CSE330: Numerical Methods

Topic: Polynomial Interpolation, Lagrange Polynomial, Interpolation Error, Maximum Error/ Error Bound

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Polynomial Interpolation

→ we will learn 4 methods of polynomial interpolation

$$P_n(x) = \sum_{k=0}^{n} a_k \, x_k^k$$

↳ natural basis

coefficient

$$\Rightarrow P_n(x) = a_0 x^0 + a_1 x^1 + \cdots + a_n x^n$$

4 nodes থেকে পাওয়া polynomial এর degree, $n = 3$

$\Rightarrow (n+1)$ সংখ্যক nodes থেকে পাওয়া polynomial এর degree হলো n.

[useless:

☐ $(x_0, f(x_0))$, $(x_1, f(x_1))$, ... $(x_n, f(x_n))$

value গুলো polynomial এ বসিয়ে,

$$P_n(x) = a_0 x^0 + a_1 x^1 + \dots + a_n x^n$$

$$P_n(x_1) = a_0 x_1^0 + a_1 x_1^1 + \dots + a_n x_1^n$$

$\vdots$

confusion: x এর value যাই হোক polynomial টা same থাকবে না? then why $P_n(x)$, $P_n(x_1)$

$\vdots$

$$P_n(x_n) = a_0 x_n^0 + a_1 x_n^1 + \dots + a_n x_n^n$$

$$\underbrace{\begin{pmatrix} x_0^0 & x_0^1 & \cdots & x_n^0 \\ x_1^0 & x_1^1 & & x_n^1 \\ \vdots & \vdots & & \vdots \\ x_n^0 & x_n^1 & & x_n^n \end{pmatrix}}_{\text{Vandermondej matrix, } V} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\text{coefficient matrix, } a} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$$\Rightarrow \quad V \cdot a = f$$

$$\Rightarrow \quad a = V^{-1} f$$

polynomial interpolation means gussing a poly-nomial using given nodes. that guessed polynomial is our function. useless ends

Method-1

Que

$x_0 = 0 \qquad x_1 = \frac{\pi}{2} \qquad x_2 = \pi$

$f(x_0) = 1 \qquad f(x_1) = 0 \qquad f(x_2) = -1$

$P_n(x) = ?$

<u>soln:</u> $\because$ 3 টা node, তাই degree হবে 2

$$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

→ (1 লাইন ফাকা রাখতে পরে এসে ans লেখার জন্যে)

$$= 1 - \frac{2}{\pi} x + 0$$

→ Ans

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0^2 \\ 1 & \pi/2 & (\pi/2)^2 \\ 1 & \pi & \pi^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \pi/2 & (\pi/2)^2 \\ 1 & \pi & \pi^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2/\pi \\ 0 \end{pmatrix}$$

Ans: $1 - \frac{2}{\pi}x + 0$

Method-2

reason: method এর matrix inversion process pc এর জন্য inefficient

↳ Lagrange Polynomial

$$P_n(x) = \sum_{k=0}^{n} \underbrace{f(x_k)}_{\text{coeffi-cient}} \, l_k(x)$$

↳ lagrange basis, instead of natural basis

where, $$l_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x - x_j}{x_k - x_j}$$

$$\Rightarrow l_k(x) = \frac{(x-x_0)}{(x_k-x_0)} \times \frac{(x-x_1)}{(x_k-x_1)} \times \cdots \times \frac{(x-x_n)}{(x_k-x_n)}$$

$\boxed{\text{Que}}$

$x_0 = -\frac{\pi}{4}$ $\quad x_1 = 0$ $\quad x_2 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ $\quad f(x_1) = 1$ $\quad f(x_2) = \frac{1}{\sqrt{2}}$

$P_n(x) = ?$

soln: 3 nodes. so degree = 2

$n = 2$

$P_2(x) = ?$

$$P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

$$+ f(x_2)\, l_2(x)$$

$$P_2(x) = \frac{1}{\sqrt{2}} \times \frac{8}{\pi^2} x\left(x - \frac{\pi}{4}\right) +$$

$$1 \times \frac{-16}{\pi^2}\left(x^2 - \frac{\pi^2}{16}\right) +$$

$$\frac{1}{\sqrt{2}} \times \frac{8}{\pi^2} x \left(x + \frac{\pi}{4}\right)$$

$k=0$ এর জন্য

$$\boxed{l_k = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}}$$

$$l_0 = \cancel{\frac{(x-x_0)}{(x_0-x_0)}} \times \frac{x-x_1}{x_0-x_1} \times \frac{x-x_2}{x_0-x_2}$$

$$= \frac{x-0}{-\frac{\pi}{4}-0} \times \frac{x-\frac{\pi}{4}}{-\frac{\pi}{4}-\frac{\pi}{4}}$$

$$= \frac{x \times \left(x-\frac{\pi}{4}\right)}{\frac{\pi}{4} \times \frac{\pi}{2}}$$

$$= \frac{8}{\pi^2}\, x\left(x-\frac{\pi}{4}\right)$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{\cancel{(x - x_1)}}{\cancel{(x_1 - x_1)}} \times \frac{x - x_2}{x_1 - x_2}$$

k=1

$$= \cdot \quad \cdot \quad \cdot$$

$$= \frac{-16}{\pi^2}\left(x^2 - \frac{\pi^2}{16}\right)$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} \times \frac{\cancel{x - x_2}}{\cancel{x_2 - x_2}}$$

k=2

$$= \frac{8}{\pi^2}\, x\left(x + \frac{\pi}{4}\right)$$

. . . . going back to this's question

$P_2(10) = ?$

interpolation error

কত হবে, $x=0$

Interpolation Error

$$\Rightarrow \left| P_2(0) - \underbrace{f(0)}_{f \text{ দেওয়া থাকবে}} \right| = ?$$

Maximum Error or Error Bound

Cauchy's theorem

$$\left| f(x) - P_n(x) \right| \le \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\dots(x-x_n) \right|$$

Que

$x_0 = \frac{-\pi}{4}$ $\quad x_1 = 0$ $\quad x_2 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ $\quad f(x_1) = 1$ $\quad f(x_2) = \frac{1}{\sqrt{2}}$

$n = 2$ $\qquad f(x) = \cos(x)$

X_i এর range $[-1, 1]$

Max error = ?

soln:

(multiple nodes, or cauchy's theorem)

$f^3(\xi_0) \Rightarrow f^3(\cos x) = \sin x;\ f^3(\xi_0) = \sin(\xi_0)$

$$\left|f(x) - P_n(x)\right| \leq \left|\frac{f^3(\xi_\omega)}{3!}(x+\frac{\pi}{4})(x-0)(x-\frac{\pi}{4})\right|$$

$$\leq \left|\frac{\sin(1)}{6} x\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)\right|$$

$$\leq \left|\frac{\sin(1)}{6} \times 0.383\right|$$

$$\leq 0.0537$$

(Ans)

let, $\omega(x) = x\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$

$$= x^3 - \frac{\pi^2}{16}x$$

$$\omega'(x) = 3x^2 - \frac{\pi^2}{16}$$

$$\omega'(x) = 0$$

$$3x^2 = \frac{\pi^2}{16}$$

$$x^2 = \frac{\pi^2}{16} \times \frac{1}{3}$$

$$x = \pm \frac{\pi}{4\sqrt{3}}$$

$\boxed{\text{Que}}$ function এর max value বের কর।

Ans:

x	$\omega(x)$
$+\frac{\pi}{4\sqrt{3}}$	-0.186
$-\frac{\pi}{4\sqrt{3}}$	$+0.186$
-1	-0.383
1	$+0.383$