MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Cauchy-Riemann Equation

Prepared by:

Saad Bin Sohan

BRAC University

Email: sohan.academics@gmail.com

GitHub: https://github.com/saad-bin-sohan

MAT215
(Machine Learning & Signal Processing)

Cauchy → Real analysis

Riemann → Complex analysis

Cauchy-Riemann theorem → related to **Analytic Function**

Analytic Function

Function with limits

conditions

→ Continuous

more conditions

→ Differentiable

some more conditions

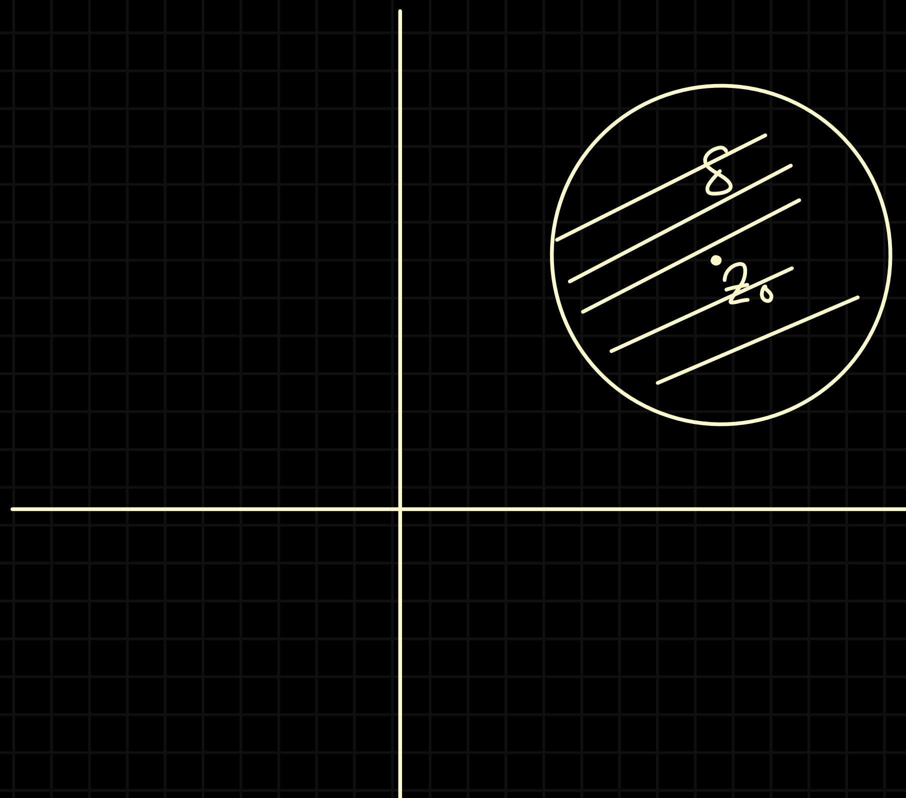
→ **Analytic function**

A function $f(z)$ is called analytic at a point $z = z_0$ if there is $\delta > 0$ such that $f(z)$ is differentiable at the disk

$$|z - z_0| < \delta$$

→ some function কোনো point এ Analytic means ঐ point এর আশের পাশেরও circular $\boxed{\text{region}}$ (ছোট/বড়) এ its differentiable

must be a region; differentiable at a single point isn't analytic function



→ if the derivative $f'(z)$ exists, at all points z of a Region R,

then $f(z)$ is said to be analytic in R

→ also called Holomorphic function

$$\text{Analytic} \underset{\text{not necessarily}}{\overset{\text{must be}}{\rightleftarrows}} \text{Differentiable} \underset{\text{not necessarily}}{\overset{\text{must be}}{\rightleftarrows}} \text{continuous} \underset{\text{not necessarily}}{\overset{\text{must be}}{\rightleftarrows}} \text{limit exists}$$

eg: $f(z) = z^2,\ e^z,\ \sin(z),\ \cos(z)$

eg of non-analytic functions: $f(z) = \frac{1}{z}$, not analytic at $z=0$

$f(z) = \frac{1}{z-1}$ " " " $z=1$

$f(z) = \tan z$ " " " $z = \pi/2$

$f(z) = e^{-1/z}$ " " " $z=0$

$f(z) = |z|^2$ " " " $z \neq 0$

Entire Function

if a function is analytic at ALL POINTS, then $f(z)$ is Entire function.

e.g: $f(z) = z^2$

$$\text{Entire function} \underset{\text{not necessary}}{\overset{\text{must be}}{\rightleftarrows}} \text{Analytic}$$

let a complex valued function has real(u) and imaginary parts(v)

$$f(z) = \underbrace{u(x,y)}_{\text{real part}} + i\,\underbrace{v(x,y)}_{\text{img part}}$$

e.g: $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{\text{real part}} + i\,\underbrace{2xy}_{\text{img part}}$

how to tell if such function is analytic or not $\Rightarrow$ using Cauchy-Riemann's equation

Cauchy-Riemann Equation

2 parts: (i) necessary part, (ii) sufficient part

let $f(z) = u(x,y) + i\,v(x,y)$

$f(z)$ is analytic if

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}}$$

→ necessary part, but vice versa not necessary

↳ says analytic হলে কী হবে

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ &\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \end{aligned} \quad + \quad \text{all partial derivatives are continuous}}$$

→ sufficient part

↳ says কী condition এ analytic হবে

Proof of the necessary part: → xm এ লাগবে না

given $f(z) = u(x,y) + i\,v(x,y)$ is analytic

so $f'(z)$ exists

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\,\Delta y}$$

in $\Delta y = 0$ direction,

$$\text{limit} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

in the $\Delta x = 0$ direction,

$$\text{limit} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

but $f'(z)$ exists $\Rightarrow$ limit also exists

$$\therefore \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial v}{\partial y}$$

so, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial y}$ [Proven] if $f(z) = u(x,y) + i\,v(x,y)$ then,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

eg math: Given $f(z) = (x^2 - y^2) + i\,(2xy)$. Find $f'(z)$

solve:

$u = x^2 - y^2 \qquad v = 2xy$

$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= 2x + i\,2y$

$= 2\,(x + iy)$

$= 2z$

$f'(z) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$

$= 2x - i\,(-2y)$

$= 2\,(x + iy)$

$= 2z$

$f(z) = z^2$

$f'(z) = 2z$

same answer

eg math Show $f(z) = z^2$ is $\underbrace{\text{analytic}}_{\text{use the sufficient part}}$.

solve:

$$f(z) = z^2$$
$$= (x+iy)^2$$
$$= (x^2-y^2) + i(2xy)$$

$u = x^2 - y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial y} = 2x$

we observe that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

so cauchy-Riemann equations are satisfied. Also all the partial derivatives are continuous.

$\therefore f(z)$ is analytic

eg math: show that $f(z) = z - \bar{z}$ is not analytic

solve:

$$f(z) = z - \bar{z}$$
$$= (x+iy) - (x-iy)$$

$$= i\,2y$$

$$\therefore \quad u = 0, \qquad v = 2y$$

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 2$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

but $$\frac{\partial v}{\partial x} \neq -\frac{\partial v}{\partial y}$$

and partial derivatives are continuous

since cauchy-riemann equation doesn't satisfy,

$f(z)$ is not analytic

Harmonic Function

A function $\phi(x,y)$ is called harmonic if it satisfies the Laplace equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

eg math: show that $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

is harmonic.

solve:

$$\frac{\partial u}{\partial x} = 6xy + 4x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 6y+4$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -6y-4$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y+4-6y-4 = 0$$

$\therefore u$ is harmonic

eg math: Given $f(z) = u + iv$ is analytic in a region R. Prove that u and v are harmonic if they have continuous second partial derivatives in R.

solve:

given $u+iv$ is analytic,
cauchy-reimann eqn holds true.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\ldots (i)$$

and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \underbrace{\frac{-\partial^2 v}{\partial y \partial x} = \frac{-\partial^2 v}{\partial x \partial y}}_{\text{swapping was possible because second derivatives are continuous}} \quad \ldots \text{ (ii)}$$

adding (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} + \left(-\frac{\partial^2 v}{\partial x \partial y}\right) = 0$$

$\therefore$ u is harmonic

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\ldots \text{iii}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x \partial y} = \frac{-\partial^2 u}{\partial x \partial y} \quad \ldots \text{iv}$$

adding (iii) and (iv)

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}$$

$$= 0$$

since laplace equation holds true, v is also harmonic

Harmonic Conjugate

if $f(z) =$ is analytic then v is called the harmonic conjugate of u.

given $u \to$ find $v \to$ conjugate of u

$\downarrow$

such v that f is analytic

<u>eg math:</u> show that $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Also find the harmonic conjugate $v(x,y)$ such that $u + iv$ is harmonic.

<u>solve:</u>

$$\frac{\partial u}{\partial x} = 6xy + 4x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 6y + 4$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -6y - 4$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$\therefore u$ is harmonic

since u+iv is analytic, cauchy-reimann eqn satisfies

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (ii)}$$

from eqn (i)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y + 2x^2 - y^3 - 2y^2\right)$$

$$\frac{\partial v}{\partial y} = 6xy + 4x$$

$$\Rightarrow v(x,y) = \int (6xy + 4x)\, dy$$

$$= 6x\frac{y^2}{2} + 4xy$$

$$= 3xy^2 + 4xy + g(x)$$

from eqn (ii)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(3xy^2+4xy+g(x)\right) = -\frac{\partial}{\partial y}\left(3x^2y+2x^2-y^3-2y^2\right)$$

$$\Rightarrow 3y^2+4y+g'(x) = -3x^2+3y^2+4y$$

$$\Rightarrow g'(x) = -3x^2$$

$$\Rightarrow g(x) = \int -3x^2\,dx$$

$$= \frac{-3x^3}{3} + c$$

$$\therefore\ g(x) = -x^3 + c$$

so, $v = 3xy^2 + 4xy - x^3 + c$

eg math:

show that $u(x,y) = x e^{-x} \sin y - e^{-x} y \cos y$ is harmonic.

Also find the harmonic conjugate $v(x,y)$ such that

$u+iv$ is analytic.

solve:

1 st part (harmonic):

we have to show

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + e^{-x} y \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y - e^{-x} \sin y + x e^{-x} \sin y - e^{-x} y \cos y$$

$$= x e^{-x} \sin y - 2 e^{-x} \sin y - e^{-x} \cos y$$

again,

$$\frac{\partial u}{\partial y} = x e^{-x} \cos y - e^{-x} \{ y(-\sin y) + 1 \cdot \cos y \}$$

$$= x e^{-x} \cos y + e^{-x} y \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -x e^{-x} \sin y + e^{-x} y \cos y + e^{-x} \sin y + e^{-x} \sin y$$

$$= -x e^{-x} \sin y + 2 e^{-x} \sin y + e^{-x} y \cos y$$

now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is harmonic [proved]

2nd part (finding v):

given, $u+iv$ is analytic

$\therefore$ C-R eqn holds true.

so,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots \text{(i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots \text{(ii)}$$

from (i)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$= e^{-x}\sin y - x e^{-x}\sin y + e^{-x} y \cos y$$

$$\therefore v(x,y) = \int \left(e^{-x}\sin y - x e^{-x}\sin y + e^{-x} y \cos y\right) dy$$

$$= e^{-x}\int \sin y \, dy - x e^{-x}\int \sin y \, dy + e^{-x}\int y \cos y \, dy$$

$$= -e^{-x}\cos y + x e^{-x}\cos y + e^{-x}\left[y\int \cos y \, dy - \int 1.\sin y \, dy\right]$$

$$= -e^{-x}\cos y + x e^{-x}\cos y + e^{-x}\left[y \sin y + \cos y\right] + g(x)$$

$$= -e^{-x}\cos y + x e^{-x}\cos y + e^{-x} y \sin y + e^{-x}\cos y + g(x)$$

$$\therefore V(x,y) = x e^{-x}\cos y + e^{-x} y \sin y + g(x)$$

now, from (ii)

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(x e^{-x}\cos y + e^{-x} y \sin y + g(x)\right) = -\left(x e^{-x}\cos y + e^{-x} y \sin y - e^{-x}\cos y\right)$$

$$\Rightarrow e^{-x}\cos y - x e^{-x}\cos y - e^{-x} y \sin y + g'(x) = -x e^{-x}\cos y - e^{-x} y \sin y + e^{-x}\cos y$$

$$\Rightarrow g'(x) = 0$$

$$\therefore \quad g(x) = \int 0 \, dx$$

$$= c$$

$\therefore v(x,y) = x e^{-x}\cos y + e^{-x} y \sin y + c$, where c is a constant complex number

hw: determine which of the following functions u are harmonic. For each harmonic function, find the conjugate harmonic function v and express $u+iv$ as an analytic function of z.

(a) $3x^2y + 2x^2 - y^3 - 2y^2$

(b) $2xy + 3xy^2 - 2y^3$

(c) $e^{-2xy}\sin(x^2-y^2)$

shortcut for integration uv formula

$$\int x^3 \sin(2x)\, dx$$

x^3	$\sin 2x$
$3x^2$	$-\frac{1}{2}\cos 2x$
$6x$	$-\frac{1}{4}\sin 2x$
6	$\frac{1}{8}\cos 2x$
0	$\frac{1}{16}\sin 2x$

$$= \left(-\frac{x^3}{2}\cos 2x\right) - \left(\frac{-3}{4}x^2 \sin 2x\right) + \left(\frac{6x}{8}\cos 2x\right)$$

$$- \left(\frac{6}{18}\sin 2x\right) + c$$

u has to be the part that becomes zero eventually