CSE330: Numerical Methods

Assignment 4

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Instructions for preparing the solution script:

- Write your name, ID#, and Section number clearly in the very front page.
- Write all answers sequentially.
- Start answering a question (not the pat of the question) from the top of a new page.
- Write legibly and in orderly fashion maintaining all mathematical norms and rules. Prepare a single solution file.
- Start working right away. There is no late submission form. If you miss the deadline, you need to use the make-up assignment to cover up the marks.

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1. Consider a function $f(x) = x^3 + x^2 - 4x - 4$.
   (a) (3 marks) Compute the minimum number of iterations required to find the root if the machine epsilon(error bound) is 1×10^{-2} and the interval is [-10,-1.5].
   (b) (4 marks) Show 5 iterations using the Bisection Method to find the root of the above function within the interval [-10,-1.5].
   (c) (2 marks) State the exact roots of $f(x)$ and construct two different fixed point functions $g(x)$ such that $f(x) = 0$.
   (d) (3 marks) Compute the convergence rate of each fixed point function $g(x)$ obtained in the previous part, and state which root it is converging to or diverging.
2. Consider the following function: $f(x) = xe^x - 1$.
   (a) (3 marks) Find solution of $f(x) = 0$ up to 5 iterations using Newton's method starting with $x_0 = 1.5$. Keep up to four significant figures.
   (b) (4 marks) Consider the fixed point function, $g(x) = \frac{2x+1}{\sqrt{x+1}}$. Show that to be super-linearly convergent, the root must satisfy x* $= \frac{-3}{2}$.
3. (a) (3 marks) Consider the function $f(x) = x^2 - x + 1$ and starting point $x_0 = 0$. Show that the sequence using Newton's method x1, x2, $\cdots$ fails to approach a root of $f(x)$.
   (b) (4 marks) Consider the function $f(x) = cos(2x) - sinx$. Compute the solution of the function, such that f(x) = 0, using Newton's method with Aitken's acceleration and starting point, $x_0 = 0$. Consider up to five decimal places.[Error bound is 1×10^{-3}]

Answer to the question no-1

1(a)

Given, $f(x) = x^3 + x^2 - 4x - 4$

$\epsilon_m = 10^{-2}$

$[a_0, b_0] = [-10, -1.5]$

max iteration count,

$$n \geq \frac{\log_{10}|b_0 - a_0| - \log_{10}(\epsilon_m)}{\log_{10} 2} - 1$$

$$n \geq \frac{\log_{10} |-1.5 + 10| - \log_{10}(10^{-2})}{\log_{10} 2} - 1$$

$$n \geq 8.73$$

$$n \geq 9$$

$$\boxed{\text{Ans: } n \geq 9}$$

1(b)

Given $f(x) = x^3 + x^2 - 4x - 4$

range $\rightarrow I = [a_0, b_0] = [-10, -1.5]$

$k = 5$

k	a_k	m_k	b_k	$f(a_k)$	$f(m_k)$	$f(b_k)$	$x_* \in [\ ]$
0	$a_0 = -10$	$m_0 = -5.75$	$b_0 = -1.5$	$f(a_0) = -864$	$f(m_0) = -138.05$	$f(b_0) = 0.875$	$x_* \in [m_0, b_0]$
1	$a_1 = -5.75$	$m_1 = -3.625$	$b_1 = -1.5$	$f(a_1) = -138.05$	$f(m_1) = -23.99$	$f(b_1) = 0.875$	$x_* \in [m_1, b_1]$
2	$a_2 = -3.625$	$m_2 = -2.5625$	$b_2 = -1.5$	$f(a_2) = -23.99$	$f(m_2) = -4.01$	$f(b_2) = 0.875$	$x_* \in [m_2, b_2]$
3	$a_3 = -2.5625$	$m_3 = -2.0312$	$b_3 = -1.5$	$f(a_3) = -4.01$	$f(m_3) = -0.1299$	$f(b_3) = 0.875$	$x_* \in [m_3, b_3]$
4	$a_4 = -2.0312$	$m_4 = 1.7656$	$b_4 = -1.5$	$f(a_4) = -0.1299$	$f(m_4) = 0.6757$	$f(b_4) = 0.875$	$x_* \in [a_4, m_4]$

approximated root, $x_* \in [a_4, m_4]$

$\therefore\ x_* \in [-2.0312, 1.7656]$

1(c)

actual root:

$$x^3 + x^2 - 4x - 4 = 0$$

$$\Rightarrow x^2(x+1) - 4(x+1) = 0$$

$$(x^2 - 4)(x+1) = 0$$

$$(x+1)(x+2)(x-2) = 0$$

$\therefore$ actual roots,

$$x_* = -1, -2, 2$$

now, $x^3 + x^2 - 4x - 4 = 0$

$$4x = x^3 + x^2 - 4$$

$$\therefore x = \frac{1}{4}\left(x^3 + x^2 - 4\right) = g_1(x)$$

again,

$$x^3 + x^2 - 4x - 4 = 0$$

$$x^3 + x^2 - 4x = 4$$

$$x\left(x^2 + x - 4\right) = 4$$

$$x = \frac{4}{\left(x^2 + x - 4\right)} = g_2(x)$$

1(d)

For $g_1(x)$,

$$g_1(x) = \frac{1}{4}(x^3 + x^2 - 4)$$

$$g_1'(x) = \frac{3}{4}x^2 + \frac{1}{2}x$$

converge rate,

$$\lambda_1 = |g_1'(x_*)| = \begin{cases} |g'(-1)| = \frac{1}{4}; & \text{linear convergence} \\ |g'(-2)| = 2; & \text{divergence} \\ |g'(2)| = 4; & \text{divergence} \end{cases}$$

For $g_2(x)$

$$g_2(x) = \frac{4}{x^2 + x - 4}$$

$$g_2'(x) = \frac{-4(2x+1)}{(x^2+x-4)^2}$$

$\therefore$ converge rate,

$$\lambda = \begin{cases} |g'(-1)| = 1/4; & \text{linear convergence} \\ |g'(-2)| = 3; & \text{divergence} \\ |g'(2)| = 5; & \text{divergence} \end{cases}$$

Answer to the question no-2

2 (a)

Given, $f(x) = x e^{x} - 1$

$f'(x) = e^{x}(x+1)$

$x_0 = 1.5$

in newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

now,

k	x_k
0	$x_0 = 1.500$
1	$x_1 = 0.0893$
2	$x_2 = 0.6789$
3	$x_3 = 0.5766$
4	$x_4 = 0.5672$
5	$x_5 = 0.5671$

solution of $f(x) = 0$

is 0.5671 (Ans)

2(b)

$$g(x) = \frac{2x+1}{\sqrt{x+1}}$$

$$g'(x) = \frac{2\sqrt{x+1} - (2x+1) \times \frac{1}{2\sqrt{x+1}}}{x+1}$$

$$= \frac{4(x+1) - 2x-1}{2\sqrt{x+1}\,(x+1)}$$

$$= \frac{2x+3}{2(x+1)^{3/2}}$$

now, $g'(x_*) = 0$

$$\Rightarrow \frac{2x_* + 3}{2(x_* + 1)^{3/2}} = 0$$

$$2x_* + 3 = 0$$

$$x_* = -3/2$$

$\therefore$ The root for $f(x)$ will be satisfied with $x_* = \frac{-3}{2}$ in order to be super linearly convergent.

[shown]

Answere to the question no-3

3(a)

given, $f(x) = x^2 - x + 1$

$f'(x) = 2x - 1$

$x_0 = 0$

in newton's method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

now,

$$x_0 = 0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{-1} = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1}{1} = 0$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0 - \frac{1}{-1} = 1$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1 - \frac{1}{1} = 0$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0 - \frac{1}{-1} = 1$$

since we see a repeating case of 0 and 1, it doesn't converge in any direction to provide with root of $f(x)$ and roots are originally complex numbers.

3 (b)

Given, $f(x) = \cos(2x) - \sin x$

$$f'(x) = -2\sin 2x - \cos x$$

$$x_0 = 0$$

$$\epsilon = 1 \times 10^{-3}$$

$$x_0 = 0$$

$$x_1 = g(x_0) = 1$$

$$x_2 = g(x_1) = 0.46686$$

$$\hat{x}_2 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

$$= 0.65225$$

$\therefore\ x_2 = \hat{x}_2$

$x_3 = g(\hat{x}_2) = 0.52600$

$x_4 = g(x_3) = 0.52355$

now,

k	x_k	$f(x_k) \le \epsilon$	$f'(x_k)$
0	$x_0 = 1$	$f(x_0) = 1$	$f'(x_0) = -1$
1	$x_1 = 1$	$f(x_1) = 1.2576$	$f'(x_1) = 2.35889$
2	$x_2 = 0.46686$	$f(x_2) = 0.14974$	
$\hat{2}$	$\hat{x}_2 = 0.65225$	$f(x_3) = -0.3834$	$f'(\hat{x}_2) = -2.72552$
3	$x_3 = 0.526$	$f(x_4) = -6.369 \times 10^{-3}$	$f'(x_3) = 2.60165$
4	$x_4 = 0.52355$	$f(x_5) = 1.21 \times 10^{-4}$	

$\therefore$ root $= x_4 = 0.52355$

(Ans)

END