MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Modulus, Arguments, their properties, Polar & Cartesian form, Square roots of complex number, De Moirre's Theorem, Euler's theorem, find nth root of complex number

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MAT215
(Machine Learning & Signal Processing)

Properties of modulus and arguments:

For modulus:

(i) $|z_1 . z_2| = |z_1| . |z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(iii) $|z^m| = |z|^m$

where $m \in \mathbb{R}$

For arguments:

(i) $\mathrm{Arg}(z_1 . z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$

(ii) $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

(iii) $\mathrm{Arg}(z^m) = m . \mathrm{Arg}(z)$

Find the modulus and arguments of the following

i) $\frac{1+i}{1-i}$

=> For modulus,

$$\left|\frac{1+i}{1-i}\right| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{1^2+1^2}}{\sqrt{1^2+(-1)^2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

For argument,

$$\arg\left(\frac{1+i}{1-i}\right) = \arg(1+i) - \arg(1-i)$$

$$= \tan^{-1}\left|\frac{1}{1}\right| - \left(-\tan^{-1}\left|\frac{-1}{1}\right|\right)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

ii) $3-4i$

mod $\Rightarrow \sqrt{3^2+(-4)^2} = 5$

$\arg(z) = -\tan^{-1}\left(\left|\frac{-4}{3}\right|\right)$

$= -\tan^{-1}(4/3)$

iii) 0

$z = 0 = 0 + 0i$

$mod(z) = |z| = \sqrt{0^2 + 0^2} = 0$

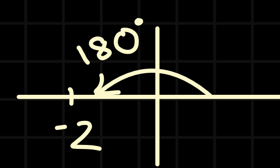
$arg(z) = 0$

e.g: $z = 2$

$= 2 + 0i$

$$\arg(z) = \tan^{-1}\left(\frac{0}{2}\right) = \tan^{-1} 0 = 0$$

e.g: let $z = -2$, $\arg(-2) = 180°$



e.g: $z = -i$

$= 0 - i$

$$\arg(-i) = \tan^{-1}\left|\frac{-1}{0}\right| = -\tan^{-1}(\text{undefined}) = -\pi/2$$

Polar Form of A Complex Number

Complex Notation

Cartesian Form/
Normal form,

$(a+ib)$

where $a,b \in \mathbb{R}$

$i = \sqrt{-1}$

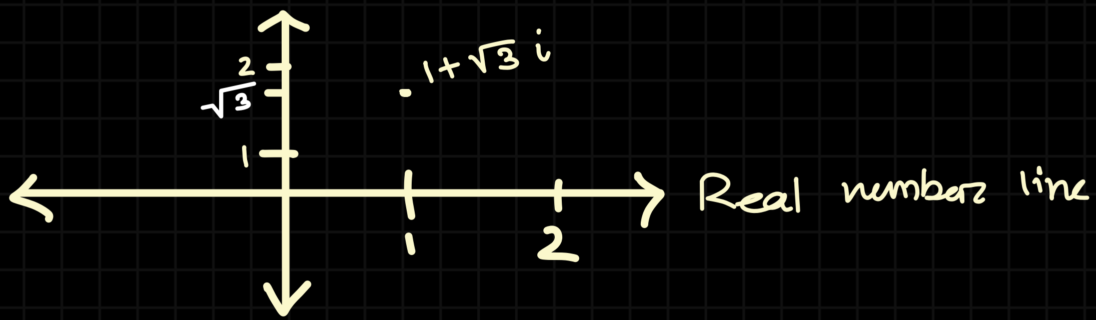
Polar form, $re^{i\theta}$

$r \geq 0$, r is the modulus, $\theta \in \mathbb{R}$

θ is an argument

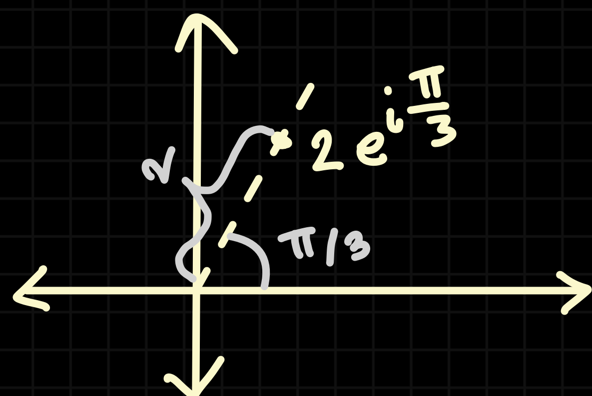
Graphical Representation of A Complex Number

let $z = 1 + \sqrt{3}\, i$



$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\theta = \frac{\pi}{3}$



Square Roots of A Complex Number

$\sqrt{4}$ $= 2$ and square roots of 4, $= 2, -2$

Find the square roots of z = a + ib

general approach:

let $\sqrt{a+ib} = x + iy$ $\quad x = ? \quad y = ?$

$\Rightarrow a + ib = (x+iy)^2$

$\Rightarrow a + ib = x^2 + 2 \cdot x \cdot iy + (iy)^2$

$\Rightarrow x^2 + y^2 + i \cdot 2xy = a + ib$

By equating coefficients,

$x^2 - y^2 = a \qquad 2xy = b$

Find the square roots of z = -7 + 24i

soln:

let $\sqrt{-7+24i} = x + iy$

$\Rightarrow -7+24i = (x+iy)^2$

$\Rightarrow -7+24i = x^2 + 2 \cdot x \cdot iy + (iy)^2$

$\Rightarrow -7+24i = x^2 - y^2 + i \cdot (2xy)$

$\therefore x^2 - y^2 = -7$ — (i) and $2xy = 24$

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= (-7)^2 + (2xy)^2$

$= (-7)^2 + 24^2$

$= 625$

$\Rightarrow x^2+y^2 = \sqrt{625}$ (addition of two squared numbers can't be negative)

$\therefore x^2 + y^2 = 25$ — (ii)

adding i and ii

$$\begin{array}{r} x^2 - y^2 = -7 \\ x^2 + y^2 = 25 \\ \hline 2x^2 = 18 \end{array}$$

$x = \pm 3$

for $x = 3$:

$$2xy = 24$$
$$2(3)y = 24$$
$$y = 4$$

for $x = -3$

$$2xy = 24$$
$$2(-3)y = 24$$
$$y = -4$$

$$x + iy = 3+4i, -3-4i$$

De Moirre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Euler's Formula/ Identity Formula :

→ finds any nth root ($\sqrt[n]{\ }$)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

let $\theta = \pi$

$\therefore e^{i\pi} = \cos\pi + i\sin\pi$

$e^{i\pi} = -1 \Rightarrow \underbrace{e^{i\pi} + 1 = 0}$

most beautiful equation in all of mathematics

$\because$ it has $e, i, \pi, 1, 0$

find the 4th root of -16

$\underbrace{\hspace{6em}}$

☆☆☆ VVI for quiz, mid

$\Rightarrow$ Modulus of $-16 = |-16| = |-16 + 0i| = \sqrt{(-16)^2 + 0^2} = 16$

Arg of $-16 = \pi + 2n\pi$

$\therefore -16 = 16\, e^{i(\pi + 2n\pi)}$

$\Rightarrow (-16)^{1/4} = \left(16\, e^{i(\pi + 2n\pi)}\right)^{1/4}$

$= 16^{1/4} \cdot e^{i \cdot \frac{\pi + 2n\pi}{4}}$

let, $n = 0, 1, 2$

<u>when $n=0$</u>

$16^{\frac{1}{4}} e^{i \cdot \frac{\pi}{4}} = 2 e^{i\frac{\pi}{4}}$

using Euler's formula,

$$2e^{i\pi/4} \equiv \underbrace{\sqrt{2} + \sqrt{2}\, i}_{\text{cartesian form}}$$

when $n=1$

$$16^{1/4} e^{i\cdot\frac{\pi}{4}} = 2e^{i\frac{3\pi}{4}} \equiv -\sqrt{2}+\sqrt{2}\,i$$

when $n=2$

$$16^{1/4} e^{i\cdot\frac{5\pi}{4}} = 2e^{i\cdot\frac{5\pi}{4}} \equiv -\sqrt{2}-\sqrt{2}\,i$$

when $n=3$

$$16^{1/4} e^{i\cdot\frac{7\pi}{4}} = 2\left(\cos\left(\tfrac{7\pi}{4}\right)+i\sin\left(\tfrac{7\pi}{4}\right)\right)$$

$$= 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

Bonus question

$$\sqrt[i]{i} = ?$$

$$\Rightarrow \sqrt[i]{i} = (i)^{1/i} = \left(e^{i\left(\frac{\pi}{2}+2n\pi\right)}\right)^{1/i}$$

$$= \underbrace{e^{\pi/2+2n\pi}}_{\text{real number}}$$

for three consecutive values of n

3rd roots of unity:

$$1 = 1\, e^{i(0+2n\pi)}$$

$$\therefore 1^{\frac{1}{3}} = e^{i \times \frac{2n\pi}{3}}$$

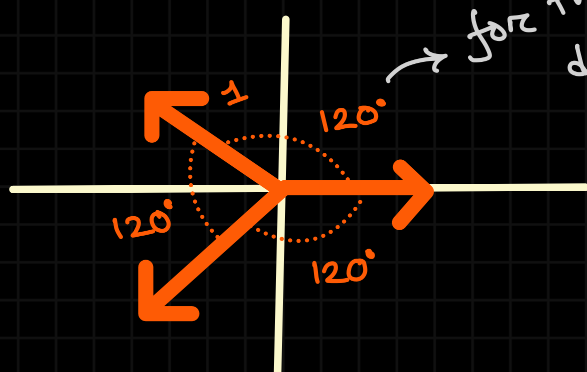
let $n = 0, 1, 2$

$$n = 0 \rightarrow e^{i(0)} = 1$$

$$n = 1 \rightarrow e^{i \times \frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$n = 2 \rightarrow e^{i \cdot \frac{4\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

in graphic representation,



for three values of n, the graph gets divided into three equal portion in terms of angle. $\therefore \frac{360°}{3} = 120°$

graphical representation of $(-16)^{1/4}$

→ VVI for assignment

$(-16)^{1/4} = \sqrt{2} + \sqrt{2}\,i,$

$-\sqrt{2} + \sqrt{2}\,i,$

$-\sqrt{2} - \sqrt{2}\,i,$

$\sqrt{2}, -\sqrt{\ }$