MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Inverse Laplace Transforms

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Inverse Laplace Transform

Let $F(s)$ be the Laplace Transform of some unknown function $f(t)$ defined for $t \geq 0$. Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds$$

where the integration is done along the vertical line $Re(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $F(s)$ and $F(s)$ is bounded on the line, for example if the contour path is in the region of convergence.

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$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t)$$

Inverse Laplace Transforms of some Algebraic functions

- $L^{-1}\left\{\frac{1}{s}\right\} = 1$

- $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

- $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

- $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

Transforms of some Trigonometric and Hyperbolic Functions

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh(at)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$$

Linearity of Inverse Laplace transformation:

$$\mathcal{L}^{-1}\left\{F(s) \pm G(s)\right\} = \mathcal{L}^{-1}\left\{F(s)\right\} \pm \mathcal{L}^{-1}\left\{G(s)\right\}$$

$$\mathcal{L}^{-1}\left\{k \cdot F(s)\right\} = k \cdot \mathcal{L}^{-1}\left\{F(s)\right\}$$

B Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= \frac{1}{6} \times t^3$$

{ans}

$$\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$$

$$= -12\,\mathcal{L}^{-1}\left\{\frac{1}{3s-4}\right\}$$

$$= -12 \times \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\}$$

$$= -4\,\mathcal{L}^{-1}\left\{\frac{1}{s-4/3}\right\}$$

$$= -4\,e^{\left(\frac{4}{3}\right)t}$$

break

$$\mathcal{L}^{-1}\left\{F(s)\,e^{-as}\right\}$$

$$= f(t-a)\cdot u(t-a)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4} e^{-2s}\right\}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= \frac{1}{3!}\mathcal{L}^{-1}\left(\frac{3!}{s^4}\right\}$$

$$f(t) = \frac{1}{6} t^3$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4} e^{-2s}\right\}$$

$$= f(t-a)\, u(t-a)$$

$$= f(t-2)\,u(t-2)$$

$$= \frac{1}{6}(t-2)^3 \cdot u(t-2)$$

(Ans)