BRAC University
Department of Mathematics and Natural Science
MAT 216: Linear Algebra & Fourier Analysis
Assignment 02
Section 09

Due Date: 07 March, 2024 | Spring 2024 | Total Mark: 170

Answer all Questions

1. **(a)** Determine the values of λ such that the following system of linear equations has: (i) no solution, (ii) more than one solution, (iii) a unique solution. **5**

$$x + \lambda y + z = 1$$
$$x + y + \lambda z = 1$$
$$\lambda x + y + z = 1$$

(b) Determine the values of λ and μ such that the following system of linear equations has: (i) no solution, (ii) more than one solution, (iii) a unique solution. **5**

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

2. (a) Describe the row picture, column picture and matrix picture of these two equations **5**

$$x + y = 3$$
$$x - 2y + = -3$$

(b) Describe the possible types of row pictures of these two equations depending on the parameters **5**

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

3. (a) Calculate the inverse of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$ (i.e. A^{-1}) using Gauss-Jordan elimination. **6**

(b) Consider $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 7 \\ 3 & 7 & 5 \end{bmatrix}$. Justify the statement " A is not invertible". **4**

(c) Use Gauss-Jordan elimination on $[UI]$ to find the U^{-1} **10**

$$UI = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. (a) Check whether the set $\mathbb{R}$ (Set of real numbers) with the usual addition and scalar multiplication is a vector space or not over the field (i) $\mathbb{F} = \mathbb{R}$ (set of real numbers), (ii) $\mathbb{F} = \mathbb{Q}$ (set of rational numbers), (iii) $\mathbb{F} = \mathbb{C}$ (set of complex numbers). Explain with details. **5*3=15**

   (b) Consider the set $\mathbb{R}^2$. A generic element of $\mathbb{R}^2$ is given by the pair (x, y), where $x, y \in \mathbb{R}$. The operations addition and scalar multiplication is defined on $\mathbb{R}^2$ by, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k\,(x_1, y_1) = (kx_1, 0)$ respectively. Is the set $\mathbb{R}^2$ with the above-defined operations a vector space over the field $\mathbb{R}$? If it is a vector space, show that it is closed under the above-defined addition and Scalar multiplication and there exists an additive identity. If it is not, give a counterexample, i.e., give an example that violates any axiom of the vector space. **10**

   (c) Let $\mathbb{R}^+$ be the set of all positive real numbers. Define vector addition and scalar multiplication by **15**

$$x \oplus y = xy \quad \text{for all } x, y \in \mathbb{R}^+$$

$$k \circ x = x^k \quad \text{for all } x \in \mathbb{R}^+ \text{ and } k \in \mathbb{R}$$

   Show that $\mathbb{R}^+$ is a vector space over the field $\mathbb{R}$.

5. Determine which of the following are subspaces of the vector space V. For a subspace, you need to show all three required axioms of subspaces are fulfilled. If it is not a subspace single counter-example will suffice as proof.

   (a) All vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x + 2y + 3z = 5$, where the total vector space $V = \mathbb{R}^3$ over the scalar field $\mathbb{F} = \mathbb{R}$, with usual addition and scalar multiplication. **5**

   (b) Consider the vector space $V = M^{2\times2}$ over the field $\mathbb{R}$ with usual matrix addition and scalar multiplication, where $M^{2\times2}$ is the set of all 2×2 matrices. Now consider a subset consisting of all 2×2 symmetric matrices of the vector space $M^{2\times2}$. Is this subset a subspace over the field $\mathbb{R}$? (Hints: For symmetric matrix $A = A^T$.) **5**

   c) Let S be the subset of R^3 consisting of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying if $x_1 + 2x_3 = 0$. Determine if S is a subspace or not. **5**

   d) Let V be a subset of the vector space R^n consisting only of the zero vector of R^n, Namely $V = \{0\}$. Then prove that V is a subspace of R^n. **5**

6. Determine which sets are vector spaces under the given operations. For those that are not vector spaces, list at least one axiom that fail to hold.

   a) The set of all triples of real numbers (x, y, z) with the operations $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$ and $k(x, y, z) = (0, 0, 0)$. **5**

   b) The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on $\mathbb{R}^2$. **5**

   c) The set of all pairs of real numbers of the form (x, y), where $x^2 + y^2 <= 1$, with the standard operations on $\mathbb{R}^2$. **5**

   d) The set of all pairs of real numbers (x, y) with the operations $(x, y) + (x', y') = (x + x' + 1, y + y' + 1)$ and $k(x, y) = (kx, ky)$. **5**

e) The set of all pairs of real numbers of the form $(1, y)$ with the operations $(1, y)+(1, y') = (1, y + y')$ and $k(1, y) = (1, ky)$. **5**

7. Consider the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 7 \\ 3 & 7 & 5 \end{bmatrix}$

   (a) Find the Row Space and Rank of A. **7**

   (b) Find the Null Space and Nullity of A. **7**

   (c) Show that Rank(A) + Nullity(A) = Number of Columns of A **1**

8. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -5 & -3 \\ 0 & 5 & 15 \end{bmatrix}$

   (i) Calculate the Colums Space of A. **5**

   (ii) Do the vector $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$ lies is in the column space of A ? (Justify) **5**

9. Find the Transformation Matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where, T is defined by,

$$T(x, y, z, w) = (4x - 6y + 5z, 3x - 2y - 3w, 2y - 3x + 4w)$$

   (a) Show that T is linear **5**

   (b) Find the Matrix of T. **2**

   (c) Find the Kernel of T. **6**

   (d) Find the Image of T. **7**

Best wishes