MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Continuity

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MAT215
(Machine Learning & Signal Processing)

Quiz-1

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syllabus: chapter 1 and 2 (আজকের class পর্যন্ত)

important: finding nth root

Continuity

A complex valued function $f(z)$ is called continuous at $z = z_0$ if the limit, $\lim_{z \to z_0} f(z)$ exists and the value of limit is $f(z_0)$.

i.e: $\lim_{z \to z_0} f(z) = f(z_0)$

$f(z) = z^2 - 2z + 5$; check the continuity at $z = 2i$

ans:

$$\text{limit} = \lim_{z \to 2i} (z^2 - 2z + 5)$$

$$= (2i)^2 - 2(2i) + 5$$

$$= 1 - 4i$$

$$f(2i) = (2i)^2 - 2 \cdot 2i + 5 = 1 - 4i$$

since $f(z_0) = \lim_{z \to z_0} f(z)$, it is continuous at $z_0 = 2i$

$f(z) = \dfrac{2z+3}{z^2+4}$; check continuity at $z = 2i$

limit, $\lim_{z \to 2i} \dfrac{2z+3}{z^2+4} = \dfrac{2(2i)+3}{(2i)^2+4} = \dfrac{4i+3}{0} \Rightarrow$ limit doesn't exist

$\therefore f(z)$ is not continuous at $z = 2i$

$f(z) = \frac{z-2i}{z^2+4}$, check continuity at $z = 2i$

$\Rightarrow$ limit, $\lim_{z \to 2i} \frac{z-2i}{z^2+4} = \lim_{z \to 2i} \frac{1}{2z} = \frac{1}{4i}$

$f(2i) = \frac{0}{0}$ = undefined/indeterminate

$\therefore f(z)$ isn't continuous at $z = 2i$

$f(z) = \begin{cases} \frac{z-2i}{z^2+4} & \text{when } z \neq 2i \\ z+4i & \text{when } z = 2i \end{cases}$

check continuity at $z = 2i$

ans:

$$\lim_{z \to 2i} f(z) = \lim_{z \to 2i} \frac{z-2i}{z^2+4} = \frac{1}{4i}$$

$f(z_0) = f(2i) = 2i + 4i = 6i$

$\therefore f(z)$ isn't continuous at $z = 2i$

$$\# \; f(z) = \begin{cases} \dfrac{z-2i}{z^2+4} & \text{when } z \neq 2i \\ \dfrac{-i}{4} & \text{when } z = 2i \end{cases}$$

<u>ans:</u>

limit, $\lim\limits_{z \to 2i} f(z) = \lim\limits_{z \to 2i} \dfrac{z-2i}{z^2+4} = \dfrac{1}{2z} = \dfrac{1}{4i} = \dfrac{i}{4i^2} = \dfrac{-i}{4}$

$f(2i) = {}^{-i}/_{4}$ $\quad \therefore f(z)$ isn't continuous at $z = 2i$

Shade the regions/plot the curve

① $|z+1-i| = 5$

(ii) $|z-2i| > 3$

(iii) $|z-2i| \leq 3$

(iv) $1 < |z+2i| < 3$

① $|z+1-i| = 5$:

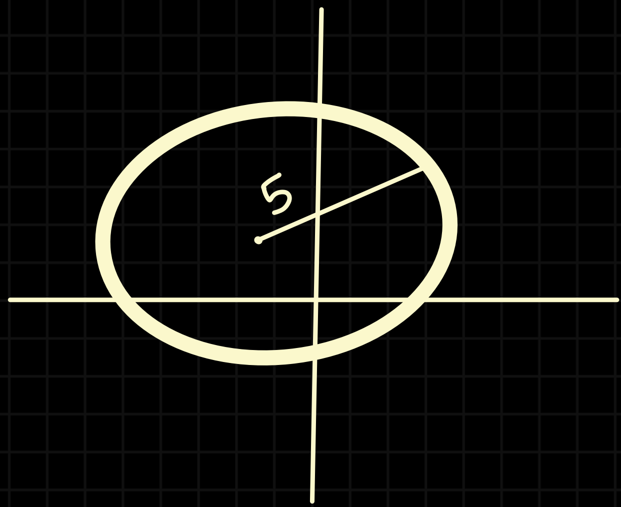
$z = x+iy$

$|(x+1)+(y-1)i| = 5$

$\Rightarrow \sqrt{(x+1)^2 + (y-1)^2} = 5$

$\Rightarrow (x+1)^2 + (y-1)^2 = 5^2$

that's a circle with center at $(-1,1)$ and radius $=5$



(ii) $|z - 2i| > 3$:

$|x + iy - 2i| > 3$

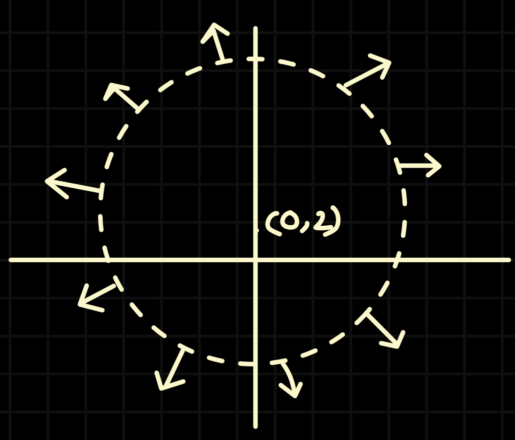
$\Rightarrow |(x) + (y-2)i| > 3$

$\Rightarrow \sqrt{x^2 + (y-2)^2} > 3$

$\Rightarrow (x-0)^2 + (y-2)^2 = 3^2$

let $(x, y) = (0, 3)$

$\therefore (0-0)^2 + (3-2)^2 > 3^2$, not true so outside the circle border should be shaded



(iii) $|z - 2i| \le 3$:

$$|(x+iy) - 2i| \le 3$$

$$\Rightarrow |(x) + (y-2)i| \le 3$$

$$\Rightarrow \sqrt{x^2 + (y-2)^2} \le 3$$

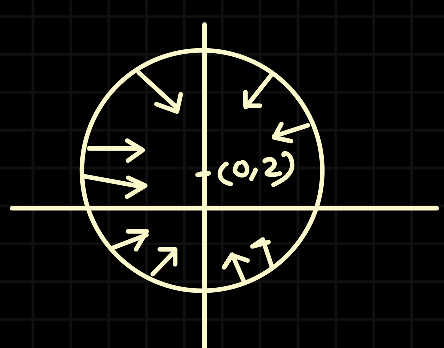
$$\Rightarrow x^2 + (y-2)^2 \le 3^2$$

$$\Rightarrow (x-0)^2 + (y-2)^2 \le 3$$

let $(x, 0) = (12, 0)$

$\therefore 12^2 + (-2)^2 \le 3$, not true

$\therefore$ inner area of circle



(iv) $1 < |z+2i| < 3$

ans:

$$1 < \sqrt{x^2 + (y+2)^2} \leq 3$$

$$\Rightarrow 1^2 < (x-0)^2 + (y+2)^2 \leq 3^2$$

let $(x, y) = (0, 10)$

$1 < 0^2 + 12^2 \leq 9$, $\therefore$ outer part not included

let $(x, y) = (0, 0)$

$1 < 0 + 2^2 \leq 3$; $\therefore$ middle area is included

Let $(x, y) = (0, -2)$; $\therefore$ inner part of inner circle isnt included

(i) $\text{Real}\{z\} > 2$

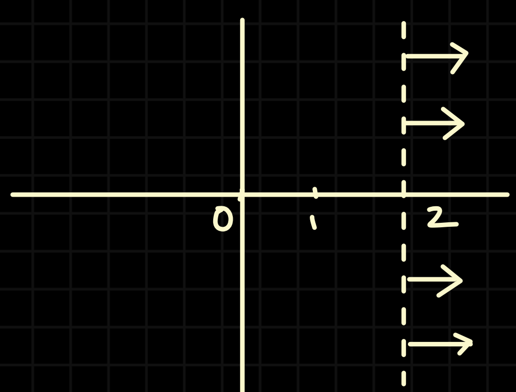
(ii) $\text{Im}\{z\} < 3$

(i) $\text{Real}\{z\} > 2$:

$$z = x + iy$$

$$\text{Real}\{z\} = x$$

$$\Rightarrow \quad x > 2$$

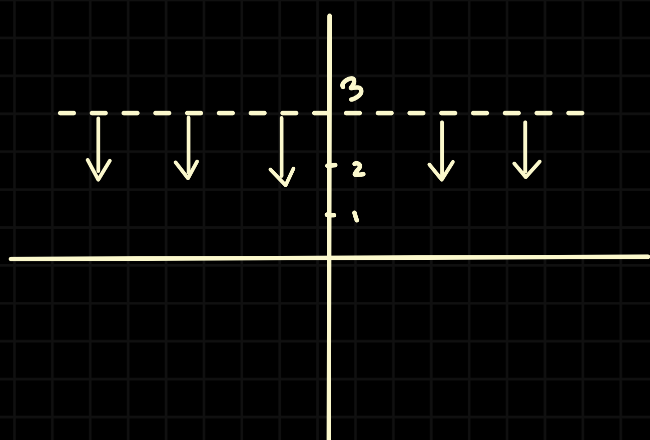


(ii) $\text{Im}\{z\} < 3$:

$$z = x + iy$$

$$\text{Im}\{z\} = y$$

$$y < 3$$



✓✓ quiz এর syllabus

Triangle Inequality

$$|z_1+z_2| \le |z_1| + |z_2|$$

now, $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$|z_1+z_2|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2$$

$$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1 \cdot \bar{z}_2)$$

$$|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2\,|z_1||z_2|$$

$$|z_1+z_2|^2 \le (|z_1| + |z_2|)^2$$

$\therefore\ |z_1+z_2| \le |z_1| + |z_2|$

$$z \cdot \bar{z} = |z|^2$$

$z = x+iy$; $\bar{z} = x-iy$

$$z \cdot \bar{z} = x^2+y^2$$

$$= \left(\sqrt{x^2+y^2}\right)^2$$

$$= |z|^2$$

$z_1 = x_1 + iy_1$, $\bar{z}_1 = x_1 - iy_1$

$z_2 = x_2 + iy_2$, $\bar{z}_2 = x_2 - iy_2$

$\therefore\ z_1\bar{z}_2 + \bar{z}_1 z_2$

$$= \underbrace{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}_{z_1\bar{z}_2}$$

$$+ x_1x_2 + ix_1y_2 - ix_2y_1 + y_1y_2$$

$$= 2x_1x_2 + 2y_1y_2$$

$$= 2(x_1x_2 + y_1y_2)$$

$$= 2 \times \text{Real}(z_1 \cdot \bar{z}_2)$$