CSE330: Numerical Methods

Topic: Least Square Approximation

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Least Square Approximation

number of equation > number of variables

→ over-determinded system

e.g:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 9x_2 + 7x_3 = 2$$
$$x_1 + 3x_2 + 5x_3 = 4$$
$$2x_1 + 11x_2 - 9x_3 = 5$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -9 & 7 \\ 1 & 3 & 5 \\ 2 & 11 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

$$A \qquad x \quad = \quad b$$

$A \rightarrow m \times n$

where $m > n$

$n \rightarrow$ # variables

$m \rightarrow$ # equations

how we solve this math then?

$\Rightarrow$ let $A^T =$ transpose matrix of A

$$Ax = b$$

$$\Rightarrow \underbrace{A^T}_{n\times m} \underbrace{A}_{m\times n} x = A^T b$$

$$\boxed{x = (A^T A)^{-1} (A^T b)}$$

distance least square

e.g. $x_0 = -3$ $x_1 = 0$ $x_2 = 6$

$f(x_0) = 0$ $f(x_1) = 0$ $f(x_2) = 2$

we want to fit a straight line passing through the nodes

soln: দুটোর বেশি nodes দিয়ে straight line draw করা possible না (theres

One exception, that's a line tentatively trying to reach all nodes, remember regression line from sta201)

we solve this problem using vanderwall matrix

<u>soln:</u> $\because$ straight line, so power = 1

$\therefore P_1(x) = a_0 + a_1 x$

since nodes are 3 but

vanderwall matrix,

$$V = \begin{pmatrix} x_0^0 & x_0^1 & x_0^2 & \cdots \\ x_1^0 & x_1^1 & x_1^2 & \cdots \\ x_2^0 & x_2^1 & x_2^2 & \cdots \end{pmatrix}$$

and

$Va = f$

$\Rightarrow a = V^{-1} f$

row সংখ্যা $\rightarrow$ node সংখ্যা
column সংখ্যা $\rightarrow$ power

$$\therefore\ V = \begin{pmatrix} -3^0 & -3^1 \\ 0^0 & 0^1 \\ 6^0 & 6^1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{pmatrix}$$

$$\therefore \quad V \quad a \quad = f$$

$$\begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$\Rightarrow$ now multiply with adjacent matrix of V.

$$Va = f$$

$$V^T V a = V^T f$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 45 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 45 \end{pmatrix}^{-1} \times \begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 3/7 \\ 5/21 \end{pmatrix}$$

$$\therefore P_1(x) = a_0 x^0 + a_1 x^1$$

$$= a_0 + a_1 x$$

$$= \frac{3}{7} x + \frac{5}{21} x$$

[to avoid matrix inversion, a new method called "QR decomposition" arrives]

Prerequisite knowledge for QR decomposition:

Orthonormality

<u>vector dot products</u>

2 types of notation → vector notation $\vec{x}.\vec{y}$ ✗

matrix notation $x^T.y$ ✓

e.g. $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\underbrace{x^T y}_{\text{dot product/ inner product}} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$= 1 \times 4 + 2 \times 5 + 3 \times 6$$

$$= 32$$

$$\underbrace{x^T \cdot x}_{l_2 \text{ norm}}$$ → নিজের সাথে নিজের dot product

magnitude

$$\underbrace{\vec{a} \cdot \vec{b}}_{a^T \cdot b = 0} = |a||b| \cos 90^\circ$$

$$= 0$$

↳ a and b are orthogonal

let $S = \{ \vec{a}, \vec{b}, \vec{c} \}$

S is an orthogonal set if

$\vec{a} \cdot \vec{b} = 0$

$\vec{b} \cdot \vec{c} = 0$

$\vec{c} \cdot \vec{a} = 0$

Orthonormality

two properties of orthonormality:

(i) vector গুলো orthognal হবে
(dot product = 0)

(ii) the length of the vector = 1
(unit vector হবে)

e.g. check if a and b are ortho-normal where

$$a = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

solⁿ: (i) orthognality check:

$$a^T b = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$$= 0 \quad (\therefore \text{ orthognal check})$$

(ii) magnitude check:

$$|a| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$$

$$|b| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$$

a and b are orthogonal but not orthonormal since their unit is not zero.

Normalization:

→ process of transforming non-orthonormal matrix to orthonormal

→ divide the matrix by its magnitude

$$\therefore \hat{a} = \frac{a}{|a|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix}$$

$$|\hat{a}| = \sqrt{\left(\frac{4}{\sqrt{21}}\right)^2 + \left(\frac{2}{\sqrt{21}}\right)^2 + \left(1/\sqrt{21}\right)^2} = 1$$

$$\hat{b} = \frac{b}{|b|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}$$

Que:

$$s = \left\{ \frac{1}{\sqrt{5}}(2,1)^T , \frac{1}{\sqrt{5}} (1,-2)^T \right\}$$

is s orthonormal set?

<u>sol:</u>

<u>step-1:</u> matrix representation:

$$s = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

$$s = \left\{ \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} , \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix} \right\}$$

$$\therefore\ u = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \qquad v = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

<u>step-2:</u> orthonormality check

orthogonality check →

$$u^T v = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$= 2 - 2 = 0$$

$\therefore$ orthogonal ✓

unit chek →

$$|u| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

$$||v|| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{-2}{\sqrt{5}}\right)^2} = 1$$

$\therefore$ the set of vectors is

orthonormal