MAT216: Linear Algebra & Fourier Analysis

Topic: Eigenvalue & Eigenvector

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topics:

i) linear transformation

ii) eigenvalue and eigenvector (Idea)

iii) characteristic equation

iv) Finding eigenvalue

v) Finding eigenvector

Matrix as Linear Transformation:

function

$$f(x,y) = (x+y,\ 2x-y)$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}}_{f} \begin{pmatrix} x \\ y \end{pmatrix}$$

linear algebra → matrix works as a function

$$\underset{2\times 3}{\begin{pmatrix} 3 & 6 & 5 \\ 2 & 8 & 9 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 3x+6y+5z \\ 2x+8y+9z \end{pmatrix}}$$

input output

$$f(x, y, z) = (3x+6y+5z,\ 2x+8y+9z)$$

how to represent a given transforma-tion in matrix:

[Q] consider, $T(x,y) = (3x-2y, 2x-2y)$

<u>ans:</u>

step-1: write T as column

step-2: split the variables and coffecients into different matrices.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y \\ 2x-2y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\therefore$ the matrix representation of T is,

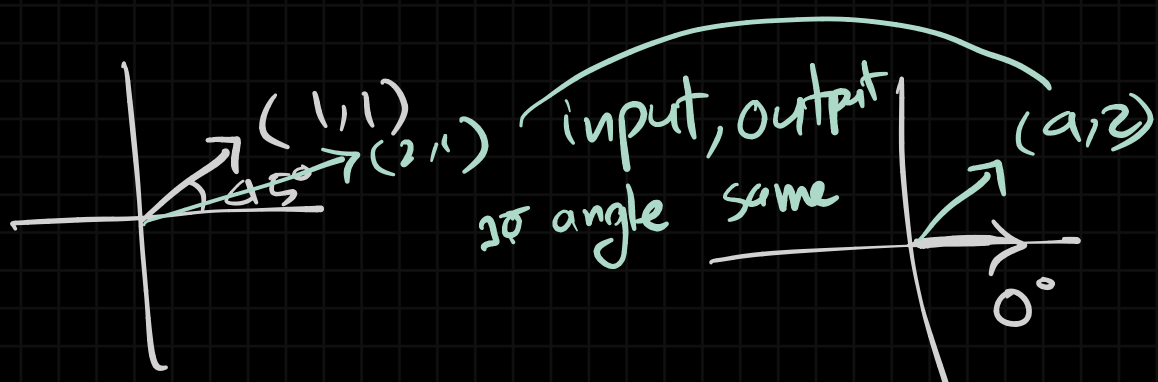
$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

Idea of Eigenvalue and Eigenvector:

☐ consider the matrix form of

$$T(x,y) = (3x - 2y,\ 2x - 2y)$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$



input output

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix} = \overset{\text{eigenvalue}}{2}\begin{pmatrix}2\\1\end{pmatrix}$$

input/eigenvector

a vector which doesn't change the direction (or angle)

eigenvalue: multiplier value

↳ can be positive or negative

$$\underbrace{T\begin{pmatrix}8\\4\end{pmatrix}}_{\text{eigenvector}} = \begin{pmatrix}16\\8\end{pmatrix} = \overset{\text{eigenvalue}}{2}\begin{pmatrix}8\\4\end{pmatrix}$$

$$T\underbrace{\begin{pmatrix}3\\6\end{pmatrix}}_{\text{eigenvector}} = \begin{pmatrix}-3\\-6\end{pmatrix} = \underbrace{-1}_{\text{same line; eigenvalue}}\begin{pmatrix}3\\6\end{pmatrix}$$

$$T\underline{\begin{pmatrix}1\\1\end{pmatrix}} = \begin{pmatrix}1\\0\end{pmatrix} \stackrel{?}{=} ? \begin{pmatrix}1\\1\end{pmatrix}$$

not eigenvector bcz input এর কোনো multiplier এর সাথে output পাওয়া যাচ্ছে না

Idea of Characteristic Polynomial:

eigenvector

$$Av = \lambda v$$

eigenvector

eigenvalue

$$\Rightarrow \quad Av - \lambda v = 0$$

$$v(A - \lambda I) = 0$$

to give a value its matrix form

$$\boxed{\det(A - \lambda I) = 0}$$

Finding Eigenvalue

Let A be a square matrix of size $n \times n$. If l is an eigenvalue of A then for any vector v of size $n \times 1$ then

$$Av = \lambda v$$

$$\det(A - \lambda I) = 0$$

finding λ

$\boxed{4}$ Find the characteristic equation and eigenvalues of the matrix A.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$\boxed{\text{ans}}$

$$A - \lambda I = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix}$$

now,

$$\det(A - \lambda I) = 0$$

$$\therefore (3-\lambda)(2-\lambda) - 2 = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\lambda^2 - \lambda - 4\lambda + 4 = 0$$

$$\lambda(\lambda - 1) - 4(\lambda - 1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, 4$$

[1] Find the characteristic equation and eigenvalues of matrix A.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$\boxed{\text{ans}}$

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

now, $\det(A - \lambda I) = 0$

$$\Rightarrow (1-\lambda)\{(2-\lambda)(3-\lambda)-2\} - 1\cdot 2$$

$$+ 2(2-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\{6-2\lambda-3\lambda+\lambda^2-2\} - 2+4-2\lambda = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4) + 2-2\lambda = 0$$

$$\lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda + 2 - 2\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

[1] Find the characteristic equation and eigenvalues of matrix A.

$$A = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$$

$\boxed{\text{ans}}$

$$A - \lambda I = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 2 & -5 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(2-\lambda)\{(7-\lambda)(-4-\lambda) - 2\cdot(-15)\}$$

$$-2\{3\cdot(-4-\lambda) - 1\cdot(-15)\} - 5\{6 - 1\cdot(7-\lambda)\}$$

$$= 0$$

$$\Rightarrow \quad \lambda = 1, 1, 3$$

Properties of Eigenvalues

Let A be a square matrix and $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ be the eigenvalues of A then

- ❑ $Trace(A) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$.
- ❑ $Determinant(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \cdots \cdot \lambda_n$.

Trace = sum of the main diagonal elements

NB: $n \times n$ matrix এর জন্য n সংখ্যক eigenvalue থাকে

Verify the previous statement for

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

ans

Trace = sum of elements in main diagonal

$= 1+2+3 = 6$

$\Sigma \lambda = 1+2+3 = 6$

$\vdots$

and det $= 6$

Eigenvalues of a Triangular Matrix

Let A be an Upper or Lower Triangular Matrix.

$$A = \begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ or } A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ * & \lambda_2 & 0 \\ * & * & 3 \end{pmatrix}$$

Then the diagonal elements are the Eigenvalues.

e.g.: Find the eigenvalues of,

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \text{upper triangular matrix}$$

eigenvalues $= 6, 2, 4$

(2) Find the eigenvalues of,

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 7 & 2 \end{pmatrix}$$

eigenvalues: 5, 3, 2

3 types of math

Finding Eigenvector

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Let A be a square matrix of size $n \times n$. If λ is an eigenvalue of A then the corresponding eigenvector v satisfies

$$Av = \lambda v$$

$$(A - \lambda I)v = 0$$

That means v is an non-trivial solution of $\boxed{(A - \lambda I)v = 0.}$

Type-1

Find the eigenvectors of A where 1 and 4 are the eigenvalues of the matrix A. Also find the Basis of eigenspace of A.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

ans

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix}$$

condition for eigenvector

$$\boxed{(A - \lambda I) v = 0}$$

For $\lambda = 1$ let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector.

$\rightarrow A - \lambda I$ 2×2

size থেকে, ওটা

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\therefore \begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

since 2nd row is entirely zero,

x_2 is a free variable

now, $2x_1 + x_2 = 0$

free variable $\rightarrow$ $x_2 = t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t/2 \\ t \end{pmatrix}$$

$$= \frac{1}{2} t \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{\text{eigenvector}}$$

for $\lambda = 4$ let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the

eigenvector

now, $$\begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2+2(-1) & -2+2(1) & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\therefore -x_1 + x_2 = 0$$

free variable $= x_2$

$\therefore \quad x_1 = t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$$

eigenvector $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

testing by $\begin{pmatrix}1\\1\end{pmatrix}$

$$AV = \begin{pmatrix}3 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix}$$

λv

now with $\begin{pmatrix}-1\\2\end{pmatrix}$

$$\begin{pmatrix}3 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix} = 1\begin{pmatrix}-1\\2\end{pmatrix}$$

λv

[b] Find the eigenvectors of A where 1, 2 and 3 are the eigenvalues of the matrix A. Also find the Basis of the eigenspace of A.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

ans: for $\lambda = 1$ let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector

$\therefore (A-\lambda)\, v_1$

$$= \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$-x_1 + x_2 + 2x_3 = 0$$

$$x_2 + 2x_3 = 0$$

free variable $= x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} t$$

For $\lambda = 2$, let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector.

$\therefore \quad (A - \lambda I) v = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1+\frac{1}{2}\times 2 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

free variable $\Rightarrow x_3 = t$

$$\therefore \quad -x_1 + x_2 + 2x_3 = 0$$
$$2x_3 = 0$$

free variable $\rightarrow x_2 = t$

★ 모든 pivot element 제외 다 free variable

$x_2 = t$

$x_3 = 0$

$x_1 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda = 3$, let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector.

$\therefore (A - \lambda I) v = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

$\Rightarrow \quad r_1' = r_1 - r_2$

$$\Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & 4 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

$$r_1' = r_1 / (-2) \quad ; \quad r_2 = -r_2$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

$$\Rightarrow r_3' = r_3 - r_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right)$$

$r_3' = r_3 + r_1$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_3 = t \rightarrow$ free variable

$$x_1 - 2x_3 = 0$$

$$x_2 - 2x_3 = 0$$

$$x_3 = t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ 2t \\ t \end{pmatrix}$$

$$= t \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Complete Eigenspace

$A_{n\times n} \Rightarrow$ n eigenvalues.

if we can find n independent eigenvectors $\Rightarrow$ its called complete eigenspace

Rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\theta = 90 \rightarrow \; = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$