CSE330: Numerical Methods

Assignment 3

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Instructions for preparing the solution script:

- Write your name, ID#, and Section number clearly in the very front page.
- Write all answers sequentially.
- Start answering a question (not the pat of the question) from the top of a new page.
- Write legibly and in orderly fashion maintaining all mathematical norms and rules. Prepare a single solution file.
- Start working right away. There is no late submission form. If you miss the deadline, you need to use the make-up assignment to cover up the marks.

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1. A function is given by $f(x) = 2x - e^{-6x}$. Now answer the following:
   (a) (3 marks) Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using the forward difference method up to 5 significant figures.
   (b) (3 marks) Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using the central difference method up to 6 significant figures.
   (c) (4 marks) Calculate the upper bound of truncation error of $f(x)$ at $x_0 = 2$ using $h = 0.1$ in both of the above mentioned methods for the interval $[2.4, 2.7]$.
   (d) (5 marks) Compute $D_{0.5}^{(1)}$ at $x_0 = 0.2$ using Richardson extrapolation method up to 6 significant figures and calculate the truncation error.

2. During the class, we derived in detail the first order Richardson extrapolated derivative, by using $h \to h/2$,

$$D_h^{(1)} \equiv f'(x_0) - \frac{h^4}{480} f^{(5)}(x_0) + \mathcal{O}(h^6) \ .$$

   (a) (4 marks) Using $h \to h/2$, derive the expression for $D_h^{(2)}$ which is the second order Richardson extrapolation.
   (b) (5 marks) Now starting from the definition of D_h and using $h \to h/3$, derive the expression for $D_h^{(1)}$.
   (c) (3 marks) Now identify the Error Part of the expression found in the previous part, and also find the Error Bound of the expression found in the previous part.
   (d) (3 marks) If $f(x) = \ln x$, $x_0 = 1$, $h = 0.1$, find the upper bound of error for $D_h^{(1)}$.

Motto: Mathematics is NOT difficult, but what is difficult is to believe that mathematics is NOT difficult.

Answer to question no-1

1 (a)

Given, $f(x) = 2x - e^{-6x}$

$$x_0 = 0.5$$

$$h = 0.2$$

forward differentiation,

$$= \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \frac{2(x_0+h) - e^{-6x(x_0+h)} - 2x_0 + e^{-6x_0}}{h}$$

$$= \frac{2\times 0.7 - e^{6\times 0.7} - 2\times 0.5 + e^{-6\times 0.5}}{0.2}$$

$$= 2.1740$$

(Answer)

1(b)

given, $f(x) = 2x - e^{-6x}$

$x_0 = 0.5$

$h = 0.2$

central differentiation,

$$= \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{2(x_0+h) - e^{-6\times(x_0+h)} - 2(x_0-h) + e^{-6(x_0-h)}}{2h}$$

$$= \frac{2\times 0.7 - e^{-6\times 0.7} - 2\times 0.3 + e^{-6\times 0.3}}{2\times 0.2}$$

$$= 2.37576$$

$\boxed{\text{Answer}}$

1(c)

given $f(x) = 2x - e^{-6x}$

$x_0 = 2$

$h = 0.1$

$$f'(x) = \frac{d}{dx}\{f(x)\}$$

$$= 2 + 6e^{-6x}$$

$$f''(x) = \frac{d}{dx}\{f'(x)\} = -36e^{-6x}$$

$$f'''(x) = \frac{d}{dx}\{f''(x)\} = 216e^{-6x}$$

now upper bound error given

the certain forward

differentiation $\leq \left| \frac{f''(\xi) \times (-h)}{2!} \right|$

$f''(\xi = 2.4)$ $= -36e^{-6\times2.4}$ $= -2.007\times10^{-5}$
$f''(\xi = 2.7)$, $= -36\, e^{-6\times2.7}$ $= -3.317\times10^{-6}$
$\therefore$ max value, $= -2.007\times10^{-5}$

" $\leq \left| \frac{(-2.007\times10^{-5})\times(-0.1)}{2} \right|$

" $\leq 1.004 \times10^{-6}$

$\boxed{\text{Answer}}$

Again

upper bound error for the central

differentian $\le \left| \dfrac{f'''(\xi) \times h^2}{3!} \right|$

$f'''(\xi = 2.4)$, $= 216e^{-6\times 2.4}$ $= 1.204\times 10^{-4}$
$f'''(\xi = 2.7)$, $= 216e^{-6\times 2.7}$ $= 1.990\times 10^{-5}$
$\therefore$ max value, $= 1.204\times 10^{-4}$

" $\le \left| \dfrac{(1.204\times 10^{-4})\times(0.1)^2}{6} \right|$

" $\le 2.007\times 10^{-7}$

$\boxed{\text{Answer}}$

1 (d)

Given, $f(x) = 2x - e^{-6x}$

$x_0 = 2$

$h = 0.5$

$\Rightarrow h/2 = 0.25$

$$D_h^{(1)} = \frac{4 D_{h/2} - D_h}{3}$$

$$D_h = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$D_{h/2} = \frac{f(x_0+\frac{h}{2}) - f(x_0-\frac{h}{2})}{2h}$$

now,

$$D_{0.5} = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{f(0.7) - f(-0.3)}{2 \times 0.5}$$

$$= \frac{2 \times .7 - e^{-6 \times 0.7} + 2 \times 0.3 + e^{6 \times 0.3}}{2 \times 0.5}$$

$$= 8.03465$$

$$D_{0.25} = \frac{f(x_0 + \frac{h}{2}) - f(x_0 - \frac{h}{2})}{2h}$$

$$= \frac{f(0.45) - f(-0.05)}{2 \times 0.25}$$

$$= \frac{2\times 0.45 - e^{-6\times 0.45} + 2\times 0.05 + e^{6\times 0.05}}{2\times 0.25}$$

$$= 4.56531$$

$$\therefore D_{0.5}^{(1)} = \frac{4D_{H/2} - D_h}{3}$$

$$= \frac{4\times 4.56531 - 8.03465}{3}$$

$$= 3.40886$$

now,

$$f'(x) = \frac{d}{dx}\{f(x)\} = 2 + 6e^{-6x}$$

actual value when $x_0 = 0.2$ is

$$f'(0.2) = 2 + 6e^{-6\times 0.2}$$

$$= 3.80717$$

$\therefore$ Truncation error,

$= \text{actual value} - D^{(1)}_{0.5}$

$= 3.80717 - 3.40886$

$= 0.39831$

$\boxed{\text{Answer}}$

Answer to question no-2

2 (a)

given, $D_h^{(1)} = f'(x_0) - \frac{h^4}{480} f^{v}(x_0) + O(h^6)$

now,

$$D_{h/2}^{(1)} = f'(x_0) - \frac{h^4}{7680} f^{v}(x_0) + O(h^6)$$

$$\Rightarrow 16 D_{h/2}^{(1)} = 16 f'(x_0) - \frac{h^4}{480} f^{v}(x_0) + O(h^6)$$

$$\Rightarrow 16 D_{h/2}^{(1)} - D_h^{(1)} = 15 f'(x_0) + O(h^6)$$

$$\therefore \quad \frac{16 D_{h/2}^{(1)} - D_h^{(1)}}{15} = f'(x_0) + O(h^6) = D_h^{(2)}$$

Answer

2 (b)

$$D_h = f'(x_0) + \frac{f'''(x_0)}{3!} h^2 + \frac{f^{v}(x_0)}{5!} h^4 + O(h^6)$$

$$D_{h/3} = f'(x_0) + \frac{f'''(x_0)}{54} h^2 + \frac{f^{v}(x_0)}{9720} h^4 + O(h^6)$$

$$9D_{h/3} = 9f'(x_0) + \frac{f'''(x_0)}{6} + \frac{f^{v}(x_0)}{1080}h^4 + O(h^6)$$

$$\Rightarrow 9D_{h/3} - D_h = 8f'(x_0) - \frac{f^{v}(x_0)}{135}h^4 + O(h^6)$$

$$\therefore \frac{9D_{h/3} - D_h}{8} = f'(x_0) - \frac{f^{v}(x_0)}{1080}h^4 + O(h^6)$$

$$= D_h^{(1)}$$

$\boxed{\text{Answer}}$

$\boxed{2(c)}$

Error part of $D_n^{(1)}$,

$$= \frac{-f^{V}(x_0)}{1080} h^4 + O(h^6)$$

$\boxed{\text{Answer}}$

Error bound of $D_n^{(1)} = -\frac{f^{V}(x_0)}{1080} h^4$

$\boxed{\text{Answer}}$

$\boxed{2(d)}$

Given, $f(x) = \ln(x)$

$x_0 = 1$

$h = 0.1$

now, upper bound error for

$$D_h^{(1)} = \left| -\frac{f^{v}(x_0)}{1080} h^4 \right|$$

$$= \left| -\frac{1}{45 x_0^5} h^4 \right|$$

$$= \left| -\frac{1}{45 \times 1} \times (0.1)^4 \right|$$

$$= 2.22222 \times 10^{-6}$$

$\boxed{\text{Answer}}$

$f(x) = \ln(x)$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$f'''(x) = -2/x^3$

$f^{iv}(x) = -6/x^4$

$f^{v}(x) = 24/x^5$