# MAT216: Linear Algebra & Fourier Analysis

## Topic: Orthogonality

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# topic : Inner Product Spaces

→ inner product spaces

→ Inner Products

→ Orthogonality মানে কি

→ Gram - schmidt process

# Inner Product

(vector) + (vector) = vector

scalar × vector = vector

vector · vector = scalar

inner product          dot product

Inner product axioms/conditions:

i) $\langle u, u \rangle \geq 0$ ;

$\underbrace{\langle u, u \rangle}_{\text{inner product}} = 0$ if and only if $u = 0$

ii) $\langle u \cdot v \rangle = \langle v . u \rangle$

iii) $\langle k\, u, v \rangle = k \langle u, v \rangle$

iv) $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle +$

$\langle u_2, v)$

## Inner Product in different spaces:

### $\overbrace{\text{Inner Product}}^{\text{dot product}}$ on $\mathbb{R}^3$:

$u = (2, 3, 5)$

$v = (6, -1, 2)$

$$\langle u, v \rangle = 12 - 3 + 10$$

$$= 19$$

$$\langle v, u \rangle = 19$$

\# $\langle (a, b, c), (x, y, z),$

$$= ax + by + cz$$

$\boxed{4}$ $LHS = \langle k\, u, v \rangle$

$$= \langle (6, 9, 15), (6, -1, 2) \rangle$$

$$= 36 - 9 + 30$$

$$= 57$$

$$RHS = k \langle u, v \rangle$$

$$= 3 \times 19$$

$$= 57$$

# Inner Product on Matrix

Space of 2x2 matrix:

A, B দুটো matrix হলে,

$$\langle A \cdot B \rangle = \mathrm{Trace}\left(A^T B\right)$$

$$\Rightarrow \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right\rangle =$$

$$\mathrm{Trace}\left( \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right)$$

$$= \text{Trace}\begin{pmatrix} ax+cz & ay+cw \\ bx+dz & by+dw \end{pmatrix}$$

$$= ax+cz+by+dw$$

# Norm

↳ length

norm of vector $v$ = $\|v\|$

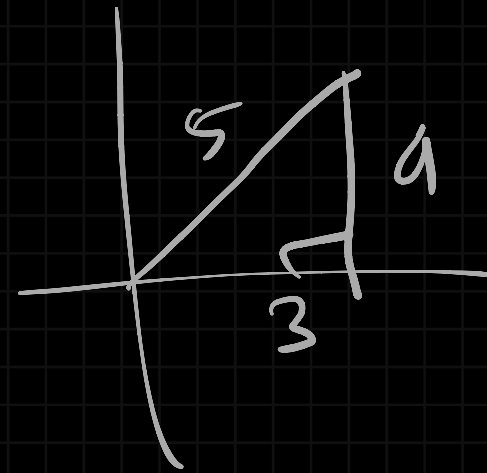
and $\|v\| = \sqrt{\langle v . v \rangle}$

eg: $v = (3, 4)$

$\langle v . v \rangle = \langle (3,4), (3,4) \rangle$

$= 25$

$$\|v\| = \sqrt{25} = 5$$




$$\|e^x\| = \sqrt{\langle e^x \cdot e^x \rangle}$$

# Orthogonality

$\langle u \cdot v \rangle = 0 \Rightarrow u$ and $v$ are orthogonal / $90°$ / লম্ব / perpendicular

eg: $\langle (1,-2), (a,2) \rangle$

$= 0$

# Orthogonal set and orthonormal set :

$$S = \{(0,2,0), (0,0,3), (5,0,0)\}$$

$$\left.\begin{aligned} \langle v_1, v_2 \rangle &= 0 \\ \langle v_1, v_3 \rangle &= 0 \\ \langle v_2, v_3 \rangle &= 0 \end{aligned}\right\}$$

$\therefore$ S is an orthogonal set (zero zero)

Orthogonal set: A set $S$ is called orthogonal set if for any $u, v \in S$ $\langle u, v \rangle = 0$ where $u \neq v$.

orthonormal set: A set $S$ is called orthonormal if

$\langle u, v \rangle = 0$ for $u \neq v$

$\langle u, u \rangle = 1$

que: $v_1 = (0, 1, 0)$; $v_2 = \left(\frac{-4}{5}, 0, \frac{3}{5}\right)$

$v_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$

show that $S = \{v_1, v_2, v_3\}$ is ortho-normal.

$\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_1, v_3 \rangle = 0$

$\therefore$ s is orthogonal

$\langle v_1, v_1 \rangle = 0$

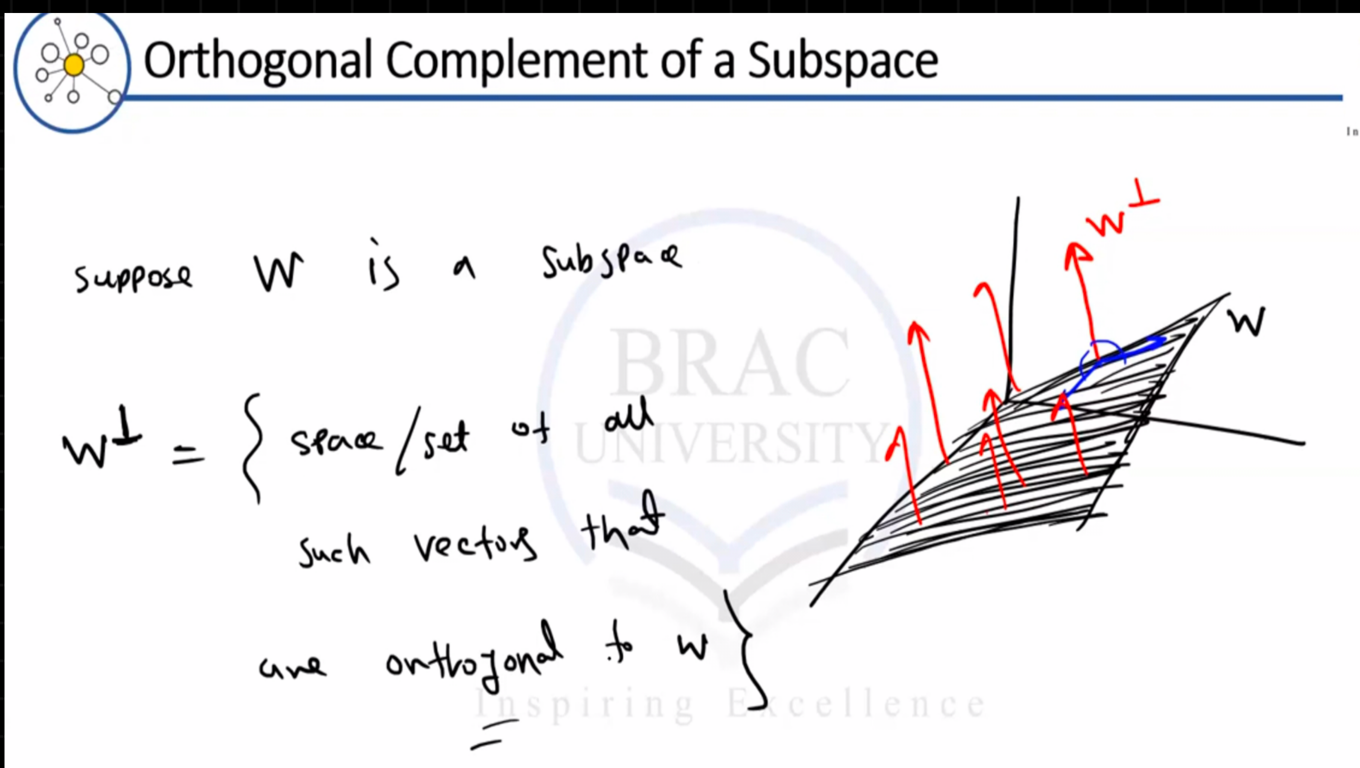
$\langle v_2 \cdot v_2 \rangle = 0$

$\langle v_3 \cdot v_3 \rangle = 0$

so s is orthonormal

# Orthogonal Complement of a Subspace

plane এর সাথে perpendicular vector গুলোর collection is ~.



$\boxed{\text{qm}}$

$W =$ span $\{(1, 2, 3, 4), (2, -1, 3, 1),$

$(3, 1, 6, 5)\}$ → এই তিনটার সাথে ৯০° যে হলো $W^{\perp}$.

find the orthogonal complement of $W^{\perp}$.

$W^{\perp} : \{(x, y, z, w) :$ is orthogonal to $w\}$

ans:

acc to def.

$<(1,2,3,4),(x,y,z,w)> \Rightarrow$

$$x+2y+3z+4w=0 \quad (i)$$

$<(2,-1,3,1),(x,y,z,w)> = 0 \Rightarrow$

$$2x-y+3z+w=0 \quad - ii$$

$<(3,1,6,5),(w,x,y,z)> \Rightarrow$

$$(3x+y+6z+5w)=0$$

Augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 9 & 0 \\ 2 & -1 & 3 & 1 & 6 \\ 3 & 6 & 6 & 5 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 9 & 0 \\ 0 & -5 & -3 & -7 & 0 \\ 0 & -5 & -3 & -7 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 9 & 0 \\ 0 & -5 & -3 & -7 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 5 & 0 & 9 & 6 & 0 \\ 0 & -5 & -3 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad 5x + 9z + 6w = 0$$

$$-5y - 3z - 7w = 0$$

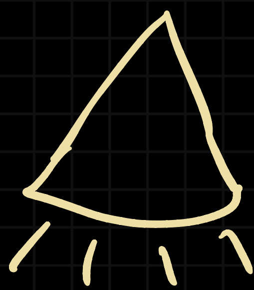
$$z = t_1, \quad w = t_2$$

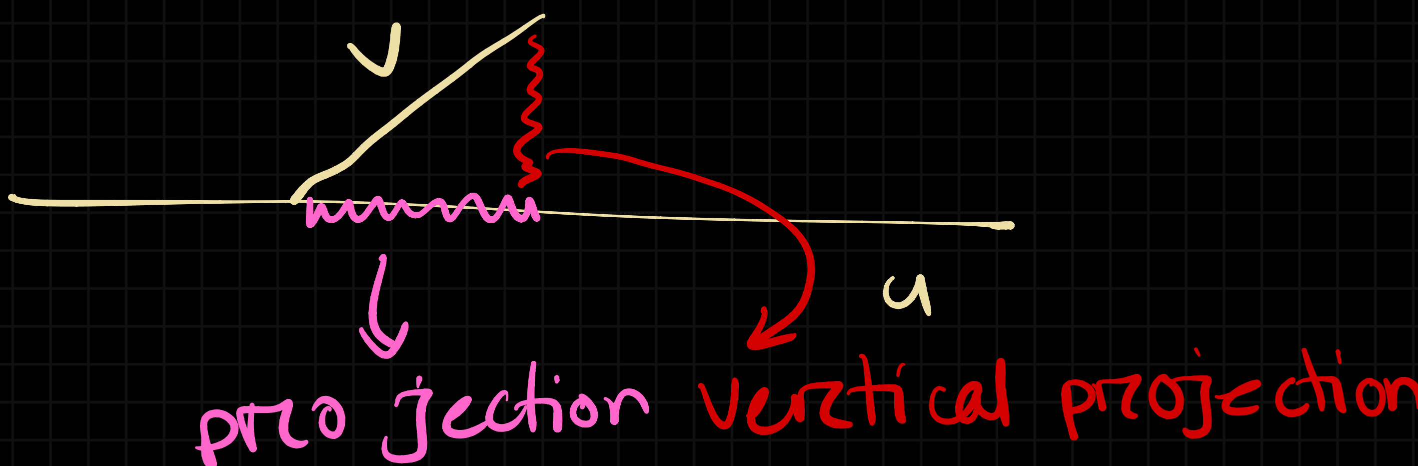
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{9}{5}t_1 - \frac{6}{5}t_2 \\ -\frac{3}{5}t_1 - \frac{7}{5}t_2 \\ t_1 \\ t_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 \\ -3/5 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} -6/5 \\ -7/5 \\ 0 \\ 1 \end{pmatrix} t_2$$

$$W^{\perp} = \text{span}\left\{ \left(-\frac{2}{5}, -\frac{3}{5}, 1, 0\right), \left(-\frac{6}{5}, -\frac{7}{5}, 0, 1\right) \right\}$$

# Projection






2 orthogonal projections

projection of v on u:

$$\frac{\langle v, u \rangle \cdot u}{\|u\|^2}$$

vertical projection:

$$v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u$$

# Basis

$\mathbb{R}^2$ plane এর একটা vector কে কয়েকটা vector এর combo হিসেবে লেখা যায়।

eg: $(2,5) = x\underbrace{(2,0)}_{\text{basis}} + y\underbrace{(0,3)}_{\text{basis}}$

$\mathbb{R}^2 = \text{span}\{(2,0),(0,3)\}$

↳ এই span দিয়ে পুরো $\mathbb{R}^2$ space এর যেকোনো place এ যাওয়া possible

যে basis দেওয়া থাকবে তাকে orthonormal basis বানানো লাগবে → Gram-Schmidt process

# Gram Schmidt Process

Consider the vector space $R^3$ with the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_1 = (0, 0, 1)$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

প্রথমটা ($u_1$) কে change

করবোনা, পরেরগুলো change করবো

<u>steps:</u> i) $v_1 = u_1$

ii) $$v_2 = u_2 - \frac{\langle u_2 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$$

iii) $$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

For the math, ans:

i) $v_1 = u_1 = (1,1,1)$

ii) $$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_2 = (0,1,1) - \frac{\langle (0,1,1), (1,1,1) \rangle}{\langle (1,1,1), (1,1,1) \rangle} \times (1,1,1)$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

(iii) $$v_3 = (0,0,1) - \frac{\langle (0,0,1), (1,1,1) \rangle}{\langle (1,1,1), (1,1,1) \rangle} \times$$

$$(1,1,1) - \frac{\langle (0,0,1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \rangle}{\langle \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \rangle} \times$$

$$\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} x$$

$$\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$\therefore$ orthogonal set:

$v_1$ | $v_2$ | $v_3$

$(1,1,1)$ $\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$ $\left(0, -\frac{1}{2}, \frac{1}{2}\right)$

orthogonal $\rightarrow$ orthonormal:

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}}}$$

$$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4}}}$$

$$= \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$q_1, q_2, q_3 \Rightarrow$ orthonormal

# Converting a basis to orthonormal basis

<u>steps:</u> i) $v_1 = u_1$

ii) $$v_2 = u_2 - \frac{\langle u_2 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$$

iii) $$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$v_3 = u_3 - v_1 - v_2$$

# Least Square Approximation

$$Ax = b \;\; 2\overline{\zeta n},$$

$\rightarrow$ no solution

$$A^T A x = A^T b$$

$$x = (ATA)^{-1} \cdot A^T b$$

$A^T A x = A^T b \rightarrow$ consistent

((columns of

$\leftarrow$ unique

many