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# **Vectors and matrices: review**

- vectors
- vector operations
- matrices
- matrix operations
- determinant and inverse
- linear equations

### Vector

a column vector is an ordered list of scalars or numbers, represented by:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{or} \quad a = (a_1, \dots, a_n)$$

- $\bullet$   $a_i$  is the the *i*th *entry* (or element, coefficient, component) of vector a
- i is the *index* of the ith element  $a_i$
- number of elements n is the size (length, dimension) of the vector
- a vector of size n is called an n-vector;  $\mathbb{R}^n$  denote the set of real vectors of size n
- two vectors a, b are equal, denoted a = b, if the have the same size and corresponding entries are all equal

# **Example**

$$a = \begin{bmatrix} 1 \\ -2 \\ 3.3 \\ 0.3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -2 \\ 3.3 \end{bmatrix}$$

- *a* is a 4-vector, *b* is a 3-vector
- third component of a is  $a_3 = 3.3$
- $\bullet$   $a_5, b_4$  does not make sense
- lack a is not equal to b since their dimension is different

## Row vector and transpose

an *row* vector b of size n with entries  $b_1, \ldots, b_n$  has the form:

$$b = [ b_1 \quad b_2 \quad \dots \quad b_n ]$$

- all vectors are column vectors unless otherwise stated
- other notation exists, e.g.,  $b = [b_1, b_2, \dots, b_n]$

**Transpose**: the *transpose* of an *n*-column vector a is the row vector  $a^T$ :

$$a^{T} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^{T} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

- $\blacksquare$  (.)  $^T$  is transpose operation
- $(a^T)^T = a$  (transpose of row vector is a column vector)

### Block vectors, subvectors

### Stacking

- vectors can be stacked (concatenated) to create larger vectors
- stacking vectors b, c, d of size m, n, p gives an (m + n + p)-vector

$$a = \begin{bmatrix} b \\ c \\ d \end{bmatrix} = (b, c, d) = (b_1, \dots, b_m, c_1, \dots, c_n, d_1, \dots, d_p)$$

- we say that b, c, and d are subvectors or slices of a
- example: if b = 1, c = (2, -1), d = (4, 2, 7), then (b, c, d) = (1, 2, -1, 4, 2, 7)

### Subvectors slicing

- colon (:) notation can be used to define subvectors (slices) of a vector
- for vector a, we define  $a_{r:s} = (a_r, \dots, a_s)$
- example: if a = (1, -1, 2, 0, 3), then  $a_{2:4} = (-1, 2, 0)$

# **Special vectors**

#### Zero vector and ones vector

$$0 = (0, 0, \dots, 0), \quad \mathbf{1} = (1, 1, \dots, 1)$$

size follows from context (if not, we add a subscript and write  $0_n, 1_n$ )

#### **Unit vectors**

• there are *n* unit vectors of size *n*, written  $e_1, e_2, \ldots, e_n$ 

$$(e_i)_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

- the *i*th unit vector is zero except its *i*th element which is 1
- example: for n=3,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

• the size of  $e_i$  follows from context (or should be specified explicitly)

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### **Addition and subtraction**

for n-vectors a and b,

$$a+b = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}, \quad a-b = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}$$

### Example

$$\left[\begin{array}{c} 0\\7\\3 \end{array}\right] + \left[\begin{array}{c} 1\\2\\0 \end{array}\right] = \left[\begin{array}{c} 1\\9\\3 \end{array}\right]$$

**Properties:** for vectors a, b of equal size

• commutative: a + b = b + a

■ associative: a + (b + c) = (a + b) + c

# Scalar-vector multiplication

for vector  $a \in \mathbb{R}^n$  and scalar  $\beta$ :

$$\beta a = (\beta a_1, \beta a_2, \dots, \beta a_n)$$

**Properties:** for vectors a, b of equal size, scalars  $\beta, \gamma$ 

• commutative:  $\beta a = a\beta$ 

■ associative:  $(\beta \gamma)a = \beta(\gamma a)$ , we write as  $\beta \gamma a$ 

• distributive with scalar addition:  $(\beta + \gamma)a = \beta a + \gamma a$ 

• distributive with vector addition:  $\beta(a+b) = \beta a + \beta b$ 

#### Linear combination

a *linear combination* of vectors  $a_1, \ldots, a_m$  is a sum of scalar-vector products

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_m a_m$$

- scalars  $\beta_1, \ldots, \beta_m$  are the *coefficients* of the linear combination
- example: any *n*-vector *b* can be written as

$$b = b_1 e_1 + \dots + b_n e_n$$

# Inner product

if a and b are n-vectors, then the *inner product* or *dot product* of a, b is

$$\langle a, b \rangle = a^T b = a_1 b_1 + \dots + a_n b_n$$

- a scalar
- for example

$$\begin{bmatrix} 1 \\ -2 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} -2 \\ 6 \\ 4 \end{bmatrix} = (2)(-2) + (-2)(6) + (0.5)(4) = -14$$

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### **Matrix**

a matrix is a rectangular array of scalars or elements written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- numbers in array are the elements (entries, coefficients, components)
- a horizontal set of elements is called a row and a vertical set is called a column
- $a_{ij}$  is the i, j element of A (i is row index, j is column index)
- *size* (*dimensions*) of the matrix is  $m \times n = (\#rows) \times (\#columns)$
- $\blacksquare$  a matrix of size  $m \times n$  is called an  $m \times n$  matrix
- $\blacksquare$   $\mathbb{R}^{m \times n}$  is set of  $m \times n$  matrices with real elements
- elements  $a_{ii}$  are called principal or main diagonal of the matrix

# **Example**

$$A = \begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -3 \\ 12 & 0 \end{bmatrix}$$

- $\blacksquare$  A is a  $3 \times 4$  matrix, B is  $2 \times 2$
- the matrix A has four columns; B has two rows
- for example,  $a_{23} = -0.1$ ,  $a_{22} = 4$ , but  $a_{41}$  is meaningless
- in A, the row index of the entry with value -2.3 is 1; its column index is 3

# **Matrix shapes**

**Scalar:** a  $1 \times 1$  matrix is a scalar

#### Row and column vectors

- $\blacksquare$  a  $1 \times n$  matrix is called a row vector
- an  $n \times 1$  matrix is called a column vector (or just vector)

### **Tall, wide, square matrices:** an $m \times n$ matrix is

- tall if m > n
- wide if m < n
- square if m = n

# **Matrix equality**

#### A = B means:

- A and B have the same size
- the corresponding entries are equal

for example,

$$\begin{bmatrix} -2\\3.3 \end{bmatrix} \neq \begin{bmatrix} -2 & -3.3 \end{bmatrix}$$

since the dimensions don't agree

•

$$\begin{bmatrix} -2\\3.3 \end{bmatrix} \neq \begin{bmatrix} -2\\3.1 \end{bmatrix}$$

since the 2nd components don't agree

### Columns and rows

an  $m \times n$  matrix can be viewed as a matrix with row/column vectors

### Columns representation

$$A = \begin{bmatrix} a_1 \ a_2 \cdots a_n \end{bmatrix}$$
 each  $a_j$  is an  $m$ -vector (the  $j$ th column of  $A$ ) 
$$a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

### **Rows** representation

$$A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$b_i = [a_{i1} \cdots a_{in}]$$

each  $b_i$  is a  $1 \times n$  row vector (the *i*th row of A)

### **Block matrix and submatrices**

- a block matrix is a rectangular array of matrices
- elements in the array are the *blocks* or *submatrices* of the block matrix

**Example:** a  $2 \times 2$  block matrix

$$A = \left[ \begin{array}{cc} B & C \\ D & E \end{array} \right]$$

- $\blacksquare$  submatrices can be referred to by their block row and column (C is 1, 2 block of A)
- dimensions of the blocks must be compatible
- if the blocks are

$$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 4 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \end{bmatrix}, \quad E = \begin{bmatrix} -1 & 6 & 0 \end{bmatrix}$$

then

$$A = \left[ \begin{array}{cccc} 2 & 0 & 2 & 3 \\ 1 & 5 & 4 & 7 \\ 1 & -1 & 6 & 0 \end{array} \right]$$

### Slice of matrix

$$A_{p:q,r:s} = \begin{bmatrix} a_{pr} & a_{p,r+1} & \cdots & a_{ps} \\ a_{p+1,r} & a_{p+1,r+1} & \cdots & a_{p+1,s} \\ \vdots & \vdots & & \vdots \\ a_{qr} & a_{q,r+1} & \cdots & a_{qs} \end{bmatrix}$$

- an  $(q p + 1) \times (s r + 1)$  matrix
- obtained by extracting from A elements in rows p to q and columns r to s
- from last page example, we have

$$A_{2:3,3:4} = \begin{bmatrix} 1 & 4 \\ 5 & 4 \end{bmatrix}$$

# **Special matrices**

#### Zero matrix

- matrix with  $a_{ij} = 0$  for all i, j
- notation: 0 or  $0_{m \times n}$  (if dimension is not clear from context)
- example:

$$0_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### **Identity matrix**

- square matrix with  $a_{ij} = 1$  if i = j and  $a_{ij} = 0$  if  $i \neq j$
- notation: I or  $I_n$  (if dimension is not clear from context)
- columns of  $I_n$  are unit vectors  $e_1, e_2, \ldots, e_n$ ; for example,

$$I_3 = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[ \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \right]$$

### Structured matrices

matrices with special patterns or structure arise in many applications

### **Diagonal matrix**

- square with  $a_{ij} = 0$  for  $i \neq j$
- represented as  $A = \operatorname{diag}(a_1, \dots, a_n)$  where  $a_i$  are diagonal elements

$$\operatorname{diag}(0.2, -3, 1.2) = \left[ \begin{array}{ccc} 0.2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1.2 \end{array} \right]$$

**Lower triangular matrix:** square with  $a_{ij} = 0$  for i < j

$$\begin{bmatrix} 4 & 0 & 0 \\ 3 & -1 & 0 \\ -1 & 5 & -2 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

**Upper triangular matrix:** square with  $a_{ij} = 0$  for i > j

(a triangular matrix is **unit** upper/lower triangular if  $a_{ii} = 1$  for all i)

# Transpose of a matrix

*transpose* of an  $m \times n$  matrix A is the  $n \times m$  matrix  $(A^T)_{ij} = a_{ji}$ :

example:

$$\begin{bmatrix} 0 & 4 & 7 \\ 0 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

lacktriangleright rows and columns of A are transposed in  $A^T$ 

### **Properties**

- $(A^T)^T = A$
- the transpose of a block matrix (shown for a 2 × 2 block matrix)

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]^T = \left[\begin{array}{cc} A^T & C^T \\ B^T & D^T \end{array}\right]$$

- A, B, C, and D are matrices with compatible sizes
- concept holds for any number of blocks

# Symmetric matrices

a square matrix A is symmetric if

$$A = A^T$$

- $a_{ij} = a_{ji}$
- examples:

$$\begin{bmatrix} 4 & 3 & -2 \\ 3 & -1 & 5 \\ -2 & 5 & 0 \end{bmatrix}, \begin{bmatrix} 4+3j & 3-2j & 0 \\ 3-2j & -j & -2j \\ 0 & -2j & 3 \end{bmatrix}$$

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### Matrix addition

sum of two  $m \times n$  matrices A and B

$$A+B=\left[\begin{array}{ccccc} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{array}\right]$$

### Example

$$\begin{bmatrix} 0 & 4 & 7 \\ 0 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 9 \\ 3 & 3 & 5 \end{bmatrix}$$

matrix subtraction is similar:

$$\begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix} - I = \begin{bmatrix} 0 & 6 \\ 9 & 2 \end{bmatrix}$$

(here we had to figure out that I must be  $2 \times 2$ )

# Properties of matrix addition

- commutativity: A + B = B + A
- associativity: (A + B) + C = A + (B + C), , so we can write as A + B + C
- addition with zero matrix: A + 0 = 0 + A = A
- transpose of sum:  $(A + B)^T = A^T + B^T$

# Scalar-matrix multiplication

scalar-matrix product of  $m \times n$  matrix A with scalar  $\beta$  is entry-wise

$$\beta A = \begin{bmatrix} \beta a_{11} & \beta a_{12} & \cdots & \beta a_{1n} \\ \beta a_{21} & \beta a_{22} & \cdots & \beta a_{2n} \\ \vdots & \vdots & & \vdots \\ \beta a_{m1} & \beta a_{m2} & \cdots & \beta a_{mn} \end{bmatrix}$$

for example,

$$(-2)\begin{bmatrix} 1 & 6 & 9 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -12 & -18 \\ -6 & -12 & 0 \end{bmatrix}$$

**Properties:** for matrices A, B, scalars  $\beta, \gamma$ 

- transposition:  $(\beta A)^T = \beta A^T$
- associativity:  $(\beta \gamma)A = \beta(\gamma A)$
- distributivity:  $(\beta + \gamma)A = \beta A + \gamma A$  and  $\beta(A + B) = \beta A + \beta B$

$$0 \cdot A = 0$$
;  $1 \cdot A = A$ 

# **Matrix-vector product**

product of  $m \times n$  matrix A with n-vector x

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1^Tx \\ \vdots \\ b_m^Tx \end{bmatrix}$$

- $\bullet$   $b_i^T$  is *i*th row of A
- dimensions must be compatible (number of columns of A equals the size of x)
- Ax is a linear combination of the columns of A:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

each  $a_i$  is an m-vector (ith column of A)

# Properties of matrix-vector multiplication

for matrix A, vectors u, v and scalar  $\alpha$ 

- associativity:  $(\alpha A)u = A(\alpha u) = \alpha(Au)$  (we write  $\alpha Au$ )
- distributivity: A(u + v) = Au + Av and (A + A)u = Au + Au
- transposition:  $(Au)^T = u^T A^T$

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## **General examples**

- 0x = 0, *i.e.*, multiplying by zero matrix gives zero
- Ix = x, *i.e.*, multiplying by identity matrix does nothing
- inner product  $a^Tb$  is matrix-vector product of  $1 \times n$  matrix  $a^T$  and n-vector b
- $lacksquare Ae_j=a_j$  , the jth column of  $A\left[(A^Te_i)^T=e_i^TA \text{ is } i\text{th row}\right]$
- the m-vector A1 is the sum of the columns of A

# Matrix multiplication

product of  $m \times n$  matrix A and  $n \times p$  matrix B

$$C = AB$$

is the  $m \times p$  matrix with i, j element

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

- to get  $c_{ij}$ : move along i th row of A, j th column of B
- dimensions must be compatible:

#columns in 
$$A =$$
#rows in  $B$ 

- to find i, j entry of the product C = AB, you need the ith row of A and the jth column of B
  - form product of corresponding entries, e.g., third component of ith row of A and third component of jth column of B
  - add up all the products

## **Examples**

example 1:

$$\begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 11 \\ -3 & -3 \end{bmatrix}$$

for example, to get 1, 1 entry of product:

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} = (1)(0) + (6)(-1) = -6$$

example 2:

$$\begin{bmatrix} -1.5 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3.5 & -4.5 \\ -1 & 1 \end{bmatrix}$$

# Special cases of matrix multiplication

- scalar-vector product (with scalar on right!)  $x\alpha$
- matrix-vector multiplication Ax
- outer product of *m*-vector *a* and *n*-vector *b*

$$ab^{T} = \begin{bmatrix} a_{1}b_{1} & a_{1}b_{2} & \cdots & a_{1}b_{n} \\ a_{2}b_{1} & a_{2}b_{2} & \cdots & a_{2}b_{n} \\ \vdots & \vdots & & \vdots \\ a_{m}b_{1} & a_{m}b_{2} & \cdots & a_{m}b_{n} \end{bmatrix}$$

• multiplication by identity AI = A and IA = A

## **Matrix powers**

- if matrix A is square, then product AA makes sense, and is denoted  $A^2$
- more generally, k copies of A multiplied together gives  $A^k$ :

$$A^k = \underbrace{AA\cdots A}_k$$

by convention we set  $A^0 = I$ 

- $\ \ \,$  (non-integer powers like  $A^{1/2}$  are tricky that's an advanced topic)
- we have  $A^k A^l = A^{k+l}$

# Properties of matrix-matrix product

- associativity: (AB)C = A(BC) so we write ABC
- associativity with scalar multiplication:  $(\gamma A)B = \gamma (AB) = \gamma AB$
- distributivity with sum:

$$A(B+C) = AB + AC$$
,  $(A+B)C = AC + BC$ 

- transpose of product:  $(AB)^T = B^T A^T$
- **not** commutative:  $AB \neq BA$  in general; for example,

$$\left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right] \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \neq \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right]$$

order of multiplication is important

- 0A = 0, A0 = 0 (here 0 can be scalar, or a compatible matrix)
- IA = A, AI = A

### Product of block matrices

block-matrices can be multiplied as regular matrices

**Example:** product of two  $2 \times 2$  block matrices

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \left[\begin{array}{cc} W & Y \\ X & Z \end{array}\right] = \left[\begin{array}{cc} AW + BX & AY + BZ \\ CW + DX & CY + DZ \end{array}\right]$$

if the dimensions of the blocks are compatible

# Column and row representations

### Column representation

■  $A ext{ is } m \times p$ ,  $B ext{ is } p \times n ext{ with columns } b_i$ 

$$AB = A \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix}$$

lacksquare so AB is 'batch' multiply of A times columns of B

### Row representation

lacksquare with  $a_i^T$  the rows of A

$$AB = \left[ \begin{array}{c} a_1^T B \\ a_2^T B \\ \vdots \\ a_m^T B \end{array} \right] = \left[ \begin{array}{c} \left( B^T a_1 \right)^T \\ \left( B^T a_2 \right)^T \\ \vdots \\ \left( B^T a_m \right)^T \end{array} \right]$$

 $\blacksquare$  row i is  $(B^Ta_i)^T$ 

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### Matrix determinant

if A is an  $n \times n$  matrix, then the ijth submatrix of A, denoted by  $A_{ij}$ , is the  $(m-1) \times (m-1)$  obtained by deleting row i and column j of A; for example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

**Determinant:** the *determinant* of a matrix is computed a follows; pick any value of i (i = 1, 2, ..., n) and compute

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij}) a_{ij}$$

- $\det(A_{ij})$  is called the *minor* of element  $a_{ij}$
- $lacksquare (-1)^{i+j}\det(A_{ij})$  is called the *cofactor* of element  $a_{ij}$

# **Example**

- a) for a scalar matrix  $A = [a_{11}]$ , we have  $det(A) = a_{11}$
- b) for a  $2 \times 2$  matrix, the determinant is

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- c) for the matrix  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ 
  - we have for i = 1

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

- thus, the determinant is

$$\det(A) = (-1)^2 a_{11} \det(A_{11}) + (-1)^3 a_{12} \det(A_{12}) + (-1)^4 a_{13} \det(A_{13})$$
$$= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$$
$$= 1(-3) - 2(-6) + 3(-3) = 0$$

### **Properties of determinants**

■ multiplication of a single row/column by a constant: if a single row or column of a matrix, A, is multiplied by a constant, k, forming the matrix,  $\tilde{A}$ , then

$$\det \tilde{A} = k \det A$$

multiplication of all elements by a constant

$$\det(kA) = k^n \det A$$

■ transpose

$$\det A^T = \det A$$

determinant of the product of square matrices

$$\det AB = \det A \det B$$
$$\det AB = \det BA$$

#### Inverse

the matrix  $A^{-1}$  is said to be the **inverse** of the  $n \times n$  matrix A if it satisfies

$$AA^{-1} = A^{-1}A = I_n$$

- if A has an inverse, it is called *invertible* or *nonsingular*
- invertible matrices must be square
- for a non-zero scalar a, the inverse is the number x such that ax = 1, which we denote by  $x = 1/a = a^{-1}$
- a square matrix A is invertible if and only if  $det(A) \neq 0$
- if A doesn't have an inverse, it's called singular or noninvertible

# **Example**

- a) the identity matrix I is invertible, with inverse  $I^{-1} = I$  since (I)I = I
- b) any  $2 \times 2$  matrix A is invertible if and only if  $a_{11}a_{22} \neq a_{12}a_{21}$ , with inverse

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

for example

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

the matrix

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

does not have an inverse; let's see why:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} a-2b & -a+2b \\ c-2d & -c+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

. . . but you can't have a - 2b = 1 and -a + 2b = 0

### c) a diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

is invertible if and only if  $d_{ii} \neq 0$  for i = 1, ..., n, and

$$D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix}$$

## **Properties of inverse**

- $(A^{-1})^{-1} = A$ , *i.e.*, inverse of inverse is original matrix (assuming A is invertible)
- $(AB)^{-1} = B^{-1}A^{-1}$  (assuming A, B are invertible)
- $(A^T)^{-1} = (A^{-1})^T$  (assuming A is invertible)
- $(\alpha A)^{-1} = (1/\alpha)A^{-1}$  (assuming A invertible,  $\alpha \neq 0$ )
- if y = Ax, where  $x \in \mathbb{R}^n$  and A is invertible, then  $x = A^{-1}y$ :

$$A^{-1}y = A^{-1}Ax = Ix = x$$

■ let A be a square invertible matrix, then

$$(A^p)^{-1} = (A^{-1})^p$$

for any integer p

### **Outline**

- vectors
- vector operations
- matrices
- matrix operations
- determinant and inverse
- linear equations

### **Linear functions**

- $f: \mathbb{R}^n \to \mathbb{R}^m$  means f is a function mapping n-vectors to m-vectors
- value is an m-vector  $f(x) = (f_1(x), \dots, f_m(x))$
- example:  $f(x) = (x_1^2, x_2 x_1, x_2)$  is  $f: \mathbb{R}^2 \to \mathbb{R}^3$

**Linear function**  $f: \mathbb{R}^n \to \mathbb{R}^m$  is *linear* if it satisfies *superposition* properties:

- homogeneous (scaling): for any *n*-vector x, any scalar  $\alpha$ ,  $f(\alpha x) = \alpha f(x)$
- **additive:** for any *n*-vectors u and v, f(u+v)=f(u)+f(v)

**Example:** f(x) = y, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} x_3 - 2x_1 \\ 3x_1 - 2x_2 \end{bmatrix}$$

let's check scaling property:

$$f(\alpha x) = \begin{bmatrix} (\alpha x_3) - 2(\alpha x_1) \\ 3(\alpha x_1) - 2(\alpha x_2) \end{bmatrix} = \alpha \begin{bmatrix} x_3 - 2x_1 \\ 3x_1 - 2x_2 \end{bmatrix} = \alpha f(x)$$

# Matrix multiplication and linear functions

general example: f(x) = Ax, where A is  $m \times n$  matrix

- scaling:  $f(\alpha x) = A(\alpha x) = \alpha Ax = \alpha f(x)$
- superposition: f(u+v) = A(u+v) = Au + Av = f(u) + f(v)

so, matrix multiplication is a linear function

#### Converse

- every linear function y = f(x), with y an m-vector and x and n-vector, can be expressed as y = Ax for some  $m \times n$  matrix A
- you can get the coefficients of A from  $a_{ij} = y_i$  when  $x = e_j$

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### **Linear equations**

an equation in the variables  $x_1, \ldots, x_n$  is called *linear* if each side consists of a sum of multiples of  $x_i$ , and a constant, *e.g.*,

$$1 + x_2 = x_3 - 2x_1$$

is a linear equation in  $x_1, x_2, x_3$ 

**Systems of linear equations:** m linear equations in n variables  $x_1, \ldots, x_n$ :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

- can express compactly as Ax = b
- $a_{ij}$  are the coefficients;  $A \in \mathbb{R}^{m \times n}$  is the coefficient matrix
- $b \in \mathbb{R}^m$  is called the *right-hand side*
- may have no solution, a unique solution, infinitely many solutions

## Classification of linear equations

$$Ax = b$$

- under-determined if m < n (A wide; more unknowns than equations)
- square if m = n (A square)
- *over-determined* if m > n (A tall; more equations than unknowns)

# **Example**

two equations in three variables  $x_1, x_2, x_3$ :

$$1 + x_2 = x_3 - 2x_1$$
,  $x_3 = x_2 - 2$ 

step 1: rewrite equations with variables on the lefthand side, lined up in columns, and constants on the righthand side:

$$2x_1 + x_2 - x_3 = -1$$
  
 $0x_1 - x_2 + x_3 = -2$ 

(each row is one equation)

step 2: rewrite equations as a single matrix equation:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

- ith row of A gives the coefficients of the ith equation
- jth column of A gives the coefficients of  $x_i$  in the equations
- ith entry of b gives the constant in the ith equation

## Solving square linear equations

- suppose we have n linear equations in n variables  $x_1, \ldots, x_n$
- compact matrix form: Ax = b, where A is an  $n \times n$  matrix, and b is an n-vector
- suppose A is invertible, *i.e.*, its inverse  $A^{-1}$  exists
- multiply both sides of Ax = b on the left by  $A^{-1}$ :

$$A^{-1}(Ax) = A^{-1}b$$

■ lefthand side simplifies to  $A^{-1}Ax = Ix = x$ , so we've solved the linear equations:

$$x = A^{-1}b$$

## **Square linear equation**

set or system of n linear equations with n variables  $x_1, \ldots, x_n$ :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
  
 $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$   
 $\vdots$   
 $a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$ 

- scalars  $a_{ij}$  are called *coefficients*
- the numbers  $b_i$  are called *right-hand-sides*

#### Matrix notation

$$Ax = b$$

- the  $n \times n$  matrix A is called the *coefficient matrix*
- the *m* vector *b* is called the *right-hand side*

### Cramer's rule

if  $det(A) \neq 0$ , then the square linear system Ax = b has a unique solution

$$x = A^{-1}b$$

we can find the solution using Cramer's formula

$$x_k = \frac{|D_k|}{|A|}, \quad k = 1, 2, \dots, n$$

- lacksquare  $D_k$  is the matrix obtained replacing the kth column of A by b
- from Cramer's formula (with some algebra), we have

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \det A_{11} & \det A_{21} & \cdots & \det A_{n1} \\ \det A_{12} & \det A_{22} & \cdots & \det A_{n1} \\ \vdots & \vdots & \cdots & \vdots \\ \det A_{1n} & \det A_{2n} & \cdots & \det A_{nn} \end{bmatrix}$$

 $A_{i,i}$ , is the  $(m-1) \times (m-1)$  obtained by deleting row i and column j of A

# Example: Cramer's rule

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

the determinant can be written as

$$|A| = \begin{vmatrix} 0.3 & 0.52 & 1\\ 0.5 & 1 & 1.9\\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

the minors are:

$$A_{11} = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 1(0.5) - 1.9(0.3) = -0.07$$

$$A_{12} = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.5(0.5) - 1.9(0.1) = 0.06$$

$$A_{13} = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.5(0.3) - 1(0.1) = 0.05$$

### **Example: Cramer's rule**

$$|A| = 0.3(-0.07) - 0.52(0.06) + 1(0.05) = -0.0022$$

### Solution using Cramer's rule

$$x_{1} = \frac{\begin{vmatrix} -0.01 & 0.52 & 1\\ 0.67 & 1 & 1.9\\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_{2} = \frac{\begin{vmatrix} 0.3 & -0.01 & 1\\ 0.5 & 0.67 & 1.9\\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_{3} = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01\\ 0.5 & 1 & 0.67\\ -0.0022 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

### Linear equations with non-inveretible matrix

when A isn't invertible, i.e., inverse doesn't exist

- one or more of the equations is redundant (i.e., can be obtained from the others)
- the equations are inconsistent or contradictory

in practice:  $\boldsymbol{A}$  isn't invertible means you've set up the wrong equations, or don't have enough of them

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## Solving linear equations in practice

- to solve Ax = b (i.e., compute  $x = A^{-1}b$ ) by computer, we don't compute  $A^{-1}$ , then multiply it by b (but that would work!)
- practical methods compute  $x = A^{-1}b$  directly, via specialized methods (studied in numerical linear algebra)
- standard methods, that work for any (invertible) A, require about  $n^3$  multiplies & adds to compute  $x = A^{-1}b$
- but modern computers are very fast, so solving say a set of 1000 equations in 1000 variables takes only a second or so, even on a small computer
- . . . which is simply amazing

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