APPM 4440 HW 9

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Problem	Self-Grade	Grade
#1	5	
#2	5	
#3	5	
#4	5	
#5	5	
#6	5	
#7	5	
#8	5	
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e #10	5	
Tot/50	50/50	

1. $\boxed{\text{6.1.1(c)}}$ Let $P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ be a partition of $[0, 1]$. Consider the function $f(x) = -x^2$. Consider any $u, v \in [0, 1]$ s.t. $u < v$. Observe that $u < v \implies u^2 < v^2 \implies -v^2 < u^2$, so $f(x)$ is strictly decreasing on the $[0, 1]$. Then

$$\begin{aligned} &m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \text{ for } 1 \le i \le n,\ n = 3 \\ \implies &m_1 = \inf\{f(x) \mid x \in [0, 1/4]\} = f(1/4) = -1/16, \\ &m_2 = \inf\{f(x) \mid x \in [1/4, 1/2]\} = f(1/2) = -1/4, \\ &m_3 = \inf\{f(x) \mid x \in [1/2, 1]\} = f(1) = -1 \end{aligned}$$

Similarly,

$$\begin{aligned} &M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \text{ for } 1 \le i \le n,\ n = 3 \\ \implies &M_1 = \sup\{f(x) \mid x \in [0, 1/4]\} = f(0) = 0, \\ &M_2 = \sup\{f(x) \mid x \in [1/4, 1/2]\} = f(1/4) = -1/16, \\ &M_3 = \sup\{f(x) \mid x \in [1/2, 1]\} = f(1/2) = -1/4 \end{aligned}$$

We can then compute the lower and upper Darboux sums of f on the partition P of $[a, b]$:

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= -\frac{1}{16}(\frac{1}{4} - 0) - \frac{1}{4}(\frac{1}{2} - \frac{1}{4}) - 1(1 - \frac{1}{2}) \\ &= -\frac{1}{16}(\frac{1}{4}) - \frac{1}{4}(\frac{1}{4}) - 1(\frac{1}{2}) \\ &= -\frac{1}{64} - \frac{4}{64} - \frac{32}{64} = -\frac{37}{64} \\ U(f, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= 0(\frac{1}{4}) - \frac{1}{16}(\frac{1}{4}) - \frac{1}{4}(\frac{1}{2}) \\ &= -\frac{1}{64} - \frac{8}{64} = -\frac{9}{64} \end{aligned}$$

Darboux sums are correct and have been calculated correctly. 5/5

2. $\boxed{\text{6.1.5}}$

Proof. Suppose two bounded functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ have the property that $g(x) \le f(x)$ for all $x \in [a, b]$. We will show that (1) for a partition P of $[a, b]$, $L(g, P) \le L(f, P)$ and (2) $\underline{\int_a^b} g \le \underline{\int_a^b} f$. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$. Suppose that $m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$ and $m'_i = \inf\{g(x) \,|\, x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$ Suppose for any i that $m'_i > m_i$. Then $\exists x_0, x'_0 \in [x_{i-1}, x_i]$ s.t. $m_i = f(x_0) < g(x'_0) = m'_i$. Since $g(x'_0) = m'_i = \inf\{g(x) \,|\, x \in [x_{i-1}, x_i]\}$, it follows that $\forall x \in [x_{i-1}, x_i]$, $g(x'_0) \le g(x)$. But $x_0 \in [x_{i-1}, x_i]$ and by assumption, $g(x_0) \le f(x_0)$, which implies $g(x'_0)$ is not actually an infimum of $g(x)$ for $x \in [x_{i-1}, x_i]$. By contradiction, $m'_i \le m_i$. Then the lower Darboux sums can be related:

$$L(g, P) = \sum_{i=1}^{n} m'_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = L(f, P) \implies L(g, P) \le L(f, P)$$

Now consider any two partitions of $[a, b]$ P_1 and P_2 s.t. $\underline{\int_a^b} g = L(g, P_1) = \sup\{L(g, P) \,|\, P \text{ is a partition of } [a, b]\}$ and $\underline{\int_a^b} f = L(f, P_2) = \sup\{L(f, P) \,|\, P \text{ is a partition of } [a, b]\}$. Suppose $\underline{\int_a^b} f < \underline{\int_a^b} g$. By definition, $\underline{\int_a^b} f$ is the supremum of all lower Darboux sums of f over $[a, b]$, so given any partition of $[a, b]$ P, it is true that $L(f, P) \le L(f, P_2)$. Then $\underline{\int_a^b} f < \underline{\int_a^b} g \implies L(f, P_2) = \underline{\int_a^b} f < \underline{\int_a^b} g = L(g, P_1) \le L(f, P_1)$, which implies that $L(f, P_2)$ is not actually an upper bound on all lower Darboux sums of f on $[a, b]$. By contradiction, $\underline{\int_a^b} g \le \underline{\int_a^b} f$. QED

Proceeds in slightly different way than key but still appears to be correct. 5/5

3. $\boxed{6.1.6}$

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $L(f, P) = U(f, P)$. We will show that f is constant. We know that $L(f, P) = U(f, P) \implies \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, where $m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$. Since each m_i and M_i are lower and upper bounds of f on a particular interval, it follows that $m_i \le M_i$. Assume for some i that $M_i > m_i$. We know that

$$m_1(x_1 - x_0) + \cdots + m_n(x_n - x_{n-1}) = M_1(x_1 - x_0) + \cdots + M_n(x_n - x_{n-1}),$$

so

$$(M_1 - m_1)(x_1 - x_0) + \cdots + (M_i - m_i)(x_i - x_{i-1}) + \cdots + (M_n - m_n)(x_n - x_{n-1}) = 0.$$

Subtracting $(M_i - m_i)(x_i - x_{i-1})$ from both sides yields

$$(M_1 - m_1)(x_1 - x_0) + \cdots + (M_n - m_n)(x_n - x_{n-1}) = -(M_i - m_i)(x_i - x_{i-1}),$$

which implies that the remaining sum is negative. If all terms are zero, then this sums to zero, which is nonnegative. If at least one term is positive and the rest are nonnegative, then positivity and zero being the additive identity imply that the sum will be positive, which is also nonnegative. Regardless, it is clear that all terms cannot be nonnegative, so at least one must be negative. Then for at least one other $j \ne i$, it will be true that $(M_j - m_j)(x_j - x_{j-1}) < 0$. Since P is a partition of $[a, b]$, $x_j - x_{j-1} > 0$, so $M_j - m_j < 0 \implies m_j > M_j$. But $m_j, M_j \in \{f([x_{j-1}, x_j])\}$, which implies M_j is not an upper bound of $\{f([x_{j-1}, x_j])\}$. So the assumption that $m_i < M_i$ was incorrect. By way of contradiction, it is therefore true for all $1 \le i \le n$ that $m_i = M_i \implies \inf\{f([x_{i-1}, x_i])\} = \sup\{f([x_{i-1}, x_i])\}$. Choose any $y \in f([x_{i-1}, x_i])$. It follows that $m_i \le y \le M_i$, but $m_i = M_i \implies y = m_i = M_i$, so the function is constant on each interval $[x_{i-1}, x_i]$. We can show that $f(x) = c$ across all interval via a form of induction. It follows directly from our previous result that $f([x_0, x_1]) = c$. Now suppose $f([x_{i-1}, x_i]) = c$ for some $1 \le i \le n$. We'll show $f([x_i, x_{i+1}]) = c$. We know that $x_i \in [x_{i-1}, x_i]$. By our inductive hypothesis $f(x_i) = c$. But we asserted earlier that $f(x)$ must be constant on any partition interval. Since $x_i \in [x_i, x_{i+1}]$, it follows that $f([x_i, x_{i+1}]) = c$. Thus f is constant over every partition interval. Since the partition covers the domain of f, we can conclude that f is constant over its domain. QED

Proceeds in nearly same way as proof with very similar steps. 5/5

4. $\boxed{6.2.4}$

(a) *Proof.* We will show that for $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. We will proceed by induction. When $n = 1$, we see that $\sum_{i=1}^{1} i = 1 = \frac{2}{2} = \frac{1(1+1)}{2}$. Now suppose for some $k \in \mathbb{N}$ that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. We will show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$:

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1) + 2(k+1)}{2}
\end{aligned}$$

$$= \frac{(k+2)(k+1)}{2}$$

Thus concludes the inductive step, proving the initial claim. QED

(b) *Proof.* Consider the function $f : [a, b] \to \mathbb{R}$ where $f(x) = x$. We claim that $\int_a^b f = \frac{b^2-a^2}{2}$. Let P_n be the regular partition of $[a, b]$ into n equally-sized intervals such that $x_i - x_{i-1} = \frac{b-a}{n}$. Then,

$$L(f, P_n) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$U(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Since $m_i = \inf\{f([x_{i-1}, x_i])\}$, $M_i = \sup\{f([x_{i-1}, x_i])\}$, and f is strictly increasing, it follows that $m_i = f(x_{i-1}) = x_{i-1}$ and $M_i = f(x_i) = x_i$. Since P_n is a regular partition, we can write $x_j = a + j\frac{b-a}{n}$:

$$\begin{aligned}
L(f, P_n) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \left(a + (i-1)\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^{n} a\frac{b-a}{n} + (i-1)\left(\frac{b-a}{n}\right)^2 \\
&= a(b-a) + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n}(i-1) \\
&= a(b-a) + \left(\frac{b-a}{n}\right)^2 \left(\frac{n(n+1)}{2} - n\right) \\
U(f, P_n) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^{n} a\frac{b-a}{n} + i\left(\frac{b-a}{n}\right)^2 \\
&= a(b-a) + \left(\frac{b-a}{n}\right)^2 \left(\frac{n(n+1)}{2}\right)
\end{aligned}$$

Then we can show that

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \left(\frac{b-a}{n}\right)^2 \cdot n \\
&= \frac{(b-a)^2}{n} \implies \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0
\end{aligned}$$

The Archimedes-Riemann Theorem thus establishes that f is integrable and that $\int_a^b f = \lim_{n\to\infty} U(f, P_n)$:

$$\begin{aligned}
\lim_{n\to\infty} U(f, P_n) &= \lim_{n\to\infty} a(b-a) + \left(\frac{b-a}{n}\right)^2 \left(\frac{n(n+1)}{2}\right) \\
&= a(b-a) + \lim_{n\to\infty} \frac{(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2}
\end{aligned}$$

$$= a(b-a) + \frac{(b-a)^2}{2} \lim_{n\to\infty} \frac{(n+1)}{n}$$

Pick any $\epsilon > 0$. Observe that $|\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n}$. By the Archimedean principle there exists an N s.t. $\forall n \geq N$, $|\frac{n+1}{n} - 1| < \epsilon$. So $\frac{n+1}{n} \to 1$ and it follows that $\lim_{n\to\infty} U(f, P_n) = a(b-a) + \frac{(b-a)^2}{2} = \frac{2ab-2a^2}{2} + \frac{b^2-2ab+a^2}{2} = \frac{b^2-a^2}{2}$. Therefore $\int_a^b x\,dx = \int_a^b f = \frac{b^2-a^2}{2}$. QED

Proof by induction appears to be correct. Proof for part (b) similarly uses the limit of the Darboux sum to find the integral. 5/5

5. $\boxed{\text{6.2.5(b)}}$ We want to find $\int_0^1 [4x+1]\,dx$. Let us denote $f(x) = 4x+1$ on the domain $[0,1]$. Note that f is strictly increasing, so if we construct a regular partition P_n of $[0,1]$ such that $x_i - x_{i-1} = \frac{1-0}{n} = \frac{1}{n}$, we see that $m_i = 4x_{i-1}+1$ and $M_i = 4x_i + 1$ for any set $f([x_{i-1}, x_i])$. Then

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} (4x_{i-1}+1)(x_i - x_{i-1}) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(\frac{i-1}{n}\right)+1\right] \\ &= \frac{4}{n^2}\sum_{i=1}^{n}(i-1) + \sum_{i=1}^{n}\frac{1}{n} \\ &= \frac{4}{n^2}\left(\frac{n(n-1)}{2}\right)+1 \\ &= 2\frac{n-1}{n}+1 \end{aligned}$$

Similarly,

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} (4x_i+1)(x_i - x_{i-1}) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(\frac{i}{n}\right)+1\right] \\ &= \frac{4}{n^2}\sum_{i=1}^{n} i + \sum_{i=1}^{n}\frac{1}{n} \\ &= \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right)+1 \\ &= 2\frac{n+1}{n}+1 \end{aligned}$$

Observe that

$$\begin{aligned} \lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] &= \lim_{n\to\infty}\left[2\frac{n+1}{n} + 1 - \left(2\frac{n-1}{n}+1\right)\right] \\ &= \lim_{n\to\infty}\left[2\frac{n+1}{n} - 2\frac{n-1}{n}\right] \\ &= 2\lim_{n\to\infty}\frac{2}{n} = 0 \end{aligned}$$

The Archimedes-Riemann Theorem thus establishes that $f(x) = 4x+1$ is integrable over $[0,1]$ and that $\int_0^1[4x+1]\,dx = \lim_{n\to\infty} U(f,P_n)$. Observe that $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty}\left[2\frac{n+1}{n}+1\right] = 1 + 2\lim_{n\to\infty}\frac{n+1}{n}$. We found in 4(b) that $\lim_{n\to\infty}\frac{n+1}{n} = 1$, so $1 + 2\lim_{n\to\infty}\frac{n+1}{n} = 3 \implies \int_0^1[4x+1]\,dx = 3$. Uses the Darboux sum to find the correct integral value. 5/5

6. $\boxed{\text{6.2.9}}$

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are integrable. We will show that there is a sequence $\{P_n\}$ of partitions of $[a,b]$ that is an Archimedean sequence of partitions for f on $[a,b]$ and also an Archimedean sequence of partitions for g on $[a,b]$. Since f and g are integrable, by the Archimedes-Riemann Theorem there exist sequences $\{A_n\}$ and $\{B_n\}$ of partitions such that $\lim_{n\to\infty}[U(f,A_n) - L(f,A_n)] = 0$ and $\lim_{n\to\infty}[U(g,B_n) - L(g,B_n)] = 0$. Now, let P_n

be the common refinement of A_n and B_n. By Lemma 6.2, $L(f,A_n) \leq L(f,P_n)$ and $U(f,P_n) \leq U(f,A_n)$. It follows that $L(f,A_n)+U(f,P_n) \leq L(f,P_n)+U(f,A_n) \implies U(f,P_n)-L(f,P_n) \leq U(f,A_n)-L(f,A_n)$. Since $L(f,P) \leq U(f,P)$ for any partition P, the LHS and RHS of the inequality are both nonnegative and thus $U(f,P)-L(f,P) = |U(f,P)-L(f,P)|$:

$$|U(f,P_n) - L(f,P_n)| \leq |U(f,A_n) - L(f,A_n)|$$

The Comparison Lemma thus establishes that $\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)] = 0$, so P_n is an Archimedean sequence of f on $[a,b]$. Similarly, Lemma 6.2 (Refinement Lemma) can be used to show that $|U(g,P_n)-L(g,P_n)| \leq |U(f,B_n)-L(f,B_n)|$. By the same logic as with f and P_n, it follows that P_n is an Archimedean sequence of g on $[a,b]$. QED

Similarly uses the refinement lemma to construct an Archimedean sequence for both f and g. 5/5

7. $\boxed{6.2.12}$

Proof. Suppose that the function $f : [a,b] \to \mathbb{R}$ is Lipschitz, i.e. $\exists c \geq 0$ s.t. $|f(u)-f(v)| \leq c|u-v|$ for all points u,v in $[a,b]$. For a partition P of $[a,b]$, we will show that $0 \leq U(f,P)-L(f,P) \leq c[b-a]\cdot \operatorname{gap} P$. Let us use any partition of $[a,b]$, P. Because $m_i = \inf\{f([x_{i-1}, x_i])\}$ and $M_i = \sup\{f([x_{i-1}, x_i])\}$, it follows that $m_i \leq M_i$. Since $L(f,P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ and $U(f,P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$, it follows directly that $L(f,P) \leq U(f,P) \implies 0 \leq U(f,P)-L(f,P)$.

Observe that $0 < [x_i - x_{i-1}] \leq \max\{x_i - x_{i-1}\} = \operatorname{gap} P \implies [x_i - x_{i-1}]^2 \leq [x_i - x_{i-1}]\cdot \operatorname{gap} P$. Taking the summation of both sides yields $\sum_{i=1}^n [x_i - x_{i-1}]^2 \leq \operatorname{gap} P \cdot \sum_{i=1}^n [x_i - x_{i-1}] = \operatorname{gap} P \cdot [b-a]$. So $\sum_{i=1}^n [x_i - x_{i-1}]^2 \leq [b-a]\cdot \operatorname{gap} P$.

Since f is Lipschitz, it is also continuous. For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{c}$. Then, for any $u,v \in [a,b]$, given $|u-v| < \delta$, we can say $|f(u)-f(v)| \leq c|u-v| < c\frac{\epsilon}{c} = \epsilon$. Therefore f is continuous over each subinterval $[x_{i-1}, x_i] \subseteq [a,b]$. Then EVT tells us that f attains a minimum and maximum in each subinterval. Therefore $m_i = \inf\{f([x_{i-1}, x_i])\}$ and $M_i = \sup\{f([x_{i-1}, x_i])\}$ both exist and it follows that $M_i - m_i = |M_i - m_i|$ If we consider the preimages x_{m_i} and x_{M_i} s.t. $m_i = f(x_{m_i})$ and $M_i = f(x_{M_i})$, we know that $x_{m_i}, x_{M_i} \in [x_{i-1}, x_i] \implies |x_{M_i} - x_{m_i}| \leq |x_i - x_{i-1}|$ because x_{m_i} and x_{M_i} are at least x_{i-1} and at most x_i. Then

$$\begin{aligned}
U(f,P)-L(f,P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n |M_i - m_i|(x_i - x_{i-1}) \\
&= \sum_{i=1}^n |f(x_{M_i}) - f(x_{m_i})|(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n c|x_{M_i} - x_{m_i}|(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n c|x_i - x_{i-1}|(x_i - x_{i-1}) \\
&= c\sum_{i=1}^n [x_i - x_{i-1}]^2 \\
&\leq c[b-a]\cdot \operatorname{gap} P
\end{aligned}$$

Hence $0 \leq U(f,P)-L(f,P) \leq c[b-a]\cdot \operatorname{gap} P$. QED

Proceeds using very similar steps as proof to arrive at same conclusion. 5/5

8. $\boxed{6.3.1}$ Suppose that f, g, f^2, g^2, and fg are integrable on the closed bounded interval $[a,b]$. We will first show that $[f-g]^2$ is also integrable on $[a,b]$ and that $\int_a^b [f-g]^2 \geq 0$. We know that $[f-g]^2 = f^2 - 2fg + g^2$. Since each term is integrable on $[a,b]$ by assumption, it follows from the linearity of the integral that $\int_a^b [f-g]^2 = \int_a^b [f^2 - 2fg + g^2]$ is also integrable, with coefficients 1, -2, and 1. By definition we know $[f-g]^2 \geq 0$ and the Archimedes-Riemann Theorem tells us that there is a sequence of partitions of $[a,b]$ s.t. $\int_a^b [f-g]^2 = \lim_{n\to\infty} U([f-g]^2, P_n)$. For any $P_n = \{x_0, \ldots, x_k\}$, we know $U([f-g]^2, P_n) = \sum_{i=1}^k M_i(x_i - x_{i-1})$, where $M_i \in \{h(x) \,|\, x \in [x_{i-1}, x_i]\}$ (given $h(x) = [f-g]^2 \geq 0$). Thus $M_i \geq 0 \implies U([f-g]^2, P_n) \geq 0$. So the sequence $\{U([f-g]^2, P_n)\}$ is always nonnegative. Supposing it converges to some negative number $I < 0$, we see that there must be Darboux sums in the neighborhood of $(I-\epsilon, I+\epsilon)$ for any $\epsilon > 0$. When $\epsilon < |I| = -I$, $I + \epsilon < I + |I| = 0$ and we see that no Darboux sums will occur in this neighborhood. This implies that the Darboux sums must converge to a nonnegative value, hence $\int_a^b [f-g]^2 \geq 0$.

Now, since $\int_a^b [f-g]^2 = \int_a^b [f^2 - 2fg + g^2] = \int_a^b f^2 - 2\int_a^b fg + \int_a^b g^2 \geq 0$, we see that $2\int_a^b fg \leq \int_a^b f^2 + \int_a^b g^2$. Therefore $\int_a^b fg \leq \frac{1}{2}\left[\int_a^b f^2 + \int_a^b g^2\right]$. Proceeds using nearly identical steps as key. 5/5

9. $\boxed{6.3.6}$

Proof. Suppose that the function $f : [a,b] \to \mathbb{R}$ is bounded and let $a < c < b$. We will show that if f is integrable on $[a,c]$ and on $[c,b]$, then it is integrable on $[a,b]$. By the Archimedes-Riemann Theorem, we can let $P_n' = \{u_0, \ldots, u_{n_1}\}$ and $P_n'' = \{v_0, \ldots, v_{n_2}\}$ be Archimedean sequences of partitions for f on $[a,c]$ and $[c,b]$, respectively. It follows that $\lim_{n\to\infty}[U(f,P_n') - L(f,P_n')] = 0$ and $\lim_{n\to\infty}[U(f,P_n'') - L(f,P_n'')] = 0$. We know that $U(f,P_n') = \sum_{i=1}^{n_1} M_i(u_i - u_{i-1})$ and $U(f,P_n'') = \sum_{i=1}^{n_2} M_i(v_i - v_{i-1})$. Now let $P_n = P_n' \cup P_n''$, and let us reindex the elements as $P_n = \{x_0, \ldots, x_{n_3}\}$, where $n_3 = n_1 + n_2$. Since $u_{n_1} = v_0 = c$, it follows that P_n is a partition of $[a,b]$ and thus $U(f,P_n') + U(f,P_n'') = U(f,P_n)$ and $L(f,P_n') + L(f,P_n'') = L(f,P_n)$. Then

$$\begin{aligned}
U(f,P_n) - L(f,P_n) &= U(f,P_n') + U(f,P_n'') - [L(f,P_n') + L(f,P_n'')] \\
U(f,P_n) - L(f,P_n) &= [U(f,P_n') - L(f,P_n')] + [U(f,P_n'') - L(f,P_n'')] \\
&\implies \lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0 \text{ by the sum property of convergent sequences}
\end{aligned}$$

The Archimedes-Riemann Theorem thus establishes that P_n is an Archimedean sequence of partitions for f on $[a,b]$ and additionally that f is integrable on $[a,b]$: $\int_a^b f = \lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)$. QED

Proceeds in slightly different way than key but still appears to be correct. 5/5

10. (a) *Proof.* Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, i.e. there exists $T > 0$ such that $f(x) = f(x+T)$ for all $x \in \mathbb{R}$. We will show that $\int_0^T f = \int_a^{a+T} f$ for all $a \in \mathbb{R}$. We know that f will be continuous on any closed, bounded interval. It is therefore uniformly continuous on any closed interval $[a,b] \in \mathbb{R}$. Suppose $a < b$. For any $\epsilon > 0$, $\exists \delta > 0$, s.t. given $u, v \in [a,b]$, $|u - v| < \delta \implies |f(u) - f(v)| < \frac{\epsilon}{b-a}$. The Archimedean property lets us choose some $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$. Let $P_n = \{x_0, \ldots, x_n\}$ be a regular partition of $[a,b]$ into n intervals. It follows for any $1 \le i \le n$ that $\frac{b-a}{n} = x_i - x_{i-1} < \delta$. So, $|f(x_i) - f(x_{i-1})| < \frac{\epsilon}{b-a}$. Since $m_i, M_i \in f([x_{i-1}, x_i])$, it follows that $|M_i - m_i| = M_i - m_i < \frac{\epsilon}{b-a}$. Then,

$$\begin{aligned}
U(f,P_n) - L(f,P_n) &= \sum_{i=1}^n (M_i - m_i)\frac{b-a}{n} \\
&< \sum_{i=1}^n \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} = \epsilon \\
&\implies |U(f,P_n) - L(f,P_n)| < |\epsilon| \\
&\implies \lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0, \text{ by the Comparison Lemma.}
\end{aligned}$$

By the Archimedes-Riemann Theorem, $\int_a^{a+T} f$ therefore exists and consequently $\int_0^T f$ exists. Then we see that if $a > 0$,

$$\int_a^{a+T} f = \int_a^T f + \int_T^{a+T} f$$

Then $\int_T^{a+T} f = \lim_{n\to\infty} U(f,P_n)$, where $P_n = \{x_0, \ldots, x_n\}$ is a regular partition of $[T, a+T]$. Now, define another regular partition $P_n' = \{x_0 - T, \ldots, x_n - T\}$ of $[0,a]$. We expect each M_i to have some pre-image $x_{M_i} \in [x_{i-1}, x_i] \subseteq [T, a+T]$. If we let $x_{M_i} \in [T, a+T]$ and $u_{M_i} = x_{M_i} - T \in [0,a]$:

$$U(f,P_n) = \sum_{i=1}^n f(x_{M_i})\frac{a}{n} = \sum_{i=1}^n f(u_{M_i} + T)\frac{a}{n} = U(f \circ g, P_n'), \text{ where } g(x) = x + T$$

But $f(x) = f(x+T) \implies f = f \circ g$, which implies that $U(f,P_n) = U(f,P_n') \implies \int_T^{a+T} f = \int_0^a f$. Therefore $\int_a^{a+T} f = \int_a^T f + \int_0^a f = \int_0^T f$, under the assumption that a is positive. Now suppose $a < 0$. Then,

$$\int_a^{a+T} f = \int_a^0 f + \int_0^{a+T} f$$

Then $\int_a^0 f = \lim_{n\to\infty} U(f,P_n)$, where $P_n = \{x_0, \ldots, x_n\}$ is a regular partition of $[a,0]$. Now, define another regular partition $P_n'' = \{x_0 + T, \ldots, x_n + T\}$ of $[a+T, T]$. If we let $x_{M_i} \in [a+T, T]$ and $u_{M_i} = x_{M_i} + T \in [0,a]$:

$$U(f,P_n) = \sum_{i=1}^n f(x_{M_i})\frac{a}{n} = \sum_{i=1}^n f(u_{M_i} - T)\frac{a}{n} = U(f \circ g, P_n''), \text{ where } g(x) = x - T$$

But $f(x-T) = f((x-T)+T) \implies f(x-T) = f(x) \implies f = f \circ g$, which implies that $U(f, P_n) = U(f, P_n'') \implies \int_a^0 f = \int_{a+T}^T f$. Therefore $\int_a^{a+T} f = \int_{a+T}^T f + \int_0^{a+T} f = \int_0^T f$, under the assumption that a is negative. When $a = 0$, this equality follows directly. So we have shown that $\int_0^T f = \int_a^{a+T} f$. QED

(b) *Proof.* Suppose f is differentiable and periodic. We will show that f' is periodic also. Since f is differentiable, it is true that $f'(x_0) = \lim_{x_n \to x_0} \frac{f(x_n)-f(x_0)}{x_n - x_0}$. Then it is true that for any ϵ, $\exists N$ s.t. $|x_n - x_0| < \epsilon$ for $n \geq N$. If we let $h_n = x_n - x_0$, this implies that $\lim_{n\to\infty} h_n = 0$. Then we can rewrite $f'(x_0) = \lim_{x_n \to x_0} \frac{f(x_0+h_n)-f(x_0)}{h_n + x_0 - x_0} \lim_{h_n \to 0} \frac{f(x_0+h_n)-f(x_0)}{h_n}$. If we now consider $f'(x+T)$, we see that, by the periodicity of f, $f'(x_0+T) = \lim_{h_n\to 0} \frac{f((x_0+h_n)+T)-f(x_0+T)}{h_n} = \lim_{h_n \to 0} \frac{f(x_0+h_n)-f(x_0)}{h_n}$. So $f'(x_0+T) = f'(x_0)$, which shows that f' is also periodic with the same period T as f. QED

Uses additivity to prove (a) like in key. Uses the definition of the derivative to prove (b) like in key. 5/5