MAT216: Linear Algebra & Fourier Analysis

Topic: Fourier Part I

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Period of a function

যে difference পরপর সে repeat হয়।

$\sin x, \cos x, \sec x, \operatorname{cosec} x \rightarrow p = 2\pi$

$\tan x, \cot x = \pi$

eg: $f(t) = 1952 \cos^{1971}(2022x + 2023)$

এর period কত?

e.g.:

period এর ভূমিকা → b, c এর only

$$f(x) = A \sin^b(cx+d) + e$$

b এর ভূমিকা:

$\sin x, \cos x, \sec x, \operatorname{cosec} x \rightarrow p = 2\pi$

$\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$

$n =$ even / জোড় হলে, $p = \pi$

$n =$ odd হলে, $p = 2\pi$

$\tan^n x$, $\cot^n x$

$n =$ even or odd

period $= \pi$

c এর ভূমিকা:

$$\text{period} = \frac{\text{Main Period}}{c}$$

$\boxed{\text{Que}}$

$$f(x) = 1952 \cos^{1971}(2022x + 2023)$$

$\boxed{\text{ans}}$:

$$\cos x \rightarrow \text{period} = 2\pi$$

odd

$$\cos^{1971} x \rightarrow \text{period} = 2\pi$$

$$\cos^{1971}(2022x) = \frac{2\pi}{2022}$$

$$1952 \cos^{1971}(2022x + 2023)$$

$$\text{final period} = \frac{2\pi}{2022}$$

$\boxed{\text{Que}}$ $y = 2023 \sin^{2022} 11x + 12$

ans:

$\sin x \rightarrow$ period $= 2\pi$

$\sin^{2022} x \rightarrow$ period $= \pi$

$\sin^{2022}(11x) \Rightarrow$ period $= \frac{\pi}{11}$

finally,

$2023 \sin^{2022}(11x) + 12 \rightarrow$ period $= \frac{\pi}{11}$

$\boxed{\text{Que}}$ $y = \tan^{2022}(2022x + 2023)$

$\tan x \rightarrow \text{period} = \pi$

$\tan^{2022}(x) \rightarrow p = \pi$

$\tan^{2022}(2022x) \rightarrow p = \dfrac{\pi}{2022}$

$\tan^{2022}(2022x + 2023) \rightarrow p = \dfrac{2\pi}{2022}$

Sketch of functions

plot ≠ sketch

plot → point

sketch → guess

eg: $f(x) = \begin{cases} x & ; \quad 0 < x < \pi \\ -x & ; \quad -\pi < x < 0 \end{cases}$

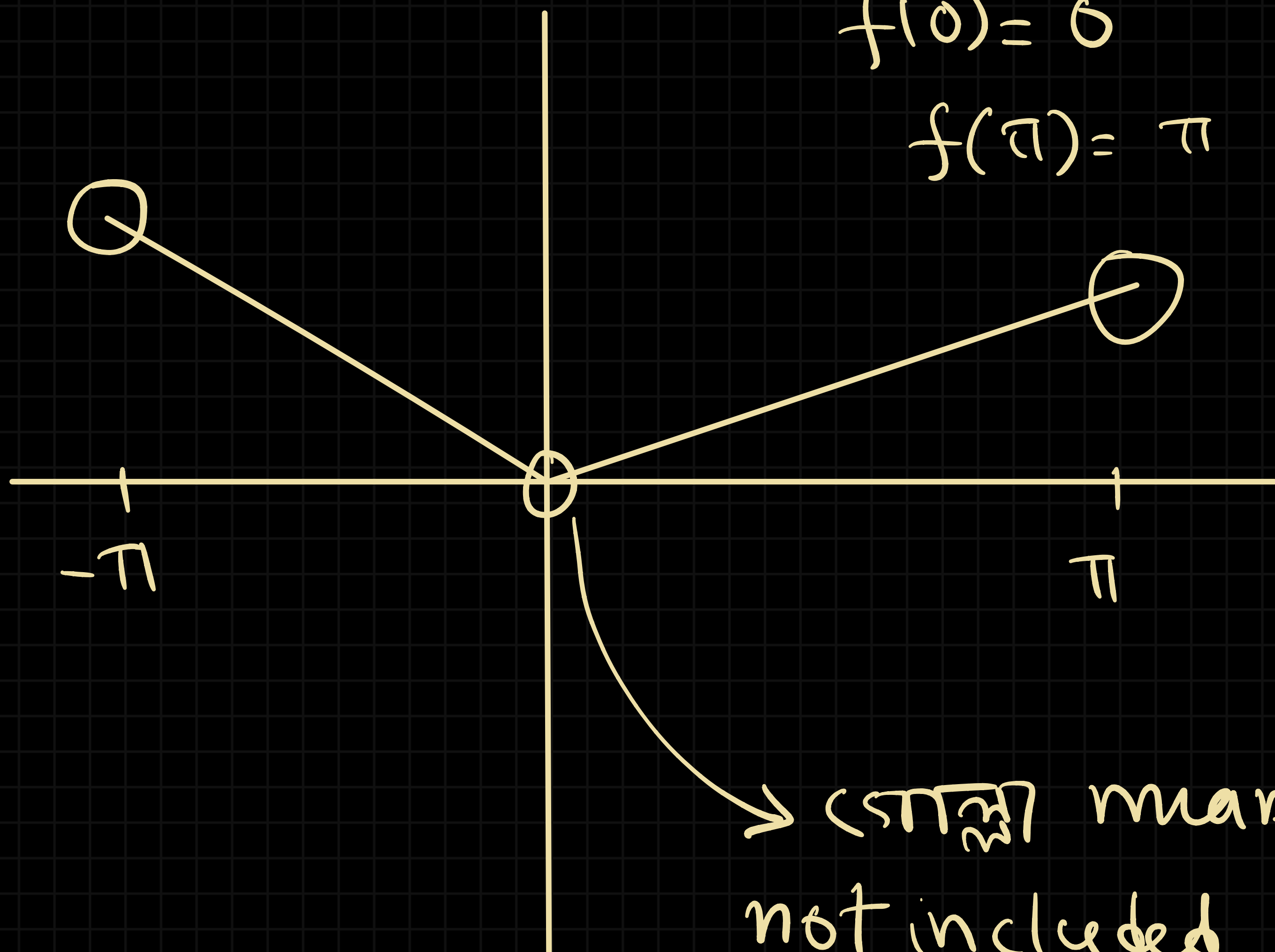
x এর $\underbrace{\text{highest power}}_{\text{1 বা এর কম হলে easy}} \rightarrow 1 \rightarrow$ linear f

$$f(x) = \begin{cases} x & ; \quad 0 < x < \pi \\ -x & ; \quad -\pi < x < 0 \end{cases}$$

$\hookrightarrow f(-\pi) = -(-\pi) = \pi$

$f(0) = 0$

$f(\pi) = \pi$



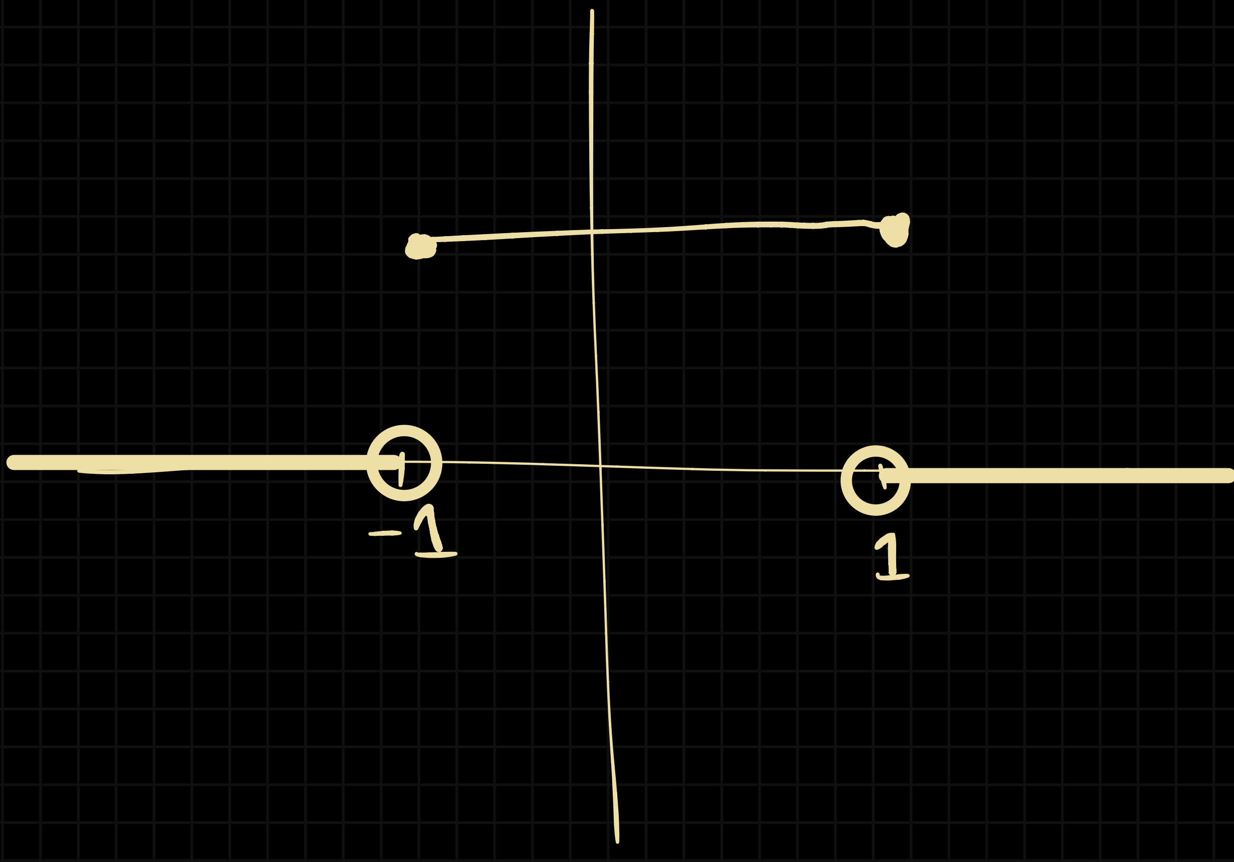
গোল্লা means not included (equals to নাই)

$\boxed{\text{que}}$ $f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0 & \text{else} \end{cases}$

$\boxed{\text{ans}}$:

$|x| \leq 1$ mans

$-1 \leq x \leq 1$



Que

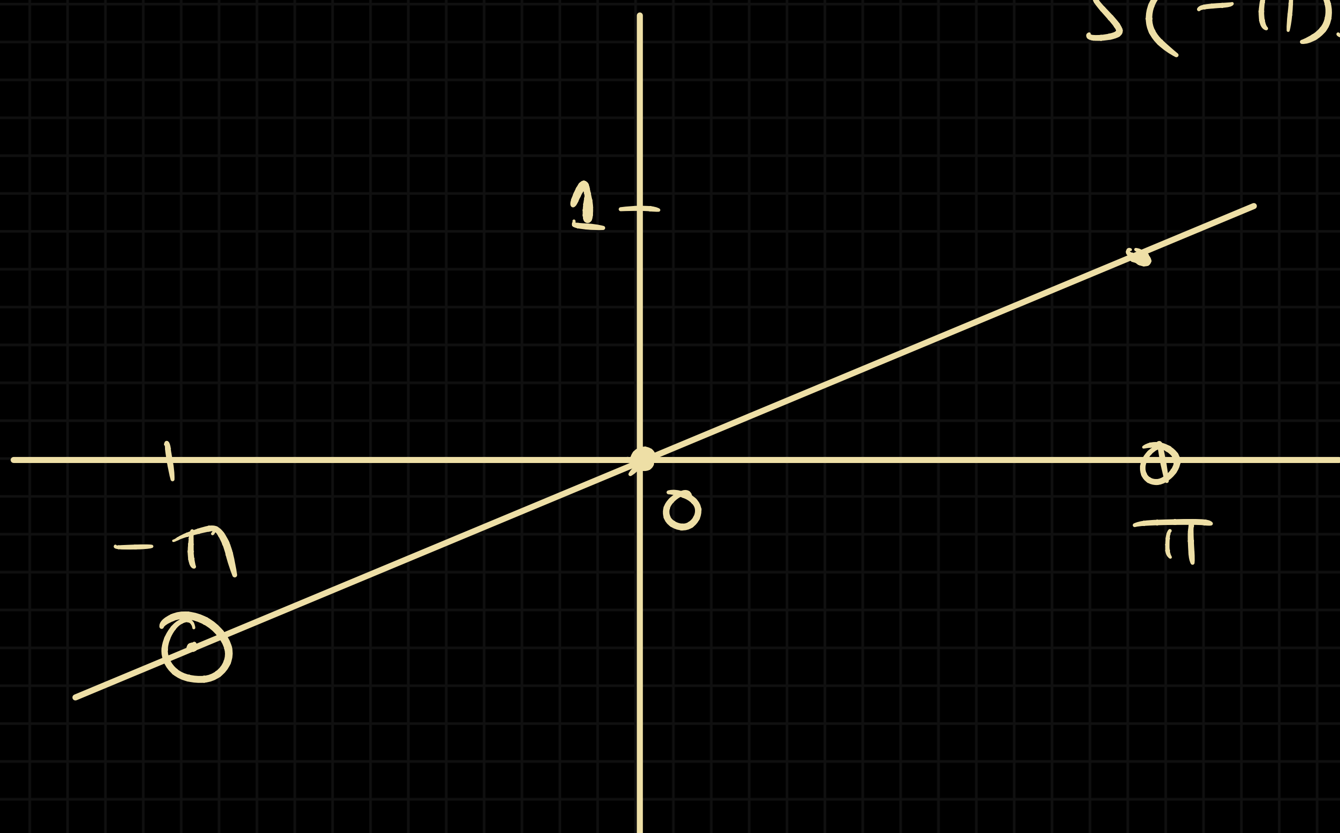
$$S(x) = \frac{x}{\pi} \; ; \quad -\pi < x < \pi$$

ans:

$S(0) = 0$

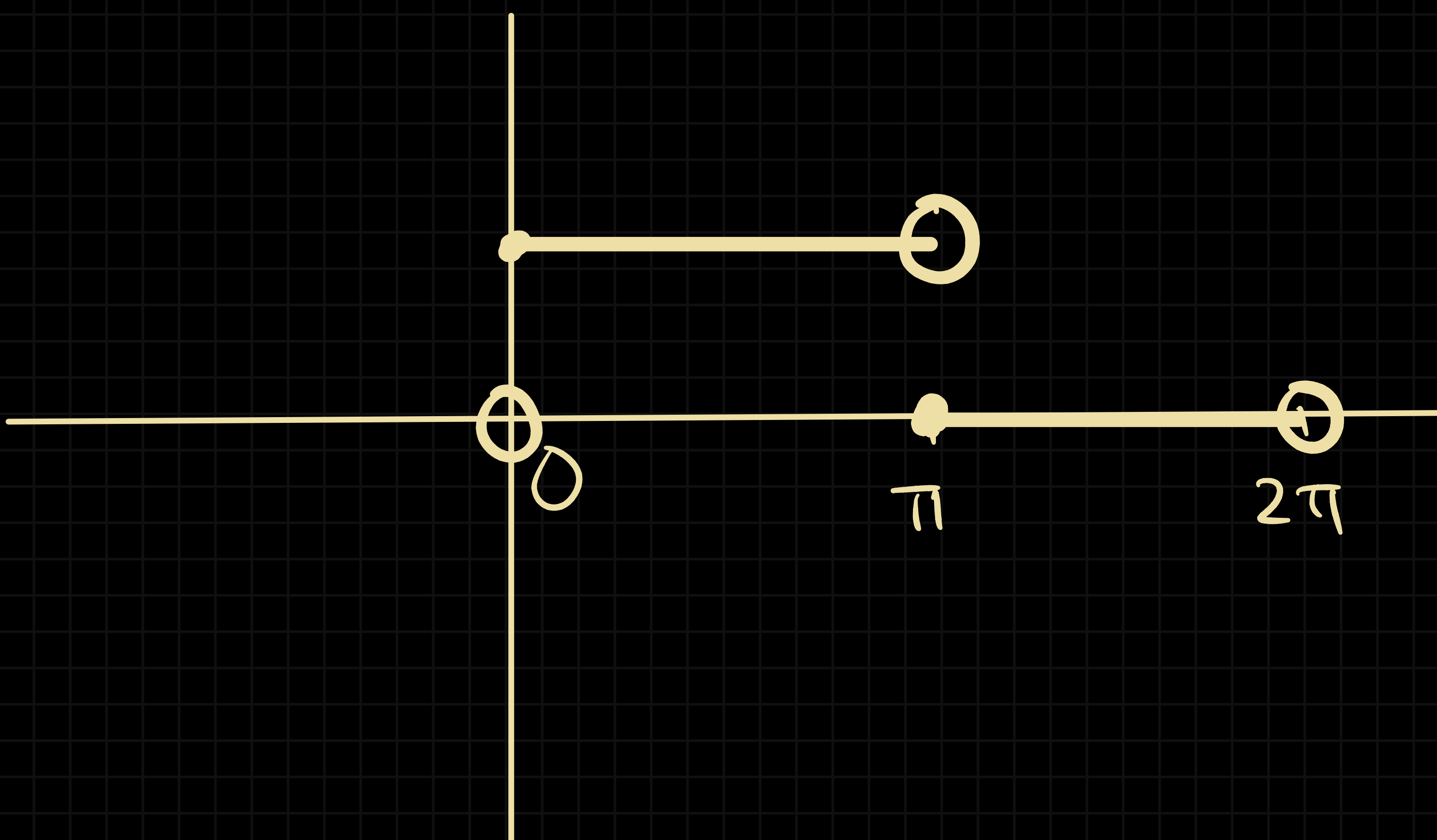
$S(\pi) = 1$

$S(-\pi) = -1$



Que

$$f(x) = \begin{cases} 0 ; & 0 \le x < \pi \\ 1 ; & \pi < x \le 2\pi \end{cases}$$

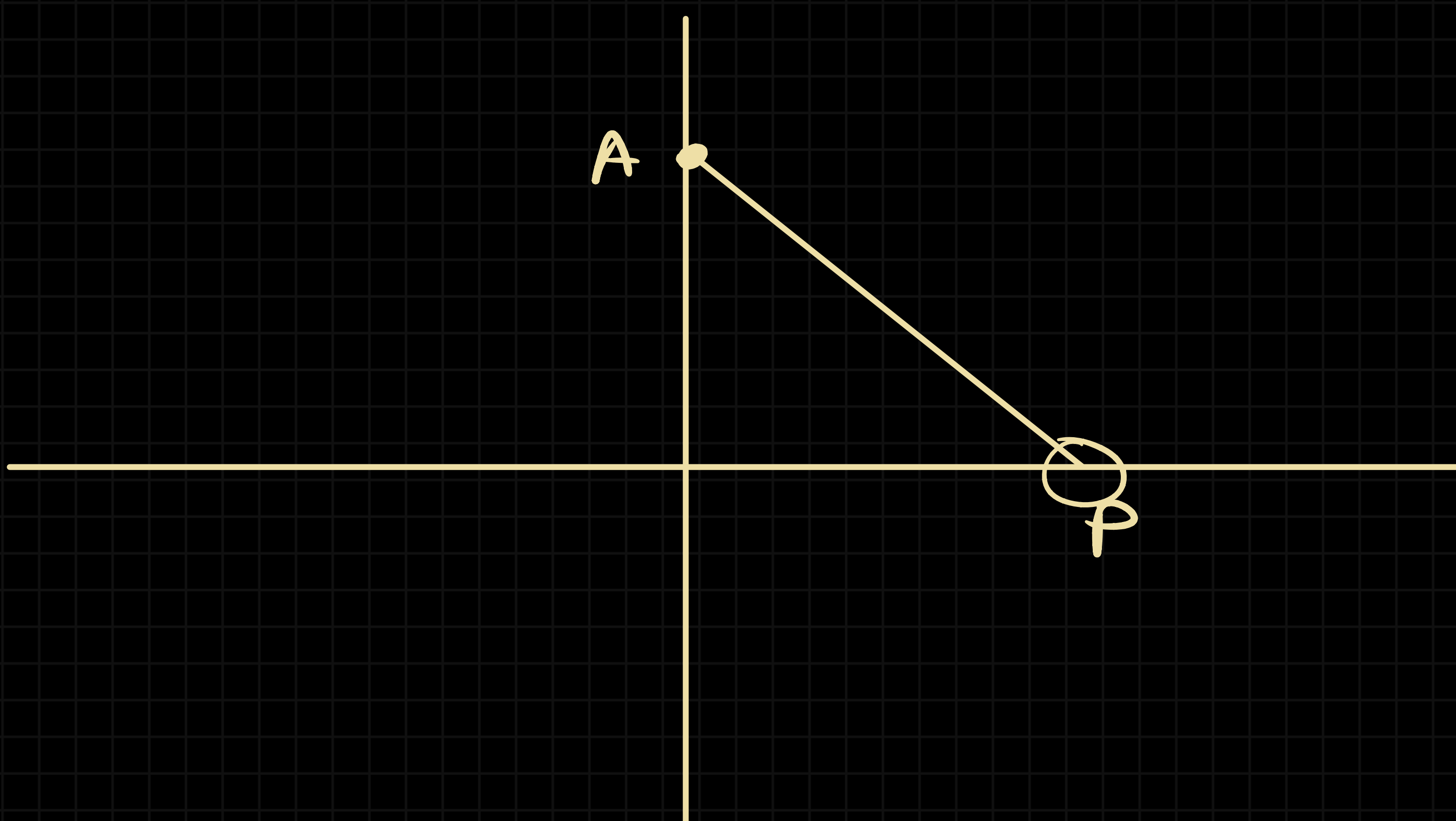


Que $s(x) = A - \frac{Ax}{P}\ ;\ 0 \le x < P$

ans

$s(0) = A$

$s(P) = 0$



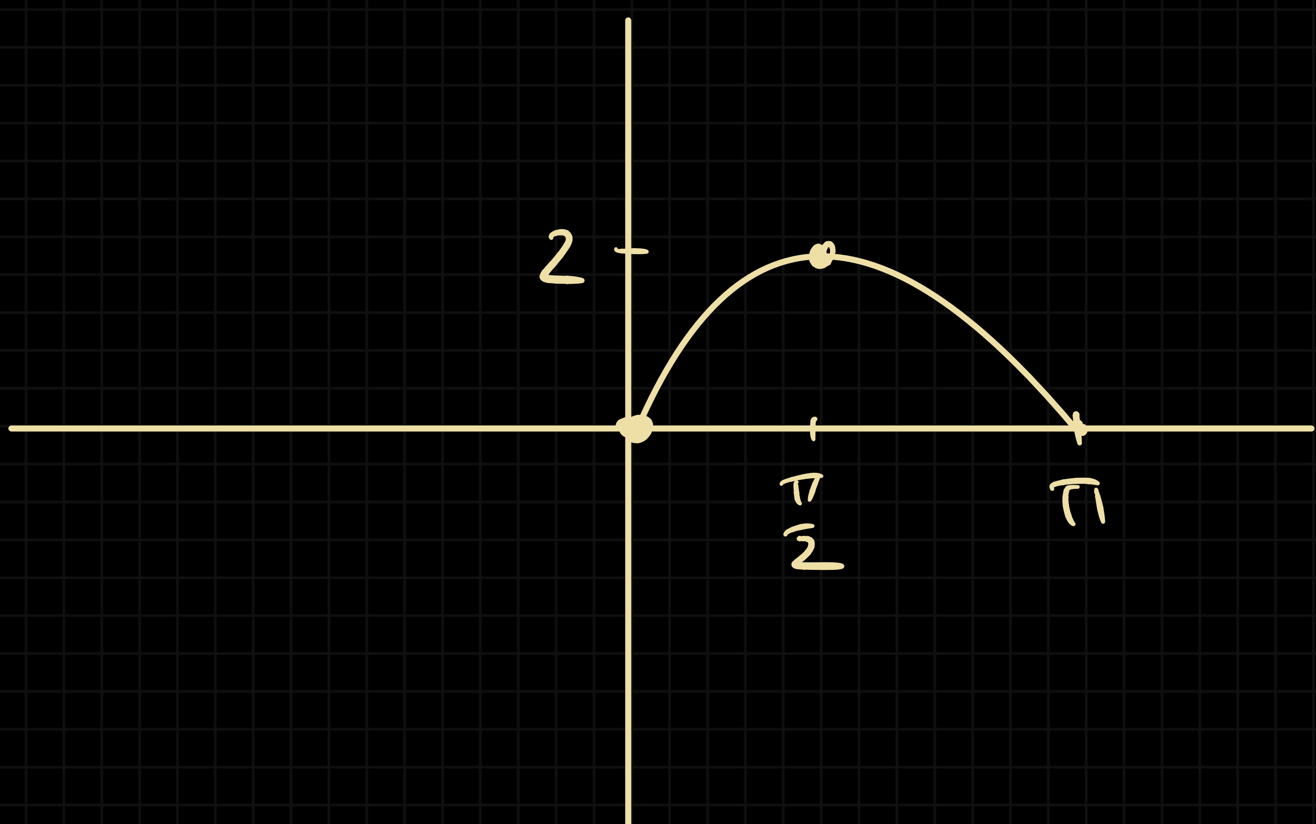
© que

$$f(x) = \begin{cases} 2\sin x ; & 0 \leq x < \pi \\ 0 ; & \pi \leq x < 2\pi \end{cases}$$

$f(0) = 0$

$f(\pi/2) = 2$

$f(\pi) = 0$



$\boxed{\text{b. que}}$

$s(x) = 3|\sin x| \quad ; \quad 0 \leq x \leq 2\pi$

ans:

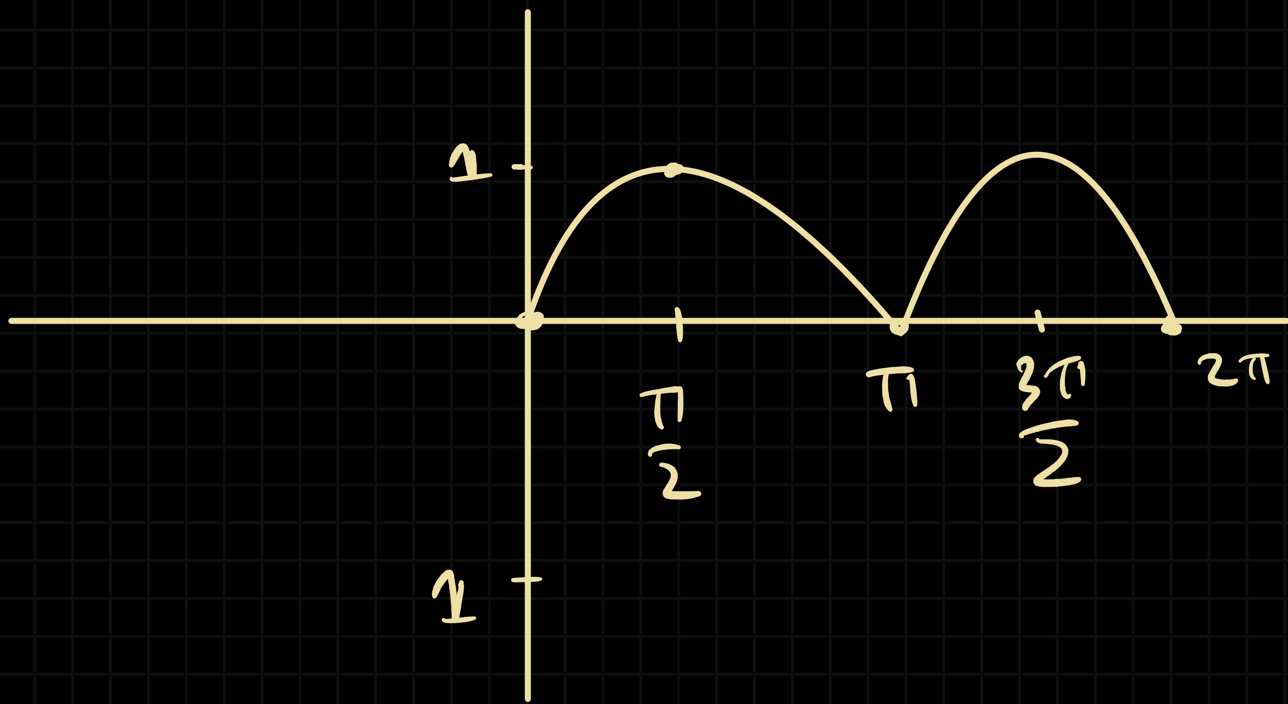
$S(0) = 0$

$S(\frac{\pi}{2}) = 1$

$S(\pi) = 0$

$S(\frac{3\pi}{2}) = 1$

$S(2\pi) = 0$



1
π/2
π
3π/2
2π
1

Odd Even Functions

1. $f(-x) = f(x)$; even

2. $f(-x) = -f(x)$; odd

eg: $f(x) = x^4 \rightarrow$ even

$f(-x) = (-x)^4 = f(x) = x^4$

if $f(x) = \cos x \longrightarrow$ even

$f(x) = x \sin x \longrightarrow$ even

$$f(-x) = (-x)\sin(-x) = x\sin x$$

$\hookrightarrow$ even

odd

$$f(-x) = -f(x)$$

$f(x) = x^3 \rightarrow$ odd

$f(x) = \sin x \rightarrow$ odd

b) Determine if the following functions are odd, even or neither.

a. $f(x) = \sin x(\sin x - 1)$

b. $f(x) = e^{-x}(x-1)^{\frac{1}{3}}$

c. $f(x) = \frac{9x}{|x|}$

a. $f(x) = 2x^3 - 5x - 1$

b. $f(x) = x(x-1)^{\frac{1}{3}}$

(b)

$$f(x) = x(x-1)^{1/3}$$

$$f(-x) = (-x)(-x-1)^{1/3}$$

$$= x(x+1)^{1/3} \neq f(x)$$

$$= -f(x)$$

neither even, nor odd

(a)

$f(x) = 2x^3 - 5x - 1$

$f(-x) = 2(-x)^3 - 5(-x) - 1$

$= -2x^3 + 5x - 1$

$= f(x)$; not even

$= -(2x^3 - 5x + 1) \neq -f(x)$;

not odd either

(c)

$$f(x) = \frac{9x}{|x|}$$

$$f(-x) = \frac{9(-x)}{|-x|} = \frac{-9x}{|x|}$$

$$= -f(x)$$

so odd

(b)

$$f(x) = e^{-x} \times (x-1)^{1/3}$$

$$f(-x) = e^{-(-x)} (-x-1)^{1/3}$$

$$= e^{x} (x+1)^{1/3}$$

$\neq f(x)$; not even

$\neq -f(x)$; not odd

(a)

$f(x) = \sin x(\sin x - 1)$

$f(-x) = \sin(-x) \times \{\sin(-x) - 1\}$

$= -\sin x \times (-\sin x - 1)$

$= \sin x(\sin x + 1)$

$\neq f(x)$, not even

$\neq -f(x)$, not odd

Fourier Series

কোনো continuous periodic function কে sin এবং cosine এর series দিয়ে লেখা যায়।

eg: $f(x) \rightarrow$ periodic.

let period $= 2l$

domain $-L \leq x \leq L$

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{2}\right) +$$

$$a_2 \cos\left(\frac{2\pi x}{2}\right) + a_3 \cos\left(\frac{3\pi x}{2}\right) + \cdots$$

$$\cdots + b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right)$$

$$+ b_3 \sin\left(\frac{3\pi x}{2}\right) + \cdots$$

$$\boxed{= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right]}$$

Fourier series,

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right]$$

where,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \; ; \; n = 0, 1, 2 \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \; ; \; n = 1, 2, 3 \ldots$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} \int f(x)\, dx$$

Que $f(x) = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$

fourier expansion করো।

ans $L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$a_n = ?$ $b_n = ?$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$\rightarrow n = 0, 1, 2, 3, 4$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad f(x) = \begin{cases} x, & 0 \le x < \pi \\ -x, & -\pi < x < 0 \end{cases}$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{\pi} \times \int_{0}^{\pi} x\,dx$$

$$= \frac{-1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{-1}{\pi}\left[-\frac{\pi^2}{2}\right] + \frac{1}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos(nx)\,dx +$$

$$\frac{1}{\pi}\int_{0}^{\pi} x\cos x \,dx$$

→ unfinished

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,(\sin nx)\;dx$$

$$+\frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\;dx$$

$\boxed{\text{Que}}$ if

$$x(t) = t - t^2 ; \ -\pi < t < \pi$$

and $x(t)$ is periodic over 2π, then find the fourier series of $x(t)$. From general form of Fourier Series deduce the expression of a_n and a_0.

$\boxed{\text{ans}}$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right) \right]$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx$$

$n = 1, 2, 3 \ldots$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

Draw sketches and determine the a_0, a_n, b_n and therefore determine the Fourier series (up to five terms at least) for the following two functions. [5+5=10]

a) $f(x) = \begin{cases} -2 \; if \; 0 < x < \pi \\ +2 \; if \; \pi < x < 2\pi \end{cases}$, for $-\pi < x < +\pi$

b) $f(x) = \frac{x}{\pi}$ for $-\pi < x < \pi$

(b)

$$f(x) = \frac{x}{\pi} \; ; \; -\pi < x < \pi \; , \; L = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi) + b_n \times \sin(n\pi) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{\pi}\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad , n = 1, 2, 3,$$

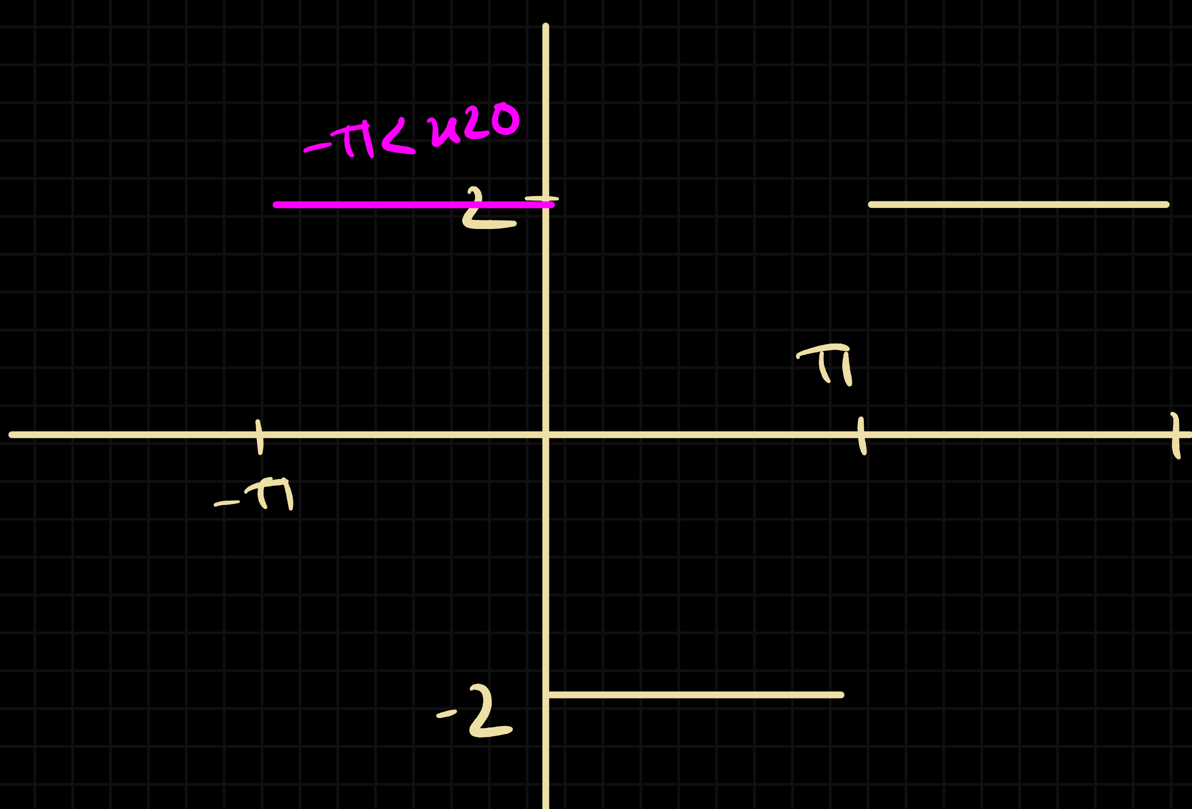
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{\pi} \cos(nx)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{\pi} \sin(nx)\,dx$$

(a)

$$f(x) = \begin{cases} -2, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$$

$-\pi < x < \pi$ নেওয়া হল।



so,

$$f(x) = \begin{cases} -2, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$$

$$= \begin{cases} -2, & 0 < x < \pi \\ 2, & -\pi < x < 0 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$n = 1, 2, 3 \ldots$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (2)\, dx + \frac{1}{\pi} \times \int_{0}^{\pi} (-2)\, dx$$

$$= \frac{1}{\pi} \left[2x\right]_{-\pi}^{0} + \frac{1}{\pi} \times \left[-2x\right]_{0}^{\pi}$$

$$= \frac{1}{\pi} \left(2\pi\right) + \frac{1}{\pi} \times \left(-2\pi\right)$$

$$= \quad 2 - 2 = 0$$

Draw sketches and determine the Fourier Series for the following functions.

a. $s(x) = \frac{x}{\pi}$, for $-\pi < x < +\pi$

b. $s(x) = 3|\sin x|$ for $0 \leq x < 2\pi$

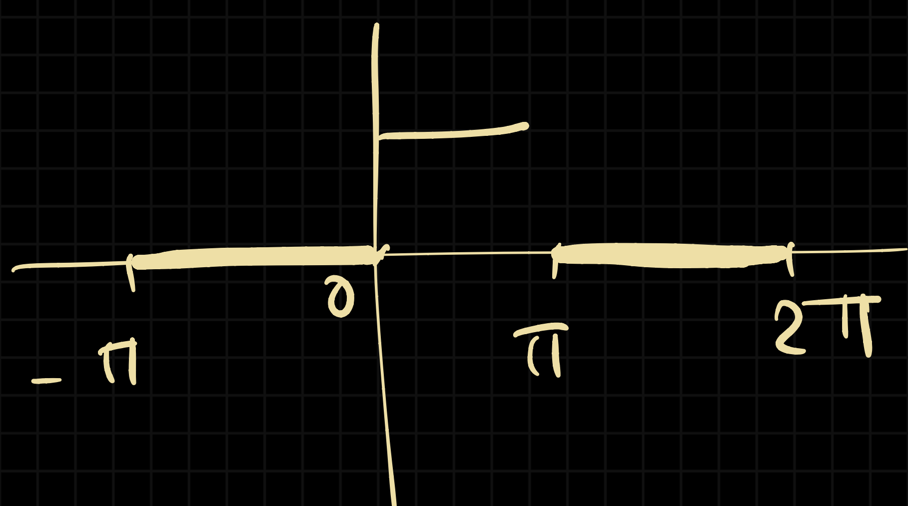
c. $s(x) = \begin{cases} 2\sin x & \text{for } 0 \leq x < \pi \\ 0 & \text{for } \pi \leq x < 2\pi \end{cases}$

d. $s(x) = \begin{cases} 1 & \text{for } 0 \leq x < \pi \\ 0 & \text{for } \pi \leq x < \pi \end{cases}$

e. $s(x) = A - \frac{Ax}{P}$ for $0 \leq x < P$

(d)

$$s(x) = \begin{cases} 1 \ ; & 0 \leq x < \pi \\ 0 & \pi \leq x < 2\pi \end{cases}$$



$$f = \begin{cases} 0, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} 0 + \frac{1}{\pi} \int_{0}^{\pi} 1 \cdot \sin(nx)\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} (\sin nx)\, dx$$

$$= \frac{1}{\pi} \times \left[-\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n\pi)}{n} - \frac{-\cos 0}{n} \right]$$

$$= \frac{1}{\pi} \times \left[-\frac{(-1)^n}{n} + \frac{1}{n} \right]$$

NB: $\cos(n\pi) = (-1)^n$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

Half Fourier Series

$f(x)$ even function $\Longrightarrow$ $b_n = 0$

so no sine, only cosine

$f(x)$ odd function $\Longrightarrow$ $a_n = 0$

so no cosine, only sine

Que

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0 \end{cases}$$

ans $f(\pm 1/2)$ দিয়ে দেখ even/odd নাকি।

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$f\left(-\frac{1}{2}\right) = -\left(\frac{-1}{2}\right) = \frac{1}{2}$$

↳ even function

since $f(x)$ is even,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi}$$

even/odd chenar upay

even → y axis e symmetric

odd → origin er respect e

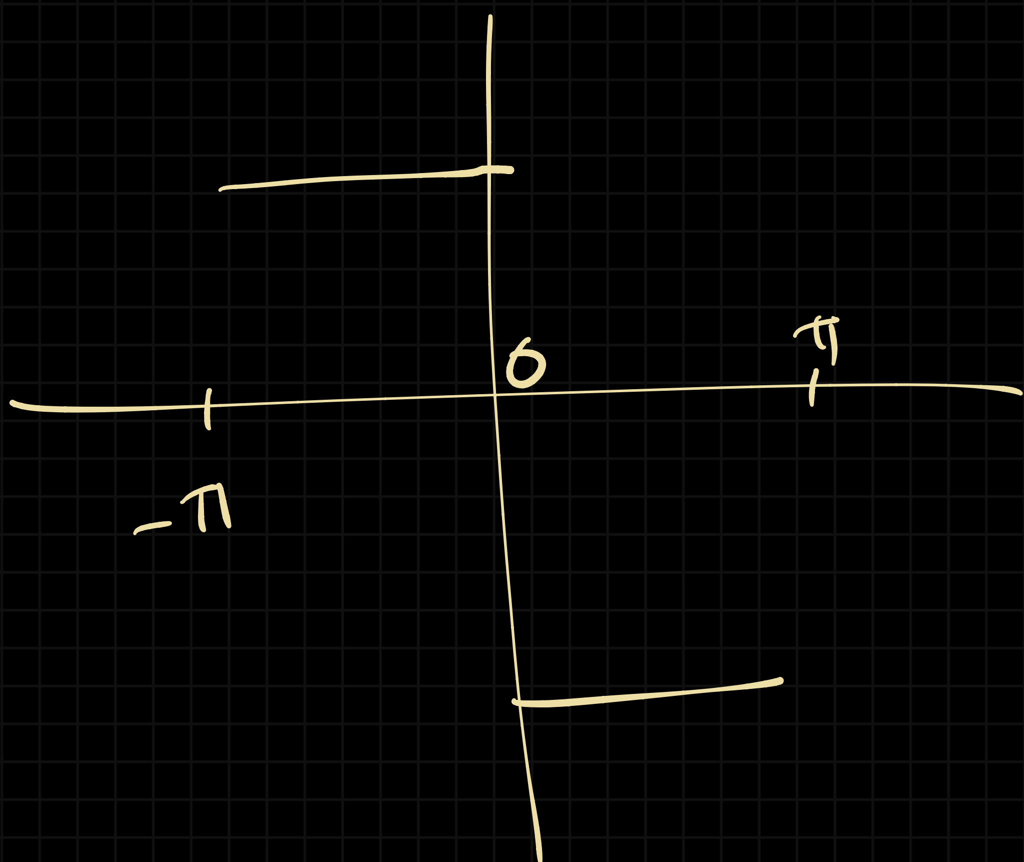
symmetric

y and also x

eg: a) $f(x) = \begin{cases} -2, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

$-\pi < x < \pi$

odd



b) $f(x) = \frac{x}{\pi}$ for $-\pi < x < \pi$

$$f(-x) = \frac{-x}{\pi} \rightarrow \text{odd function}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \qquad n = 1, 2, 3$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{\pi} \sin(nx)\, dx$$

Draw sketches and determine the Fourier Series for the following functions.

a. $s(x) = \frac{x}{\pi}$, for $-\pi < x < +\pi$

b. $s(x) = 3|\sin x|$ for $0 \leq x < 2\pi$

$-\pi$

(b)

$$s(x) = 3|\sin x| \quad , \quad -\pi < x < \pi$$

$$s(-x) = 3|\sin(-x)|$$

$$= 3\sin x$$

$$= s(x)$$

$\rightarrow$ even

Calculate the Fourier Sine series of the function $f(x) = \cos x$, $0 \leq x \leq \pi$
[Hint: The function is defined in the half range.]

sine series => only sine terms থাকে

$f(x) \to$ odd ($\because$ sin থাকে)

(so now জোর করে cos টাকে odd হিসা লাগাবো)

ans since we have to calculate sine series,

=> $f(x)$ must be odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$L = \pi$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \cos(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin(nx)\cos x\,dx$$

$2\cos A \sin B$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin(nx)\cos(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(nx+x) + \sin(nx-x)$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin\{x(n+1)\}\,dx + \frac{1}{\pi}\times$$

$$\int \sin\{x(n-1)\}\,dx$$

$$= \frac{1}{\pi}\,\frac{-\cos\{x(n+1)\}}{(n+1)} - \frac{1}{\pi}\times \frac{-\cos\{x(n-1)\}}{(n-1)}$$

$$= \frac{-1}{\pi(n+1)} \cos\left\{x(n+1)\right\} + \frac{1}{\pi(n-1)} \cos\left(x(n-1)\right)$$

Integration Formulas

i) $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

ii) $$\int \sin(mx) = \frac{-\cos(mx)}{m} + c$$

iii) $$\int \cos(mx) = \frac{\sin(mx)}{m} + c$$

(v) $\int \underbrace{x}_{u} \underbrace{\sin(2x)\,dx}_{dv}$

$$\int u\,dv = uv - \int v\,du$$

$$\int uv\,du = u\int v\,du - \int\left(u'\int v\,du\right)du$$

Trigonometric Identities

i) $2\,sA\,sB = c(A-B) - c(A+B)$

ii) $2\,cA\,cB = c(A+B) + c(A-B)$

iii) $2\,sA\,cB = s(A+B) + s(A-B)$

Half fourier series

$\boxed{\text{even}}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$\boxed{\text{Odd}}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$