+ \end{aligned}$$

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## Consensus problem

$$\text{minimize } f(x) = \sum_{i=1}^N f_i(x)$$

- variable  $x \in \mathbb{R}^n$
- $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  represents the  $i$ th component of the objective function
- $f_i$  is available only on machine processor  $i$
- goal is to solve this problem with  $f_i$  handled by processor  $i$  only

## Example

many classification or regression problems can be formulated as:

$$\text{minimize} \quad \sum_{j=1}^m l(x; \xi_j)$$

- $l(x; \xi_j)$  represent the loss function for data  $\xi_j$
- for large  $m$ , storing the data on a single machine may not be feasible
- the problem can be solved by distributing the data across multiple machines,

$$f_i(x) = \sum_{j \in \mathcal{J}_i} l(x; \xi_j),$$

where  $\mathcal{J}_i$  is the set of training data indices at machine  $i$

## Equivalent formulation

$$\begin{array}{ll}\text{minimize} & \sum_{i=1}^N f_i(x_i) \\ \text{subject to} & x_i - z = 0, \quad i = 1, \dots, N\end{array}$$

- $x_i \in \mathbb{R}^n$  handled by processing unit  $i$
- $z$  is a global variable  $z$  handled by central processing unit called *central server*
- the constraints ensure that all local variables are equal
- objective is now separable in the variables  $x_i$
- the augmented Lagrangian is

$$L_\rho(x_1, \dots, x_N, z, \lambda) = \sum_{i=1}^N (f_i(x_i) + (\lambda_i)^T(x_i - z) + \frac{\rho}{2} \|x_i - z\|^2)$$

## ADMM updates

$$x_i^{(k+1)} = \underset{x_i}{\operatorname{argmin}} (f_i(x_i) + \lambda_i^{(k)T}(x_i - z^{(k)}) + \frac{\rho}{2} \|x_i - z^{(k)}\|^2)$$

$$z^{(k+1)} = \frac{1}{N} \sum_{i=1}^N (x_i^{(k+1)} + \frac{1}{\rho} \lambda_i^{(k)})$$

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} + \rho(x_i^{(k+1)} - z^{(k+1)})$$

- the first and last steps are updated independently by each machine  $i$
- central server updates  $z$  after it receives all  $x_i$  and then send it back to machines

## Equivalent simpler update

using overline to denote the average of a vector, we can express the  $z$ -update as:

$$z^{(k+1)} = \bar{x}^{(k+1)} + \frac{1}{\rho} \bar{\lambda}^{(k)}$$

by taking the average of the  $\lambda$ -update, we get:

$$\bar{\lambda}^{(k+1)} = \bar{\lambda}^{(k)} + \rho(\bar{x}^{(k+1)} - z^{(k+1)})$$

substituting the first equation into the subsequent one, we obtain  $\bar{\lambda}^{(k+1)} = 0$  for all  $k$

hence  $z^{(k)} = \bar{x}^{(k)}$  and ADMM can be reformulated as:

$$\begin{aligned} x_i^{(k+1)} &= \underset{x_i}{\operatorname{argmin}} (f_i(x_i) + \lambda_i^{(k)T} (x_i - \bar{x}^{(k)}) + \frac{\rho}{2} \|x_i - \bar{x}^{(k)}\|^2) \\ \lambda_i^{(k+1)} &= \lambda_i^{(k)} + \rho(x_i^{(k+1)} - \bar{x}^{(k+1)}) \end{aligned}$$

## Regularized consensus problem

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^N f_i(x_i) + g(z) \\ & \text{subject to} && x_i - z = 0, \quad i = 1, \dots, N, \end{aligned}$$

- objective term  $g$  is a constraint or regularization (e.g.,  $g(z) = \|z\|_1$ )
- for this case, the ADMM method is:

$$x_i^{(k+1)} = \underset{x_i}{\operatorname{argmin}} (f_i(x_i) + \lambda_i^{(k)T}(x_i - z^{(k)}) + \frac{\rho}{2} \|x_i - z^{(k)}\|^2)$$

$$z^{(k+1)} = \underset{z}{\operatorname{argmin}} (g(z) + \sum_{i=1}^N (-\lambda_i^{(k)T} z + \frac{\rho}{2} \|x_i^{(k+1)} - z\|^2))$$

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} + \rho(x_i^{(k+1)} - z^{(k+1)})$$



- collecting linear and quadratic terms, the  $z$ -update can be expressed as:

$$z^{(k+1)} = \underset{z}{\operatorname{argmin}} \left( g(z) + \frac{N\rho}{2} \|z - \bar{x}^{(k+1)} - \frac{1}{\rho} \bar{\lambda}^{(k)}\|^2 \right)$$

- when  $g$  is nonzero, we don't typically get that  $\bar{\lambda}^{(k)} = 0$
- hence  $\lambda_i$  terms cannot be eliminated as in the non-regularized case
- using the above update form for  $z$ , ADMM is:

$$x_i^{(k+1)} = \underset{x_i}{\operatorname{argmin}} \left( f_i(x_i) + \lambda_i^{(k)T} (x_i - z^{(k)}) + \frac{\rho}{2} \|x_i - z^{(k)}\|^2 \right)$$

$$z^{(k+1)} = \underset{z}{\operatorname{argmin}} \left( g(z) + \frac{N\rho}{2} \|z - \bar{x}^{(k+1)} - \frac{1}{\rho} \bar{\lambda}^{(k)}\|^2 \right)$$

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} + \rho (x_i^{(k+1)} - z^{(k+1)})$$

## Examples

- for  $g(z) = \eta \|z\|_1$ , the  $z$ -update translates into a soft threshold operation:

$$z^{(k+1)} = S_{\eta/N\rho}(\bar{x}^{(k+1)} - \frac{1}{\rho}\bar{\lambda}^{(k)})$$

- considering  $g$  as the indicator function of  $\mathbb{R}_+^n$ , then

$$z^{(k+1)} = (\bar{x}^{(k+1)} - \frac{1}{\rho}\bar{\lambda}^{(k)})_+$$

for this problem, the scaled variant of ADMM, exhibited below, is often more streamlined and manageable compared to its unscaled counterpart:

$$x_i^{(k+1)} = \underset{x_i}{\operatorname{argmin}} (f_i(x_i) + \frac{\rho}{2} \|x_i - z^{(k)} + u_i^{(k)}\|^2)$$

$$z^{(k+1)} = \underset{z}{\operatorname{argmin}} (g(z) + \frac{N\rho}{2} \|z - \bar{x}^{(k+1)} - \bar{u}^{(k)}\|^2)$$

$$u_i^{(k+1)} = u_i^{(k)} + x_i^{(k+1)} - z^{(k+1)}$$

## References and further readings

- I. Griva and S. G. Nash and A. Sofer. *Linear and Nonlinear Optimization*. SIAM, 2009.
- E. K.P. Chong, Wu-S. Lu, and S. H. Zak. *An Introduction to Optimization: With Applications to Machine Learning*. John Wiley & Sons, 2023. (chapter 14)
- S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. *Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers*. Foundations and Trends in Machine learning, 2011.