APPM 4440 HW 10

Siraaj Sandhu

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Problem	Self-Grade	Grade
#1	5	
#2	5	
#3	4	
#4	5	
#5	5	
#6	5	
#7	2	
#8	4	
#9	5	
#10	5	
Tot/50	45/50	

1. $\boxed{\text{8.1.1(b,d)}}$

(b) Let $f(x) = e^{x^2}$, $g(x) = 1 + 2x^2$, and $x_0 = 0$. Then

$$\begin{aligned} f(x_0) &= f(0) = 1 = g(0) = g(x_0) \\ f'(x_0) &= 2x_0 e^{x_0^2} = 0 = 4x_0 = g'(x_0) \\ f''(x_0) &= 2e^{x_0^2} + 4x_0 e^{x_0^2} = 2(1) = 2 \neq 4 = g''(x_0) \end{aligned}$$

So the highest order of contact for f and g is one.

(d) Let $f(x) = \ln x$, $g(x) = (x-1)^{200} + \ln x$, and $x_0 = 1$. Then

$$\begin{aligned} f(x_0) &= f(1) = 0 = g(1) = g(x_0) \\ f'(x_0) &= 1/x_0 = 1 = 200(0)^{199} + 1 = 200(x_0 - 1)^{199} + 1/x_0 = g'(x_0) \\ f''(x_0) &= -x_0^{-2} = -1 = (199 \cdot 200)(x_0 - 1)^{198} + -x_0^{-2} = g''(x_0) \end{aligned}$$

At this point, note that for any nonnegative integers l and k, $\frac{d^k}{dx^k}[(x - x_0)^l] = 0$ unless $k = l$. So the derivatives of this quantity will consistently vanish up until the 200th derivative. It thus suffices to compare the logarithmic terms of g and f, but since these are equal, it follows that their derivatives will match and thus the order of contact will be 199.

The order of contact and reasoning is correct for both parts. 5/5

2. $\boxed{\text{8.1.4}}$ Suppose that $f : \mathbb{R} \to \mathbb{R}$ has 3 derivatives and that the third Taylor polynomial at $x_0 = 0$ is $p_3(x) = 1+4x-x^2+x^3/6$. We claim hat there is a neighborhood of the point 0 s.t. $f : I \to \mathbb{R}$ is positive, strictly increasing, and has a strictly decreasing derivative.

Proof. The third Taylor polynomial, assuming $x_0 = 0$, is given by the formula $p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3$. From this definition and the premise we can deduce that $f(0) = 1$, $f'(0) = 4$, $f''(0) = -1$, and $f^{(3)}(0) = 1/6$. Since f, f', and f'' are all differentiable at zero, they are all continuous at zero. It thus follows for any positive ϵ that $\exists\delta_1$, δ_2, δ_3 all greater than zero s.t. if $x \neq x_0$, then

$$\begin{aligned} f(x) &\in (1 - \epsilon, 1 + \epsilon) \text{ if } x \in (x_0 - \delta_1, x_0 + \delta_1) \\ f'(x) &\in (4 - \epsilon, 4 + \epsilon) \text{ if } x \in (x_0 - \delta_2, x_0 + \delta_2) \\ f''(x) &\in (-1 - \epsilon, -1 + \epsilon) \text{ if } x \in (x_0 - \delta_3, x_0 + \delta_3) \end{aligned}$$

Then, if we choose $\epsilon < 1$ and let I be the intersection $(x_0 - \delta_1, x_0 + \delta_1) \cup (x_0 - \delta_2, x_0 + \delta_2) \cup (x_0 - \delta_3, x_0 + \delta_3)$, we see that it will be simultaneously true that $f(0) > 0$, $f'(0) > 0$, and $f''(0) < 0$ on this neighborhood I of zero. QED

Proceeds via different strategy that key, but still appears to be correct. 5/5

3. $\boxed{8.1.5}$ Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a second derivative and that $\begin{cases} f''(x) + f(x) = e^{-x}, & \text{for all } x \\ f(0) = 0 \text{ and } f'(0) = 2 \end{cases}$. We intend to find the fourth Taylor polynomial for f given $x_0 = 0$. It follows from the given definition that $f''(x) = e^{-x} - f(x)$. Then we see that $f''(0) = e^0 - f(0) = 1 - 0 = 1$ Since the exponential function and f are both differentiable at zero, we see that $f^{(3)}(x) = \frac{d}{dx}f''(x) = -e^{-x} - f'(x)$ when $x = 0$. Because f'' exists at zero, we see that $f^{(4)}(x) = \frac{d}{dx}f^{(3)}(x) = e^{-x} - f''(x)$ when $x = 0$. It follows that $f^{(3)}(0) = -1 - 2 = -3$ and $f^{(4)}(0) = 1 - (-3) = 4$. Using the definition of the Taylor polynomial centered around $x_0 = 0$, we see that

$$\begin{aligned} p_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 \\ &= 0 + 2x + \frac{1}{2}x^2 - \frac{3}{6}x^3 + \frac{4}{24}x^4 \\ &= 2x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 \end{aligned}$$

Minor calculation error when finding $f^{(4)}$. Accidentally substituted f''' instead of f'' when finding the fourth derivative. 4/5

4. $\boxed{8.2.2}$ We intend to prove that $1 + x/3 - x^2/9 < (1+x)^{1/3} < 1 + x/3$ if $x > 0$.

Proof. Consider the function $f(x) = (1+x)^{1/3}$. This function has at least 3 derivatives:

$$\begin{aligned} f(x) &= (1+x)^{1/3} \\ f'(x) &= \frac{1}{3}(1+x)^{-2/3} \\ f''(x) &= -\frac{2}{9}(1+x)^{-5/3} \\ f^{(3)}(x) &= \frac{10}{27}(1+x)^{-8/3} \end{aligned}$$

The Lagrange Remainder Theorem dictates that $\exists c_1, c_2 \in (0, x)$ s.t.

$$\begin{aligned} f(x) &= \sum_{2}^{k=0} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(3)}(c_1)}{3!}x^3 \\ &= 1 + x/3 - x^2/9 + \frac{\frac{10}{27}(1+c_1)^{-8/3}}{3!}x^3 \\ &= 1 + x/3 - x^2/9 + \frac{10}{162}(1+c_1)^{-8/3}x^3 \\ \text{and } f(x) &= \sum_{1}^{k=0} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(2)}(c_2)}{2!}x^2 \\ &= 1 + x/3 + \frac{-\frac{2}{9}(1+c_2)^{-5/3}}{2!}x^2 \\ &= 1 + x/3 - \frac{1}{9}(1+c_2)^{-5/3}x^2 \end{aligned}$$

Since c_1 and c_2 are positive by definition, we expect the quantities $(1+c_2)^{-8/3}$ and $(1+c_2)^{-5/3}$ to both be positive. It directly follows that

$$\begin{aligned} & 0 < \frac{10}{162}(1+c_1)^{-8/3}x^3 \\ & \implies \left[1 + x/3 - x^2/9\right] < \left[1 + x/3 - x^2/9\right] + \frac{10}{162}(1+c_1)^{-8/3}x^3 = f(x) \\ \text{and } & -\frac{1}{9}(1+c_2)^{-5/3}x^2 < 0 \\ & \implies f(x) = [1 + x/3] - \frac{1}{9}(1+c_2)^{-5/3}x^2 < [1 + x/3] \end{aligned}$$

At which point the claim immediately follows: $1 + x/3 - x^2/9 < f(x) = (1+x)^{1/3} < 1 + x/3$ if $x > 0$. QED

Proceeds using similar method as key. 5/5

5. $\boxed{8.2.12}$ Let I be a neighborhood of the point x_0 and suppose that $f : I \to \mathbb{R}$ has a continuous third derivative with $f'''(x) > 0$ for all $x \in I$.

(a) We intend to prove that if $x_0 + h \neq x_0$ is in I, there is a unique number $\theta = \theta(h)$ in the interval $(0,1)$ s.t.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0 + \theta h)\frac{h^2}{2}$$

Proof. The claim requires that h be a nonzero real. We also know that $x_0 + h \in I$. Since we know f has at least three continuous derivatives, we can construct a degree-1 Taylor polynomial as follows:

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

If we let $x = x_0 + h \in I$, then we see that

$$p_1(x_0 + h) = f(x_0) + f'(x_0)h$$

By the Lagrange Remainder Theorem, we know that $\exists c$ strictly between x_0 and $x_0 + h$ s.t.

$$f(x_0 + h) = p_1(x_0 + h) + f''(c)\frac{h^2}{2!}$$

Let us now define the function

$$H(\theta) = \left[f(x_0) + f'(x_0)h + f''(x_0 + \theta h)\frac{h^2}{2}\right] - f(x_0 + h)$$

Note that for this proof it will thus suffice to show that H has a unique root θ in $(0,1)$. By the result from the Remainder Theorem above we have $f(x_0+h) = f(x_0) + f'(x_0)h + f''(c)\frac{h^2}{2}$, where c is strictly between x_0 and $x_0 + h$. Observe that, regardless of the sign of h, we get $\left.\begin{array}{l} h > 0 \implies x_0 < c < x_0 + h \implies 0 < \frac{c - x_0}{h} < 1 \\ h < 0 \implies x_0 + h < c < x_0 \implies 0 < \frac{c - x_0}{h} < 1 \end{array}\right\} \implies \theta = \frac{c - x_0}{h} \in (0,1)$. If we evaluate H at this theta, we get

$$\begin{aligned} H(\theta) &= H([c - x_0]/h) \\ &= \left[f(x_0) + f'(x_0)h + f''(x_0 + [c - x_0]/h \cdot h)\frac{h^2}{2}\right] - f(x_0 + h) \\ &= \left[f(x_0) + f'(x_0)h + f''(c)\frac{h^2}{2}\right] - f(x_0 + h) = 0 \end{aligned}$$

which follows directly from the result of the Remainder Theorem. So $\theta = [c - x_0]/h$ is a root in $(0,1)$, satisfying existence. Consider $H'(\theta) = \frac{d}{d\theta}H(\theta) = \frac{d}{d\theta}f''(x_0 + \theta h)\frac{h^2}{2}$. The chain rule dictates that $H'(\theta) = \frac{h^3}{2}f'''(x_0 + \theta h)$. Since f''' is always positive and h is nonzero, we expect that $H'(\theta)$ is either strictly negative or strictly positive, according to the sign of h. In both cases, this indicates that $H(\theta)$ is strictly monotone, and therefore the θ root we derived must be unique. Suppose not. Then there could be another, distinct root in $(0,1)$. But this would require that H is either monotone (not strictly), or it has local extrema, which would violate strict monotonicity. So θ is unique. By its definition, assuming a fixed x_0, we see also that $\theta = [c - x_0]/h = \theta(h)$, as c is also constrained by h. Hence

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0 + \theta h)\frac{h^2}{2}$$

QED

(b) We intend to prove that $\lim_{h \to 0} \theta(h) = 1/3$.

Proof. We can construct a degree-2 Taylor polynomial for f:

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$x = x_0 + h \implies p_2(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2}$$

By the Lagrange Remainder Theorem, assuming $h \neq 0$, there exists c strictly between x_0 and $x_0 + h$ s.t. $f(x_0 + h) = p_2(x_0 + h) + f'''(c)\frac{h^3}{6}$. Using the result we obtained in part (a), we have

$$\begin{aligned} p_2(x_0 + h) + f'''(c)\frac{h^3}{6} &= f(x_0) + f'(x_0)h + f''(x_0 + \theta h)\frac{h^2}{2} \\ f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(c)\frac{h^3}{6} &= f(x_0) + f'(x_0)h + f''(x_0 + \theta h)\frac{h^2}{2} \end{aligned}$$

$$f''(x_0)\frac{h^2}{2} + f'''(c)\frac{h^3}{6} = f''(x_0 + \theta h)\frac{h^2}{2}$$

where $\theta \in (0, 1)$ has also been obtained via the Lagrange Remainder Theorem. Further rearrangement yields

$$\begin{aligned}
f''(x_0)\frac{h^2}{2} + f'''(c)\frac{h^3}{6} &= f''(x_0 + \theta h)\frac{h^2}{2} \\
f'''(c)\frac{h^3}{6} &= [f''(x_0 + \theta h) - f''(x_0)]\frac{h^2}{2} \\
f'''(c)\frac{1}{3} &= \frac{f''(x_0 + \theta h) - f''(x_0)}{h} \\
f'''(c)\frac{1}{3} &= \left[\frac{f''(x_0 + \theta h) - f''(x_0)}{\theta h}\right]\theta \\
\lim_{h\to 0} f'''(c)\frac{1}{3} &= \lim_{h\to 0}\left[\frac{f''(x_0 + \theta h) - f''(x_0)}{\theta h}\right]\theta
\end{aligned}$$

Since θ is a bounded quantity, convergence of h to zero will be preserved in the product θh: $0 < \theta h < h \implies \theta h \to 0$ as $h \to 0$. So the RHS is just the limit of the difference quotient of f'' at x_0. As for the LHS, note that c is strictly between x_0 and $x_0 + h$. So, as $h \to 0$, we expect $c \to x_0$ as it is essentially squeezed. Then the continuity of f''' implies that $\lim_{h\to 0} f'''(c) = f'''(\lim_{h\to 0} c) = f'''(x_0)$. So we obtain

$$\begin{aligned}
f'''(x_0)\frac{1}{3} &= f'''(x_0)\lim_{h\to 0}\theta \\
\lim_{h\to 0}\theta &= \frac{1}{3}
\end{aligned}$$

QED

Uses nearly similar proof as key for first part and uses limit like in key to find θ. 5/5

6. $\boxed{8.3.2}$ Define $f(x) = 1/x$ if $0 < x < 2$.

(a) . We want to find p_n, the nth Taylor polynomial at $x_0 = 1$. We can expect the kth derivative of $1/x$ to be $k!(-1)^k x^{-(k+1)}$. Note that, with $x_0 = 1$,

$$\begin{aligned}
p_n &= \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\
&= \sum_{k=0}^{n}\frac{k!(-1)^k x_0^{-(k+1)}}{k!}(x - x_0)^k \\
&= \sum_{k=0}^{n}\frac{k!(-1)^k (1)^{-(k+1)}}{k!}(x - 1)^k \\
&= \sum_{k=0}^{n}(-1)^k(x - 1)^k \\
&= \sum_{k=0}^{n}(1 - x)^k \\
&= \frac{1 - (1 - x)^{n+1}}{1 - (1 - x)}, \text{ by the partial geometric sum formula} \\
&= \frac{1 - (1 - x)^{n+1}}{x}
\end{aligned}$$

(b) we can measure the error in p_n:

$$\begin{aligned}
f(x) - p_n(x) &= \frac{1}{x} - \frac{1 - (1 - x)^{n+1}}{x} \\
&= \frac{1 - 1 + (1 - x)^{n+1}}{x} \\
&= \frac{(1 - x)^{n+1}}{x}
\end{aligned}$$

(c) We know $f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(1)}{k!}(x - 1)^k$ holds iff $\lim_{n\to\infty}[f(x) - p_n(x)] = 0$. We assume $0 < x < 2$, or $|x - 1| < 1$ equivalently. So we can evaluate the quantity we found in (b):

$$\lim_{n\to\infty}[f(x) - p_n(x)] = \lim_{n\to\infty}\frac{(1 - x)^{n+1}}{x}$$

$$= \frac{1}{x} \lim_{n\to\infty} (1-x)^{n+1}$$

Note that $|1-x| < 1$, so additional powers of $(1-x)$ approach 0. For any ϵ, we seek some N s.t. for $n \geq N$, $|1-x|^{n+1} < \epsilon$. Consider the sequence $a_{n+1} = |1-x|a_n$, $a_1 = |1-x|^2$. Since $0 \leq |1-x| < 1$, we expect $a_{n+1} \leq a_n$. Since a_n is nonnegative and decreasing, a_n, we expect the limit $\lim_{n\to\infty} a_{n+1} = |1-x| \lim_{n\to\infty} a_n$ to yield $\lim_{n\to\infty} a_n = 0$ and thus $|1-x|^{n+1} \to 0$. So the equality holds: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$.

Proceeds using nearly similar strategy as key. 5/5

7. $\boxed{8.3.4}$ Suppose $F : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders and that $\begin{cases} F''(x) - F'(x) - F(x) = 0 \text{ for all } x \\ F(0) = 1 \text{ and } F'(0) = 1 \end{cases}$. We intend to find a recursive formula for the coefficients of the nth Taylor polynomial for F at $x = 0$.

We see that $F''(x) = F'(x) + F(x) \implies F''(0) = 2$. The first rule can then be differentiated: $F'''(x) - F''(x) - F'(x) = 0 \implies F'''(0) = 2 + 1 = 3$. So the next derivative is the sum of the previous two (fibonacci). Let us index coefficients of the nth Taylor polynomial using integers $0, 1, 2, \ldots,$ such that the highest index matches the order of the Taylor polynomial. We know a typical kth Taylor coefficient is $\frac{F^{(k)}(x_0)}{k!}$. We also know that $F^{(k)}(0) = F^{(k-1)}(0) + F^{(k-2)}(0)$. Let the sequence a_n denote the nth coefficient, s.t. $F(x) = \sum_{n=0}^{\infty} a_n x^n$. We expect that $a_0 = F(0) = 1$ and $a_1 = F'(0) = 1$. Then $a_n = \frac{(n-1)!a_{n-1} + (n-2)!a_{n-2}}{n!}$. We can simplify: $a_n = \frac{(n-2)![(n-1)a_{n-1} + a_{n-2}]}{n!}$. Thus $a_n = \frac{(n-1)a_{n-1} + a_{n-2}}{n(n-1)}$.

Found the recursive formula for the Taylor coefficient but was unable to prove convergence. 2/5

8. $\boxed{8.5.4}$ We intend to prove that if the functions $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ are continuous,with $h(x) \geq 0$ for all $x \in [a, b]$, then there is a point c in (a, b) such that $\int_a^b h(x)g(x)dx = g(c)\int_a^b h(x)dx$.

Proof. Since g is continuous, by EVT it a minimum m and a maximum M on the interval $[a, b]$. Then it follows that

$$\begin{aligned} m \leq& g(x) \leq M \\ mh(x) \leq& g(x)h(x) \leq Mh(x) \\ m\int_a^b h(x)dx \leq& \int_a^b g(x)h(x) \leq M\int_a^b h(x)dx \\ m \leq& \frac{\int_a^b g(x)h(x)dx}{\int_a^b h(x)dx} \leq M \end{aligned}$$

Since he ratio above lies in the image of g on $[a, b]$, IVT yields some $c \in (a, b)$ s.t. $\int_a^b h(x)g(x)dx = g(c)\int_a^b h(x)dx$ after some additional rearrangement. QED

The application of IVT appears to be flawed upon reviewing the answer because the range is inclusive of m and M. 4/5

9. $\boxed{8.5.5}$ The cauchy integral remainder formula dictates that for f with $n+1$ derivatives, and $f^{(n+1)}$ being continuous, we have for each $x \in I$ $f(x) = p_n(x) + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt$, where p_n denotes an nth-order Taylor polynomial. If we apply the result from the previous exercise, we see that if we assume $f^{(n+1)}$ to be continuous, we can say that there exists some c between x and x_0 s.t. $\int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt = f^{(n+1)}(c)\int_{x_0}^x (x-t)^n dt$. But now the integral in the RHS is relatively straightforward to evaluate, and we get $\int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(c)}{(n+1)}(x-x_0)^{n+1}$. If we substitute this back into the result from Cauchy integral remainder formula, we see that $f(x) = p_n(x) + \frac{f^{(n+1)!}(c)}{(n+1)}(x-x_0)^{n+1}$, which is the same as the Lagrange remainder. So we have shown that the result of the cauchy integral remainder formula may be considered more general than the lagrange remainder theorem.

Proceeds using similar strategy as key. 5/5

10. $\boxed{8.6.3}$ For a natural n, f is n-times continuously differentiable if f has an nth derivative and $f^{(n)}$ is continuous. Let $h(x) = \int_0^x |t|dt$ for all x. We will show that h is once continuously differentiable but not twice continuously differentiable.

We see that $h'(x)$, by the FTC, is just $|x|$, which is continuous. But now $h'(x)$ is not differentiable as it is the absolute value function, even though it is continuous, so it is only once continuously differentiable.

For an arbitrarily continuously-differentiable function, consider $f(x) = x^n \ln x$. When $n = 1$, $f(x) = x \ln x$, which is continuous on $[0, \infty)$ if we make a continuity correction at the origin: let $f(0) = 0$. It is also differentiable: $f'(x) = \ln x + 1$. But obviously $f''(x)$ will have a $1/x$ term which means that f' is not differentiable at 0. Similarly, when $n = 2$, we have $f(x) = x^2 \ln x$ and $f'(x) = 2x \ln x + x$. The nth power factor serves to cancel out the x in the denominator that comes from natural log's derivative, but only n times until x appears in the denominator again and the function becomes discontinuous at zero.

Demonstrates that h is only once-differentiable. Definition of function is different but still appears to be valid. 5/5