CSE330: Numerical Methods

Topic: LU Decomposition, Pivoting

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LU Decomposition, Pivoting

Process:

we know $Ax = b$

→ decompose A and get $A = LU$

lower triangular matrix

upper triangular matrix

so, $Ax = b$

$\Rightarrow LUx = b$

let $Ux = y$, so, $Ly = b$

$Ly = b$ থেকে forward substitution method use করলে we get y.

now, $Ux = y$

and backward substitution use করে x এর value বের করবো

e.g.

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

find x_1, x_2, x_3 using LU decomposition.

[Frobinuous matrix,

F → Unit lower triangular matrix

lower tri. matrix এর কর্ণের সবাই zero

3x3 frobinous matrix = $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

target এদের zero বানানো

F^{7} → কত number column নিয়ে কাজ করছি

ঐটা বাদে বাকি সব column এর কোণ

[illegible])]

soln:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ 4 \\ 9 \end{pmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$$

$$F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}$$

$$F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

F আগে

A পরে

$$A^{(2)} = F^{(1)} \times A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{pmatrix}$$

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{8}{-4} = -2$$

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$A^{(3)} = F^{(2)} \times A^{(2)}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$U = A^{(3)}$$

now find the L matrix.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

$\rightarrow \quad Ly = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$$

using forward substitution,

$$y_1 = 0$$

$$y_1 + y_2 = 4$$

$$y_2 = 4$$

$$2y_1 - 2y_2 + y_3 = 4$$

$$y_3 = 4 - 2(0) + 2(4)$$

$$= 12$$

$\rightarrow Ux = y$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix}$$

using backward substitution,

$$-2x_3 = 12$$

$$x_3 = -6$$

$$-4x_2 + x_3 = 4$$

$$x_2 = -2.5$$

$$x_1 + 2x_2 + x_3 = 0$$

$$\Rightarrow x_1 = 11$$

NB: LU decomposition এর সুবিধা:

→ A, x matrix same কিন্তু b এর value গুলো different হলেও pc নতুন করে L, U করা লাগবে না।

(reusability)

Pivoting

let,

$$\underbrace{\begin{pmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{0} \text{ (undefined)}$$

কর্ণ (primary diagonal) এর a_{11}, a_{22}, a_{33} কোনো value ই zero হওয়া যাবে না (to avoid to be undefined)

we solve the issue by → pivoting

pivoting => row swap

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

swap row1 and row2

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

NB: i) row swap করলে A, b matrix এ change আসবে। x matrix same থাকে।

লাগবেনা xm এ → ii) column swap করলে, A, x এ change হবে, b same থাকে