APPM 4440 HW 10

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Problem	Self-Grade	Grade
#1	5	
#2	5	
#3	5	
#4	3	
#5	5	
#6	5	
#7	5	
#8	5	
#9	5	
#10	5	
Tot/50	48/50	

1. $\boxed{6.4.1}$

(a) $\boxed{\text{False}}$. Consider $f(x) = x$ on the interval $[-1, 1]$. Let P_n be a regular partition of $[-1, 1]$. Then $m_i = x_{i-1}$ and $M_i = x_i$. So,

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{i-n}{n}\right] \\ &= \frac{n+1}{2n} - 1 \\ L(f, P_n) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{i-1-n}{n}\right] \\ &= \frac{1}{n^2}\left[\frac{n(n-1)}{2} - n\right] \\ &= \frac{n-1}{2n} - 1 \end{aligned}$$

So $\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty}\frac{1}{n} = 0 \implies f$ is integrable on $[-1, 1]$. So $\int_{-1}^{1} f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty}\left[\frac{n+1}{n} - 1\right] = 0$. Yet it is clear that $f(x) \neq 0$ for all $x \in [-1, 1]$.

(b) $\boxed{\text{False}}$ Consider the step function $f(x) = 0$ for $x < 1$ and $f(x) = 1$ for $x \geq 1$. It is straightforward to show that f is integrable on $[1, 2]$ because it is simply constant on that entire interval. Let P_n be a regular partition of $[0, 1]$. Then $m_i = M_i = 0$ for all partition intervals except the last, for which $M_i = 1$. So we expect $L(f, P_n) = 0$ and $U(f, P_n) = \frac{1}{n}\sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n}(1)$. We see that $[U(f, P_n) - L(f, P_n)] = \frac{1}{n}$ so f is clearly integrable on $[0, 1]$. As was shown in HW#9, it follows that f is integrable on $[0, 2]$ and yet f is not continuous on $[0, 2]$.

(c) $\boxed{\text{True}}$ We assume f is integrable and nonnegative on its interval $[a, b]$. Then $\int_a^b f = \lim_{n\to\infty} U(f, P_n)$ where P_n is an Archimedean sequence of partitions, e.g. a regular partition. But $U(f, P_n) = \frac{1}{n}\sum_{i=1}^{n} M_i$. Since it follows from the nonnegativity of f that each $M_i \geq 0$, we see that $U(f, P_n) \geq 0$ and therefore $\int_a^b f \geq 0$.

(d) $\boxed{\text{False}}$ Consider $f(x) = \frac{1}{x}$ on the interval $(0, 1)$. We know f is continuous on this interval, because x is trivially continuous and therefore its reciprocal $\frac{1}{x}$ is continuous for nonzero x. Then suppose f is bounded, i.e. $\exists M \geq 0$ s.t. $|f(x)| \leq M \implies f(x) < M$. Then consider $x_n = \frac{1}{n}$. Obviously $x_n \to 0$ but $f(x_n) = n \leq M$ is a contradiction because it implies n is bounded. So f is continuous and unbounded.

(e) $\boxed{\text{True}}$ Assume f is continuous on the closed interval $[a, b]$. Then f is uniformly continuous on $[a, b]$. It was proven on $HW\#6$ that a uniformly continuous function on an open interval (a, b) is bounded. Then it follows that f will be bounded on the closed interval $[a, b]$ because the values of f at a and b must be finite, as it is continuous at both points.

Got same T/F values for each part with similar line of reasoning for each. 5/5

2. $\boxed{6.4.5}$ Suppose the continuous function $f : [a, b] \to \mathbb{R}$ has the property that $\int_c^d f \leq 0$ whenever $a \leq c < d \leq b$. We intend to prove that $f(x) \leq 0$ for all $x \in [a, b]$. By way of contradiction, suppose $\exists x_0 \in [a, b]$ s.t. $f(x_0) > 0$. Since f is continuous, it is true that $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. given $x \in [a, b]$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. In other words, if $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$. Choose some $0 < \epsilon < f(x_0)$. Then the neighborhood $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ consists of only positive values because $0 < f(x_0) - \epsilon$. Or, the value of f is positive over the interval $(x_0 - \delta, x_0 + \delta)$. Then it follows that f is positive over the closed interval $[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$. If x_0 is a boundary point of the domain, then we see that f is still positive over the truncated intervals $(a, a + \frac{\delta}{2})$, $(b - \frac{\delta}{2}, b)$, or (a, b) (in case the neighborhood covers the entire interval). Denote $c = x_0 - \delta/2$ and $d = x_0 + \delta/2$ and adjust them to fit within the domain if necessary (i.e. if x_0 is a boundary point or the neighborhood covers the entire interval). We know $\int_c^d f = \lim_{n\to\infty}[U(f, P_n)] = \lim_{n\to\infty} \sum_{i=1}^n [M_i(x_i - x_{i-1})]$. We also know $M_i > 0$ on $[c, d]$ because we found that f is positive for all $x \in [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$. It follows that the Darboux sum is always positive and thus $\int_c^d f > 0$. But we assumed that $\int_c^d f \leq 0$ for any c, d, which is a contradiction. So it follows that $f(x) \leq 0$ for all $x \in [a, b]$.

   This does not necessarily hold if we only assume that f is integrable. For instance, let $f(x) = 1$ for $x = 0$ and $f(x) = 0$ otherwise. Clearly f is not continuous on any interval containing 0. Consider any closed interval $[a, b]$ s.t. $0 \in [a, b]$. We see that $\int_a^b f = 0$ and the integral of f will also be zero over any subset of this interval. If we use a regular partition P, we see that $L(f, P) = 0$ always and $U(f, P) = \frac{1}{n}$ or $\frac{2}{n}$ depending on whether 0 is a partition point or not. In both cases the limit of the difference of Darboux sums is zero, so f is integrable on $[a, b]$ and $\int_a^b f = 0$. For any subset $[c, d]$ of $[a, b]$ the same reasoning may be applied if $0 \in [c, d]$, and $\int_c^d f = 0$ trivially if $0 \notin [c, d]$. So it is true $\int_c^d f \leq 0$ for $a \leq c < d \leq b$ (assuming $0 \in [a, b]$). But it is not true that $f(x)$ is nonpositive for all $x \in [a, b]$ because we defined $f(0) = 1$. So continuity is reguired for the prior assertion.

   Proceeds using similar proof by contradiction as key. Uses similar counterexample as well to arrive at same conclusion. 5/5

3. $\boxed{6.4.9}$ Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous. We intend to prove that $\int_a^b |f + g| \leq \int_a^b |f| + \int_a^b |g|$. We know that the absolute value function $h(x) = |x|$ is continuous for all real numbers. Since the images of f and g are necessarily contained by the real numbers, it follows that the compositions $|f|$ and $|g|$ are also continuous. Since $f + g$ is continuous, $|f + g|$ will also be continous. By the continuity of each composition we also know that $\int_a^b |f + g|$, $\int_a^b |f|$, and $\int_a^b |g|$ all exist. Choose any Archimedean sequence of partitions P_n of $[a, b]$. Denote the supremums of each composition over each partition interval as A_i for $|f+g|$, B_i for $|f|$, and C_i for $|g|$. Then choose x_{A_i}, x_{B_i}, and x_{C_i} s.t. $A_i = |f(x_{A_i}) + g(x_{A_i})|$, $B_i = |f(x_{B_i})|$, and $C_i = |g(x_{C_i})$. By definition, $B_i = |f(x_{B_i})| = \sup |f([x_{i-1}, x_i])|$ and $C_i = |g(x_{C_i})| = \sup |g([x_{i-1}, x_i])|$, so it follows that $|f(x_{A_i})| \leq B_i$ and $|g(x_{A_i})| \leq C_i$. Then we see that

$$
\begin{aligned}
U(|f + g|, P_n) &= \sum_{i=1}^n A_i(x_i - x_{i-1}). \\
&= \sum_{i=1}^n |f(x_{A_i}) + g(x_{A_i})|(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n (|f(x_{A_i})| + |g(x_{A_i})|)(x_i - x_{i-1}), \text{ by the triangle inequality} \\
&\leq \sum_{i=1}^n (|f(x_{B_i})| + |g(x_{C_i})|)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n |B_i|(x_i - x_{i-1}) + \sum_{i=1}^n |C_i|(x_i - x_{i-1}) \\
&= U(|f|, P_n) + U(|g|, P_n)
\end{aligned}
$$

   So it follows that $\lim_{n\to\infty} U(|f+g|, P_n) \leq \lim_{n\to\infty} U(|f|, P_n) + \lim_{n\to\infty} U(|g|, P_n) \implies \int_a^b |f+g| \leq \int_a^b |f| + \int_a^b |g|$. Uses triangle inequality and continuity to reach the desired conclusion, much like key does. 5/5

4. $\boxed{6.5.6}$ We intend to show that in the First Fundamental Theorem, it is necessary to assume that the function $F : [a, b] \to \mathbb{R}$ is continuous at the endpoints of the interval. When we construct a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and apply the Mean Value Theorem to each partition interval, we see that the leftmost and rightmost partition intervals contain the endpoints: $x_0 = a$ and $x_n = b$. Moreover, MVT requires continuity over a closed interval, so it follows that f must be continous at both endpoints because they are contained in the leftmost and rightmost partition intervals.

   Identifies the correct step but neglects to show necessity. 3/5

5. $\boxed{6.6.4}$ Suppose $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Consider the function $g(x) = f(0) + f'(0)x + \int_0^x (x - t) f''(t)\, dt$. By assumption $f''(t)$ is continuous and $(x - t)$ is trivially continuous, so the product $(x - t)f''(t)$ is also continuous. Then,

by the second fundamental theorem, we expect that

$$\begin{aligned}
\frac{d}{dx}\left[\int_0^x (x-t)f''(t)\,dt\right] &= \frac{d}{dx}\left[x\int_0^x f''(t)\,dt - \int_0^x tf''(t)\,dt\right] \\
&= \left(\frac{d}{dx}x\right)\int_0^x f''(t)\,dt + x\left(\frac{d}{dx}\int_0^x f''(t)\,dt\right) - xf''(x) \\
&= \int_0^x f''(t)\,dt + xf''(x) - xf''(x) \\
&= f'(x) - f'(0)
\end{aligned}$$

It follows that $g'(x) = f'(0) + f'(x) - f'(0) = f'(x)$. By the identity criterion f and g therefore differ by a constant. We see that $g(0) = f(0) + f'(0)(0) + \int_0^0 (x-t)f''(t)\,dt$. If we treat the integral term as a limit of a Darboux sum, we see that that the partition consists of a single point and thus the length of the partition interval is zero, implying that the $\int_0^0 (x-t)f''(t)\,dt$ term is zero. So we expect $g(0) = f(0)$. By the identity criterion, it follows that f and g are identically equal, i.e. $f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)\,dt$ for all x.

Uses identity criterion and fundamental theorem to reach desired conclusion like in key. 5/5

6. $\boxed{6.6.8}$ For numbers $a_1, \ldots, a_n$, define $p(x) = a_1x + a_2x^2 + \cdots + a_nx^n$. Suppose also that $\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0$. We will show that $\exists x \in (0,1)$ s.t. $p(x) = 0$. By proposition 4.4, the antiderivative of a_ix^i is $\frac{a_i}{i+1}x^{i+1}$. So we expect the antiderivative P of p to be $P(x) = \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1}$. We therefore assume that $P(1) = 0$. So it will suffice to show that $\exists x \in (0,1)$ s.t. $P'(x) = p(x) = 0$. By the continuity of polynomials, $P(0) = 0$. So $P(0) = P(1) = 0$. Since P is differentiable, it is continuous, and by Rolle's Theorem we can say $\exists x \in (0,1)$ at which $P'(x) = p(x) = 0$.

   Proceeds using nearly similar proof as key to reach desired conclusion. 5/5

7. $\boxed{7.3.1}$ For a fixed number β, we intend to find $\lim_{n\to\infty}\left[\frac{1^\beta+2^\beta+\cdots+n^\beta}{n^{\beta+1}}\right]$. Observe that

$$\begin{aligned}
\lim_{n\to\infty}\left[\frac{1^\beta+2^\beta+\cdots+n^\beta}{n^{\beta+1}}\right] &= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1^\beta+2^\beta+\cdots+n^\beta}{n^\beta}\right] \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^\beta
\end{aligned}$$

   Now consider the function $f(x) = x^\beta = e^{\beta\ln x}$ (the second definition is valid only for positive x). We see that for $x > 0$, $u < v \implies \ln u < \ln v \implies f(u) < f(v)$ so f is strictly increasing for positive x. Because the exponential function is always positive, this holds for all nonnegative x as well because $f(0) = 0 < f(u), u > 0$. So it follows that f is bounded on $(0,1]$ because $f(1) = 1$ and the positivity and monotonicity of f means it cannot become arbitrarily large in magnitude as x approaches zero. Since $f(0) = 0^\beta = 0$, it follows that f is bounded on $[0,1]$. Since it is a composition of continuous functions, we know that f is continuous on $(0,1)$. By Theorem 6.19 f is integrable on $[0,1]$, and the Riemann Sum Convergence Theorem thus states that if $R(f,P_n,C_n)$ is a Riemann sum of f, then $\lim_{n\to\infty} R(f,P_n,C_n) = \int_a^b f$. Then the summation above can be seen as a Riemann sum of f on a regular partition P_n of $[0,1]$ with each $c_i = \frac{i}{n}$. Therefore $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^\beta = \int_0^1 f$.
   We seek now to extend the continuity of f to 0. When $|x-0| < \delta$, we want to show that $|f(x) - f(0)| = |f(x)| < \epsilon$ for $\epsilon > 0$. Since we are now considering nonnegative x only, It will suffice to show that $x < \delta \implies f(x) < \epsilon$, because f is nonnegative for nonnegative x. When $x = 0$ this is trivially true. Otherwise, for $x > 0$, we can pick $\delta = \epsilon^{\frac{1}{\beta}} \implies x < \epsilon^{\frac{1}{\beta}} = x < e^{\frac{1}{\beta}\ln\epsilon} \implies e^{\beta\ln x} = x^\beta < \epsilon$. So f is continuous on $[0,1]$ It follows that $F(x) = \frac{1}{\beta+1}x^{\beta+1}$ is continuous on $[0,1]$ as it is a product of continuous functions. By proposition 5.3, $\frac{d}{dx}F(x) = F'(x) = f(x) = x^\beta$. We know that f is continous on $(0,1)$ and bounded on $(0,1)$. Therefore we can apply the first fundamental theorem: $\int_0^1 F' = \int_0^1 f = F(1) - F(0) = \frac{1}{\beta+1}$. The Riemann Sum Convergence Theorem thus asserts that $\lim_{n\to\infty}\left[\frac{1^\beta+2^\beta+\cdots+n^\beta}{n^{\beta+1}}\right] = \frac{1}{\beta+1}$.

   Proceeds using similar proof as key and uses DSC Theorem to reach the final conclusion. 5/5

8. $\boxed{7.3.5}$ Let $b > 1$. We intend to find the value of the Riemann sum for $\int_1^b [1/\sqrt{x}]\,dx$ that one obtains for the partition $P = \{x_0, \ldots, x_n\}$ of $[1,b]$, choosing $c_i = [(\sqrt{x_i} + \sqrt{x_{i-1}})/2]^2$ for $1 \le k \le n$.

   We know $\sqrt{x}$ is continuous and nonzero for positive x so $\frac{1}{\sqrt{x}}$ is continuous on $[1,b]$, thus the integral exists. We can thus obtain the desired Riemann sum as follows, letting $f(x) = \frac{1}{\sqrt{x}}$:

$$\begin{aligned}
\int_1^b f = \lim_{n\to\infty} R(f,P,C) &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1/\left[(\sqrt{x_i}+\sqrt{x_{i-1}})/2\right]\right)(x_i - x_{i-1})
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{\sqrt{x_i} + \sqrt{x_{i-1}}} \right)(x_i - x_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{\sqrt{x_i} + \sqrt{x_{i-1}}} \right)(\sqrt{x_i} - \sqrt{x_{i-1}})(\sqrt{x_i} + \sqrt{x_{i-1}}) \\
&= 2 \lim_{n\to\infty} \sum_{i=1}^{n} (\sqrt{x_i} - \sqrt{x_{i-1}}) \\
&= 2 \lim_{n\to\infty} [\sqrt{x_1} - \sqrt{x_0} + \sqrt{x_2} - \sqrt{(}x_1) + \cdots + \sqrt{x_{n-1}} - \sqrt{x_{n-2}} + \sqrt{x_n} - \sqrt{x_{n-1}}]
\end{aligned}
$$

We can see that the simplification is a telescoping sum in which all terms will cancel except $-\sqrt{x_0}$ and $\sqrt{x_n}$. But we know P is a partition of $[1, b]$ so $\lim_{n\to\infty} R(f, P, C) = 2[\sqrt{b} - 1] = \int_1^b [1/\sqrt{x}]\, dx$.

Shows that the Riemann sum is telescoping and simplifies it to find the desired integral, much like key. 5/5

9. $\boxed{7.3.9}$ Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and let P be any partition of its domain $[a, b]$. We intend to show that there is a Riemann sum $R(f, P, C)$ that equals $\int_a^b f$.

   Since f is continuous it is integrable on $[a, b]$. The Mean Value Theorem for Integrals also states that $\exists c \in [a, b]$ s.t. $\frac{1}{b-a} \int_a^b f = f(c)$. Then $\int_a^b f = f(c)(b - a)$. At this point we can create a trivial partition $P = \{x_0, x_1\}$ with $x_0 = a$ and $x_1 = b$, and set $C = \{c\}$. Then $R(f, P, C) = \sum_{i=1}^n f(c_i)(x_i - x_{i-1}) = f(c)(x_1 - x_0) = f(c)(b - a) = \int_a^b f$. So the desired Riemann sum exists.

   Uses the mean value theorem for integrals to reach the desired conclusion directly. 5/5

10. Suppose $\alpha(x)$ is monotone increasing. Then we can define the Riemann-Stieltjes integral as $\int_a^b f\, d\alpha$, whose Darbobux sums are also defined in a similar manner as for Riemann integrals:

$$
\begin{aligned}
L(f, \alpha, P) &= \sum_{i=1}^{n} m_i[\alpha(x_i) - \alpha(x_{i-1})],\ m_i = \inf\{f([x_{i-1},\, x_i])\} \\
U(f, \alpha, P) &= \sum_{i=1}^{n} M_i[\alpha(x_i) - \alpha(x_{i-1})],\ M_i = \sup\{f([x_{i-1},\, x_i])\}
\end{aligned}
$$

We will first define f as Riemann-Stieltjes integrable given $\underline{\int}_a^b f\, d\alpha = \bar{\int}_a^b f\, d\alpha$, with $\underline{\int}_a^b f\, d\alpha \equiv \sup\{L(f,\, \alpha,\, P)\}$ and $\bar{\int}_a^b f\, d\alpha \equiv \inf\{U(f,\, \alpha,\, P)\}$.

Then, generalizing the Archimedes-Riemann Theorem, we claim that if α is monotone increasing and f is bounded on $[a, b]$, then f is Riemann-Stieltjes integrable iff there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ s.t. $\lim_{n\to\infty}[U(f, \alpha, P_n) - L(f, \alpha, P_n)] = 0$. Moreover, $\lim_{n\to\infty} L(f, \alpha, P_n) = \lim_{n\to\infty} U(f, \alpha, P_n) = \int_a^b f\, d\alpha$.

Proof. We will show both directions.

($\implies$) Suppose that P_n exists. Then $\lim_{n\to\infty}[U(f, \alpha, P_n) - L(f, \alpha, P_n)] = 0$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|U(f, \alpha, P_n) - L(f, \alpha, P_n)| < \epsilon$ for $n \geq N$. It follows from the definition of each Darboux sum that $U(f, \alpha, P_n) \geq L(f, \alpha, P_n)$ because each $M_i \geq m_i$ and $(\alpha(x_i) - \alpha(x_{i-1})) \geq 0$, since $x_i > x_{i-1}$ and α is monotone increasing. Then $U(f, \alpha, P_n) - L(f, \alpha, P_n) < \epsilon$. So, appyling the definitions of the lower and upper integrals, we have

$$
\begin{aligned}
&\bar{\int}_a^b f\, d\alpha \leq U(f, \alpha, P_n) < \epsilon + L(f, \alpha, P_n) \leq \epsilon + \underline{\int}_a^b f\, d\alpha \\
\implies &\bar{\int}_a^b f\, d\alpha - \underline{\int}_a^b f\, d\alpha \leq U(f, \alpha, P_n) - \underline{\int}_a^b f\, d\alpha < \epsilon + L(f, \alpha, P_n) - \underline{\int}_a^b f\, d\alpha \leq \epsilon \\
\implies &\bar{\int}_a^b f\, d\alpha - \underline{\int}_a^b f\, d\alpha < \epsilon
\end{aligned}
$$

We can now adapt the refinement lemma for Riemann-Stieltjes integrals. Given a refinement $P_n^\star$ of $P_n = \{x_0,\, \ldots,\, x_n\}$, Let $P_{n_i} = \{y_0,\, \ldots,\, y_k\}$ be the partition of $[x_{i-1}, x_i]$ that is induced by $P_n^\star$. Assuming m_i is a lower bound of f on $[x_{i-1}, x_i]$ and m_j' is the lower bound on each induced partition interval in $[x_{i-1}, x_i]$, we expect that $m_i(\alpha(x_i) - \alpha(x_{i-1})) = m_i(\alpha(y_k) - \alpha(y_0)) = m_i \sum_{i=1}^n (y_i - y_{i-1})$ because the summation is a telescoping sum. It follows that $l \leq m_i \implies \sum_{i=1}^k l(\alpha(y_i) - \alpha(y_{i-1})) = m_i(\alpha(y_k) - \alpha(y_0)) \leq L(f, \alpha, P_{n_i})$. Summing these over all partition intervals in P_n, we get $L(f, \alpha, P_n) \leq L(f, \alpha, P_n^\star)$. The same line of reasoning can be used to show that $U(f, \alpha, P_n) \geq U(f, \alpha, P_n^\star)$.

Thus for any two partitions of $[a,b]$, X and Y, we can say that $L(f,\alpha,X) \leq U(f,\alpha,Y)$, because we can take the common refinement $Z = X \cup Y$ and $L(f,\alpha,X) \leq L(f,\alpha,Z) \leq U(f,\alpha,Z) \leq U(f,\alpha,Y) \implies L(f,\alpha,X) \leq U(f,\alpha,Y) \implies \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$. So, returning to the inequality from before, we can say that $0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon \implies |\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha| < \epsilon$ which means $\lim_{n\to\infty} \left[\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha\right] = 0 \implies \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$. So f is Riemann-Stieltjes integrable.

($\Longleftarrow$) Conversely, suppose f is Riemann-Stieltjes integrable. Then $\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$. Because $\underline{\int_a^b} f\,d\alpha$ and $\overline{\int_a^b} f\,d\alpha$ are infimums and supremums, respectively, it follows that for any $\epsilon > 0$ we should be able to produce two partitions of $[a,b]$, X and Y, s.t. $U(f,\alpha,X) < \underline{\int_a^b} f\,d\alpha + \frac{\epsilon}{2}$ and $L(f,\alpha,Y) > \overline{\int_a^b} f\,d\alpha - \frac{\epsilon}{2}$. By the refinement lemma, taking $Z = X \cup Y$, we have $L(f,\alpha,X) \leq L(f,\alpha,Z)$ and $U(f,\alpha,Y) \geq U(f,\alpha,Z) \implies U(f,\alpha,Z) - L(f,\alpha,Z) \leq U(f,\alpha,X) - U(f,\alpha,Y) < \underline{\int_a^b} f\,d\alpha + \frac{\epsilon}{2} - \left[\overline{\int_a^b} f\,d\alpha - \frac{\epsilon}{2}\right] = \epsilon$. So we can create a sequence P_n of these Z s.t. for any positive ϵ, $\exists N$ s.t. $U(f,\alpha,P_n) - L(f,\alpha,P_n) = |U(f,\alpha,P_n) - L(f,\alpha,P_n)| < \epsilon$. By the comparison lemma, since we can let $\epsilon \to 0$ it follows that $\lim_{n\to\infty}[U(f,\alpha,P_n) - L(f,\alpha,P_n)] = 0$ QED

In the above proof there was no requirement for strict monotonicity, continuity, or boundedness of α.

Proceeds via similar proof as key and also states that additional assumptions on α are not needed. 5/5