MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Residue Theorem

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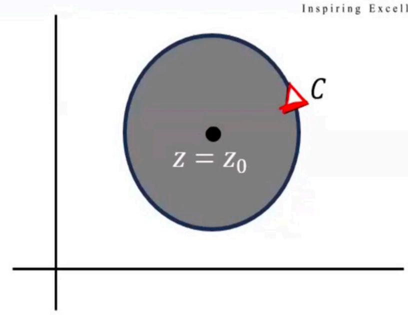
The Residue Theorem

→ it can solve any closed integral

Residues

Residues

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z = z_0$ a chosen as the center of C. Then, $f(z)$ has a Laurent series about $z = z_0$ given by



$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$where \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad for\ n = 0, \pm 1, \pm 2, \ldots$$

we call a_{-1} the residue of $f(z)$ at $z = z_0$, denoted by $Res(f, z_0)$

→ # যেখানে $f(z)$ আছে

closed curve আছে

closed curve এর ভেতর there's a singular point "z_0".

⇒ এই singular point এর জন্য we can find a residue value.

for the residue value:

let

$$f(z) = \overbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}^{\text{taylor part / analytic part}} + \underbrace{\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \dots}_{\text{lauren part / principal part}}$$

laurent expansion

where $$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

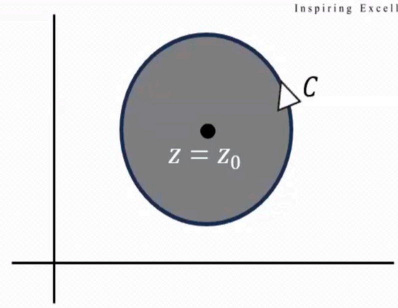
for $n = 0, \pm 1, \pm 2, \dots$

residue value $\operatorname{Res}(f, z_0) = a_{-1}$

↳ $(z - z_0)^{-1}$ এর coefficient

Why a_{-1} is so special?

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z = z_0$ a chosen as the center of C. Then, $f(z)$ has a Laurent series about $z = z_0$ given by



$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$\boxed{\text{Now the interesting fact is } \oint_C f(z)dz = 2\pi i \cdot a_{-1}}$$

proof:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$\oint f(z) = \cancelto{0}{\oint \left\{ a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \right\} dz}$$

$$+ \oint \frac{a_{-1}}{z-z_0} + \oint \frac{a_{-2}}{(z-z_0)^2} + \cdots$$

$$= \oint_C \frac{a_{-1}}{z-z_0}\, dz + \cancelto{0}{\underbrace{\frac{a_{-2}}{\cdots} \cdots}}$$

$$= a_{-1} \oint_C \frac{1}{z-z_0}\, dz$$

$$= a_{-1} \cdot 2\pi i$$

Calculation of a Residue at a Pole

The residue of a function $f(z)$ at $z = z_0$, is the constant a_{-1}

However, in the given case, where $z = z_0$ is a pole of order n, then for a_{-1},

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z) \right) \right\}$$

if order of pole = n = 1 (simple pole),

then,

$$a_{-1} = \lim_{z \to z_0} \left\{ (z - z_0) f(z) \right\}$$

1. Calculate the residue of $f(z)$ at $z=0$

$$f(z) = e^{-\frac{1}{z}}$$

[NB: ✓pole হলে → use formula

x pole না হলে → use series/expansion

e.g: $e^{-\frac{1}{0}}$ is not a pole]

solve:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{(-1/z)^2}{2!} + \frac{(-1/z)^3}{3!} + \dots$$

$$= 1 + \underbrace{(-1)}_{a_{-1}} z^{-1} + \frac{1}{2!} z^{-2} + \left(-\frac{1}{3!}\right) z^{-3} + \dots$$

now, $(z-z_0)^{-1}$ এর coefficient is the residue

$\therefore$ residue $= -1$

☐ Calculate the residue of $f(z)$ at $z=1$

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

↳ pole এর order 2

সরল pole এর order 1

<u>solve:</u>

$z=1$ is a pole of order 1 or simple pole.

$$\therefore \operatorname{Res}(f, z=1) = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} (z-1) \times \frac{z}{(z-1)(z+1)^2}$$

$$= \lim_{z \to 1} \frac{z}{(z+1)^2}$$

$$= \frac{1}{4}$$

[ans]

$\boxplus$ Calculate the residue of $f(z)$

at $z=-1$

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

$\leftarrow z=-1$

<u>solve:</u>

$z=-1$ is a pole of order $=2$

$\mathrm{Res}(f, z=-1) =$

$$\frac{1}{(2-1)!} \times \lim_{z \to -1} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left((z+1)^2 \times \frac{z}{(z-1)(z+1)^2} \right) \right\}$$

$$= \lim_{z \to -1} \left\{ \frac{d}{dz} \left(\frac{z}{z-1} \right) \right\}$$

$$= \lim_{z \to -1} \frac{(z-1) - z}{(z-1)^2}$$

$$= \frac{-1-1-(-1)}{(-1-1)^2}$$

$$= \frac{-1}{4}$$

[Ans]

vvv I→ order 2 type

The Residue Theorem

The Residue Theorem

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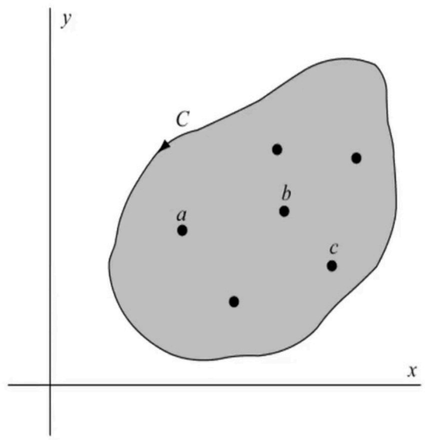
Inspiring Excellence

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C, which have residues given by a_{-1}, b_{-1}, $c_{-1}, \ldots$. Then, the residue theorem states that



$$\oint_C f(z)dz = 2\pi i \cdot (a_{-1} + b_{-1} + c_{-1} + \cdots)$$

→ ফাংশন $f(z)$ আছে

→ inside C, multiple singularity আছে

∴ $f(z)$ এর integral over C,

$= 2\pi i \times$ (singularity গুলোর residue এর যোগফল)

☐ The Cauchy-goursat theorem and the Cauchy integral formula are special cases of the Residue Theorem

Residue theorem → deals with multiple singularities

Cauchy Integral formula → only one singularity/pole at a time

cauchy goursat theorem → no singularity at all

(4) Evaluate

$$\oint_C \frac{z^2}{2z^2+5z+2}\, dz$$

using the residue at the poles, where

C is the unit circle, $|z|=1$

solve:

$$f(z) = \frac{z^2}{2z^2+5z+2}$$

singularities are at,

$$2z^2 + 5z + 2 = 0$$

$$\Rightarrow z = \frac{-1}{2}, -2$$

$$\therefore f(z) = \frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$\text{Res}\left(f, z = -\frac{1}{2}\right),$$

$$= \lim_{z \to -\frac{1}{2}} \left(z - \left(-\frac{1}{2}\right)\right) f(z)$$

$$= \lim_{z \to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \times \frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \lim_{z \to -\frac{1}{2}} \frac{z^2}{2(z+2)}$$

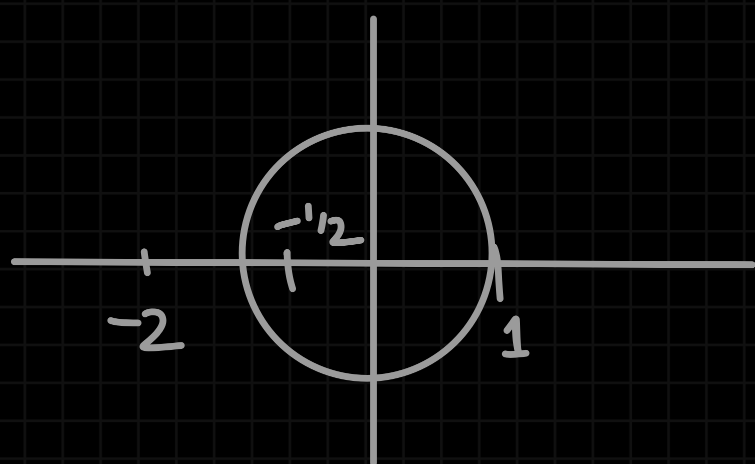
$$= \frac{(-0.5)^2}{2(-0.5+2)}$$

$$= \frac{0.25}{2 \cdot (1.5)}$$

$$= \frac{1}{12}$$

we dont need residue at $z = -2$

cz its outside C.



$$\therefore \oint_C \frac{z^2}{2z^2+5z+2}\,dz = 2\pi i\,(\text{sum of residues})$$

$$= 2\pi i \times \frac{1}{12}$$

$$= \frac{\pi i}{6} \quad \text{[Ans]}$$

☐ Evaluate

$$\oint_C \frac{z^2+4}{z^3+2z^2+2z}\,dz$$

using the residue at the poles, around the circle, $|z|=3$

solve:

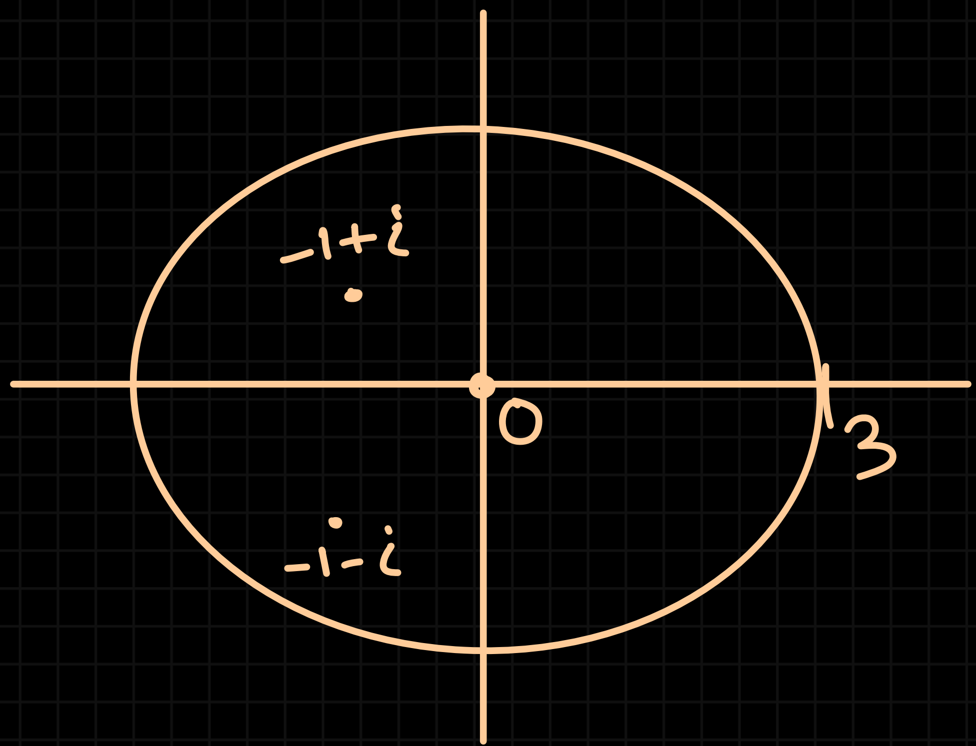
$$f(z)=\frac{z^2+4}{z^3+2z^2+2z}$$

singularities are at,

$$z^3 + 2z^2 + 2z = 0$$

$$\Rightarrow z = -1+i, -1-i, 0$$

$$\therefore f(z) = \frac{z^2 + 9}{(z-0)(z+1-i)(z+1+i)}$$



all three singularities are inside.

$\boxed{\text{for} \quad z_0 = 0}$:

$z = -1+i$ is a pole of order 1

$\therefore \operatorname{Res}(f, z=0)$,

$$= \lim_{z \to 0} \left\{ (z-0) f(z) \right\}$$

$$= \lim_{z \to 0} \left\{ z \times \frac{(z^2+4)}{z(z+1-i)(z+1+i)} \right\}$$

$$= \lim_{z \to 0} \frac{z^2+4}{(z+1-i)(z+1+i)}$$

$$= \frac{4}{(1-i)(1+i)}$$

$$= \frac{4}{1+i-i-i^2}$$

$$= \frac{4}{1+1}$$

$$= 2$$

$$\boxed{\text{for } z = -1+i}$$

simple pole

$\operatorname{Res}(f, z=-1+i)$,

$$= \lim_{z \to -1+i} \left\{ (z+1-i) \times \frac{z^2+4}{z(z+1-i)(z+1+i)} \right\}$$

$$= \lim_{z \to -1+i} \left\{ \frac{z^2+4}{z(z+1+i)} \right\}$$

$$= \frac{(-1+i)^2 + 4}{(-1+i)(-1+i+1+i)}$$

$$= \frac{1 - 2i + i^2 + 4}{(-1+i)(2i)}$$

$$= \frac{4 - 2i}{-2i + 2i^2}$$

$$= \frac{2-i}{-i+i^2} = \frac{2-i}{-i-1}$$

$$= \frac{i-2}{1+i}$$

$$= \frac{(i-2)(1-i)}{(1+i)(1-i)}$$

$$= \frac{i - i^2 - 2 + 2i}{1 - i^2}$$

$$= \frac{3i - 1}{2}$$

$$= \frac{3}{2} i - \frac{1}{2}$$

$$\boxed{\text{for } z = -1-i}$$ simple

$\text{Res}(f, z = -1-i)$

$$= \lim_{z \to -1-i} \left\{ (z+1+i) \times f(z) \right\}$$

$$= \lim_{z \to -1-i} \left\{ (z+1+i) \times \frac{z^2+9}{z(z+1+i)(z+1-i)} \right\}$$

$$= \lim_{z \to -1-i} \left\{ \frac{z^2+9}{z(z+1-i)} \right\}$$

$$= \frac{(-1-i)^2 + 4}{(-1-i)\,(-1-i+1-i)}$$

$$= \frac{1+2i+i^2+4}{-(1+i)\,(-2i)}$$

$$= \frac{4+2i}{2i+2i^2}$$

$$= \frac{4+2i}{2i-2}$$

$$= \frac{2+i}{i-1}$$

$$= \frac{(2+i)(i+1)}{(i-1)(i+1)}$$

$$= \frac{2i+2+i^2+i}{i^2+i-i-1}$$

$$= \frac{3i+1}{-1-1}$$

$$= \frac{3i+1}{-2}$$

$$= -\frac{3i}{2} - \frac{1}{2}$$

now,

$$\oint_C \frac{z^2+4}{z^3+2z^2+z}\, dz,$$

$= 2\pi i \times$ (sum of residues)

$$= 2\pi i \times \left(2 + \frac{3i}{2} - \frac{1}{2} - \frac{3i}{2} - \frac{1}{2}\right)$$

$$= 2\pi i \times 1$$

$= 2\pi i$

(Ans)

☐ Evaluate

$$\oint_C \frac{2+3\sin \pi z}{z(z-1)^2}\, dz$$

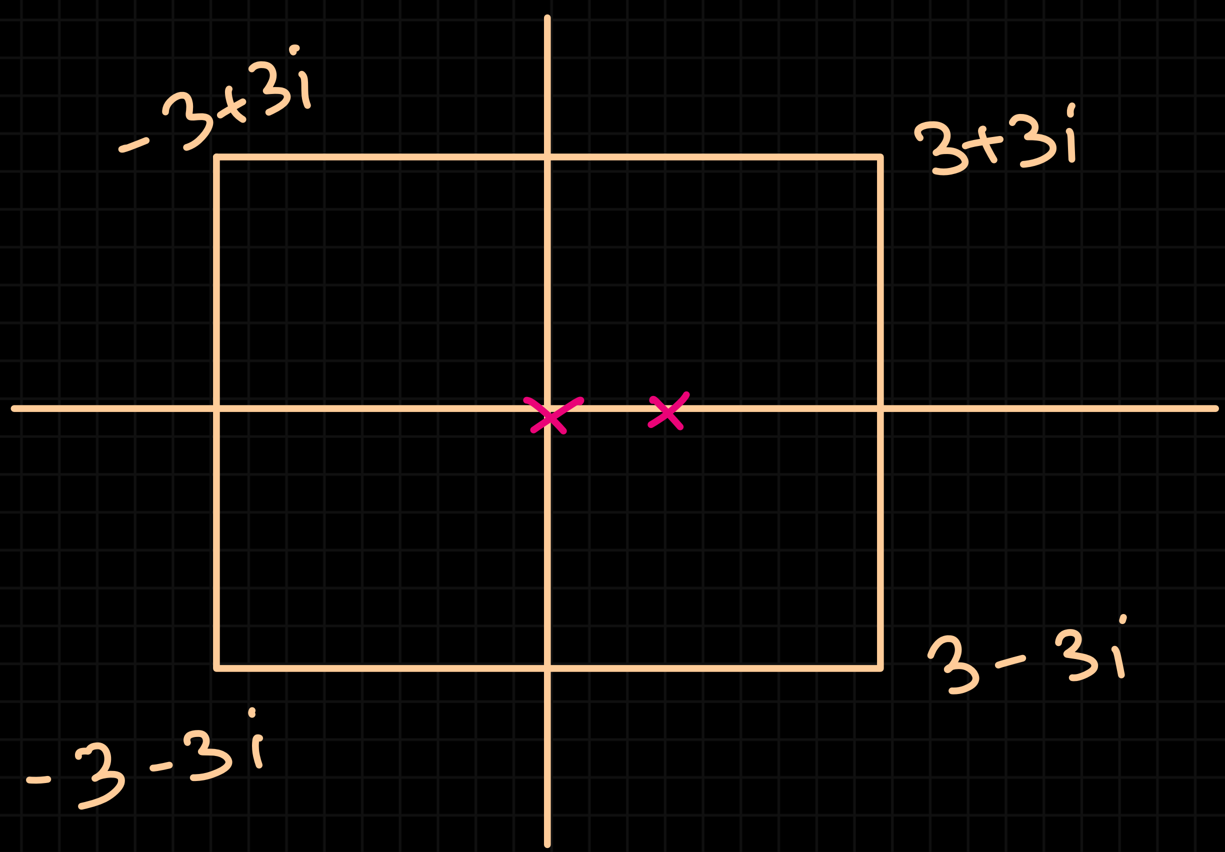
where c is a square having verties at $3+3i$, $3-3i$, $-3+3i$, $-3-3i$

solve:

$$f(z) = \frac{2+3\sin \pi z}{z\,(z-1)^2}$$

singularities are at,

$$z = 0 \quad \left| \begin{array}{l} (z-1)^2 = 0 \\ z^2 - 2z + 1 = 0 \\ \Rightarrow z = 1 \end{array} \right.$$



both the singularities are inside C.

for $z = 0$:

simple pole.

$\therefore \operatorname{Res}(f, z = 0)$,

$$= \lim_{z \to 0} \left\{ (z-0) \cdot f(z) \right\}$$

$$= \lim_{z \to 0} \left\{ z \times \frac{(2+3\sin \pi z)}{z(z-1)^2} \right\}$$

$$= \lim_{z \to 0} \left\{ \frac{2+3\sin \pi z}{(z-1)^2} \right\}$$

$$= \frac{2}{1}$$

$$= 2$$

$$\boxed{\text{for } z = 1}$$

order of pole $= 2 = n$

$\therefore$ Res$(f, z = 1)$,

$$= \frac{1}{(n-1)!} \times \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n \cdot f(z) \right) \right\}$$

$$= \frac{1}{(2-1)!} \times \lim_{z \to 1} \left\{ \frac{d}{dz} \left((z-1)^2 \times \frac{(2+3\sin \pi z)}{z(z-1)^2} \right) \right\}$$

$$= \lim_{z \to 1} \left\{ \frac{d}{dz} \left(\frac{2+3\sin \pi z}{z} \right) \right\}$$

$$= \lim_{z \to 1} \left\{ \frac{z \cdot 3(\cos \pi z) \cdot \pi - (2+3\sin \pi z) \cdot 1}{z^2} \right\}$$

$$= \lim_{z \to 1} \left\{ \frac{3\pi z (\cos \pi z) - 2 - 3\sin \pi z}{z^2} \right\}$$

$$= 3\pi \cos \pi - 2 - 3 \sin \pi$$

$$= -3\pi - 2$$

$\therefore$ now,

$$\oint_C \frac{2 + 3\sin \pi z}{z(z-1)^2} \, dz$$

$$= 2\pi i \times (\text{sum of residues})$$

$$= 2\pi i \times \{ 2 + (-3\pi - 2) \}$$

$$= 2\pi i \cdot (-3\pi)$$

$$= -6\pi^2 i$$

[Ans]

Evaluate

$$\frac{1}{2\pi i}\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\, dz$$

using the residue at the poles, around the circle C with the equation $|z| = 4$

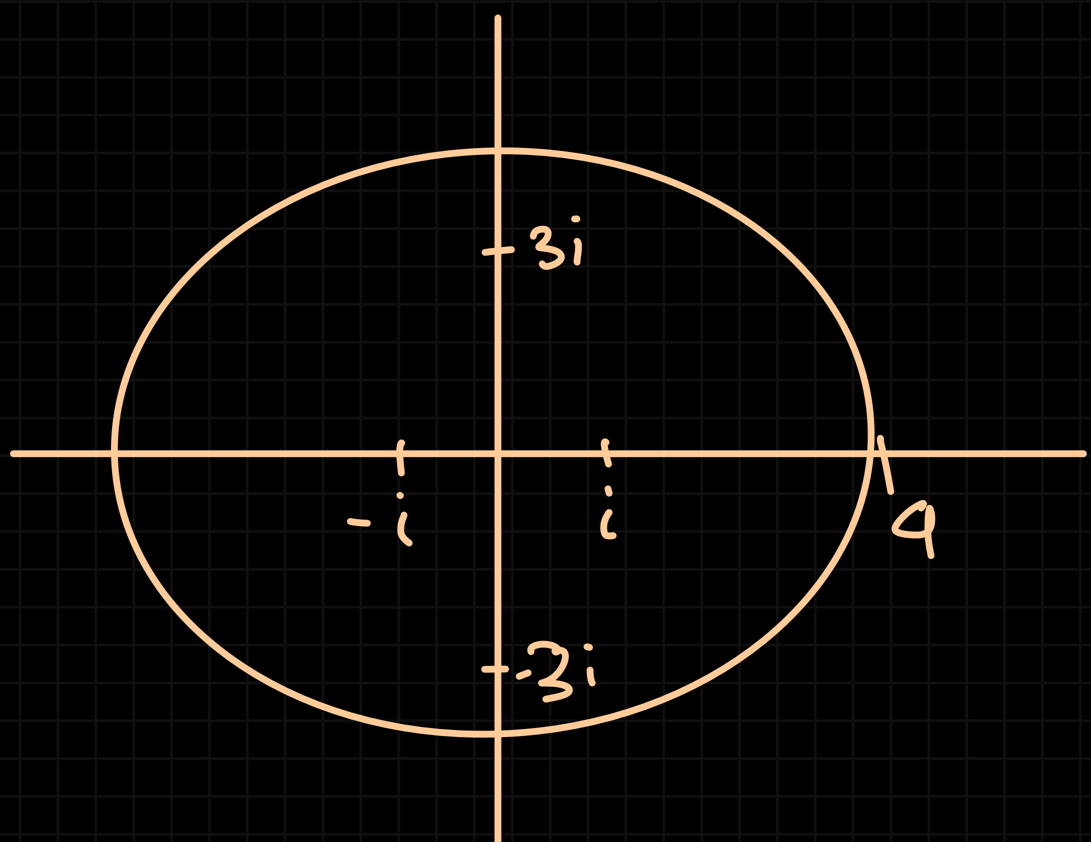
ans:

$$f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$

f(z) has singularities at,

$$z^4 + 10z^2 + 9 = 0$$

$$\Rightarrow z = 3i,\ i,\ -i,\ -3i$$



all four singularities are inside

$$\therefore f(z) = \frac{z^2 - z + 2}{(z+3i)(z-3i)(z+i)(z-i)}$$

$\boxed{\text{for } z = 3i}$:

simple pole.

$\therefore \text{Res}(f, z = 3i),$

$$= \lim_{z \to 3i} \left\{ (z - 3i) \times f(z) \right\}$$

$$= \lim_{z \to 3i} \left\{ (z-3i) \times \frac{(z^2 - z + 2)}{(z+3i)(z-3i)(z+i)(z-i)} \right\}$$

$$= \lim_{z \to 3i} \left\{ \frac{z^2 - z + 2}{(z+3i)(z+i)(z-i)} \right\}$$

$$= \frac{(3i)^2 - 3i + 2}{(3i+3i)(3i+i)(3i-i)}$$

$$= \frac{9i^2 - 3i + 2}{6i \cdot 4i \cdot 2i}$$

$$= \frac{-7 - 3i}{-48i}$$

$$= \frac{7+3i}{48i} = \frac{7i + 3i^2}{48i^2}$$

$$= \frac{7i + 3i^2}{-48} = -\frac{7i}{48} + \frac{1}{16}$$

<u>for $z = -3i$!</u>

simple pole.

$\therefore$ $\text{Res}(f, z = -3i)$,

$$= \lim_{z \to -3i} \left\{ (z+3i) \times f(z) \right\}$$

$$= \lim_{z \to -3i} \left\{ \frac{z^2 - z + 2}{(z+i)(z-i)(z-3i)} \right\}$$

$$= \frac{(-3i)^2 - (-3i) + 2}{(-2i)(-4i)(-6i)}$$

$$= \frac{9i^2 + 3i + 2}{-48i^3} = \frac{-9 + 3i + 2}{-48i^2 \cdot i}$$

$$= \frac{-7 + 3i}{48i} = \frac{(-7+3i)(i)}{48i^2}$$

$$= \frac{-7i + 3i^2}{-48} = \frac{-7i - 3}{-48}$$

$$= \frac{7i}{48} + \frac{1}{16}$$

for $z = i$:

simple pole

$\therefore \ \mathrm{Res}(f, z = i),$

$$= \lim_{z \to i} \left\{ (z-i) \times f(z) \right\}$$

$$= \lim_{z \to i} \left\{ (z-i) \times \frac{(z^2 - z + 2)}{(z+i)(z-i)(z+3i)(z-3i)} \right\}$$

$$= \lim_{z \to i} \left\{ \frac{z^2 - z + 2}{(z+i)(z+3i)(z-3i)} \right\}$$

$$= \frac{i^2 - i + 2}{2i \cdot 4i \cdot (-2i)}$$

$$= \frac{1 - i}{-16i}$$

$$= \frac{-i + i^2}{16}$$

$$= \frac{-1 - i}{16} = \frac{-1}{16} - \frac{i}{16}$$

for $z = -i$:

simple pole.

$\text{Res}(f, z=-i)$,

$$= \lim_{z \to -i} \left\{ (z+i) \times f(z) \right\}$$

$$= \lim_{z \to -i} \left\{ \frac{z^2 - z + 2}{(z-i)(z-3i)(z+3i)} \right\}$$

$$= \frac{(-i)^2 + i + 2}{(-2i) \cdot (-4i) \cdot (2i)}$$

$$= \frac{i^2 + i + 2}{-16i}$$

$$= \frac{1+i}{-16i}$$

$$= \frac{-i - i^2}{16}$$

$$= \frac{1-i}{16}$$

$$= \frac{1}{16} - \frac{i}{16}$$

finally,

$$\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\, dz$$

$$= 2\pi i \times \left\{ \cancel{-\frac{7}{48}} + \cancel{\frac{i}{16}} + \cancel{\frac{7}{48}} + \cancel{\frac{i}{16}} - \cancel{\frac{1}{16}} - \cancel{\frac{i}{16}} + \cancel{\frac{1}{16}} - \cancel{\frac{i}{16}} \right\}$$

$$= 0$$

now,

$$\frac{1}{2\pi i} \oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} \, dz$$

$$= \frac{1}{2\pi i} \times 0$$

$$= 0$$

[Ans]

4) Evaluate

$$\frac{1}{2\pi i}\oint_C \frac{e^{-zt}}{(z^2+1)^2}\,dz$$

if $t>0$ and C is the circle $|z|=3$

solve:

$$f(z)=\frac{e^{zt}}{(z^2+1)^2}$$

$f(z)$ has singularities at,

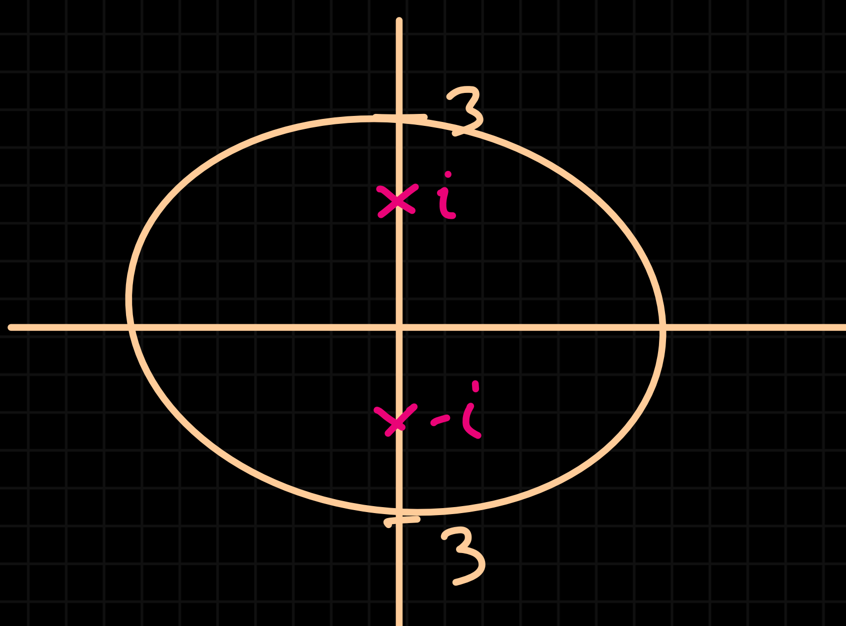
$$(z^2+1)^2 = 0$$

$$z^2+1 = 0$$

$$z^2 = -1$$

$$z = \pm\sqrt{-1}$$

$$z = \pm i$$



all of the singularities are inside.

$$f(z) = \frac{e^{zt}}{(z+i)^2 (z-i)^2}$$

for $z = i$:

order of pole $= 2 = n$

$\therefore$ $\text{Res}(f, z=2)$,

$$= \frac{1}{(n-1)!} \cdot \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n \cdot f(z) \right) \right\}$$

$$= \frac{1}{(2-1)!} \lim_{z \to i} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left((z-i)^2 \times \frac{e^{zt}}{(z-i)^2 (z+i)^2} \right) \right\}$$

$$= \lim_{z \to i} \left\{ \frac{d}{dz} \left(\frac{e^{zt}}{(z+i)^2} \right) \right\}$$

$$= \lim_{z \to i} \frac{(z+i)^2 \cdot t e^{zt} - e^{zt} \cdot 2(z+i) \cdot 1}{(z+i)^4}$$

$$= \frac{(2i)^2 \cdot t e^{it} - e^{it} \, 2 \cdot 2i}{(2i)^4}$$

$$= \frac{e^{it}\left(-4t - 4i\right)}{16}$$

$$= e^{it} \cdot \left(-\frac{t}{4} - \frac{i}{4}\right)$$

for $z = -i$:

order of pole, $n = 2$

$\therefore \text{Res}(f, z = -i),$

$$= \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n \cdot f(z) \right) \right\}$$

$$= \frac{1}{(2-1)!} \lim_{z \to -i} \left\{ \frac{d}{dz} \left((z+i)^2 \cdot \frac{e^{zt}}{(z+i)^2 (z-i)^2} \right) \right\}$$

$$= \lim_{z \to -i} \left\{ \frac{d}{dz} \left(\frac{e^{zt}}{(z-i)^2} \right) \right\}$$

$$= \lim_{z \to -i} \left\{ \frac{(z-i)^2 \cdot t e^{zt} - e^{zt} \cdot 2(z-i)}{(z-i)^4} \right\}$$

$$= \frac{(-2i)^2 \cdot t \cdot e^{-it} - e^{-it} \cdot 2 \cdot (-2i)}{(-2i)^4}$$

$$= \frac{e^{-it}\left(-4t + 4i\right)}{16}$$

$$= e^{-it}\left(\frac{-t}{4} + \frac{i}{4}\right)$$

finally,

$$\oint_C \frac{e^{zt}}{(z^2+1)^2}\, dz ,$$

$$= 2\pi i \cdot \left\{ e^{it}\left(-\frac{t}{4}-\frac{i}{4}\right) + e^{-it}\left(-\frac{t}{4}+\frac{i}{4}\right) \right\}$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^2}\, dz$$

$$= e^{it}\left(-\frac{t}{a}-\frac{i}{a}\right)+e^{-it}\left(-\frac{t}{a}+\frac{i}{a}\right)$$

⑩ Evaluate

$$\oint_C \frac{e^{2z}}{(z^2+\pi^2)^2}\, dz$$

where c is the circle $|z| = 4$

solve:

$$f(z) = \frac{e^{2z}}{(z^2+\pi^2)^2}$$

$f(z)$ has singularities at,

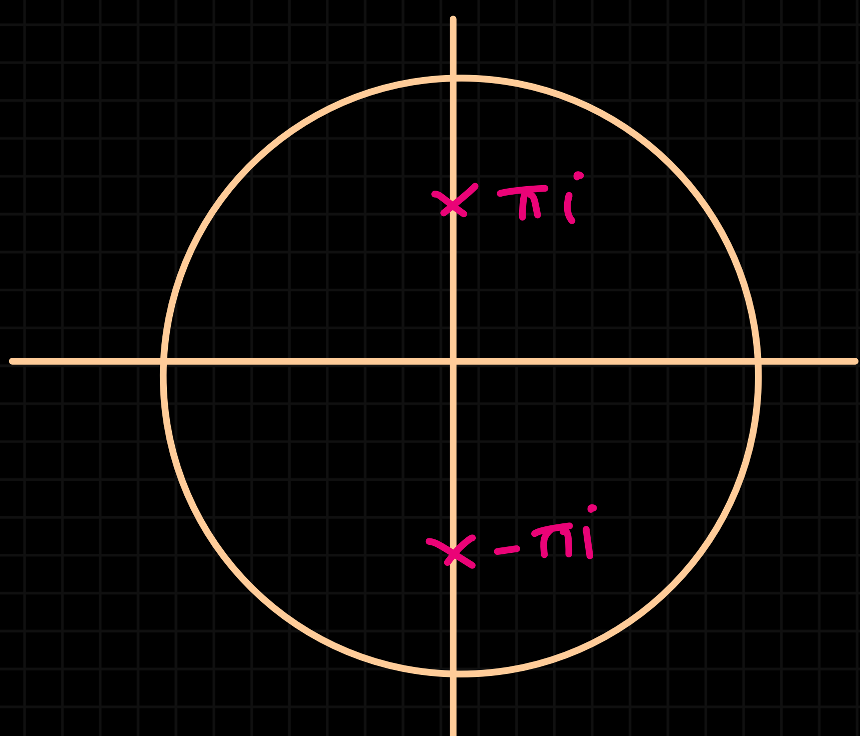
$$(z^2+\pi^2)^2 = 0$$

$$z^2+\pi^2 = 0$$

$$z^2 = -\pi^2$$

$$z = \pm\sqrt{-1}\cdot\sqrt{\pi^2}$$

$$= \pm\pi i$$



both the singularities are inside

C.

$$\therefore f(z) = \frac{e^{2z}}{(z+\pi i)^2 (z-\pi i)^2}$$

for $z = \pi i$:

order of pole $= n = 2$

$\therefore$ Res $(f, z = \pi i)$,

$$= \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n \cdot f(z) \right) \right\}$$

$$= \frac{1}{(2-1)!} \lim_{z\to\pi i} \left\{ \frac{d}{dz} \left((z-\pi i)^2 \cdot \frac{e^{2z}}{(z-\pi i)^2 (z+\pi i)^2} \right) \right\}$$

$$= \lim_{z\to\pi i} \frac{(z+\pi i)^2 \cdot 2e^{2z} - e^{2z} \cdot 2(z+\pi i) \cdot 1}{(z+\pi i)^4}$$

$$= \frac{(2\pi i)^2 \cdot 2 \cdot e^{2\pi i} - e^{2\pi i} \cdot 2(2\pi i)}{(2\pi i)^4}$$

$$= \frac{-8\pi^2 e^{2\pi i} - e^{2\pi i} \cdot 4\pi i}{16\pi^4}$$

$$= \frac{e^{2\pi i}\left(-2\pi - i\right)}{4\pi^3}$$

$$= \left\{\cos(2\pi) + i\sin(2\pi)\right\}\left(\frac{-2\pi - i}{4\pi^3}\right)$$

$$= \frac{-2\pi i - i}{4\pi^3}$$

for $z = -\pi i$:

order of pole, $n = 2$

now, $\operatorname{Res}(f, z = -\pi i)$,

$$= \lim_{z \to -\pi i} \left\{ \frac{d}{dz} \left((z+\pi i)^2 \cdot f(z) \right) \right\}$$

$$= \lim_{z \to -\pi i} \left\{ \frac{d}{dz} \left(\frac{e^{2z}}{(z-\pi i)^2} \right) \right\}$$

$$= \lim_{z \to -\pi i} \left\{ \frac{(z-\pi i)^2 \cdot 2e^{2z} - e^{2z} \cdot 2(z-\pi i)\cdot 1}{(z-\pi i)^4} \right\}$$

$$= \frac{(-2\pi i)^2 \cdot 2 \cdot e^{-2\pi i} - e^{-2\pi i} \cdot 2(-2\pi i)}{(-2\pi i)^4}$$

$$= e^{-2\pi i} \left\{ \frac{-8\pi^2 + 4\pi i}{16\pi^4} \right\}$$

$$= \left\{ \cos(-2\pi) + i\sin(-2\pi) \right\} \times \left\{ \frac{-2\pi + i}{4\pi^3} \right\}$$

$$= \frac{-2\pi + i}{4\pi^3}$$

now, $\displaystyle\oint_C \frac{e^{2z}}{(z^2+\pi^2)^2}$

$$= 2\pi i \cdot \left\{ \frac{-2\pi - i}{4\pi^3} + \frac{-2\pi + i}{4\pi^3} \right\}$$

$$= 2\pi i \left\{ \frac{-4\pi}{4\pi^3} \right\}$$

$$= \frac{-8\pi^2 i}{9\pi^3}$$

$$= \frac{-2i}{\pi}$$

[Ans]