

Divide-and-Conquer: Master Theorem

Neil Rhodes

Department of Computer Science and Engineering
University of California, San Diego

Data Structures and Algorithms
Algorithmic Toolbox

Outline

- 1 What is the Master Theorem
- 2 Proof of Master Theorem

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$



$$T(n) = O(\log n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$



$$T(n) = O(n^2)$$

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$



$$T(n) = O(n^{\log_2 3})$$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

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$$T(n) = O(n \log n)$$

Master Theorem

Theorem

If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$

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If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ (for constants $a > 0, b > 1, d \geq 0$), then:

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$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \end{cases}$$

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Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4$$

$$b = 2$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n^1)$$

$$a = 4$$

$$b = 2$$

$$d = 1$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4$$

$$b = 2$$

$$d = 1$$

Since $d < \log_b a$, $T(n) = O(n^{\log_b a}) = O(n^2)$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$a = 3$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$a = 3$$

$$b = 2$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n^{\textcolor{red}{1}})$$

$$a = 3$$

$$b = 2$$

$$d = \textcolor{red}{1}$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$a = 3$$

$$b = 2$$

$$d = 1$$

Since $d < \log_b a$,

$$T(n) = O(n^{\log_b a}) = O(n^{\log_2 3})$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$a = 2$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$a = 2$$

$$b = 2$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^{\textcolor{red}{1}})$$

$$a = 2$$

$$b = 2$$

$$d = \textcolor{red}{1}$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$a = 2$$

$$b = 2$$

$$d = 1$$

Since $d = \log_b a$,

$$T(n) = O(n^d \log n) = O(n \log n)$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

Master Theorem Example 4

$$T(n) = 1T\left(\frac{n}{2}\right) + O(1)$$

$$a = 1$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$a = 1$$

$$b = 2$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(n^0)$$

$$a = 1$$

$$b = 2$$

$$d = 0$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$a = 1$$

$$b = 2$$

$$d = 0$$

Since $d = \log_b a$, $T(n) = O(n^d \log n) = O(n^0 \log n) = O(\log n)$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

$$b = 2$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

$$b = 2$$

$$d = 2$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

$$b = 2$$

$$d = 2$$

Since $d > \log_b a$, $T(n) = O(n^d) = O(n^2)$

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- ① What is the Master Theorem
- ② Proof of Master Theorem

Master Theorem

Theorem

If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ (for constants $a > 0, b > 1, d \geq 0$), then:

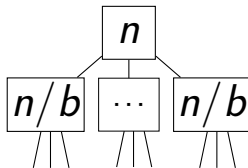
$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

$$\boxed{n}$$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

level

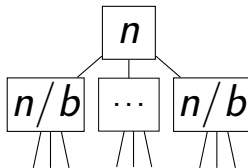


$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

level

0

1



$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

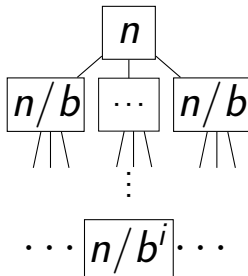
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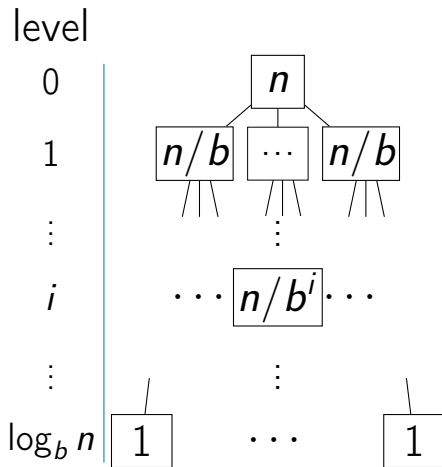
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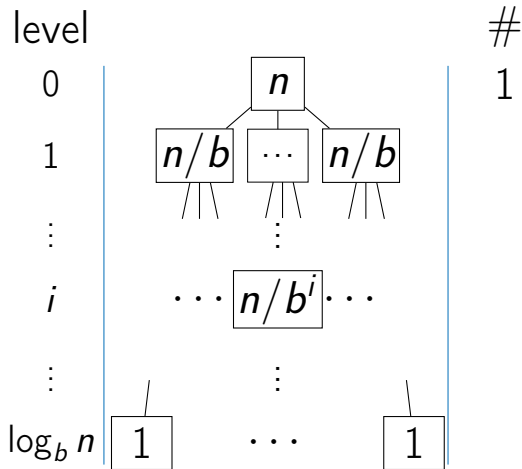
i



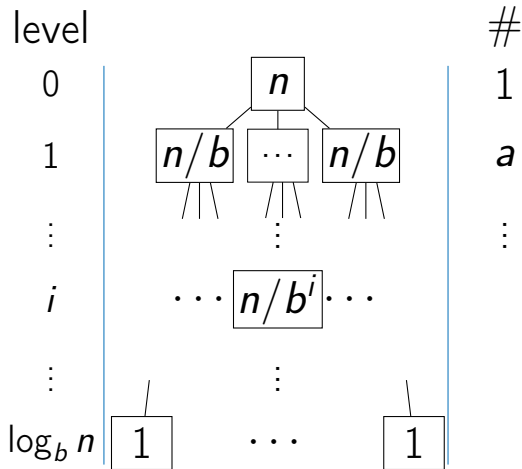
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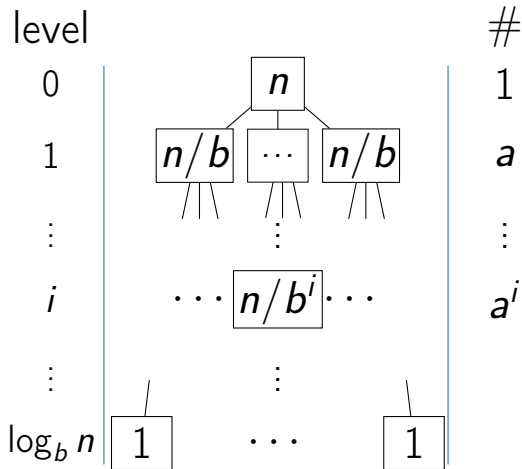
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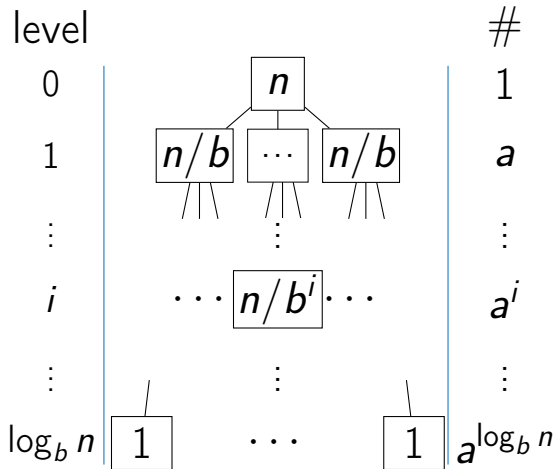
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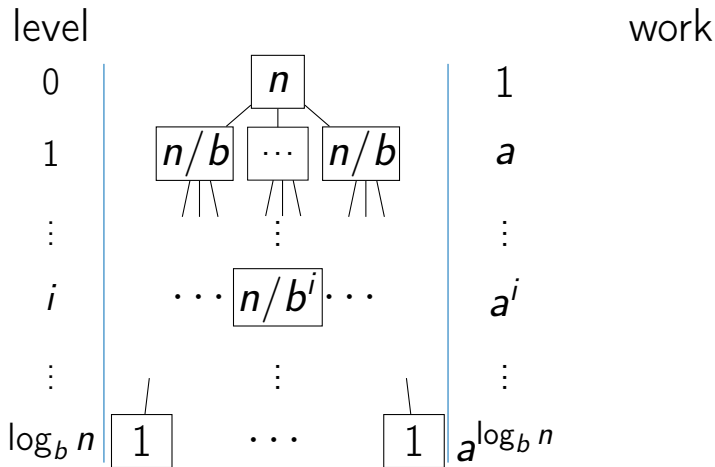
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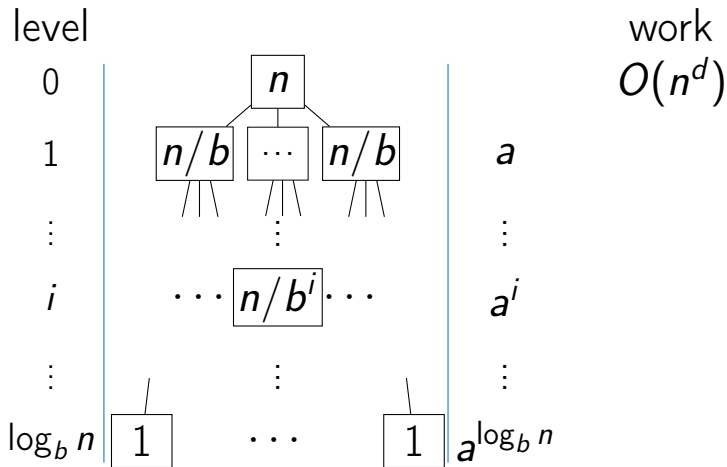
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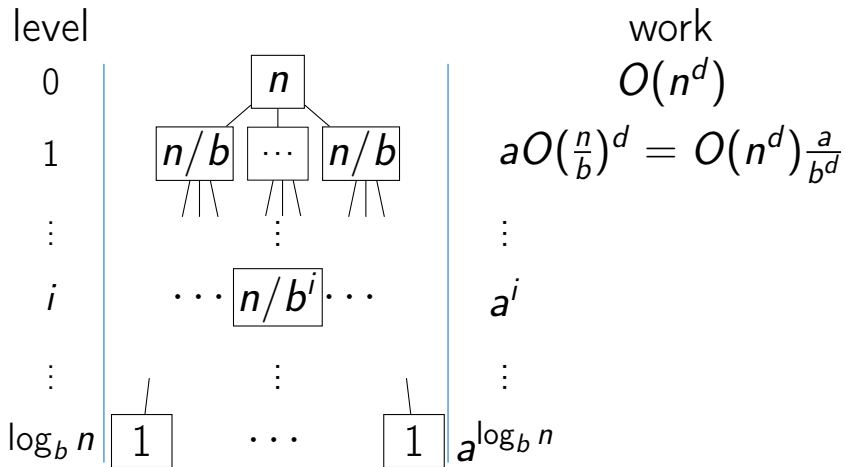
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level

0

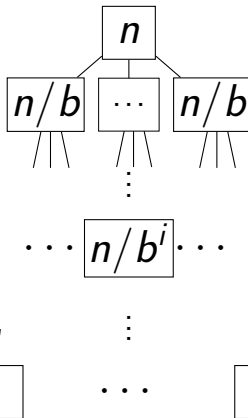
1

\vdots

i

\vdots

$\log_b n$



work

$$O(n^d)$$

$$aO\left(\frac{n}{b}\right)^d = O(n^d) \frac{a}{b^d}$$

\vdots

$$a^i O\left(\frac{n}{b^i}\right)^d = O(n^d) \left(\frac{a}{b^d}\right)^i$$

\vdots

$$a^{\log_b n}$$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

level

0

1

\vdots

i

\vdots

$\log_b n$

n

n/b

\dots

n/b

$\dots n/b^i \dots$

1

\dots

1

work

$O(n^d)$

$$aO\left(\frac{n}{b}\right)^d = O(n^d) \frac{a}{b^d}$$

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\vdots

$$a^{\log_b n} = O(n^{\log_b a})$$

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level

0

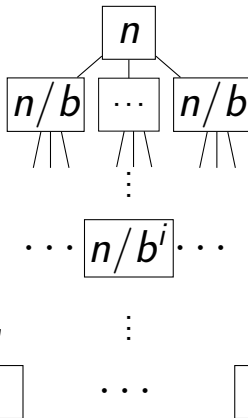
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\vdots

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\vdots

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work

$$O(n^d)$$

$$aO\left(\frac{n}{b}\right)^d = O(n^d) \frac{a}{b^d}$$

\vdots

$$a^i O\left(\frac{n}{b^i}\right)^d = O(n^d) \left(\frac{a}{b^d}\right)^i$$

\vdots

$$a^{\log_b n} = O(n^{\log_b a})$$

$$\text{Total: } \sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d}\right)^i$$

Geometric Series

For $r \neq 1$:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

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For $r \neq 1$:

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Case 1: $\frac{a}{b^d} < 1$ ($d > \log_b a$)

$$\sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i$$

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$$\sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i \\ = O(n^d)$$

Case 2: $\frac{a}{b^d} = 1$ ($d = \log_b a$)

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$$\begin{aligned} & \sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i \\ &= \sum_{i=0}^{\log_b n} O(n^d) \end{aligned}$$

Case 2: $\frac{a}{b^d} = 1$ ($d = \log_b a$)

$$\begin{aligned} & \sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i \\ &= \sum_{i=0}^{\log_b n} O(n^d) \\ &= (1 + \log_b n) O(n^d) \end{aligned}$$

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Case 3: $\frac{a}{b^d} > 1$ ($d < \log_b a$)

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$$\sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i$$
$$= O \left(O(n^d) \left(\frac{a}{b^d} \right)^{\log_b n} \right)$$

Case 3: $\frac{a}{b^d} > 1$ ($d < \log_b a$)

$$\begin{aligned} & \sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i \\ &= O \left(O(n^d) \left(\frac{a}{b^d} \right)^{\log_b n} \right) \\ &= O \left(O(n^d) \frac{a^{\log_b n}}{b^{d \log_b n}} \right) \end{aligned}$$

Case 3: $\frac{a}{b^d} > 1$ ($d < \log_b a$)

$$\begin{aligned}& \sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i \\&= O \left(O(n^d) \left(\frac{a}{b^d} \right)^{\log_b n} \right) \\&= O \left(O(n^d) \frac{a^{\log_b n}}{b^{d \log_b n}} \right) \\&= O \left(O(n^d) \frac{n^{\log_b a}}{n^d} \right)\end{aligned}$$

Case 3: $\frac{a}{b^d} > 1$ ($d < \log_b a$)

$$\begin{aligned}& \sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i \\&= O \left(O(n^d) \left(\frac{a}{b^d} \right)^{\log_b n} \right) \\&= O \left(O(n^d) \frac{a^{\log_b n}}{b^{d \log_b n}} \right) \\&= O \left(O(n^d) \frac{n^{\log_b a}}{n^d} \right) \\&= O(n^{\log_b a})\end{aligned}$$

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