CSE330: Numerical Methods

Topic: Integration

Prepared by:
Saad Bin Sohan
BRAC University

Email: sohan.academics@gmail.com
GitHub: https://github.com/saad-bin-sohan

3 methods

$$\text{Relative error} = \frac{\text{actual} - \text{numerical}}{\text{actual}}$$

actual integration, $I(f) = \int_a^b f(x)\,dx$

numerical integration, $I_n(f) = \int_a^b P_n(x)\,dx$

[we will use langrange]

$$P_n(x) = \sum_{i=0}^{n} l_k(x)\, f(x_k)$$

$$I_n(f) = \int_a^b \sum_{k=0}^{n} l_k(x)\, f(x_k)\, dx$$

$$= \sum_{k=0}^{n} f(x_k) \underbrace{\int_a^b l_k(x)\, dx}_{\sigma_k \text{ (weigthed factors)}}$$

$$\boxed{I_n(f) = \sum_{k=0}^{n} \sigma_k\, f(x_k)}$$

Newton's cote's formula

closed interval

open interval

n=1

Traperium/

trapozoid rule

n=2

simpson's rule

Trapezium Rule

$$I_n(f) = \int_a^b P_n(x)\,dx$$

$n=1$ $$I_1(f) = \int_a^b P_1(x)\,dx$$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$I_1(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) \right] dx$$

$$= \underbrace{\int_a^b l_0(x) f(x_0)\,dx}_{\delta_0} + \underbrace{\int_a^b l_1(x) f(x_1)\,dx}_{\delta_1}$$

$$I_1(f) = \delta_0 f(x_0) + \delta_1 f(x_1)$$

let $a = x_0$

$b = x_1$

$$\gamma_0 = \int_a^b l_0(x)\,dx$$

$$= \int_a^b \frac{x-x_1}{x_0-x_1}\,dx$$

$$= \int_a^b \frac{x-b}{a-b}\,dx$$

$$= \frac{1}{a-b}\int_a^b (x-b)\,dx$$

$$= \frac{1}{a-b}\left[\frac{x^2}{2} - bx\right]_a^b$$

$$= \frac{1}{a-b}\left[\frac{b^2}{2} - b^2 - \frac{a^2}{2} - ab\right]$$

$$= \frac{b-a}{2}$$

$$\gamma_1 = \int_a^b l_1(x)\,dx$$

$$\vdots \qquad \vdots$$

$$\gamma_1 = \frac{b-a}{2}$$

$$I_1(f) = \gamma_0 f(x_0) + \gamma_1 f(x_1)$$

$$= \gamma_0 f(a) + \gamma_1 f(b)$$

$$= \frac{b-a}{2} \times f(a) + \frac{b-a}{2} f(b)$$

$$= \frac{b-a}{2} \left[f(a) + f(b) \right]$$

Trapezium Rule

$$\boxed{I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]}$$

que: $f(x) = e^x \quad [0,2]$

$I_1(f) = ? \longrightarrow$ numerical

$I(f) = ? \longrightarrow$ actual

relative error $= ?$

ans:

$$I(f) = \int_0^2 e^x \, dx$$

$$= \left[e^x \right]_0^2 = e^2 - e^0$$

$$= 6.389 \quad \text{(actual)}$$

$$I_1(f) = \frac{b-a}{2} \left[f(a) + f(b) \right]$$

$$= \frac{2-0}{2} \left[f(0) + f(2) \right]$$

$$= 1 \times \left[e^0 + e^2 \right]$$

$$= 8.389 \quad \text{(numerical)}$$

Relative percent error,

$$= \frac{I - I_1}{I} \times 100\%$$

$$= \left| \frac{6.389 - 8.309}{6.389} \right| \times 100\%$$

$$= 31.3\%$$

Upper bound error

$\hookrightarrow$ cauchy's theorem

$$\left| I - I_n \right| \leq \frac{f^{(n+1)}(\xi)}{(n+1)!} \int_a^b \left| (x-x_0)(x-x_1)\cdots(x-x_n) \right| dx$$

$f(x) = e^x$

$a = 0$ $\quad n = 1 \quad I_1(f)$

$b = 2$

Error upper bound = ?

ans:

$$|I - I_1| \le \left| \frac{f^{(1+1)}(\xi)}{(1+1)!} \int_a^b (x - x_0)(x - x_1)\, dx \right|$$

$$\left| \frac{f^2(\xi)}{2!} \right| \quad \text{within } [0, 2]$$

ξ এর কোন valueর জন্য এই value max হবে

its when $\xi = 2$

$$\left| \frac{f^2(2)}{2!} \right| = \left| \frac{e^{\xi}}{2} \right| = \frac{e^2}{2}$$

now,

$$\left| \int_a^b (x-x_0)(x-x_1)\,dx \right|$$

$$= \left| \int_0^2 (x^2 - 2x)\,dx \right|$$

$$= \left| \left[\frac{x^3}{3} - x^2 \right]_0^2 \right|$$

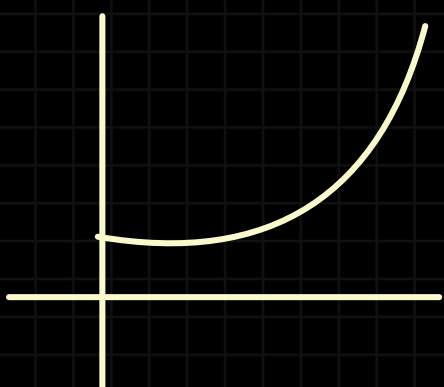
$$= \frac{4}{3}$$

$\therefore$ upper bound of error,

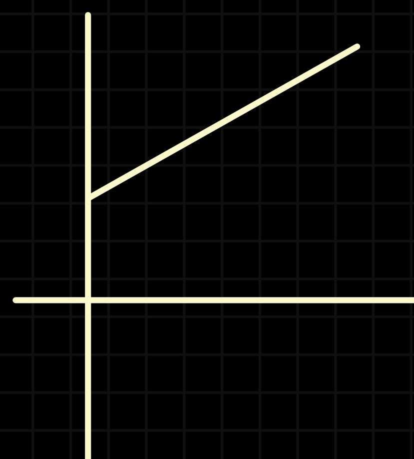
$$= \frac{e^2}{2} \times \frac{4}{3} = 4.926$$ (4 significant)

Composite Newton-cotes Formula

→ $n=1$

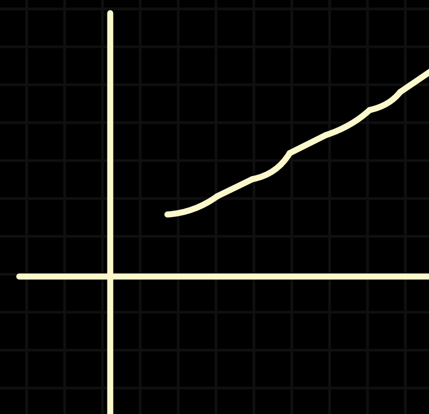


$I(f)$



$n=1$

newton cote's forward, $I_1(f)$



composite newton cotes formula

$n=1$

segmant $=5$

$C_{1,m}(f)$

$$C_{1,m}(f) = \frac{h}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{m-1}) + f(x_m) \Big]$$

que

$$f(x) = e^x$$

$$a = 0$$

$$b = 2$$

$$I(f) = 6.389$$

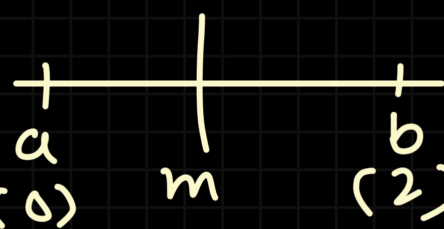
composite newton rotes formula use करो

where, $m = 2$

সমান দুইটা ভাগের sub interval

ans:

sub interval এর distance, $h = \frac{b-a}{2}$



$$h = \frac{2-0}{2} = 1$$

now we need to know the x values in those subintervals.

$m=2$, so 3 points

$x_0 = a = 0$

$x_1 = x_0 + h = 0 + 1 = 1$

$x_2 = x_1 + h = 1 + 1 = 2$

$$C_{1/2}(f) = \frac{h}{2}\left[f(x_0) + 2 \times f(x_1) + f(x_2)\right]$$

$$= \frac{1}{2}\left[e^0 + 2 \times e^1 + e^2\right]$$

$$= 6.91281$$

if $m = 3 \Rightarrow$

$$h = \frac{b-a}{m} = \frac{2-0}{3} = \frac{2}{3}$$

$x_0 = a = 0$

$x_1 = x_0 + h = 0 + \frac{2}{3} = 2/3$

$x_2 = x_1 + h = 2/3 + \frac{2}{3} = 4/3$

$x_3 = x_2 + h = 4/3 + 2/3 = 2$

$x_4 = x_3 + h = 2 + 2/3 = 8/3$

$$C_{1,3}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{2/3}{2}\left[e^0 + 2e^{2/3} + 2e^{4/3} + e^2\right]$$

$$= 6.62355$$

$m \uparrow$ accuracy $\uparrow$ Error $\downarrow$

[NB: $n=2$ $2(n)$,

Newton cotes formula

Simpson's Rule

$$I_1(f) = \gamma_0 f(x_0) + \gamma_1 f(x_1) + \gamma_2 f(x_2)$$

$$\gamma_0 = \frac{b-a}{6}, \quad \gamma_1 = \frac{2}{3}(b-a), \quad \gamma_2 = \frac{b-a}{6}$$

$x_0 = a$

$x_1 = \frac{a+b}{2}$

$x_2 = b$

simpson rule ఈ

ఉప nodes

$$\therefore I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$