CSE330: Numerical Methods

Topic: QR Decomposition

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QR Decomposition

→ over determine system $(m > n)$

কে solve করা without using inversion

$$\underbrace{Ax = b}_{m \times n \ (m > n)}$$

now, $\quad A = QR$

A: $m \times n$

Q: orthonormal matrix $(m \times n)$

R: upper triangular matrix $(n \times n)$

$$Ax = b$$

$$A^T A x = A^T b$$

$$(QR)^T (QR) x = (QR)^T b$$

$$R^T (Q^T Q) R x = R^T Q^T b$$

$$\Rightarrow \quad R x = Q^T b$$

<u>formation of Q matrix:</u>

$$u_1, u_2, u_3$$

$$\Big\downarrow \text{ Gram-schmidt process}$$

$$\underbrace{p_1, p_2, p_3}_{\text{orthogonal}} \longrightarrow \underbrace{q_1, q_2, q_3}_{\text{orthonormal}}$$

orthonormal matrix: যে ম্যাট্রিক্সের

কলামসমূহগুলোকে vector বিবেচনা করলে ভেক্টরগুলোর

set orthonormal set হবে।

Gram Schmidt Process

e.g. $x_0 = -3$ $\quad x_1 = 0$ $\quad x_2 = 6$

$f(x_0) = 0$ $\quad f(x_1) = 0$ $\quad f(x_2) = 2$

ans:

$$\underbrace{\begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{b}$$

u_1 $\quad u_2$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$$

$$P_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = u_2 - \frac{u_2 \cdot P_1}{\underbrace{P_1 \cdot P_1}_{\text{vector dot product}}} \times P_1$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - \frac{(-3\times1) + (1\times0) + (1\times6)}{(1\times1) + (1\times1) + (1\times1)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$\cancel{P_3 = u_3 - \frac{u_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{u_3 P_2}{P_2 P_2} P_2}$$

we got two vectors P_1, P_2 who are orthogonal vectors.

now we make them orthonormal.

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & \frac{5}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & \frac{-1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 5\sqrt{2}/\sqrt{21} \end{bmatrix}$$

using backward substitution,

$$\sqrt{42}\ a_1 = \frac{5\sqrt{2}}{\sqrt{21}}$$

$$a_1 = \frac{5}{21}$$

$$\sqrt{3}\, a_0 + \sqrt{3}\, a_1 = \frac{2}{\sqrt{3}}$$

$$\therefore \quad a_0 = \frac{3}{7}$$

$$P_1(x) = a_0 + a_1 x$$

$$= \frac{3}{7} + \frac{5}{21} x$$

que:

$$2x_1 - 5x_2 = 1$$
$$5x_1 + x_2 = 10$$
$$x_1 - 7x_2 = 5$$

sol:

$$\underbrace{\begin{pmatrix} 2 & -5 \\ 5 & 1 \\ 1 & -7 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix}}_{b}$$

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
u_1 \arrow[d] & & u_2 \arrow[d] \\
p_1 \arrow[d] & & p_2 \arrow[d] \\
q_1 \arrow[dr] & & a_2 \arrow[dl] \\
 & Q \arrow[d] & \\
 & R \arrow[r] & x
\end{tikzcd}$$