MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Assignment 2

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BRAC University
Department of Mathematics and Natural Science
MAT 215: Mathematics for Machine Learning and Signal Processing
Assignment - 02

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Answer all Questions

1. Show that:
   (a) $\sin^{-1}(z) = -i\ln\left(iz + \sqrt{1-z^2}\right)$ **5**
   (b) $\cot^{-1}(z) = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$ **5**
   (c) $\sinh^{-1}(z) = \ln\left(z + \sqrt{z^2-1}\right)$ **5**

2. If $f(z) = \frac{2z-1}{3z+2}$, prove that $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{7}{(3z_0+2)^2}$ provided $z_0 \neq -\frac{2}{3}$ **6**

3. Using the definitions, find the derivative of each function at the indicated points:
   (i) $f(z) = \frac{2z-i}{z+2i}$ at $z = -i$ **4**
   (ii) $f(z) = 3z^{-2}$ at $z = 1+i$ **3**

4. Evaluate the following Limits using L' Hospital's rule:
   (i) $\lim_{z\to 2i}\frac{z^2+4}{2z^2+(3-4i)z-6i}$ **3**
   (ii) $\lim_{z\to 0}\frac{z-\sin z}{z^3}$. **3**
   (iii) $\lim_{z\to 0}\frac{\tan z-\sin z}{z^3}$ **4**
   (iv) $\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$ **5**

5. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u+iv$ as an analytic function of z
   (i) $u = 3x^2y + 2x^2 - y^3 - 2y^2$ **5**
   (ii) $u = xe^x\cos y - ye^x\sin y$ **6**
   (iii) $u = e^{-x}(x\sin y - y\cos y)$ **6**

Best wishes

$$\boxed{\text{Answer to the question 1}}$$

1(a)

let $\omega = \sin^{-1} z$

$\Rightarrow \sin\omega = z$

$\Rightarrow z = \sin(\omega)$

$$z = \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

$$\Rightarrow \boxed{2zi = e^{i\omega} - e^{-i\omega}} \quad \text{— i}$$

$$\Rightarrow (e^{iw} + e^{-iw})^2 = (e^{iw} - e^{-iw})^2 + 4 \cdot e^{iw} \cdot e^{-iw}$$

$$\Rightarrow (e^{iw} + e^{-iw})^2 = (2zi)^2 + 4$$

$$\Rightarrow (e^{iw} + e^{-iw})^2 = -4z^2 + 4$$

$$= 4(1-z^2)$$

$$\boxed{e^{iw} + e^{-iw} = \sqrt{4(1-z^2)}}$$

$$\text{—— (ii)}$$

adding (i) and (ii)

$$e^{iw} + e^{-iw} = 2\sqrt{1-z^2}$$

$$e^{iw} - e^{-iw} = 2zi$$

$$2e^{iw} = 2\left(\sqrt{1-z^2} + zi\right)$$

$$\Rightarrow e^{iw} = \sqrt{1-z^2} + zi$$

$$\Rightarrow iw = \ln\left(iz + \sqrt{1-z^2}\right)$$

$$\Rightarrow w = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right)$$

$$\Rightarrow \sin^{-1}(z) = -i\ln\left(iz + \sqrt{1-z^2}\right)$$

[shown]

$\underline{1(b)}$

let $w = \cot^{-1} z$

$\Rightarrow z = \cot w$

$\Rightarrow z = \dfrac{\cos w}{\sin w}$

$$\Rightarrow z = \frac{\dfrac{e^{iw} + e^{-iw}}{2}}{\dfrac{e^{iw} - e^{-iw}}{2i}}$$

$$= \frac{e^{iw} + e^{-iw}}{2} \times \frac{2i}{e^{iw} - e^{-iw}}$$

$$z = i\left(\frac{e^{iw}+e^{-iw}}{e^{iw}-e^{-iw}}\right)$$

$$\Rightarrow \quad z e^{iw} - z e^{-iw} = i e^{iw} + i e^{-iw}$$

$$\Rightarrow \quad z e^{iw} - i e^{iw} = z e^{-iw} + i e^{-iw}$$

$$\Rightarrow \quad e^{iw}(z-i) = e^{-iw}(z+i)$$

$$\Rightarrow \quad e^{2iw} = \frac{z+i}{z-i}$$

$$\Rightarrow \quad 2iw = \ln\left(\frac{z+i}{z-i}\right)$$

$$\Rightarrow \quad w = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

$$\Rightarrow \quad \cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right)$$

1 (iii)

let $w = \sinh^{-1} z$

$$\Rightarrow \quad \sinh(w) = z$$

$$\Rightarrow \quad \frac{e^{w} - e^{-w}}{2} = z$$

$$\Rightarrow \quad e^{w} - e^{-w} = 2z$$

$$\Rightarrow \quad e^{w} - \frac{1}{e^{w}} = 2z$$

$$\Rightarrow \quad \frac{e^{2w} - 1}{e^{w}} = 2z$$

$$\Rightarrow e^{2w} - 2z\,e^{w} - 1 = 0$$

$$\Rightarrow 1.\,(e^{w})^2 + (-2z)\,e^{w} + (-1) = 0$$

$$\Rightarrow e^{w} = \frac{2z \pm \sqrt{4z^2+4}}{2}$$

$$\Rightarrow e^{w} = \frac{2z \pm \sqrt{4\,(z^2+1)}}{2}$$

$$\Rightarrow e^{w} = z + \sqrt{z^2+1}$$

[logarithm can't take negative values]

$$\Rightarrow \quad w = \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\Rightarrow \quad \sinh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\boxed{\text{Answere to the question-2}}$$

$$f(z)=\frac{2z-1}{3z+2}$$

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{2(z_0+h)-1}{3(z_0+h)+2}-\dfrac{2z_0-1}{3z_0+2}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{2z_0+2h-1}{3z_0+3h+2}-\dfrac{2z_0-1}{3z_0+2}}{h}$$

$$= \lim_{h \to 0} \frac{(2z_0+2h-1)(3z_0+2) - (-2z_0+1)(3z_0+3h+2)}{h(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h \to 0} \frac{\cancel{6z_0^2} + \cancel{4z_0} + \cancel{6hz_0} + 4h - \cancel{3z_0} - \cancel{2} - \cancel{6z_0^2} - \cancel{6hz_0} - \cancel{4z_0} + \cancel{3z_0} + 3h + \cancel{2}}{h(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h \to 0} \frac{7h}{h(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h \to 0} \frac{7}{(3z_0+3h+2)(3z_0+2)}$$

$$= \frac{7}{(3z_0+0+2)(3z_0+2)}$$

$$= \frac{7}{(3z_0+2)^2}$$

here $3z_0+2 \neq 0$

$$\Rightarrow z_0 \neq \frac{-2}{3}$$

Answer to the question-3

3 (i)

$$f(z) = \frac{2z - i}{z + 2i}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(-i) = \lim_{\Delta z \to 0} \frac{f(-i + \Delta z) - f(-i)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{2(-i+\Delta z) - i}{-i + \Delta z + 2i} - \dfrac{2(-i) - i}{-i + 2i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{-3i + 2\Delta z}{i + \Delta z} - \dfrac{-3i}{i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{-3i + 2\Delta z}{i + \Delta z} + 3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-3i + 2\Delta z + 3i + 3\Delta z}{\Delta z\,(i + \Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{5\Delta z}{\Delta z\,(i + \Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{5}{i + \Delta z}$$

$$= \frac{5}{i+0}$$

$$= \frac{5}{i}$$

$$= -5i$$

3(ii)

$$f(z) = 3z^{-2} = \frac{3}{z^2}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(1+i) = \lim_{\Delta z \to 0} \frac{f(1+i+\Delta z) - f(1+i)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{3}{(1+i+\Delta z)^2} - \frac{3}{(1+i)^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{3}{(1+i+\Delta z)^2} - \dfrac{3}{2i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{6i - 3(1+i^2+\Delta z^2+2i+2i\Delta z+2\Delta z)}{2i\,(1+i+\Delta z)^2 \times \Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{6i} - \cancel{3} + \cancel{3} - 3(\Delta z)^2 - \cancel{6i} - 6i\Delta z - 6\Delta z}{2i\,(1+i+\Delta z)^2\,\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-3(\Delta z)^2 - 6i\Delta z - 6\Delta z}{2i\Delta z\,(1+i+\Delta z)^2}$$

$$= \lim_{\Delta z \to 0} \frac{-3\Delta z - 6i - 6}{2i(1+i+\Delta z)^2}$$

$$= \frac{-6i-6}{2i(1+i)^2}$$

$$= \frac{-6i-6}{2i \times 2i} = \frac{-6i-6}{-4}$$

$$= \frac{3}{2}i + \frac{3}{2}$$

Answer to the question-4

4(i)

$$\lim_{z \to 2i} \frac{z^2+4}{2z^2+(3-4i)z-6i}$$

using L'hospital's rule

$$\Rightarrow \lim_{z \to 2i} \frac{2z}{4z+3-4i}$$

$$= \frac{2 \times 2i}{4 \times 2i + 3 - 4i}$$

$$= \frac{4i}{4i+3}$$

4(ii)

$$\lim_{z \to 0} \frac{z - \sin z}{z^3}$$

using L'hospital's rule

$$= \lim_{z \to 0} \frac{1 - \cos z}{3z^2}$$

again using L'hospital's rule

$$= \lim_{z \to 0} \frac{\sin z}{6z}$$

again using L'hospital's rule

$$\Rightarrow \lim_{z \to 0} \frac{\cos z}{6}$$

$$= \frac{\cos 0}{6}$$

$$= \frac{1}{6}$$

4 (iii)

$$\lim_{z \to 0} \frac{\tan z - \sin z}{z^3}$$

using L'hospital's rule

$$\lim_{z \to 0} \frac{\sec^2 z - \cos z}{3z^2}$$

again using L'hospital's rule

$$\Rightarrow \lim_{z \to 0} \frac{2\sec z \times \sec z \tan z + \sin z}{6z}$$

$$= \lim_{z \to 0} \frac{2\sec^2 z \tan z + \sin z}{6z}$$

again using L'hospital's rule

$$= \lim_{z \to 0} \frac{2\{\sec^2 z \cdot \sec^2 z + \tan z \cdot 2\sec^2 z \tan z\} + \cos z}{6}$$

$$= \lim_{z \to 0} \frac{2(\sec^4 z + 2\sec^2 z \tan^2 z) + \cos z}{6}$$

$$= \frac{2(1 + 2 \cdot 1 \cdot 0^2) + 1}{6}$$

$$= \frac{2+1}{6}$$

$= \frac{1}{2}$

Ans

4 (iv)

$$\lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{1/z^2}$$

let $\quad w = \left(\frac{\sin z}{z} \right)^{1/z^2}$

$$\Rightarrow \ln(w) = \ln \left\{ \left(\frac{\sin z}{z} \right)^{1/z^2} \right\}$$

$$\therefore \ln w = \frac{\ln \left(\frac{\sin z}{z} \right)}{z^2}$$

$$\therefore \lim_{z \to 0} (\ln w) = \lim_{z \to 0} \left\{ \frac{\ln \left(\frac{\sin z}{z} \right)}{z^2} \right\}$$

here $\frac{\sin z}{z} = 1$, $\ln(1) = 0$ and $z = 0$

which gives us $\frac{0}{0}$

so using L'hospital's rule

$$\lim_{z \to 0} \{\ln(w)\} = \lim_{z \to 0} \frac{\ln(\sin z) - \ln(z)}{z^2}$$

$$= \lim_{z \to 0} \frac{\frac{1}{\sin z} \cdot \cos z - \frac{1}{z}}{2z}$$

$$= \lim_{z \to 0} \frac{\frac{\cos z}{\sin z} - \frac{1}{z}}{2z}$$

$$= \lim_{z \to 0} \frac{z\cos z - \sin z}{2z^2 \sin z}$$

using L'hospital's rule

$$= \lim_{z \to 0} \frac{z(-\sin z) + \cos z - \cos z}{2\{z^2 \cos z + 2z \sin z\}}$$

$$= \lim_{z \to 0} \frac{-z \sin z}{2z^2 \cos z + 4z \sin z}$$

$$= \lim_{z \to 0} \frac{-\sin z}{2z \cos z + 4 \sin z}$$

using L'hospital's rule

$$= \lim_{z \to 0} \frac{-\cos z}{-2z\sin z + 2\cos z + 4\cos z}$$

$$= \frac{-1}{0 + 2 + 4}$$

$$= \frac{-1}{6}$$

now $\lim_{z \to 0} \left(\ln(w) \right) = \frac{-1}{6}$

$\therefore \lim_{z \to 0} w = e^{-1/6}$

Answere to the question-5

5(i)

harmonic functions satisfy the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$$

$$\frac{\partial u}{\partial x} = 6xy + 4x$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4$$

$$= 0$$

$\therefore$ u is harmonic

now lets find v.

since $u+iv$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad 6xy + 4x = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad v(x,y) = \int (6xy+4x)\, dy$$

$$= 6x \times \frac{y^2}{2} + 4xy + g(x)$$

$$= 3xy^2 + 4xy + g(x)$$

$$\text{———— } (1)$$

since they are analytic,

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(3xy^2 + 4xy + g(x)\right) = -3x^2 + 3y^2 + 4y$$

$$\Rightarrow 3y^2 + 4y + g'(x) = -3x^2 + 3y^2 + 4y$$

$$\Rightarrow g'(x) = -3x^2$$

$$\Rightarrow g(x) = \int -3x^2 \, dx$$

$$g(x) = -3 \cdot \frac{x^3}{3} + c$$

$$= -x^3 + c$$

$$\therefore v(x,y) = 3xy^2 + 4xy - x^3 + c$$

5(ii)

harmonic functions satisfy the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,y) = x e^x \cos y - y e^x \sin y$$

$$\frac{\partial u}{\partial x} = \cos y \;\; x e^x + \cos y \, e^x - y e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = \cos y \;\; x e^x + e^x \cos y + \cos y \, e^x$$
$$- y e^x \sin y$$

$$\frac{\partial u}{\partial y} = -x e^x \sin y - e^x \, y \cos y - e^x \sin y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x \cos y + e^x y \sin y - e^x \cos y$$
$$- e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xe^x \cos y + 2e^x \cos y - ye^x \sin y$$
$$- xe^x \cos y - 2e^x \cos y + ye^x \sin y$$

$$= 0$$

$\therefore$ u is harmonic

since $u+iv$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = \cos y \; x e^{x} + \cos y \, e^{x} - y e^{x} \sin y$$

$$v(x,y) = \int \left(\cos y \, x e^{x} + \cos y \, e^{x} - y e^{x} \sin y\right) dy$$

$$= x e^{x} \sin y + e^{x} \sin y - e^{x}\left(\int y \sin y \, dy\right)$$

$$= x e^{x} \sin y + e^{x} \sin y - e^{x}\left\{ y(-\cos y) - \int (-\cos y) \, dy \right\}$$

$$= x e^{x} \sin y + \cancel{e^{x} \sin y} + e^{x} y \cos y - \cancel{e^{x} \sin y} + g(x)$$

$$\therefore\ v(x, y) = x e^{x} \sin y + e^{x} y \cos y + g(x)$$

again $\qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial x}\left(x e^{x} \sin y + e^{x} y \cos y + g(x)\right) =$$

$$x e^{x} \sin y + e^{x} y \cos y + e^{x} \sin y$$

$$\Rightarrow \cancel{\sin y}\, x e^{x} + e^{x} \cancel{\sin y} + e^{x} \cancel{y \cos y} + g'(x)$$

$$= x e^{x} \cancel{\sin y} + e^{x} \cancel{y \cos y} + e^{x} \cancel{\sin y}$$

$$\therefore \quad g'(x) = 0$$

$$\therefore \quad g(x) = \int 0 \, dx$$

$$g(x) = c$$

$$\therefore\ v(x,y) = xe^{x}\sin y + e^{x}y\cos y + c$$

where c is a constant complex number

$\boxed{\text{Answer to the question-5}}$

5 (iii)

$$u = e^{-x} x \sin y - e^{-x} y \cos y$$

harmonic functions satisfy the laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = \sin y\, e^{-x} - x e^{-x} \sin y + e^{-x} y \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y + \sin y\, x e^{-x} - \sin y\, e^{-x}$$

$$- e^{-x} y \cos y$$

$$u = e^{-x} x \sin y - e^{-x} y \cos y$$

$$\frac{\partial u}{\partial y} = x e^{-x} \cos y - e^{-x} \left\{ -y \sin y + \cos y \right\}$$

$$= x e^{-x} \cos y + e^{-x} y \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} \Rightarrow$$

$$= -x e^{-x} \sin y + e^{-x} y \cos y + \sin y \, e^{-x}$$

$$+ e^{-x} \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\cancel{\sin y} + \cancel{\sin y}\, x e^{-x} - \cancel{\sin y}\, e^{-x} - e^{-x}\cancel{y}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -x e^{-x}\cancel{\sin y} + e^{-x}\cancel{y}\cos y + \cancel{\sin y}\, e^{-x} + e^{-x}\cancel{\sin y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is harmonic.

since they are analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\sin y\, e^{-x} - x e^{-x} \sin y + e^{-x} y \cos y = \frac{\partial v}{\partial y}$$

$$v(x,y) = \int \left(\sin y\, e^{-x} - x e^{-x} \sin y + e^{-x} y \cos y\right) dy$$

$$= -e^{-x} \cos y + x e^{-x} \cos y +$$

$$e^{-x}\left\{y(\sin y) + \cos y\right\} + g(x)$$

$$= -e^{-x} \cos y + x e^{-x} \cos y + e^{-x} y \sin y$$

$$+ e^{-x} \cos y + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial x}\left\{-e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\{y(\sin y) + \cos y\} + g(x)\right\}$$

$$= -xe^{-x}\cos y - e^{-x}y\sin y + e^{-x}\cos y$$

$$\Rightarrow e^{-x}\cancel{c}\text{os}y + \cos y\{x(-e^{-x}) + e^{-x}\} - e^{-x}\{y\sin y + \cos y\} + g'(x) =$$

$$-xe^{-x}\cos y - e^{-x}y\sin y + e^{-x}\cancel{\cos y}$$

$$\Rightarrow -x\cancel{e^{-x}}\cos y + e^{-x}\cancel{c}\text{os}y - e^{-x}\cancel{y}\sin y$$

$$-e^{-x}\cancel{c}\text{osy} + g'(x) = -xe^{-x}\cancel{c}\text{osy} - e^{-x}\cancel{y}\sin y$$

$$\Rightarrow g'(x) = 0$$

$$\Rightarrow g(x) = \int 0 \, dx$$

$$= c$$

$$\therefore v(x,y) = -\cancel{e^{-x}\cos y} + xe^{-x}\cos y + e^{-x}y\sin y + \cancel{e^{-x}\cos y} + c$$

$$\therefore v(x,y) = xe^{-x}\cos y + e^{-x}y\sin y + c$$