CSE330: Numerical Methods

Topic: Secant Method, Aitjen Acceleration

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Newton's method এ,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \ldots \text{(i)}$$

$f'(x_k)=0$ হলে arisen problemটা avoid করতে

new method ⇒ secant method / Quasi-Newton method

newton's method টার আরেকটা version

$\underbrace{\text{secant method / Quasi-Newton method}}_{\text{not important for final}}$

new method টাকে অর্থাৎ equationএ

$f'(x_k) \rightarrow$ replaced with backward difference

$$\text{backward difference} = \frac{f(x) - f(x-h)}{h}$$

$\because$ ch 4 এ h এর concept not included, we replace with x from previous iteration,

$\Rightarrow$ backward difference,

$$f'(x) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

eqn (i) এ $f'(x)$ বসিয়ে

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}}$$

এই method we need two nodes (x_k, x_{k-1}) instead of one:

$$\boxed{x_0, x_1}$$

difference with newton's method

Que $f(x) = \frac{1}{x} - 0.5$

$x_0 = 0.25$

$x_1 = 0.5$

task is to, new method এ next iteration এ x এর value বের করা

soln:

K	x_K	$\lvert f(x_K) \leq \epsilon \rvert$
0	$x_0 = 0.25$	
1	$x_1 = 0.5$	
2	$x_2 = 0.6875$ ↳ x_0, x_1 use করে secant method থেকে x_2 use করা	
3	$x_3 = 1.01652$	
	⋮	
	stop when $f(x)=0$ or ϵ_m reached হলে	

New technique: Aitken Acceleration

(bisection বাদে) previous 3টা method এ root খুঁজে পাওয়ার process কে boost/accelerate করে

Aitken Acceleration

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

যেই x এর?
জন্য accelerate করলো।

$\boxed{\text{Que}}$ $f(x) = \frac{1}{x} - 0.5$ $[x_* = 2]$

fixed point iteration apply করে $f(x)$ এর root বের করো (যেখানে additionally $g(x)$ দেওয়া) while using Aitken acceleration.

$$g(x) = x + \frac{1}{16}\left(\frac{1}{x} - 0.5\right)$$

$x_0 = 1.5$

soln:

lets see without using aikhen acceleration

$$x_0 = 1.5$$

$$x_1 = g(x_0) = g(1.5) = 1.510417$$

$$x_2 = g(x_1) = g(1.510417)$$

$$= 1.520546$$

$$x_3 = g(1.520546)$$

$$= 1.530400$$

$$\vdots$$

$$x_{818} = 2$$

এগুলো iteration avoid করতে aikhen acceleration now using aikhen acceleration,

NB: এই acceleration সব node এ apply হবেনা। there's a pattern for those nodes.

$x_0 = 1.5$

$x_1 = g(x_0) = g(1.5) = 1.510417$

$x_2 = g(x_1) = 1.520546$

pattern: 2টা node বের করো using fixed point iteration and then apply aitken acceleration. আবার 2টা node বের করো, then aikhen → pattern

apply aikhen →

$$\hat{x}_2 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

$$\hat{x}_2 = 1.877604$$

notice, $x_2 = 1.52\ldots$ থেকে $\hat{x}_2 = 1.87\ldots$

actual root 2 এর অনেক কাছাকাছি চলে গেছে।

that's the accleration.

and now new x_2 will be $\hat{x}_2$

$$\therefore \left.\begin{aligned} x_3 &= g(\hat{x}_2) \\ &= 1.87964 \\ x_4 &= g(x_3) \\ &= 1.881642 \end{aligned}\right\} \text{2 বার fixed point iteration. then again aikhen}$$

$$\hat{x}_4 = \hat{x}_2 - \frac{(x_3 - \hat{x}_2)^2}{x_4 - 2x_3 + \hat{x}_2}$$

$$\hat{x}_4 = 1.092634$$

$$x_5 = g(\hat{x}_4)$$

$$x_6 = g(x_5)$$

$$\hat{x}_6 =$$

$$x_7 = g(\hat{x}_6)$$

$$x_8 = g(x_7)$$

$$\hat{x}_8 = 2$$

fixed point এ aitken ছাড়া 818 টা iteration.

aitken এর 6 টা iteration

same math টা newton's method use করতে বললে (instead of floating point iteration), তখন 2টা node বের করবো using newton's method and then aitken and so on.

Ch-5: Solving Linear Eqn(s)

2 methods we will use

- → Gaussian Elimination
- → LU Decomposition

e.g. of linear eqn,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

task is to make matrix out of these equations

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_{\substack{A \\ n \times n}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\substack{x \\ n \times 1}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\substack{b \\ n \times 1}}$$

$$Ax = b$$

$$\Rightarrow x = A^{-1} b$$

remember pc

avoids matrix inversion

Basic properties of A:

(i) A square matrix হতে হবে

(ii) A will be non-singular ($\det(A) \neq 0$)

otherwise no unique solution $\therefore$ method টা use করা যাবে না

Lower triangular matrix: কর্ণের উপরে সব zero

e.g.:

$$\begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

lower triangular matrix

primary diagonal

matrix থেকে equation formation,

$$l_{11}x_1 = b_1$$

$$\therefore x_1 = \frac{b_1}{l_{11}}$$

1 div

again $$l_{21}x_1 + l_{22}x_2 = b_2$$

$$\therefore x_2 = \frac{b_2 - l_{21}x_1}{l_{22}}$$

$\underbrace{\text{1 div, 1 sub, 1 mutl}}_{\text{3 operations}}$

$\vdots$

এই process কে বলে forward sobstitution

$$x_3 = \frac{b_3 - l_{31} x_1 - l_{32} x_2}{l_{33}}$$

5 operation

$$x_4 = \frac{b_4 - l_{41} x_1 - l_{42} x_2 - l_{43} x_3}{l_{44}}$$

7 operation

x_1, x_2, x_3, x_4 বের করতে total 16 টা operation.

$\therefore$ complexity $\Rightarrow n^2$

where n = number of variables

backward substitution এও complexity n^2

NB: calculator দিয়ে determinant বের করা জেনে রাখবে।