MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Laplace 02 - Inverse Laplace Transformations

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Inverse Laplace Transformation

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t)$$

Inverse Laplace Transforms of some Algebraic Functions

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Transforms of some Trigonometric and Hyperbolic Functions

1. $\mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$

2. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$

3. $\mathcal{L}^{-1}\left\{\dfrac{a}{s^2-a^2}\right\} = \sinh(at)$

4. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh(at)$

if $\mathcal{L}\{f(t)\} = F(s)$

then

- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
- $\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds}\left(F(s)\right)$
- $\mathcal{L}^{-1}\{F(s)\} = e^{at} \cdot \mathcal{L}^{-1}\{F(s+a)\}$

also,

- $\mathcal{L}\{f(t) \cdot u(t-a)\} = \mathcal{L}\{f(t+a)\} e^{-as}$
- $\mathcal{L}^{-1}\{F(s) e^{-as}\} = f(t-a) \cdot u(t-a)$

Transformation of derivaties

$\mathcal{L}\{f(t)\} = F(s)$ হলে,

$\boxed{1}$ $\mathcal{L}\{f'(t)\} = s \cdot F(s) - f(0)$

$\boxed{2}$ $\mathcal{L}\{f''(t)\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$

Linearity of Inverse Laplace transformation:

$$\mathcal{L}^{-1}\left\{F(s) \pm G(s)\right\} = \mathcal{L}^{-1}\left\{F(s)\right\} \pm \mathcal{L}^{-1}\left\{G(s)\right\}$$

$$\mathcal{L}^{-1}\left\{k \cdot F(s)\right\} = k \cdot \mathcal{L}^{-1}\left\{F(s)\right\}$$

D Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= \frac{1}{6} \times t^3$$

{ans}

$$\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$$

$$= -12\,\mathcal{L}^{-1}\left\{\frac{1}{3s-4}\right\}$$

$$= -12 \times \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\}$$

$$= -4\,\mathcal{L}^{-1}\left\{\frac{1}{s-4/3}\right\}$$

$$= -4\,e^{\left(\frac{4}{3}\right)t}$$

Evaluate

i) $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+9}\right\}$

$= 5 \cdot \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3^2}\right\}$

$= \dfrac{5}{3} \cdot \mathcal{L}^{-1}\left\{\dfrac{3}{s^2+3^2}\right\}$

$= \dfrac{5}{3}\sin 3t$

B $\mathcal{L}^{-1}\left\{\dfrac{23s-15}{s^2+4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{23s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{15}{s^2+4}\right\}$$

$$= 23 \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 15\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= 23 \cdot \cos(2t) - \frac{15}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 23\cos(2t) - \frac{15}{2}\sin(2t)$$

iii) $$\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-9}\right\}$$

$$= 2\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\} - \frac{5}{3}\cdot\mathcal{L}^{-1}\left\{\frac{3}{s^2-3^2}\right\}$$

$$= 2\cosh(3t) - \frac{5}{3}\sinh(3t)$$

First translation theorem for inverse laplace transformation

⊡ if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and

a is any real number, then,

$$\mathcal{L}^{-1}\{F(s)\} = e^{at} \cdot \mathcal{L}^{-1}\{F(s+a)\}$$

⑦ Find the inverse laplace transform of

$$\frac{6}{(s-2)^6}$$

solve:

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

here $F(s) = \frac{6}{(s-2)^6}$

replacing s with what will give simply s^6 ? $\Rightarrow s \to s+2$

now,

$$\therefore \mathcal{L}^{-1}\left\{F(s)\right\} = e^{at}\, \mathcal{L}^{-1}\left\{F(s+a)\right\}$$

$\therefore$ replacing s with $s+2$,

$$\therefore \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$= e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{6}{(s+2-2)^6}\right\}$$

$$= e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{6}{s^6}\right\}$$

$$= e^{2t} \cdot 6 \cdot \frac{1}{5!} \quad \mathcal{L}^{-1}\left\{ \frac{5!}{s^{5+1}} \right\}$$

$$= e^{2t} \cdot \frac{6}{120} \cdot t^5$$

$$= \frac{e^{2t} \cdot t^5}{20}$$

① $\mathcal{L}^{-1}\left\{\dfrac{5}{(s-3)^2+25}\right\}$

solve:

$s \to s+3$

$\therefore \mathcal{L}^{-1}\left\{\dfrac{5}{(s-3)^2+25}\right\}$

$= e^{3t} \cdot \mathcal{L}^{-1}\left\{\dfrac{5}{(s+3-3)^2+5^2}\right\}$

$= e^{3t} \cdot \mathcal{L}^{-1}\left\{\dfrac{5}{s^2+5^2}\right\}$

$= e^{3t} \cdot \sin(5t)$

B) $\mathcal{L}^{-1}\left\{ \dfrac{4s}{(s-3)^2+25} \right\}$

solve:

replacing s with s+3,

$$4 \cdot e^{3t} \cdot \mathcal{L}^{-1}\left\{ \frac{s+3}{s^2+5^2} \right\}$$

$$= 4 \cdot e^{3t}\left[\mathcal{L}^{-1}\left\{ \frac{s}{s^2+5^2} \right\} + \mathcal{L}^{-1}\left\{ \frac{3}{s^2+5^2} \right\}\right]$$

$$= 4e^{3t}\left[\cos(5t) + \frac{3}{5} \cdot \mathcal{L}^{-1}\left\{ \frac{5}{s^2+5^2} \right\}\right]$$

$$= 9\, e^{3t} \left[\cos 5t + \frac{3}{5} \cdot \sin(5t) \right\}$$

⑭ $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-8s-9}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-2\cdot s\cdot 4+4^2-25}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{6s-4}{(s-4)^2-5^2}\right\}$

replacing s with $s+4$,

$= e^{4t}\cdot\mathcal{L}^{-1}\left\{\dfrac{6(s+4)-4}{s^2-5^2}\right\}$

$$= e^{4t}\left[\mathcal{L}^{-1}\left\{\frac{6s}{s^2-5^2}\right\}+\mathcal{L}^{-1}\left\{\frac{20}{s^2-5^2}\right\}\right]$$

$$= e^{at}\left(6\cdot\cosh(5t)+\frac{20}{5}\cdot\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\}\right]$$

$$= e^{at}\left[6\cosh(5t)+4\quad\sinh(5t)\right\}$$

B $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+2s+5}\right\}$

$$= 5 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1^2+2^2}\right\}$$

$$= 5 \ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

replacing s with $s-1$

$$= 5 \cdot e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= \frac{5}{2} \cdot e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{5}{2} \cdot e^{-t} \cdot \sin(2t)$$

Second translation theorem for Inverse laplace

if $\mathcal{L}\{f(t)\} = F(s)$ then,

$$\mathcal{L}\{f(t) \cdot u(t-a)\} = \mathcal{L}\{f(t+a)\} \cdot e^{-as}$$

if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and a is any real number then,

$$\mathcal{L}^{-1}\{F(s)\, e^{-as}\} = f(t-a) \cdot u(t-a)$$

NB: e^{-as} type কিছু L^{-1} এ থাকলে

use 2nd transl. then.

1) $\mathcal{L}^{-1}\left\{\frac{1}{s^4} e^{-2s}\right\}$

solve:

let $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$

$= \frac{1}{3!} \cdot \mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$

$= \frac{1}{6} \cdot t^3$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{1}{s^4} e^{-2s} \right\}$$

$$= f(t-a) \cdot u(t-a)$$

$$= \frac{1}{6} \cdot (t-2)^3 \cdot u(t-2)$$

[Ans]

$\boxed{D}$ $\mathcal{L}^{-1}\left\{\frac{1}{s-3} \cdot e^{-2s}\right\}$

solve:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= e^{3t}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s-3} e^{-2s}\right\}$$

$$= f(t-a) \cdot u(t-a)$$

$$= e^{3(t-2)} \cdot u(t-2)$$

Ans.

⑬ $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\, e^{-\pi s}\right\}$

solve!

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{2} \cdot \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{2}\sin(2t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\, e^{-\pi s}\right\}$$

$$= \frac{1}{2}\sin\{2(t-\pi)\} \cdot u(t-9)$$

$$= \frac{1}{2}\sin(2t) \cdot u(t-9)$$

NB:

$$\sin(x \pm \text{odd}.\pi) = -\sin x$$

$$\sin(x \pm \text{even}.\pi) = \sin x$$

$$\cos(x \pm \text{odd}.\pi) = -\cos x$$

$$\cos(x \pm \text{even}\pi) = \cos x$$

$$\sin(-x) = \sin x$$

$$\cos(-x) = \cos(x)$$

$\boxed{4}$ $\mathcal{L}^{-1}\left\{ \frac{s}{s^2+9} \cdot e^{-\pi s} \right\}$

solve:

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+9} \right\}$$

$$= \cos 2t$$

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+9} \cdot e^{-\pi s} \right\}$$

$$= f(t-a) \cdot u(t-a)$$

$$= \cos 2\cdot(t-\pi) \cdot u(t-a)$$

$$= \cos 2t \cdot u(t-a)$$

$\boxed{B}$ $\mathcal{L}^{-1}\left\{ \dfrac{6s-3}{s^2+9} \cdot e^{-\pi s} \right\}$

ans:

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{6s-3}{s^2+9} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{6s}{s^2+9} \right\} - \mathcal{L}^{-1}\left\{ \frac{3}{s^2+9} \right\}$$

$$= 6 \cdot \cos(2t) - \frac{3}{2} \cdot \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= 6\cos 2t - \frac{3}{2}\sin(2t)$$

now,

$a = \pi$

$$\mathcal{L}^{-1}\left\{\frac{6s-3}{s^2+9} \cdot e^{-\pi s}\right\}$$

$$= f(t-a) \cdot u(t-a)$$

$$= \left\{6\cos 2(t-\pi) - \frac{3}{2}\sin 2(t-\pi)\right\}$$

$$\cdot\, u(t-\pi)$$

$$= \left(6\cos 2t - \frac{3}{2}\sin 2t\right) \cdot u(t-\pi)$$

$\boxed{\square}$ $\mathcal{L}^{-1}\left\{\dfrac{6s-9}{s^2-8s+25}\cdot e^{-\pi s}\right\}$

ans:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6s-9}{s^2-8s+25}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6s}{s^2-8s+25}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s^2-8s+25}\right\}$$

$$= 6\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2-2.s.4+4^2+3^2}\right\}$$

$$- 4.\ \mathcal{L}^{-1}\left\{\frac{1}{s^2-2.s.4+4^2+3^2}\right\}$$

$$= 6\, \mathcal{L}^{-1}\left\{ \frac{s}{(s-9)^2 + 3^2} \right\}$$

$$- 9\, \mathcal{L}^{-1}\left\{ \frac{1}{(s-9)^2 + 3^2} \right\}$$

replacing s with $s+9$

$$= 6 \cdot e^{9t} \cdot \mathcal{L}^{-1}\left\{ \frac{s+9}{s^2+3^2} \right\} -$$

$$\frac{9}{3} \cdot e^{9t} \mathcal{L}^{-1}\left\{ \frac{3}{s^2+3^2} \right\}$$

$$= 6\, e^{9t} \left[\mathcal{L}^{-1}\left\{ \frac{s}{s^2+3^2} \right\} + \frac{9}{3} \mathcal{L}^{-1}\left\{ \frac{3}{s^2+3^2} \right\} \right]$$

$$-\frac{a}{3}e^{at}\sin(3t)$$

$$= 6e^{at}\left[\cos 3t + \frac{a}{3}\sin 3t\right]$$

$$- \frac{a}{3}e^{at}\sin 3t$$

now,

$$\mathcal{L}^{-1}\left\{\frac{6s-a}{s^2-8s+25}e^{-\pi s}\right\}$$

$a=\pi$

$$= f(t-a) \cdot u(t-a)$$

$$= \left[6e^{9(t-\pi)} \left[\cos 3(t-\pi) + \frac{9}{3} \sin 3(t-\pi) \right] - \frac{9}{3} e^{9(t-\pi)} \sin 3(t-\pi) \right]$$

$$\cdot\, u(t-\pi)$$

$$= \left[6e^{9t-9\pi} \left[-\cos 3t - \frac{9}{3} \sin 3t \right] + \frac{9}{3} e^{9t-9\pi} \cdot \sin 3t \right] \cdot u(t-\pi)$$

$$= e^{9t-9\pi} \left[6 \left\{ -\cos 3t - \frac{9}{3} \sin 3t \right\} \right.$$

$$+\frac{9}{3}\sin 3t\Big]\cdot u(t-\pi)$$

$$= e^{9t-9\pi}\Big[-6\cos 3t - 8\sin 3t$$

$$+\frac{9}{3}\sin 3t\Big]\cdot u(t-\pi)$$

$$= e^{9t-9\pi}\Big[-6\cos 3 - \frac{20}{3}\sin 3t\Big]\cdot$$

$$u(t-9)$$

[Ans]

Partial Fraction Method

$$\frac{\cdots}{(s-2)^3(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3} + \frac{D}{(s+1)}$$

$$\frac{\cdots}{(s-1)^2(s^2+9)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{(s^2+9)}$$

$$\frac{\cdot \quad - \quad - \quad \cdot}{(s-1)^3 (s^2+9)^2 (s^2+1)^2}$$

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{Dx+C}{s^2+9}$$

$$\frac{Fs+G}{(s^2+9)^2} + \frac{Hs+I}{s^2+1} + \frac{Js+k}{(s^2+1)^2}$$

(Q) Decompose into partial fractions and then find the inverse laplace transform of,

$$\frac{1}{s(s+1)}$$

solve:

$$\frac{1}{s(s+1)} \equiv \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 1 \equiv A(s+1) + Bs$$

when $s=0$:

$$1 = A$$

$s=-1$:

$$1 = B(-1)$$

$$B = 1$$

$$\therefore \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - \mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}$$

$$= 1 - e^{-t}$$

Q) Decompose into partial fractions and then find the inverse laplace transfom m of,

$$\frac{1}{s(s+1)}e^{-s}$$

solve:

$$\frac{1}{s(s+1)} \equiv \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 1 \equiv A(s+1) + Bs$$

when $s = 0$:

$1 = A$

$s = -1$:

$1 = B(-1)$

$B = 1$

$$\therefore \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - \mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}$$

$$= 1 - e^{-t}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\ e^{-1 \cdot s}\right\}$$

$a=1$

$$= f(t-a) \cdot u(t-a)$$

$$= \left\{ 1 - e^{-(t-1)} \right\} \cdot u(t-1)$$

$$= \left(1 - e^{1-t} \right) \cdot u(t-1)$$

[Ans]

1) Decompose into partial fractions and then find the inverse laplace transform of,

$$\frac{9s^2-5s}{(s+1)(s-2)^2}$$

Solve:

$$\frac{9s^2-5s}{(s+1)(s-2)^2} = \frac{A}{s+1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$$\Rightarrow 4s^2 - 5s = A(s-2)^2 + B(s+1)(s-2)$$
$$+ C(s+1)$$

$s = -1$:

$4 + 5 = 9A$

$A = 1$

$s = 2$:

$16 - 10 = 3C$

$C = 2$

$$4s^2 - 5s = A(s-2)^2 + B(s+1)(s-2) + C(s+1)$$

$$\Rightarrow 4s^2 - 5s = A(s^2 - 2s + 4) + B(s^2 - s - 2) + C(s+1)$$

$$\Rightarrow 4s^2 - 5s = s^2(A+B) + s\{-2A - B + C\} + 4A - 2B + C$$

now, equating the coefficients

$$4 = A + B$$

$$4 = 1 + B$$

$\Rightarrow B = 3$

$$\therefore \frac{9s^2 - 5s}{(s+1)(s-2)^2} = \frac{1}{s+1} + \frac{3}{s-2} + \frac{2}{(s-2)^2}$$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{9s^2 - 5s}{(s+1)(s-2)^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{3}{s-2} \right\} + \mathcal{L}^{-1}\left\{ \frac{2}{(s-2)^2} \right\}$$

$$= e^{-t} + 3 \cdot e^{2t} + 2 \cdot e^{2t} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{(s+2-2)^2} \right\}$$

$$= e^{-t} + 3e^{2t} + 2e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-t} + 3e^{2t} + 2e^{2t}\,t$$

Ⓓ Decompose into partial fractions and then find the inverse laplace transform of,

$$\frac{-s}{(s^2+1)(s+1)}$$

solve:

$$\frac{-s}{(s^2+1)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1}$$

$$\Rightarrow -s = (As+B)(s+1) + C(s^2+1)$$

$\underline{s = -1}$:

$1 = 2C$

$\Rightarrow C = \frac{1}{2}$

now,

$-s = As^2 + As + Bs + B + Cs^2 + C$

$\Rightarrow -s = s^2(A+C) + s(A+B) + B+C$

equating,

$A + C = 0$

$$A = -C$$

$$\therefore A = -\frac{1}{2}$$

again,

$$A + B = -1$$

$$\Rightarrow B = \frac{1}{2} - 1$$

$$B = -\frac{1}{2}$$

so,

$$\frac{-s}{(s^2+1)(s+1)} = \frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1} + \frac{1/2}{s+1}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{-s}{(s^2+1)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1/2}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$+ \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -\frac{1}{2} \cos t - \frac{1}{2} \sin t + \frac{1}{2} e^{-t}$$

now,

$$f(t) = -\frac{1}{2} \cos t - \frac{1}{2} \sin t + \frac{1}{2} e^{-t}$$

$$\therefore \mathcal{L}^{-1} \left\{ \frac{-s}{(s^2+1)(s+1)} e^{-\pi s} \right\}$$

$a = \pi$

$$= f(t-a) \cdot u(t-a)$$

$$= \left\{ -\frac{1}{2} \cos(t-\pi) - \frac{1}{2} \sin(t-\pi) + \frac{1}{2} e^{-(t-\pi)} \right\}$$

$$\cdot u(t-a)$$

$$= \left(\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2} e^{\pi - t} \right),$$

$$u(t-a)$$

Q) Decompose into partial fractions and then find the inverse laplace transform of,

$$\frac{1}{(s^2+1)(s^2+9)}$$

ans:

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow 1 = (As+B)(s^2+9) + (Cs+D)(s^2+1)$$

$$\Rightarrow 1 = As^3 + 9As + Bs^2 + 9B + Cs^3 + Cs + Ds^2 + D$$

$$\Rightarrow 1 = s^3(A+C) + s^2(B+D) + s(9A+C) + 9B+D$$

equating,

$$A + C = 0$$

$$\Rightarrow \boxed{A = -C}$$

$$B + D = 0$$

$$\Rightarrow \boxed{B = -D}$$

$$4A + C = 0$$

$$4A = -C$$

$$A = \frac{-C}{4}$$

$$\therefore \Rightarrow -C = \frac{-C}{4}$$

$$\Rightarrow -4C = -C$$

$$\Rightarrow -3C = 0$$

$\Rightarrow \boxed{C = 0}$

since, $A = -C$, $\boxed{A = 0}$

$B = -D$

now,

$$9B + D = 1$$

$$\Rightarrow 9(-D) + D = 1$$

$$-3D = 1$$

$$\boxed{D = -1/3}$$

$$\therefore \boxed{B = 1/3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/3}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{3}\times\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin(2t)$$

Ⓓ Decompose into partial fractions and then find the inverse laplace transform of,

$$\left\{ \frac{1}{(s^2+1)(s^2+9)} e^{-2\pi s} \right\}$$

ans:

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow 1 = (As+B)(s^2+9) + (Cs+D)(s^2+1)$$

$$\Rightarrow 1 = As^3 + 9As + Bs^2 + 9B +$$
$$Cs^3 + Cs + Ds^2 + D$$

$$\Rightarrow 1 = s^3(A+C) + s^2(B+D) +$$
$$s(9A+C) + 9B + D$$

equating,

$$A + C = 0$$

$$\Rightarrow \boxed{A = -C}$$

$$B + D = 0$$

$$\Rightarrow \boxed{B = -D}$$

$$4A + C = 0$$

$$4A = -C$$

$$A = \frac{-C}{4}$$

$$\therefore \Rightarrow -C = \frac{-C}{4}$$

$$\Rightarrow -4C = -C$$

$$\Rightarrow -3C = 0$$

$$\Rightarrow \boxed{C = 0}$$

since, $A = -C$, $\boxed{A = 0}$

$B = -D$

now,

$$4B + D = 1$$

$$\Rightarrow 4(-D) + D = 1$$

$$-3D = 1$$

$$\boxed{D = -1/3}$$

$$\therefore \boxed{B = 1/3}$$

$$\therefore \alpha^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$$

$$= \alpha^{-1}\left\{\frac{1/3}{s^2+1}\right\} + \alpha^{-1}\left\{\frac{-1/3}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{3}\times\frac{1}{2}\,\alpha^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin(2t)$$

$$\therefore f(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin(2t)$$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s} \right\}$$

$a = 2\pi$

$$= f(t-a) \cdot u(t-a)$$

$$= \left\{ \frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi) \right\}$$

$$\cdot u(t-a)$$

$$= \left\{ \frac{1}{3} \sin t - \frac{1}{6} \sin(2t) \right\} \cdot u(t-9)$$

Q) Decompose into partial fractions and then find the inverse laplace transform of,

$$\frac{s^2 + 2s + 3}{(s^2+2s+2)(s^2+2s+5)}$$

ans:

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\Rightarrow s^2 + 2s + 3 = (As + B)(s^2 + 2s + 5) +$$

$$(Cs + D)(s^2 + 2s + 2)$$

$A = 0$

$B = 1/3$

$C = 0$

$D = \frac{2}{3}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2/3}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2.s.1+1+1^2}\right\} +$$

$$\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2.s.1+1^2+2^2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

replacing s with s-1,

$$= \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1^2} \right\} + \frac{2}{3} \cdot e^{-t} \cdot$$

$$\frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{e^{-t}}{3} \sin 2t$$

[Ans]

D) Decompose into partial fractions and then find the inverse laplace transform of,

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}e^{-3\pi s}$$

ans:

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\Rightarrow s^2 + 2s + 3 = (As+B)(s^2+2s+5) +$$

$$(Cs+D)(s^2+2s+2)$$

$$A = 0$$

$$B = 1/3$$

$$C = 0$$

$$D = \frac{2}{3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2/3}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2.s.1+1+1^2}\right\} +$$

$$\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2.s.1+1^2+2^2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

replacing s with s-1,

$$= \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1^2} \right\} + \frac{2}{3} \cdot e^{-t} \cdot \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{e^{-t}}{3} \sin 2t$$

now,

$$\mathcal{L}^{-1}\left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} e^{-3\pi s} \right\}$$

using 2nd translation theorem,

$a = 3\pi$

$$= f(t-a) \cdot u(t-a)$$

$$= \left\{ \frac{1}{3} e^{-(t-3\pi)} \sin(t-3\pi) + \frac{e^{-(t-3\pi)}}{3} \sin 2(t-3\pi) \right\}$$

$$\cdot u(t-a)$$

$$= \left\{ -\frac{1}{3} e^{3\pi - t} \sin t + \frac{1}{3} e^{3\pi - t} \sin 2t \right\}$$

$$\cdot u(t-9)$$

1st vs 2nd Transl. Thm.

1st

$$\mathcal{L}\left\{ e^{at} f(t) \right\}$$

$$\mathcal{L}^{-1}\left\{ F(s-a) \right\}$$

NB: exponential in time, shift in frequency

2nd

$$\mathcal{L}\left\{ f(t)\, u(t-a) \right\} = \mathcal{L}\{ f(t+a) \} e^{-as}$$

$$\mathcal{L}^{-1}\left\{ F(s)\, e^{-as} \right\} = f(t-a) \cdot u(t-a)$$

* shift in time, exponential in frequency