APPM 4440 HW 6

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Problem	Self-Grade	Grade
#1	5	
#2	5	
#3	5	
#4	5	
#5	5	
#6	5	
#7	5	
#8	5	
#9	5	
#10	5	
Tot/50	50/50	

1. $\boxed{3.4.3}$

Proof. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous. For any sequences $\{u_n\}, \{v_n\} \subset D$ s.t. $\lim_{n\to\infty}[u_n - v_n] = 0$, it is thus true that $\lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$ and $\lim_{n\to\infty}[g(u_n) - g(v_n)] = 0$. In other words, the sequences $F_n = f(u_n) - f(v_n)$ and $G_n = g(u_n) - g(v_n)$ both converge to zero. The sum of a convergent sequence is also convergent: $\lim_{n\to\infty}[F_n + G_n] = 0 + 0 = 0$. If we substitute the formulae for F_n and G_n, we get

$$\begin{aligned}
\lim_{n\to\infty}[(f(u_n) - f(v_n)) + g(u_n) - g(v_n)] &= 0\\
\lim_{n\to\infty}[f(u_n) + g(u_n) - f(v_n) - g(v_n)] &= 0\\
\lim_{n\to\infty}[f(u_n) + g(u_n) - f(v_n) - g(v_n)] &= 0\\
\lim_{n\to\infty}[f(u_n) + g(u_n) - (f(v_n) + g(v_n))] &= 0\\
\lim_{n\to\infty}[(f + g)(u_n) - (f + g)(v_n)] &= 0
\end{aligned}$$

We know $\lim_{n\to\infty}[u_n - v_n] = 0$ and have just shown that $\lim_{n\to\infty}[(f + g)(u_n) - (f + g)(v_n)] = 0$. By definition, $f + g : D \to \mathbb{R}$ is uniformly continuous. QED

Used properties of convergent sequences (Sum rule) rather than providing a direct $\epsilon - N$ proof of convergence, but the Sum rule encapsulates the same reasoning. 5/5

2. $\boxed{3.4.6}$

Proof. Suppose that $f : D \to \mathbb{R}$ $g : D \to \mathbb{R}$ are uniformly continuous. For any sequences $\{u_n\}, \{v_n\} \subset D$ s.t. $\lim_{n\to\infty}[u_n - v_n] = 0$, it is thus true that $\lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$ and $\lim_{n\to\infty}[g(u_n) - g(v_n)] = 0$. We will show that the product $fg : D \to \mathbb{R}$ is not necesarily uniformly continuous. Suppose $f(x) = x$, $g(x) = x$, and $D = \mathbb{R}$. We can show that f and g are both uniformly continuous. Choose any $\{u_n\}, \{v_n\} \subset D = \mathbb{R}$ s.t. $\lim_{n\to\infty}[u_n - v_n] = 0$. Since f and g are the identity functions, it follows directly from this choice that $\lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$ and $\lim_{n\to\infty}[g(u_n) - g(v_n)] = 0$, so both are uniformly continuous over $\mathbb{R}$. However, $fg(x) = x^2$. We can now choose $u_n = n$ and $v_n = n - \frac{1}{n}$. Clearly, $\lim_{n\to\infty}[u_n - v_n] = \lim_{n\to\infty}\frac{1}{n} = 0$. However, $\lim_{n\to\infty}[fg(u_n) - fg(v_n)] = \lim_{n\to\infty}[n^2 - (n^2 - 2 + \frac{1}{n^2})] = \lim_{n\to\infty}[2 - \frac{1}{n^2}] = 2 \neq 0$. So the product fg is not necessarily uniformly continuous. QED

Provided the same example as on the key with the same reasoning. 5/5

3. $\boxed{3.4.10}$

Proof. Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous. We claim f is bounded on (a, b). Suppose not. Then $\forall M > 0$, $\exists x_0 \in (a, b)$ s.t. $|f(x_0)| > M$. Since $\mathbb{N} \subset \mathbb{R}^+$, we can say $\forall n \in \mathbb{N}, \exists x_n \in (a, b)$ s.t. $|f(x_n)| > n$.

Lemma 1. *We claim $f(x_n)$ diverges. Suppose not. Then $\exists L$ s.t. $\forall\epsilon > 0, \exists N > 0$ s.t. $|f(x_n) - L| < \epsilon$ for $n \geq N$. It could be true that $f(x_n) > n$. Then $n - L < f(x_n) - L < \epsilon$. This requires $n < \epsilon + L$, but if $\epsilon + L \leq 0$, this is trivially false because $n \in \mathbb{N}$. Otherwise, if $\epsilon + L > 0$, the Archimedean property tells us $\exists N' \in \mathbb{N}$ s.t. $n > \epsilon + L$ for $n \geq N'$. Clearly, $[N', \infty) \cap [N, \infty) \neq \emptyset$, implying there is some n for which $n < \epsilon + L$ and $n > \epsilon + L$, which is a contradiction. If $f(x_n) \not> n$, it must be true that $-f(x_n) > n$. For convergence it suffices to show $-\epsilon < f(x_n) - L < -n - L$, requiring $-\epsilon < -n - L \implies n < \epsilon - L$. Similarly to the first case, if $\epsilon - L \leq 0$ this is trivially false because $n \in \mathbb{N}$. Otherwise, if $\epsilon - L > 0$, the Archimedean property tells us $\exists N' \in \mathbb{N}$ s.t. $n > \epsilon - L$ for $n \geq N'$. Clearly, $[N', \infty) \cap [N, \infty) \neq \emptyset$, implying there is some n for which $n < \epsilon - L$ and $n > \epsilon - L$, which is a contradiction. In either case, given $|f(x_n)| > n$, $f(x_n)$ diverges.*

By Lemma 1, $f(x_n)$ diverges and all subsequences $f(x_{n_k})$ diverge because they conform to our original assumption: $|f(x_{n_k})| > n_k$. However, $x_n \subset (a, b)$ so $|x_n| \leq \max\{|a|, |b|\}$, i.e. x_n is bounded. Thus x_n has a convergent subsequence x_{n_k}, which is equivalently Cauchy. Since f is uniformly continuous, $\forall x, y \in (a, b), \forall\epsilon > 0, \exists\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Since x_{n_k} is Cauchy, $\forall\delta > 0, \exists N > 0$ s.t. $\forall n_k, n_j \geq N, |x_{n_k} - x_{n_j}| < \delta$. It follows from uniform continuity that $|f(x_{n_k}) - f(x_{n_j})| < \epsilon$ for $n_k, n_j \geq N$. In other words, $f(x_{n_k})$ is Cauchy and therefore convergent. But Lemma 1 tells us that $f(x_{n_k})$ must diverge. So the original assumption that f is not bounded is incorrect, i.e. f must be bounded.

QED

Employ similar reasoning but use a sequence s.t. $f(x_n) > n$ and claims that subsequences of $f(x_n)$ will diverge. Also uses the fact that x_n contains a convergent subsequence (i.e. contains a Cauchy subsequence) to establish a contradiction because uniformly continuous functions preserve Cauchy sequences (convergent sequences). 5/5

4. $\boxed{3.4.11}$

Proof. Suppose $f : D \to \mathbb{R}$ is Lipschitz, i.e. $\exists C \geq 0$ s.t. $\forall u, v \in D, |f(u) - f(v)| \leq C|u - v|$. We claim f is uniformly continuous. If $C = 0$, we see that $|f(u) - f(v)| \leq 0$. Thus for any positive ϵ, we can choose any positive δ. If $u, v \in D$ and $|u - v| < \delta$, we can say that $|f(u) - f(v)| \leq 0 < \epsilon$. By definition f is thus uniformly continuous. If $C > 0$, we see that $|f(u) - f(v)| \leq C|u - v|$. For any positive ϵ, choose $\delta = \frac{\epsilon}{C} > 0$. Given $u, v \in D$ and $|u - v| < \delta$, we see that $|f(u) - f(v)| \leq C|u - v| < C\delta = \epsilon$. Since δ depends only on ϵ, f is uniformly continuous by definition. QED

Uses similar reasoning but proceeds using ϵ-δ proof instead. Believe the proof is still correct. 5/5

5. $\boxed{3.5.3}$

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. We will verify the ϵ-δ criterion for continuity at each point x_0. We must show that $\forall x_0 \in \mathbb{R}, \forall\epsilon > 0, \exists\delta > 0$ s.t. $x \in D$ and $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Suppose $|x - x_0| < \delta$. By the triangle inequality we can say $|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < \delta + |x_0|$. We know that

$$\begin{aligned}
|f(x) - f(x_0)| &= |x^3 - x_0^3| \\
&= |(x - x_0)(x^2 + xx_0 + x_0^2)| \\
&= |x - x_0||x^2 + xx_0 + x_0^2|
\end{aligned}$$

By the triangle inequality we can say $|x^2 + xx_0 + x_0^2| \leq |x|^2 + |x||x_0| + |x_0|^2$. We know that $|x| < \delta + |x_0|$. By definition, $|x| \geq 0$. Suppose $|x| = 0$, then because $\delta > 0 \wedge |x_0| \geq 0 \implies \delta + |x_0| > 0$, it is true by positivity that $0^2 = 0 < (\delta + |x_0|)^2 \implies |x|^2 < (\delta + |x_0|)^2$. Otherwise, if $|x| > 0$, we can say directly that $|x|^2 < (\delta + |x_0|)^2$. If $|x| = 0$ and $|x_0| > 0$, then by positivity $|x||x_0| = 0 < (\delta + |x_0|)|x_0|$. If $|x_0| = 0$, we see trivially that $|x||x_0| = 0 = (\delta + |x_0|)|x_0|$. If $|x| > 0, |x_0| > 0$, then $|x| < \delta + |x_0| \implies |x||x_0| < (\delta + |x_0|)|x_0|$. Thus $|x||x_0| \leq (\delta + |x_0|)|x_0|$. Summing these inequalities gives $|x|^2 + |x||x_0| + |x_0|^2 < (\delta + |x_0|)^2 + (\delta + |x_0|)|x_0| + |x_0|^2$. Suppose for the following that $0 < |x - x_0|$ (if $x = x_0$, we see trivially that $\forall\epsilon > 0, \forall\delta > 0, |x - x_0| = 0 < \delta \implies |f(x) - f(x_0)| = 0 < \epsilon$). At this point, we could choose $\delta = \min\{1, \frac{\epsilon}{1+3|x_0|+3|x_0|^2}\} > 0$. Then

$$\begin{aligned}
|f(x) - f(x_0)| &= |x - x_0||x^2 + xx_0 + x_0^2| \\
&\leq |x - x_0|(|x|^2 + |x||x_0| + |x_0|^2) \\
&< |x - x_0|((\delta + |x_0|)^2 + (\delta + |x_0|)|x_0| + |x_0|^2) \\
&< \delta((\delta + |x_0|)^2 + (\delta + |x_0|)|x_0| + |x_0|^2) \\
&= \delta(\delta^2 + 3\delta|x_0| + 3|x_0|^2) \\
&\leq \delta(1 + 3|x_0| + 3|x_0|^2), \text{ because } \delta \leq 1 \\
&\leq \epsilon, \text{ because } \delta \leq \frac{\epsilon}{1 + 3|x_0| + 3|x_0|^2}
\end{aligned}$$

So $\forall x_0 \in \mathbb{R}, \forall\epsilon > 0, \exists\delta > 0$ s.t. $x \in D \wedge |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ given we choose $\delta = \min\{1, \frac{\epsilon}{1+3|x_0|+3|x_0|^2}\}$.
QED

The proof arrives at the same choice of δ ($\min\{1, \frac{\epsilon}{1+3|x_0|+3|x_0|^2}\}$) for an ϵ-δ proof of uniformy continuity with same reasoning for choosing such a δ. 5/5

6. $\boxed{3.5.7}$

   (a) *Proof.* We claim $f : [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is continuous. We will show first that f is continuous at each $x_0 \in (0, 1]$ and second that f is continuous at 0. For the former claim, it suffices to show $\forall x_0 \in (0, 1], \forall \epsilon > 0, \forall \delta > 0, |x-x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Suppose $|x - x_0| < \delta$. By assumption $x_0 > 0$, so we can say that $\sqrt{x_0} > 0$. Since $\sqrt{x} \geq 0$, it follows that $\sqrt{x} + \sqrt{x_0} > 0$. Also, $\sqrt{x} \geq 0 \implies \sqrt{x} + \sqrt{x_0} \geq \sqrt{x_0} \implies \frac{1}{\sqrt{x_0}} \geq \frac{1}{\sqrt{x}+\sqrt{x_0}}$. With this in mind, let us choose $\delta = \epsilon\sqrt{x_0}$. Then

$$
\begin{aligned}
|f(x) - f(x_0)| &= |\sqrt{x} - \sqrt{x_0}| \\
&= \left| \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{(\sqrt{x_n} + \sqrt{x_0})} \right| \\
&= \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \\
&\leq \frac{|x - x_0|}{\sqrt{x_0}} \\
&< \frac{\delta}{\sqrt{x_0}} \\
&= \epsilon, \text{ after substituting for } \delta
\end{aligned}
$$

   So $|f(x) - f(x_0)| < \epsilon \implies f$ is continuous on $(0, 1]$. At 0, we need to show that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - 0| < \delta \implies |\sqrt{x} - 0| < \epsilon$. For any positive ϵ, choose $\delta = \epsilon^2$. Then $|x| < \delta \implies x < \delta \implies \sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon$. So $|f(x) - f(0)| < \epsilon \implies f$ is continuous at 0. Thus f is continuous at each $x_0 \in [0, 1]$. QED

   (b) *Proof.* We claim $f : [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous. We know from (a) that f is continuous on $[0, 1]$. The interval $[0, 1]$ is closed and bounded so by Theorem 3.17 f is uniformly continuous. QED

   (c) *Proof.* We claim $f : [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is not a Lipschitz function. Suppose not, i.e. f is a Lipschitz function. Then by definition, $\exists C \geq 0$ s.t. $\forall u, v \in [0, 1], |f(u) - f(v)| \leq C|u - v|$. So $|\sqrt{u} - \sqrt{v}| \leq C|u - v|$. We could then choose $u = \frac{1}{n^2}$ and $v = 0$ and expect that $|\frac{1}{n} - 0| \leq C|\frac{1}{n^2} - 0|$. Restrict n to $n \in \mathbb{N}$ and we get $\frac{1}{n} \leq C\frac{1}{n^2} \implies \forall n \in \mathbb{N}, n \leq C$. By definition n will diverge (converge to infinity) and thus a finite bound C does not exist, which is a contradiction. Thus f is not a Lipschitz function. QED

   Uses nearly identical ϵ-δ proof for (a) with same choice of δ. Uses same theorem for (b). Uses similar reasoning as key to show that no finite C satisfies criterion for uniform continuity. 5/5

7. $\boxed{3.5.8}$

Proof. We claim that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ is periodic, then it is uniformly continuous. By periodic we mean $\exists p > 0$ s.t. $\forall x, f(x + p) = f(x)$. We know that f is continuous so f is uniformly continuous on the compact interval $[0, p]$. That is, $\forall x, x_0 \in [0, p], \forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

Lemma 2. *We claim for any real number y that $\exists k \in \mathbb{Z}$ s.t. $y \in [k, k + 1)$. Suppose $y > 0$. The Archimedean principle tells us that $\exists n \in \mathbb{N}$ s.t. $y < n$. That is, the set $A = \{x \in \mathbb{N} | y < x\}$ is nonempty and bounded below by construction. So $\exists n' = \inf A$ and we claim $n' \in A$. Suppose not. Then the next greatest number that could belong to A is $n' + 1$, i.e. $\forall a \in A, n' + 1 \leq a$, but this implies that n' is not the greatest lower bound, because $n' < n' + 1$. Thus $n' \in A$. So given $n' = \min A$, we know $n' - 1 \notin A \implies n' - 1 \leq y \implies n' - 1 \leq y < n'$. Choose $k = n' - 1 \in \mathbb{Z}$. We see that $y \in [k, k + 1)$. Now suppose $y = 0$. Choose $k = 0$ and we see that $y \in [k, k + 1) = [0, 1)$. Now suppose $y < 0$. Since $0 \in \mathbb{Z}$, we know the set $A = \{x \in \mathbb{Z} | y < x\}$ is nonempty (it at least contains 0) and bounded below by construction. Using the same logic we applied in the $y > 0$ case, we know $\exists n' \in A$ s.t. $n' = \min A$. Thus $n' - 1 \notin A \implies n' - 1 \leq y$. Again, we can choose $k = n' - 1 \implies k \leq y < k + 1 \implies y \in [k, k + 1)$.*

By Lemma 2, we can choose any $x \in \mathbb{R}$ and say that $\exists k \in \mathbb{Z}$ s.t. $\frac{x}{p} \in [k, k + 1)$. So $k \leq \frac{x}{p} < k + 1 \implies kp \leq x < (k + 1)p \implies 0 \leq x - kp < p$. Now, choose any $x', x'_0 \in \mathbb{R}$. We know $\exists j, k \in \mathbb{Z}$ s.t. $x' - jp \in [0, p]$ and $x'_0 - kp \in [0, p]$. Observe that if $j, k = 0$, then $x, x_0 \in [0, p]$ and f is uniformly continuous at x_0. Otherwise, if $j < 0$ we see that

$$
\begin{aligned}
f(x' - jp) &= f(x' - (j + 1)p + p) = f(x' - (j + 1)p) \\
&= f(x' - (j + 2)p + p) = f(x' - (j + 2)p) \\
&= \cdots \\
&= f(x' - (j + n)p + p) = f(x' - (j + n)p), \text{ where } n \in \mathbb{N} \\
&= \cdots \\
&= f(x'), \text{ because eventually } n = -j
\end{aligned}
$$

The same logic demonstrates that $f(x_0' - kp) = f(x_0')$ if $k < 0$. Observe also that $x = x - p + p \implies f(x-p+p) = f(x-p) = f(x)$. Then, if $j > 0$,

$$\begin{aligned}
f(x' - jp) &= f(x' - (j-1)p - p) = f(x' - (j-1)p) \\
&= f(x' - (j-2)p - p) = f(x' - (j-2)p) \\
&= \cdots \\
&= f(x' - (j-n)p - p) = f(x' - (j-n)p), \text{ where } n \in \mathbb{N} \\
&= \cdots \\
&= f(x'), \text{ because eventually } n = -j
\end{aligned}$$

The same logic demostrates that $f(x_0' - kp) = f(x_0')$ if $k > 0$. Now suppose $x = x' - jp \in [0,p]$, $x_0 = x_0' - kp \in [0,p]$. We want to show that $\forall x', x_0' \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0$ s.t. $|x' - x_0'| < \delta \implies |f(x') - f(x_0')| < \epsilon$. Let us restrict $j = k$, then $|x' - x_0'| < \delta$ whenever $|x - x_0| < \delta$, because $|x - x_0| = |x' - jp - (x_0' - kp)| = |x' - x_0' + kp - jp| = |x' - x_0'|$. Since f is uniformly continous on $[0,p]$, we can say that $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ s.t. $|x - x_0| = |x' - x_0'| < \delta \implies |f(x) - f(x_0)| = |f(x' - jp) - f(x_0' - kp)| = |f(x') - f(x_0')| < \epsilon$. By definition, f is therefore uniformly continuous for all reals.

QED

Uses slightly more involved reasoning than key but main points of the proof are nearly identical with the key. 5/5

8. $\boxed{3.6.2}$

(a) Observe that $2x - 1$ and $x^2 - x$ are continuous on $(0,1)$ because they are both polynomials. Since $x^2 - x = x(x-1) \neq 0$ on the interval $(0,1)$, the function $f(x) = \frac{2x-1}{x(x-1)}$ is continuous on $(0,1)$. Now consider the interval $[\frac{1}{2n}, 1 - \frac{1}{2n}]$. . Observe that $f(\frac{1}{2n}) = \frac{1-n}{1-2n}4n$, and that $f(\frac{1}{2n}) - (n-1) = \frac{-((n-1)^2 + n(n+1) - 2)}{1-2n}$. Since $n \in \mathbb{N}$, $n \geq 1$, and thus the numerator and denominator will always be negative (since $(n-1)^2 \geq 0$ and $n(n+1) - 2 \geq 0$), indicating that this quantity is positive, i.e. $f(\frac{1}{2n}) \geq n - 1$. It is clear that $n-1$ is unbounded above, so it follows that $f(\frac{1}{2n})$ is unbounded above as $n \to \infty$. Similarly, we see that $f(1 - \frac{1}{2n}) = -\frac{1-n}{1-2n}4n \implies f(1 - \frac{1}{2n}) \leq 1 - n$, using the logic from above. Clearly $1 - n$ is unbounded below, so it follows that $f(1 - \frac{1}{2n})$ is unbounded below as $n \to \infty$. So $f(1 - \frac{1}{2n}) \leq f(\frac{1}{2n})$. They are only equal when $n = 1$, so let us consider f on intervals $[\frac{1}{2n}, 1 - \frac{1}{2n}]$ where $n > 1$. We can choose any $c \in \mathbb{R}$ and we know, because $f(\frac{1}{2n})$ is not bounded above and $f(1 - \frac{1}{2n})$ is not bounded below, that there will exist an n large enough s.t. $f(1 - \frac{1}{2n}) < c < f(\frac{1}{2n})$. IVT tells us there must be an $x_0 \in \bigcup_{n=1}^{\infty} [\frac{1}{2n}, 1 - \frac{1}{2n}] = (0,1)$ s.t. $f(x_0) = c$, i.e. the image of f is $\mathbb{R}$.

(b) We know $\sin x$ is continuous and bounded: $|\sin x| \leq 1$. So, $-1 \leq \sin x \leq +1 \implies 0 \leq \sin x + 1 \leq 2 \implies 0 \leq \frac{1}{2}(\sin x + 1) \leq 1$. If we introduce some scaling factor $k = 2\pi$, then we see that $f(x) = \frac{1}{2}(\sin 2\pi x + 1)$ will map from $(0,1)$ to $[0,1]$. For $x = \frac{1}{4}, f(x) = 1$, and for $x = \frac{3}{4}, f(x) = 0$. IVT tells us for any number c s.t. $c \in (f(\frac{3}{4}), f(\frac{1}{4})) = (0,1), \exists x_0 \in (\frac{1}{4}, \frac{3}{4}) \subset (0,1)$ s.t. $f(x_0) = c$. So we can say f maps from $(0,1)$ to $[0,1]$, as $0 \leq f(x) \leq 1$ and IVT showed us that the image is an interval. Moreover, f is continuous because it was defined via compositions, sums, and products of continuous functions.

(c) We suggest the inverse of the function $f : (-1,1) \to \mathbb{R}, f(x) = \frac{x}{\sqrt{1-x^2}}$. We suggset f is continuous because $1 - x^2 > 0$ on the interval $(-1,1)$ and thus the composition $\sqrt{1-x^2}$ will be continuous, and thus the quotient $\frac{x}{\sqrt{1-x^2}}$ will be continuous. We can show that f is strictly increasing. Suppose $u > v$. Then

$$\begin{gathered}
u^2 > v^2 \\
u^2 - u^2v^2 > v^2 - u^2v^2 \\
u^2(1 - v^2) > v^2(1 - u^2) \\
u\sqrt{1 - v^2} > v\sqrt{1 - u^2} \\
\frac{u}{\sqrt{1-u^2}} > \frac{v}{\sqrt{1-v^2}} \\
f(u) > f(v)
\end{gathered}$$

We can show that $f((-1,1)) = \mathbb{R}$. Consider the interval $[-1 + \frac{1}{n+1/2}, 1 - \frac{1}{n+1/2}]$. We see that $f(1 - \frac{1}{n+1/2}) = \frac{n-1/2}{\sqrt{2n}} = \sqrt{n/2} - \frac{1}{2\sqrt{2n}}$. The second term converges to zero but we see that the first term is unbounded above, thus $f(1 - \frac{1}{n+1/2})$ is unbounded above as $n \to \infty$. We also see that $f(-1 + \frac{1}{n+1/2}) = \frac{1/2-n}{\sqrt{2n}} = \frac{1}{2\sqrt{2n}} - \sqrt{n/2}$. The first term converges to zero but we see that the second term is unbounded below, thus $f(-1 + \frac{1}{n+1/2})$ is unbounded below as $n \to \infty$. So we can choose any $c \in \mathbb{R}$ and say that there exists n large enough s.t. $f(-1 + \frac{1}{n+1/2}) < c < f(1 - \frac{1}{n+1/2})$. IVT tells us there must be an $x_0 \in \bigcup_{n=1}^{\infty} [-1 + \frac{1}{n+1/2}, 1 - \frac{1}{n+1/2}] = (-1,1)$ s.t. $f(x_0) = c$, i.e. the image of f is $\mathbb{R}$. Theorem 3.29 tells us that since f is strictly increasing over the interval

$(-1,1)$, its inverse $f^{-1}: \mathbb{R} \to (-1,1)$ is continuous, strictly increasing (shown in class), and maps $\mathbb{R}$ to $(-1,1)$. We see that $f^{-1}(x) = \frac{x}{\sqrt{1+x^2}}$.

Uses different examples than key but I believe an adequate justification is provided for each. 5/5

9. $\boxed{3.6.4}$

Proof. Define

$$f(x) = \begin{cases} x-1 & x<0 \\ x+1 & x \geq 0 \end{cases}$$

We will show that f is strictly increasing. We must show that given $u, v \in \mathbb{R}$ s.t. $u > v$, $f(u) > f(v)$. If we restrict both $u, v < 0$ or both $u, v \geq 0$, it follows directly that $f(u) > f(v)$ because $u > v \implies u-1 > v-1$ and $u > v \implies u+1 > v+1$. Now suppose $u \geq 0$ and $v < 0$. Then $f(u) = u+1$ and $f(v) = v-1$. Clearly $v-1 < v < u < u+1$, so $f(u) > f(v)$. Note that $u < 0$ and $v \geq 0$ is not possible because it contradicts our assumption that $u > v$. Since f is strictly increasing and maps $\mathbb{R}$, an interval, to $\mathbb{R}$, we know by Theorem 3.29 that $f^{-1}: f(\mathbb{R}) \to \mathbb{R}$ is continuous. We see that $f(\mathbb{R}) = (-\infty, -1) \cap [1, \infty)$. Since $1 \in [1, \infty)$, i.e. 1 is in the image of f (its preimage is $x = 0 \implies f(x) = f(0) = 0+1 = 1$), we can say that f^{-1} is continuous at 1. QED

We use similar reasoning as key to prove strictly increasing property. Uses Theorem 3.29 instead of directly proof of continuity to establish continuity of the inverse, but I believe the reasoning is still valid. 5/5

10. $\boxed{3.6.13}$

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous and one-to-one s.t. $f(a) < f(b)$. Let c be a point in the open interval (a, b). We will show that $f(a) < f(c) < f(b)$. Suppose this is not the case, i.e. $f(c) \leq f(a)$ or $f(c) \geq f(b)$. Since f is one-to-one, $f(c) = f(a) \implies c = a$, which is a contradiction, since $c \in (a, b)$. The same follows for $f(b)$, and thus $f(c) \neq f(a)$ and $f(c) \neq f(b)$. So it must be true that $f(c) < f(a)$ or $f(c) > f(b)$. Consider the case where $f(c) < f(a)$. Choose any d s.t. $f(c) < d < f(a)$. Then it is also true that $f(c) < d < f(b)$. By IVT, we know $\exists x_0, x_0'$ s.t. $x_0 \in (a, c)$ and $f(x_0) = d$, and $x_0' \in (c, b)$ and $f(x_0') = d$. Therefore $f(x_0) = f(x_0') \implies x_0 = x_0'$ because f is one-to-one. But $(a, c) \cap (c, b) = \emptyset$ because $a < c < b$. Therefore $x_0 \neq x_0'$, violating the premise that f was one-to-one. Therefore $f(a) < f(c) < f(b)$. QED

Uses nearly identical proof by contradiction with same cases as key, and arrives at same conclusion. 5/5