MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Chapter 4 Part 1 (Complex Integration)

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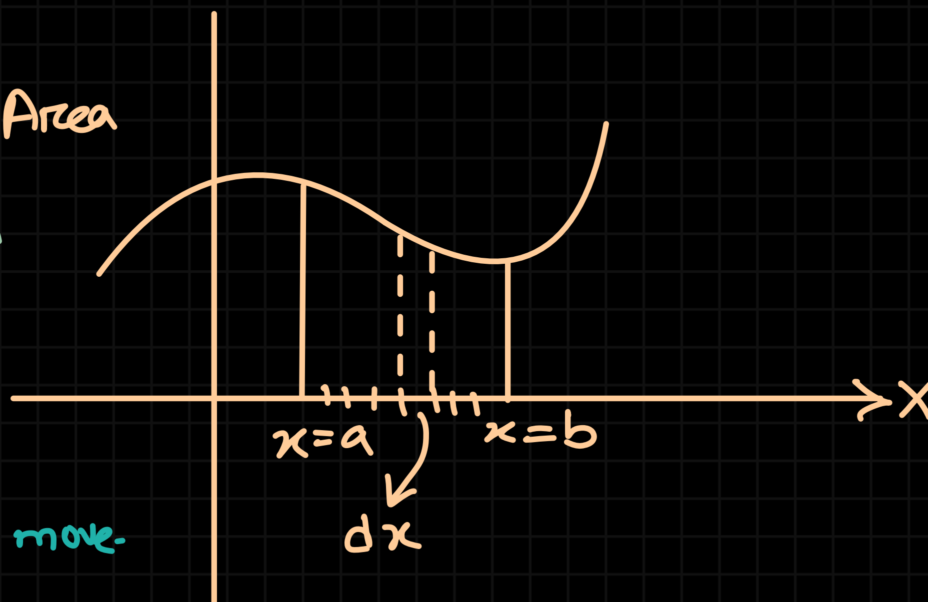
Single Integration on Real Line

Single integral deals with integrating functions of a single variable and calculates the area under a curve along one dimension

e.g: $\int_a^b \underbrace{f(x)}_{\text{height}}\,\underbrace{dx}_{\text{width}} = \text{Area}$



since its a one dimensional movement, a থেকে b তে যাওয়ার only one way

Single/Line/Path Integration on $\mathbb{R}^2$

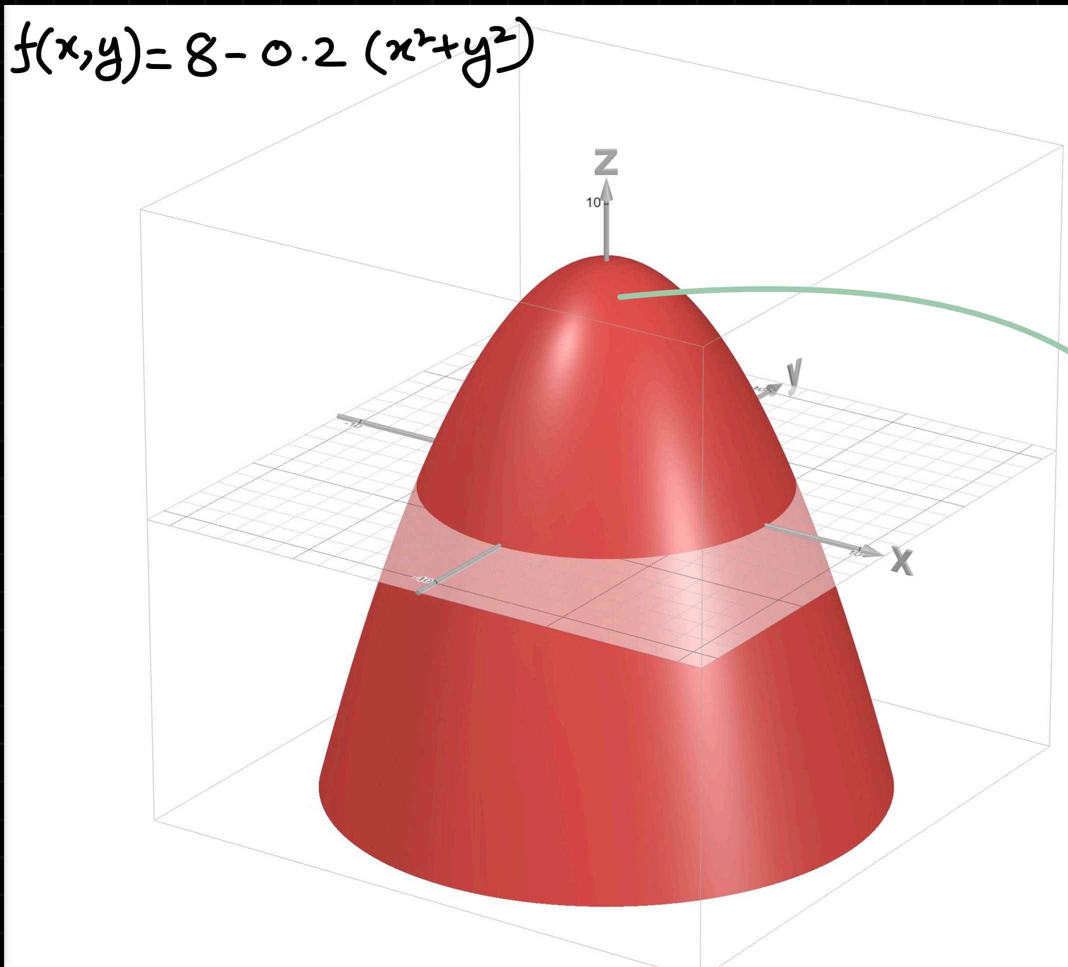
$\hookrightarrow$ 2d space

or XY plane

$$\underbrace{\int f(x,y)\,dx \quad \int f(x,y)\,dy \quad \int f(x,y)\,ds}_{\text{2 variable function এর integration}}$$

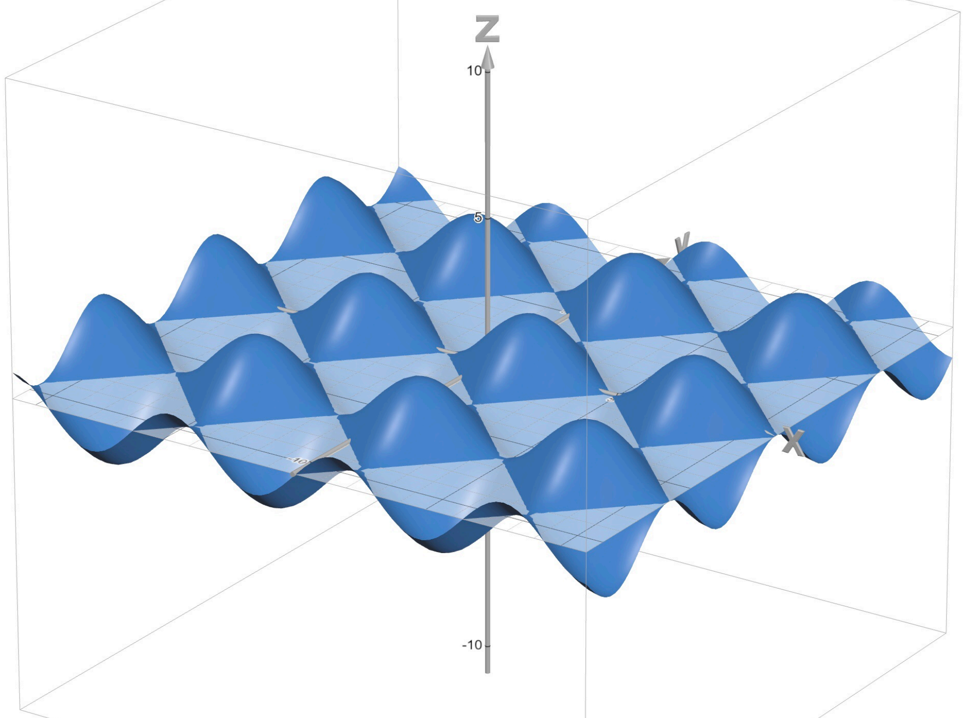
$$\underbrace{\text{one input, one output}}_{\text{2d plane}}$$

$$\underbrace{\text{2 inputs, 1 output}}_{\text{3d space}}$$

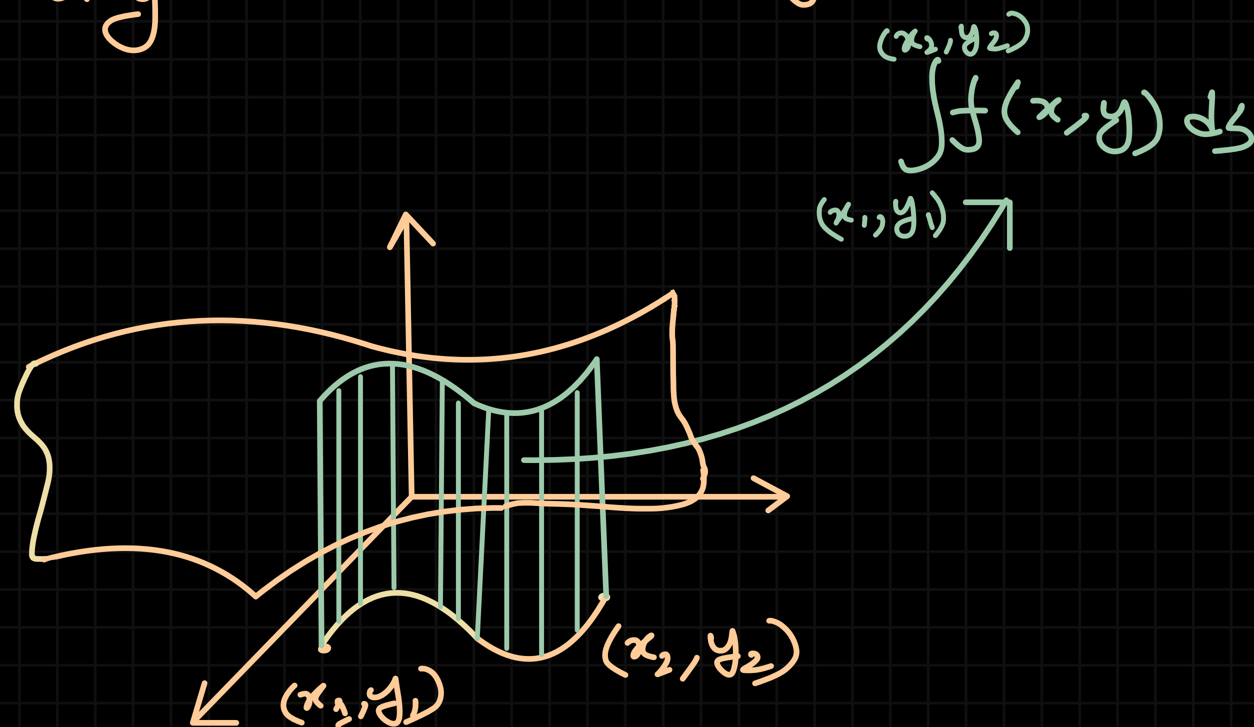


→ x, y এর any valueর জন্য যে height,
thats the function value.

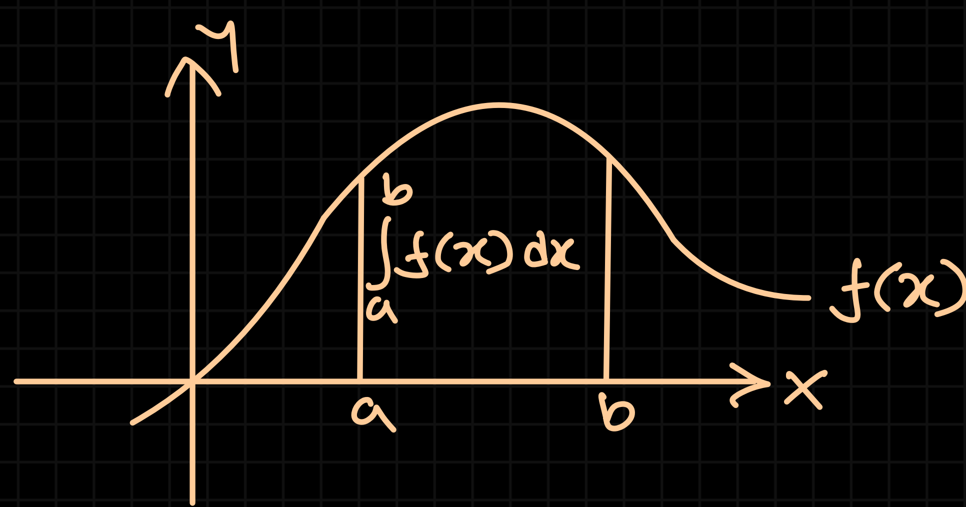
$f(x,y) = \sin x + \cos y$



for any double variable function



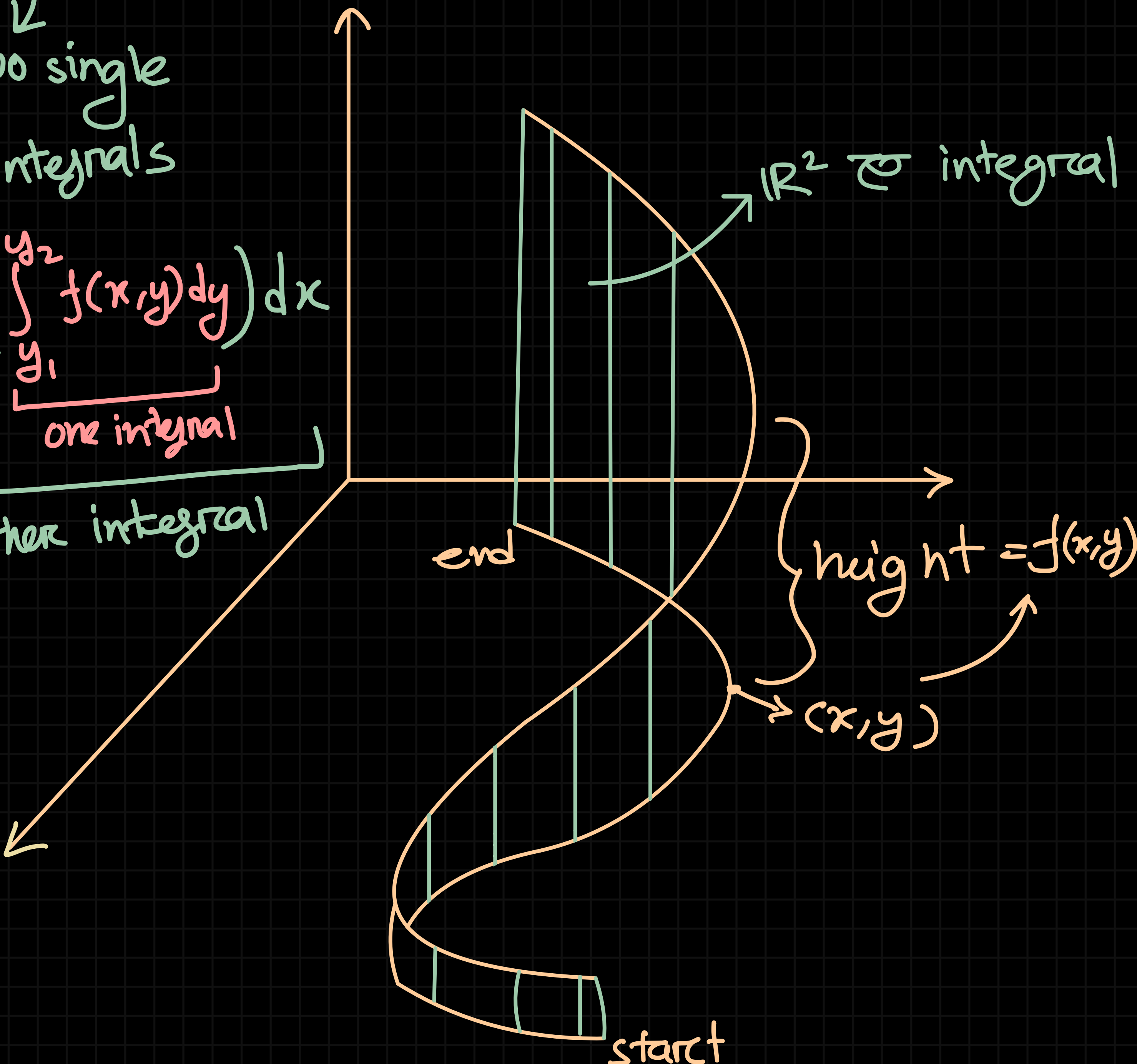
single integral outputs be like:

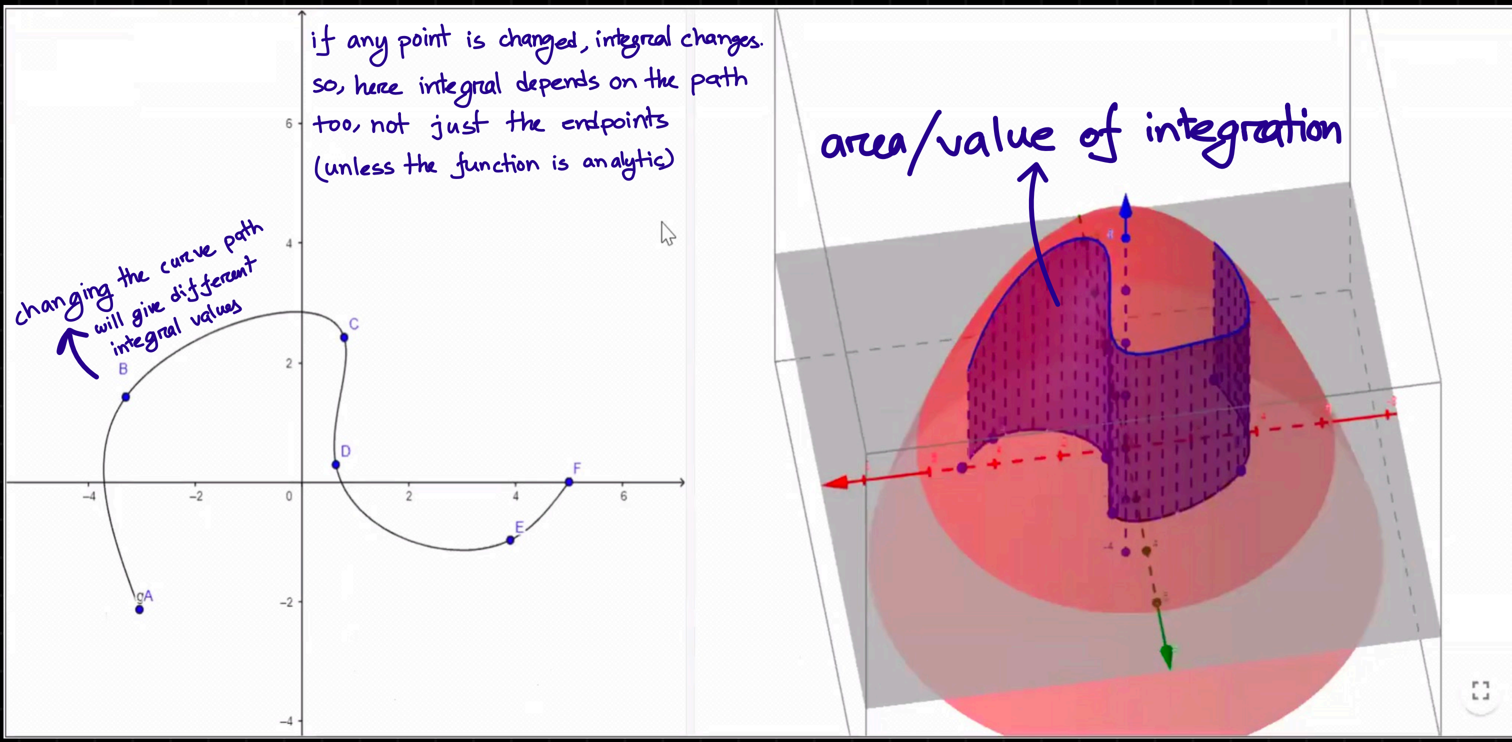


double integral outputs be like:

two single integrals

$$\underbrace{\int_{x_1}^{x_2}\left(\underbrace{\int_{y_1}^{y_2} f(x,y)\,dy}_{\text{one integral}}\right)dx}_{\text{another integral}}$$



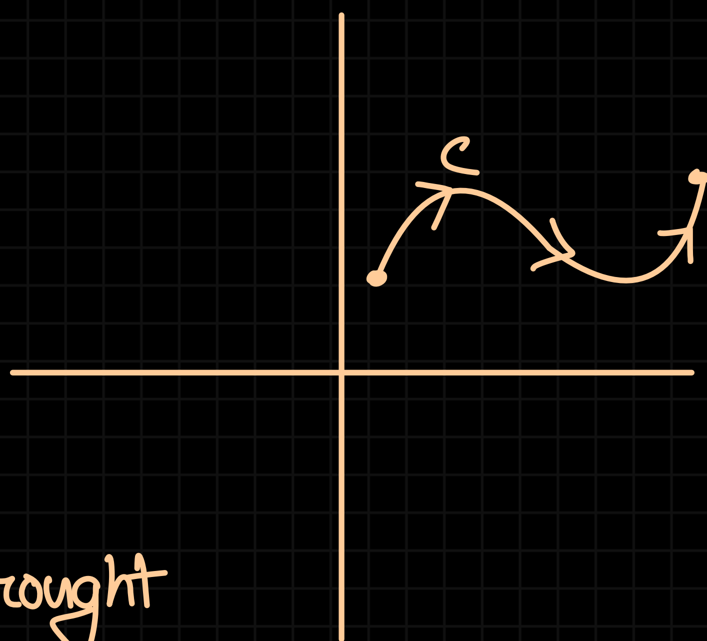


if any point is changed, integral changes.
so, here integral depends on the path
too, not just the endpoints
(unless the function is analytic)
changing the curve path
will give different
integral values
A
B
C
D
E
F
area/value of integration

Contour

→ the path, over which the integration is performed

$$\int_C f(z)$$



→ contour must be $\underbrace{\text{differentiable}}_{\text{differentiation} \neq 0}$ throught

Evaluate $\displaystyle\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$

along —

ⓐ the curve $y = x^2+1$

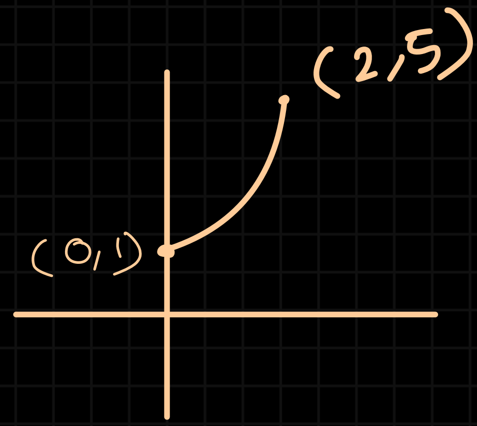
ⓑ the straight line joining $(0,1)$ and $(2,5)$

ⓒ the straight lines from $(0,1)$ to $(0,5)$ and then $(0,5)$ to $(2,5)$

Q. Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$

along the curve $y = x^2 + 1$

$\boxed{\text{solve}}$



let $x = t$, $dx = dt$

$\therefore y = t^2 + 1$

$dy = 2t\,dt$

start range $x=0,\ y=1$

and $x=0,\ y=1$ when $\boxed{t=0}$

$x=t$

$y=t^2+1$

end point $\Rightarrow$ $x=2,\ y=5$

its when $\boxed{t=2}$

$$\int_0^2 (3t+t^2+1)\,dt + (2t^2+2-t)\,2t\,dt$$

$$= \left[3\times\frac{t^2}{2}+\frac{t^3}{3}+t\right]_0^2 +$$

$$\int_0^2 (4t^3+4t-2t^2)\,dt$$

$$= 3\times\frac{4}{2} + \frac{8}{3} + 2 +$$

$$\left[4\times\frac{t^4}{4} + 4\times\frac{t^2}{2} - 2\times\frac{t^3}{3}\right]_0^2$$

$$= 6 + \frac{8}{3} + 2 + \left(2^4 + 2\times2^2 - 2\times\frac{8}{3}\right)$$

$$= 8 + \frac{8}{3} + 16 + 8 - \frac{16}{3}$$

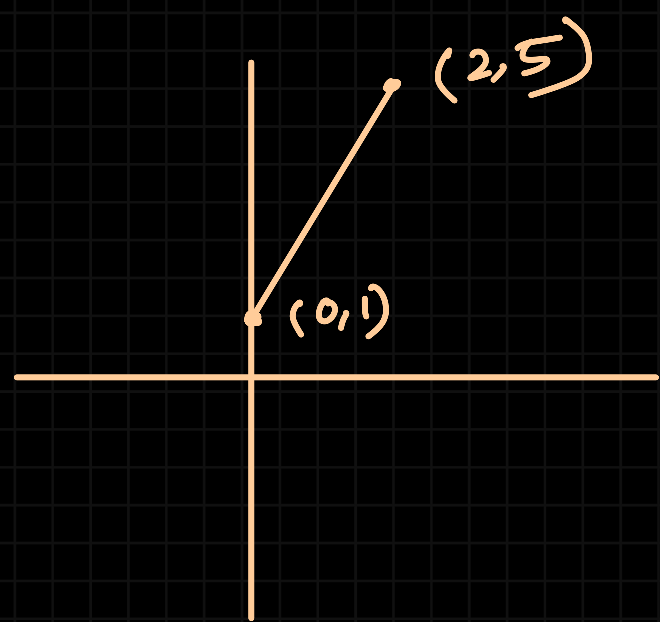
$$= \frac{88}{3}$$

[Ans]

Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$

along the straight line joining $(0,1)$ and $(2,5)$

solve:



slope of the line joining $(0,1)$ and $(2,5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-1}{2-0} = 2$$

equation of a straight line joining (x_1, y_1) and (x_2, y_2),

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

$$\Rightarrow \frac{y-1}{1-5} = \frac{x-0}{0-2}$$

$$\Rightarrow \frac{y-1}{-4} = \frac{x}{-2}$$

$$\Rightarrow \frac{y-1}{2} = x$$

$$\Rightarrow y-1 = 2x$$

$$\Rightarrow y = 2x+1$$

let $x = t$, $dx = dt$

$\therefore$ $y = 2t + 1$, $dy = 2dt$

$(x, y) = (0, 1)$ when $t = 0$

$(x, y) = (2, 5)$ when $t = 2$

now

$$\int_0^2 (3t + 2t + 1)\,dt + (4t + 2 - t)\,2dt$$

$$= \left[3 \times \frac{t^2}{2} + 2 \times \frac{t^2}{2} + t\right]_0^2 +$$

$$\int_0^2 (8t + 4 - 2t)\,dt$$

$$= 3\times\frac{2^2}{2} + 2^2 + 2 +$$

$$\left[8\times\frac{t^2}{2} + 4t - 2\times\frac{t^2}{2}\right]_0^2$$

$$= 6 + 4 + 2 + \left[4\times 2^2 + 4\times 2 - 2^2\right]$$

$$= 12 + \left[16 + 8 - 4\right]$$

$$= 32$$

(Ans)

Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$

along the straight lines from $(0,1)$ to $(0,5)$ and then $(0,5)$ to $(2,5)$

solve:

$z_1(t) = i + (5i - i)t = i + 4ti$

$= i(1+4t)$

$x = 0$	$dx = 0$	$t \to 0$
$y = 1+4t$	$dt = 4dt$	$\to 1$

now,

$$\int_0^1 0 + (2+8t)\,4\,dt$$

$$= \int_0^1 (8+32t)\,dt$$

$$= \left[8t + 32 \times \frac{t^2}{2} \right]_0^1$$

$$= 8 + 16 = 24$$

again,

$$z_2(t) = 5i + (2 + 5i - 5i)t$$
$$= 5i + 2t$$

$x = 2t$	$dx = 2dt$	$t \to 0$
$y = 5$	$dy = 0$	$t \to 1$

$$\int_0^1 (6t + 5)\,2dt + 0$$

$$= \int_0^1 (12t + 10)\,dt$$

$$= \left[12\frac{t^2}{2} + 10t\right]_0^1$$

$$= \left[6t^2 + 10t \right]_0^1$$

$$= 16$$

$\therefore$ Cumulative integral,

$$= 24 + 16 = 40$$

Ans.

Evaluate $\oint_C (x+2y)\,dx + (y-2x)\,dy$

around the eclipse C defined by

$x = 4\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$

if C is described in a counter-clockwise direction

solve:

$$x = 4\cos\theta$$

$$dx = -4\sin\theta\, d\theta$$

$$y = 3\sin\theta$$

$$dy = 3\cos\theta\, d\theta$$

now,

$$\int_0^{2\pi} (4\cos\theta + 6\sin\theta)(-4\sin\theta\, d\theta) + (3\sin\theta - 8\cos\theta)\, 3\cos\theta\, d\theta$$

$$= \int_0^{2\pi} (-16\sin\theta\cos\theta - 24\sin^2\theta)\, d\theta + (9\sin\theta\cos\theta - 24\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-16\sin\theta\cos\theta - 24\sin^2\theta + 9\sin\theta\cos\theta - 24\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \{-7\sin\theta\cos\theta - 24(\sin^2\theta + \cos^2\theta)\}\, d\theta$$

$$= \int_0^{2\pi} (-7 \sin\theta \cos\theta - 24)\, d\theta$$

$$= \int_0^{2\pi} \left(-\frac{7}{2} \times 2\sin\theta\cos\theta - 24\right) d\theta$$

$$= \int_0^{2\pi} \left(-\frac{7}{2} \sin 2\theta - 24\right) d\theta$$

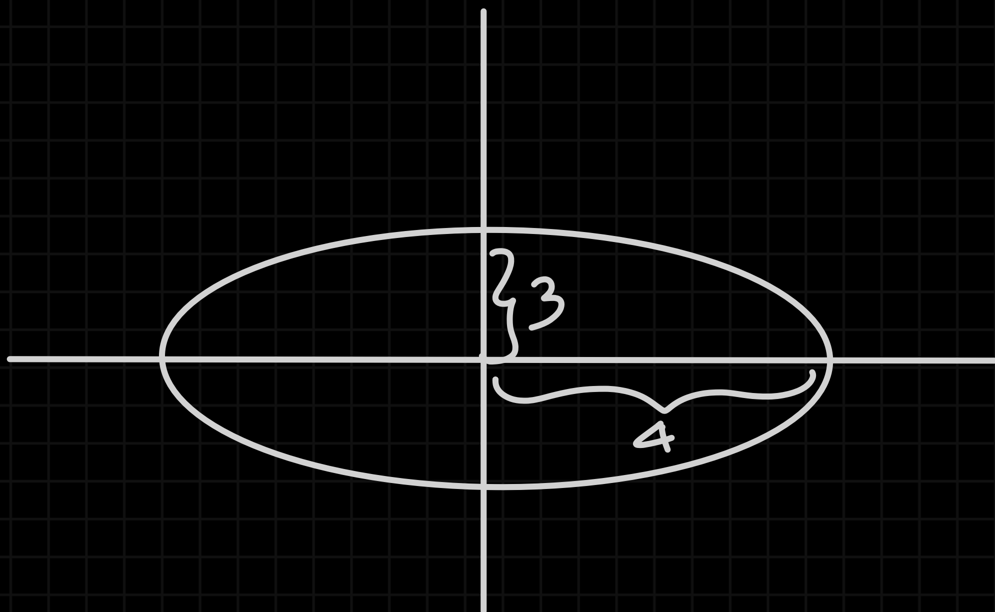
$$= \left[\frac{7}{2} \frac{\cos 2\theta}{2} - 24\theta \right]_0^{2\pi}$$

$$= -48\pi$$

NB:

eclipse with $x = 4\cos\theta$

$y = 3\sin\theta$



real line integration এ closed curved

এর জন্য integral $= 0$.

complex line এ integral $\neq 0$.

Complex Line/Contour Integral

$$\underbrace{\int_c f(z)\,dz}_{\text{contour integral}} \qquad \underbrace{\oint_c f(z)\,dz}_{\text{closed contour integral}}$$

PARAMETRIZATION OF A STRAIGHT LINE

Parametrization from z_1 to z_2

$$z(t) = \underbrace{z_1}_{\text{start point}} + (\underbrace{z_2 - z_1}_{\text{end point}})\,t$$

Evaluate $\int_C (x^2 - iy^2)\,dz$ along —

(a) the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$

(b) the straight lines from $(1,1)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$

ⓒ the straight line from $(1,1)$ to $(2,8)$

☐ Evaluate $\int_C (x^2 - iy^2)\, dz$ along the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$

solve:

let $x = t$

$y = 2t^2$

$z = x + iy$

$\therefore z = t + 2t^2 i$

$$dz = (1 + 4ti)\, dt$$

(x, y) will be $(1, 2)$ when $t = 1$

(x, y) will be $(2, 8)$ when $t = 2$

now,

$$\int_C (x^2 - iy^2)\, dz$$

$$= \int_1^2 \left\{ t^2 - i\,(2t^2)^2 \right\} (1 + 4ti)\, dt$$

$$= \int_1^2 (t^2 - 4t^4 i) \quad (1 + 4t\, i) \quad dt$$

$$= \int_1^2 \left(t^2 - 4t^4 i + 4t^3 i - 16t^5 i^2\right) dt$$

$$= \int_1^2 \left(16t^5 - 4t^4 i + 4t^3 i + t^2\right) dt$$

$$= \left[16 \times \frac{t^6}{6} - 4i \times \frac{t^5}{5} + 4 \times \frac{t^4}{4} i + \frac{t^3}{3}\right]_1^2$$

$$= 16 \times \frac{64}{6} - 4i \times \frac{32}{5} + 16i + \frac{8}{3} -$$

$$\left(16 \times \frac{1}{6} - 4i \times \frac{1}{5} + i + \frac{1}{3}\right)$$

$$= \frac{512}{3} - \frac{128}{5} i + 16i + \cancel{\frac{8}{3}} - \cancel{\frac{8}{3}} + \frac{4i}{5}$$

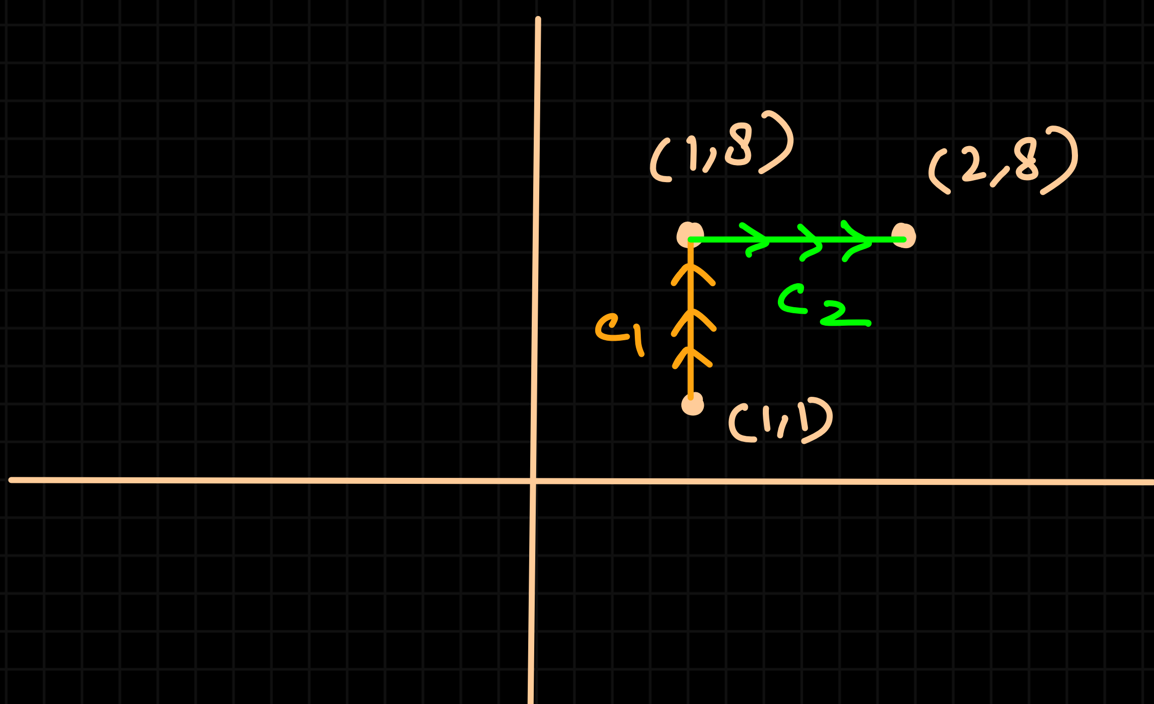
$$- i - \frac{1}{3}$$

$$= \frac{511}{3} - \frac{49}{5} i$$

(Ans)

Evaluate $\int_C (x^2 - iy^2)\, dz$ along the straight lines from $(1,1)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$

solve:



$$\therefore \int_C (x^2 - iy^2)\, dz$$

$$= \int_{C_1} (x^2 - iy^2)\, dz + \int_{C_2} (x^2 - iy^2)\, dz$$

for c_1:

$(1,1) \equiv (1+i)$

$(1,8) \equiv (1+8i)$

for straight line path

$z(t) = z_1 + (z_2 - z_1)t$

$= 1+i + \left(1+8i - (1+i)\right)t$

$= 1+i + 7it$

$\therefore z = 1 + (7t+1)i$

$\therefore x = 1, \quad y = 7t+1$

$t \to \text{start} = 0$

$t \to \text{end} = 1$

$$\int_{C_1} (x^2 - iy^2)\, dz$$

$$= \int_0^1 \left(1^2 - i(1+7t)^2\right) 7i\, dt$$

$$= \int_0^1 \left(7i + 7(1+14t+49t^2)\right) dt$$

$$= \int_0^1 (7i + 7 + 98t + 343t^2)\, dt$$

$$= \left[7it + 7t + 98 \times \frac{t^2}{2} + 343 \times \frac{t^3}{3} \right]_0^1$$

$$= 7i + 7 + 49 + \frac{343}{3}$$

$$= \frac{511}{3} + 7i$$

for c_2:

from $(1+8i)$ to $(2+8i)$

for straight path,

$$z(t) = (1+8i) + (1)t$$

$$= (1+t) + (8)i$$

$$dz = 1\, dt$$

$$x = 1+t$$

$$y = 8$$

$t \to$ start $= 0$

$t \to$ end $= 1$

$$\int_{C_2} (x^2 - iy^2)\, dz$$

$$= \int_0^1 \left((1+t)^2 - i\, 8^2 \right) dt$$

$$= \int_0^1 \left(1 + 2t + t^2 - 64i \right) dt$$

$$= \left[t + 2 \times \frac{t^2}{2} + \frac{t^3}{3} - 64i \times t \right]_0^1$$

$$= 1 + 1 + \frac{1}{3} - 64i$$

$$= \frac{7}{3} - 64i$$

now,

$$\int_c (x^2 - iy^2)\, dz$$

c: (1,1) to (1,8)

then (1,8) to (2,8)

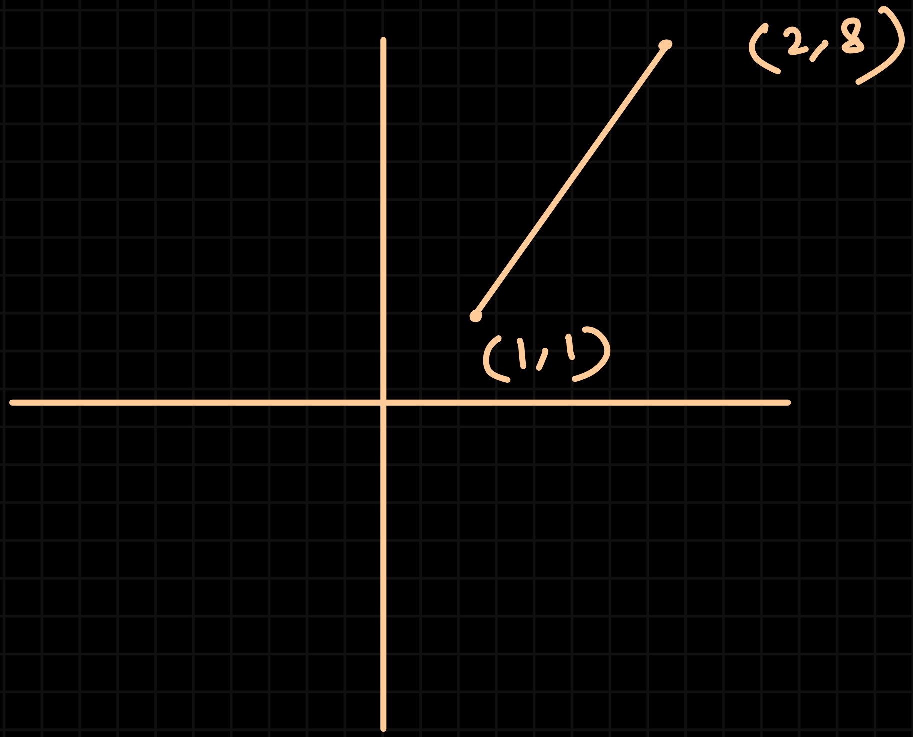
$$= \int_{c_1} (x^2 - iy^2)\, dz + \int_{c_2} (x^2 - iy^2)\, dz$$

$$= \left(\frac{511}{3} + 7i\right) + \left(\frac{7}{3} - 64i\right)$$

$$= \frac{518}{3} - 57i$$

Evaluate $\int_C (x^2 - iy^2)\, dz$ along the straight line from $(1,1)$ to $(2,8)$

solve:



$$z(t) = 1+i + (2+8i-1-i)t$$

$$= 1+i+(1+7i)t$$

$$= 1+i+t+7it$$

$$= (1+t) + i(1+7t)$$

$$dz = (1+7i)\,dt$$

$$x = 1+t$$

$$y = 1+7t$$

$$t \begin{cases} \to 0 \\ \to 1 \end{cases}$$

$$\int_C (x^2 - iy^2)\,dz$$

$$= \int_0^1 \left\{(1+t)^2 - i(1+7t)^2\right\}(1+7i)\,dt$$

$$= (1+7i)\int_0^1 \left\{1+2t+t^2 - i\,(1+14t+49t^2)\right\} dt$$

$$= (1+7i)\int_0^1 \left(1+2t+t^2 - i - 14ti - 49t^2 i\right) dt$$

$$= (1+7i)\left[t + 2\times\frac{t^2}{2} + \frac{t^3}{3} - it - 14\frac{t^2}{2}i - 49i\,\frac{t^3}{3}\right]_0^1$$

$$= (1+7i)\left[1+1+\frac{1}{3} - i - 7i - \frac{49i}{3}\right]$$

$$= (1+7i)\left(\frac{7}{3} - \frac{73}{3}i\right)$$

$$= \frac{7}{3} - \frac{73}{3}i + \frac{49}{3}i - \frac{511}{3}i^2$$

$$= \frac{7}{3} - 8i + \frac{511}{3}$$

$$= \frac{518}{3} - 8i$$

2. Evaluate $\int_{i}^{2-i} (3xy + iy^2)\, dz$ along —

(a) the straight line joining $z = i$ and $z = 2-i$

(b) along the parabola $x = 2t-2$, $y = 1 + t - t^2$

Evaluate $\displaystyle\int_{i}^{2-i} (3xy + iy^2)\, dz$ along the straight line joining $z = i$ and $z = 2 - i$

solve 2:

$$z(t) = z_1 + (z_2 - z_1)t$$

$$= i + (2 - i - i)t$$

$$= i + (2 - 2i)t$$

$$= i + 2t - 2it$$

$$= 2t + i(1 - 2t)$$

$$dz = (2 - 2i)\,dt$$

$x = 2t$

$y = 1 - 2t$

$t \to 0 \to i$

$t \to 1 \to 2 - i$

$$\int_{i}^{2-i} (3xy + iy^2)\, dz$$

$$= \int_0^1 \left\{3 \cdot 2t \cdot (1-2t) + i \cdot (1-2t)^2\right\} (2-2i)\,dt$$

$$= (2-2i) \int_0^1 \left\{6t\,(1-2t) + i\,(1-4t+4t^2)\right\} dt$$

$$= (2-2i) \int_0^1 \left\{6t - 12t^2 + i - 4t\,i + 4t^2 i\right\} dt$$

$$= (2-2i) \left[6 \times \frac{t^2}{2} - 12\,\frac{t^3}{3} + it - 4\,\frac{t^2}{2}\,i + 4\,\frac{t^3}{3}\,i\right]_0^1$$

$$= (2-2i)\left[3-4+i-2i+\frac{9}{3}i\right]$$

$$= (2-2i)\left(-1+\frac{1}{3}i\right)$$

$$= -2+\frac{2}{3}i+2i-\frac{2}{3}i^2$$

$$= -2+\frac{2}{3}+i\left(2+\frac{2}{3}\right)$$

$$= -\frac{4}{3}+\frac{8}{3}i$$

① Evaluate $\int_{i}^{2-i} (3xy + iy^2)\, dz$ along

along the parabola $x = 2t-2$,

$y = 1 + t - t^2$

solve:

$$z = x + iy$$

$$z = (2t-2) + i(1 + t - t^2)$$

$$= 2t - 2 + i + it - it^2$$

$$dz = (2 + i - 2ti)\, dt$$

$$t \nearrow 1 \rightarrow (x,y) = (0,1)$$
$$t \searrow 2 \rightarrow (x,y) = (2,-1)$$

$$\int_{i}^{2-i} (3xy + iy^2)\, dz$$

$$= \int_{1}^{2} \left\{ 3 \cdot (2t-2) \cdot (1+t-t^2) + i(1+t-t^2)^2 \right\}$$

$$(2+i-2+i)\, dt$$

$c = (-t^2)$

$$= \int_{1}^{2} \Big[(6t-6)(1+t-t^2) + i\{1+t^2+t^4+2t$$
$$+ 2t(-t^2) + 2(-t^2)\} \Big] (2+i-2+i)\, dt$$

$$= \int_1^2 \left\{ \left(6t + 6t^2 - 6t^3 - 6 + 6t + 6t^2\right) + i\left(1 + t^2 + t^4 + 2t - 2t^3 - 2t^2\right) \right\} (2 + i - 2 + i)\, dt$$

$$= \int_1^2 \left\{ \left(12t + 12t^2 - 6t^3 - 6\right) + i\left(1 - t^2 + 2t - 2t^3 + t^4\right) \right\} (2 + i - 2 + i)\, dt$$

$\vdots$

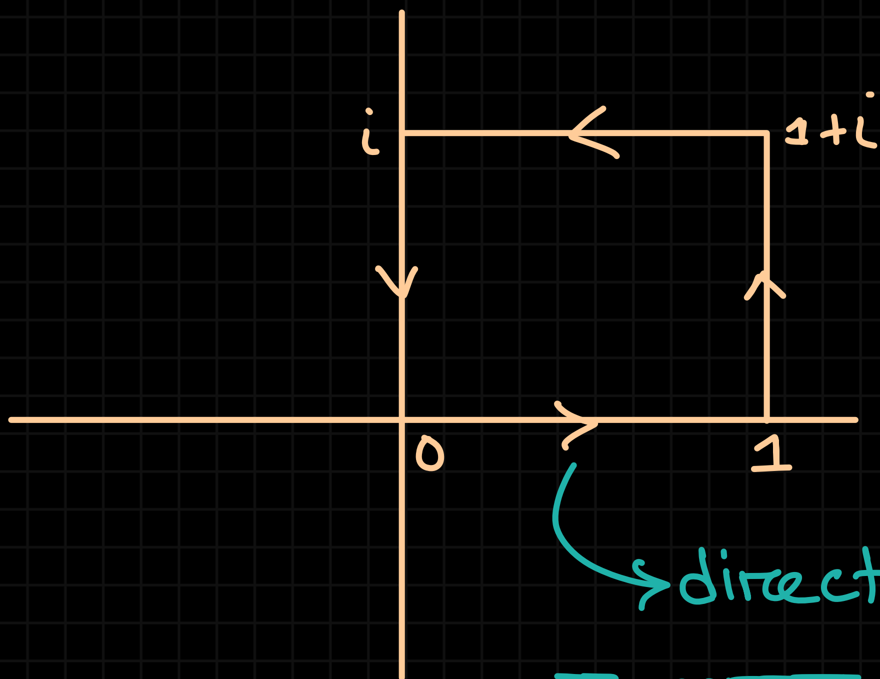
Evaluate $\oint_C |z|^2 \, dz$ around the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$

solve:

$$|z| = \sqrt{x^2+y^2}$$

$$|z|^2 = x^2+y^2$$

$$\oint_C |z|^2 \, dz = \oint_C (x^2+y^2) \, dz$$



direction mention না থাকলে take anti clockwise

for c_1: from 0 to 1 along the straight line,

$$z(t) = 0 + (1-0)t = t = t + 0i$$

$$dz = dt$$

$x = t$

$y = 0$

$$t \xrightarrow{(x,y)=(0,0)} \text{start} = 0$$

$$t \xrightarrow[(x,y)=(1,0)]{} \text{end} = 1$$

$$\int_{C_1} (x^2 + y^2)\, dz$$

$$= \int_0^1 t^2 \, dt$$

$$= \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3}$$

for c_2:

from 1 to $1+i$,

$$z(t) = 1 + (1 + i - 1)t$$

$$= 1 + it$$

$$dz = i\,dt$$

$$x = 1$$

$$y = t$$

$$t \xrightarrow{(x,y)=(1,0)} \text{start} = 0$$

$$t \xrightarrow[(x,y)=(1,1)]{} \text{end} = 1$$

$$\int_{C_2} (x^2+y^2)\,dz$$

$$= \int_0^1 (1^2+t^2)\, i\, dt$$

$$= \int_0^1 (i + it^2)\, dt$$

$$= \left[it + i \times \frac{t^3}{3} \right]_3^1$$

$$= i + \frac{i}{3} \qquad = \frac{4i}{3}$$

for c_3:

from $(1+i)$ to i,

$$z(t) = (1+i) + (i-1-i)\,t$$

$$= (1+i) - t \quad = (1-t) + i$$

$$dz = -dt$$

$$x = 1-t$$

$$y = 1$$

$$t \xrightarrow{(1,1)} t=0$$

$$t \xrightarrow[(0,1)]{} t=1$$

$$\int_{C_3} (x^2 + y^2)\, dz$$

$$= \int_0^1 \left\{ (1-t)^2 + 1^2 \right\} (-dt)$$

$$= \int_0^1 (1 - 2t + t^2 + 1)(-dt)$$

$$= \int_0^1 (-2 + 2t - t^2)\, dt$$

$$= \left[-2t + 2 \times \frac{t^2}{2} - \frac{t^3}{3} \right]_0^1$$

$$= -2 + 1 - \frac{1}{3}$$

$$= \frac{-4}{3}$$

<u>for c_4:</u>

from i to 0

$$z(t) = i + (0 - i)t$$

$$= i - it$$

$$= 0 + i(1 - t)$$

$$dz = -i\,dt$$

$$x = 0$$
$$y = 1 - t$$

$$t \xrightarrow{(0,1)} t = 0$$
$$t \xrightarrow{(0,0)} t = 1$$

$$\int_{C_4} (x^2 + y^2)\, dz$$

$$= \int_0^1 (1-t)^2 \, (-i\,dt)$$

$$= \int_0^1 -i(1 - 2t + t^2)\, dt$$

$$= \left[-it + 2 \times \frac{t^2}{2} i - \frac{t^3}{3} i \right]_0^1$$

$$= i\left(-1 + 1 - \frac{1}{3}\right) = -i/3$$

now adding integrals through c_1, c_2, c_3, c_4 give us

$$\frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= -1 + i$$

Parametrization of Circle

in cartesian form:

center at (α, β) and radius $= r$

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

in polar form:

center $\alpha + \beta i$, radius $= r$

$$|z - (\alpha+\beta i)| = r$$

$$z - (\alpha + \beta i) = r e^{i\theta}$$

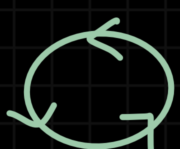
A circle centered at z_0 with radius r can be expressed as

$$|z - z_0| = r$$

$$\Rightarrow \; z - z_0 = r e^{i\theta}$$

θ 2π limit!

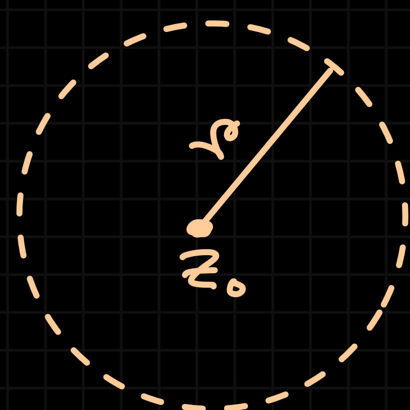
(i) $\underbrace{\text{anticlockwise}}_{\text{positive orientation / counter clockwise}}$



2π

$\theta \to$ start $= 0$

$\theta \to$ end $= 2\pi$



(ii) clockwise 2π

$\theta \to$ start $= 2\pi$

$\theta \to$ end $= 0$

$$\underbrace{|z - z_0| = r}_{\text{indicates circle}}$$

Evaluate $\oint_C (\bar{z})^2 \, dz$ around the circles —

(a) $|z| = 1$

(b) $|z-1| = 1$

Evaluate $\oint_C (\bar{z})^2 \, dz$ around the circles $|z|=1$

solve: $|z| = 1$

$$z = 1 \cdot e^{i\theta}$$

$$z = e^{i\theta}$$

$\bar{z} = e^{-i\theta}$ | $dz = i e^{i\theta} d\theta$ | $\theta \to 0$ to 2π

$$\oint_C (\bar{z})^2 \, dz$$

$$= \int_0^{2\pi} \left(e^{-i\theta}\right)^2 \, i e^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} i \cdot e^{-2i\theta} \cdot e^{i\theta} \, d\theta$$

$$= i \int_0^{2\pi} e^{-i\theta} \, d\theta$$

$$= i \cdot \frac{1}{-i} \left[e^{-i\theta} \right]_0^{2\pi}$$

$$= -\left[e^{-i2\pi} - e^{0}\right]$$

$$= -\left[\cos(-2\pi) + i\sin(-2\pi) - 1\right]$$

$$= 1 - i \cdot 0 - 1$$

$$= 0$$

(Ans)

Evaluate $\oint_C (\bar{z})^2 \, dz$ around the circles $|z-1| = 1$

solve:

$$|z-1| = 1$$

$$z-1 = 1 \cdot e^{i\theta}$$

$$z = 1 + e^{i\theta}$$

$\bar{z} = 1 + e^{-i\theta}$ | $dz = i e^{i\theta} \, d\theta$ | $\theta \to 0$, $\theta \to 2\pi$

$$\oint_C (\bar{z})^2 \, dz$$

$$= \int_0^{2\pi} (1+e^{-i\theta})^2 \cdot i \cdot e^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} \left(1 + 2e^{-i\theta} + e^{-2i\theta}\right) i e^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} \left(i e^{i\theta} + 2i + i e^{-i\theta}\right) d\theta$$

$$= \left[i \cdot \frac{1}{i} e^{i\theta} + 2\theta i + i \cdot \frac{1}{-i} e^{-i\theta} \right]_0^{2\pi}$$

$$= \left[e^{i\theta} + 2\theta i - e^{-i\theta} \right]_0^{2\pi}$$

$$= \left[\cos\theta + i\sin\theta + 2\theta i - \left(\cos(-\theta) + i\sin(-\theta) \right) \right]_0^{2\pi}$$

$$= \left[\cos\theta + i\sin\theta + 2\theta i - \cos\theta - i\sin\theta \right]_0^{2\pi}$$

$$= \left[2\theta i \right]_0^{2\pi}$$

$$= 2 \times 2\pi \times i$$

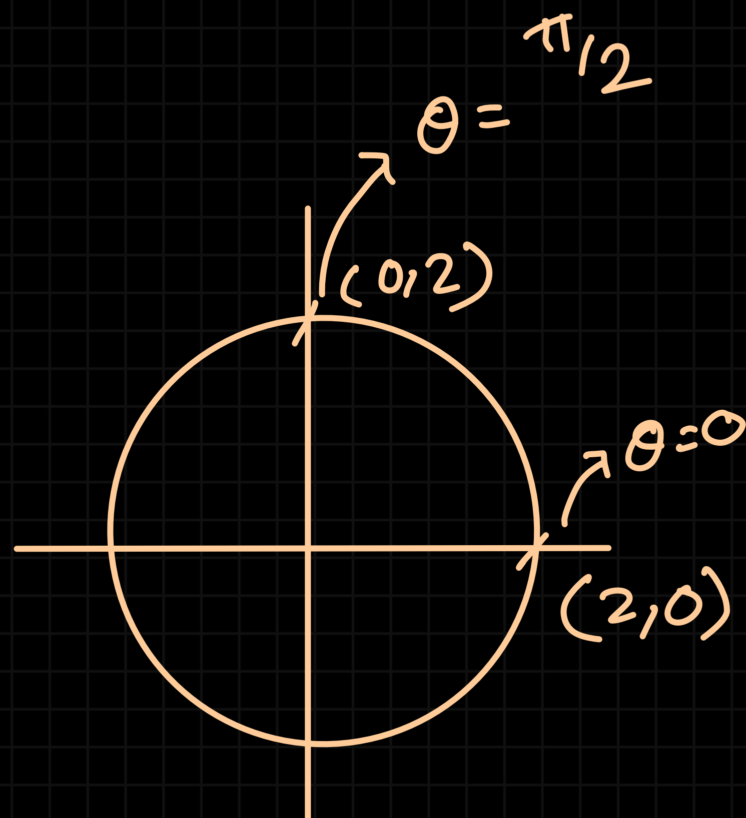
$$= 4\pi i$$

(Ans)

Evaluate $\int_C (z^2 + 3z)\, dz$

along the circle $|z| = 2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction.

solve:



$|z| = 2$

$z = 2e^{i\theta}$

$dz = 2ie^{i\theta}$

$\theta \to 0$ to $\pi/2$

$$\oint_C (z^2+3z)\, dz$$

$$= \int_0^{\pi/2} (4e^{2i\theta} + 6e^{i\theta})\, 2ie^{i\theta}\, d\theta$$

$$= 2i \times 2 \int_0^{\pi/2} (2e^{2i\theta} + 3e^{i\theta})e^{i\theta}\, d\theta$$

$$= 4i \int_0^{\pi/2} (2e^{3i\theta} + 3e^{2i\theta})\, d\theta$$

$$= 4i \int_0^{\pi/2} \Big\{ 2\cos(3\theta) + i \times 2\sin(3\theta) + 3\cos(2\theta) + i \times 3\sin(2\theta) \Big\} d\theta$$

$$= 4i \Big[2 \times \frac{1}{3} \sin 3\theta + i \times 2 \times \frac{1}{3} \times (-\cos 3\theta) + 3 \times \frac{1}{2} \times \sin(2\theta) + i \times 3 \times \frac{1}{2} \times (-\cos 2\theta) \Big]_0^{2\pi}$$

$$= 4i \Big[\Big\{ \frac{2}{3} \sin(6\pi) - \frac{2i}{3} \cos(6\pi) + \frac{3}{2} \times \sin(4\pi) - \frac{3i}{2} \cos(4\pi) \Big\} - \Big\{ -\frac{2i}{3} \cos 0 - \frac{3i}{2} \cos 0 \Big\}$$

$$= 4i\left[0 - \cancel{\frac{2i}{3}} + 0 - \cancel{\frac{3i}{2}} + \cancel{\frac{2i}{3}} + \cancel{\frac{3i}{2}}\right]$$

$$= 0$$