MAT215: Machine Learning & Signal Processing

Topic: 1) Functions of complex variables
2) Single & Multivalued functions
3) Complex representation of functions
4) Limit
5) Continuity

Former Title: Complex variables & Laplace Transformations

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MAT215
(Machine Learning & Signal Processing)

$f(x) = \sqrt{x}$; where x is a real number

$f(9) = \sqrt{9} = 3$

$f(-9) =$ undefined in Real number system

$f(2i) = \sqrt{2i}$
$=$ undefined in Real number system

$f(z) = \sqrt{z}$

$f(9) = \sqrt{9}$
$= 3, -3$

$f(-9) = \sqrt{-9}$
$= 3i, -3i$

$f(2i) = \sqrt{2i}$
$= 1+i, -1-i$

$f(z) = 3z^2+1$

$f: \mathbb{C} \to \mathbb{C}$

complex number set

Single-valued and Multi-valued functions

function এর single input এ single output is called Single-valued functions

$$\begin{cases} f(z) = 3z^2+1 \\ f(2i) = 3(2i)^2+1 = -12+1 = -11 \\ f(1+i) = 3(1+i)^2+1 = 3(1+2i-1) = 6i-1 \end{cases}$$

single input valueর জন্য multiple-values of output is Multi-valued function

$$\begin{cases} f(z) = z^{1/2},\ f(9) = \sqrt{9} = 3, -3 \\ f(2i) = \sqrt{2i} = 1+i,\ -1-i \end{cases}$$

<u>examples</u>

1) $f(z) = z^{1/2}$

2) $f(z) = z^{1/6} \rightarrow 6$ outputs

3) $f(z) = z^{1/n} \rightarrow n$ outputs for $n \geq 2,\ n \in \mathbb{N}$

4) $\ln(z) = \log_e^{z}$

$\ln(-1) = ?$

$$\Rightarrow \ln\left(1 \cdot e^{i(\pi+2n\pi)}\right) \longrightarrow z = -1 \text{ (cartesian form)}$$

$$= -1 + 0i$$

$$\text{in polar } (re^{i\theta}) = 1 \cdot e^{i(\pi+2n\pi)}$$

$$\Rightarrow \log_e\left(e^{i(\pi+2n\pi)}\right)$$

$$\Rightarrow i(\pi+2n\pi) \times \log_e e$$

$$= i(\pi+2n\pi) \quad ; n \in \mathbb{Z}$$

$\ln(1+i) = ?$

$$\Rightarrow \ln\left[\sqrt{2}\, e^{i\left(\frac{\pi}{4}+2n\pi\right)}\right]$$

$$\Rightarrow \ln(\sqrt{2}) + \log_e e^{i\left(\frac{\pi}{4}+2n\pi\right)} \qquad [\because \log(MN) = \log M + \log N]$$

$$= \ln(\sqrt{2}) + i\left(\frac{\pi}{4}+2n\pi\right)$$

Complex representation of some important functions

$$e^{i\theta} = \cos\theta + i\sin\theta$$

let $\theta = -\theta \Rightarrow e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

adding, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$\boxed{\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$\rightarrow$ vvi for integration

e.g.: $\int \cos^3\theta \, d\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

subtracting, $e^{i\theta} - e^{-i\theta} = i2\sin\theta$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

circular function (vvI)

e.g.: $x^2 + y^2 = 1$

$x = \cos\theta, \quad y = \sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{(e^{i\theta} - e^{-i\theta})/2i}{(e^{i\theta} + e^{-i\theta})/2} = \frac{(e^{i\theta} - e^{-i\theta})/2i}{(e^{i\theta} + e^{-i\theta})/2}$$

similarly,

$$\underbrace{\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}}_{\text{hyperbolic functions}}$$

$$x = \cosh\theta, \quad y = \sinh\theta$$

$$x^2 - y^2 = 1 \Rightarrow (\cosh\theta)^2 - (\sinh\theta)^2 = 1$$

$$\left(\frac{e^{\theta} + e^{-\theta}}{2}\right)^2 + \left(\frac{e^{\theta} - e^{-\theta}}{2}\right)^2 = 1$$

Problems on Complex representation of some important functions

if $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; find $\cos^{-1}x$.

<u>ans:</u> let $\cos^{-1}x = y$

$\Rightarrow x = \cos y$

$\Rightarrow x = \dfrac{e^{iy} + e^{-iy}}{2}$

$\Rightarrow e^{iy} + e^{-iy} = 2x$ ----- ①

$$(e^{iy} - e^{-iy})^2 = (e^{iy} + e^{-iy})^2 - 4 \cdot e^{iy} \cdot e^{-iy}$$

$$= (2x)^2 - 4$$

$$= 4(x^2 - 1)$$

$e^{iy} - e^{-iy} = 2\sqrt{x^2 - 1}$ ---- ②

adding ① and ②

$$2e^{iy} = 2x + 2\sqrt{x^2-1}$$

$$e^{iy} = x + \sqrt{x^2-1}$$

$$\ln e^{iy} = \ln(x+\sqrt{x^2-1})$$

$$\Rightarrow y = \frac{1}{i} \ln(x+\sqrt{x^2-1})$$

$$\Rightarrow \cos^{-1} x = \frac{1}{i} \ln(x+\sqrt{x^2-1})$$

Ans.

NB: skip δ ϵ definition in relation or Relation maths in the book

Find the limits

(i) $\lim\limits_{z \to 1+i} (z^2 - 5z + 10)$

(ii) $\lim\limits_{z \to 2i} \dfrac{(2z+3)(z-1)}{z^2 - 2z + 4}$

(iii) $\lim\limits_{z \to 1+i} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

Undefined : $\dfrac{\text{nonzero}}{\text{zero}}$ => undefined state, limit does not exist

indeterminate: $\dfrac{0}{0}$, 1^∞, $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$

indeterminate হলে use L'hospital's Rule (differentiate both হর and লব)

① $\lim_{z \to 1+i} (z^2 - 5z + 10)$

soln: $\lim_{z \to 1+i} (z^2 - 5z + 10) = (1+i)^2 - 5(1+i) + 10$

$= 2i - 5 - 5i + 10$

$= 5 - 3i$

no indeterminate or
undefined situation
after putting z's value

(ii) $\lim_{z \to 2i} \dfrac{(2z+3)(z-1)}{z^2 - 2z + 4}$

soln:

$$\lim_{z \to 2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4} = \frac{\{2(2i)+3\}(2i-1)}{(2i)^2 - 2(2i) + 4}$$

$$= \frac{(4i+3)(2i-1)}{4i^2 - 4i + 4} = \frac{8i^2 - 4i + 6i - 3}{-4 - 4i + 4}$$

$$= \frac{-8 + 2i - 3}{-4i} = \frac{(-11+2i)i}{-4i^2} = \frac{-11i + 2i^2}{4}$$

$$= \frac{-11i - 2}{4} = \frac{-1}{2} - \frac{11}{4} i$$

(3)

(iii) $\lim_{z \to 1+i} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

ans:

$$\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$$

L'hospital's rule $\Rightarrow$ $\lim_{z \to 1+i} \dfrac{2z - 1}{2z - 2} = \dfrac{2(1+i) - 1}{2(1+i) - 2}$

$$= \frac{1+2i}{2i} = \frac{(1+2i)i}{2i \cdot i} = \frac{i + 2i^2}{2i^2}$$

$$= \frac{i-2}{-2} = 1 - \frac{i}{2}$$

(iv) $\lim_{z \to 2i} \dfrac{2z-2}{z^2+4} = \dfrac{\text{nonzero}}{\text{zero}} \Rightarrow \therefore$ limit does not exist

Some special maths from limit (VVI exam)

① $$\lim_{z \to 0} \frac{\bar{z}}{z} = \frac{\text{not differntiable}}{1}$$

so L'hospital's rule,

$$\lim_{x+iy \to 0} \frac{\overline{x+iy}}{x+iy} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x-iy}{x+iy}$$

(NB: for complex limit, we consider limit value to be approached from all directions)

in the direction, $x=0$,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x-iy}{x+iy} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{-iy}{iy} = -1$$

in the direction $y=0$,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x-0.i}{x+0.i} = \frac{x}{x} = 1$$

since coming from direction doesn't give same value,

limit doesn't exist.

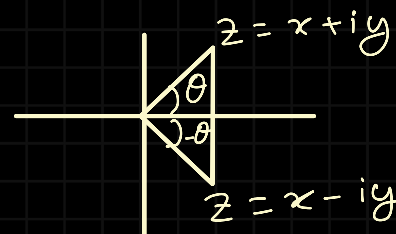
$\lim_{z \to 0} \frac{\bar{z}}{z} = ?$, do it in the polar form

ans:

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

$$= \lim_{re^{i\theta} \to 0} \frac{\overline{re^{i\theta}}}{re^{i\theta}}$$

$$= \lim_{re^{i\theta} \to 0} \frac{re^{-i\theta}}{re^{i\theta}}$$



$$= \lim_{\substack{r \to 0 \\ \theta \to \alpha}} e^{-2\theta}$$

$re^{i\theta} \hookrightarrow 0 \cdot e^{i\alpha} = 0$

$\Rightarrow 0 = 0 \cdot e^{i\alpha}$

$\arg(0) =$ anything, therefore indeterminate

$= e^{-2i\alpha}$ $\Rightarrow$ its not a unique value for every value of α. so limit doesn't exist

② $\lim_{z \to 0} \frac{xy}{x^2+y^2}$

$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$

$z \to 0$

$x + iy \to 0$, this is only possible when $x \to 0$ and $y \to 0$

in the $x = 0$ direction:

$\lim_{y \to 0} \frac{0}{y^2} = 0$

in the $y=0$ direction:

$\lim_{x \to 0} \frac{0}{x^2} = 0$

in the $y = x$ direction:

$\lim_{x \to 0} \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2}$

$= \frac{1}{2}$

$\therefore$ since all values dont match,

limit doesnt exist.

② $\lim_{z \to 0} \frac{xy}{x^2+y^2}$

<u>ans:</u>

$$\lim_{z \to 0} \frac{xy}{x^2+y^2} =$$

$$= \lim_{\substack{r \to 0 \\ \theta \to \alpha}} \frac{(r\cos\theta)(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$z = x+iy = re^{i\theta}$

$= \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}$

$$= \lim_{\substack{r \to 0 \\ \theta \to \alpha}} \frac{r^2\cos\theta\sin\theta}{r^2}$$

$$= \lim_{\substack{r \to 0 \\ \theta \to \alpha}} \sin\theta\cos\theta$$

limit doesn't exist since for different values of θ, we don't get unique limit values

③ $\lim_{z \to 0} \frac{x^2 y^2}{x^2+y^2}$

$$\Rightarrow \lim_{\substack{r \to 0 \\ \theta \to \alpha}} \frac{(r\cos\theta)^2 \cdot (r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \lim_{\substack{r \to 0 \\ \theta \to \alpha}} \frac{r^4 \sin^2\theta \cos^2\theta}{r^2}$$

$$= \lim_{\substack{r \to 0 \\ \theta \to \alpha}} r^2 \sin^2\theta \cos^2\theta$$

$$= 0^2 \times \sin^2\alpha \times \cos^2\alpha$$

$$= 0$$