CSE330: Numerical Methods

Topic: Fixed Point Representation, Newton's Method

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কত fast root খুঁজে পাওয়া যায় (convergence rate) সেই calculation এ bisection method very inefficient. এজন্য better convergence rate এর জন্য more efficient algorithm জানা দরকার।

e.g: Fixed point iteration

Fixed Point Representation

Process idea:

→ $f(x) = 0$ দেওয়া থাকবে।

→ $f(x) = 0$ থেকে $g(x) = x$ বানাবো।

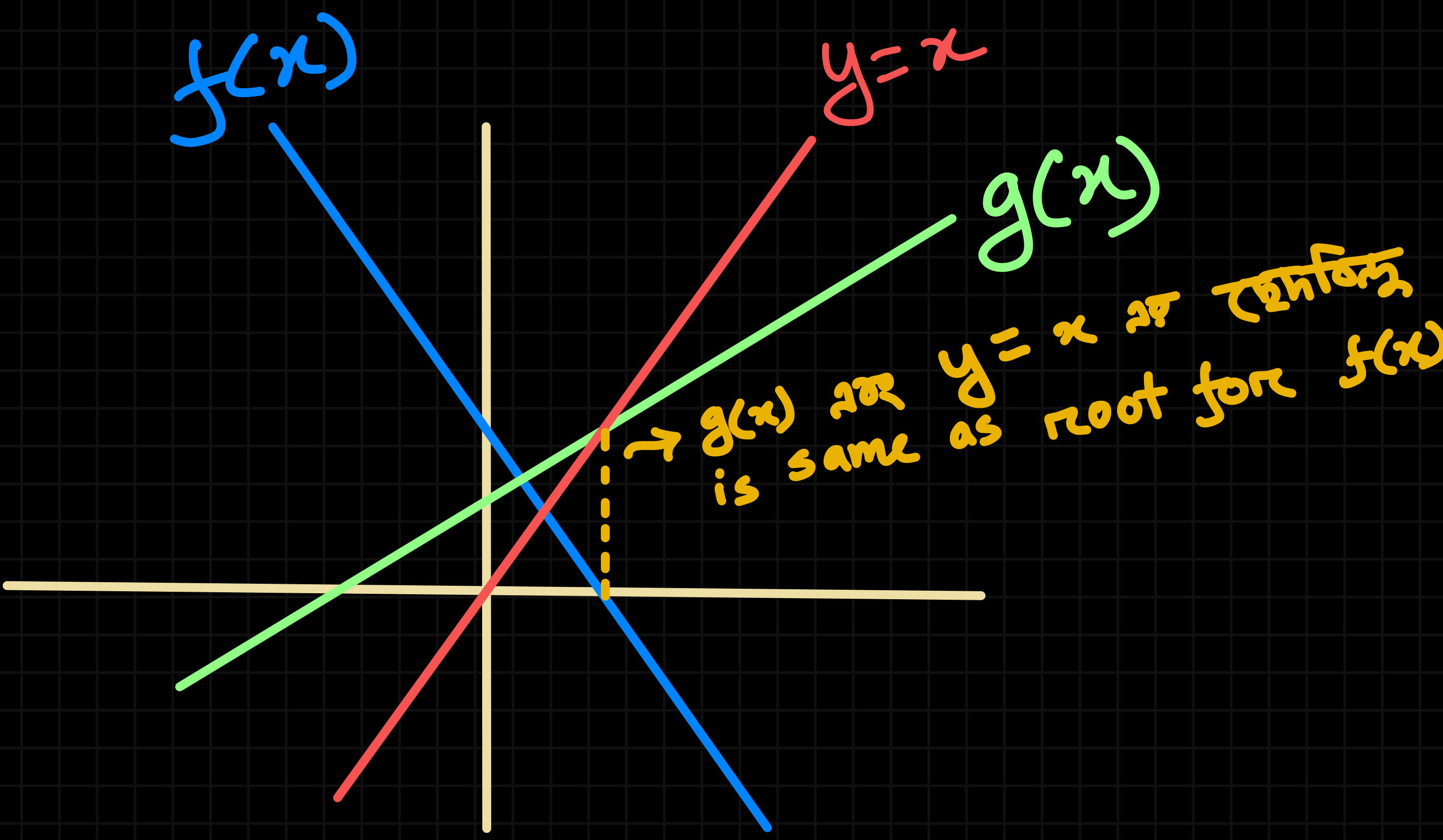
→ x এর যে value (x_*) এর জন্য $g(\underset{x_*}{x}) = x$ হবে, that

x_* is the root for $f(x)$

Que $f(x) = -\frac{1}{2}x + 1 = 0$

converted $g(x)$,

$$g(x) \Rightarrow \frac{x+2}{2} = x$$



how to find $g(x)$:

There can be multiple ways.

e.g. $f(x) = x^2 - 2x - 3 = 0$

(i) $x^2 - 2x - 3 = 0$

$$\Rightarrow \quad x^2 = 2x+3$$

$$x = \underbrace{\sqrt{2x+3}}_{g(x)}$$

(ii) $x^2 - 2x - 3 = 0$

$$x(x-2) - 3 = 0$$

$$x = \underbrace{\frac{3}{x-2}}_{g(x)}$$

(iii) $x^2 - 2x - 3 = 0$

$$x^2 - x - x - 3 = 0$$

$$x = \underbrace{x^2 - x - 3}_{g(x)}$$

(iv) $x^2 - 2x - 3 = 0$

$$\Rightarrow 2x^2 - x^2 - 2x - 3 = 0$$

$$2x^2 - 2x = x^2 + 3$$

$$x(2x-2) = x^2 + 3$$

$$\therefore x = \underbrace{\frac{x^2+3}{2x-2}}_{g(x)}$$

g(x) যেহেতু multiple ways এ বানানো possible, exam এ সবার g(x) আলাদা হওয়াও possible.

এটার disadvantage: for some g(x)

you can get all the roots, some $g(x)$ can give you no root at all. so the task is to find the fittest $g(x)$. so different $g(x)$ shows different behaviour. now we find the fitness of each $g(x)$

(i) $g(x) = \sqrt{2x+3}$, $x_0 = 0$

for any function, a guess value needs to be given.

$$g(x_0) = g(0) = 1.73$$

$x = 0$ but $g(0) \neq 0 \therefore g(x_0) \neq x_0$

now $g(1.73) = 2.54$, still not same so we continue

$g(2.54) = 2.84$

$g(2.84) = 2.95$

$g(2.95) = 2.98$

$g(2.98) = 2.99$

$g(2.99) = 3$

$g(3) = 3$ BANG ON!

$g(x) = x$ for $x = 3$

$\therefore$ root, $x = 3$

(iii) $g(x) = x^2 - x - 3$

$x_0 = 0$

$g(x_0) = g(0) = -3$

$g(-3) = 9$

$g(9) = 69$

$\vdots$

এটাতে we will never find root

যার জন্য $g(x) = x$

(4) $g(x) = \dfrac{x^2+3}{2x-2}$

$x_0 = 0$

$g(0) = -1.5$

$g(-1.5) = -1.05$

$g(-1.05) = -1$

$g(-1) = -1$ BOOM!

$\hookrightarrow$ fixed point reached

now we do ① again

① $g(x) = \sqrt{2x+3}$

$x_0 = 42 \rightarrow$ any random value

$g(42) = 9.33$

$g(9.33) = 4.65$

$g(4.65) = 3.51$

$g(3.51) = 3.17$

$g(3.17) = 3.06$

$g(3.06) = 3.02$

$g(3.02) = 3.01$

$g(3.01) = 3$

$g(3) = 3$

for 3 again,

$g(x) = x^2 - x - 3$

$x_0 = 42$

$g(x_0) = 1.72 \times 10^3$

$g(1.72 \times 10^3) = 2.95 \times 10^6$

$g(2.95 \times 10^6) = 8.72 \times 10^{12}$

$\hookrightarrow$ divergence

for (iv)

$$g(x) = \frac{x^2+3}{2x-2}$$

$$x_0 = 42$$

$$g(42) = 21.6$$

$$g(21.6) = 11.4$$

$$g(11.4) = 6.39$$

$$g(6.39) = 4.07$$

$$g(4.07) = 3.19$$

$$g(3.19) = 3.01$$

$$g(3.01) = 3$$

$$g(3) = 3$$

but same random value (42),
some $g(x)$ converges, some
divergences, some show better
convergence rate.

Contraction Mapping

→ g(x) এর quality check

→ initial value isn't a concern

→ actual root এর কাছে হয় ঘুরাতে

→ $\lambda = |g'(\text{root})|$ এর জন্য for each actual root

λ → convergence rate

g' → derivative

if $\lambda < 1$, g(x) will be convergent

otherwise g(x) will be divergent

(i) $g(x) = \sqrt{2x+3}$

root
↳ $-1, 3$

$$g'(x) = \frac{1}{\sqrt{2x+3}}$$

$|g'(-1)| = 1$ ✗ diverge করবে

$|g'(3)| = \frac{1}{3}$ ✓ converge করবে

(ii) $g(x) = x^2 - x - 3$

$$g'(x) = 2x - 1$$

$\lambda = |g'(-1)| = 3$ ✗ divergent হবে

$\lambda = |g'(3)| = 5$ ✗ divergent হবে

↳ since $\lambda > 1$

③ $g(x) = \dfrac{x^2+3}{2x-2}$

$\lambda = |g'(-1)| = 0$ ✓ converge

$\lambda = |g'(3)| = 0$ ✓ converge

NB: দুইটা g(x) ই যদি সঠিক root দেয়, কোনটা নিবে?

$\Rightarrow$ যেহেতু λ এর value 0 এর বেশি কাছে, 3টা নিবো।

$\therefore$ λ zero'র কাছে means higher convergence rate.

Orders of convergence

$\lambda = 0 \rightarrow$ super linear convergence

$\hookrightarrow$ fastest convergence

$0 < \lambda < 1 \Rightarrow$ linear convergence

$\lambda \geq 1 \Rightarrow$ divergence

For different initial guess values, we can get different roots (applies only $g(x)$ that gives all the roots)

we can make a $\underbrace{\text{guess}}$ out of it.

$\hookrightarrow$ but its not a fullproof guess e.g. previously (iii) এ

initial value zero ? দিলে → root -1

initial value 42 ? দিলে → root 3

because

0 is closer to -1 than 3

42 হলো 3 এর বেশি কাছে than -1

Newton's method

↳ extension of fixed point iteration

↳ converge করলেই এর converge rate high (tho it got its own dis advatages)

↳ method -↳

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Que: $f(x) = \frac{1}{x} - 0.5 = 0$

$x_0 = 1$

root $\rightarrow x_* = 2$

soln:

$k = 0 \quad \therefore \quad x_0 = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{\frac{1}{1} - 0.5}{\frac{d}{dx=1}\left(\frac{1}{x} - 0.5\right)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$\vdots$

K	xK
0	$x_0 = 1$
1	$x_1 = 1.5$
2	$x_2 = 1.875$
3	$x_3 = 1.9921875$
4	$x_4 = 1.999969482$
5	$x_5 = 2$
6	$x_6 = 2$

$\rightarrow$ ☐ এবার x same আসা শুরু হলে, we know its the root

root within a range (ϵ_m) এর হতে চলে আসে। SO, we keep an extra column. let $\epsilon_m = 10^{-5}$

k	x_k	$\lvert f(x_k) \rvert \le 10^{-5}$
0	$x_0 = 1$	×
1	$x_1 = 1.5$	×
2	$x_2 = 1.875$	×
3	$x_3 = 1.9921875$	×
4	$x_4 = 1.99996948 2$	✓
5	$x_5 = 2$	
6	$x_6 = 2$	

এক্ষেত্রে accuracy reach হলে no need of further iteration.