MAT216: Linear Algebra & Fourier Analysis

Topic: Diagonalisation

Prepared by:
Saad Bin Sohan

BRAC University

Email: sohan.academics@gmail.com
GitHub: https://github.com/saad-bin-sohan

Diagonalisation

$P^{-1} A P =$ diagonal

diagonal matrix এ diagonal অন্য সব element zero.

i.e: $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P^{-1} A P = D$

$\Rightarrow \boxed{A = P D P^{-1}}$ ✓

P = usual matrix, D = Diagonal matrix

$$\Rightarrow \quad P^{-1} A P = D$$

$$\Rightarrow \quad P^{-1} A P \times P^{-1} = D \times P^{-1}$$

$$\Rightarrow \quad P^{-1} A = D P^{-1}$$

$$\Rightarrow \quad P \times P^{-1} A = P \times D P^{-1}$$

$$\Rightarrow \quad A = P D P^{-1}$$

$\boxed{\text{Que}}$ show A is diagnolisable matrix. find basis of eigenspace of A.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

<u>ans:</u>

step-1: eigenvalue, eigenvector বের করা

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

now, $\det(A - \lambda I) = 0$

$$\Rightarrow (1-\lambda)\left\{(2-\lambda)(3-\lambda)-2\right\} - 1\cdot 2$$

$$+2(2-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\left\{6-2\lambda-3\lambda+\lambda^2-2\right\}-2+4-2\lambda = 0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2-5\lambda+4\right)+2-2\lambda = 0$$

$$\lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda+2-2\lambda = 0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6 = 0$$

$$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

eigenvector:

$$v_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda = 1$ | $\lambda = 2$ | $\lambda = 3$

NB: D, P calculation

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0 \\ 0 & 1 & 2 \\ -0.5 & 0.5 & 1 \end{pmatrix}$$

$$A = P D P^{-1}$$

$$= \begin{pmatrix} 1 & -5 & 2 \\ 0 & -4 & 2 \\ -1 & -2 & 3 \end{pmatrix}$$

$$= \ldots$$

unfinished math

Que show that A is diagonalisable.

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

NB: $n \times n$ matrix এ n সংখ্যক independent eigen-vector না থাকলে diagonalisable না

Power of a diagonal matrix

e.g.: $A^5 = ?$

$A^5 = A \cdot A \cdot A \cdot A \cdot A$

$\boxed{\text{Que}}$ $A^{100} = ?$

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

ans:

step-1: diagonalize

$$A-\lambda I = \begin{pmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}$$

$$\det = (6-\lambda)(3-\lambda)+2=0$$

$$\Rightarrow 18-6\lambda-3\lambda+\lambda^2+2=0$$

$$\Rightarrow \lambda^2-9\lambda+20=0$$

$$\Rightarrow \lambda=4,5$$

$$(A - \lambda I)v = 0$$

for $\lambda = 4$:

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

$$2x_1 - x_2 = 0$$

$$x_2 = t$$

$$x_1 = x_2/2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t/2 \\ t \end{pmatrix} = \frac{t}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$ ✓

for $\lambda = 5$:

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_2 = t$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow P \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= A$$ (A is diagonalisable)

$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ قطرية

$$D^{100} = \begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix}$$

→ diagonal قطرية

✓

$$\boxed{A^{100} = P \times D^{100} \times P^{-1}}$$

$$\therefore A^{100} = P \times D^{100} \times P^{-1}$$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

$$D^{100} = \begin{pmatrix} 5^{100} & 0 \\ 0 & 9^{100} \end{pmatrix}$$

$$\therefore A^{100} = P D^{100} P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5^{100} & 0 \\ 0 & 9^{100} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.57 \times 10^{70} & -7.8 \times 10^{69} \\ 1.57 \times 10^{70} & -7.8 \times 10^{69} \end{pmatrix}$$

find the value of $\sqrt{A}$,

$$\sqrt{A} = \sqrt{\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}}$$

$$\sqrt{A} = P \, D^{1/2} \, P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5^{1/2} & 0 \\ 0 & 4^{1/2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{Ans}}$$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$$e^{\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix} +$$

$$\frac{\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}^2}{2} + \cdots$$