MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Laplace 03 - Solving Differential Equations - Integrals

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Laplace-3 is about

→ Laplace transformation এর Laplace inverse use করে linear constant coefficient differential eqn solve করা

<u>linear:</u>

e.g: $y''' + \cdot\cdot\, y'' + \cdot\cdot\, y' + \cdot\cdot\, y = e^t \sin t$

$y y' \rightarrow$ non linear

$y^2 / y^3 \ldots\ldots \rightarrow$ nonprocess of solving differential equation:

1) Laplace transformation of the entire equation (both side)

2) rearrange করে $Y(s)$ বের করা

3) inverse laplace entire eqn.

Voilà. You got the answer.

Laplace transform of Derivatives

if $\mathcal{L}\{y(t)\} = Y(s)$ then,

$$\mathcal{L}\{y'(t)\} = s \cdot Y(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2 \cdot Y(s) - s \cdot y(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3 \cdot Y(s) - s^2 \cdot y(0) - s \cdot y'(0) - y''(0)$$

proof:

(i) $\mathcal{L}\{y''(t)\} = s^2 Y(s) - s.y(0) - y'(0)$

ans:

let $\quad \mathcal{L}\{y(t)\} = Y(s)$

$\therefore \mathcal{L}\{y''(t)\} = \int_0^\infty e^{-st} \cdot y''(t)\, dt$

$\left[\int uv\, dt = u \int v\, dt - \int \left(\frac{du}{dt} \int v\, dt \right) dt \right.$

$$= \left[e^{-st} \cdot y'(t) \right]_0^\infty - \int_0^\infty \left((-s) e^{-st} \cdot y'(t) \right) dt$$

$$= 0 - e^0 \cdot y'(0) + s \int_0^\infty e^{-st} \cdot y'(t)\, dt$$

$$= -y'(0) + s \cdot \left[\left[e^{-st} \cdot y(t) \right]_0^\infty - \int_0^\infty -s \cdot e^{-st} y(t)\, dt \right]$$

$$= -y'(0) + s \left[0 - e^0 \cdot y(0) + s \underbrace{\int_0^\infty e^{-st} y(t)\, dt}_{\mathcal{L}\{y(t)\}} \right]$$

$$= -y'(0) + s \left[-y(0) + s \cdot Y(s) \right]$$

$$= s^2 Y(s) - s\, y(0) - y'(0)$$

Ⓑ solve the given differential equation:

$$y'' - 4y' + 4y = t^3 e^{2t}, \quad y'(0) = 0, \quad y(0) = 0$$

ans:

let $\mathcal{L}\{y(t)\} = Y(s)$

applying the laplace transform operator,

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3 e^{2t}\}$$

$$\Rightarrow s^2 \cdot Y(s) - s \cdot y(0) - y'(0) - 4\{s \cdot Y(s) - y(0)\}$$

$$+4Y(s) = \frac{3!}{(s-2)^4}$$

$a = 2$

$$\mathcal{L}\{f(t)e^{at}\}$$

$$= \mathcal{L}\{f(t)\}\Big|_{s \to s-a}$$

$$= \mathcal{L}\{t^3\}\Big|_{s \to s-a}$$

$$= \frac{3!}{s^4}\Big|_{s \to s-2}$$

$$= \frac{3!}{(s-2)^4}$$

value বসিয়ে,

$$\Rightarrow s^2 Y(s) - 4sY(s) + 4Y(s)$$

$$= \frac{6}{(s-2)^4}$$

$$\Rightarrow (s^2 - 4s + 4)\, Y(s) = \frac{6}{(s-2)^4}$$

$$\Rightarrow (s-2)^2 \cdot Y(s) = \frac{6}{(s-2)^4}$$

$$\Rightarrow Y(s) = \frac{6}{(s-2)^6}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$\Rightarrow y(t) = 6 \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^6}\right\}$$

replacing s with $s+2$,

$$\Rightarrow y(t) = 6e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\}$$

$$\Rightarrow y(t) = 6e^{2t} \cdot \frac{1}{5!} \cdot \mathcal{L}^{-1}\left\{\frac{5!}{s^{5+1}}\right\}$$

$$\Rightarrow y(t) = \frac{e^{2t}}{20} \cdot t^5 = \frac{1}{20}e^{2t}t^5$$

[Ans]

Solve the given differential equation:

$$y''' - 3y'' + 3y' - y = e^t t^2,$$

$y(0) = 0$

$y'(0) = 1$

$y''(0) = -2$

ans:

let $\mathcal{L}\{y(t)\} = Y(s)$

how,

$$\mathcal{L}\{y''' - 3y'' + 3y' - y\} = \mathcal{L}\{e^t t^2\}$$

$$\Rightarrow s^3 \cdot Y(s) - s^2 \cdot \cancel{y(0)}^{0} - s \cdot \cancel{y'(0)}^{1} - y''(0)$$

$$-3\{s^2 \cdot Y(s) - s \cdot \cancel{y(0)}^{0} - \cancel{y'(0)}^{1}\} + 3$$

$$\{s \cdot Y(s) - \cancel{y(0)}^{0}\} - Y(s) = \frac{2}{(s-1)3}$$

$$\mathcal{L}\{f(t)e^{at}\} = \mathcal{L}\{f(t)\}\Big|_{s \to s-a}$$

$$= \mathcal{L}\{t^2\}\Big|_{s \to s-1} = \frac{2!}{s^3}\Big|_{s \to s-1} = \frac{2}{(s-1)^3}$$

$$\Rightarrow s^3 \cdot Y(s) - s + 2 - 3\{s^2 \cdot Y(s) - 1\}$$
$$+ 3s\,Y(s) - Y(s) = \frac{2}{(s-1)^3}$$

$$\Rightarrow s^3 \cdot Y(s) - s + 2 - 3s^2 Y(s) + 3$$
$$+ 3s\,Y(s) - Y(s) = \frac{2}{(s-1)^3}$$

$$\Rightarrow s^3 \cdot Y(s) - 3s^2 \cdot Y(s) + 3s\,Y(s) - Y(s)$$

$$= \frac{2}{(s-1)^3} + s - 5$$

$$\Rightarrow (s^3 - 3s^2 + 3s - 1) \cdot Y(s) = \frac{2}{(s-1)^3} + s - 5$$

$$\Rightarrow \left\{s^3 - 3 \cdot s^2 \cdot 1 + 3 \cdot s \cdot (-1)^2 - (1)^3\right\} \cdot Y(s)$$

$$= \frac{2}{(s-1)^3} + s - 5$$

$$\Rightarrow (s-1)^3 \cdot Y(s) = \frac{2}{(s-1)^3} + s - 5$$

$$\Rightarrow Y(s) = \frac{2}{(s-1)^6} + \frac{s}{(s-1)^3} - \frac{5}{(s-1)^3}$$

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^6}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s-1)^3}\right\}$$

$$- 5\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

replacing s with s+1,

$$y(t) = 2 \cdot e^{t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} + e^{t} \cdot \mathcal{L}^{-1}\left\{\frac{s+1}{s^3}\right\}$$

$$-5\,e^{t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 2e^{t} \cdot \frac{1}{5!} \cdot t^{5} + e^{t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + e^{t} \cdot$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - 5\,e^{t} \times \frac{1}{2!} \cdot t^{2}$$

$$= \frac{e^{t} t^{5}}{60} + e^{t}\,t + e^{t} \cdot \frac{1}{2} \cdot t^{2} - \frac{5}{2} e^{t} t^{2}$$

$$= \frac{e^t t^5}{60} + e^t t - 2e^t \cdot t^2$$

[Ans]

Solve the given differential equation:

$$y'' + 2y' + 5y = e^{-t} \sin(t),$$

$$y(0) = 0$$

$$y'(0) = 1$$

ans:

let, $\mathcal{L}\{y(t)\} = Y(s)$

$$\therefore \mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{e^{-t}\sin t\}$$

$$\Rightarrow s^2 \cdot Y(s) - s \cdot \cancel{y(0)}^{0} - \cancel{y'(0)}^{1} + 2 \cdot \{ s \cdot Y(s)$$

$$- \cancel{y(0)}^{0} \} + 5\,Y(s) = \frac{1}{(s+1)^2 + 1}$$

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\} \Big|_{s \to s-a}$$

$$\mathcal{L}\{e^{-t}\sin t\} = \mathcal{L}\{\sin t\}_{s \to s+1} = \frac{1}{s^2+1} \Big|_{s \to s+1}$$

$$= \frac{1}{(s+1)^2 + 1}$$

$$\Rightarrow s^2 \cdot Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow Y(s)\left\{s^2+2s+5\right\} = \frac{1}{s^2+2s+2} + 1$$

$$\Rightarrow Y(s)\left\{s^2+2s+5\right\} = \frac{1+s^2+2s+2}{s^2+2s+2}$$

$$\Rightarrow Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{ \quad . \quad . \quad . \quad \right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

let, $s^2+2s=u$.

using partial fraction,

$$\frac{u+3}{(u+2)(u+5)} = \frac{A}{u+2} + \frac{B}{u+5}$$

$$\Rightarrow u+3 = A(u+5) + B(u+2)$$

$\underline{u=-5:}$

$-2 = -3B$

$\Rightarrow B = \frac{2}{3}$

$\underline{u=-2:}$

$1 = 3A$

$\Rightarrow A = \frac{1}{3}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2/3}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2.s.1+1+1^2}\right\} +$$

$$\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2.s.1+1^2+2^2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

replacing s with s-1,

$$= \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1^2} \right\} + \frac{2}{3} \cdot e^{-t} \cdot$$

$$\frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{e^{-t}}{3} \sin 2t$$

$$\therefore y(t) = \frac{1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t$$

solve the given differential equation,

$$y' + y = f(t), \quad y(0) = 5$$

$$f(t) = \begin{cases} 0 & , 0 \le t < \pi \\ \cos t & , t \ge \pi \end{cases}$$

ans:

step-1: piecewise function to a single unit function.

$$f(t) = \cos t \left[u(t-\pi) - u(t-\infty) \right]$$

$$= \cos t \cdot u(t-\pi)$$

$$\therefore \; y + y' = \cos t \cdot u(t-\pi) \,, \quad y(0) = 5$$

<u>step-2: solving</u>

$$\mathcal{L}\{y + y'\} = \mathcal{L}\{\cos t \cdot u(t-\pi)\}$$

$$Y(s) + s \cdot Y(s) - \cancel{y(0)}^{\,5} = \frac{-s e^{-\pi s}}{s^2+1}$$

$$\mathcal{L}\{\cos t \cdot u(t-\pi)\}$$

$$\mathcal{L}\{f(t)\cdot u(t-a)\} = \mathcal{L}\{f(t+a)\}\cdot e^{-as}$$

$$\mathcal{L}\{\cos t \cdot u(t-\pi)\} = \mathcal{L}\{\cos(t+\pi)\}\cdot e^{-\pi s}$$

$$= -\mathcal{L}\{\cos t\}\cdot e^{-\pi s}$$

$$= -\frac{s\cdot e^{-\pi s}}{s^2+1^2}$$

$$\Rightarrow Y(s)(s+1) = 5 - \frac{s e^{-\pi s}}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{5}{s+1} - \frac{s e^{-\pi s}}{(s^2+1)(s+1)}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{Y(s)\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s e^{-\pi s}}{(s^2+1)(s+1)}\right\}$$

$$\Rightarrow y(t) = 5e^{-t} - \mathcal{L}^{-1}\left\{\underbrace{\frac{s}{(s^2+1)(s+1)}}_{\text{partial fraction}} \cdot e^{-\pi s}\right\}$$

$$\frac{-s}{(s^2+1)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1}$$

$$\Rightarrow -s = (As+B)(s+1) + C(s^2+1)$$

$\underline{s=-1:}$

$$1 = 2C$$

$$C = \frac{1}{2}$$

again,

$$-s = As^2 + As + Bs + B + Cs^2 + C$$

$$-s = s^2(A+C) + s(A+B) + (B+C)$$

$$A + C = 0$$

$$A = -C$$

$$\boxed{A = -\frac{1}{2}}$$

$$A + B = -1$$

$$B = -1 - A$$

$$B = -\frac{1}{2}$$

$$\frac{-s}{(s^2+1)(s+1)} = \frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1} + \frac{1/2}{s+1}$$

$$= \frac{-\frac{1}{2}(s+1)}{s^2+1} + \frac{1/2}{s+1}$$

$$\mathcal{L}^{-1}\left\{-\frac{1}{2}\,\frac{(s+1)}{s^2+1} + \frac{1/2}{s+1}\right\}$$

$$= -\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$+ \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}e^{-t}$$

$$\Rightarrow y(t) = 5e^{-t} + \mathcal{L}^{-1}\left\{ \frac{-s e^{-\pi s}}{(s^2+1)(s+1)} \right\}$$

now,

$$\mathcal{L}^{-1}\left\{ \frac{-s\, e^{-\pi s}}{(s^2+1)\,(s+1)} \right\}$$

$$= f(t-a) \cdot u(t-a)$$

$$a = \pi$$

$$= \left\{ -\frac{1}{2}\cos(t-\pi) - \frac{1}{2}\sin(t-\pi) + \frac{1}{2} e^{-(t-\pi)} \right\} \cdot u(t-\pi)$$

$$= \left(\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}e^{\pi - t} \right) \cdot u(t-\pi)$$

$$y(t) = 5e^{-t} + \left(\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}e^{\pi - t}\right) \cdot u(t-\pi)$$

Q) solve the given differential equation,

$$y'' + 4y = \sin t \cdot u(t-2\pi)$$

$$y(0) = 1$$

$$y'(0) = 0$$

ans:

let, $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{\sin t \cdot u(t-2\pi)\}$$

$$s^2 \cdot Y(s) - s \cdot \cancel{y(0)}^{\,1} - \cancel{y'(0)}^{\,0} + 9Y(s)$$

$$= \frac{1}{s^2+1} \cdot e^{-2\pi s}$$

$$\mathcal{L}\{\sin t \cdot u(t-2\pi)\}$$

$$\mathcal{L}\{f(t) \cdot u(t-a)\}$$

$$= \mathcal{L}\{f(t+a)\} \cdot e^{-as}$$

$$= \mathcal{L}\{\sin(t+2\pi)\} \cdot e^{-2\pi s}$$

$$= \mathcal{L}\{\sin t\} \cdot e^{-2\pi s}$$

$$= \frac{1}{s^2+1} \cdot e^{-2\pi s}$$

now,

$$Y(s) \cdot (s^2+9) = \frac{1}{s^2+1} \cdot e^{-2\pi s} + s$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)(s^2+9)} \cdot e^{-2\pi s} + \frac{s}{s^2+9}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)} \cdot e^{-2\pi s}\right\}$$

$$+ \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+9)} \cdot e^{-2\pi s} \right\}$$

$$+ \cos 2t$$

now,

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow 1 = (As+B)(s^2+9) + (Cs+D)\cdot(s^2+1)$$

$$\Rightarrow 1 = As^3 + 9As + Bs^2 + 9B + Cs^3 + Cs + Ds^2 + D$$

$$\Rightarrow 1 = s^3(A+C) + s^2(B+D) + s(9A+C) + (9B+D)$$

equating,

$A + C = 0$

$A = -C$

$B + D = 0$

$B = -D$

$4B + D = 1$

$\Rightarrow 4(-D) + D = 1$

$\Rightarrow -3D = 1$

$\Rightarrow D = -1/3$

Also,

$4A + C = 0$

$4A = -C$

$A = -\frac{C}{4}$

$$\therefore B = -D = \frac{1}{3}$$

$$A = -c \quad \Big| \quad A = -\frac{c}{a}$$

$$\therefore -\frac{c}{a} = -c$$

$$-c = -ac$$

$$\Rightarrow 3c = 0$$

$$\Rightarrow c = 0$$

$$\therefore A = 0$$

$$A=0,\ B=\frac{1}{3},\ C=0,\ D=\frac{-1}{3}$$

$$\therefore \frac{1}{(s^2+1)(s^2+9)}=\frac{1/3}{(s^2+1)}-\frac{1/3}{s^2+9}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}=\mathcal{L}^{-1}\left\{\frac{1/3}{(s^2+1)}\right\}$$

$$-\mathcal{L}^{-1}\left\{\frac{1/3}{s^2+9}\right\}$$

$$=\frac{1}{3}\sin t-\frac{1}{3}\cdot\frac{1}{2}\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t$$

now

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+9)} \cdot e^{-2\pi s} \right\}$$

$$+ \cos 2t$$

$$\mathcal{L}^{-1}\left\{ F(s) \cdot e^{-as} \right\} = f(t-a)\, u(t-a)$$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+9)} \cdot e^{-2\pi s} \right\}$$

$$= f(t-2\pi) \cdot u(t-2\pi)$$

$$= \left\{ \frac{1}{3} \sin(t-2\pi) - \frac{1}{6} \sin 2(t-2\pi) \right\} .$$

$$u(t-2\pi)$$

$$= \left(\frac{1}{3} \sin t - \frac{1}{6} \sin 2t \right) . u(t-2\pi)$$

now

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+1)(s^2+4)} \cdot e^{-2\pi s} \right\}$$

$$+ \cos 2t$$

$$\therefore y(t) = \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right) \cdot u(t-2\pi)$$

$$+ \cos 2t$$

2) solve the given differential equation,

$$y'' + 9y = \cos 2t,$$

$$y(0) = 1$$

$$y\left(\frac{\pi}{2}\right) = -1$$

ans:

let, $\mathcal{L}\{y(t)\} = Y(s)$

and let $y'(0) = c$

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{\cos 2t\}$$

$$\Rightarrow s^2 \cdot Y(s) - s \cdot \overset{1}{\cancel{y(0)}} - \overset{c}{\cancel{y'(0)}} + 9\,Y(s) = \frac{s}{s^2+2^2}$$

$$\Rightarrow Y(s)\left(s^2 + 9\right) = \frac{s}{s^2+2^2} + c + s$$

$$\Rightarrow Y(s) = \frac{s}{(s^2+2^2)(s^2+9)} + \frac{c}{s^2+9} + \frac{s}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+2^2)(s^2+9)}\right\}$$

$$+ c \cdot \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+2^2)(s^2+9)}\right\}$$

$$+ c \cdot \frac{1}{3}\sin(3t) + \cos 3t$$

now,

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow 1 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$\Rightarrow 1 = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs$$
$$+ Ds^2 + 4D$$

$$\Rightarrow 1 = s^3(A+C) + s^2(B+D) +$$
$$s(9A+4C) + (9B+4D)$$

$$A + C = 0$$

$$A = -C$$

$$B + D = 0$$

$$B = -D$$

$$9A + 4C = 0$$

$$\Rightarrow 9(-C) + 4C = 0$$

$$\Rightarrow -5C = 0$$

$$\Rightarrow \boxed{C = 0}$$

$$\boxed{A = 0}$$

$$9B + 4D = 1$$

$$\Rightarrow 9(-D) + 4D = 1$$

$$\Rightarrow -5D = 1$$

$$\boxed{D = \frac{-1}{5}}$$

$$\boxed{B = \frac{1}{5}}$$

$$\therefore \frac{1}{(s^2+4)(s^2+9)} = \frac{1/5}{s^2+4} - \frac{1/5}{s^2+9}$$

now,

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s}{(s^2+2^2)(s^2+9)} \right\}$$

$$+ \; c \cdot \frac{1}{3} \sin(3t) \; + \cos 3t$$

$$\mathcal{L}^{-1}\left\{ \frac{s}{(s^2+4)(s^2+9)} \right\} = \mathcal{L}^{-1}\left\{ \frac{s/5}{s^2+4} \right\}$$

$$- \mathcal{L}^{-1}\left\{ \frac{s/5}{s^2+9} \right\}$$

$$= \frac{1}{5} \cdot \cos 2t - \frac{1}{5} \cos 3t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{ \frac{s}{(s^2+2^2)(s^2+9)} \right\}$$

$$+ \; c \cdot \frac{1}{3}\sin(3t)$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t +$$

$$\frac{c}{3}\sin 3t$$

<u>finding c:</u>

$$y\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos(\pi) - \frac{1}{5}\cos\left(3\frac{\pi}{2}\right)$$

$$+ \cos\left(3\frac{\pi}{2}\right) + \frac{c}{3}\sin\left(3\frac{\pi}{2}\right)$$

$$\Rightarrow -1 = -\frac{1}{5} - 0 + 0 - \frac{c}{3}$$

$$\Rightarrow -\frac{9}{5} = -\frac{c}{3}$$

$$\Rightarrow c = \frac{12}{5}$$

$$\therefore\ y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t$$

$$+ \frac{12}{5} \times \frac{1}{3}\sin 3t$$

$$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$

L-1

degree

যত বেশি

∞

hard

L-2

laplace math difficulty level for choosing math

hard

$\sin(2t) \cdot u(t-a)$

$\sin(2t) \cdot e^{at}$ অথবা $\cos 3t \, e^{at} \rightarrow$ medium

$t^n e^{at}$

easy

L-3

$y(0) = ?$, $y'(0) = ?$

$y''(0)$

$y(0) = 0$

easy

$y \neq 0$ hard

Evaluation of some Improper Integrals

Q) Evaluate

$$\int_0^\infty \sin 3t \cdot e^{-2t} \, dt$$

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\Rightarrow \int_0^\infty e^{-st} \sin 3t \, dt = \frac{3}{s^2+9}$$

$$\Rightarrow \int_0^\infty e^{-2t} \sin 3t \, dt = \frac{3}{2^2+9} = \frac{3}{13}$$

$\boxed{4}$ $\displaystyle\int_0^\infty t \sin 2t \, e^{-t} \, dt$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{4s}{(s^2+4)^2}$$

$$\int_0^\infty e^{-st} \, t \sin 2t \, dt = \frac{4s}{(s^2+4)^2}$$

$$\int_0^\infty e^{-t} t \sin 2t \, dt = \frac{9}{25}$$

$$\int_0^\infty \frac{\sin t}{t} \, dt$$

ans:

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1} \, du$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}} = \left[\tan^{-1} u\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\therefore \mathcal{L}\left\{ \frac{\sin t}{t} \right\} = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \cdot \frac{\sin t}{t} \, dt = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{0} \frac{\sin t}{t} \, dt = \frac{\pi}{2} - \tan^{-1} 0$$

$$\int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2}$$

distinction between 1st & 2nd transl. thm.

1st

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{at}\,\mathcal{L}^{-1}\{F(s+a)\}$$

2nd

$$\mathcal{L}\{f(t)\cdot u(t-a)\} = \mathcal{L}\{f(t+a)\}\cdot e^{-as}$$

$$\mathcal{L}^{-1}\{F(s)\,e^{-as}\} = f(t-a)\cdot u(t-a)$$