CSE330: Numerical Methods

Assignment 2

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Instructions for preparing the solution script:

- Write your name, ID#, and Section number clearly in the very front page.
- Write all answers sequentially.
- Start answering a question (not the pat of the question) from the top of a new page.
- Write legibly and in orderly fashion maintaining all mathematical norms and rules. Prepare a single solution file.
- Start working right away. There is no late submission form. If you miss the deadline, you need to use the make-up assignment to cover up the marks.

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1. Let $f(x) = \tan(x)$. In the following we would like to calculate the truncation errors.
   (a) (3 marks) First write down the approximate polynomial, $p_3(x)$, for the function $f(x)$ and identify the Taylor coefficients, $a_0, \cdots, a_3$.
   (b) (2 marks) Compute the percentage relative error at $x = \pi/4$ if $f(x)$ is approximated by $p_3(x)$ polynomial.
   (c) (5 marks) Use the Lagrange reminder form to evaluate the upper bound of truncation error at $x = \pi/4$ for some $\xi \in [0, \pi/4]$.

2. Consider the function $f(x) = e^x - e^{-x}$ and the nodes are at -1, 0, and 1. Now answer the following questions using 3 significant figures:
   (a) (1 mark) Write down the matrices b and V used in Vandermonde method.
   (b) (2 marks) Compute the determinant of the Vandermonde matrix V .
   (c) (3 marks) Using The results of the previous two parts, calculate the Taylor coefficients a_0, a_1 and a_2; and finally find the interpolating polynomial.
   (d) (4 marks) Evaluate the upper bound of interpolation error for the given function for the interval $\xi \in [-2.1, 2.1]$.

3. Consider the function $f(x) = e^x + e^{-x}$ and the nodes are at -1, 0, and 1. Now answer the following questions using 3 significant figures:
   (a) (4 marks) Evaluate the Lagrange bases for the given function and nodes.
   (b) (3 marks) Compute the Lagrange interpolation polynomial for the given function, and express the result in the natural basis. Also use this polynomial to find an approximate value of $f(6)$.
   (c) (3 marks) Evaluate the relative error in percentage form at $x = 1.5$.

4. Consider the function $f(x) = e^x - e^{-x}$ and the nodes are at -2, 0, and 2. Now answer the following questions using 3 significant figures:
   (a) (4 marks) Evaluate the Newton coefficients $a_k = f[x_0, \cdots, x_k]$ using Newton's divided-difference method for the given function and nodes.
   (b) (3 marks) Compute the Newton interpolation polynomial for the given function, and express the result in the natural basis. Also use this polynomial to find an approximate value of $f(6)$.
   (c) (3 marks) Evaluate the relative error in percentage form at $x = 1.5$.

Answer to que - 1

1(a)

given, $f(x) = \tan(x)$

$$f(x) = \frac{f(x_0)\, f(x-x_0)^0}{0!} + \frac{f'(x_0)(x-x_0)^1}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!}$$

So,

$f'(x) = \sec^2 x$; $f''(x) = 2\sec^2 x \tan x$;

$f'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$

let $x_0 = 0$, so $f'(x_0) = f'(0) = 1$

$f''(0) = 0$; $f'''(0) = 2$; $f(0) = 0$

so $f(x) = \frac{0 \times (x-0)^0}{0!} + \frac{1 \times (x-0)^1}{1!} + \frac{0 \times (x-0)^2}{2!}$

$+ \frac{2 \times (x-0)^3}{3!} + \cdots$

$= 0 . x^0 + 1 \times x^1 + 0 \times x^2 + \frac{1}{3} x^3$

$= x + \frac{1}{3} x^3 \Rightarrow$ taylor expansion of $\tan x$

so $P_3(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

comparing basis coefficients in $f(x)$ and $P_3(x)$

$a_0 = 0$; $a_1 = 1$, $a_2 = 0$, $a_3 = \frac{1}{3}$

$\boxed{\text{Ans}}$

$\boxed{1(b)}$

$x = \frac{\pi}{4}$ $f(x) = \tan x$

Now, $f(\pi/4) = \tan \pi/4 = 1$

$P_3(x) = x + (1/3) x^3$ [From 1(a)]

so $P_3(\pi/4) = \pi/4 + \frac{1}{3}(\pi/4)^3 = 0.9469$

$\therefore$ Relative error $= \frac{|f(x) - P_3(x)|}{f(x)}$

$= \frac{|1 - 0.9469|}{0.9469} = 0.0531 = 5.31\%$

(Ans)

$\boxed{1(c)}$

we have $x = \pi/4$; $\xi_x \in [0, \pi/4]$

and $f(x) = P_3(x) + \frac{f^{(3+1)}(\xi_x)}{(3+1)!}(x - x_0)^{3+1}$

Now, $f'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$

$\Rightarrow f'''(x) = 2\sec^4 x + 4(1 - \sec^2 x)\sec^2 x$

$= 6\sec^4 x - 4\sec^2 x$

so $f^{iv}(x) = 6 \times 4 \times \sec^3 x \sec x \tan x -$
$4 \times 2 \times \sec x \times \sec x \tan x$

$= 24\sec^4 x \tan x - 8\sec^2 x \tan x$

$f^{iv}(\xi = 0) = 24 \times 1 \times 0 - 8 \times 1 \times 0$

$f^{iv}(\xi = \pi/4) = 24(\sqrt{2})^4 \times 1 - 8(\sqrt{2})^2 \times 1$
$= 80$

Lagrange form of remainder,

$= \frac{f^{iv}(\xi_x)}{4!}\left(\frac{\pi}{4} - 0\right)^4$

$= \frac{80}{24} \times (\pi/4)^4 = 1.26835$

$\boxed{\text{Ans}}$

Ans to que-2

given $f(x) = e^x - e^{-x}$

nodes: $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$f(x_0) = -2.35$, $f(x_1) = 0$, $f(x_2) = 2.35$

now,

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} -2.35 \\ 0 \\ 2.35 \end{pmatrix}$$

2(b)

$$\det(V) = 1 \times (0 \times 1 - 0 \times 1) + 1\,(1-0) + 1\,(1-0)$$
$$= 1 \times 0 + 1 \times 1 + 1 \times 1 = 2 \quad \text{(Ans)}$$

2(c)

we know, $V \times a = b$

$$\Rightarrow a = V^{-1} b$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} -2.35 \\ 0 \\ 2.35 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix} \times \begin{pmatrix} -2.35 \\ 0 \\ 2.35 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2.35 \\ 0 \end{pmatrix}$$

$\therefore a_0 = 0, \quad a_1 = 2.35, \quad a_2 = 0$

$$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

$$= 0 \times x^0 + (2.35 \times x) + 0 \times x^2$$

$$= 2.35x \quad \boxed{\text{Ans}}$$

$\boxed{2(d)}$

$$|f(x) - P_2(x)| \leq \left| \frac{f^{(2+1)}(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right|$$

$$\leq \left| \frac{f^3(\xi)}{3!} (x+1)\, x\, (x-1) \right|$$

$$\leq \left| \frac{f'''(\xi)}{6} (x^3 - x) \right|$$

let

$w(x) = x^3 - x$

$w'(x) = 3x^2 - 1$

now,

$3x^2 - 1 = 0$

$\Rightarrow x = \pm 0.577$

now

x	$w(x)$
$+0.577$	-0.385
-0.577	$+0.385$
$+2.1$	$+7.161$
-2.1	-7.161

as to get the maximum value,

$w(x) = +7.161$

given, $f(x) = e^x - e^{-x}$

$f'(x) = e^{-x} - (-1)e^{-x} = e^{-x} + e^{-x}$

$f''(x) = e^x - e^{-x}$; $f'''(x) = e^x + e^{-x}$

Now,

$f'''(\xi = -2.1) = e^{-2.1} + e^{-(-2.1)} = 8.29$

$f'''(\xi = 2.1) = e^{2.1} + e^{-2.1} = 8.29$

so, $|f(x) - P_2(x)| = \left| \frac{8.29}{6} \times 7.16 \right|$

$= 9.90$ $\boxed{\text{Ans}}$

Ans to the que-3

3 (a)

Given, $f(x) = e^x + e^{-x}$

nodes: $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$

$$l_0(x) = \frac{(x - x_0)}{(x_0 - x_0)} \times \frac{(x - x_1)}{(x_0 - x_1)} \times \frac{(x_1 - x_2)}{(x_0 - x_2)}$$

$$= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}(x^2 - x$$

$$L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} \times \frac{(x - x_2)}{(x_1 - x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = 1 - x^2$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}(x^2+x)$$

3 (b)

$$P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, L_1(x) + f(x_2)\, l_2(x)$$

now,

$f(x_0) = 3.09$, $f(x_1) = 2$, $f(x_2) = 3.09$

$$P_2(x) = 3.09 \times \frac{1}{2}(x^2 - x) + 2(1 - x^2) + 3.09 \times \frac{1}{2} \times (x^2 + x)$$

$$= 1.545 \{(x^2 - x) + (x^2 + x)\} + 2 - 2x^2$$

$$= 3.09x^2 + 2 - 2x^2$$

$$= 2 + 1.09x^2$$

now, $P_2(6) = 2 + 1.09 \times (6)^2$

$$= 41.2$$ $\boxed{\text{Ans}}$

$\boxed{3(c)}$

given, $f(x) = e^{-x} + e^{x}$; $P_2(x) = 2 + 1.09x^2$

now, $f(1.5) = 4.70$; $P_2(x) = 4.45$

$$\text{Relative error} = \left| \frac{f(x) - P_2(x)}{f(x)} \right|$$

$$= \left| \frac{4.70 - 4.45}{4.70} \right|$$

$$= 0.0532 = 5.32\ \% \quad \boxed{\text{Ans}}$$

Ans to the que-4

4(a)

given, $f(x) = e^{x} - e^{-x}$

nodes: $x_0 = -2,\ x_1 = 0,\ x_2 = 2$

now, $f(x_0) = -7.25$, $f(x_1) = 0$

$f(x_2) = 7.25$

$x_0 = -2,$

$f(x_0) = -7.25$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0 + 7.25}{0 + 2}$$

$$= 3.63$$

$x_1 = 0$

$f(x_1) = 0$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{7.25 - 0}{2 - 0}$$

$$= 3.63$$

$x_2 = 2$

$f(x_2) = 7.25$

$f[x_0, x_1] = 3.63$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{3.63 - 3.63}{2+2} = 0$$

$f[x_1, x_2] = 3.63$

therefore, $a_0 = f[x_0] = -7.25$

$a_1 = f[x_0, x_1] = 3.63$

$a_2 = f[x_0, x_1, x_2] = 0$

$$\boxed{4(b)}$$

$P_2(x) = a_0 n_0(x) + a_1 n_1(x) + a_2 n_2(x)$

$= a_0 \times 1 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$

$= -7.25 + 3.63(x+2) + 0\times(x+2)(x-0)$

$= 3.63x + 0.01$

$= 3.63x + 0.01$

so, $P_2(6) = 3.63\times 6 + 0.01$

$= 21.8$ $\boxed{\text{Ans}}$

$$\boxed{4(c)}$$

$f(x) = e^x - e^{-x}$ so $f(1.5) = 4.26$

$P_2(x) = 3.63x + 0.01$ $\quad P_2(1.5) = 5.46$

Relative error $= \left| \dfrac{f(x) - P(x)}{f(x)} \right|$

$= \left| \dfrac{4.26 - 5.46}{4.26} \right|$

$= 0.282 = 28.2\%$ (Ans)