MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Chapter 4 Part 2 (Cauchy-Goursat Theorem)

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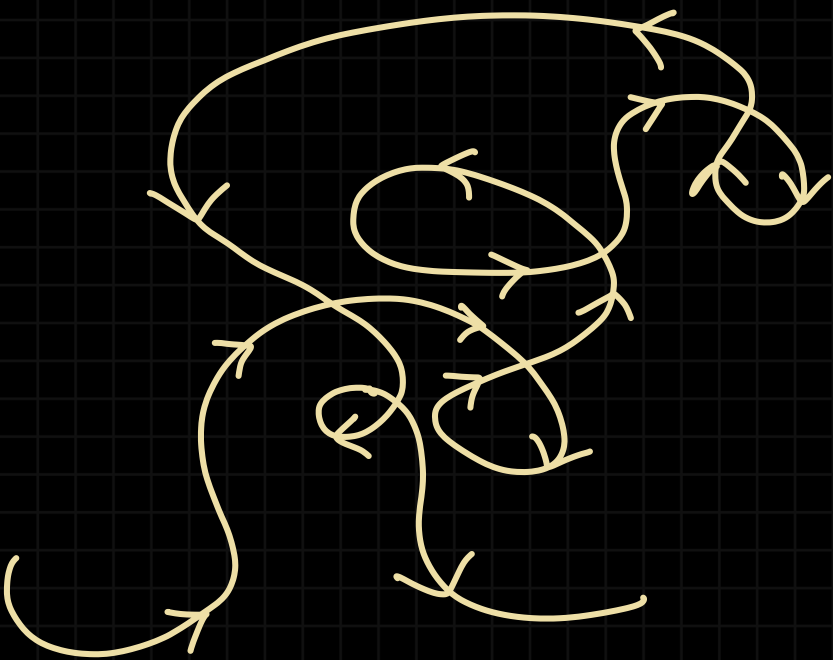
Cauchy-Goursat Theorem

two prerequisite concepts

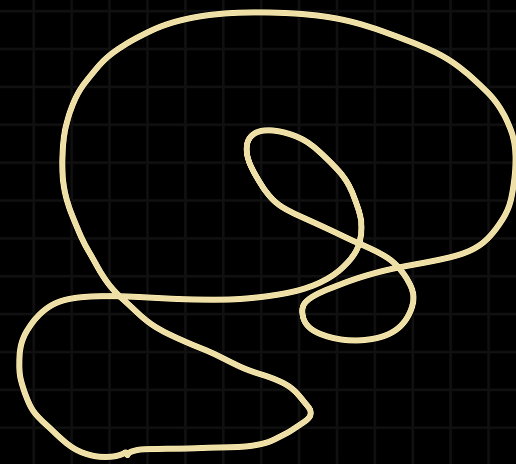
(i) simple closed contour

(ii)

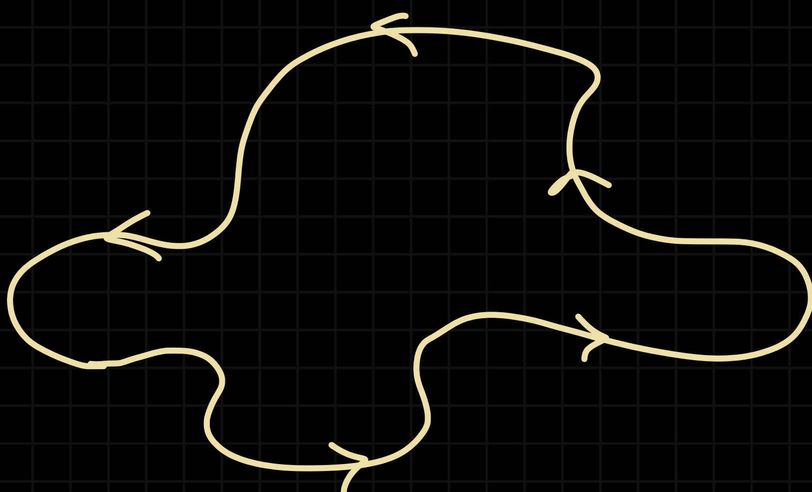
Simple Closed Contour



contour



closed contour

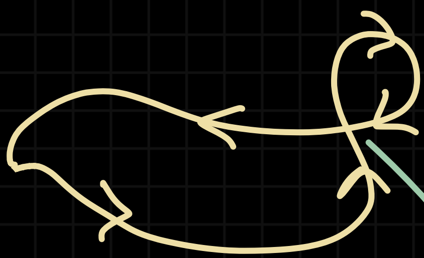


simple closed contour

no self-intersection

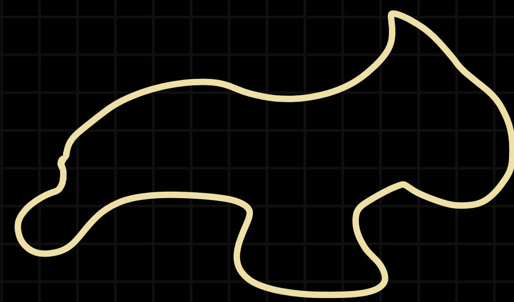
→ doesn't cross itself

→ ends where it started

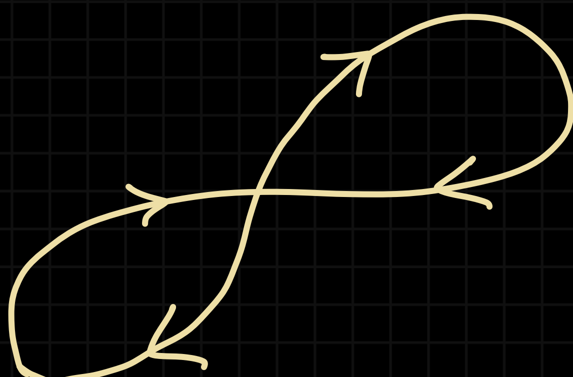


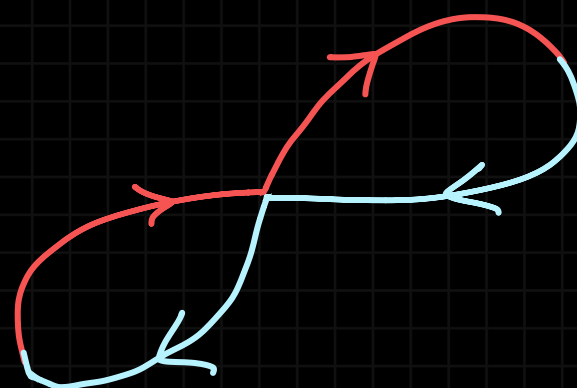
closed but

not simple



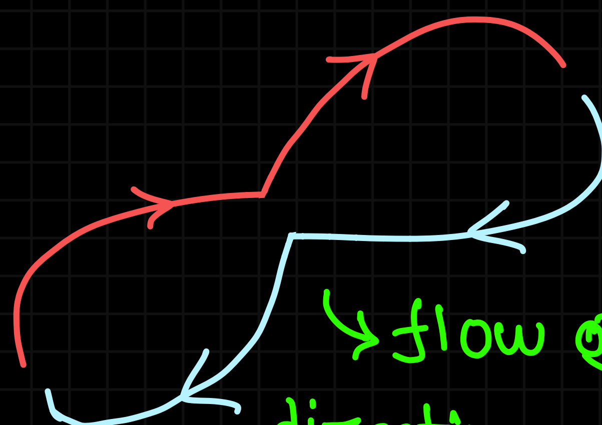
simple and closed





simple and

closed. because

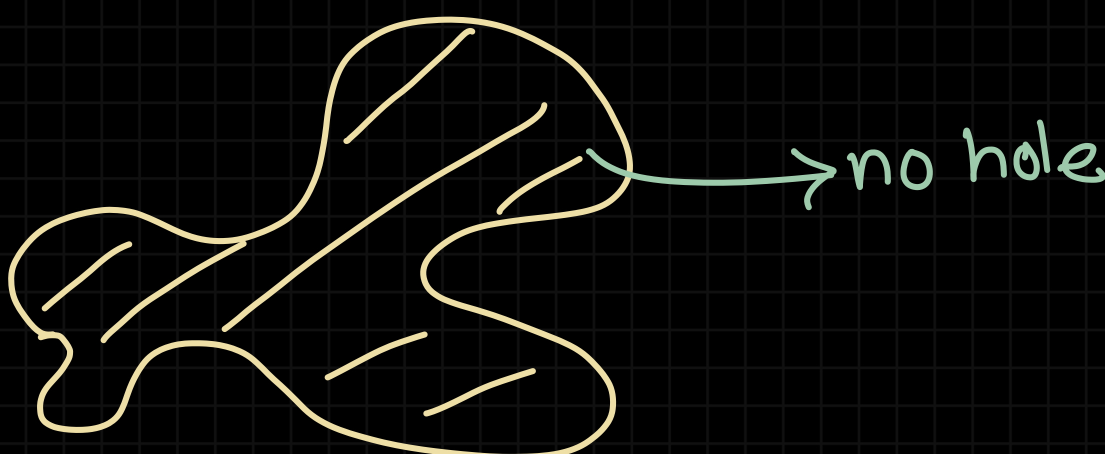


↳ flow of direction makes them touch themselves, but not cross each other

Simply and Multiply Connected Regions

A region R is called simply-connected if ~~any~~ all the possible simple closed curve, in that region which lies in R, can be shrunk to a point without leaving R. A region R, which is not simply-connected, is called multiply connected.

Simply Connected Region



Multiply Connected Region

Cauchy-Goursat Theorem

Cauchy-Goursat Theorem

Let $f(z)$ be analytic in a region R and on its boundary C. Then

$$\oint_C f(z)dz = 0$$

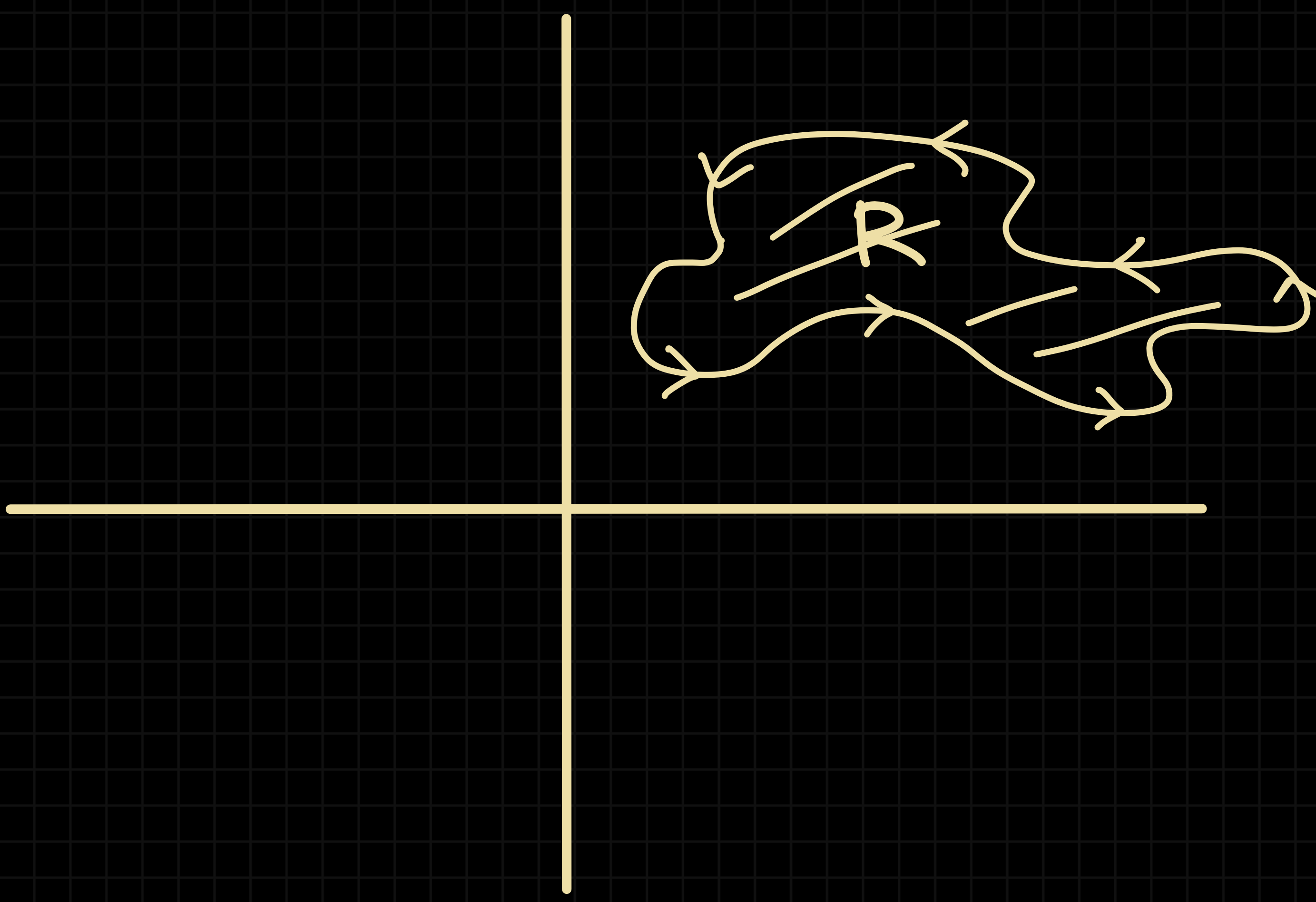
This fundamental theorem, often called Cauchy's integral theorem or simply Cauchy's theorem, is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in R. However, Goursat gave a proof which removed this restriction. For this reason, the theorem is sometimes called the Cauchy–Goursat theorem when one desires to emphasize the removal of this restriction.

Cauchy-Goursat theorem:

let a function $f(z)$ be analytic

on a region R and on its boundary c,

then, $\oint_c f(z)\, dz = 0$



4. Evaluate $\oint (5z^4 - z^3 + 2)\, dz$

around the circle $|z| = 1$

☆ closed integral → solve with cauchy theorem

solve:

step-1: check if analytic

$f(z) = 5z^4 - z^3 + 2$

$|z| = 1$

There is no singularity inside and on the boundaries of $|z|=1$

$\therefore f(z)$ is analytic inside and on the boundary of $c: |z|=1$

$$\therefore \oint f(z)\,dz = 0$$

(by using cauchy-goursat theorem)

$$\therefore \oint_c (5z^4 - z^3 + 2)\,dz = 0$$

Evaluate

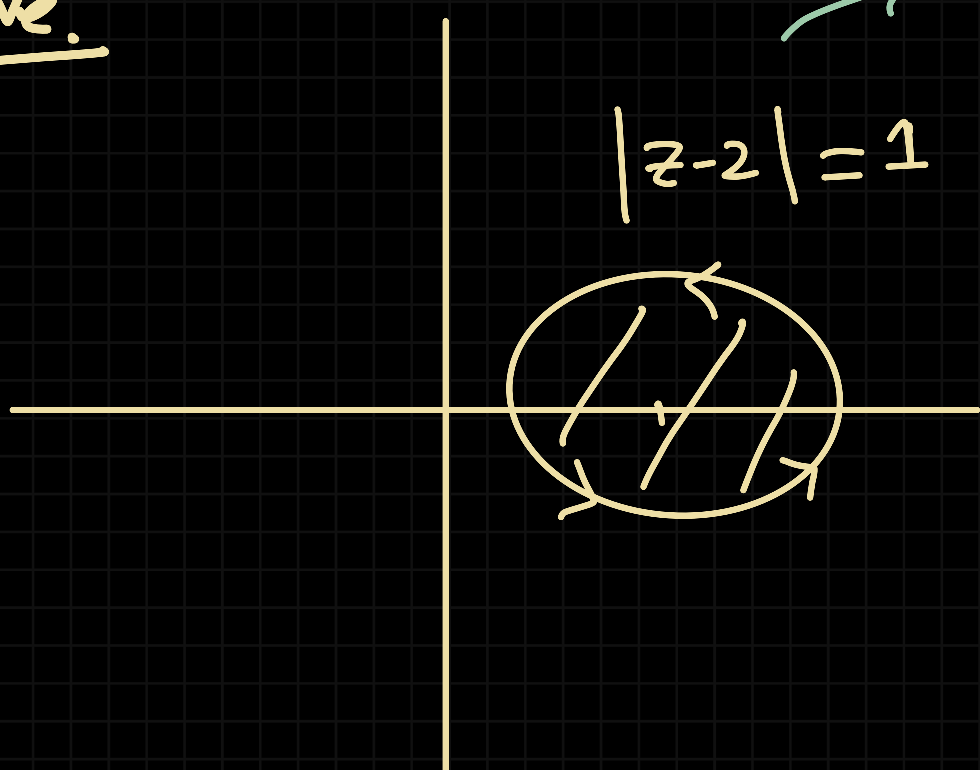
$$\oint_C \frac{e^{3z}\cos(z)}{(z^2+\pi^2)^3(z-5)}\,dz$$

where C is the circle $|z-2|=1$

solve:

center $(2,0)$
radius 1

$|z-2|=1$



never undefined

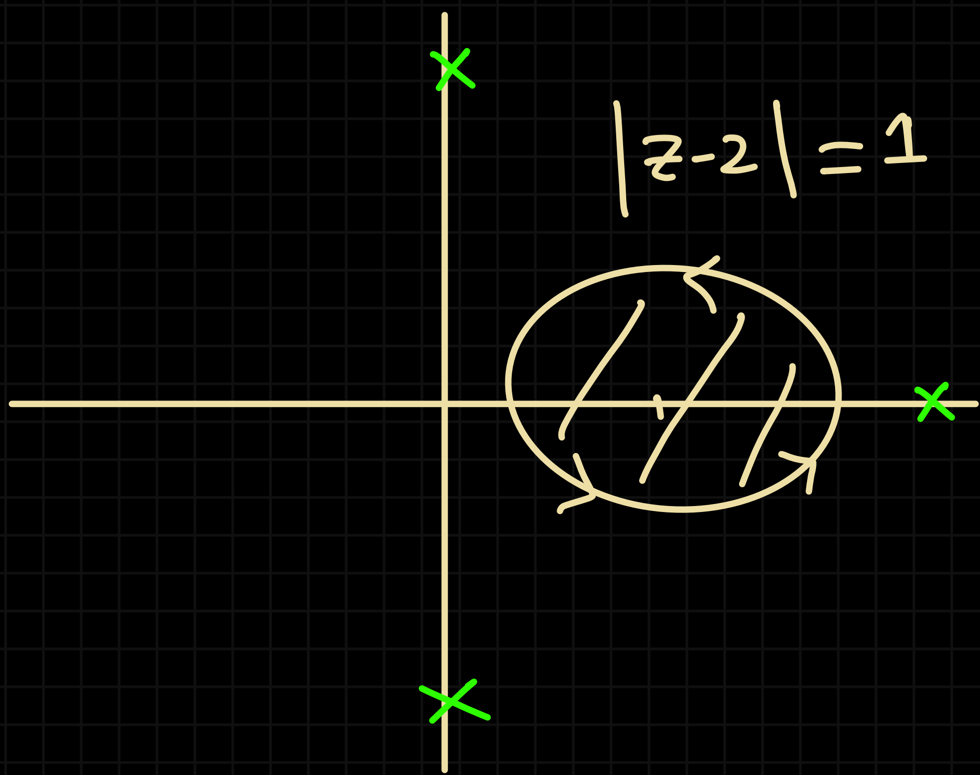
$$f(z) = \frac{e^{3z} \cos(z)}{(z^2+\pi^2)^3 (z-5)} dz$$ has

singularity at

$z - 5 = 0$	$(z^2 + \pi^2)^3 = 0$
$z = 5$	$z^2 + \pi^2 = 0$
	$z^2 = -\pi^2$
	$z = \pm i\pi$

the singularities are at

$$z = 5, \pi i, -\pi i$$



$\therefore f(z)$ has no singularity inside and on the boundary of $c: |z-2| = 1$

$\therefore f(z)$ is analytic inside and on the boundary of c.

$\therefore$ according to cauchy-goursat theorem,

$$\oint_c f(z)\,dz = 0$$

$$\oint \frac{e^{3z}\cos(z)}{(z^2+\pi^2)^3(z-5)} = 0$$

4 Verify the cauchy-goursat theorem for

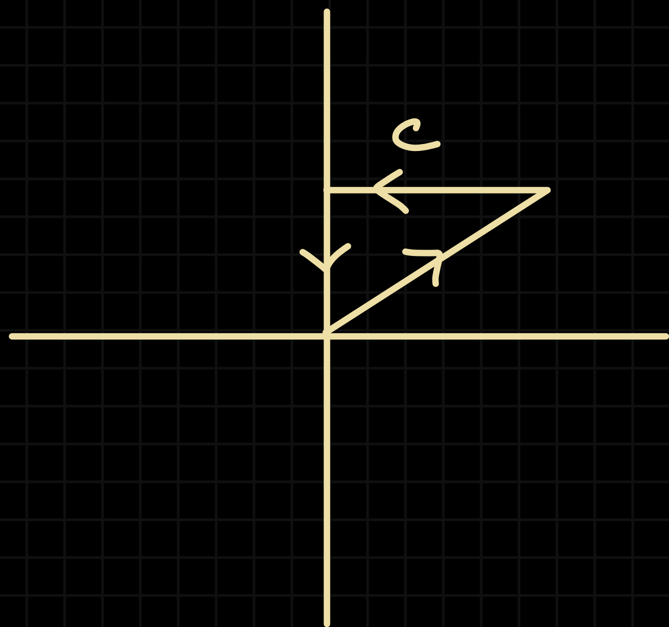
$$\oint z^2 \, dz$$

where c is the boundary of the triangle with vertices (0,0), (1,1) and (0,1)

solve:

1st part:

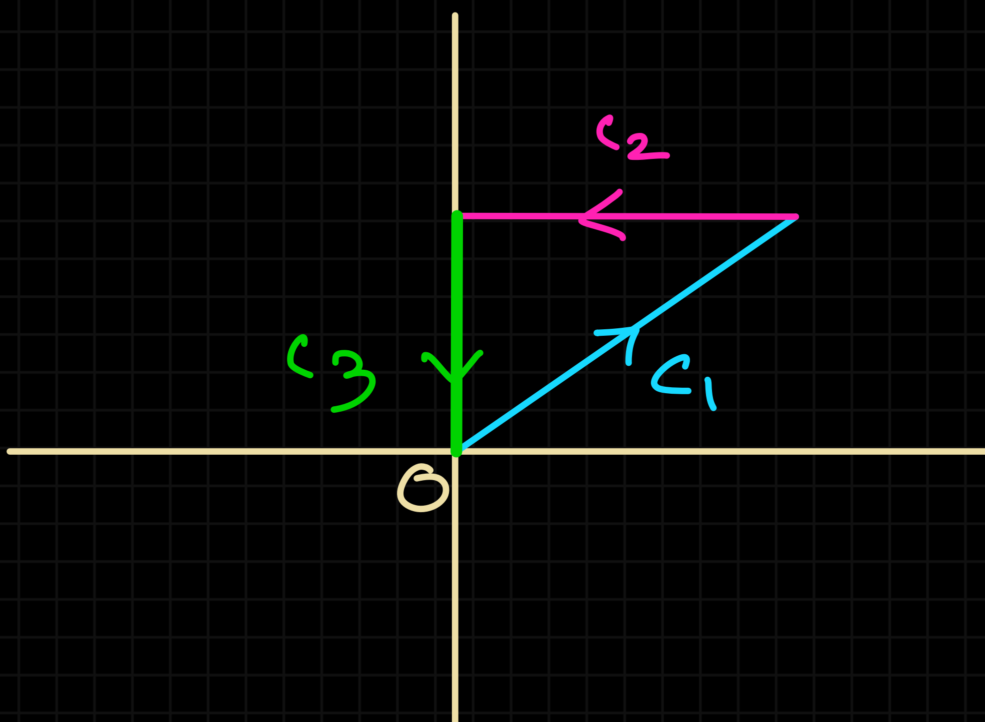
$$f(z) = z^2$$



$f(z) = z^2$ is analytic inside and on the boundary of the triangular contour c.

$$\therefore \oint f(z)\,dz = 0$$

2nd part:



$$\oint_C z^2\,dz = \oint_{c_1} z^2\,dz + \int_{c_2} z^2\,dz + \int_{c_3} z^2\,dz$$

for c_1: $\quad z(t) = 0 + \{(1+i) - 0\}\,t$

$$= t + it$$

$$= (t) + i\,(t)$$

$x = t$

$y = t$

$dz = (1 + i)\,dt$

$t \to$ start $t = 0$

$t \to$ end $t = 1$

$$\oint_{C_1} z^2\,dz = \int_0^1 (t + it)^2 (1+i)\,dt$$

$$= (1+i) \int_0^1 (t^2 + 2it^2 - t^2)\, dt$$

$$= (1+i) \left[\frac{t^3}{3} + 2i \times \frac{t^3}{3} - \frac{t^3}{3} \right]_0^1$$

$$= (1+i) \left(\frac{1}{3} + \frac{2i}{3} - \frac{1}{3} \right)$$

$$= (1+i) \times \frac{2i}{3}$$

$$= \frac{2i}{3} - \frac{2}{3}$$

for c_2:

$$z(t) = (1+i) + \left\{ i - (1+i) \right\} t$$

$$= (1+i) - t$$

$$= (1-t) + i$$

$$dz = -dt$$

$x = 1-t$

$y = 1$

$$t \begin{cases} \to \text{start} = 0 \\ \to \text{end} = 1 \end{cases}$$

$$\int_{C_2} z^2 \, dz = \int_0^1 \{(1-t)+i\}^2 \, (-dt)$$

$$= -\int_0^1 (1-t+i)^2 \, dt$$

$$i^2 + (-t)^2$$

$$= -\int_0^1 \{1+t^2+i^2+2(-t)+2(-t)i + 2i\} \, dt$$

$$= -\int_0^1 (t^2 - 2t - 2ti + 2i) \, dt$$

$$= -\left[\frac{t^3}{3} - 2 \times \frac{t^2}{2} - 2 \times \frac{t^2}{2} i + 2it \right]_0^1$$

$$= -\left(\frac{1}{3} - 1 - i + 2i \right)$$

$$= -\left(-\frac{2}{3} + i \right)$$

$$= \frac{2}{3} - i$$

for c_3:

$$z(t) = i + \{0 - (i)\} t$$

$$= i - it$$

$$= 0 + i(1-t)$$

$$dz = -i\,dt$$

$x = 0$

$y = 1 - t$

$t \rightarrow$ start $= 0$

$t \rightarrow$ end $= 1$

$$\int_{C_3} z^2\, dz = \int_0^1 \{i(1-t)\}^2 \times (-i\,dt)$$

$$= i\int_0^1 (1-t)^2\, dt$$

$$= i\int_0^1 (1-2t+t^2)\, dt$$

$$= i\left[t - 2\times\frac{t^2}{2} + \frac{t^3}{3}\right]_0^1$$

$$= i\left(1 - 1 + \frac{1}{3}\right)$$

$$= \frac{i}{3}$$

$$\oint_C z^2\,dz = \oint_{C_1} z^2\,dz + \int_{C_2} z^2\,dz + \int_{C_3} z^2\,dz$$

$$= \frac{2i}{3} - \frac{2}{3} + \frac{2}{3} - i + \frac{i}{3}$$

$$= 0$$

$\therefore$ cauchy goursat theorem is verified

Consequence of Cauchy Goursat's theorem:

Consequences of Cauchy's Theorem

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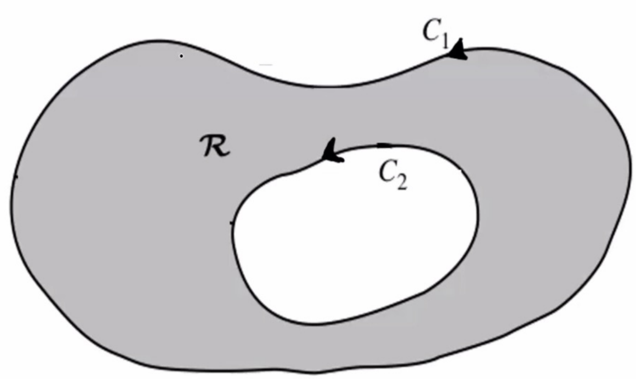
Inspiring Excellence

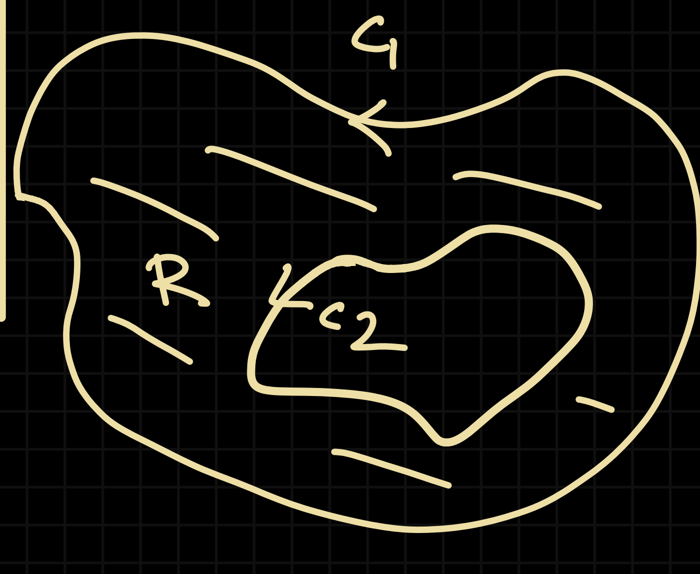
Let $f(z)$ be analytic in a region R bounded by two simple closed curves C_1 and C_2 and also on C_1 and C_2. Then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

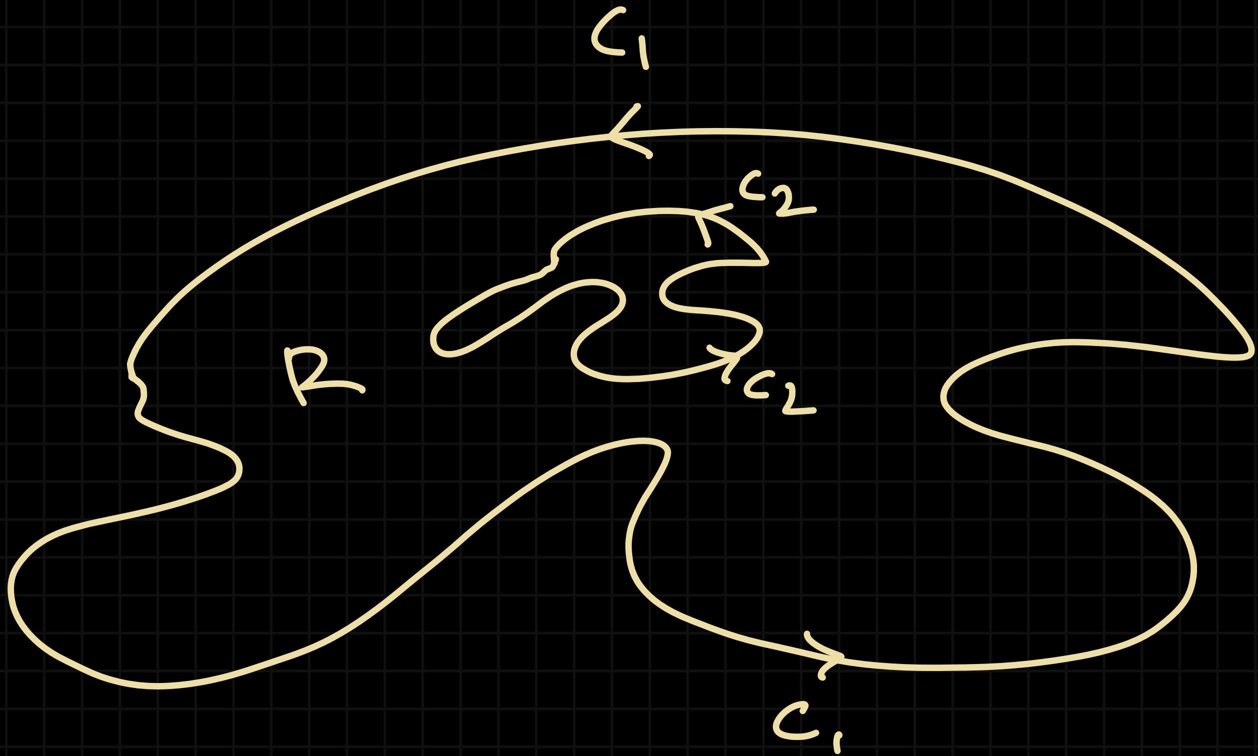


where C_1 and C_2 are both traversed in the positive sense.

$$\boxed{\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz}$$



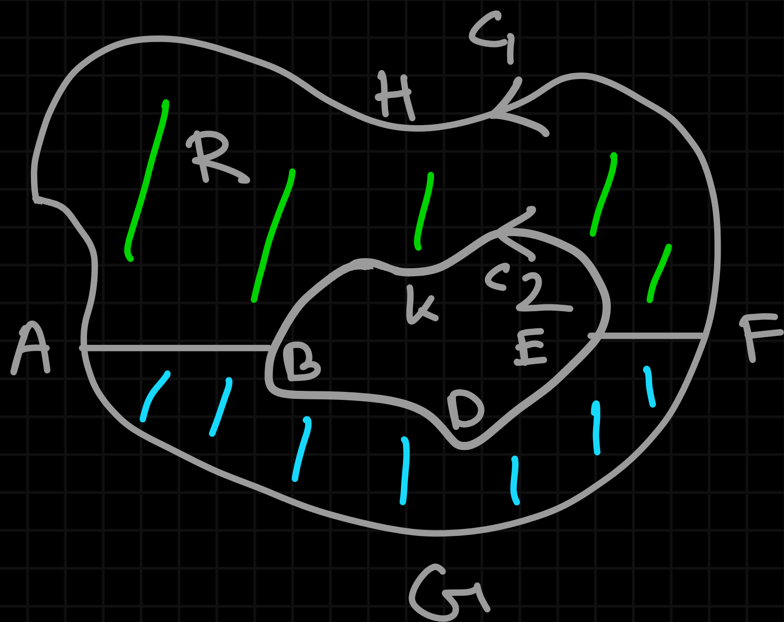
↳ c_1, c_2 same direction এ থাকা must



C_1
C_2
R
C_2
C_1

$\rightarrow$ not imp

proof:



$$\oint f(z)\,dz = 0$$

AGFEDBA

$$\int_{AGF} + \int_{FE} + \int_{EDB} + \int_{BA} = 0$$

$$\oint f(z)\,dz = 0$$

ABKEFHA

$$\int_{AB} + \int_{BKE} + \int_{EF} + \int_{FHA} = 0$$

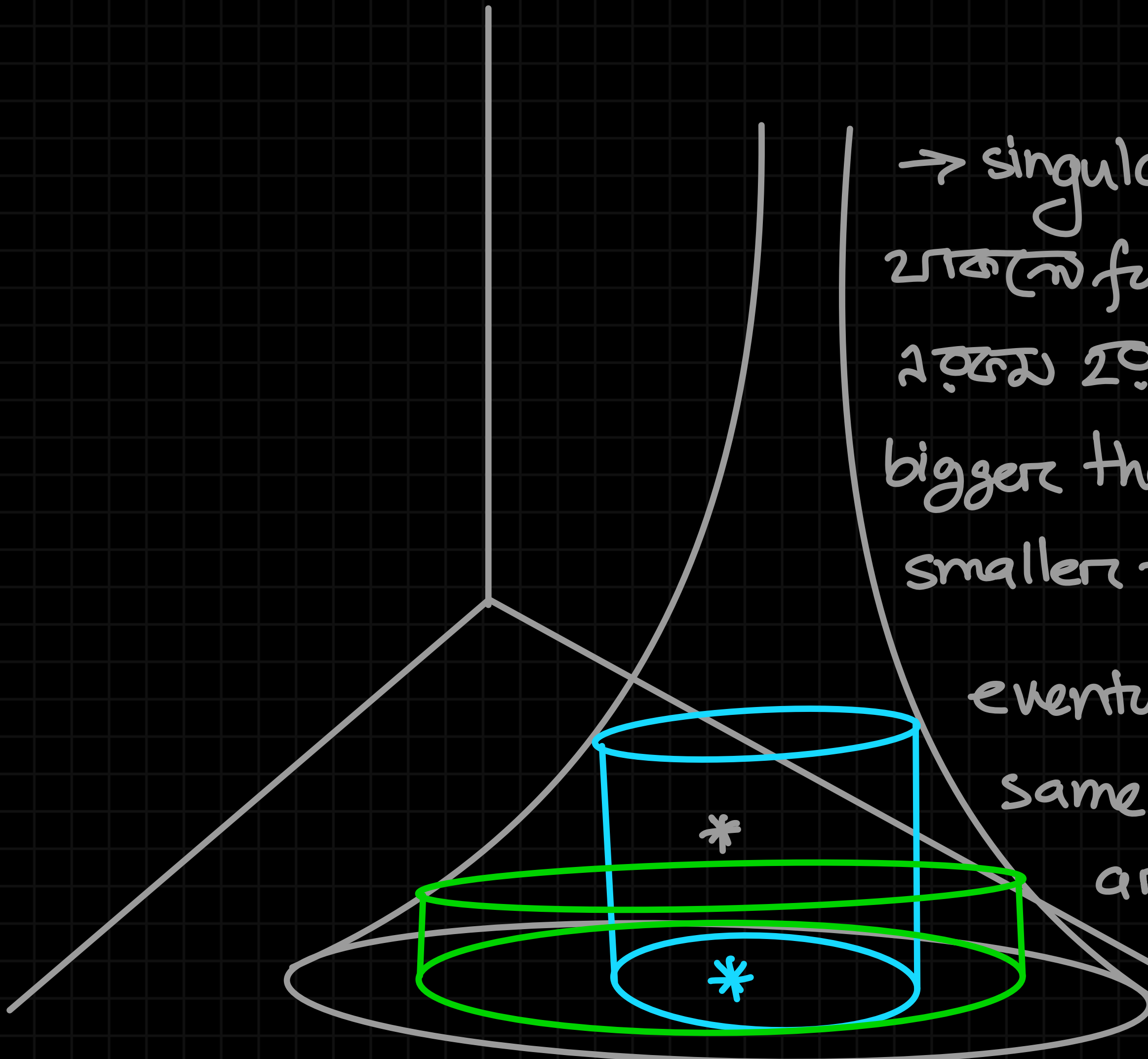
$\therefore$

$$\int_{AGF} + \int_{FE} + \int_{EDB} + \int_{BA} + \int_{AB} + \int_{BKE} + \int_{EF} + \int_{FHA} = 0$$

$$\Rightarrow \int_{AGF} + \int_{FHA} = -\int_{EDB} - \int_{BKE}$$

$$\Rightarrow \oint_{C_1} = \int_{BDE} + \int_{EKB}$$

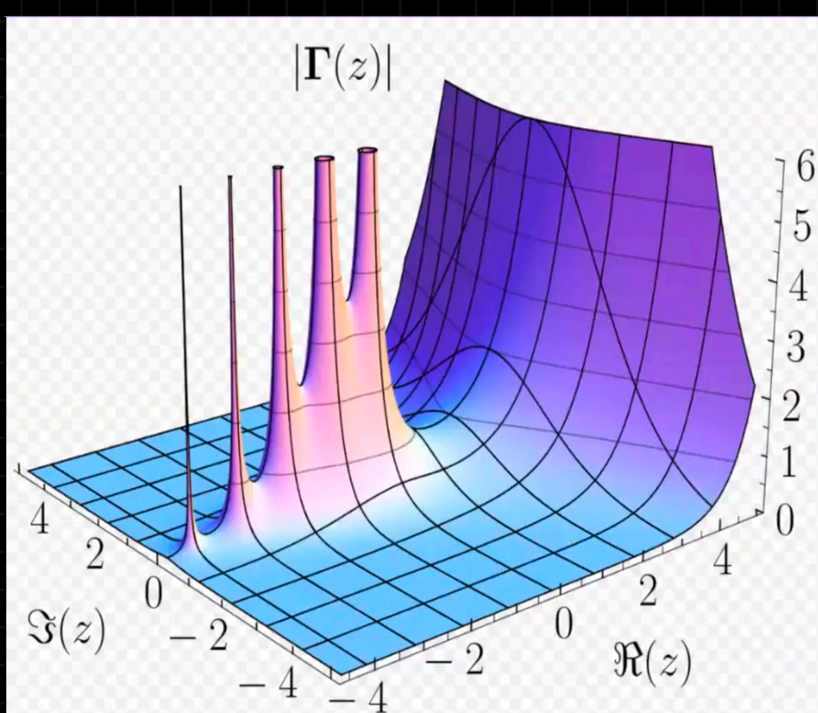
$$\therefore \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



→ singularity থাকলে function এরকম হয়।

bigger the length, smaller the height.

eventually gives same surface area for a singularity.



→ capital gamma

$\Gamma(n) = (n-1)!$

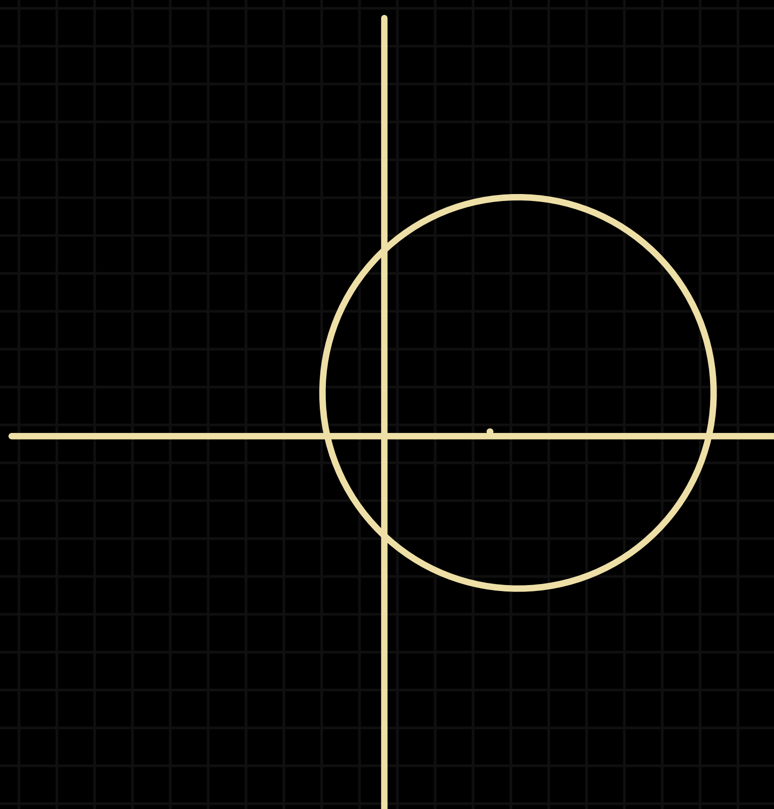
④ Evaluate

$$\oint_c \frac{z^2}{z-2}\, dz$$

where c is the circle $|z-2|=3$

Solve:

$|z-2|=3$



$z-2=3e^{i\theta}$

$z=2+3e^{i\theta}$

$dz=3ie^{i\theta}$

$$\oint_C \frac{z^2}{z-2}\, dz$$

$$= \int_0^{2\pi} \frac{(2+3e^{i\theta})^2}{3e^{i\theta}}\, i \times 3e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} \left(4 + 12 e^{i\theta} + 9 e^{2i\theta}\right) d\theta$$

$$= i \left[4\theta + 12 \times \frac{1}{i} e^{i\theta} + \frac{9}{2i} e^{2i\theta} \right]_0^{2\pi}$$

$$= i\left[8\pi + \frac{12}{i}e^{i2\pi} + \frac{9}{2i}e^{4\pi i} - \right.$$

$$\left. 0 - \frac{12}{i}e^{i\times 0} - \frac{9}{2i}e^{i\times 0}\right]$$

$$= i\left\{8\pi + \frac{12}{i}\cos(2\pi) + i\times\frac{12}{i}\sin(2\pi)\right.$$

$$\left. + \frac{9}{2i}\times\cos(4\pi) + i\times\frac{9}{2i}\sin(4\pi) - \frac{12}{i} - \frac{9}{2i}\right\}$$

$$= i\left(8\pi + \cancel{\frac{12}{i}} + \cancel{\frac{9}{2i}} - \cancel{\frac{12}{i}} - \cancel{\frac{9}{2i}}\right)$$

$$= 8\pi i$$

Evaluate

$$\oint_c \frac{z^2}{z-1}\,dz$$

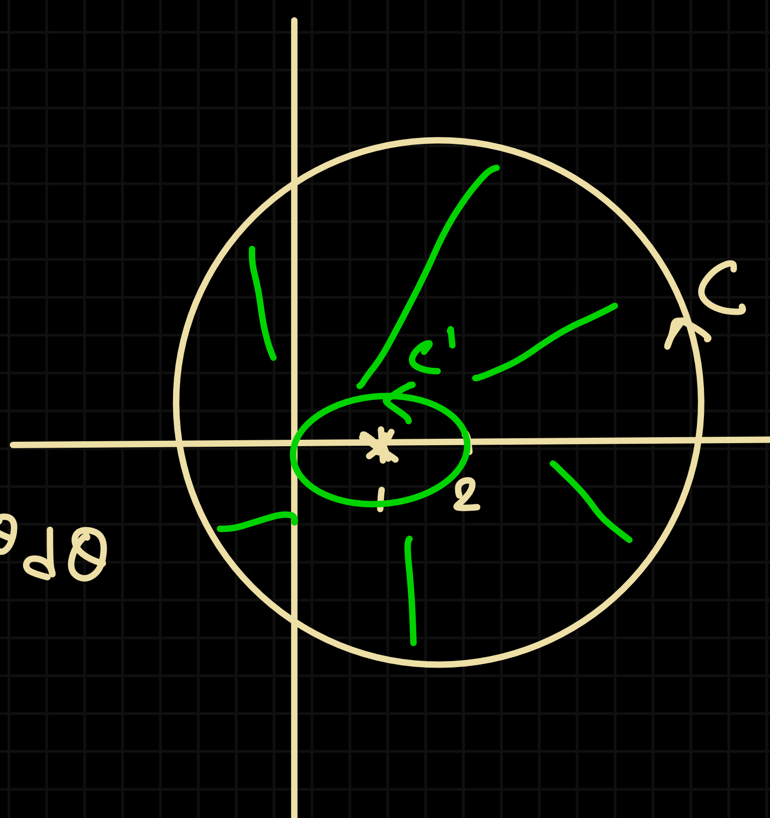
where c is the circle $|z-2|=3$

solve:

$$|z-2| = 3$$

$$z-2 = 3e^{i\theta}$$

$$\int \frac{(2+3e^{i\theta})^2}{(1+3e^{i\theta})} \times i3e^{i\theta}\,d\theta$$



$$f(z) = \frac{z^2}{z-1}$$ has a singularity at

$z = 1$

max 2

$c: |z-2| = 3$, $\quad c': |z-1| = 1$

$f(z)$ is analytic in between c and c'.

$$\therefore \oint_c f(z)\,dz = \oint_{c'} f(z)\,dz$$

NB: যে point এ singularity, curve কে সেই point কে center করে move করে circle বানাতে হবে। even if its an ellipse or any other shape. create circle centered at the singularity

$$C' : |z-1| = 1$$

$$z-1 = e^{i\theta}$$

$$dz = ie^{i\theta}$$

$$\oint_{C'} f(z)\,dz \quad = \oint_{C} \frac{z^2}{z-1}\,dz$$

$$= \int_0^{2\pi} \frac{(1+e^{i\theta})^2}{1+e^{i\theta}-1} \times ie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} \left(1 + 2e^{i\theta} + e^{2i\theta}\right) d\theta$$

$$= i\left[\theta + \frac{2}{i} e^{i\theta} + \frac{1}{2i} \times e^{2i\theta}\right]_0^{2\pi}$$

$$= i\left[2\pi + \frac{2}{i}\cos(2\pi) + i \times \frac{2}{i}\sin(2\pi) + \right.$$

$$\frac{1}{2i} \times \cos(4\pi) + i \times \frac{1}{2i}\sin(4\pi)$$

$$\left. - 0 - \frac{2}{i} e^{0} - \frac{1}{2i} e^{0}\right]$$

$$= i\left(2\pi + \cancel{\frac{2}{i}} + 0 + \cancel{\frac{1}{2i}} + 0 - \cancel{\frac{2}{i}} - \cancel{\frac{1}{2i}}\right)$$

$$= 2\pi i$$

Evaluate

$$\oint_C \frac{1}{z-a}\, dz$$

where C is any simple closed curve

and

(i) $z = a$ is outside C

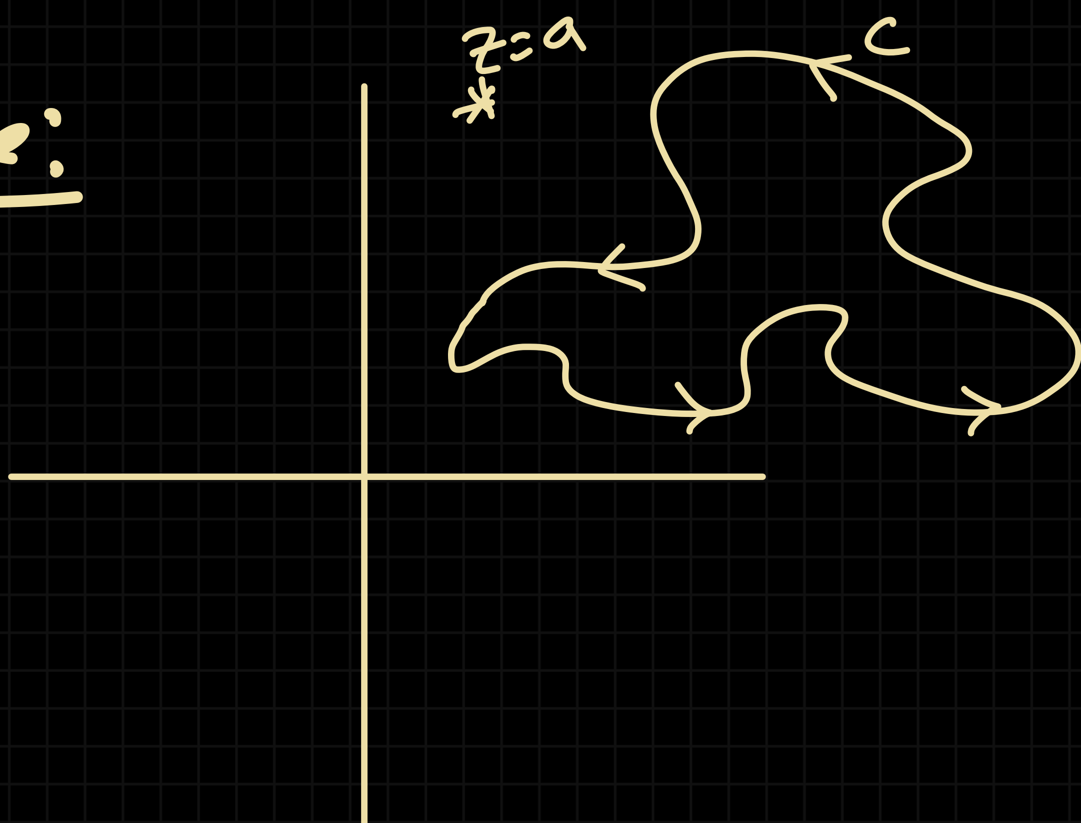
(ii) $z = a$ is inside C

Evaluate

$$\oint_C \frac{1}{z-a}\, dz$$

where C is any simple closed curve and $z=a$ is outside C

solve:



$f(z)$ has the only singularity at $z = a$ which is outside of C.

$\Rightarrow f(z)$ is analytic inside and on the boundary of C.

$$\therefore \oint_C f(z)\,dz = 0$$

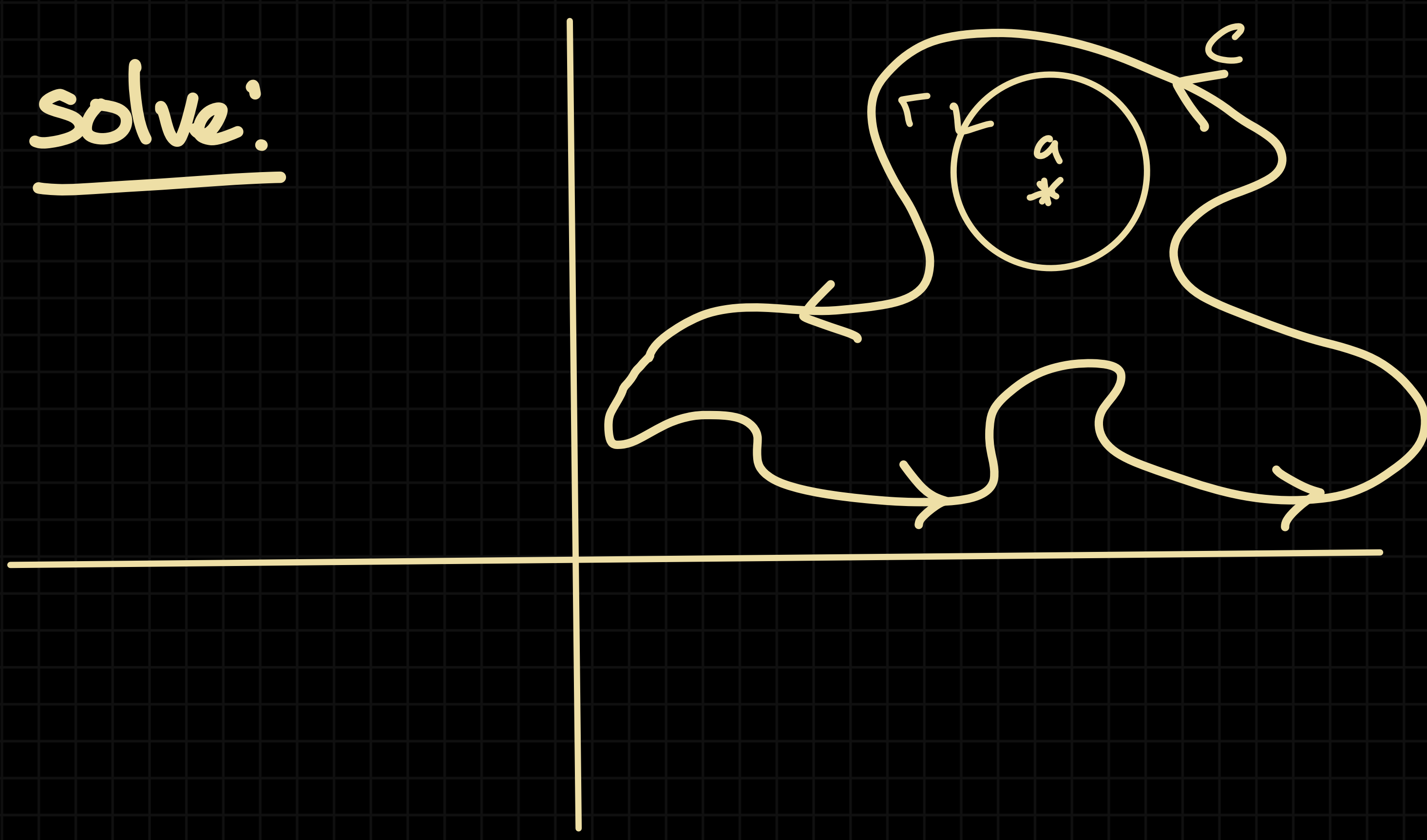
[using cauchy-goursat theorem]

☐ Evaluate

$$\oint_C \frac{1}{z-a}\,dz$$

where C is any simple closed curve and $z=a$ is inside C

solve:



$$f(z) = \frac{1}{z-a}$$

since $z=a$ is an interior point, we can construct a circle centered at $z=a$ with radius $r(\text{or } \epsilon)$ sufficiently small such that the circle lies entirely inside c.

$$\Gamma : |z-a| = \epsilon$$

$$\Rightarrow z-a = \epsilon e^{i\theta}$$

$$dz = i\epsilon e^{i\theta}$$

Now $f(z)$ is analytic in between C and Γ

$$\Rightarrow \oint_C f(z)\,dz = \oint_\Gamma f(z)\,dz$$

$$\therefore \oint_\Gamma f(z)\,dz$$

$$= \oint_\Gamma \frac{1}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{1}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta$$

$$= i \int_0^{2\pi} d\theta$$

$$= i \left[\theta \right]_0^{2\pi}$$

$$= 2\pi i$$

$\boxed{\square}$ Evaluate $\longrightarrow$ গুরুত্ব

$$\oint_C \frac{1}{(z-a)^n}\,dz$$

where $n \in \mathbb{N}$, C is any simple closed curve and

(i) $z = a$ is outside C

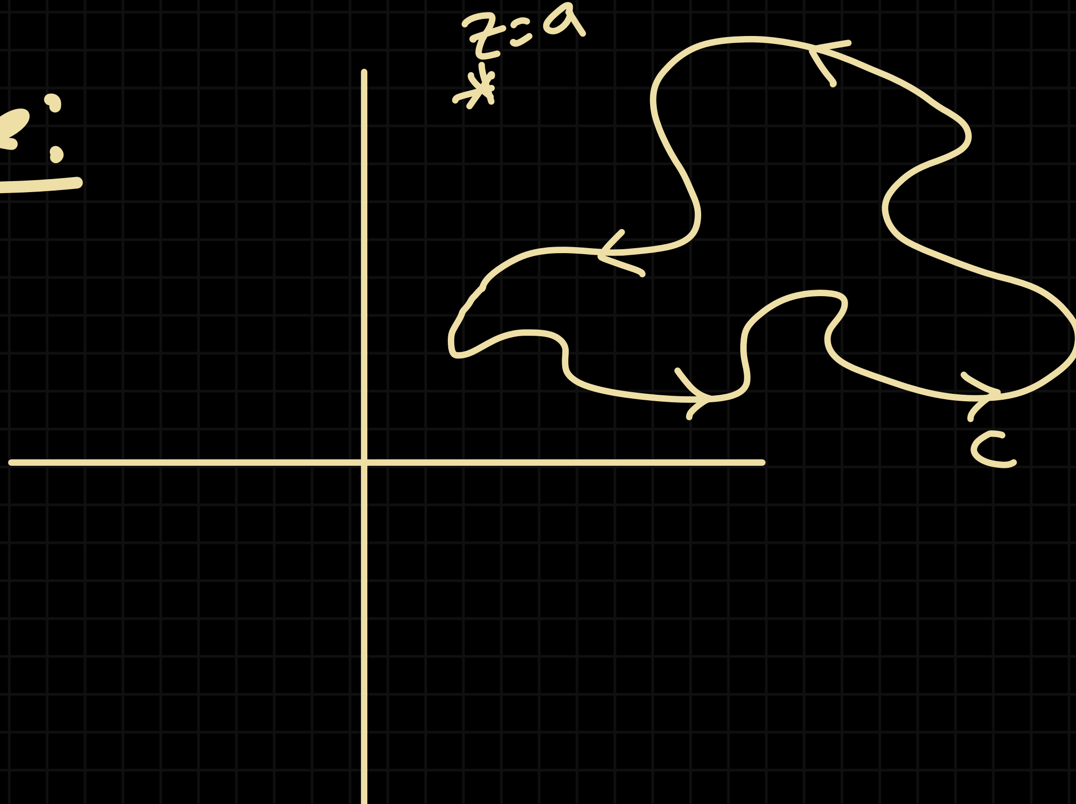
(ii) $z = a$ is inside C

Evaluate

$$\oint_C \frac{1}{(z-a)^n}\, dz$$

where $n \in \mathbb{N}$, C is any simple closed curve and $z=a$ is outside C

solve:



$f(z)$ has the only singularity at $z = a$ which is outside of C.

$\Rightarrow f(z)$ is analytic inside and on the boundary of C.

$$\therefore \oint_C f(z)\,dz = 0$$

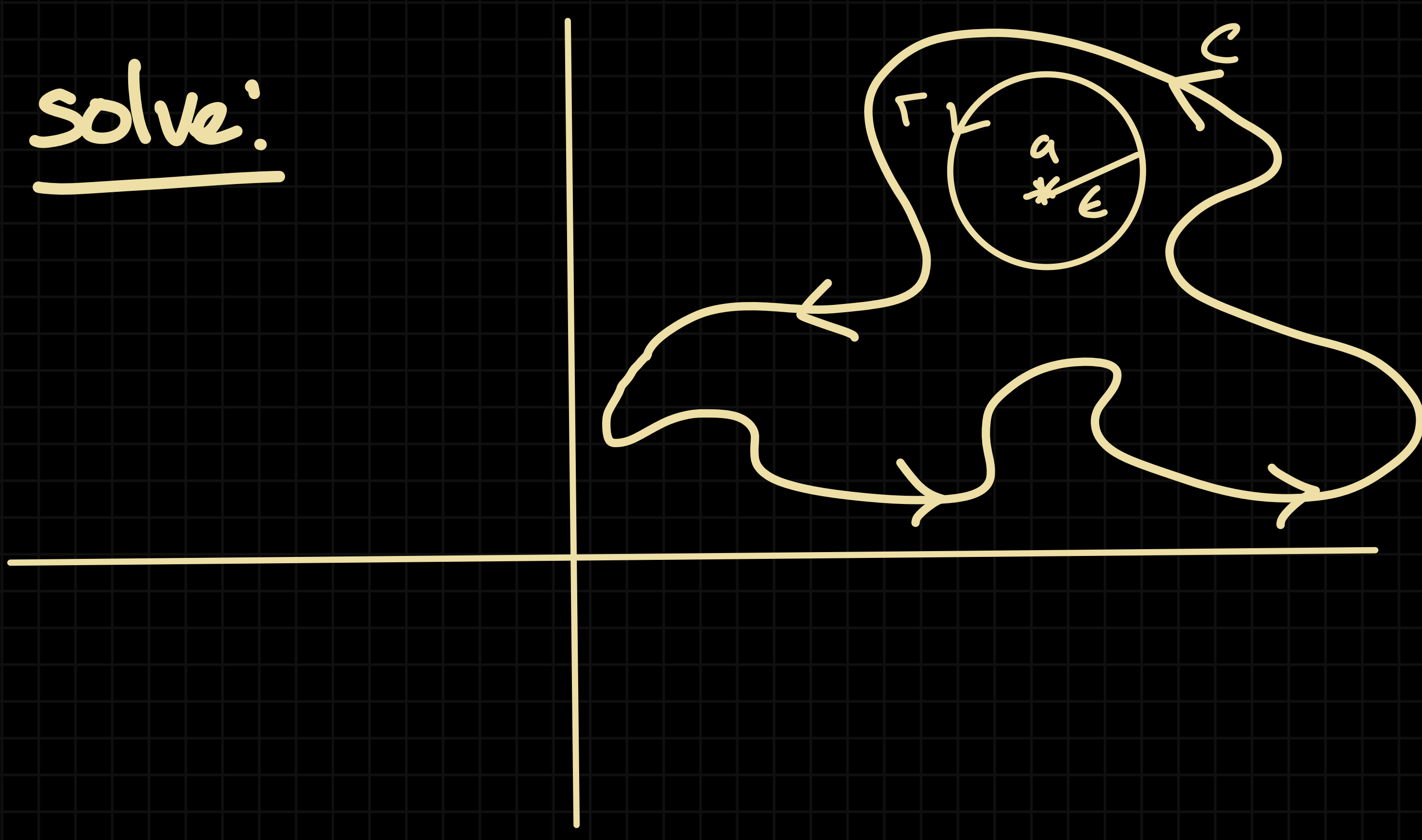
[using cauchy-goursat theorem]

☐ Evaluate

$$\oint_C \frac{1}{(z-a)^n}\,dz$$

where $n \in \mathbb{N}$, C is any simple closed curve and $z=a$ is inside C

solve:



$$f(z) = \frac{1}{z-a}$$

since $z=a$ is an interior point, we can construct a circle centered at $z=a$ with radius $r(=\epsilon)$ sufficiently small such that the circle lies entirely inside c.

$$\Gamma : |z-a| = \epsilon$$

$$\Rightarrow z - a = \epsilon e^{i\theta}$$

$$dz = i\epsilon e^{i\theta}$$

Now $f(z)$ is analytic in between C and Γ

$$\Rightarrow \oint_C f(z)\,dz = \oint_\Gamma f(z)\,dz$$

$$\therefore \oint_\Gamma f(z)\,dz$$

$$= \oint_\Gamma \frac{1}{(z-a)^n}\,dz$$

$$= \int_0^{2\pi} \frac{1}{(\epsilon e^{i\theta})^n}\, i\epsilon e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} \epsilon^{1-n} e^{i\theta(1-n)} \, d\theta$$

<u>**fon $n=1$:**</u>

$$\oint_\Gamma f(z)\,dz = i \int_0^{2\pi} \epsilon^0 e^0 \, d\theta$$

$$= i \int_0^{2\pi} d\theta$$

$$= 2\pi i$$

for $n>1$:

$$\oint_\Gamma f(z)\,dz = i\epsilon^{1-n}\left[\frac{e^{i\theta(1-n)}}{i(1-n)}\right]_0^{2\pi}$$

$$= i\epsilon^{1-n}\left[\frac{\overbrace{e^{i(1-n)2\pi}}^{\text{for any } n\text{, the value is } 1}}{i(1-n)} - \frac{e^{0}}{i(1-n)}\right]$$

$$= i\epsilon^{1-n} \times 0$$

$$= 0$$

$$\boxed{\oint \frac{1}{(z-a)^n}\,dz = \begin{cases} 2\pi i, & \text{at } n=1 \\ 0, & \text{at } n\neq 1 \end{cases}}$$

variations of the previous math can be

Evaluate

$$\oint_C \frac{1}{(z-a)^{n+1}}\, dz$$

where $n = 0, 1, 2, 3 \ldots$ and C is any simple closed curve and

(i) $z = a$ is outside $C \Rightarrow$ ans $= 0$

(ii) $z = a$ is inside C

ans:

VVI

$$\boxed{\oint \frac{1}{(z-a)^{n+1}}\, dz = \begin{cases} 2\pi i, & \text{at } n=0 \\ 0, & \text{at any other integer} \end{cases}}$$

as another variation if $n<0$

e.g:

$$\oint \frac{1}{(z-a)^{-6}}\,dz = \oint_c \underbrace{(z-a)^6}_{\text{analytic}}\,dz$$

$$= 0$$

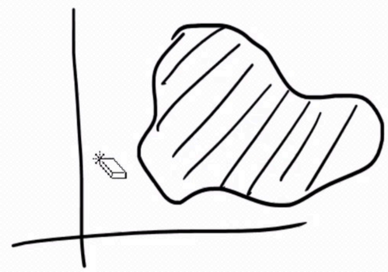
NB: power $= 1 \Rightarrow$ nonzero answer

power any other integer $\Rightarrow$ zero answer

Morera's Theorem

Let $f(z)$ be continuous in a simply-connected region R and suppose that

$$\oint_C f(z)dz = 0$$



around every simple closed curve C in R. **Then $f(z)$ is analytic in R.** This theorem, due to Morera, is often called the converse of Cauchy's theorem. It can be extended to multiply-connected regions.