MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Differentiation and Differentiability in Complex Valued Function

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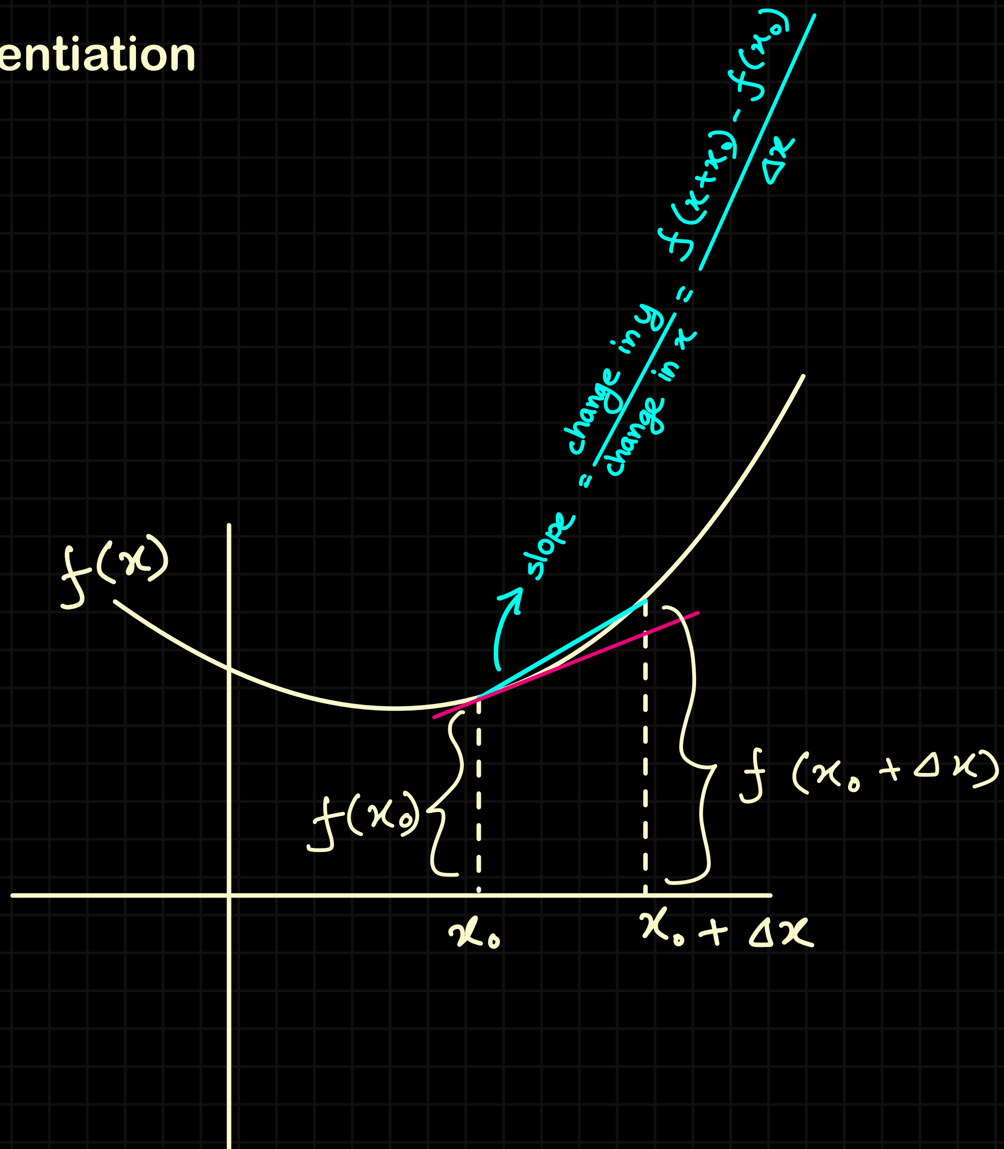
MAT215
(Machine Learning & Signal Processing)

→ starts chapter 3

What is Differentiation



differentiation (in real number system)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Differentiation in Complex Valued Function

→ if $f(z)$ is a complex valued function, $f(z)$ is called

differentiable at some point $z = z_0$ if the limit

$$\lim_{\Delta z \to 0} \frac{f(z + z_0) - f(z_0)}{\Delta z} \text{ exists.}$$

then we can say $f(z)$ is differentiable at $z = z_0$ And

the values of the differentiation is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

eg math: Using definition show that $f(z) = \frac{2z-3i}{3z-2i}$ is differentiable at $z = -i$

solve:

Using definition

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$f'(-i) = \lim_{\Delta z \to 0} \frac{f(-i+\Delta z) - f(-i)}{\Delta z}$$

$$= \frac{\frac{2(-i+\Delta z)-3i}{3(-i+\Delta z)-2i} - \frac{2(-i)-3i}{3(-i)-2i}}{\Delta z}$$

$$= \frac{\frac{-2i+2\Delta z-3i}{-3i+3\Delta z-2i} - \frac{-2i-3i}{-3i-2i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{2\Delta z - 5i}{3\Delta z - 5i} - 1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2\Delta z - 5i - 3\Delta z + 5i}{(3\Delta z - 5i)\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-\Delta z}{\Delta z(3\Delta z - 5i)}$$

$$= \lim_{\Delta z \to 0} \frac{-1}{3\Delta z - 5i}$$

$$= \frac{-1}{3 \times 0 - 5i}$$

$$= \frac{1}{5i}$$

eg math: Using Definition show that $f(z) = z^2$ is differentiable at $\underbrace{\text{all points}}_{\text{we take any arbitary value}}$.

solve: let $z = z_0$ be an arbitary point. Using definition

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0\Delta z + (\Delta z)^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z_0 + \Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z)$$

$$= 2z_0$$

so limit exists for any value of z_0

$\therefore$ $f(z)$ is differentiable at any z_0 and therefore

at all points.

[for value of differentiation:

$f(z_0) = 2z_0$

$f(z) = 2z)$]

eg math: Using definition show that $f(z) = \bar{z}$ is NOT differentiable at $z = 0$

solve:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z} - \bar{0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{z \to 0} \frac{\bar{z}}{z}$$

as limit doesnt exist, so we take it to the cartesian coordinates

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{\Delta x + i \Delta y}}{\Delta x + i \Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

(keep checking limits until 'limit doesn't exist' situation appears)

in the $\Delta x = 0$ direction,

$$\text{limit} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} -1$$

$$= -1$$

in the $\Delta y = 0$ direction,

$$\text{limit} = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta x}$$

$$= \lim_{\Delta y \to 0} 1$$

$$= 1$$

eg math Using definition show that $f(z) = |z|^2$ is differentiable at $z = 0$

solve: using definition,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z \cdot \overline{\Delta z}}{\Delta z}$$

$|z|^2 = z \cdot \overline{z}$

$$= \lim_{\Delta z \to 0} \overline{\Delta z}$$

$$= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to \alpha}} \overline{\Delta r e^{i\Delta\theta}}$$

$\Delta z \to 0$

$\Delta r \to 0$ ✓

$\Delta\theta \to 0$ ✗

rather $\Delta\theta \to \alpha$

$$= 0$$

eg math Using definition show that $f(z) = |z|^2$ is not differentiable other than $z = 0$

solve:

consider an arbitary point $z_0 \neq 0$

using definition.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z_0 + z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0 + \Delta z}) - z_0 \overline{z_0}}{\Delta z}$$

$|z|^2 = z \cdot \overline{z}$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0 \overline{z_0}}{\Delta z}$$

$\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$$= \lim_{\Delta z \to 0} \frac{z_0\overline{z_0} + z_0\overline{\Delta z} + \overline{z}_0\Delta z + \overline{\Delta z}\,\Delta z - z_0\overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\overline{\Delta z} + \overline{z}_0\Delta z + \Delta z\,\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z}_0 + \overline{\Delta z} \right)$$

$$= z_0 \lim_{\Delta z \to 0} \left(\frac{\overline{\Delta z}}{\Delta z} \right) + \overline{z_0} + \lim_{\Delta z \to 0} \left(\overline{\Delta z} \right)$$

$$= z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y} + \overline{z}_0 + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\overline{\Delta x + i\Delta y} \right)$$

$$= z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} + \overline{z}_0 + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\Delta x - i\Delta y \right)$$

in the $\Delta x = 0$ direction:

$$\text{limit} = z_0 \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} + \overline{z_0} + \lim_{\Delta y \to 0} (-i\Delta y)$$

$$= z_0 \lim_{\Delta y \to 0} (-1) + \overline{z_0} + (-i \times 0)$$

$$= -z_0 + \overline{z_0}$$

in the $\Delta y = 0$ direction:

$$\text{limit} = z_0 \lim_{\Delta x \to 0} \left(\frac{\Delta x}{\Delta x}\right) + \overline{z_0} + \lim_{\Delta x \to 0} (\Delta x)$$

$$= z_0 \lim_{\Delta x \to 0} (1) + \overline{z_0} + 0$$

$$= z_0 + \overline{z_0}$$

now for $z_0 \neq 0$,

$$-z_0 + \overline{z_0} \neq \overline{z_0 + z_0}$$

$\Rightarrow$ for $z \neq 0$, $f(z)$ is not differentiable other than zero

eg math using definition show that $f(z) = |z|^2$

is $\boxed{\text{only}}$ differentiable at $z=0$

combine the solution of

i) $f(z)$ is differentiable at $z=0$

ii) $f(z)$ is not differentiable at values other than

zero

solve: using definition,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z \cdot \overline{\Delta z}}{\Delta z}$$

$|z|^2 = z \cdot \bar{z}$

$$= \lim_{\Delta z \to 0} \overline{\Delta z}$$

$$= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to \alpha}} \overline{\Delta r e^{i\Delta\theta}}$$

$\Delta z \to 0$

$\Delta r \to 0$ ✓

$\Delta\theta \to 0$ ✗

rather $\Delta\theta \to \alpha$

$$= 0$$

now consider an arbitary point $z_0 \neq 0$

using definition,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z_0 + z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\overline{z_0+\Delta z}) - z_0 \overline{z_0}}{\Delta z}$$

$|z|^2 = z \cdot \overline{z}$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0 \overline{z_0}}{\Delta z}$$

$\because \overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

$$= \lim_{\Delta z \to 0} \frac{z_0\overline{z_0} + z_0\overline{\Delta z} + \overline{z_0}\Delta z + \overline{\Delta z}\,\Delta z - z_0\overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \overline{z_0} \Delta z + \Delta z \, \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z} \right)$$

$$= z_0 \lim_{\Delta z \to 0} \left(\frac{\overline{\Delta z}}{\Delta z} \right) + \overline{z_0} + \lim_{\Delta z \to 0} \left(\overline{\Delta z} \right)$$

$$= z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{\Delta x + i \Delta y}}{\Delta x + i \Delta y} + \overline{z_0} + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\overline{\Delta x + i \Delta y} \right)$$

$$= z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i \Delta y}{\Delta x + i \Delta y} + \overline{z_0} + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\Delta x - i \Delta y \right)$$

in the $\Delta x = 0$ direction:

$$\text{limit} = z_0 \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} + \overline{z_0} + \lim_{\Delta y \to 0} (-i\Delta y)$$

$$= z_0 \lim_{\Delta y \to 0} (-1) + \overline{z_0} + (-i \times 0)$$

$$= -z_0 + \overline{z_0}$$

<u>in the $\Delta y = 0$ direction:</u>

$$\text{limit} = z_0 \lim_{\Delta x \to 0} \left(\frac{\Delta x}{\Delta x}\right) + \overline{z_0} + \lim_{\Delta x \to 0} (\Delta x)$$

$$= z_0 \lim_{\Delta x \to 0} (1) + \overline{z_0} + 0$$

$$= z_0 + \overline{z_0}$$

now for $z_0 \neq 0$,

$$-z_0 + \overline{z_0} \neq z_0 + \overline{z_0}$$

$\Rightarrow$ for $z \neq 0$, $f(z)$ is not differentiable other than zero

$f(z) = |z|^2$ is only differentiable at $z = 0$

$\boxed{\text{eg math}}$ if $f(z) = u(x,y) + i\,v(x,y)$ is differentiable, then find $f'(z)$

$\boxed{\text{solve}}$

using definition,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\,\Delta y}$$

$f(z)$
$\downarrow$
$u(x,y) + i\,v(x,y)$

$\therefore f(z+\Delta z)$
$\downarrow$
$u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$

in the $\Delta y = 0$ direction,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x,y) - i\,v(x,y)}{}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

in the $\Delta x = 0$ direction,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - u(x,y) - i v(x,y)}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i \Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{i}{i} \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

coming from both direction we get $\underbrace{\text{same value}}_{\text{as cauchy-Reimann eqn suggests}}$