MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Assignment 1

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BRAC University
Department of Mathematics and Natural Science
MAT 215: Mathematics for Machine Learning and Signal Processing
Assignment - 01
Section: 11

Total Time : | Spring 2024 | Total Mark: 20

Answer all Questions

1. Express each of the following complex numbers in polar form: (a) $2-2i$, (b) $-1+\sqrt{3}i$, (c) $2\sqrt{2}+2\sqrt{2}i$, (d) $-i$, (e) -4 , (f) $-2\sqrt{3}-2i$, (g) $\sqrt{2}i$, (h) $\sqrt{3}/2-3i/2$.

2. Describe and graph the locus represented by each of the following: (a) $|z-i|=2$, (b) $|z+2i|+|z-2i|=6$, (c) $|z-3|-|z+3|=4$, (d) $z(\bar{z}+2)=3$, (e) $\operatorname{Im}\{z^2\}=4$.

3. Describe graphically the region represented by each of the following: (a) $1<|z+i|\leq 2$, (b) $\operatorname{Re}\{z^2\}>1$, (c) $|z+3i|>4$, (d) $|z+2-3i|+|z-2+3i|<10$.

4. Find each of the indicated roots and locate them graphically. (a) $(2\sqrt{3}-2i)^{1/2}$, (b) $(-4+4i)^{1/5}$, (c) $(2+2\sqrt{3}i)^{1/3}$, (d) $(-16i)^{1/4}$, (e) $(64)^{1/6}$, (f)$(i)^{2/3}$.

5. Find all the indicated roots and locate them in the complex plane. (a) Cube roots of 8 , (b) square roots of $4\sqrt{2}+4\sqrt{2}i$, (c) fifth roots of $-16+16\sqrt{3}i$, (d) sixth roots of $-27i$.

6. Solve the equations (a) $z^4+81=0$, (b) $z^6+1=\sqrt{3}i$.

7. Evaluate using theorems on limits. In each case, state precisely which theorems are used. (a) $\lim_{z\to 2i}(iz^4+3z^2-10i)$, (c) $\lim_{z\to i/2}\frac{(2z-3)(4z+i)}{(iz-1)^2}$, (b) $\lim_{z\to e^{\pi/4}}\frac{z^2}{z^4+z+1}$, (d) $\lim_{z\to i}\frac{z^2+1}{z^6+1}$, (e) $\lim_{z\to 1+i}\left\{\frac{z-1-i}{z^2-2z+2}\right\}^2$

8. Find $\lim_{z\to e^{\pi i/3}}\left(z-e^{\pi i/3}\right)\left(\frac{z}{z^3+1}\right)$.

9. Let $f(z)=\frac{z^2+4}{z-2i}$ if $z\neq 2i$, while $f(2i)=3+4i$. (a) Prove that $\lim_{z\to i}f(z)$ exists and determine its value. (b) Is $f(z)$ continuous at $z=2i$? Explain. (c) Is $f(z)$ continuous at points $z\neq 2i$? Explain.

10. Find all points of discontinuity for the following functions. (a) $f(z)=\frac{2z-3}{z^2+2z+2}$, (b) $f(z)=\frac{3z^2+4}{z^4-16}$ (c) $f(z)=\cot z$, (d) $f(z)=\frac{1}{z}-\sec z$

Best wishes

Answer to the question no 1

1(a)

$z = 2 - 2i$

$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

$\theta = -\tan^{-1}\left|\frac{-2}{2}\right| + 2n\pi$

$= -\frac{\pi}{4} + 2n\pi$

$\therefore$ in polar form,

$$re^{i\theta} = 2\sqrt{2}\; e^{i\left(-\frac{\pi}{4} + 2n\pi\right)}$$

1(b)

$z = -1 + \sqrt{3}\, i$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| + 2n\pi$

$= \pi - \frac{\pi}{3} + 2n\pi$

$= \frac{2\pi}{3} + 2n\pi$

in polar form,

$re^{i\theta} \Rightarrow 2\, e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}$

1(c)

$$z = 2\sqrt{2} + 2\sqrt{2}\, i$$

$$r = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$$

$$\theta = \tan^{-1}\left|\frac{2\sqrt{2}}{2\sqrt{2}}\right| + 2n\pi$$

$$= \frac{\pi}{4} + 2n\pi$$

in polar form,

$$re^{i\theta} = 4e^{i\left(\frac{\pi}{4} + 2n\pi\right)}$$

1(d)

$$z = -i = 0 - i$$

$$r = \sqrt{0^2 + (-1)^2} = 1$$

$$\theta = -\tan^{-1}\left|\frac{-1}{0}\right| + 2n\pi$$

$$= -\frac{\pi}{2} + 2n\pi$$

in polar form,

$$re^{i\theta} = e^{i\left(-\frac{\pi}{2} + 2n\pi\right)}$$

1(e)

$z = -4 \quad = -4 + 0i$

$r = \sqrt{(-4)^2 + 0^2} \quad = 4$

$\theta = \pi - \tan^{-1}\left|\frac{0}{-4}\right| + 2n\pi$

$= \pi + 2n\pi$

in polar form,

$re^{i\theta} \Rightarrow 4e^{i(\pi + 2n\pi)}$

1(f)

$$z = -2\sqrt{3} - 2i$$

$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = 4$$

$$\theta = -\pi + \tan^{-1}\left|\frac{-2}{-2\sqrt{3}}\right| + 2n\pi$$

$$= -\pi + \frac{\pi}{6} + 2n\pi$$

$$= \frac{-5\pi}{6} + 2n\pi$$

in polar form,

$$re^{i\theta} \Rightarrow 4e^{i\left(\frac{-5\pi}{6} + 2n\pi\right)}$$

1(g)

$z = \sqrt{2}\, i$

$r = \sqrt{0^2 + (\sqrt{2})^2} = \sqrt{2}$

$\theta = \tan^{-1}\left|\frac{\sqrt{2}}{0}\right| + 2n\pi$

$= \frac{\pi}{2} + 2n\pi$

in polar form,

$re^{i\theta} \Rightarrow \sqrt{2}\, e^{i\left(\frac{\pi}{2} + 2n\pi\right)}$

1 (h)

$$z = \frac{\sqrt{3}}{2} - \frac{3}{2} i$$

$$r = \sqrt{(\sqrt{3}/2)^2 + (-3/2)^2} = \sqrt{3}$$

$$\theta = -\tan^{-1}\left(\frac{3/2}{\sqrt{3}/2}\right) + 2n\pi$$

$$= \frac{-\pi}{3} + 2n\pi$$

in polar form,

$$re^{i\theta} \Rightarrow \sqrt{3}\, e^{i\left(\frac{-\pi}{3} + 2n\pi\right)}$$

Answer to the question no 2

2 (a)

$|z - i| = 2$

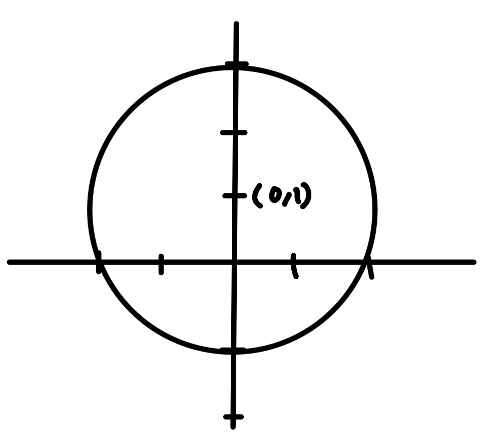
$\Rightarrow |x + iy - i| = 2$

$\Rightarrow |x + i(y-1)| = 2$

$\Rightarrow \sqrt{x^2 + (y-1)^2} = 2$

$\Rightarrow x^2 + (y-1)^2 = 2^2$

$\therefore$ center $(0, 1)$ and radius $= 2$



2 (b)

$$|z+2i| + |z-2i| = 6$$

$$\Rightarrow |x+iy+2i| + |x+iy-2i| = 6$$

$$\Rightarrow \sqrt{x^2+(y+2)^2} + \sqrt{x^2+(y-2)^2} = 6$$

$$\Rightarrow \sqrt{x^2+(y+2)^2} = 6 - \sqrt{x^2+(y-2)^2}$$

$$\Rightarrow x^2+(y+2)^2 = \left(6 - \sqrt{x^2+(y-2)^2}\right)^2$$

$$\Rightarrow x^2+y^2+4y+4 = 36 - 12\sqrt{x^2+(y-2)^2} + x^2+(y-2)^2$$

$$\Rightarrow y^2+4y = 32 - 12\sqrt{x^2+(y-2)^2} + y^2-4y+4$$

$$\Rightarrow 0 = 36 - 12\sqrt{x^2+(y-2)^2} - 8y$$

$$\Rightarrow -12\sqrt{x^2+(y-2)^2} = 8y-36$$

$$\Rightarrow 144\{x^2+(y-2)^2\} = (8y-36)^2$$

$$\Rightarrow 144x^2 + 144(y^2-4y+4) = 64y^2 - 2\times 8y\times 36 + 36^2$$

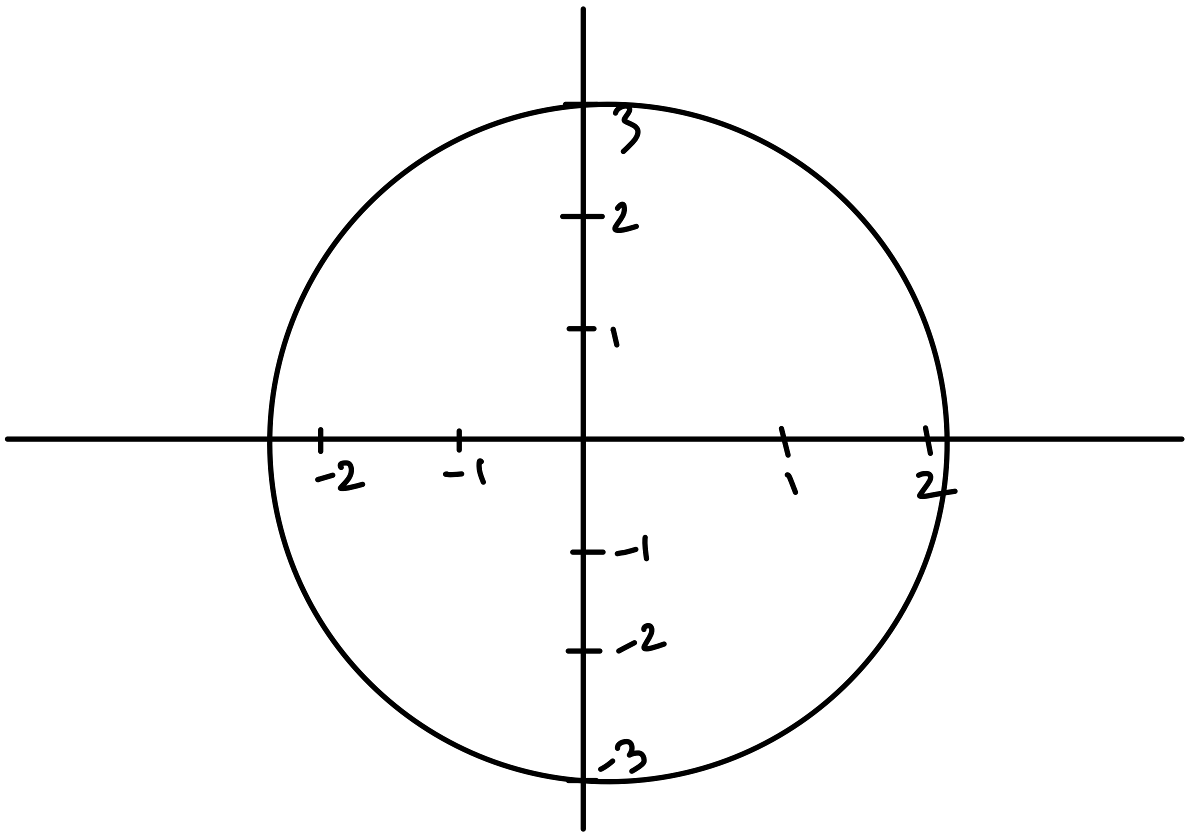
$$\Rightarrow 144x^2 + 144y^2 - 576y + 576 = 64y^2 - 576y + 1296$$

$$\Rightarrow 144x^2 + 80y^2 = 720$$

$$\Rightarrow \frac{x^2}{720/144} + \frac{y^2}{720/80} = 1$$

$$\Rightarrow \frac{x^2}{(\sqrt{5})^2} + \frac{y^2}{3^2} = 1$$

center is at (0, 0)



2(c)

$$|z-3| - |z+3| = 4$$

$$\Rightarrow |x+iy-3| - |x+iy+3| = 4$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} - \sqrt{(x+3)^2+y^2} = 4$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 4 + \sqrt{(x+3)^2+y^2}$$

$$\Rightarrow (x-3)^2+y^2 = \left\{4+\sqrt{(x+3)^2+y^2}\right\}^2$$

$$\Rightarrow x^2-6x+9+y^2 = 16+8\sqrt{(x+3)^2+y^2}+(x+3)^2+y^2$$

$$\Rightarrow \cancel{x^2}-6x+\cancel{9}+\cancel{y^2} = 16+8\sqrt{(x+3)^2+y^2}+\cancel{x^2}+6x+\cancel{9}+\cancel{y^2}$$

$$\Rightarrow 8\sqrt{(x+3)^2+y^2} = -16-12x$$

$$\Rightarrow 64\left\{(x+3)^2+y^2\right\} = 16^2+2\times16\times12x+144x^2$$

$$\Rightarrow 64(x^2+6x+9+y^2) = 256 + 384x + 144x^2$$

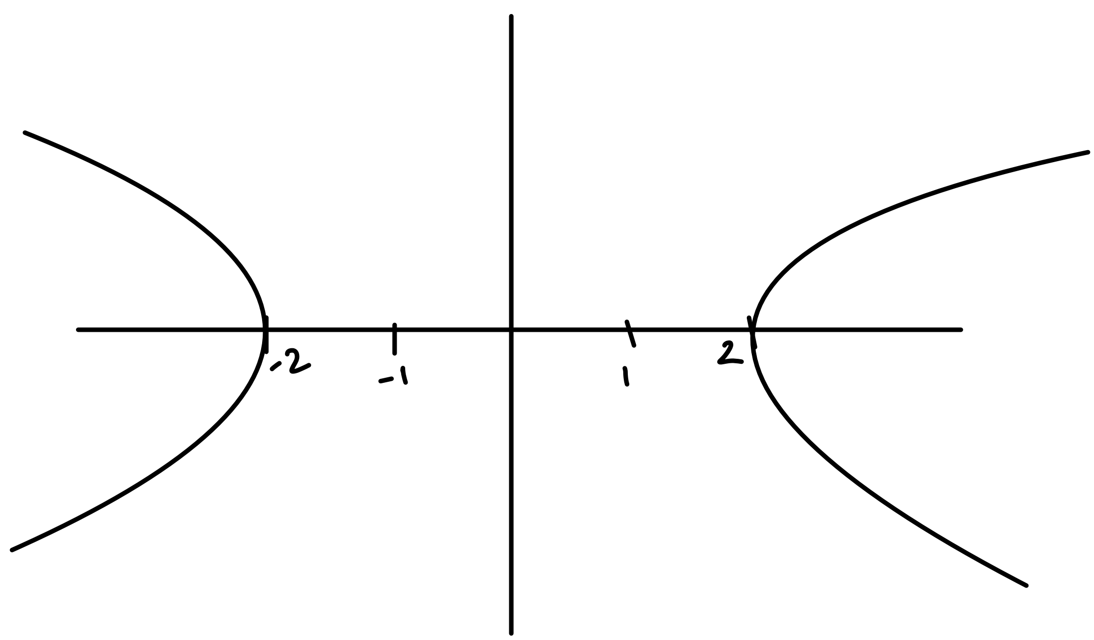
$$\Rightarrow 64x^2 + 384x + 576 + 64y^2 = 256 + 384x + 144x^2$$

$$\Rightarrow -80x^2 + 320 + 64y^2 = 0$$

$$\Rightarrow 10x^2 - 8y^2 = 40$$

$$\Rightarrow \frac{x^2}{2^2} - \frac{y^2}{(\sqrt{5})^2} = 1$$

$\therefore$ center of the hyperbola is at $(2, \sqrt{5})$



2 (d)

$$z(\bar{z}+2) = 3$$

$$\Rightarrow (x+iy)(x-iy+2) = 3$$

$$\Rightarrow x^2 - ixy + 2x + ixy - i^2y^2 + i\,2y = 3$$

$$\Rightarrow x^2 + y^2 + 2x + 2yi = 3$$

$$\Rightarrow x^2 + 2x + y^2 + 2yi = 3$$

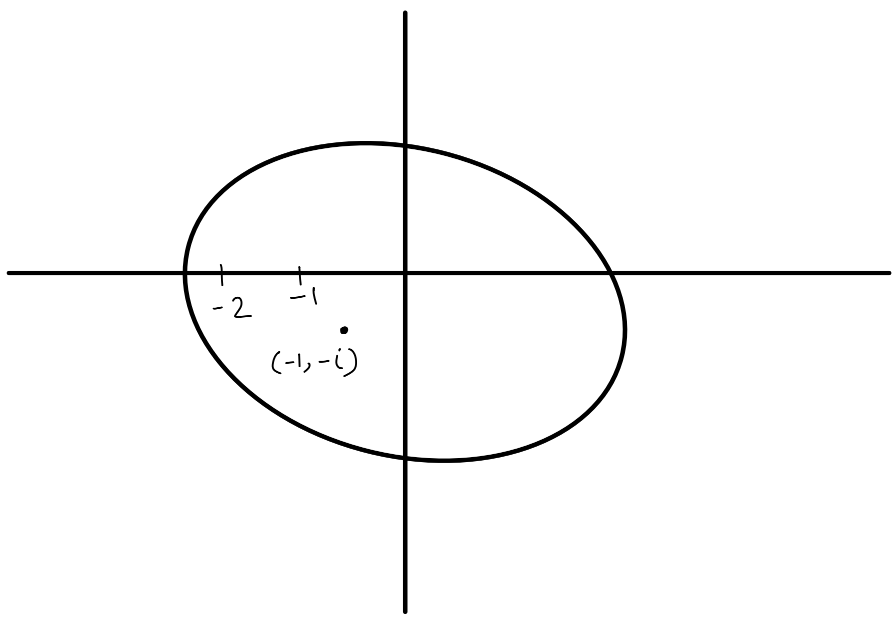
$$\Rightarrow (x^2+2x+1) - 1 + (y^2 + 2yi + i^2) + 1 = 3$$

$$\Rightarrow (x+1)^2 + (y+i)^2 = 3$$

here center $(-1, -i)$ and center $= 3$

here we have a complex plane. where x is real part,

y axis represents imaginary part



2 (e)

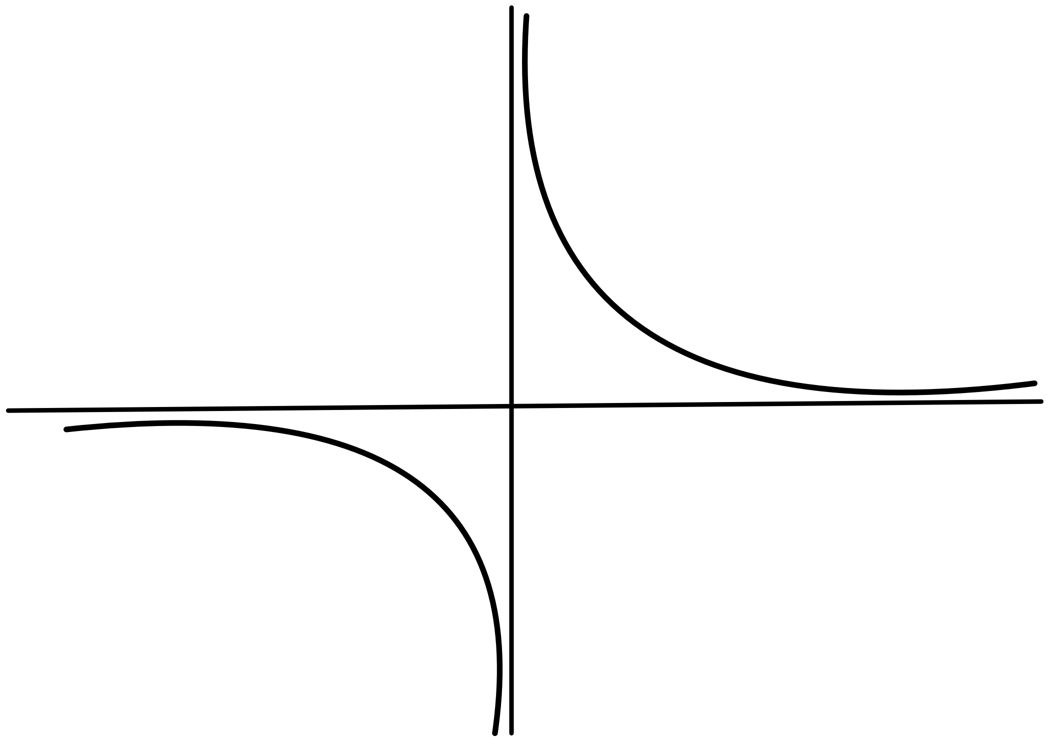
$$\operatorname{Im}\{z^2\} = 4$$

$$z^2 = (x+iy)^2 = x^2 + 2xyi + i^2y^2$$
$$= x^2 - y^2 + 2xyi$$

$$\Rightarrow \quad 2xy = 4$$

$$\Rightarrow xy = 2$$

$$\Rightarrow y = \frac{x}{2}$$



the locus is a hyperbola with a center at (0,0)

$$\boxed{\text{Answer to the question no 3}}$$

3(a)

$1 < |z + i| \leq 2$

$1 < |x + iy + i| \leq 2$

$1 < \sqrt{x^2 + (y+1)^2} \leq 2$

$1 < (x-0)^2 + (y+1)^2 \leq 2^2$

let $(x, y) = (0, -1)$

$1 < 0 + 0$ not true

but $0 + 0 \leq 2^2$ true

$\therefore$ the inner part of the inner circle is not included

let $(x, y) = (0, 0.5)$

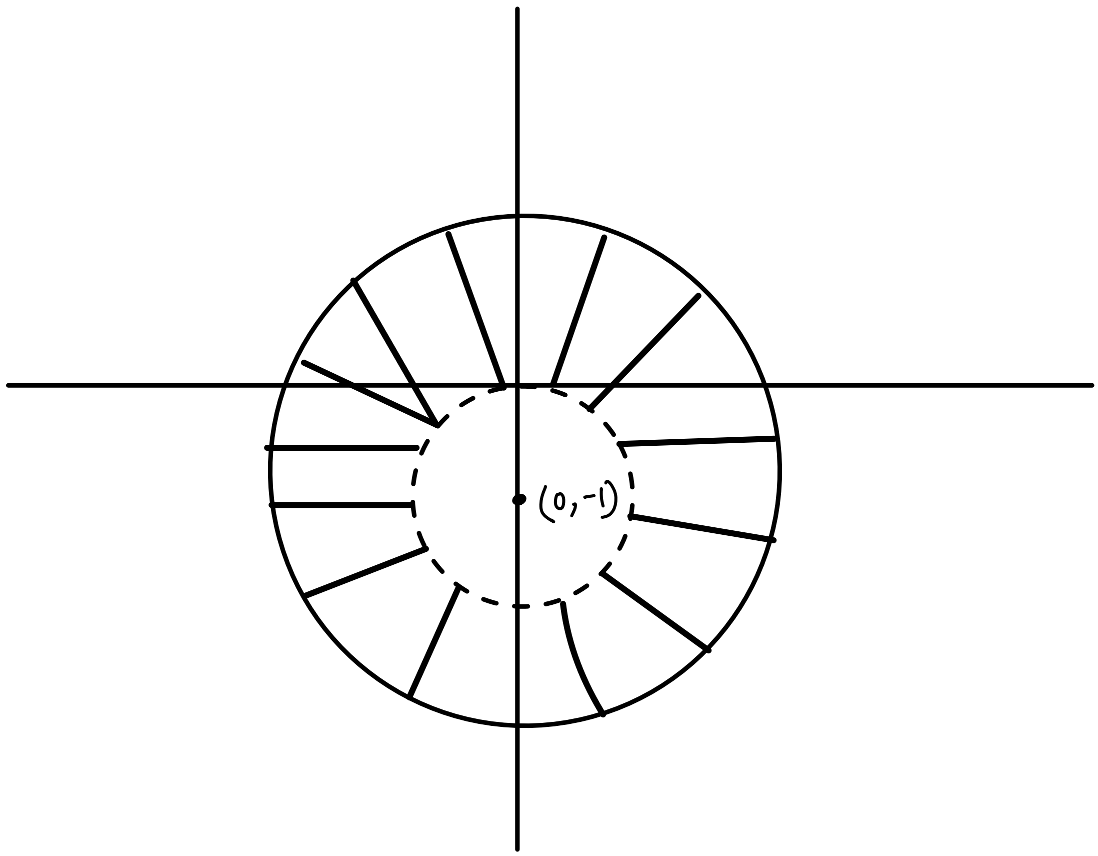
$$1 < 0^2 + 0.5^2 \leq 2^2$$

$\therefore$ the inner part of the outer circle is included.

Let $(x, y) = (10, 10)$

$\therefore$ $1^2 < 10^2 + 10^2 \leq 2^2$; not true

so the outer part of the outer circle is not included


(0, -1)

3(b)

$Re\{z^2\} > 1$

$z^2 = x^2 - y^2 + 2xy2$

$\therefore x^2 - y^2 > 1$

its a hyperbola with center at $(0,0)$

let $(x,y) = (0,0)$

$0^2 - 0^2 > 1$; not true

let $(x,y) = (10,5)$

$\therefore 10^2 - 5^2 > 0$; true. $\therefore$ the area covered

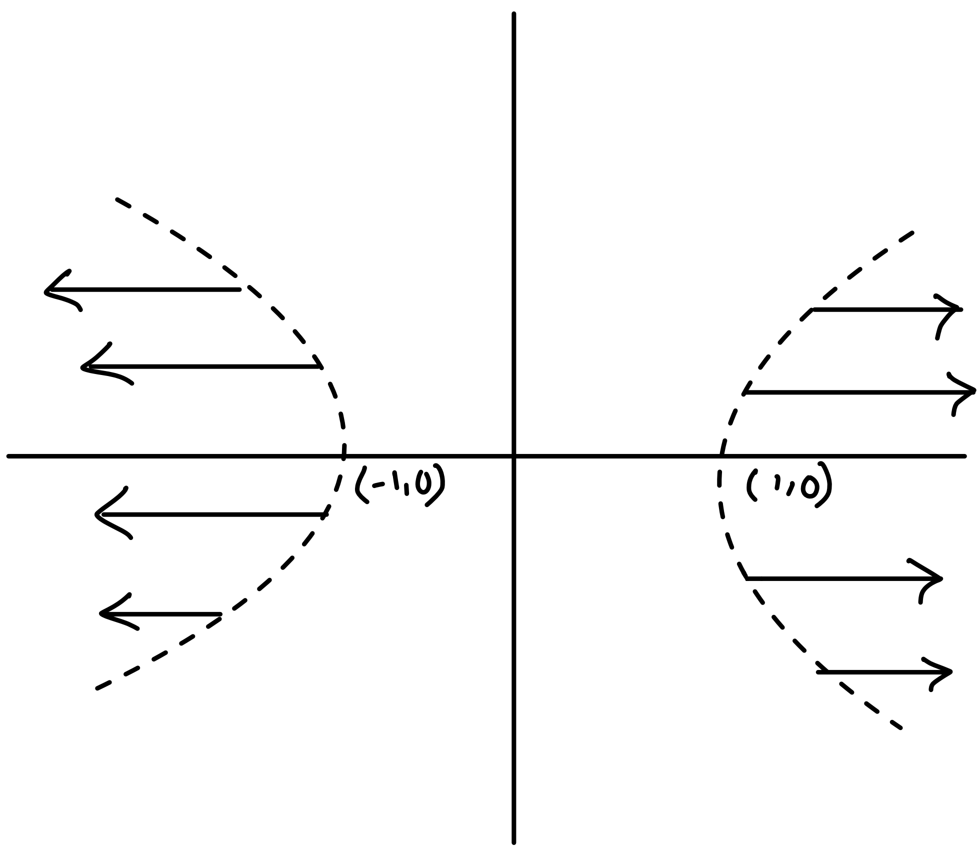
in the right half is included

let $(x,y) = (-1,0)$

$(-1)^2 - 0^2 > 1$; not true

let $(x,y) = (-10,-5)$

$(-10)^2 - (-5)^2 > 1$; true. $\therefore$ area covered by the left portion is included



3(c)

$|z+3i| > 4$

$|x+iy+3i| > 4$

$\Rightarrow \sqrt{x^2+(y+3)^2} > 4$

$\Rightarrow x^2+(y+3)^2 > 4^2$

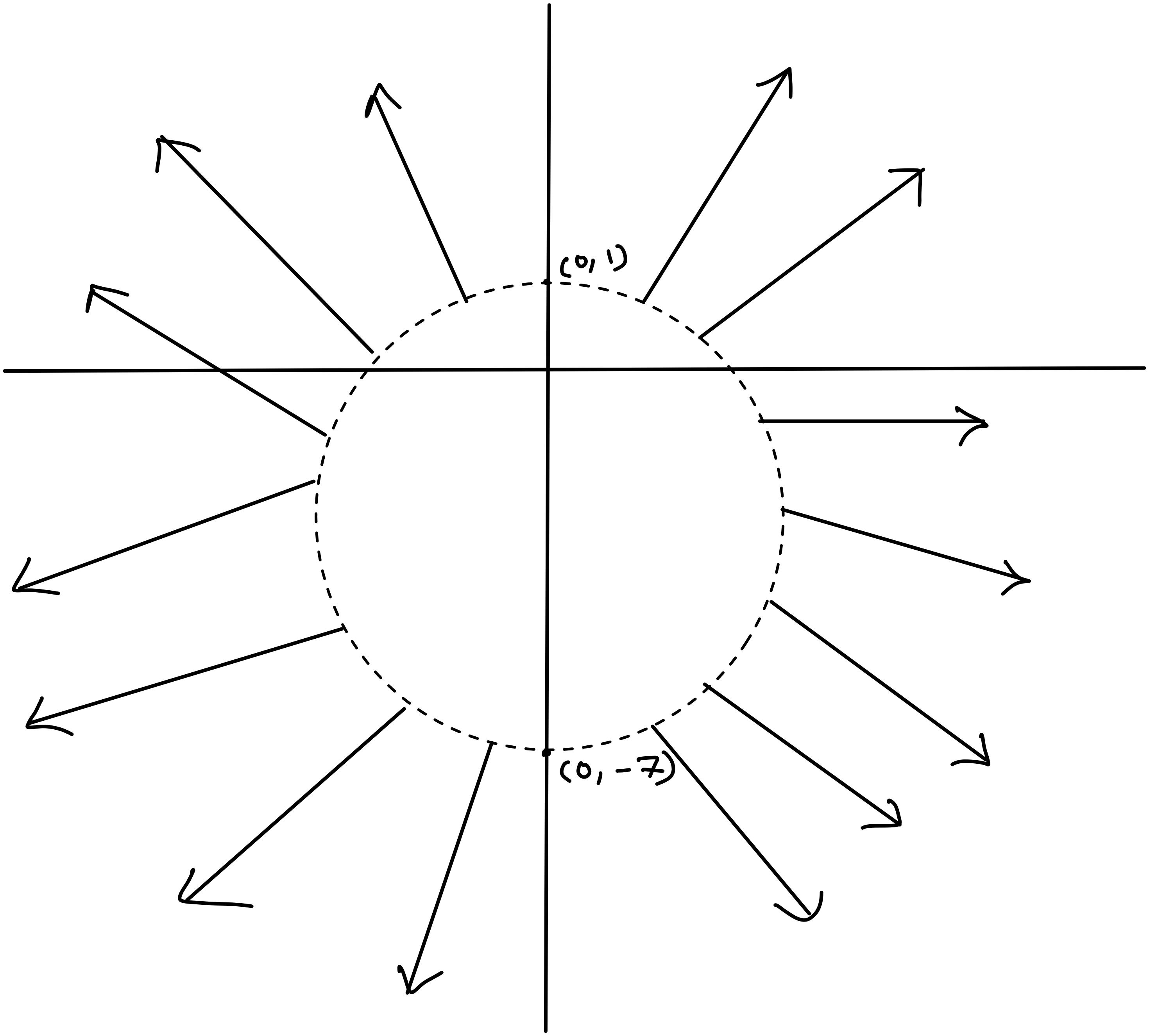
center at $(0,-3)$ radius 4

let $(x,y) = (0,-2)$

$0^2+(-2+3)^2 > 4^2$; not true

let $(x,y) = (10,10)$

$10^2+(10+3)^2 > 4$; $\therefore$ outer part of circle is included


(0, 1)
(0, -7)

3 (d)

$$|z+2-3i| + |z-2+3i| < 10$$

$$|(x+2) + (iy-3i)| + |(x-2) + i(y+3)| < 10$$

$$\Rightarrow \sqrt{(x+2)^2 + (y-3)^2} + \sqrt{(x-2)^2 + (y+3)^2} < 10$$

$$\Rightarrow (x+2)^2 + (y-3)^2 < \left\{10 - \sqrt{(x-2)^2 + (y+3)^2}\right\}^2$$

$$\Rightarrow \cancel{x^2} + 4x + \cancel{4} + \cancel{y^2} - 6y + \cancel{9} < 100 - 20\sqrt{(x-2)^2 + (y+3)^2} + \cancel{x^2} - 4x + \cancel{4} + \cancel{y^2} + 6y + \cancel{9}$$

$$\Rightarrow 8x - 12y - 100 < -20\sqrt{(x-2)^2 + (y+3)^2}$$

$$\Rightarrow 2x - 3y - 25 < -5\sqrt{(x-2)^2 + (y+3)^2}$$

$$\Rightarrow -2x + 3y + 25 > 5\sqrt{(x-2)^2 + (y+3)^2}$$

$$\Rightarrow (-2x+3y+25)^2 > 25\{x^2-4x+4+y^2+6y+9\}$$

$$\Rightarrow \{-1(2x-3y-25)\}^2 > 25\{x^2-4x+4+y^2+6y+9\}$$

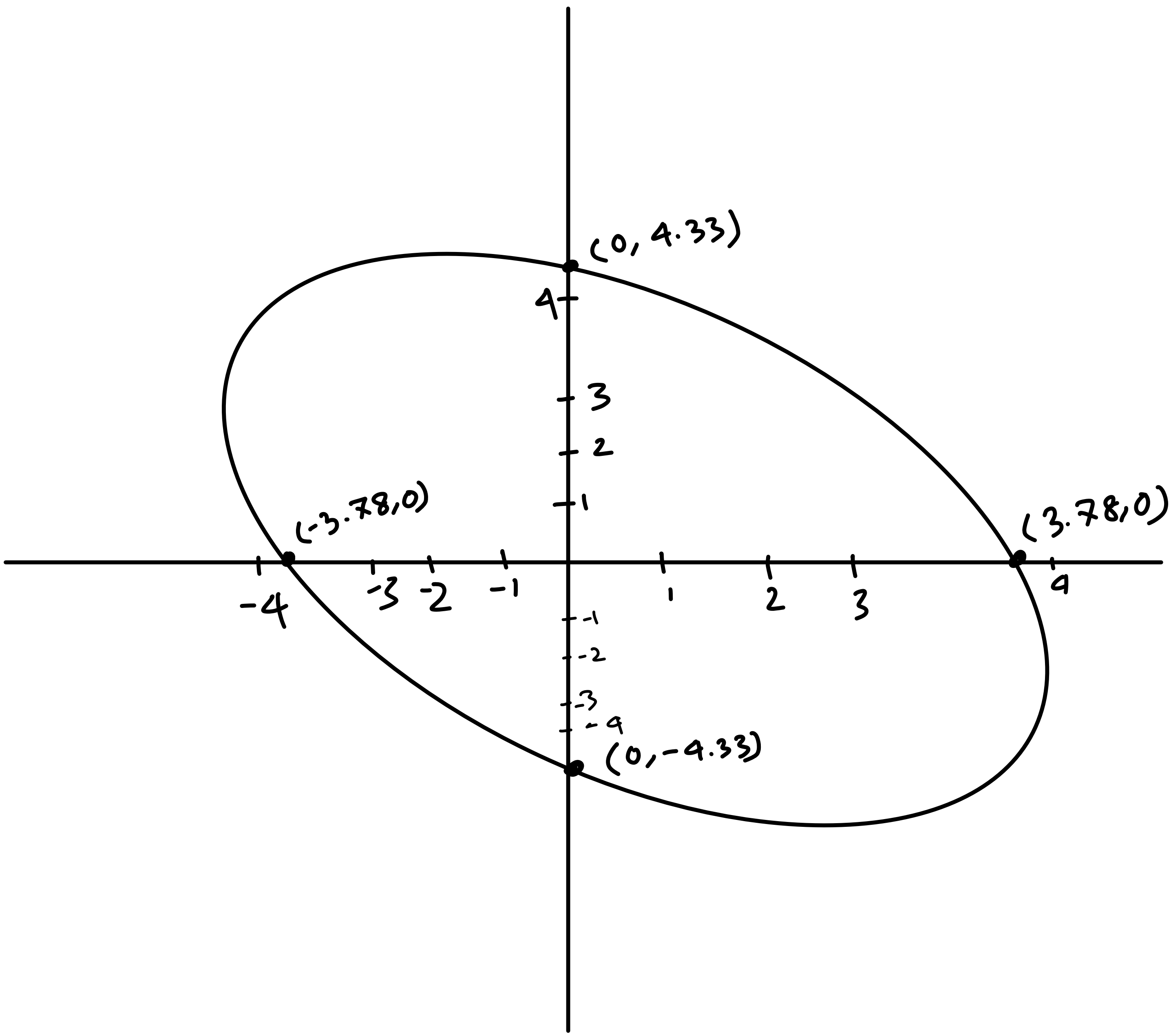
$$\Rightarrow (2x-3y-25)^2 > 25\{x^2-4x+4+y^2+6y+9\}$$

$$\Rightarrow 4x^2+9y^2+625+2(2x)(-3y)+2(-3y)(-25)$$
$$+2(2x)(-25) > 25x^2-100x+100+25y^2$$
$$+150y+225$$

$$\Rightarrow -21x^2-16y^2+300-12xy > 0$$

$$\Rightarrow -21x^2-16y^2-12xy > -300$$

$$\Rightarrow 21x^2+16y^2+12xy < 300$$


(0, 4.33)
(-3.78,0)
(3.78,0)
(0,-4.33)
4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
1
2
3
4

Answer to the question no 4

4(a)

$$(2\sqrt{3} - 2i)^{1/2}$$

$$z = 2\sqrt{3} - 2i$$

$$r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$$

$$\theta = -\tan^{-1}\left|\frac{-2}{2\sqrt{3}}\right| + 2n\pi$$

$$= -\frac{\pi}{6} + 2n\pi$$

$$\therefore z = \left(4\, e^{i\left(-\frac{\pi}{6} + 2n\pi\right)}\right)^{1/2}$$

$$z^{1/2} = 2e^{i \times \frac{\left(-\frac{\pi}{6}+2n\pi\right)}{2}}$$

$$= 2e^{i\left(-\frac{\pi}{12}+n\pi\right)}$$

when $n=0$:

$$z = 2e^{i \times -\frac{\pi}{12}}$$

$$= 2\cos\left(-\frac{\pi}{12}\right) + i\,2\sin\left(-\frac{\pi}{12}\right)$$

$$= 1.932 - 0.517i$$

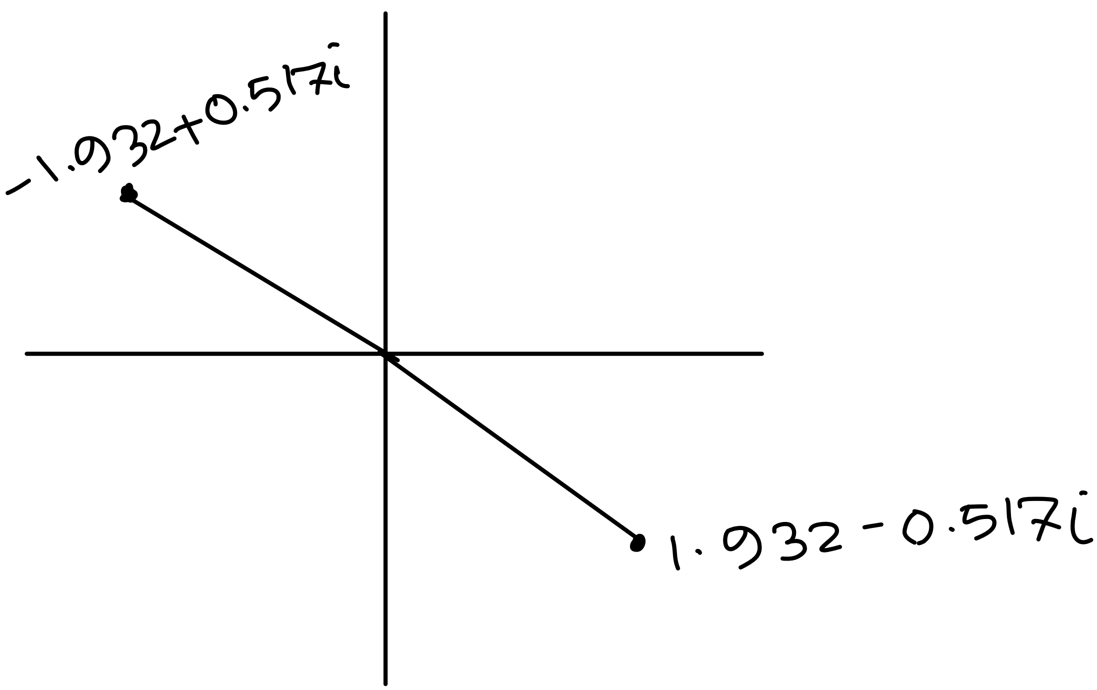
when $n=1$

$$z = 2e^{i\left(-\frac{\pi}{12}+\pi\right)}$$

$$= 2e^{i\frac{11}{12}\pi}$$

$$z = 2\cos\left(\frac{11}{12}\pi\right) + i \times 2\sin\left(\frac{11}{12}\pi\right)$$

$$= -1.032 + 0.517i$$



4(b)

$$(-4 + 4i)^{1/5}$$

let, $z = -4 + 4i$

$$r = \sqrt{(-4)^2 + 4^2}$$

$$= 4\sqrt{2}$$

$$\theta = \pi - \tan^{-1}\left|\frac{4}{-4}\right| + 2n\pi$$

$$= \pi - \frac{\pi}{4} + 2n\pi$$

$$= \frac{3\pi}{4} + 2n\pi$$

$$\therefore \; z^{1/5} = \left\{ 4\sqrt{2} \; e^{i\left(\frac{3\pi}{4} + 2n\pi\right)} \right\}^{1/5}$$

$$= \sqrt{2} \; e^{i\left(\frac{3\pi}{20} + \frac{2n\pi}{5}\right)}$$

<u>when $n=0$:</u>

we set, $\sqrt{2} \; e^{i\left(\frac{3\pi}{20}\right)}$

$$= \sqrt{2}\cos\left(\frac{3\pi}{20}\right) + i \times \sqrt{2}\sin\left(\frac{3\pi}{20}\right)$$

$$= 1.26 + 0.64\,i$$

when $n=1$:

$$\sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{2\pi}{5}\right)}$$

$$= \sqrt{2}\, e^{i \times \frac{11\pi}{20}}$$

$$= \sqrt{2}\cos\left(\frac{11\pi}{20}\right) + \sqrt{2}\, i \sin\left(\frac{11\pi}{20}\right)$$

$$= -0.22 + 1.39\, i$$

when $n=2$

$$\sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{4\pi}{5}\right)}$$

$$\Rightarrow \sqrt{2}\cos\left(\frac{19\pi}{20}\right) + i\sqrt{2}\sin\left(\frac{19\pi}{20}\right)$$

$= -1.39 + 0.22i$

when $n=3$:

$$\sqrt{2}\, e^{i\left(\frac{3\pi}{20} + \frac{6\pi}{5}\right)}$$

$= \sqrt{2} \cos(1.35\pi) + i\sqrt{2} \sin(1.35\pi)$

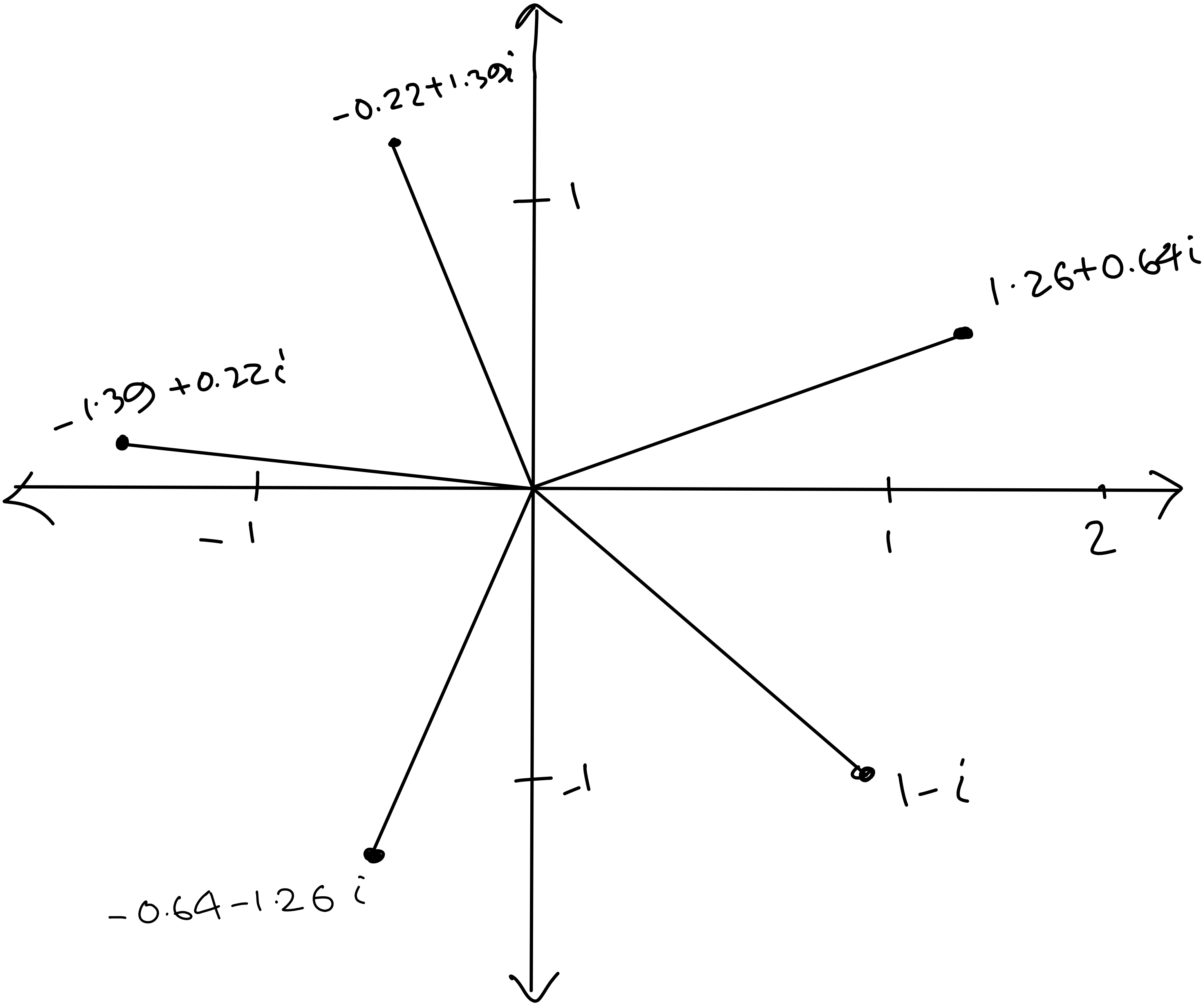
$= -0.64 - 1.26i$

when $n=4$:

$$\sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{8\pi}{5}\right)}$$

$$= \sqrt{2}\cos(1.75\pi) + i\sqrt{2}\sin(1.75\pi)$$

$$= 1-i$$



-0.22+1.39i
1
1.26+0.64i
-1.39+0.22i
-1
1
2
-1
1-i
-0.64-1.26i

<u>4(c)</u>

$$z = (2+2\sqrt{3}i)^{1/3}$$

$$x = 2+2\sqrt{3}i$$

$$r = 4$$

$$\theta = \tan^{-1}(2\sqrt{3}/2) + 2n\pi$$

$$= \frac{\pi}{3} + 2n\pi$$

$$x = 4e^{i(\pi/3 + 2n\pi)}$$

$$z = \left\{4e^{i\left(\frac{\pi}{3}+2n\pi\right)}\right\}^{1/3}$$

$$= \left\{4^{\frac{1}{3}} e^{i\left(\frac{\pi}{9} + \frac{2n\pi}{3}\right)}\right\}$$

$$n=0 \Rightarrow z = 4^{1/3} e^{i(\pi/9)}$$

$$= 4^{1/3}\cos\left(\frac{\pi}{9}\right) + i\, 4^{1/3}\sin(\pi/9)$$

$$= 1.49 + 0.54i$$

$$n=1 \Rightarrow 4^{1/3} \times e^{i\left(\frac{\pi}{9} + \frac{2\pi}{3}\right)}$$

$$\Rightarrow 4^{1/3} e^{i\,7\pi/9}$$

$$\Rightarrow 4^{1/3}\cos\left(\frac{7\pi}{9}\right) + i\,4^{1/3}\sin(7\pi/9)$$

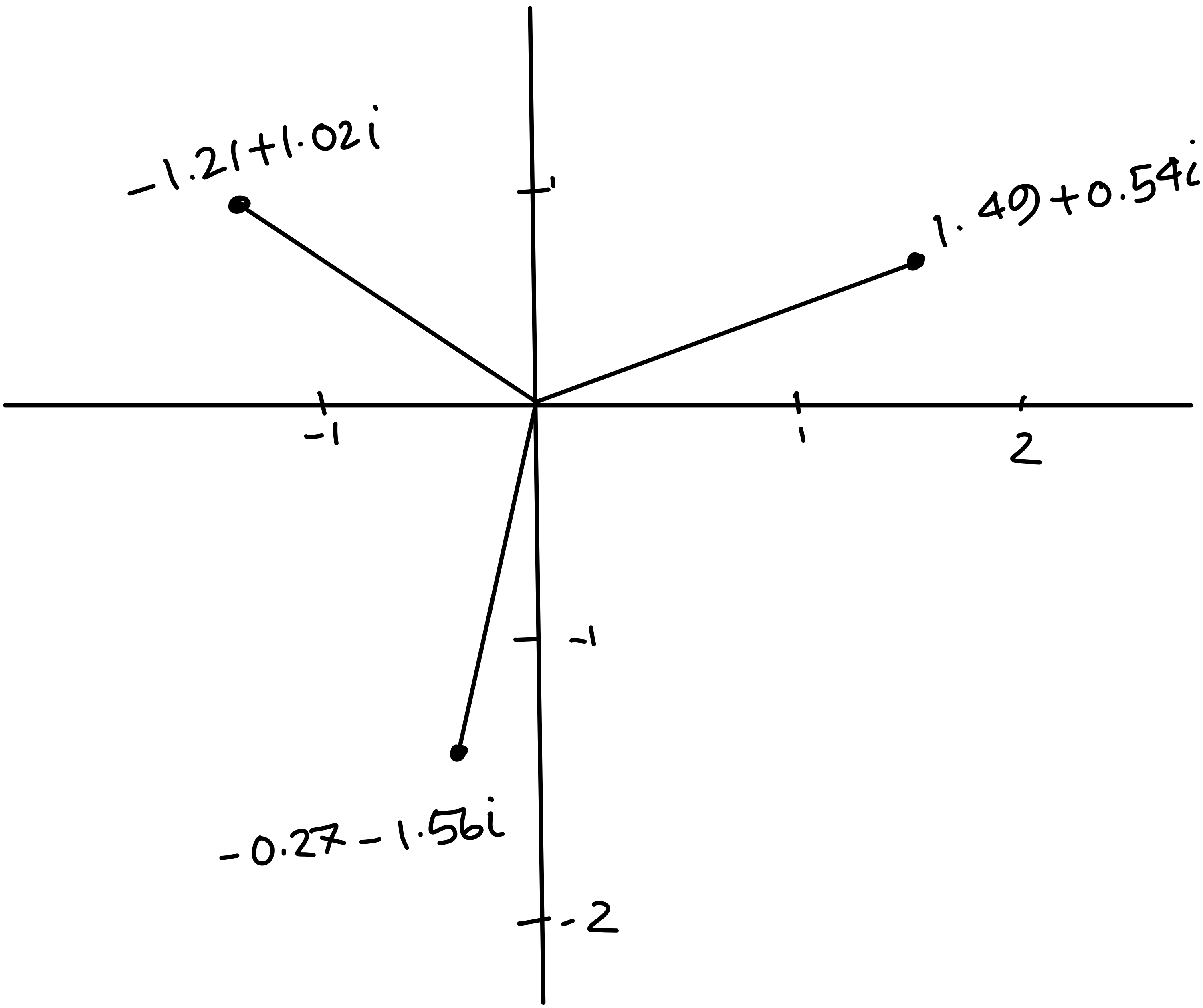
$$\Rightarrow -1.21 + 1.02i$$

$$n=2 \Rightarrow 4^{1/3} e^{i\left(\frac{\pi}{9} + \frac{4\pi}{3}\right)}$$

$$= 4^{1/3} e^{i\frac{13\pi}{9}}$$

$$= 4^{1/3}\cos\left(\frac{13\pi}{9}\right) + i\,4^{\frac{1}{3}}\sin\left(\frac{13\pi}{9}\right)$$

$$= -0.27 - 1.56i$$



4(d)

let $z = (-16i)^{1/4}$

let $n = -16i$

$\therefore r = \sqrt{(-16)^2 + 0^2} = 16$

$\theta = -\tan^{-1}\left|\frac{-16}{0}\right| + 2n\pi$

$= \frac{-\pi}{2} + 2n\pi$

$\therefore z = \left\{ 16\, e^{i\left(-\frac{\pi}{2} + 2n\pi\right)} \right\}^{1/4}$

$n = 0, 1, 2, 3$

when $n=0$:

$$z = 2 e^{i\left(-\frac{\pi}{8}\right)}$$

$$z = 2\cos\left(-\frac{\pi}{8}\right) + i2\sin\left(-\pi/8\right)$$

$$= 1.84 - 0.76i$$

when $n=1$:

$$z = 2 e^{i\left(-\frac{\pi}{2} + 2\pi\right)/4}$$

$$= 2 e^{i\frac{3\pi}{8}}$$

$$z = 2\cos\left(\frac{3\pi}{8}\right) + i\ 2\sin\left(\frac{3\pi}{8}\right)$$

$$= 0.76 + 1.84i$$

<u>when n=2:</u>

$$z = 2e^{i\left(-\frac{\pi}{2}+4\pi\right)/4}$$

$$z = 2e^{i\left(\frac{7\pi}{8}\right)}$$

$$z \Rightarrow 2\cos\left(\frac{7\pi}{8}\right) + i\,2\sin\left(\frac{7\pi}{8}\right)$$

$$= -1.84 + 0.76i$$

<u>when n=3:</u>

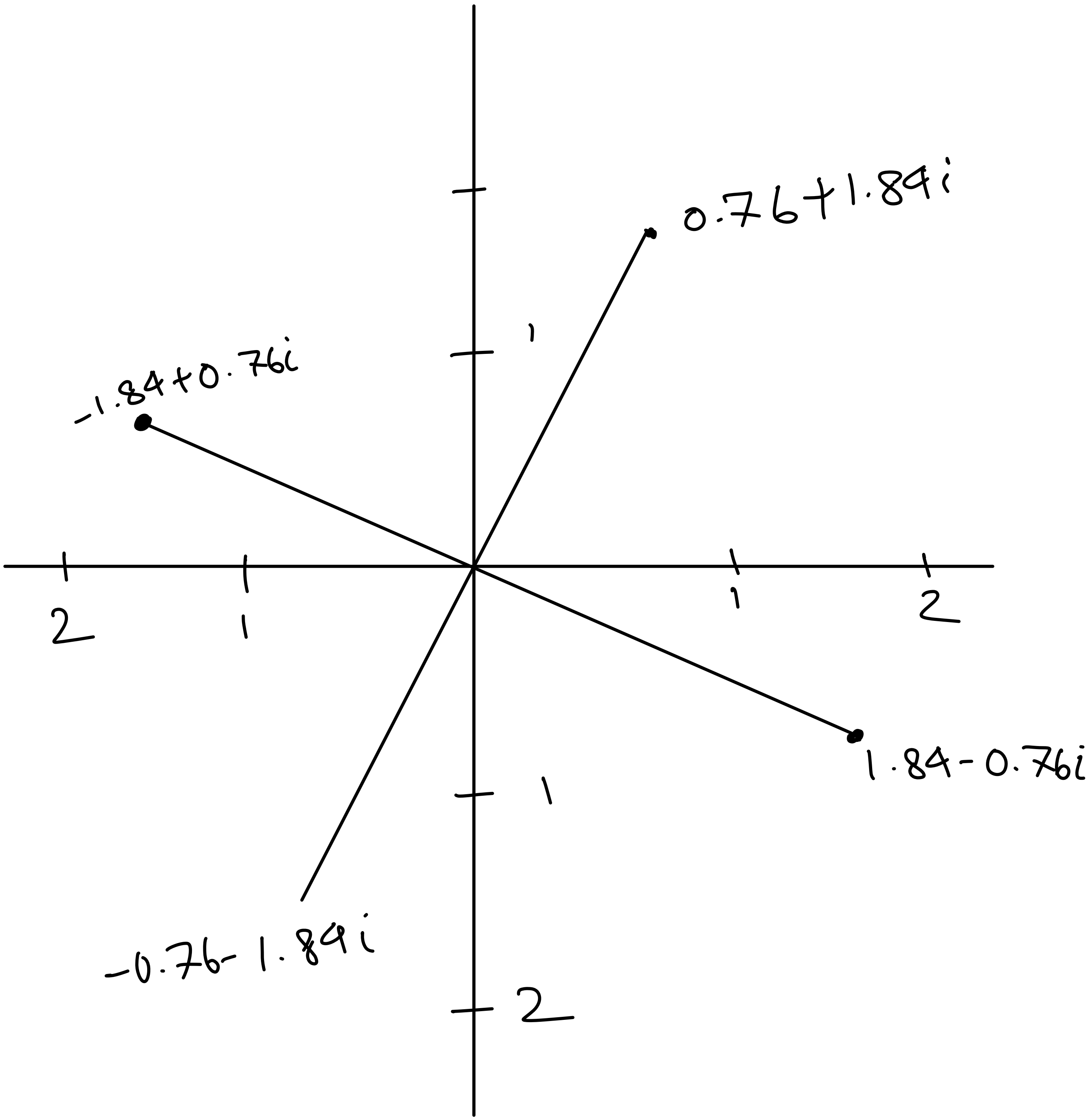
$$z = 2e^{i\left(-\frac{\pi}{2}+6\pi\right)/4}$$

$$= 2e^{i\left(\frac{11\pi}{8}\right)}$$

$$\Rightarrow 2\cos\left(\frac{11\pi}{8}\right) + i\,2\sin\left(\frac{11\pi}{8}\right)$$

$= -0.76 - 1.84i$

$= 0.76 + 1.84i$



4(e)

let $z = (64)^{1/6}$

$x = 64$

$r = \sqrt{64^2} = 64$

$\theta = \tan^{-1}\left(\frac{0}{64}\right) = 0 + 2n\pi = 2n\pi$

$x = 64\, e^{i2n\pi}$

$z = \left(64\, e^{i2n\pi}\right)^{1/6}$

$= 2\, e^{ix\, n\pi/3}$

$n = 0 \Rightarrow z = 2$

$n = 1 \Rightarrow 2e^{i \times \pi/3}$

$\Rightarrow 2\cos(\pi/3) + i2\sin(\pi/3)$

$\Rightarrow 1 + \sqrt{3}i$

$n = 2 \Rightarrow 2e^{i\,2\pi/3}$

$z = 2\cos(2\pi/3) + i\,2\sin(2\pi/3)$

$= -1 + \sqrt{3}\,i$

$n = 3 \Rightarrow 2e^{i\pi}$

$\Rightarrow 2\cos\pi + i2\sin\pi \Rightarrow -2 + 0i$

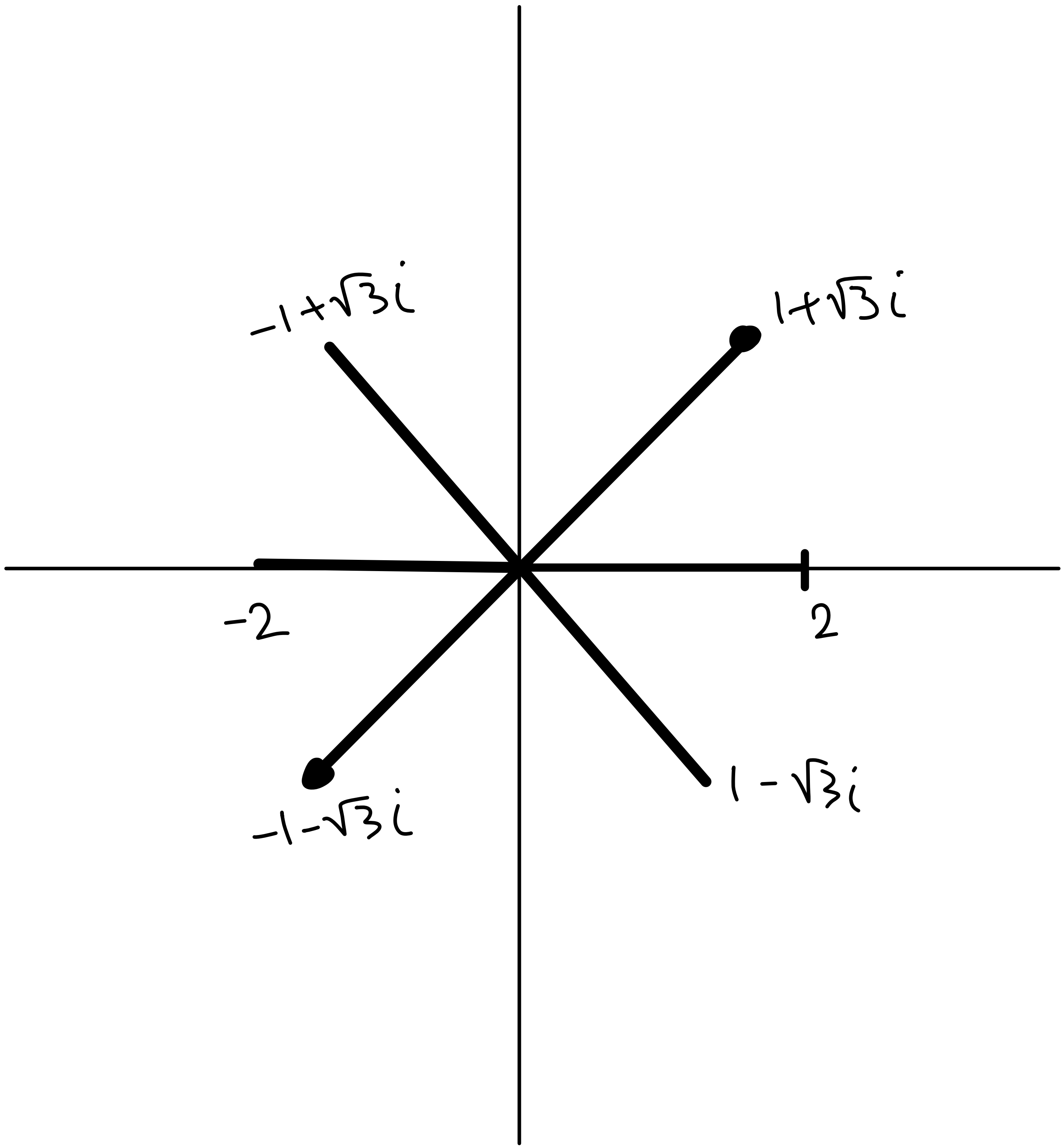
$n=4 \Rightarrow 2e^{i \times 4\pi/3}$

$= 2\cos\left(\frac{4\pi}{3}\right) + i\, 2\sin\left(4\pi/3\right)$

$= -1 - \sqrt{3}i$

$n=5 \Rightarrow 2e^{i \times 5\pi/3}$

$= 1 - \sqrt{3}i$


$-1+\sqrt{3}i$
$1+\sqrt{3}i$
-2
2
$-1-\sqrt{3}i$
$1-\sqrt{3}i$

4(f)

$$z = i^{2/3}$$

$$x = i$$

$$r = 1$$

$$\theta = \frac{\pi}{2} + 2n\pi$$

$$z = \left\{ e^{i\left(\frac{\pi}{2} + 2n\pi\right)} \right\}^{2/3}$$

$$= e^{i\left(3\pi + \frac{4}{3}n\pi\right)}$$

$$n = 0 \Rightarrow z = e^{i(3\pi)}$$

$$z = \cos 3\pi + i \sin(3\pi)$$

$$= -1$$

$$n=1 \Rightarrow e^{i\left(3\pi + \frac{4\pi}{3}\right)}$$

$$\Rightarrow \cos\left(\frac{13\pi}{3}\right) + i \sin\left(\frac{13\pi}{3}\right)$$

$$\Rightarrow \frac{1}{2} + i \frac{\sqrt{3}}{2} = 0.5 + 0.86i$$

$$n=2 \Rightarrow e^{i\left(3\pi + \frac{8}{3}\pi\right)}$$

$$\Rightarrow \cos\left(\frac{17\pi}{3}\right) + i \sin\left(\frac{17\pi}{3}\right)$$

$$\Rightarrow \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$= 0.5 - 0.86i$$

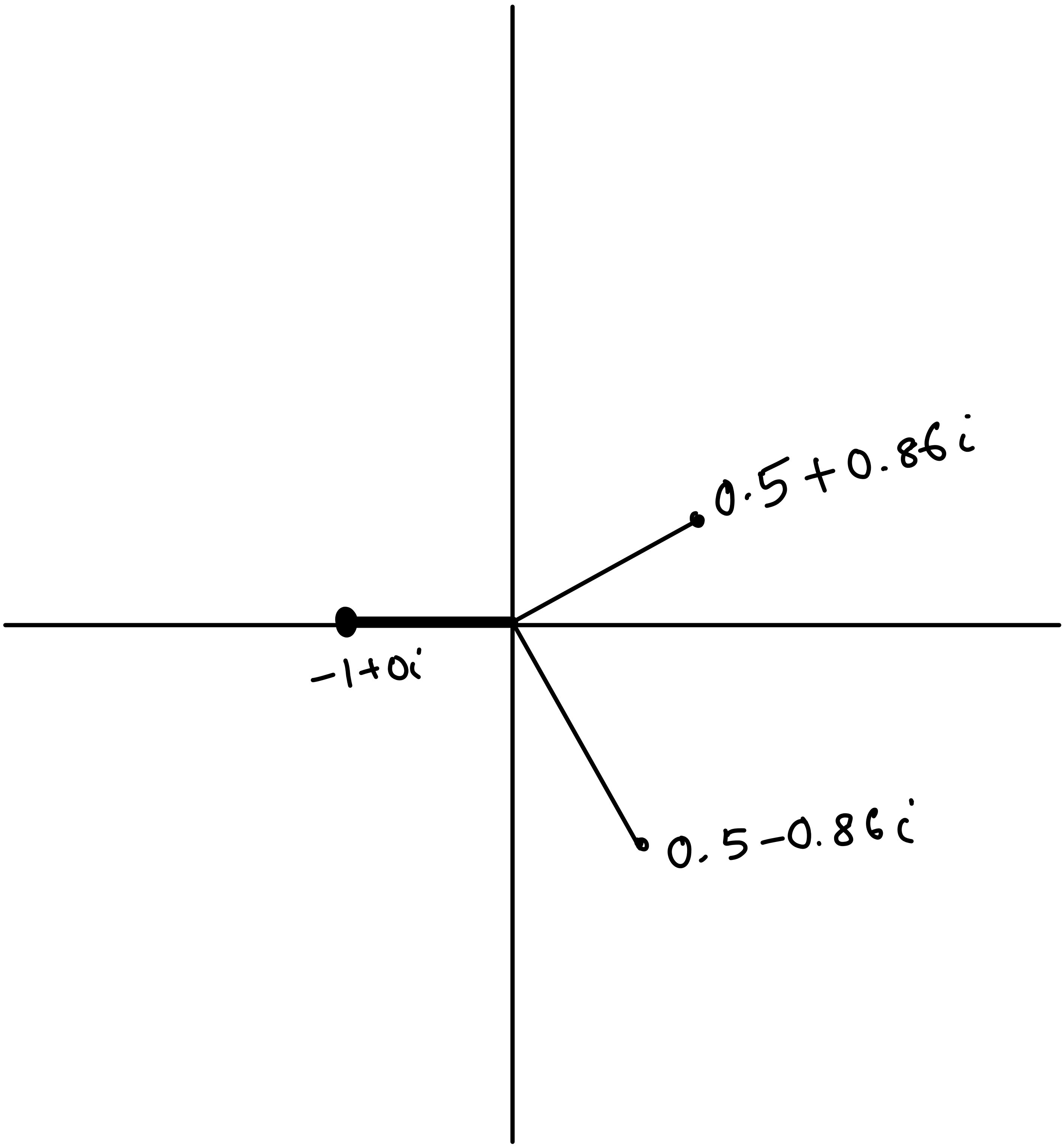
$n=3 \Rightarrow e^{i(7\pi)}$

$= \cos(7\pi) + i\sin(7\pi)$

$= -1$

and the pattern repeats from

this value of n



0.5+0.86i
-1+0i
0.5-0.86i

Answer to the question no 5

5(a)

$z = 8^{1/3}$

let $x = 8$

$r = 8$

$\theta = 2n\pi$

$x = 8e^{i(2n\pi)}$

$z = \{8e^{i(2n\pi)}\}^{1/3}$

$$z = 2e^{i(2n\pi)/3}$$

$$n=0 \Rightarrow 2e^{i \times 0}$$

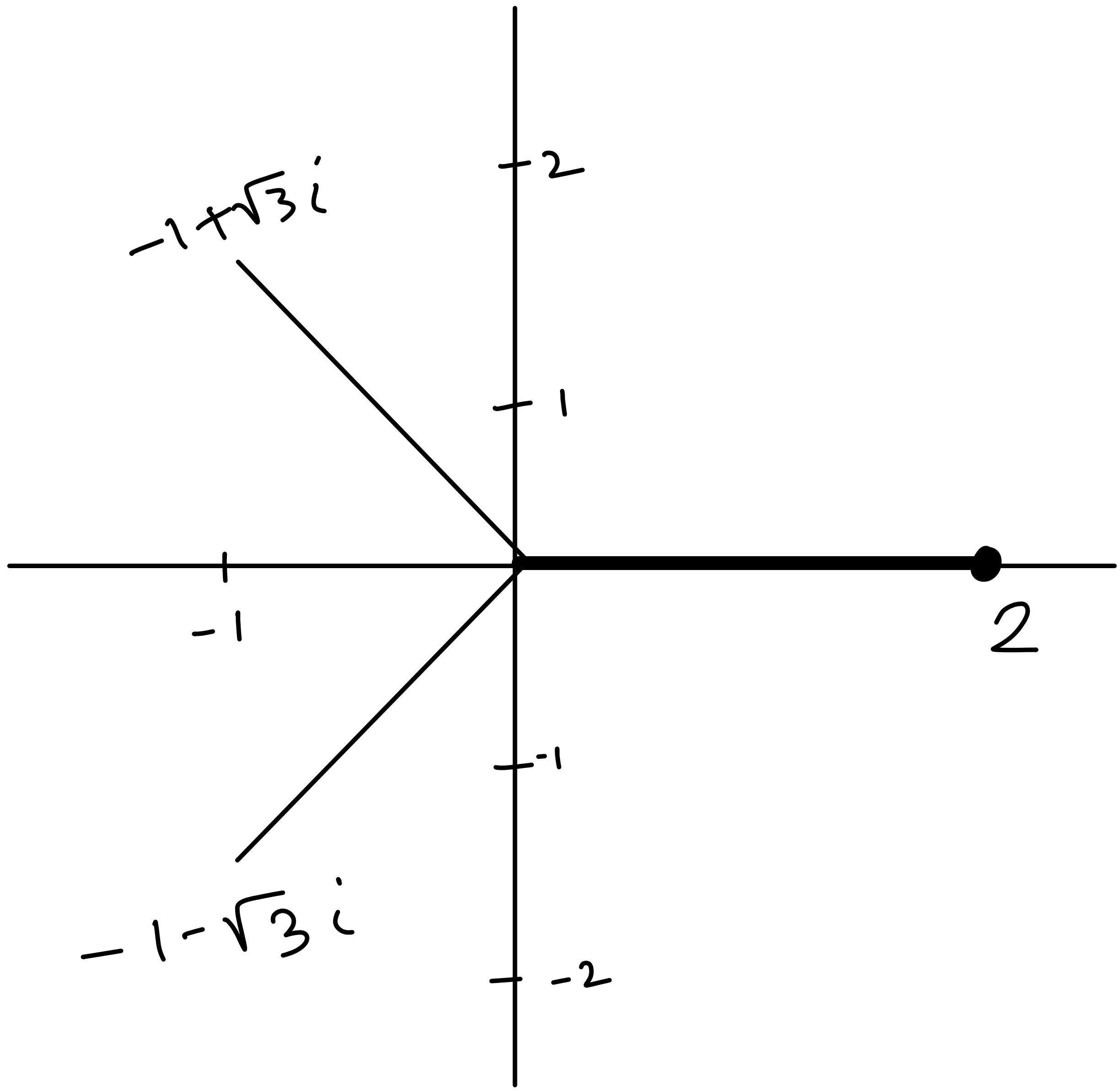
$$= 2$$

$$n=1 \Rightarrow 2e^{i \times 2\pi/3}$$

$$\Rightarrow -1 + \sqrt{3}i$$

$$n=2 \Rightarrow 2e^{i \times \frac{4\pi}{3}}$$

$$\Rightarrow -1 - \sqrt{3}i$$



$-1+\sqrt{3}i$
2
1
-1
2
-1
$-1-\sqrt{3}i$
-2

5(b)

$$z = \left\{4\sqrt{2} + 4\sqrt{2}\, i\right\}^{1/2}$$

$$r = 8$$

$$\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) + 2n\pi$$

$$= \frac{\pi}{4} + 2n\pi$$

$$\therefore z = \left\{8\, e^{i(\pi/4 + 2n\pi)}\right\}^{1/2}$$

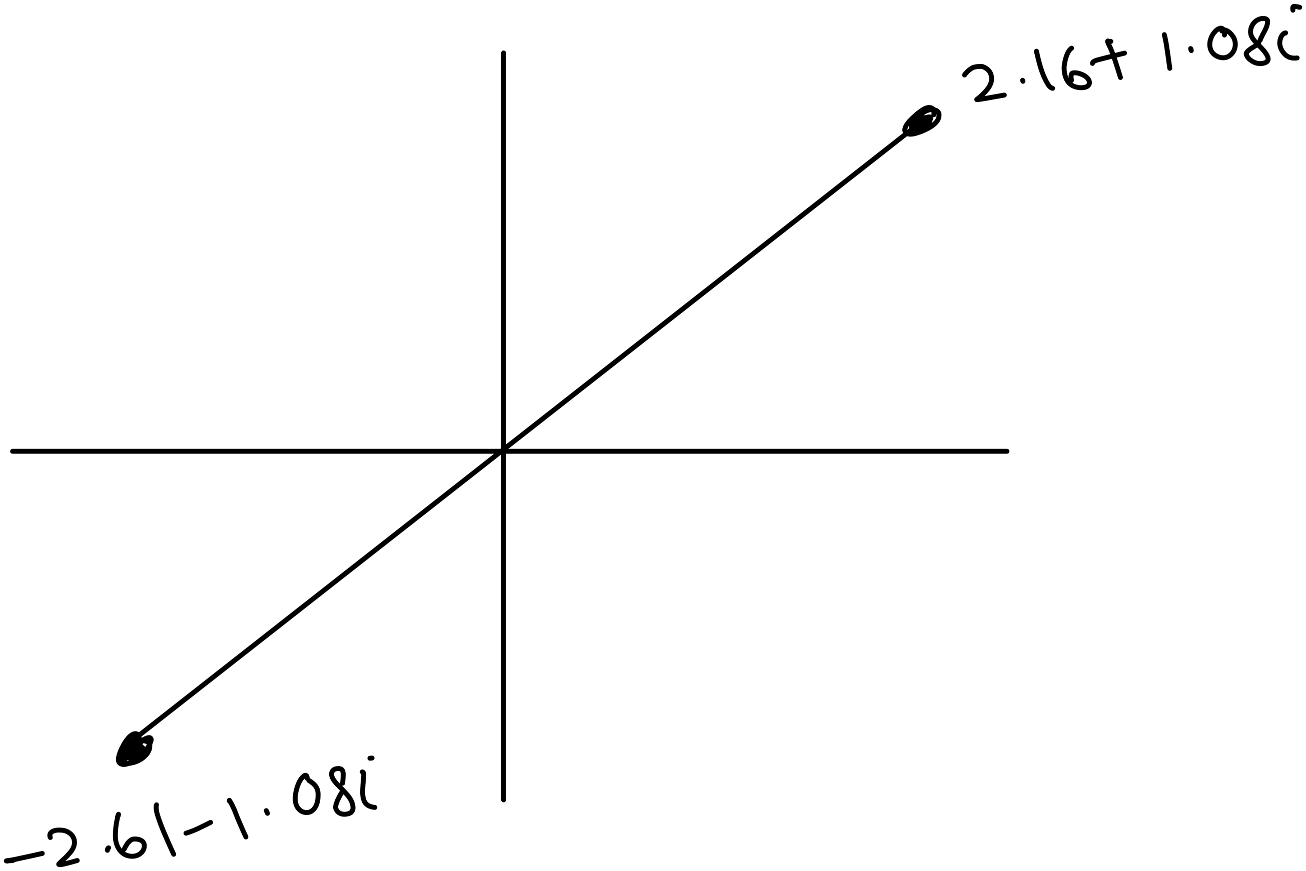
$$= 2\sqrt{2}\, e^{i\left(\frac{\pi}{8} + n\pi\right)}$$

$$n = 0 \Rightarrow z = 2\sqrt{2}\, e^{i\pi/18}$$

$$= 2.61 + 1.08i$$

$$n = 1 \Rightarrow z = 2\sqrt{2}\, e^{i(9\pi/18)}$$

$$= -2.61 - 1.08i$$



$\underline{5(c)}$

$$z = \{-16 + 16\sqrt{3}\, i\}^{1/5}$$

$$r = 32$$

$$\theta = \pi - \tan^{-1}\left|\frac{16\sqrt{3}}{-16}\right| + 2n\pi$$

$$= \frac{2\pi}{3} + 2n\pi$$

$$\therefore\ 32\, e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}$$

$$z = \left\{32\, e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}\right\}^{1/5}$$

$$= 2e^{i\left(\frac{2\pi}{15} + \frac{2n\pi}{5}\right)}$$

$$n = 0 \Rightarrow 2e^{i\,2\pi/15}$$

$$= 1.82 + 0.81i$$

$$n = 1 \Rightarrow 2e^{i\,(8\pi/15)}$$

$$= -0.21 + 1.98i$$

$$n = 2 \Rightarrow 2e^{i\left(\frac{14}{15}\pi\right)}$$

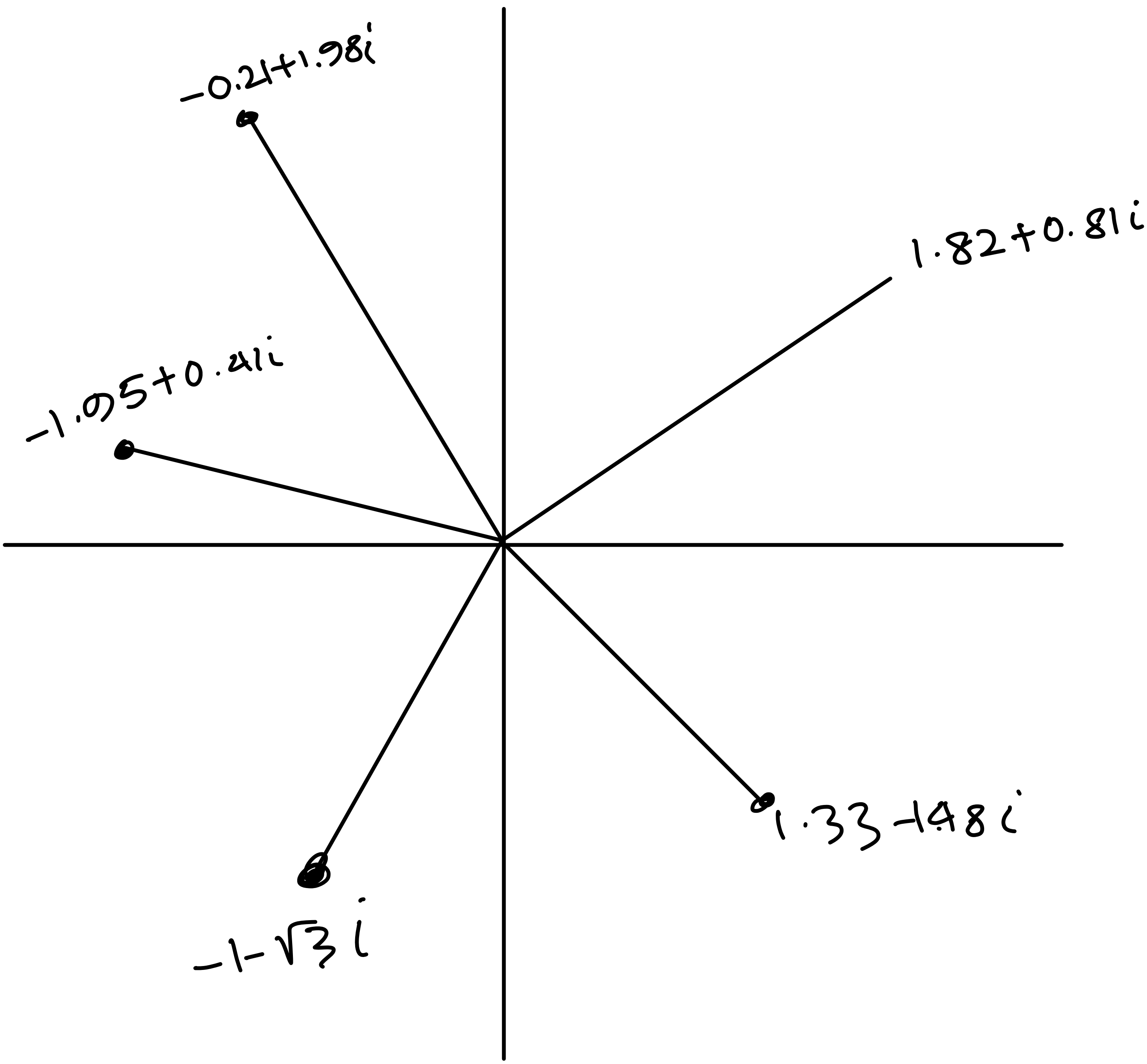
$$= -1.95 + 0.41i$$

$$n = 3 \Rightarrow 2e^{i\times\left(\frac{4}{3}\pi\right)}$$

$$= -1 - \sqrt{3}i$$

$n = 4 \Rightarrow 2e^{i\left(\frac{26}{15}\pi\right)}$

$= 1.33 - 1.48i$



5(d)

$$z = \{-27i\}^{1/6}$$

$$r = 27$$

$$\theta = -\frac{\pi}{2} + 2n\pi$$

$$\therefore\ 27e^{i(-\pi/2 + 2n\pi)}$$

$$z = \left\{27e^{i\left(-\frac{\pi}{2} + 2n\pi\right)}\right\}^{1/6}$$

$$= \sqrt{3}\ e^{i\left(-\frac{\pi}{12} + \frac{n\pi}{3}\right)}$$

$$n = 0 \Rightarrow \sqrt{3}\,e^{i(-\pi/12)} \Rightarrow 1.67 - 0.44i$$

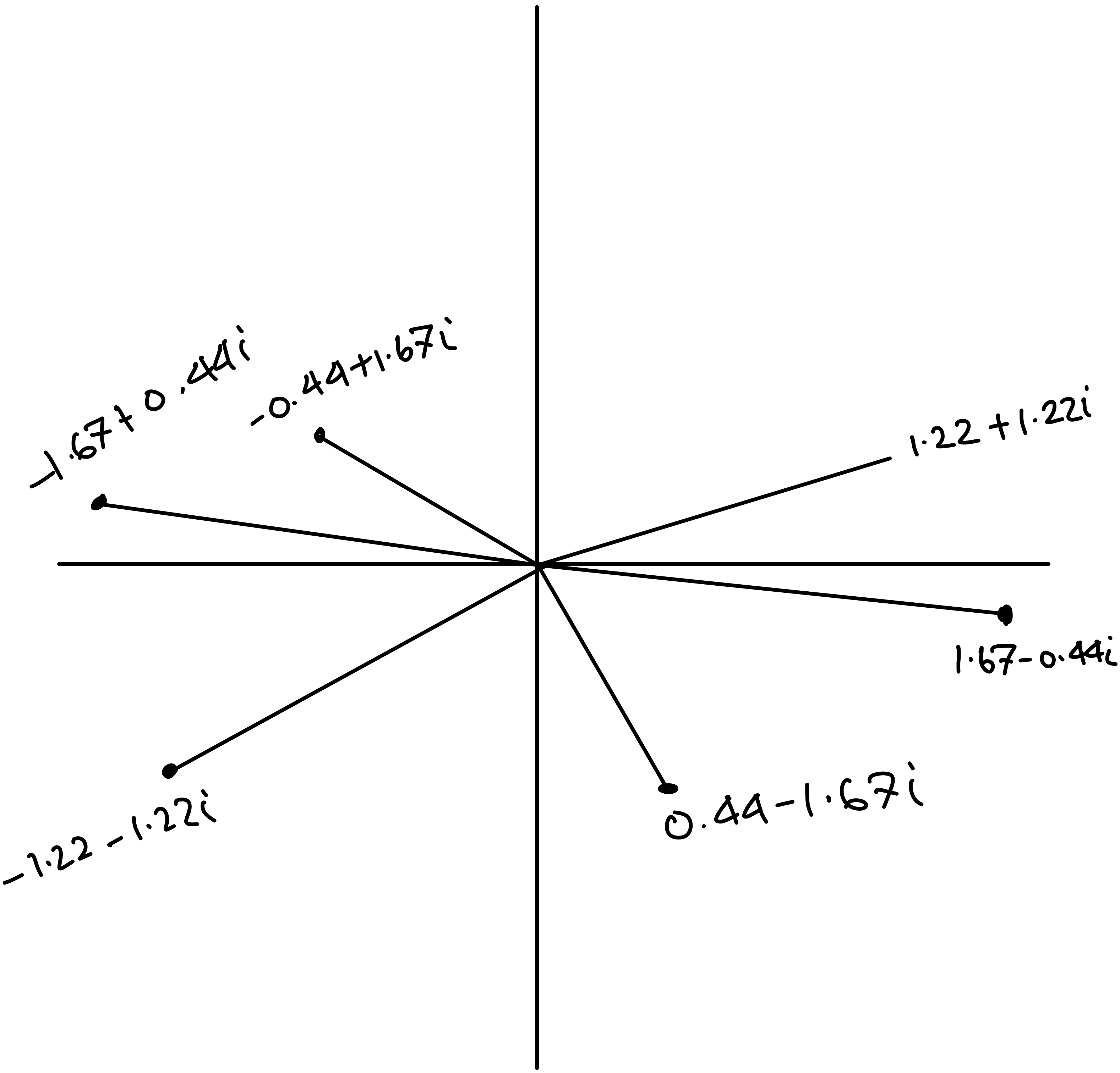
$$n=1 \Rightarrow \sqrt{3}\, e^{i\frac{\pi}{4}} \Rightarrow 1.22 + 1.22\, i$$

$$n=2 \Rightarrow \sqrt{3}\, e^{i\left(\frac{7\pi}{12}\right)} \Rightarrow -0.44 + 1.67\, i$$

$$n=3 \Rightarrow \sqrt{3}\, e^{i\left(\frac{11}{12}\pi\right)} \Rightarrow -1.67 + 0.44\, i$$

$$n=4 \Rightarrow \sqrt{3}\, e^{i\left(\frac{5}{4}\pi\right)} \Rightarrow -1.22 - 1.22\, i$$

$$n=5 \Rightarrow \sqrt{3}\, e^{i\left(\frac{19\pi}{12}\right)} \Rightarrow 0.44 - 1.67\, i$$



-1.67+0.44i
-0.44+1.67i
1.22+1.22i
1.67-0.44i
-1.22-1.22i
0.44-1.67i

Ans to que-6

6(a)

$z^4 + 81 = 0$

$\Rightarrow z^4 = -81$

$\Rightarrow z = (-81)^{1/4}$

$x = -81$

$r = 81$

$\theta = \pi - \tan^{-1}\left|\frac{0}{81}\right| + 2n\pi$

$= \pi + 2n\pi$

$$x = 81\, e^{i(\pi + 2n\pi)}$$

$$z = \left\{ 81\, e^{i(\pi + 2n\pi)} \right\}^{1/4}$$

$$= 3\, e^{i\left(\pi/4 + n\pi/2\right)}$$

$$n = 0 \Rightarrow 3\, e^{i\pi/4}$$

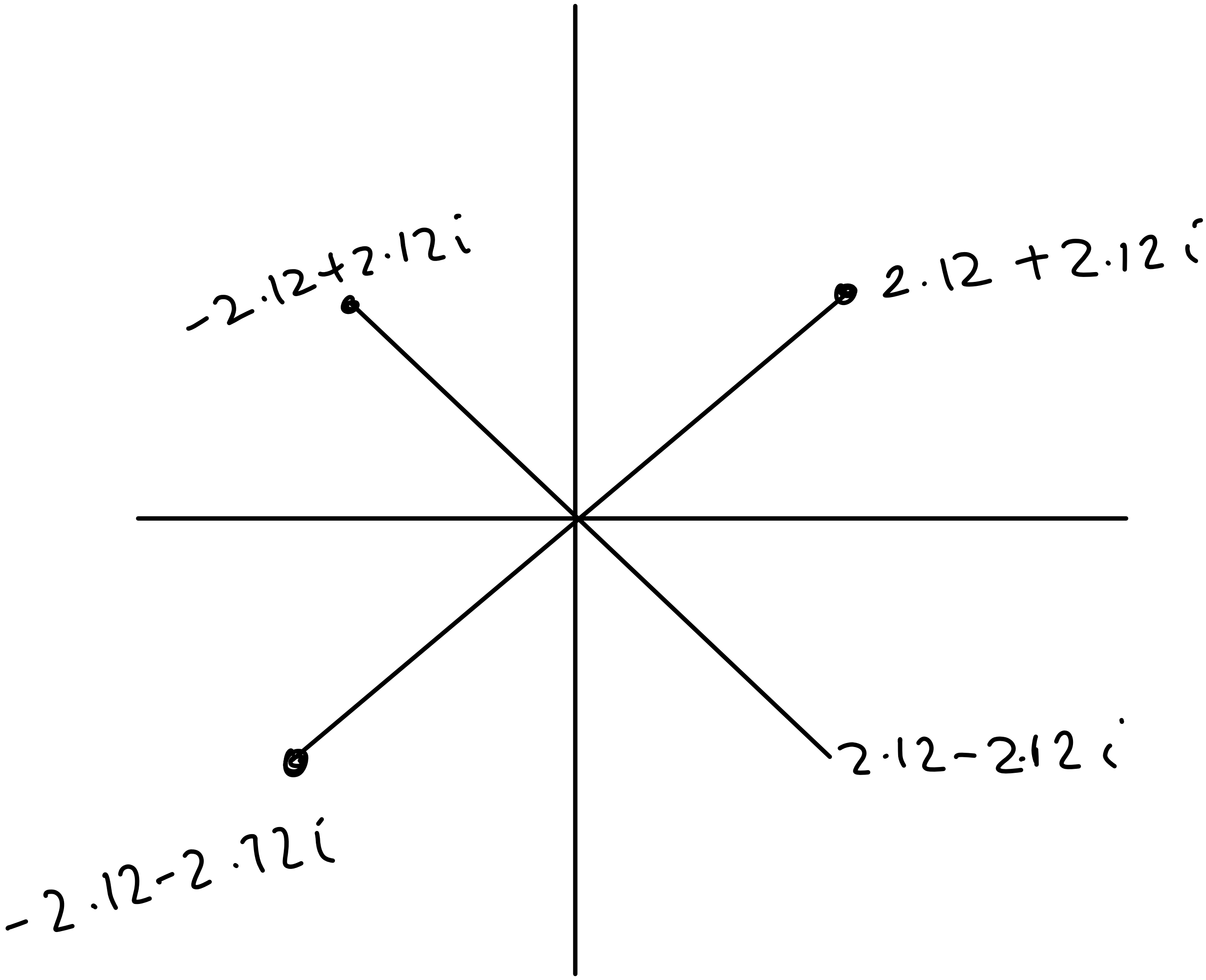
$$\Rightarrow 2.12 + 2.12i$$

$$n = 1 \Rightarrow 3\, e^{i(3\pi/4)}$$

$$= -2.12 + 2.12i$$

$$n = 2 \Rightarrow 3e^{i(5\pi/4)} \Rightarrow -2.12 - 2.12i$$

$$n = 3 \Rightarrow 3e^{i(7\pi/4)} \Rightarrow 2.12 - 2.12i$$



-2.12+2.12i
2.12 +2.12i
2.12-2.12i
-2.12-2.72i

6(b)

$$z^6 + 1 = \sqrt{3}i$$

$$z^6 = -1 + \sqrt{3}i$$

$$\Rightarrow z = \{-1 + \sqrt{3}i\}^{1/6}$$

$$r = 2$$

$$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| + 2n\pi$$

$$= \frac{2\pi}{3} + 2n\pi$$

$$z = \left\{2e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}\right\}^{1/6}$$

$$= 2^{1/6} e^{i\left(\frac{\pi}{9} + \frac{n\pi}{3}\right)}$$

$$n=0 \Rightarrow 2^{1/6}\, e^{i(\pi/9)}$$

$$= 1.05 + 0.38i$$

$$n=1 \Rightarrow 0.19 + 1.05i$$

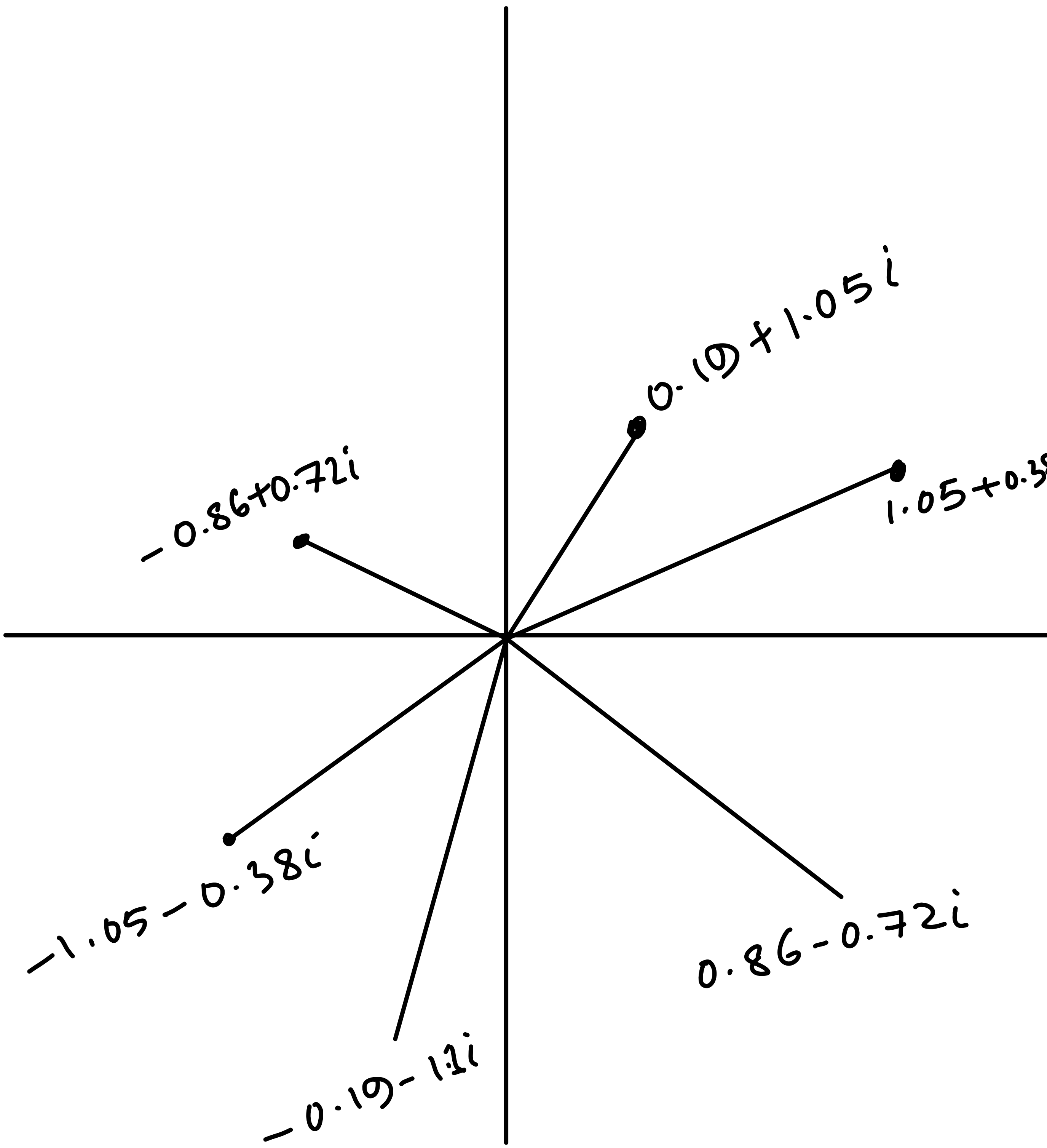
$$n=2 \Rightarrow 2^{1/6}\, e^{i\frac{7\pi}{9}} \Rightarrow -0.85 + 0.72i$$

$$n=3 \Rightarrow 2^{1/6}\, e^{i(10\pi/9)} \Rightarrow -1.05 - 0.38i$$

$$n=4 \Rightarrow 2^{1/6}\, e^{i(\frac{13}{9}\pi)} \Rightarrow -0.19 - 1.1i$$

$$n=5 \Rightarrow 2^{1/6}\, e^{i(\frac{16}{9}\pi)}$$

$$= 0.86 - 0.72i$$


0.19+1.05i
-0.86+0.72i
1.05+0.38i
-1.05-0.38i
0.86-0.72i
-0.19-1.1i

Ans to que 7

7(a)

$$\lim_{z \to 2i} \left(iz^4 + 3z^2 - 10i \right)$$

$$= i\,(2i)^4 + 3(2i)^2 - 10i$$

$$= i \times 16 - 12 - 10i$$

$$= -12 + 6i$$

7(c)

$$\lim_{z \to i/2} \frac{(2z-3)(4z+i)}{(iz-1)^2}$$

$$= \frac{\left(2 \times \frac{i}{2} - 3\right)\left(4 \times i/2 + i\right)}{\left(i \times \frac{i}{2} - 1\right)^2}$$

$$= \frac{(i-3) \times 3i}{(-3/2)^2} = -3 - 9i \times \frac{4}{9}$$

$$= -\frac{4}{3} - 4i$$

7(b)

$$\lim_{z \to e^{\pi/4}} \frac{z^2}{z^4 + z + 1}$$

$$= \frac{e^{\pi/2}}{e^{\pi} + e^{\pi/4} + 1}$$

$$= \frac{\cos \pi/2 + i \sin \pi/2}{\cos \pi + i \sin \pi + \cos \frac{\pi}{4} + i \sin \pi/4 + 1}$$

$$= \frac{i}{-1 + \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} + 1}$$

$$= \frac{i}{\frac{1}{\sqrt{2}}(1+i)}$$

$$= \frac{i^2}{\frac{1}{\sqrt{2}}(i+i^2)}$$

$$= \frac{-\sqrt{2}}{i-1}$$

7(d)

$$\lim_{z \to i} \frac{z^2+1}{z^6+1}$$

L'hospital's rule,

$$\Rightarrow \lim_{z \to i} \frac{2z}{6z^5}$$

$$= \frac{2 \times \cancel{i}}{6 \times i^2 \times i^2 \times \cancel{i}} = \frac{2}{6} = \frac{1}{3}$$

7(c)

$$\lim_{z \to 1+i} \left\{ \frac{z-1-i}{z^2-2z+2} \right\}^2$$

$$= \lim_{z \to 1+i} \left\{ \frac{z-1-i}{(z^2-2z+1)+1} \right\}^2$$

$$= \lim_{z \to 1+i} \left\{ \frac{(z-1-i)}{(z-1)^2-i^2} \right\}^2$$

$$= \lim_{z \to 1+i} \left\{ \frac{\cancel{(z-1-i)}}{\cancel{(z-1-i)}(z-1+i)} \right\}^2$$

$$= \lim_{z \to 1+i} \frac{1}{(z-1+i)^2}$$

$$= \frac{1}{(1+i-1+i)^2}$$

$$= \frac{1}{(2i)^2}$$

$$= \frac{1}{4i^2}$$

$$= -1/4$$

Ans to que 8

$$\lim_{z \to e^{\pi i/3}} \left(z - e^{i\pi/3}\right)\left(\frac{z}{z^3+1}\right)$$

$$z^3+1 = (z+1)(z^2-z+1)$$

$$= (z+1)\left(z-e^{i\pi/3}\right)\left(z-e^{-\pi i/3}\right)$$

$z^2-z+1=0$

$z = \dfrac{1+\sqrt{3}i}{2}$

$z = e^{i\pi/3},\ e^{-\pi i/3}$

$$\lim_{z \to e^{i\pi/3}} \left(z - e^{i\pi/3}\right)\frac{z}{(z+1)\left(z-e^{i\pi/3}\right)\left(z-e^{-i\pi/3}\right)}$$

$$= \lim_{z \to e^{i\pi/3}} \frac{z}{(z+1)\,(z - e^{-i\pi/3})}$$

$$= \frac{e^{i\pi/3}}{(e^{i\pi/3}+1)\,(e^{i\pi/3} - e^{i\pi/3})}$$

$$= \frac{\frac{1+\sqrt{3}i}{2}}{\left(\frac{1+\sqrt{3}\,i}{3+\sqrt{3}\,i} + 1\right) \cdot 2i \cdot \frac{\sqrt{3}}{2}}$$

$$= \frac{1+\sqrt{3}i}{(3+\sqrt{3}i)\,\sqrt{3}i}$$

$$= \frac{(1+\sqrt{3}i)}{3(-1+\sqrt{3}i)} \times \frac{-1-\sqrt{3}i}{-1-\sqrt{3}i}$$

$$= \frac{-1-\sqrt{3}i \quad -\sqrt{3} \; +3}{3 \quad (1+3)}$$

$$= \frac{2 - 2\sqrt{3}i}{12}$$

$$= \frac{1}{6} \; (1 - i\sqrt{3})$$

Ans to que 9

9(a)

$$\lim_{z \to 2i} f(z)$$

$$= \lim_{z \to 2i} \frac{z^2+4}{z-2i}$$

using L'hospital's rule,

$$= \lim_{z \to 2i} \frac{2z}{1}$$

$$= 2(2i)$$

$$= 4i$$

9(b)

$$\lim_{z \to 2i} \frac{z^2+4}{z-2i}$$

L'Hospital's rule,

$$\Rightarrow \lim_{z \to 2i} \frac{2z}{1}$$

$$= \frac{2 \times 2i}{1} = 4i$$

$f(z)$ at $2i \Rightarrow f(2i) = 3+4i$

since $\lim_{z \to 2i} f(z) \neq f(z_0)$

it is not continuous

g (c)

$$f(z) = \frac{z^2+4}{z-2i} \; ; \text{ when } z \neq 2i$$

when we evaluate limit at $2i$, we consider values approaching $2i$, but not exactly $2i$.

from g(a) we see limit approaching $2i$ exists.

so if we take values $\neq 2i$ let that number z_1

we will see that $f(z_1) = \lim_{z \to z_1} f(z)$

and therefore it is continuous

at $z \neq 2i$

Ans to que 10

10(a)

$$f(z) = \frac{2z-3}{z^2+2z+2}$$

z^2+2z+2 cant be zero

$$z^2 + 2z + 2 \neq 0$$

$$\Rightarrow (z^2 + 2 \cdot z \cdot 1 + 1) + 1 \neq 0$$

$$\Rightarrow (z+1)^2 \neq -1$$

$$\Rightarrow z+1 \neq \pm\sqrt{-1}$$

$\Rightarrow z \neq -1 \pm i$

$f(z)$ is discontinuous at $z = -1 \pm i$

10(b)

$$f(z) = \frac{3z^2 + 4}{z^4 - 16}$$

here $z^4 \neq 16$

$$z \neq (16)^{1/4}$$

$r = 16$

$\theta = 2n\pi$

$$z \neq \left\{16\, e^{i(2n\pi)}\right\}^{1/4}$$

$$z \neq 2\, e^{i\left(\frac{n\pi}{2}\right)}$$

here n can be $0, 1, 2, 3 \ldots$

$\therefore$ $f(z)$ is discontinuous at $z = 2e^{i\left(\frac{n\pi}{2}\right)}$

10(c)

$$f(z) = \cot z$$

$$= \frac{\cos z}{\sin z}$$

its discontinuous at $\sin z = 0$

$$\therefore \quad \sin z = 0$$

$$\Rightarrow \quad \frac{e^{iz} - e^{-iz}}{2i} = 0$$

$$\Rightarrow \quad e^{iz} = e^{-iz}$$

$$\Rightarrow \quad e^{2iz} = 1$$

$$\Rightarrow \quad e^{2i(x+iy)} = 1$$

$$\Rightarrow e^{2ix-2y} = 1 \cdot e^{i2n\pi}$$

$$\Rightarrow e^{2ix-2y} = e^{i2n\pi}$$

$$\Rightarrow \ln e^{2ix-2y} = \ln e^{i2n\pi}$$

$$\Rightarrow 2xi - 2y = 2n\pi$$

$\therefore 2x = 2n\pi$	$-2y = 0$
$\Rightarrow x = n\pi$	$\Rightarrow y = 0$

$\therefore z = x + iy = n\pi$, where $n \in \mathbb{Z}$

at $z = n\pi$, its discontinuous

10(d)

$$f(z) = \frac{1}{z} - \sec z$$

here z can't be zero

$\therefore z \neq 0$

or $re^{i\theta} \neq 0$

even though $e^{i\theta}$ is possible to be 1,
still r can't be zero

here $r \neq 0$ and $e^{i\theta} \neq 0$

or $x+iy \neq 0$

for $x+iy=0$, its discontinuous

$\therefore \quad x + iy = 0$

$x = 0 \quad | \quad y = 0$

$\therefore$ at $z = 0 + 0i$, $f(z)$ is discontinuous