Suppose $\lambda = 1$

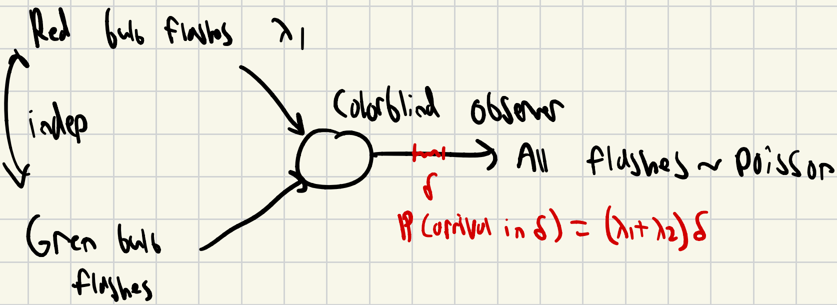
$N[0,2] \sim$ poisson w/ mean 2

indep

$N[2,5] \sim$ Poisson w/ mean 3

0 — 2 — 5

$N[0,2] + N[2,5]$ = Poisson mean = 5



Consider some δ

$$\mathbb{P}(R=1 \cap G=1) = \lambda_1\delta\lambda_2\delta$$
$$\mathbb{P}(R=1 \cap G=0) = \lambda_1\delta(1-\lambda_2\delta)$$
$$\mathbb{P}(R=0 \cap G=0) = (1-\lambda_1\delta)(1-\lambda_2\delta)$$
$$\mathbb{P}(R=0 \cap G=1) = \lambda_2\delta(1-\lambda_1\delta)$$

When $\delta \to 0$ we keep first order terms so $\lambda_1\delta\lambda_2\delta \to 0$

$$\lambda_1\delta(1-\lambda_2\delta) \to \lambda_1\delta$$
$$\lambda_2\delta(1-\lambda_1\delta) \to \lambda_2\delta$$

Given you saw an arrival, prob is red is $\frac{\lambda_1}{\lambda_1+\lambda_2}$

Lec 13 Convergence

1) Deterministic : $a_1, \dots,$ Sequence of $\mathbb{R}$ and $a \in \mathbb{R}$

$$\lim_{n\to\infty} a_n = a \quad \text{if} \quad \varepsilon > 0 \ \exists\ n_0 \text{ s.t}$$

$$|a_n - a| \le \varepsilon \quad \forall n \ge n_0$$

2) Convergence in Probability. Let $Y_1, Y_2 \dots$ seq of R.V

Let $a \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - a| \ge \varepsilon) = 0 \Rightarrow Y_n \text{ Converges to } a \text{ in probability}$$

example : X_n : $X_n \sim$ Bernoulli $\nearrow X_n = 1 : \frac{1}{n}$, $\searrow X_n = 0 : 1 - 1/n$

if $\varepsilon = 1/2$ $a = 0$

$$\mathbb{P}(|X_n - 0| > \tfrac{1}{2}) = \mathbb{P}(X_n = 1) = 1/n \qquad \lim_{n\to\infty} 1/n = 0$$

3) Convergence w/ probability 1 (almost surely)

$Y_1, \dots, Y_n$ Sequence of RVs , $c \in \mathbb{R}$

$$Y_n \to c \ \underline{\text{A.S}} \text{ if } \mathbb{P}(\lim_{n\to\infty} Y_n = c) = 1$$

Borel-Cantelli lemmas

sequence of events $A_1, A_2, \dots$ Define: A_n occurs infinitely often

Example: A_n being the nth flip is heads, this occurs i.o for fair coin

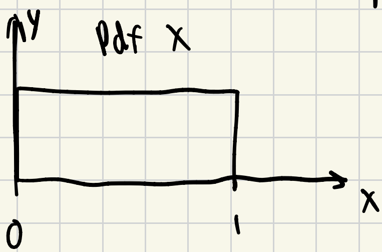
Lemma 1 if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(A_n \text{ i.o}) = 0$

Lemma 2 if A_n indp. and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(A_n \text{ i.o}) = 1$

A.S Convergence vs Deterministic Sequence

RV $X \sim \text{unif}(0,1)$

RV $Y_1, \dots Y_n$ binary $Y_n \begin{cases} \to 1 & \text{if } x \le 1/n \\ \to 0 & x > \frac{1}{n} \end{cases}$



$Y_n \to 0$ a.s but not deterministically

Weak Law of large numbers

$M_n = \dfrac{X_1 + \dots + X_n}{n}$, $\mathbb{E}(M_n) = \dfrac{n\mu}{n} = \mu$ where X_i iid

$$\text{Var}(M_n) = \frac{1}{n^2}\left(\text{var}(X_1) + \dots + \text{var}(X_n)\right) = \frac{n \text{ Var}(X_i)}{n^2} = \sigma^2/n$$

Chebyshev

$$\mathbb{P}(|M_n - \mu| \ge \varepsilon) \le \frac{\sigma^2}{n\varepsilon^2} \quad \forall \varepsilon \ge 0$$

$n \to \infty$

We have Convergence in probability: $\lim_{n \to \infty} \mathbb{P}(|M_n - \mu| \ge \varepsilon) = 0$

WLLN: convergence in prob
SLLN: Convergence of $M_n \to m$ a.s

Let $X_1, \dots, X_n$ iid $M_n = \dfrac{X_1 + \dots + X_n}{n}$

$\mathbb{P}(\lim_{n \to \infty} M_n = \mu) = 1 \leftarrow$ WTS

$\bar{X}_n = 1/n \sum_{k=1}^{n} X_k$, $\bar{X}_n \to$ a.s to p

Define event $A_n : \{|\bar{X}_n - p| > \varepsilon\}$

Goal: Show A_n occurs finitely often

$$\mathrm{Var}(\bar{X}_n) = \frac{n \cdot \mathrm{var}(X_k)}{n^2} = \frac{p(1-p)}{n} = \text{"}\sigma^2\text{"}$$

$$\mathbb{P}(|\bar{X}_n - p| > \varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2}$$ $\leftarrow$ plug into Chebyshev

As $n \to \infty$ $\frac{p(1-p)}{n\varepsilon^2}$ diverges

Trick: $n_k = k^2$ take subsequence $X_1, X_4, X_9, \ldots$

$$\mathbb{P}(A_{n_k}) \leq \frac{p(1-p)}{k^2\varepsilon^2}$$

$$\sum_{k=1}^{\infty} \mathbb{P}(A_{n_k}) \leq \sum_{k=1}^{\infty} \frac{p(1-p)}{k^2\varepsilon^2} < \infty$$

By Gorel lemma

$\mathbb{P}(A_{n_k} \text{ i.o}) = 0$

⋮

Central limit theorum

$S_n = X_1 + \ldots + X_n$ iid

$\mathbb{E}(S_n) = n\mu$

$Var(S_n) = n\sigma^2$

let

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

$\mathbb{E}(Z_n) = 0$

$Var(Z_n) = 1$

The CDF of $Z_n \to \Phi(z)$ Standard normal $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx$

$\to$ convergence in distribution

Example 100 packages each w_i indep ~uniform $(5, 50)$

$\mathbb{P}$(Total weight) > 3000 lbs

$S_n = w_1 + \ldots + w_n$

$$\mathbb{P}(S_n \geq a) \leq \frac{\mathbb{E}(S_n)}{a} \quad \text{(markov)}$$

$$\leq \frac{2750}{3000} = .917$$

$3000 - 2750$

$$\mathbb{P}(|S_n - \mu| > K) \leq \frac{\sigma^2}{K^2}$$

$$\leq \frac{10{,}875}{(250)^2} = .27 \quad \text{if Sym} \cdot \tfrac{1}{2} = .135$$

much tighter

CLT

$z = (3000 - n\mu) / \sigma\sqrt{n} = 1.92$

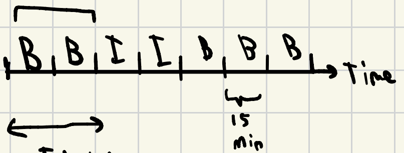
$$\mathbb{P}(S_n \geq z) = 1 - \Phi(z) = 1 - \Phi(1.92) = \boxed{.0274}$$

Lec 14 Stochastic processes

- probabilistic experiment that evolves in time, ex: stock prices
  $X_1, \ldots, X_n$ X_i is R.V from single common experiment and common sample space

Example

B: len of first busy period



Idle vs Busy, $\mathbb{P}(B) = p$
$\mathbb{P}(I) = 1-p$

T: index of first idle slot

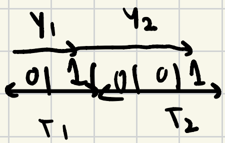
$\mathbb{P}(T=k) = p^{k-1}(1-p)$

$\mathbb{P}(B=k) =$

I: length of first idle period $\mathbb{P}(I=k) = (1-p)^{k-1}p$

Interarrival times:

Y_k: Time of kth success / arrival
T_k: kth interarrival time, $T_i \sim$ Geo



$Y_i = T_1 + \ldots + T_k$ iid geo
$\mathbb{E}(Y_k) = \mathbb{E}(T_1) + \ldots + \mathbb{E}(T_k) = k/p$
$\mathrm{Var}(Y_k) = \mathrm{Var}(T_1) + \ldots + \mathrm{Var}(T_k) = k \cdot (1-p)/p^2$

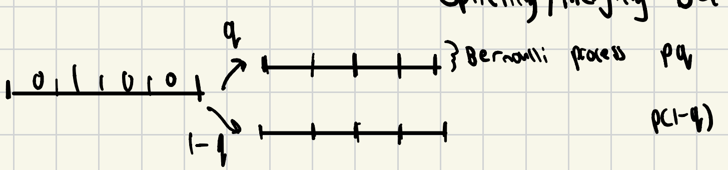
General PMF of kth arrival Y_k

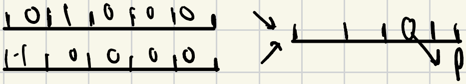
A: kth bin has an arrival, $\mathbb{P}(A) = p$
B: exactly $k-1$ arrivals in first $t-1$ trials

$$\mathbb{P}(Y_k = t) = \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = p \cdot \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} = \binom{t-1}{k-1} p^k (1-p)^{t-k}$$

prob of kth arrival taking t timesteps

Splitting / Merging Bernoulli:





$P(1) = 1-(1-p)(1-q)$

Sum processes

Let $X_1, X_2, \ldots$ be sequence of RVs $\in \{0,1\}$

Let Y be cts RV that takes values in $[0,1]$

Relate X to Y, $Y \in \mathbb{R}$, binary rep as $0.X_1X_2X_3\ldots$

$0 \cdot 1/2 \quad 0 \cdot 1/4 \quad 1/8$

$$Y = \sum_{k=1}^{\infty} 2^{-k} X_k \quad X_k \in \{0,1\}$$

If the X_is form a bernoulli process w/ parameter $p = 1/2$, Show Y is uniformly distributed

- Suppose $Y \in [0, 1/2] \rightarrow X_1$ must be $= 0$, $\mathbb{P}(Y \in [0, 1/2)) = 1/2$

$$Y = \frac{1}{2}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{16}X_4 + \ldots$$

$Y \in [1/2, 3/4) \longrightarrow X_1 = 1, X_2 = 0 \quad \mathbb{P}(X_1 = 1, X_2 = 0) = 1/4$

$$\mathbb{P}\left(Y \in \left[\frac{(i-1)}{2^k}, \frac{i}{2^k}\right)\right) = 1/2^k$$

CDF of Y: $\mathbb{P}(Y \le i/2^k) = i \cdot \left(\frac{1}{2^k}\right)$

$\mathbb{P}(Y \le y) = y$, $y \in [0,1]$, $\mathbb{P}(Y) = \text{Uniform}(0,1]$

Poisson Process

Bernoulli process discretely divides time

Poisson : $\mathbb{P}(k, \tau)$ k arrivals within cts τ $= e^{-\lambda\tau} \dfrac{(\lambda\tau)^k}{k!}$

3 properties define poisson

a) Time homogeneity : $\mathbb{P}(k, \tau_1) = \mathbb{P}(k, \tau_2)$ if $|\tau_1| = |\tau_2|$

b) Independence : # of arrivals in interval is indep of arrivals outside interval. $\mathbb{P}(k, t'-t)$ doesnt change on conditioning outside t'-t

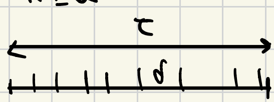
c) Small interval probabilities: τ small enough, events are binary

$\mathbb{P}(0, \tau) = 1 - \lambda\tau$

$\mathbb{P}(1, \tau) = \lambda\tau$

$\mathbb{P}(k, \tau) \approx 0$ for $k \ge 2$

Consider



$\mathbb{P}(1, \delta) = \lambda\delta$

$\mathbb{P}(0, \delta) = 1 - \lambda\delta$

As $\delta \to 0$, looks Bernoulli
$P(K,\tau)$ looks binomial ~ K successes in n bins, $n = \frac{\tau}{\delta}$
$\mathbb{E}(\text{outcomes}) = np = \left(\frac{\tau}{\delta}\right)(\delta\lambda) = \underline{\lambda\tau}$

Binomial converges to poisson for large n small p

Poisson RV: $e^{-\lambda\tau} \frac{(\lambda\tau)}{K!}$ λ: rate per unit time
τ: tim

N_τ: # of arrivals in τ
$\mathbb{E}[N_\tau] = \lambda\tau$ this is Poisson Limit thm.

Lecture 15

Bernoulli process K arrivals in n bins $P_{Y_K} = \binom{t-1}{K-1} p^K (1-p)^{t-K}$

Tricks
$Y = X_1 + \dots + X_N$ $X_i \sim$ iid $\text{Geo}(p)$, $N \sim \text{Geo}(q)$
1) MGF, $M_Y(s) = M_N(\log(M_X(s)))$
2) $P(Y=K) = \sum_{n=1}^{\infty} P(N=n) \cdot P(\sum_i^n X_i = K)$

3) Cast as Bernoulli: $X_1, X_1+X_2,$ rrival times
- X_i interarrival ~ Geo(p)
- each arrival accept/rej w/ prob = q
- first accepted arrival ~ Geo(pq) = Y Ch 6 #22,23

Poisson Process

- arrival is called a poisson process
  a) time homogeneity b) independence c) small interval prob $P(0,\tau) \approx 1-\lambda\tau$
  $P(K,\tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^K}{K!}$ $P(1,\tau) = \lambda\tau$

T: RV describing first arrival

$P(T \le t) = 1 - P(T > t) = 1 - e^{-\lambda t} \frac{(\lambda t)^0}{0!} = 1 - e^{-\lambda t}$
$\frac{d}{dt}(1-e^{-\lambda t}) = \lambda e^{-\lambda t}$ = PDF of first arrival is exponential

	Poisson Proc	Bernoulli Proc
times of arrivals	cts.	discrete
PMF of # of arrivals	Poisson	Binomial
interarrival times	exponential	geometric
arrival rate	λ: arrival / unit time	p: Prob arrival within one bin

Interarrival times

Y_K: Kth arrival $= T_1 + \dots + T_K$, PDF Y_K: erlang PDF of order K , $f_{Y_K}(k) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$

$T_K : Y_K - Y_{K-1}$

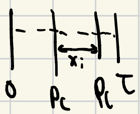
$T_i \sim$ exponential

Example

- Police arrive $\sim Poiss(\lambda)$

1) N : # of police cars in time τ , $\mathbb{E}(N) = \lambda\tau$

2)

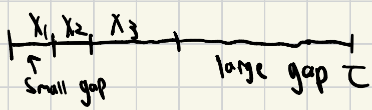


expected wait time to see an interval $\geq \tau$

X: time you wait

$X = X_1 + X_2 + X_3 + \dots$ ← interarrival times b/w Pc arrival

$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{E}(X \mid K=k) \cdot \mathbb{P}(K=k)$



k: # of interarrival $< \tau$ before first large gap

each gap is iid so

$k \cdot \mathbb{E}(X_i \mid X_i < \tau)$

$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot \mathbb{E}(X_i \mid X_i < \tau) \cdot \mathbb{P}(K=k)$

K: arrival process $\sim Geo(p)$, small gap = failure

$= \mathbb{E}(X_i \mid X_i < \tau) \sum_{k=1}^{\infty} k \cdot \mathbb{P}(K=k) = \mathbb{E}(X_i \mid X_i < \tau) \cdot \mathbb{E}(K)$ = expected duration of gaps

- expected # of small gaps

$= \dfrac{\int_0^{\tau} x_i f_{x_i}(x_i)\, dx_i}{\mathbb{P}(X_i < \tau)}$ ← plug in exp ← cdf

$\cdot \frac{1}{p}$

Splitting and merging Poisson Process

- Start w/ Poisson process rate parameter λ → keep arrival w/ prob p → Poisson λp; → discard w/ $1-p$

PP 1 $\sim \lambda_1$, PP 2 $\sim \lambda_2$ → Merged $\sim$ Poisson$(\lambda_1 + \lambda_2)$

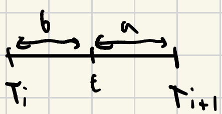
Random incedence paradox

Poisson process $\sim \lambda$

1) Expected interarrival time?
2) Choose some time t: expected length of interarrival interval in which t falls?

(1) interarival $\sim \exp(\lambda)$ $\mathbb{E}$(interarrival time) $= 1/\lambda$

(2)



$$T_{i+1} - T_i = \underbrace{(T_{i+1} - t)}_{a \sim \exp(\lambda)} + \underbrace{(t - T_i)}_{b \sim \exp(\lambda)}$$

$$= \exp(\lambda) + \exp(\lambda)$$

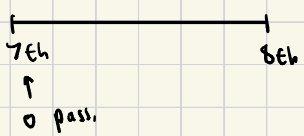
$$= \text{erlang PDF order } 2 = f_{Y_k}(t) = \lambda^2 t e^{-\lambda t}$$

$$\mathbb{E}(\lambda^2 t e^{-\lambda t}) = \frac{2}{\lambda}$$

Bart process

- train stops $\sim$ Poisson process ($\lambda = .25$ trains per min)
- passengers arrive at station $\sim$ ($\mu = 2$ pass / min)
- all waiting passengers get on

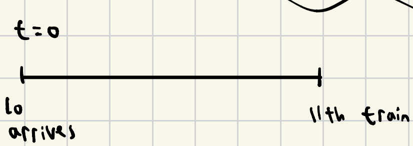
5 min



Dist of # of passengers entering 8th train

$\sim$ Poiss(k, 5 min)

Poisson discrete $(\mu\tau) = (\mu \cdot 5) = (10)$



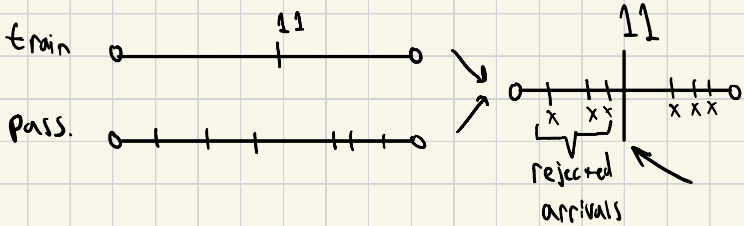
S: arrival time of 11th train

N: # of pass who get on 11th

$$\mathbb{P}(N=k) = \int_0^\infty \mathbb{P}(N=k \mid S=s) \cdot f_S(s)\, ds$$

$$\int_0^\infty e^{-\mu s} \frac{(\mu s)^k}{k!} \cdot \lambda e^{-\lambda s}$$

or merge them



each arrival ~ Poiss($\lambda+\mu$)

Recast as $\underbrace{\text{\# of rejections}}_{\text{passenger arrivals}}$ before first $\underbrace{\text{success}}_{\text{train}}$

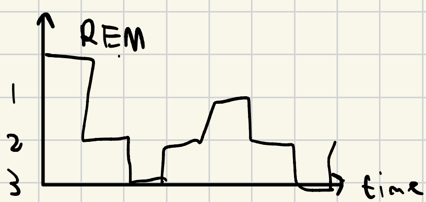
Prob of observing a train → Prob of observing a train before observing passenger

$$\mathbb{P}(\text{train first}) = \mathbb{P}(T_{train} < T_{passenger}) = \int_0^\infty \mathbb{P}(T_{train}=t)\cdot\mathbb{P}(T_{pass.}>t)\,dt$$

$$= \int_0^\infty \lambda e^{-\lambda t}\cdot e^{-\mu t}\,dt$$

$$= \lambda\int_0^\infty e^{-(\lambda+\mu)t}\,dt = \frac{\lambda}{\lambda+\mu}$$

dist of # of passengers who board the train ~ Geo($\frac{\lambda}{\lambda+\mu}$)



Wake : 0
Rem : 1
1 : 2
2 : 3
3 : 4

Discrete time Markov Chains Lec 16

- State Space $\mathcal{X}$, $\{0,1,\ldots,m\}$ ← finite
- Sequence $\underbrace{X_1,\ldots,X_n}_{\text{Process}}$ $X_i \in$ State Space
- X_1 → Corr to "State" at time bin 1
- π_0 : initial State dist, $\mathbb{P}(X_0=i)$, $i = \{0,\ldots,m$
  ↳ [1 0 0 0 0]
- $P_{ij} = \mathbb{P}(X_{n+1}=j \mid X_n=i)$ ← time invariant
- Probability transition matrix : $m \times m$

$$P \stackrel{i}{=} \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0m} \\ P_{10} & \cdots & \cdots & P_{m} \end{bmatrix}$$ Sum over all row vectors = 1

Markov Property

$$\mathbb{P}(X_{n+1}=j \mid X_n=i, X_{n-1}=i-1, \dots, X_0=i_0) = \mathbb{P}(X_{n+1}=j \mid X_n=i)$$

Example

up to date: State 1
behind: State 2

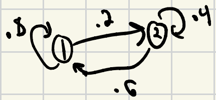
if $U \to \mathbb{P}(U_{i+1} \mid U_i) = .8$
$\to \mathbb{P}(B_{i+1} \mid U_i) = .2$

$B \to \mathbb{P}(B_{i+1} \mid B_i) = .4$
$\to \mathbb{P}(U_{i+1} \mid B_i) = .6$

$$\Pi_0 = [\underset{\substack{\uparrow \\ 1}}{1} \quad \underset{\substack{\uparrow \\ 2}}{0}]$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

Graph



X_0, X_1, X_2
what state shes in on ith week

1	1	2
U	U	B
X_0	X_1	X_2

$$\mathbb{P}(X_0=1, X_1=1, X_2=2)$$
$$= \mathbb{P}(X_0=1) \cdot \mathbb{P}(X_1=1 \mid X_0=1) \cdot \mathbb{P}(X_2=2 \mid X_1=1)$$
$$\pi_0(1) \cdot p_{1,1} \cdot p_{1,2}$$
$$= 1 \cdot (.8) \cdot (.2)$$

n-Step transition probabilites

$$r_{ij} = \mathbb{P}(X_n=j \mid X_0=i)$$

Chapman-Kolmogorov equations

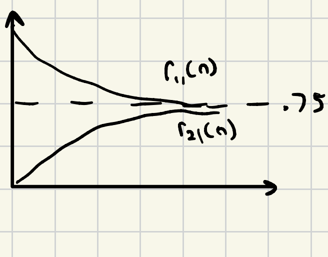
$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj}$$

$$= \sum_{\text{all } x_1, \dots, x_{n-1}} p_{ix_1} \cdot p_{x_1x_2} \cdots p_{n-1j} = p_{ij}^n$$

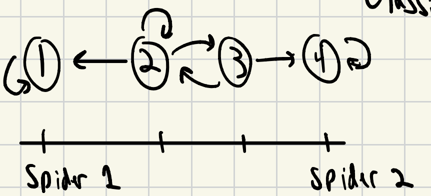
$$\underset{1\times m}{\Pi_0} \cdot \underset{m\times m}{P} = \underset{1\times m}{\Pi}$$ ← distribution over states in timestep 1

$$P = \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

$$r_{ij}(5) = \begin{bmatrix} .75 & .25 \\ .75 & .25 \end{bmatrix}$$
$$= P^5$$



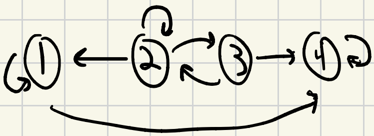
Classification of States



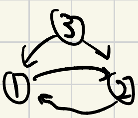
if fly hits spider, its enter and stuck

- recurrent vs. transient states
- Communicating states

Examples
Communicating Class : $A(1) = \{1,2,3,4\}$, $A(2) = \ldots$ All are $\{1,2,3,4\}$ and recurrent



Example



Communicating Class $\{1,2\}$

$A(1) = \{2\}$, $A(2) = \{1\}$ Recurrent

What about 3? Transient, once you leave 3 you can't come back

- A markov chain is irreducible if it consists only of a single communicating class
  $\forall x, y \in$ the chain, can reach y from x and x from y

$\boxed{\text{Periodicity}}$: Property of recurrent class

- a recurrent class is said to be periodic if the states can be grouped into disjoint subsets $S_1, \ldots, S_d$ $d > 1$ where you can only return to a set every k steps

Checking for Periodicity

Check every state in a recurrent class is accessible from some state i within that class at some particular n implies aperiodicity

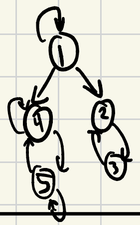
Steady State

- For a single recurrent class that is aperiodic, states j are associated with steady state probabilities π_j that have the following properties:

  a) for each j $\lim_{n\to\infty} r_{ij}(n) = \pi_j$ (convergence in distribution)

  b) the π_j are unique solutions to $\pi = \pi P$ "balance equations"

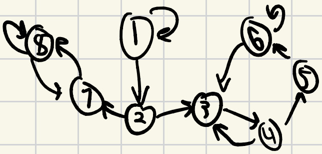
  c) $\pi_j = 0$ for all transient states j, $\pi_j > 0$ for recurrent states



Lec 16

- Communicating classes: states always in communicating classes w/ themselves

Example



Comm. classes: $\{1\}, \{7,8\}, \{3,4,5,6\}, \{2\}$
transient: $\{1\}, \{2\}$
Recurrent: $\{7,8\}, \{3,4,5,6\}$

Period of $\{3,4,5,6\}$? Aperiodic, in 6 steps from 3 we can reach $\{3,4,5,6\}$

Steady State Convergence

- Single recurrent classes that are aperiodic

$\pi_j \to$ steady state probability for state j

a) for each j $\lim_{n\to\infty} \underbrace{r_{ij}(n) = \pi_j}$ "stationary distribution"

$P(X_n = j \mid X_0 = i)$

b) π_j are unique sols to the system of equations

$$\pi_j = \sum_{k=1}^{m} \pi_k P_{kj} \quad \text{for } j = 1, \dots, m$$

$$\pi = \pi P$$