MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Cauchy Integral Formula

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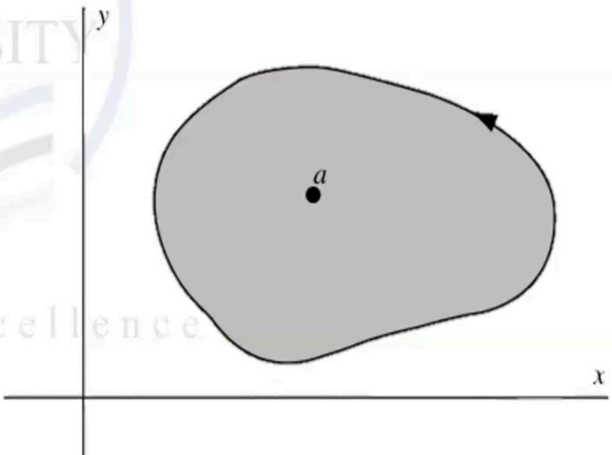
Cauchy Integral Formula

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C. Then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$



$\because$ $f(z)$ is analytic inside and on C,

$\oint f(z)\, dz = 0$

but $\oint \frac{f(z)}{z-a} dz \neq 0$ → the whole function $\frac{f(z)}{z-a}$ has a singularity at $z=a$

$$\boxed{\oint \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)}$$

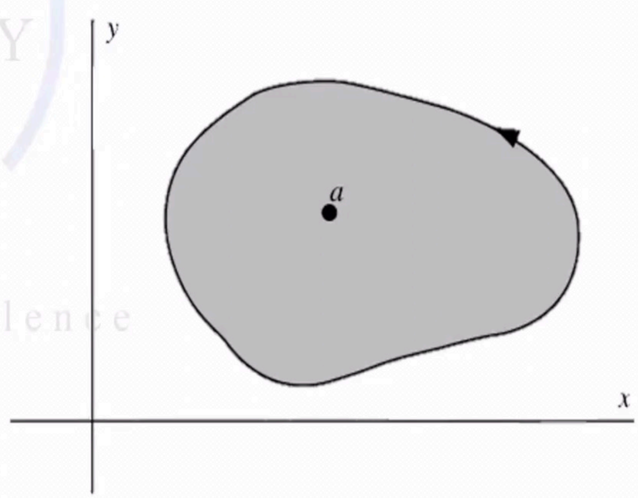
in general form,

Cauchy Integral Formula (General Version)

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C. Then

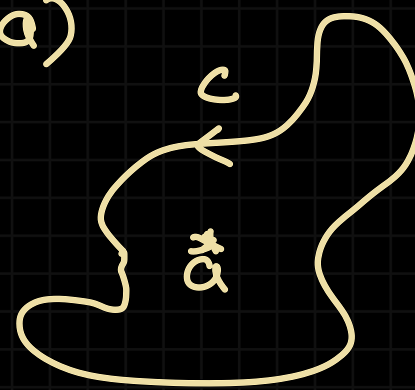
$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where $n = 1,2,3,\ldots$



so in general form,

$$\oint_c \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)$$



$$\boxed{\oint_c \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} \left[\frac{d^n}{dz^n}\big(f(z)\big)\right]_{z=a}}$$

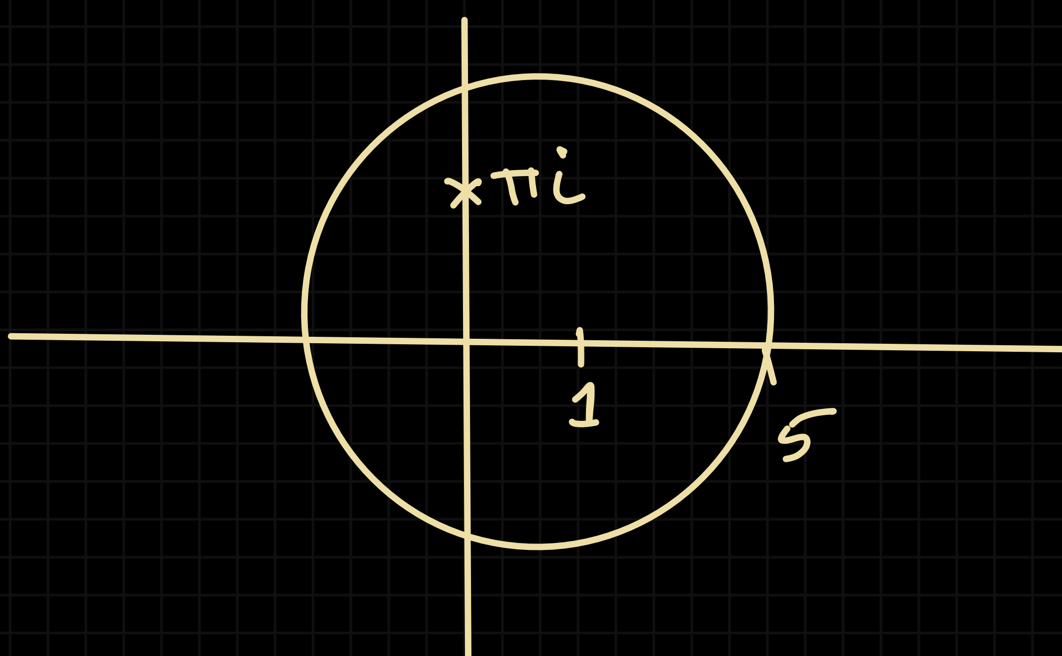
☐ Evaluate

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz$$

where C is inside the circle $|z-1|=4$

Solve:

$$|z-1|=4$$



$f(z) = e^{3z}$, $a = \pi i$

$f(z)$ is analytic inside and and on C.

Also $z = a = \pi i$ is inside C.

$$\therefore \oint \frac{f(z)}{z - \pi i}\, dz = 2\pi i \cdot f(\pi i)$$

$$= 2\pi i \times e^{3\pi i}$$

$$= 2\pi i(\cos 3\pi + i \sin 3\pi)$$

$$= -2\pi i$$

D Evaluate

$$\oint \frac{z^2+\cos^2 \pi z}{z-2}\, dz$$

if -

(a) $c: |z|=3$

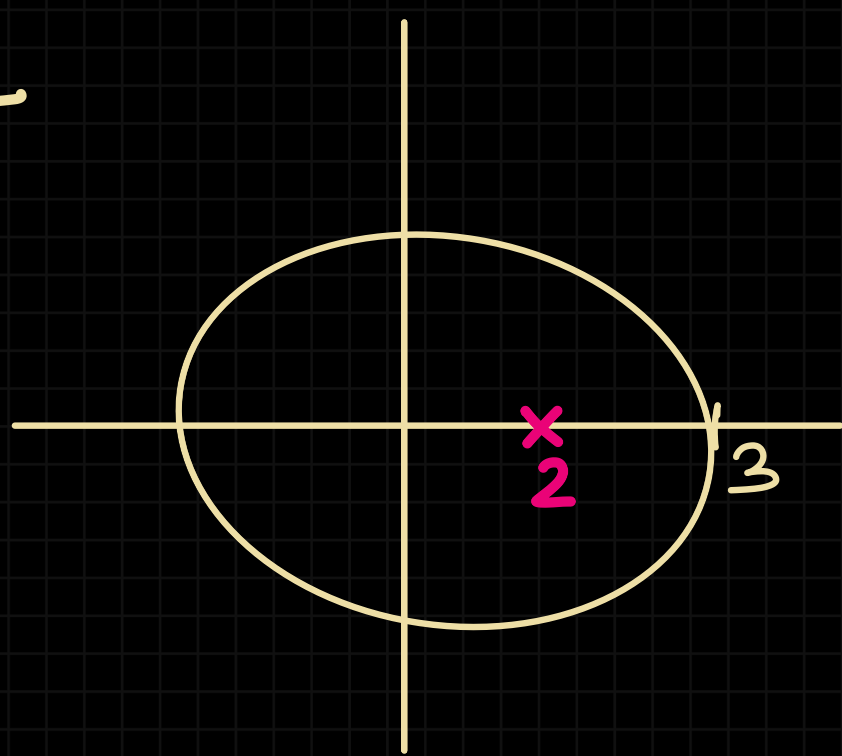
(b) $c: |z| = 1$

Evaluate

$$\oint \frac{z^2 + \cos^2 \pi z}{z-2} \, dz$$

if $C: |z| = 3$

solve:



$f(z) = z^2 + \cos^2 \pi z \qquad a = 2$

$f(z)$ is analytic inside and on

$c: |z| = 3.$

Also $z = a = 2$ is inside c.

using Cauchy integral formula

$$\oint \frac{z^2 + \cos^2 \pi z}{z-2}\, dz = 2\pi i \cdot f(2)$$

$$= 2\pi i \times \left\{ 2^2 + \cos^2 (2\pi) \right\}$$

$$= 8\pi i + 2\pi i = 10\pi i$$

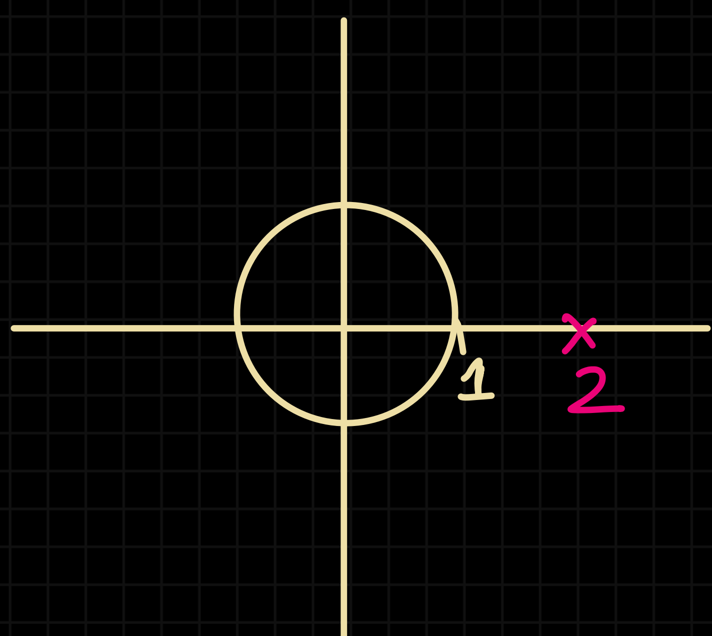
Ⓓ Evaluate

$$\oint \frac{z^2+\cos^2 \pi z}{z-2}\, dz$$

if $c: |z| = 1$

solve:

$|z| = 1$



the function $f(z) = \dfrac{z^2 + \cos^2 \pi z}{z-2}$

is analytic inside and on $C: |z| = 1$

applying cauchy-goursat theorem,

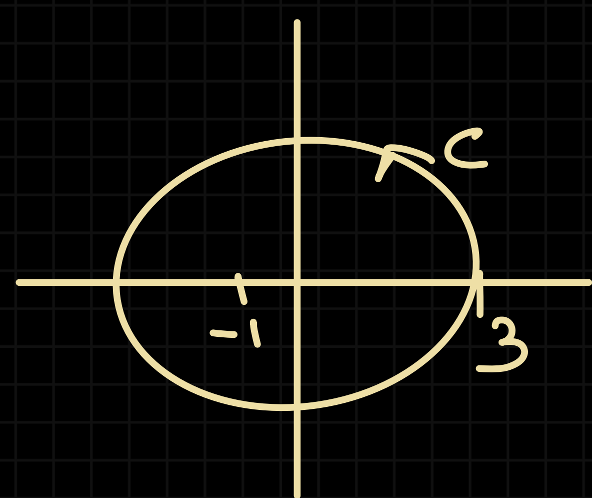
$$\therefore \oint_C \frac{z^2 + \cos^2 \pi z}{z-2}\, dz = 0 \quad \text{where}$$

$C: |z| = 1$

Evaluate

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

where C is the circle $|z|=3$



solve:

$f(z) = e^{2z}$ is

analytic inside and on C

$$(z+1)^4 = 0$$

$\Rightarrow z+1=0$

$\Rightarrow z=-1$

$\therefore a=-1$

$n+1=4$

$n=3$

Also $z=a=-1$ is inside c.

$\therefore$ Using

Cauchy integral formula,

$$\oint_c \frac{e^{2z}}{(z+1)^4} = \frac{2\pi i}{3!} \times f^3(-1)$$

$$f(z) = e^{2z}$$

$$f'(z) = 2e^{2z}$$

$$f''(z) = 4e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$f'''(-1) = 8e^{-2}$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} = \frac{2\pi i}{3!} \times 8e^{-2}$$

$$= \frac{16\pi i}{6e^2}$$

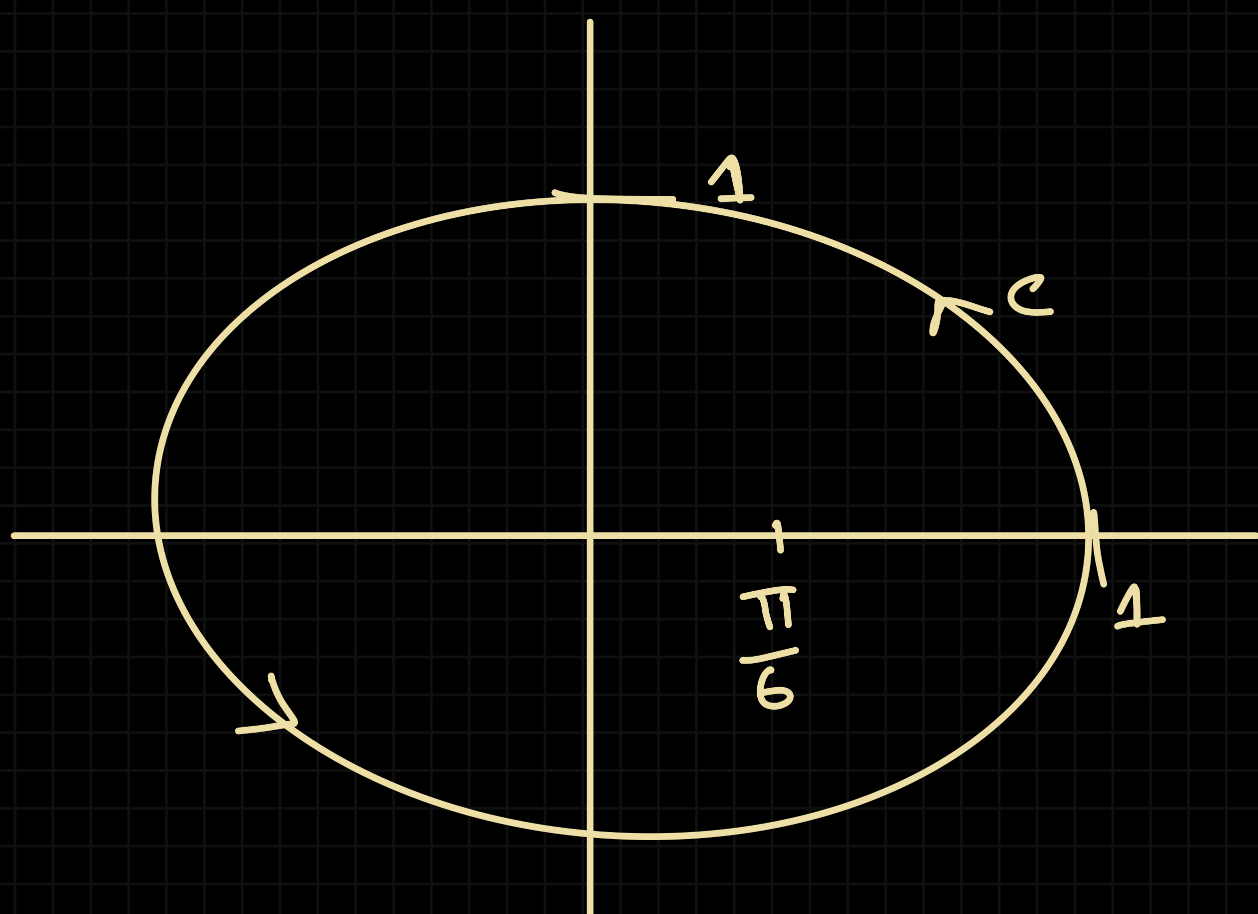
$$= \frac{8}{3}\frac{\pi i}{e^2}$$

Evaluate

$$\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^4}\,dz \qquad C: |z|=1$$

solve:

$|z|=1$



$f(z) = \sin^6 z$ is analytic inside and on C.

Also $z = a = \frac{\pi}{6}$ is inside C.

$$\therefore \oint_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^4} \, dz$$

$n+1 = 4$
$n = 3$

$$= \frac{2\pi i}{3!} \times f^3(\pi/6)$$

$f(z) = \sin^6 z$

$f'(z) = 6 \sin^5 z \times \cos z$

$$f''(z) = 6\left\{\sin^5 z \times -\sin z + \cos z \times 5\sin^4 z \times \cos z\right\}$$

$$= 6\left\{-\sin^6 z + 5\sin^4 z\cos^2 z\right\}$$

$$= -6\sin^6 z + 30\sin^4 z\cos^2 z$$

$$f'''(z) = -6 \times 6\sin^5 z\cos z + 30 \times \left\{\sin^4 z \times 2\cos z \times -\sin z + \cos^2 z \times 4\sin^3 z \times \cos z\right\}$$

$$= -36\sin^5 z\cos z - 60\sin^5 z\cos z + 120\sin^3 z\cos^3 z$$

$$\Rightarrow f'''(z) = -96\sin^5 z\cos z + 120\sin^3 z\cos^3 z$$

$$f'''\left(\frac{\pi}{6}\right) = -96 \times \frac{1}{2^5} \times \frac{\sqrt{3}}{2} + 120 \times \frac{1}{2^3} \times \left(\frac{\sqrt{3}}{2}\right)^3$$

$$= \frac{33\sqrt{3}}{8}$$

$$\therefore \oint_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^4} \, dz$$

$$= \frac{2\pi i}{\underset{6}{3!}} \times f^3\left(\pi/6\right)$$

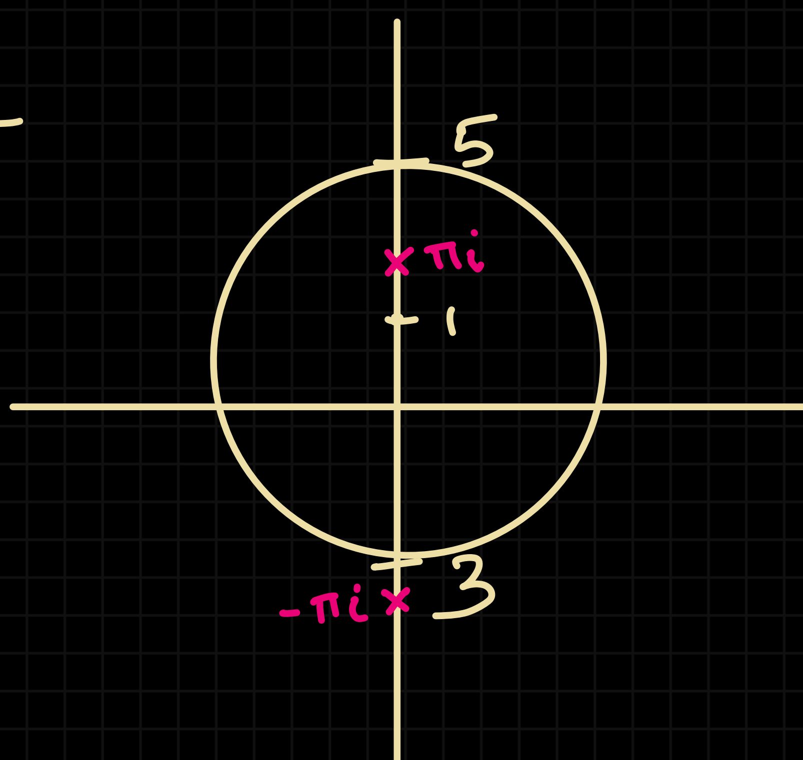
$$= \frac{2\pi i}{6} \times \frac{33\sqrt{3}}{8}$$

$$= \frac{11\sqrt{3}}{8} \pi i$$

Evaluate

$$\oint_C \frac{e^z \, dz}{(z^2+\pi^2)^2} \qquad C: |z-i|=4$$

solve:



$$(z^2+\pi^2)^2=0$$

$$z^2+\pi^2=0$$

$$z^2=-\pi^2$$

$$z=\pm\sqrt{-\pi^2}$$

$$=\pm\pi i$$

now,

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$$

$$= \int \frac{e^z}{(z^2-\pi^2 i^2)^2}\,dz$$

$$= \int \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}\,dz$$

$$= \int \frac{\frac{e^z}{(z+\pi i)^2}}{(z-\pi i)^2}\,dz$$

$f(z) = \dfrac{e^z}{(z+\pi i)^2}$ is analytic

inside and on C. and $a = \pi i$

$z = a = \pi i$ is inside C.

using cauchy integral thm,

$$\oint \frac{\dfrac{e^z}{(z+\pi i)^2}}{(z-\pi i)^2}\, dz$$

$$= \frac{2\pi i}{1!} \times f'(\pi i)$$

$$f(z) = \frac{e^{z}}{(z+\pi i)^{2}}$$

$$\therefore \quad f'(z),$$

$$= \frac{(z+\pi i)^{2} \times e^{z} - e^{z} \times \left\{2(z+\pi i)\right\}}{\left\{(z+\pi i)^{2}\right\}^{2}}$$

$\therefore f'(\pi i)$

$$= \frac{(\pi i+\pi i)^2 e^{\pi i} - e^{\pi i} \cdot 2(\pi i+\pi i)}{(\pi i+\pi i)^4}$$

$$= \frac{4\pi^2 i^2 \{\cos\pi\} - (\cos\pi) \cdot 4\pi i}{16\pi^4 i^4}$$

$$= \frac{4\pi^2 + 4\pi i}{4\pi \times 4\pi^3 i^4}$$

$$= \frac{\pi + i}{4\pi^3 i^4}$$

$$= \frac{\pi + i}{4\pi^3}$$

$$= \frac{1}{4\pi^2} + \frac{i}{\pi^3}$$

$$\oint \frac{\frac{e^z}{(z+\pi i)^2}}{(z-\pi i)^2} dz = 2\pi i \times \frac{(\pi + i)}{4\pi^3}$$

$$= \frac{\pi i + i^2}{2\pi^2}$$

$$= \frac{\pi i - 1}{2\pi^2}$$

in case there is more than

one singularity

two ways to solve

→ Partial Fraction (x)

→ Residual Theorem

→ (chapter-7)

Partial Fraction

e.g.:

$$\frac{1}{(x-1)(x-3)^2(x-5)^3}$$

$$= \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-5)}$$

$$+ \frac{E}{(x-5)^2} + \frac{F}{(x-5)^3}$$