MAT216: Linear Algebra & Fourier Analysis

Topic: Fourier Part II

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Topics:

→ evaluating those integrals

Integration Formulas

i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

ii) $\int \sin(mx) = \frac{-\cos(mx)}{m} + c$

iii) $\int \cos(mx) = \frac{\sin(mx)}{m} + c$

$\boxed{}$ $\sin(n\pi) = 0$

$\boxed{}$ $\cos(n\pi) = (-1)^n$

} for any integer n

$$\int u\,dv = uv - \int v\,du$$

u, v choose এর নিয়ম:

LIATE

~~Logarithm~~

~~In~~

Algebric

Trigonometric

Exponential

উপরে থাকলে u ধরো

Calculating Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

for $(-\pi, \pi)$ interval, $L = \pi$

and so,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

Half Series

Even

→ cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) \right]$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)$$

Odd

sine series

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \sin(nx) \right]$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

22 que:

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0 \end{cases}$$

$f(x)$ ஐ fourier expansion

$\boxed{\text{ans}}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx\,dx + \frac{1}{\pi}\times$$

$$\int_{0}^{\pi} x \times \cos nx \times dx$$

$$= \frac{-1}{\pi} \int_{-\pi}^{0} x\cos(nx)\,dx + \frac{1}{\pi}x$$

$$\int_{0}^{\pi} x\cos(nx)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$u = x$

$dv = \cos nx\,dx \Rightarrow v = \int \cos nx\,dx$

$$v = \frac{\sin nx}{n}$$

$$\therefore \int x \cos nx \, du = x \cdot \frac{\sin(nx)}{n} - \int \frac{\sin(nx)}{n} \, dx$$

$$= \frac{x}{n} \times \sin(nx) + \frac{1}{n} \frac{\cos(nx)}{n}$$

$$= \frac{x}{n} \sin(nx) + \frac{1}{n^2} \cos(nx)$$

So,

$$\frac{-1}{\pi} \int_{-\pi}^{0} x\cos(nx)\,dx + \frac{1}{\pi} \times$$

$$\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{-1}{\pi} \times \left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi} \times \left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}\left(\frac{1}{n^2} + \frac{1}{n^2} \times \cos(-n\pi)\right) + \frac{1}{\pi} \times$$

$$\left[\frac{\pi}{n} \times \underbrace{\sin(n\pi)}_{0} + \frac{1}{n^2} \times (-1)^n - \frac{1}{n^2}\right]$$

$$= -\frac{1}{\pi n^2} + \frac{1}{\pi n^2} \times (-1)^n +$$

$$\frac{1}{\pi} \times \left(\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right)$$

$$= -\frac{1}{\pi n^2} + \frac{1}{\pi n^2}(-1)^n + \frac{1}{\pi n^2}(-1)^n$$

$$- \frac{1}{\pi n^2}$$

$$= \frac{-2}{\pi n^2} + \frac{2}{\pi n^2}(-1)^n$$

$$\therefore a_n = \frac{-2}{\pi n^2} + \frac{2}{\pi n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx$$

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0 \end{cases}$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (-x) \sin(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} x \cdot \sin(nx)\, dx$$

$$= -\frac{1}{\pi} \int_{-\pi}^{0} x \sin(nx)\, dx + \frac{1}{\pi} \times \int_{0}^{\pi} x \sin(nx)\, dx$$

$$\int u\,dv = uv - \int v\,du$$

$$u = x, \quad dv = \sin(nx)\,dx$$

$$v = \frac{-\cos(nx)}{n}$$

$$\therefore \int x \sin(nx)\,dx = x \times \frac{-\cos(nx)}{n}$$

$$- \int \frac{-\cos(nx)}{n} \times dx$$

$$= \frac{-x}{n} \cos(nx) + \frac{1}{n} \int \cos(nx)\,dx$$

$$= \frac{x}{n} \cos(nx) + \frac{1}{n} \times \frac{\sin(nx)}{n}$$

$$= \frac{-1}{\pi} \int_{-\pi}^{0} x \sin(nx)\, dx + \frac{1}{\pi} \times \int_{0}^{\pi} x \sin(nx)\, dx$$

$$= -\frac{1}{\pi} \left[\frac{x}{n} \cos(nx) + \frac{1}{n} \times \frac{\sin(nx)}{n} \right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi} \left[\frac{x}{n} \cos(nx) + \frac{1}{n} \times \frac{\sin(nx)}{n} \right]_{0}^{\pi}$$

$$= \frac{-1}{\pi}\left(\frac{-\pi}{n} \times \cos(-n\pi) - \frac{1}{n^2} \times \sin(-n\pi)\right)$$

$$+ \frac{1}{\pi}\left[\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin(\pi n)\right]$$

$$= \frac{-1}{\pi}\left(\frac{\pi}{n} \times (-1)^n + \frac{1}{n^2}\sin(n\pi)\right)$$

$$+ \frac{1}{\pi}\left[\frac{\pi}{n}(-1)^n + \frac{1}{n^2} \times 0\right]$$

$$= \frac{-1}{n}(-1)^n + \frac{1}{n}(-1)^n$$

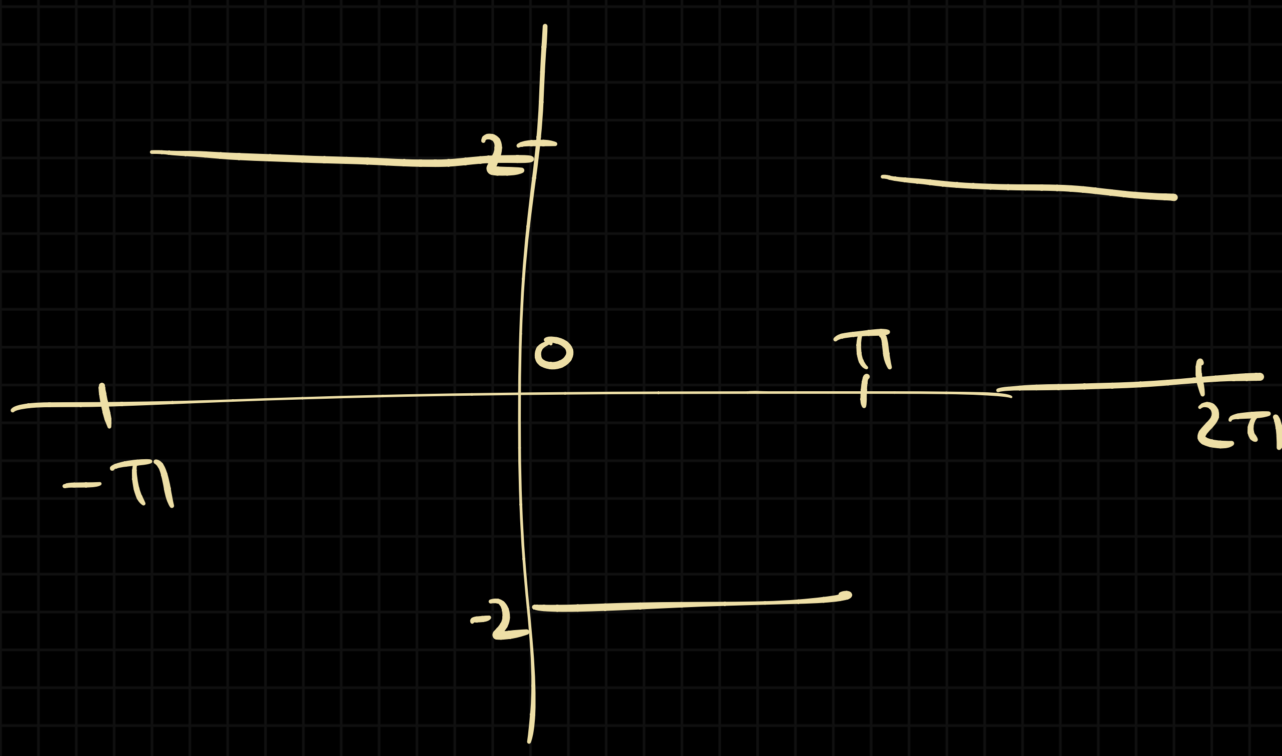
$= 0$

Draw sketches and determine the a_0, a_n, b_n and therefore determine the Fourier series (up to five terms at least) for the following two functions. [5+5=10]

a) $f(x) = \begin{cases} -2 \ if \ 0 < x < \pi \\ +2 \ if \ \pi < x < 2\pi \end{cases}$, for $-\pi < x < +\pi$

(a)

$$f = \begin{cases} -2, & 0 < x < \pi \\ & -\pi < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$$



$$\therefore f(x) = \begin{cases} -2, & 0 < x < \pi \\ 2, & -\pi < x < 0 \end{cases}$$

$\because f(x)$ is odd function,

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (-2) \sin(nx)\, dx$$

$$= \frac{-4}{\pi} \int_0^{\pi} \sin(nx)\, dx$$

$$= \frac{-4}{\pi} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi}$$

$$= \frac{4}{\pi n} \left[\cos nx \right]_0^{\pi}$$

$$= \frac{4}{\pi n}\left(\cos n\pi - \cos 0 \right)$$

$$= \frac{4}{\pi n}\left\{ (-1)^n - 1 \right\}$$

<u>$n = 1:$</u>

$$b_1 = \frac{4}{\pi}(-1-1) = \frac{-8}{\pi}$$

$$b_2 = \frac{4}{2\pi}(1-1) = 0$$

$$b_3 = \frac{4}{3\pi}(-1-1) = \frac{-8}{3\pi}$$

$$b_4 = \frac{4}{4\pi}(1-1) = 0$$

$$b_5 = \frac{4}{5\pi}(-1-1) = \frac{-8}{5\pi}$$

$$b_6 = \frac{4}{6\pi}(1-1) = 0$$

$$b_7 = \frac{4}{7\pi}(-1-1) = \frac{-8}{7\pi}$$

$$b_8 = \frac{4}{8\pi}(1-1) = 0$$

$$b_9 = \frac{4}{9\pi}(-1-1) = \frac{-8}{9\pi}$$

$$b_1 = \frac{-8}{\pi},\ b_3 = \frac{-8}{3\pi},\ b_5 = \frac{-8}{5\pi}$$

$$b_7 = \frac{-8}{7\pi},\ b_9 = \frac{-8}{9\pi}$$

$$\therefore f(x) = -\frac{8}{\pi} \sin(x) - \frac{8}{3\pi}(\sin 3x)$$

$$-\frac{8}{5\pi} \sin(5x) - \frac{8}{7\pi} \sin(7x) -$$

$$\frac{8}{9} \sin(9x)$$

If $x(t) = t - t^2$; $-\pi < t < \pi$ and x(t) is periodic over 2π then find the Fourier series of x(t). From general form of Fourier Series deduce the expression of $\mathbf{a_n}$ and $\mathbf{a_0}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (t - t^2) \cos(nt)\, dx$$

$u = t - t^2$

$dv = \cos(nt)\,dx$

$\int u\,dv = uv - \int v\,du$

$v = \int \cos(nt)\,dx$

$= \dfrac{\sin(nt)}{n}$

$$\int (t-t^2)\cos(nt)\,du,$$

$$= (t-t^2) \times \frac{\sin(nt)}{n} -$$

$$\int \frac{\sin(nt)}{n}$$

→ unfinished

ⓐ $f(x) = 1-x^2 \qquad -\pi < 0 < \pi$

even or odd?

$x = 3 \rightarrow f(x) = -8$

$x = -3 \rightarrow f(-3) = -8$

so its even.

[illegible] $f(x) = x(x-1)(x+1)$

$f(1) = 0$

$f(-1) = 0$

[illegible]

$f(3) = 3 \times 2 \times 4 = 29$

$f(-3) = -3 \times -4 \times -2 = -29$

$f(-x) = -f(x)$

$\therefore$ odd

$\boxed{\text{Q}}$ Fourier sine series বের কর :

$$f(x) = \cos x \,; \quad 0 \leq x \leq \pi$$

[NB: half range]

$\boxed{\text{ans}}$ since it is sine series,

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin(nx) \right)$$

$a_0 = a_n = 0$

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(nx) \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(nx) \cos(x) \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left\{ \sin(nx + x) + \sin(nx - x) \right\} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin\{x(n+1)\}\, dx \; +$$

$$\frac{2}{\pi} \int_0^{\pi} \sin\{x(n-1)\}\, dx$$

$$= \frac{2}{\pi} \times \left[-\frac{\cos(nx+x)}{n+1} \right]_0^{\pi} \; +$$

$$\frac{2}{\pi} \left[-\frac{\cos(nx-x)}{(n-1)} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{\frac{-1}{n+1} - \frac{1}{n-1}\right\}\right]$$

sir 1 লেখসন

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

Draw sketches and determine the Fourier Series for the following functions.

a. $s(x) = \frac{x}{\pi}$, for $-\pi < x < +\pi$

b. $s(x) = 3|\sin x|$ for $0 \le x < 2\pi$

c. $s(x) = \begin{cases} 2\sin x & \text{for } 0 \le x < \pi \\ 0 & \text{for } \pi \le x < 2\pi \end{cases}$

d. $s(x) = \begin{cases} 1 & \text{for } 0 \le x < \pi \\ 0 & \text{for } \pi \le x < \pi \end{cases}$

Hw

Orthogonality related theorem

$\langle u, v \rangle = 0 \quad . \quad . \quad .$ when $u \neq v$

function এর inner product

$< \sin(mx), \sin(nx) > = 0$ if $m \neq n$

integration ই 0

eg:

$$\int_{-\pi}^{\pi} \sin(mx) \cdot \sin(nx)\, d = \text{inner product}$$

when $m \neq n \rightarrow$ iprod $= 0$

when $m = n \Rightarrow$ non zero inner product

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos mx \, dx = 0$$

$2\cos^2 x = 1 + \cos 2x$

$2\sin^2 x = 1 - \cos 2x$

$\int_{-\pi}^{\pi} \sin(nx)\,dx$

$$= \left[\frac{-\cos(nx)}{n} \right]_{-\pi}^{\pi}$$

$$= \frac{-(-1)^n}{n} - \frac{-(-1)^n}{n}$$

5. Define Orthogonal Functions. Using $\int_{-\pi}^{+\pi} sin\,nx\,dx = 0, and\ \int_{-\pi}^{+\pi} cos\,nx\,dx = 0, where\ m, n \in Z$, prove the following identities –

b. $\int_0^{\pi} \cos nx \cos mx\,dx = \begin{cases} \pi & \text{if } n = m = 0 \\ \frac{\pi}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$

case 1 : $n = m = 0$:

$$I = \int_0^{\pi} \cos nx \cos mx\, dx$$

$$= \int_0^{\pi} 1 \cdot 1 \, dx$$

$$= \Big[x \Big]_0^{\pi}$$

$$= \pi$$

<u>case-2</u>: $n = m \neq 0$:

$$I = \int_0^{\pi} \cos(nx) \cos(mx)\, dx$$

$$= \int_0^{\pi} \cos(nx) \cos(nx)\, dx$$

$$= \int_0^{\pi} \cos^2(nx) \, dx$$

$$= \frac{1}{2} \int_0^{\pi} 2 \cos^2(nx) \, dx$$

$$= \frac{1}{2} \int_0^{\pi} (1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \left[\int_0^{\pi} dx + \int_0^{\pi} \cos 2x \, dx \right]$$

$$= \frac{1}{2}\left[x\right]_0^{\pi} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2}\pi + \frac{1}{4}\sin 2\pi$$

$$= \frac{1}{2}\pi$$

case-3: $n \neq m$

$$I = \int_0^{\pi} \cos nx \cos mx \, dx$$

$$= \int_0^{\pi} \left[\cos\{x(n+m)\} + \cos\{x(n-m)\} \right] dx$$

$$= \left[\frac{\sin(mx+nx)}{m+n} \right]_0^{\pi} +$$

$$\left[\frac{\sin(nx-mx)}{n-m}\right]_0^{\pi}$$

$$= \underbrace{\frac{\sin(m+n)\pi}{m+n}}_{0} + \underbrace{\frac{\sin(n-m)\pi}{n-m}}_{0}$$

$\because \sin(\text{something } \pi) = 0$

$= 0$ [Ans]

5. Define Orthogonal Functions. Using $\int_{-\pi}^{+\pi} \sin nx \, dx = 0$, and $\int_{-\pi}^{+\pi} \cos nx \, dx = 0$, where $m, n \in Z$, prove the following identities –

c. $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$ $\rightarrow \begin{cases} 0, & n = m = 0 \\ \pi, & n = m \neq 0 \\ 0, & n \neq m \end{cases}$

(d)

case-2: $n = m$

$$I = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$= \int_{-\pi}^{\pi} (\sin nx)^2 \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 2\sin^2 x \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[1-\cos 2x\right]dx$$

$$= \frac{1}{2}\left[\left[x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos 2x \, dx\right]$$

$$= \frac{1}{2}\left[\pi-(-\pi) - \left[\frac{\sin 2x}{2}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2} \times 2\pi = \pi$$

<u>case-3:</u> $n \neq m$

$$I = \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx$$

$$I = \frac{1}{2} \int_{-\pi}^{\pi} 2 \sin(mx) \sin(nx) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left[\cos(m-n)x + \cos(m+n)x \right] dx$$

$$= \frac{1}{2} \left[\int_{-\pi}^{\pi} \cos(m-n)x \, dx \; + \right.$$

$$\left. \int_{-\pi}^{\pi} \cos(m+n)x \, dx \right]$$

$$= \frac{1}{2}\left[\cancel{\left[\frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}}^{0} + \cancel{\left[\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}}^{0} \right]$$

$$= 0$$

i) $\sin(\text{integer } \pi) = 0$

ii) $\cos(\text{integer } \pi) = (-1)^{\text{that integer}}$