APPM 4440 HW 7

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1. $\boxed{3.6.8}$

Proof. We claim that there does not exist a strictly increasing function $f : \mathbb{Q} \to \mathbb{R}$ s.t. $f(\mathbb{Q}) = \mathbb{R}$. By way of contradiction, suppose such a function f does exist, i.e. $f : \mathbb{Q} \to \mathbb{R}$ is strictly increasing and $f(\mathbb{Q}) = \mathbb{R}$. Since f is strictly increasing, it is one-to-one and thus has an inverse, $f^{-1} : f(\mathbb{Q}) \to \mathbb{R}$. Note that f^{-1} is also strictly increasing (shown in class). Consider some $y_n, y_0 \in f(\mathbb{Q})$. Denote $x_n = f^{-1}(y_n)$. Choose any $\epsilon > 0$ and let $u = f(x_n - \epsilon)$ and $v = f(x_n + \epsilon)$. Since f is strictly increasing, $y_n - u > 0$ and $v - y_n > 0$. At this point, we can pick $\delta = \min\{y_n - u,\, v - y_n\} > 0$. Assume $|y_n - y_0| < \delta$. Then

$$\begin{aligned}
|y_n - y_0| < \delta &\implies -\delta < y_n - y_0 < \delta \\
&\implies -\delta < y_0 - y_n < \delta \\
&\implies u - y_n \le -\delta < y_0 - y_n < \delta \le v - y_n \\
&\implies f(x_n - \epsilon) = u < y_0 < v = f(x_n + \epsilon) \\
&\implies x_n - \epsilon < f^{-1}(y_0) < x_n + \epsilon, \text{ because } f^{-1} \text{ is strictly increasing} \\
&\implies f^{-1}(y_n) - \epsilon < f^{-1}(y_0) < f^{-1}(y_n) + \epsilon \\
&\implies |f^{-1}(y_0) - f^{-1}(y_n)| < \epsilon \\
&\implies |f^{-1}(y_n) - f^{-1}(y_0)| < \epsilon
\end{aligned}$$

So $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall y_n, y_0 \in f(\mathbb{Q}), |y_n - y_0| < \delta \implies |f^{-1}(y_n) - f^{-1}(y_0)| < \epsilon$, i.e. f^{-1} is continuous. However, by Corollary 3.25, since $f(\mathbb{Q}) = \mathbb{R}$, i.e. its domain is an interval, f^{-1} is not continuous because its image, $f^{-1}(f(\mathbb{Q})) = f^{-1}(\mathbb{R}) = \mathbb{Q}$ is not an interval. This is a contradiction, and thus such a function defined as f has been does not exist. QED

2. $\boxed{3.7.12}$

Proof. Suppose for $a, b \in \mathbb{R}$ that $a < b$ and let $I = (a, b)$. Suppose also that the function $f : I \to \mathbb{R}$ is monotonically increasing and bounded. We claim $\lim_{x\to a} f(x)$ exists. Consider some $\{x_n\} \subset I$ (by construction, $a \notin \{x_n\}$). Suppose also that $\lim_{n\to\infty} x_n = a$. So $\forall \delta > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N, |x_n - a| < \delta$. To prove existence of $\lim_{x\to a} f(x)$, we need to show that the convergence of x_n to a implies the convergence of $f(x_n)$ to L. Choose any $\epsilon > 0$. As f is bounded below, we can denote $L = \inf f(I)$. By definition, $-\epsilon + L < f(x_n)$. Then, since L is the infimum of $f(I)$, it follows that $\epsilon + L$ is not a lower bound on $f(I)$. So $\exists y \in f(I)$ s.t. $y < L + \epsilon \implies \exists x' \in I$ s.t. $y = f(x') < L + \epsilon$. Since f is monotonically increasing, if we restrict $x_n < x' \implies f(x_n) \le f(x')$. So, when $a < x_n < x'$, it is true that $-\epsilon + L < f(x_n) < \epsilon + L \implies |f(x) - L| < \epsilon$. So we can choose $\delta = x' - a > 0$ because $x' \in I$ and we know x' will always exist. Then, $|x_n - a| < \delta = x' - a \implies x_n - a < x' - a \implies x_n < x'$. We already know $a < x_n$, so by our reasoning from before we know that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x_n - a| < \delta \implies |f(x_n) - L| < \epsilon$, i.e. $f(x_n) \to L$ whenever $x_n \to a$, given $n \ge N$. So $\lim_{x\to a} f(x) = L$, i.e. the limit exists.

QED

3. $\boxed{4.1.5}$

   (a) We want to evaluate $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} \frac{x^2 - 0^2}{x - 0}$. Let $f(x) = x^2$ then we see the desired quantity is $f'(0)$. Prop 4.4 tells us $f'(0) = 2(0)^{2-1} = 0$.

   (b) We want to find $\lim_{x\to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$. Let $y = \sqrt{x}$. Then $x^2 = y^4$, and because y is continuous on $[0, \infty)$, we can just find $\lim_{y\to 1} \frac{y^4 - 1^4}{y - 1}$. Let $f(x) = x^4$ then we see the desired quantity is $f'(1)$. By Prop 4.4, $f'(1) = 4(1)^{4-1} = 4$.

   (c) We want to find $\lim_{x\to 1} \frac{x-1}{\sqrt{x} - 1}$. Let $y = \sqrt{x}$. Then $x = y^2$, and because y is continuous on $[0, \infty)$ we can just find $\lim_{y\to 1} \frac{y^2 - 1^2}{y - 1}$. Let $f(x) = x^2$. Then we see the desired quantity is just $f'(1)$. By Prop 4.4, $f'(1) = 2(1)^{2-1} = 2$.

   (d) Observe that $\lim_{x\to 2} \frac{x^4 - 16}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x^3 + 2x^2 + 4x + 8)}{x - 2} = \lim_{x\to 2}(x^3 + 2x^2 + 4x + 8) = 8 + 8 + 8 + 8 = 32$.

4. $\boxed{4.1.9}$

Proof. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the property that $-x^2 \le f(x) \le x^2$ for all x. We will show that f is differentiable at 0 and that $f'(0) = 0$. It suffices to show that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. This is the same as showing that given $\{x_n\} \in \mathbb{R} \setminus \{0\}$, $\lim_{n\to\infty} x_n = 0 \implies \lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n-0} = 0$. Since x^2 is differentiable (Prop 4.4) and $\lim_{n\to\infty} x_n = 0$, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $\left|\frac{x_n^2-0^2}{x_n-0}\right| < \epsilon \implies \left|\frac{-x_n^2+0^2}{x_n-0}\right| < \epsilon$. If $x_n < 0$, then $-\epsilon < \frac{x_n^2}{x_n} \le \frac{f(x_n)}{x_n} \le \frac{-x_n^2}{x_n} < \epsilon$. If $x_n > 0$, then $-\epsilon < \frac{-x_n^2}{x_n} \le \frac{f(x_n)}{x_n} \le \frac{x_n^2}{x_n} < \epsilon$. In either case, $\left|\frac{f(x_n)}{x_n}\right| < \epsilon$. Observe that $-x^2 \le f(x) \le x^2 \implies -0^2 \le f(0) \le 0^2 \implies f(0) = 0$. So $\left|\frac{f(x_n)}{x_n}\right| = \left|\frac{f(x_n)-f(0)}{x_n-0}\right| < \epsilon$. Thus $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $\left|\frac{f(x_n)-f(0)}{x_n-0}\right| < \epsilon$, or $\lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n-0} = 0$ whenever $\lim_{n\to\infty} x_n = 0$. Thus f is differentiable at zero and $f'(0) = 0$.

QED

5. $\boxed{4.1.11}$

Proof. Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable at 0 and that for each $n \in \mathbb{N}$, $g(1/n) = 0$. We will show that $g(0) = 0$ and $g'(0) = 0$. Since g is differentiable at 0, g is continuous at 0 and thus given $\{x_n\} \in \mathbb{R}$, $x_n \to 0$, we know that $g(x_n) \to g(0)$. We can pick $x_n = \frac{1}{n}$ and we know that $x_n \to 0$. It is clear that $g(x_n) = 0 \implies g(0) = 0$. We know g is differentiable at zero so it is true for any sequence $\{u_n\} \in \mathbb{R} \setminus \{0\}$ chosen s.t. $u_n \to 0$ that $g'(0) = \lim_{n\to\infty} \frac{g(u_n)-g(0)}{u_n-0}$. In this case, we may reuse $\{x_n\}$. Observe that $g'(0) = \lim_{n\to\infty} \frac{g(x_n)-g(0)}{x_n-0} = \lim_{n\to\infty} \frac{g(1/n)}{1/n} = \lim_{n\to\infty}(n \cdot 0) = 0$. QED

6. A function $f : D \to \mathbb{R}$ is Lipschitz if $\exists C \ge 0$ s.t. $\forall u, v \in D$, $|f(u) - f(v)| \le C|u - v|$.

   (a) We claim the function $f(x) = |x|$ is Lipschitz continuous everywhere but not differentiable everywhere on the real line. Observe that, given $u, v \in \mathbb{R}$, $||u| - |v|| \le |u - v|$ by the triangle inequality (proven in previous homework). If we pick $C = 1$, it is clear that f is Lipschitz continuous everywhere on the real line. However, f is not differentiable at zero. Choose $u_n = \frac{-1}{n}$ and $v_n = \frac{1}{n}$. Note that $0 \notin \{u_n\}$ and $0 \notin \{v_n\}$ but $u_n \to 0$ and $v_n \to 0$. Observe that $\lim_{n\to\infty} \frac{f(u_n)-f(0)}{u_n-0} = \lim_{n\to\infty} \frac{|\frac{-1}{n}|-0}{\frac{-1}{n}-0} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{-1}{n}} = -1$. However, $\lim_{n\to\infty} \frac{f(v_n)-f(0)}{v_n-0} = \lim_{n\to\infty} \frac{|\frac{1}{n}|-0}{\frac{1}{n}-0} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1$. So $\lim_{x\to 0} \frac{f(x)-0}{x-0}$ does not exist because all sequences in $\mathbb{R}$ do not converge to the same limit, i.e. f is not differentiable at zero.

   (b) We claim the function $f(x) = x^2$ is differentiable everywhere but not Lipschitz continuous everywhere on the real line. By Prop 4.4, f is differentiable everywhere in $\mathbb{R}$ because x^2 is a positive integral power of x. Now suppose that f is Lipschitz continuous. Then $\exists C \ge 0$ s.t. $\forall u, v \in \mathbb{R}$, $|f(u) - f(v)| \le C|u - v| \implies |u^2 - v^2| \le C|u - v|$. Choose $u_n = n$, $v_n = 0$. Then $|n^2 - 0^2| \le C|n - 0| \implies n^2 \le Cn \implies n \le C$ for all natural numbers n. The Archimedean property tells us there is no finite C that satisfies this requirement, so f is not Lipschitz continuous everywhere on the real line.

7. $\boxed{4.2.1}$ Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable and define $h \equiv f \circ g : \mathbb{R} \to \mathbb{R}$. We also know that $g(1) = 2$, $g(2) = 1$, $f'(1) = -1$, $f'(2) = 2$, $g'(1) = 3$, $g'(2) = 4$. We want to find $h'(1)$ and $h'(2)$. By the chain rule we know that $h'(x) = g'(f(x))f'(x)$. So $h'(1) = f'(g(1))g'(1) = 3f'(2) = 6$. Also, $h'(2) = f'(g(2))g'(2) = 4f'(1) = -4$.

8. $\boxed{4.2.4}$ Define $f(x) = \frac{1}{1+x}$ for $x \in I$ where $I = (0, 1)$. Observe that if we define $g(x) = 1/x$ and $h(x) = 1 + x$ then $f \equiv g \circ h$. We see that h is trivially differentiable everywhere. We claim g is differentiable for positive x. Choose any $x_0 \in \mathbb{R}^+ = (0, \infty)$ and $\{x_n\} \subset \mathbb{R}^+ \setminus \{x_0\}$ s.t. $x_n \to x_0$. We see that

$$\begin{aligned} g'(x_0) = \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} &= \lim_{n\to\infty} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\ &= \lim_{n\to\infty} \frac{1/x_n - 1/x_0}{x_n - x_0} \\ &= \lim_{n\to\infty} \frac{1/x_n - 1/x_0}{x_n - x_0} \frac{x_n x_0}{x_n x_0} \\ &= \lim_{n\to\infty} \frac{x_0 - x_n}{-(x_0 - x_n)(x_n x_0)} \\ &= \lim_{n\to\infty} \frac{-1}{x_n x_0} \\ &= \frac{-1}{x_0} \lim_{n\to\infty} \frac{1}{x_n} \\ &= \frac{-1}{x_0^2} \end{aligned}$$

So g is differentiable for positive reals. Observe that $x \in I \implies 0 < x < 1 \implies 1 < x+1 < 2 \implies h(I) = (1,2) \subset \mathbb{R}^+$ so $f \equiv g \circ h : I \to \mathbb{R}$ is differentiable on I by the chain rule. Moreover, given $x_0 \in I$, $1 < h(x_0) < 2 \implies \frac{1}{2} < \frac{1}{h(x_0)} = f(x_0) < 1$ because $1/y$ is strictly decreasing for positive y ($u < v \implies 1/v < 1/u$). Thus $f(I) = (\frac{1}{2}, 1)$. Now choose $u, v \in I$ s.t. $u < v$. Then $u < v \implies u+1 < v+1 \implies 1 < \frac{v+1}{u+1} \implies \frac{1}{v+1} < \frac{1}{u+1}$ so f is strictly decreasing on I, suggesting it has an inverse. We can now find the inverse of f directly by solving for y in $x = \frac{1}{y+1}$ given $y \in f(I) \subset \mathbb{R}^+$.

$$\begin{aligned} x = \frac{1}{y+1} &\implies x(y+1) = 1 \\ &\implies y + 1 = 1/x \\ &\implies y = 1/x - 1 \\ &\implies f^{-1}(x) = g(x) - 1 \end{aligned}$$

Then it follows that $(f^{-1})'(x) = g'(x) - 0 = \frac{-1}{x^2}$. Formula 4.6 tells us that $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(1/x-1)}$. By the chain rule, $f'(x) = g'(h(x))h'(x) = g'(x+1) = \frac{-1}{(x+1)^2}$. We see that

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{f'(1/x - 1)} \\ &= \frac{1}{\frac{-1}{[(1/x-1)+1]^2}} \\ &= \frac{1}{\frac{-1}{(1/x)^2}} \\ &= \frac{\frac{1}{x^2}}{-1} \\ &= \frac{-1}{x^2} \end{aligned}$$

Which matches the derivative we derived previously.

9. $\boxed{4.2.5}$

Proof. Let I be a neighborhood of x_0 and let $f : I \to \mathbb{R}$ be continuous, strictly monotone, and differentiable at x_0. Assume that $f'(x_0) = 0$. We will show that $f^{-1} : f(I) \to \mathbb{R}$ is not differentiable at $f(x_0)$. Define $g : f(I) \to \mathbb{R}$ as $g(x) = f^{-1}(f(x)) = x$. By way of contradiction, suppose f^{-1} is differentiable at $f(x_0)$. Then by the chain rule $g'(x_0) = (f^{-1} \circ f)'(x) = (f^{-1})'(f(x_0))f'(x_0)$. By assumption $f'(x_0) = 0$ so $g'(x_0) = (f^{-1})'(f(x_0)) \cdot 0 = 0$. But $g(x) = x$ so we can find its derivative directly. Suppose $x_0 \in I$ and $\{x_n\} \subset I \setminus \{x_0\}$ s.t. $x_n \to x_0$. Then $g'(x) = \lim_{n\to\infty} \frac{g(x_n)-g(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{x_n - x_0}{x_n - x_0}$. Since $x_0 \notin x_n$ it follows that $x_n - x_0 \neq 0$ and thus $g'(x) = 1$. But this contradicts the result of the chain rule, so the original assumption that f^{-1} is differentiable at $f(x_0)$ is false, i.e. f^{-1}, defined as such, is not differentiable at $f(x_0)$. QED

10. *Proof.* Suppose $f : D \to \mathbb{R}$ is Lipschitz and that for some $a \in D$, the function $g : f(D) \to \mathbb{R}$ is differentiable at $f(a) = b$. Assuming that $g'(b) = 0$, we will show that $g \circ f$ is differentiable at a with $(g \circ f)'(a) = 0$.

Consider the quantity $\lim_{x\to a} \frac{g(f(x))-g(f(a))}{x-a} = \lim_{n\to\infty} \frac{g(f(x_n))-g(f(a))}{x_n - a}$. By definition this is equivalent to $(g \circ f)'(a)$, and we claim this limit exists and is zero. Choose any $\{x_n\} \subset D \setminus \{a\}$ s.t. $\lim_{n\to\infty} x_n = a$. Denote $y_n = f(x_n)$, then whenever $y_n \neq b$, it is true that $\frac{y_n - b}{y_n - b} = 1$. So it follows that

$$\begin{aligned} \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(a))}{x_n - a} &= \lim_{n\to\infty} \frac{g(y_n) - g(b)}{x_n - a} \frac{y_n - b}{y_n - b} \\ &= \lim_{n\to\infty} \frac{g(y_n) - g(b)}{y_n - b} \frac{y_n - b}{x_n - a} \end{aligned}$$

is the derivative $(g \circ f)'(a)$. To handle cases where $y_n = b$, we can use the function $H(y)$:

$$H(y) = \begin{cases} \frac{g(y)-g(b)}{y-b} & y \neq b \\ g'(b) & y = b \end{cases}$$

We claim $\lim_{n\to\infty} \left[H(y_n) \frac{y_n - b}{x_n - a} \right] = (g \circ f)'(a)$. If $y_n \neq b$, this follows directly from the quantity we derived above. Otherwise, if $y_n = b$, we can see that $\frac{g(y_n)-g(b)}{x_n - a} = \frac{0}{x_n - a} = 0$, i.e. the difference quotient is zero and thus the derivative is zero. Observe that $y_n = b \implies H(y_n)\frac{y_n - b}{x_n - a} = g'(b) \cdot 0 = 0$, so $\forall x \in D,\ H(y_n)\frac{y_n-b}{x_n-a} = \frac{g(y_n)-g(b)}{x_n-a}$. To prove the existence and value of the limit, it therefore suffices to show that $\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty}\left[H(y_n)\frac{y_n-b}{x_n-a}\right] = 0$. Since f is continuous, we know $\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} y_n = b$, or equivalently, $\forall \delta' > 0,\ \exists \delta > 0$ s.t. $|x_n - a| < \delta \implies |y_n - b| < \delta'$. Since $g'(b) = 0$, assuming $y_n \neq b$, then $\forall \epsilon' > 0,\ \exists \delta' > 0$ s.t. $|y_n - b| < \delta' \implies \left|\frac{g(y_n)-g(b)}{y_n-b} - 0\right| < \epsilon'$. But $H(b) = g'(b)$ so it follows that $|y_n - b| < \delta' \implies |H(y_n)| < \epsilon'$ for all y_n. Since f is Lipschitz, $\exists C \geq 0$ s.t. $\forall u, v \in D,\ |f(u) - f(v)| \leq C|u - v|$. So it is true that $\left|\frac{y_n-b}{x_n-a}\right| \leq C$.

Case 1 If $C = 0$, then $\left|\frac{y_n-b}{x_n-a}\right| = 0$, so it is trivial that $|y_n - a| < \delta' \implies \left|H(y_n)\frac{y_n-b}{x_n-a}\right| = 0 < \epsilon'$.

Case 2 Now consider the case where $C > 0$. Suppose we choose $\epsilon' = \epsilon/C$ for any $\epsilon > 0$. Then

$$\begin{aligned}
|y_n - b| < \delta' &\implies |H(y)| < \epsilon' \\
&\implies \left|H(y_n)\frac{y_n - b}{x_n - a}\right| < C\epsilon' = C\frac{\epsilon}{C} = \epsilon
\end{aligned}$$

Considering all prior steps, we have thus established that

$$\begin{aligned}
\forall \epsilon, \delta' > 0,\ \exists \delta > 0 \text{ s.t. } |x_n - a| < \delta &\implies |y_n - b| < \delta' \\
&\implies |H(y_n)| < \frac{\epsilon}{C} \\
&\implies \left|H(y_n)\frac{y_n - b}{x_n - a}\right| < \epsilon \\
&\implies \left|\frac{g(f(x_n)) - g(f(a))}{x_n - a}\right| < \epsilon \\
&\implies \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(a))}{x_n - a} = 0
\end{aligned}$$

Or, more succintly, $\lim_{x\to a} \frac{g(f(x_n))-g(f(a))}{x_n-a} = (g \circ f)'(a) = 0$. QED