MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Topic: Laplace 01 - Laplace Transformations

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Laplace transformation,

→ f(t) থেকে

→ f(t) কে integrate করে নতুন function generate করি → F(s)

→ variable t থেকে s হয়ে যায়

t → time domain

s → frequency domain

$$\mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\,dt$$

Formal Definition of the Laplace Transformation

Let $f(t)$ be a function defined for $t \geq 0$. Then the integral

↳ time can't be negative

$$\underbrace{\mathcal{L}}_{\text{laplace transformation}} \{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{F(s)}_{\text{complex frequency}}$$

is said to be the laplace tranform of f, provided that the integral converges.

where s is a complex number.

Transforms of some Algebric and Exponential Functions

↳ final 2 will be given

$\boxed{+}$ $\mathcal{L}\{1\} = \frac{1}{s}$

$\boxed{+}$ $\mathcal{L}\{t\} = \frac{1}{s^2}$

$\boxed{+}$ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

where n is a non negative integer

$\boxed{+}$ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

where n is not an integer

$\boxed{1}$ $\mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\Rightarrow \mathcal{L}\{t\} = \int_0^\infty t\, e^{-st}\, dt$$

$$= \left[t \, \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= (0 - 0) + \frac{1}{s} \int_0^\infty \frac{e^{-st}}{-s}\, dt$$

$$= \frac{1}{s}\left[0 - \frac{e^{-0}}{s} \right] = \frac{1}{s^2}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= 0 - \frac{e^{-(s-a)\cdot 0}}{-(s-a)}$$

$$= \frac{1}{s-a}$$

Transforms of some Trigonometric and Hyperbolic Functions

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

Linearity of Laplace Transformation

$$\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{k \cdot f(t)\} = k \cdot \mathcal{L}\{f(t)\}$$

Find the laplace transform of each of the following functions:

i) $3e^{-2t}$

ii) $4t^3 - e^{-t}$

iii) $(t^2+1)^2$

iv) $7\sin 2t - 3\cos 2t$

v) $(4e^{2t} - 2)^3$

i) solve:

$$\mathcal{L}\left\{3e^{-2t}\right\}$$

$$= 3\mathcal{L}\left\{e^{-2t}\right\}$$

$$= 3 \times \frac{1}{s-(-2)}$$

$$= \frac{3}{s+2}$$

ii) $\mathcal{L}\left\{ 4t^3 - e^{-t} \right\}$

$= \mathcal{L}\left\{ 4t^3 \right\} - \mathcal{L}\left\{ e^{-t} \right\}$

$= 4\,\mathcal{L}\left\{ t^3 \right\} - \mathcal{L}\left\{ e^{-t} \right\}$

$= 4 \times \dfrac{3!}{s^{3+1}} - \dfrac{1}{s-(-1)}$

$= \dfrac{24}{s^4} - \dfrac{1}{s+4}$

$$\mathcal{L}\left\{(t^2+1)^2\right\}$$

$$\mathcal{L}\left\{(t^2+1)^2\right\} = \mathcal{L}\left\{t^4 + 2t^2 + 1\right\}$$

$$= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$$

$$= \frac{4!}{s^{4+1}} + 2 \cdot \frac{2!}{s^{2+1}} + \frac{1}{s}$$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

$$\mathcal{L}\{7\sin 2t - 3\cos 2t\}$$

$$= 7\,\mathcal{L}\{\sin 2t\} - 3\,\mathcal{L}\{\cos 2t\}$$

$$= 7\,\frac{2}{s^2+4} - 3\,\frac{s}{s^2+4}$$

$$= \frac{14-3s}{s^2+4}$$

v) $\mathcal{L}\left\{(4e^{2t}-2)^3\right\}$

$$= (4e^{2t})^3 + 3(4e^{2t})^2(-2) +$$

$$3 \cdot 4e^{2t} \cdot (-2)^2 + (-2)^3$$

$$= 64e^{6t} - 96e^{4t} + 48e^{2t} - 8$$

$$\mathcal{L}\left\{64e^{6t} - 96e^{4t} + 48e^{2t} - 8\right\}$$

$$= 64 \cdot \frac{1}{s-6} - 96 \cdot \frac{1}{s-4} + 48 \cdot$$

$$\frac{1}{s-2} - \frac{8}{s}$$

First Translation Theorem

if $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

can be written as,

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s \to s-a}$$

④ Evaluate

i) $\mathcal{L}\{t^3 e^{-3t}\}$

$\Rightarrow \mathcal{L}\{t^3\}$

$= \dfrac{3!}{s^4}$

$= \dfrac{6}{s^4}$

$\mathcal{L}\{t^3 e^{-3t}\}$

$= \left.\dfrac{6}{s^4}\right|_{s \to s-(-3)}$

$= \dfrac{6}{(s+3)^4}$

$$\mathcal{L}\{5e^{3t}\sin 4t\} = ?$$

$$\mathcal{L}\{5\sin 4t\}$$

$$\Rightarrow 5\,\mathcal{L}\{\sin 4t\}$$

$$\Rightarrow 5 \times \frac{4}{4^2+s^2}$$

$$= \frac{20}{16+s^2}$$

$\therefore \mathcal{L}\{e^{3t} \cdot 5\sin 4t\}$

$$= \left.\frac{20}{16+s^2}\right|_{s \to (s-3)}$$

$$= \frac{20}{16+(s-3)^2}$$

1 $\mathcal{L}\left\{ (t+2)^2 e^t \right\}$

$$\mathcal{L}\left\{ (t+2)^2 \right\}$$

$$= \mathcal{L}\left\{ t^2 + 4t + 4 \right\}$$

$$= \frac{2!}{s^{2+1}} + 4 \cdot \frac{1}{s^2} + \frac{4}{s}$$

$$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

$$= \frac{2 + 4s + 4s^2}{s^3}$$

$$\therefore \mathcal{L}\left\{(t+2)^2 e^{t}\right\}$$

$$= \left.\frac{2+4s+4s^2}{s^3}\right|_{s \to s-1}$$

$$= \frac{2+4(s-1)+4(s-1)^2}{(s-1)^2}$$

$$\mathcal{L}\left\{e^{-t}\left(3\sinh 2t - 5\cosh 2t\right)\right\}$$

$$\therefore \mathcal{L}\left\{3\sinh 2t - 5\cosh 2t\right\}$$

$$= 3\,\mathcal{L}\left\{\sinh 2t\right\} - 5\,\mathcal{L}\left\{\cosh 2t\right\}$$

$$= 3 \times \frac{2}{s^2 - 2^2} - 5 \times \frac{s}{s^2 - 2^2}$$

$$= \frac{6 - 5s}{s^2 - 4}$$

$$\therefore \mathcal{L}\left\{e^{-t}(3\sinh 2t - 5\cosh 2t)\right\}$$

$$= \left.\frac{6-5s}{s^2-4}\right|_{s \to s-(-1)}$$

$$= \frac{6-5(s+1)}{(s+1)^2-9}$$

$\boxed{}$ $\mathcal{L}\left\{e^{-4t}\cosh 2t\right\}$

$\mathcal{L}\left\{\cosh 2t\right\}$

$$= \frac{s}{s^2 - 2^2}$$

$$\mathcal{L}\left\{e^{-4t}\cosh 2t\right\} = \frac{s}{s^2-2^2}\Bigg|_{s \to s-(-4)}$$

$$= \frac{s+4}{(s+4)^2 - 4}$$

$$\mathcal{L}\left\{e^{2t}(3\sin 4t - 4\cos 4t)\right\}$$

$$\mathcal{L}\left\{3\sin 4t - 4\cos 4t\right\}$$

$$= 3 \times \frac{4}{s^2+4^2} - 4 \times \frac{s}{s^2+4^2}$$

$$= \frac{12}{s^2+16} - \frac{4s}{s^2+16}$$

$$= \frac{12-4s}{s^2+16}$$

$$\therefore \mathcal{L}\left\{ e^{2t}(3\sin 4t - 4\cos 4t) \right\}$$

$$= \left. \frac{12 - 4s}{s^2 + 16} \right|_{s \to s-2}$$

$$= \frac{12 - 4(s-2)}{(s-2)^2 + 16}$$

Laplace Transformation of the form

if $\mathcal{L}\{f(t)\} = F(s)$ and n is a natural number,

$$\mathcal{L}\{t^n f(t)\} = \cancel{(-1)^n \frac{d^n}{d^n}} \left(F(s)\right)$$

For $n=1$,

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\left(F(s)\right)$$

if either of the function isn't exponential like —

i) $\mathcal{L}\{t \sin 2t\} = ?$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\frac{(s^2+4)\cdot 0 - 2\cdot 2s}{(s^2+4)^2}$$

$$= \frac{4s}{(s^2+16)^2}$$

Ans.

(1) Evaluate

$$\mathcal{L}\{t \sin 3t \, e^{2t}\}$$

solve:

$$\mathcal{L}\{\sin 3t\}$$

$$= \frac{3}{s^2+9}$$

now,

$$\mathcal{L}\left\{ t \sin 3t \right\}$$

$$= - \frac{d}{ds}\left\{ \frac{3}{s^2+9} \right\}$$

$$= - \frac{(s^2+9).0 - 3 \cdot 2s}{(s^2+9)^2}$$

$$= \frac{6s}{(s^2+9)^2}$$

now,

$$\mathcal{L}\left\{t \sin t \; e^{2t}\right\}$$

$$= \left.\frac{6s}{(s^2+9)^2}\right|_{s \to s-2}$$

$$= \frac{6(s-2)}{\left\{(s-2)^2+9\right\}^2}$$

PS:

$$\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds}(F(s))$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u)\, du$$

(4) Evaluate

i) $\mathcal{L}\left\{\dfrac{\sin 2t}{t}\right\}$

ans: $\mathcal{L}\{\sin 2t\}$

$$= \frac{2}{s^2+4}$$

$$\mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{u^2+4}\, du$$

$$= 2\int_s^\infty \frac{du}{u^2+2^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \left[2 \times \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) \right]_s^{\infty}$$

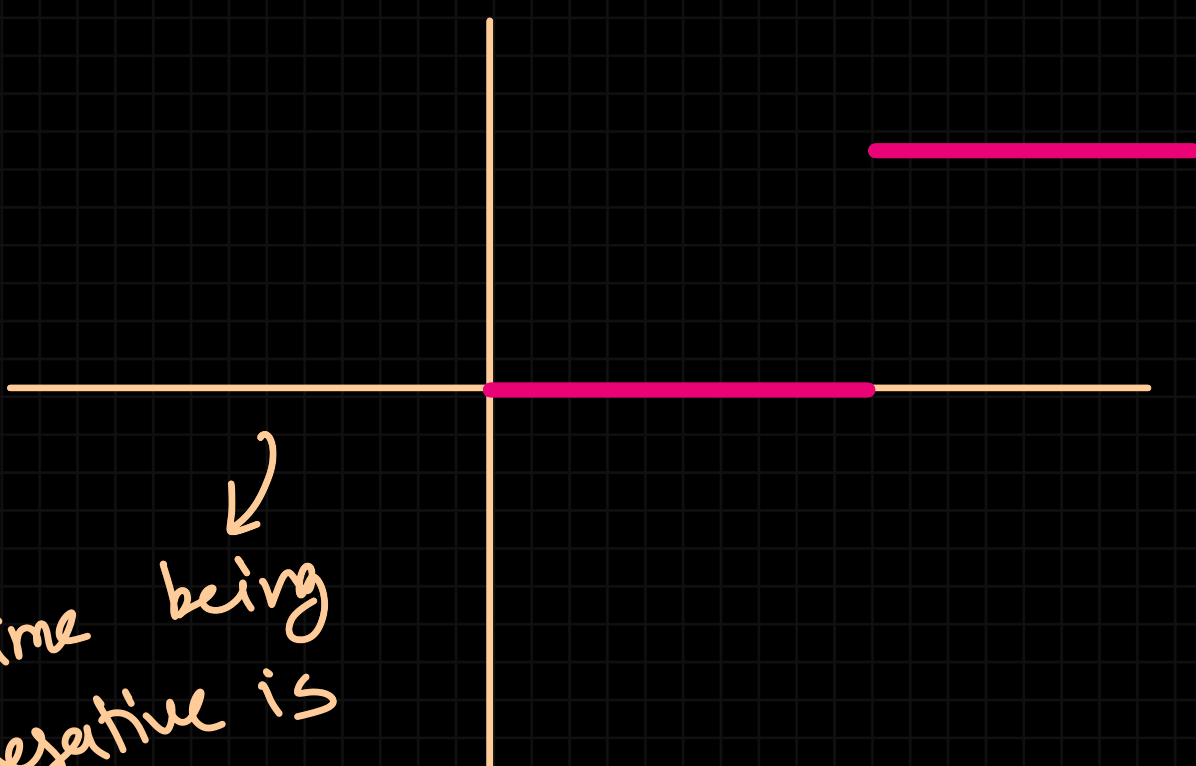
$$= \tan^{-1} \infty - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

What is Unit Step Function

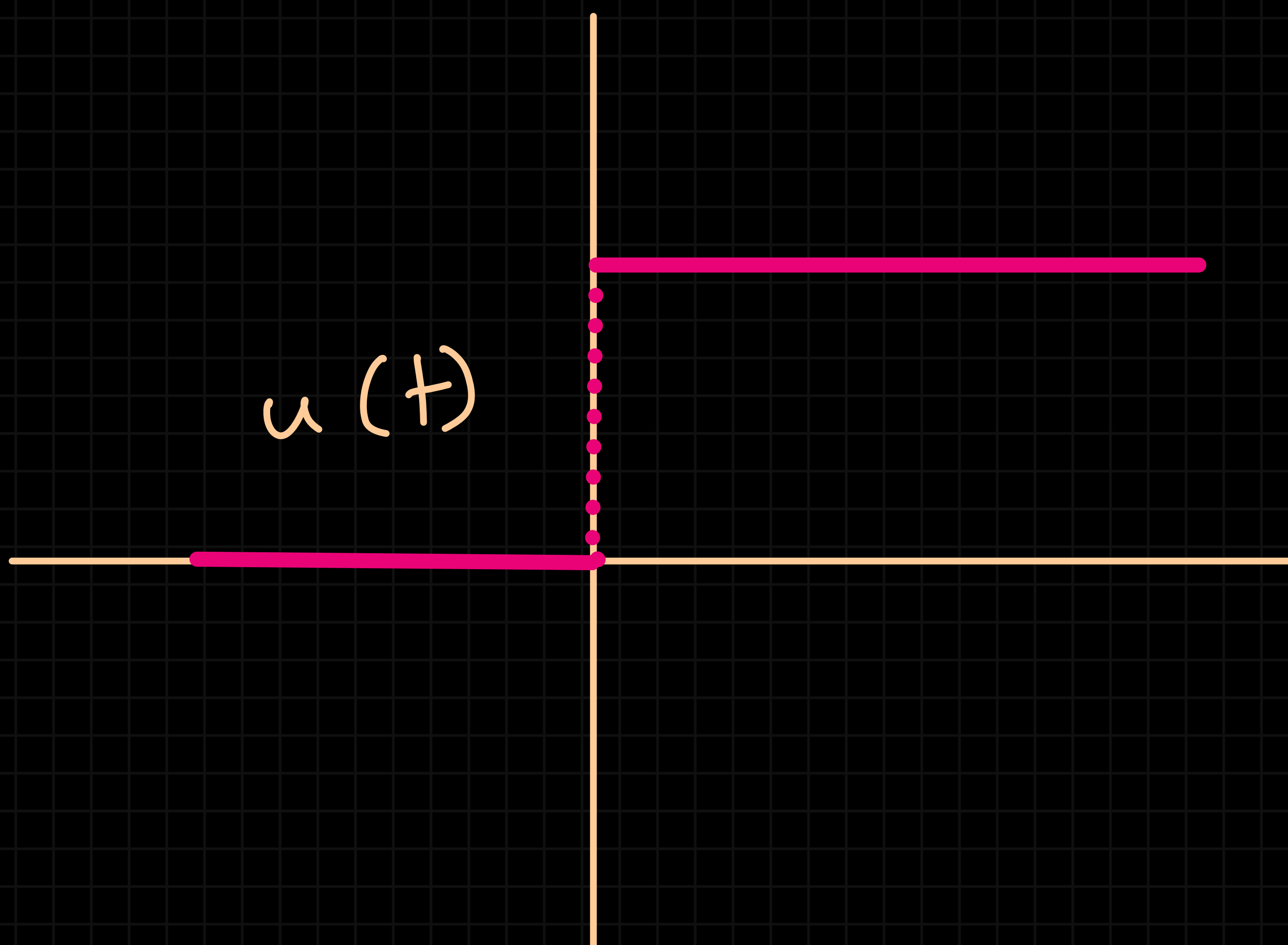
The unit step function $u(t-a)$ is defined to be

$$u(t-a) = \begin{cases} 0 , & 0 \leq t < a \\ 1 , & t \geq a \end{cases}$$

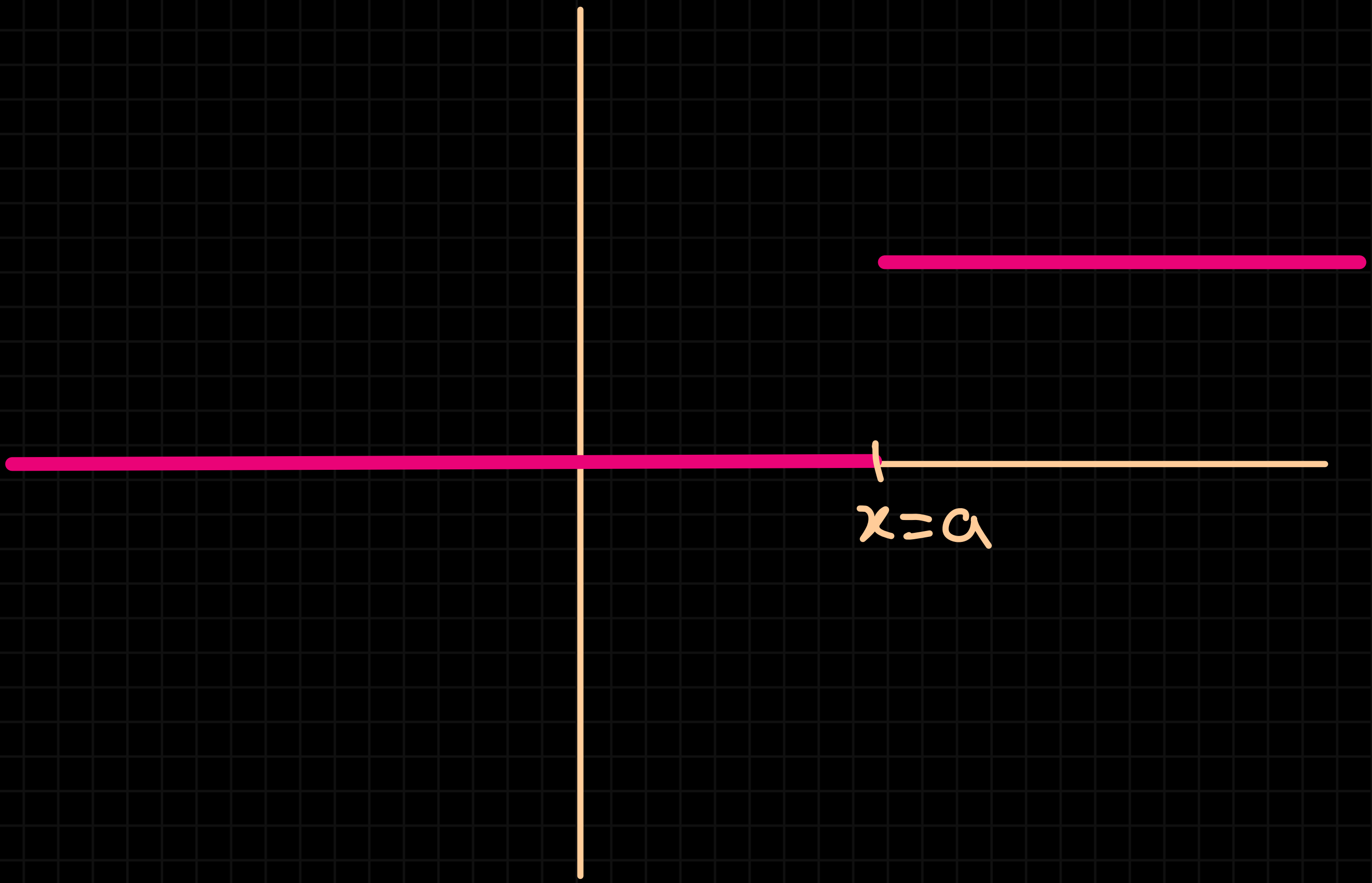


time being negative is ignored

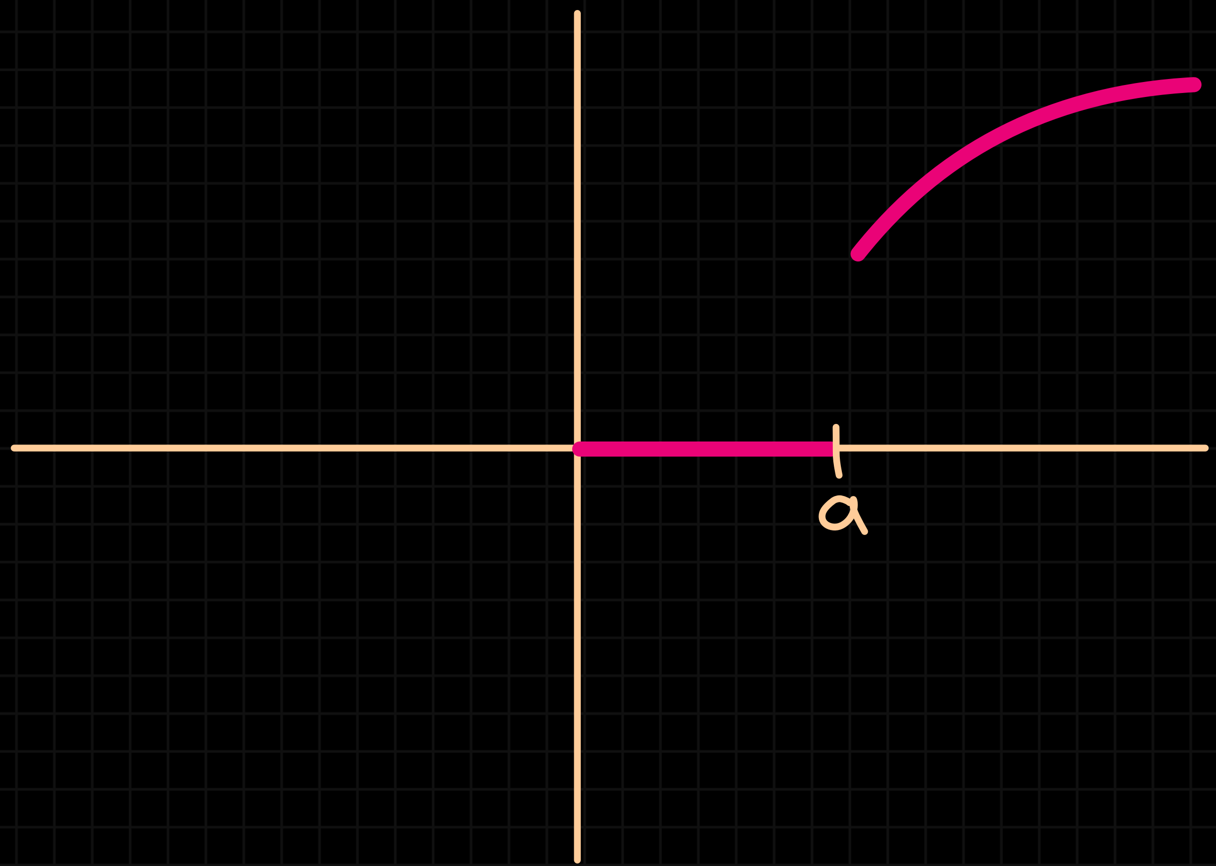
u(t) is a function with value 1 at everywhere $x \geq 0$ and 0 at every where $x < 0$ and in between is a jump. its called a unit function



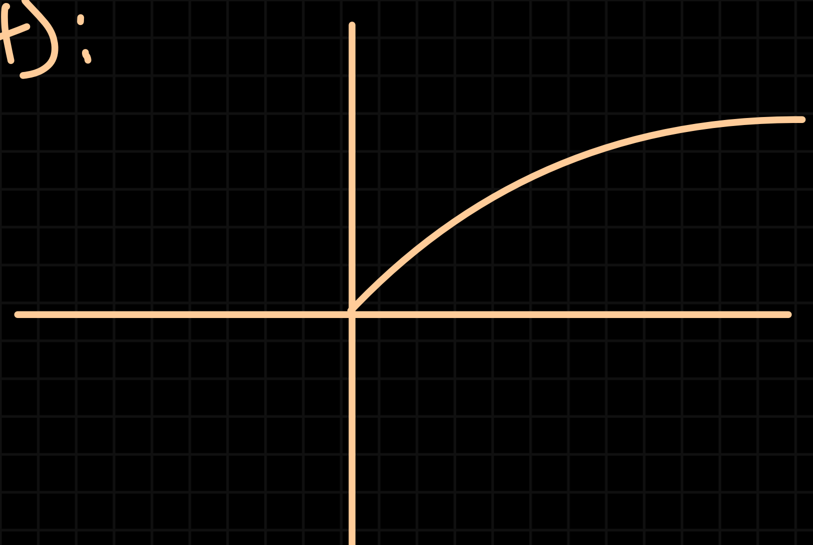
$u(t-a)$:



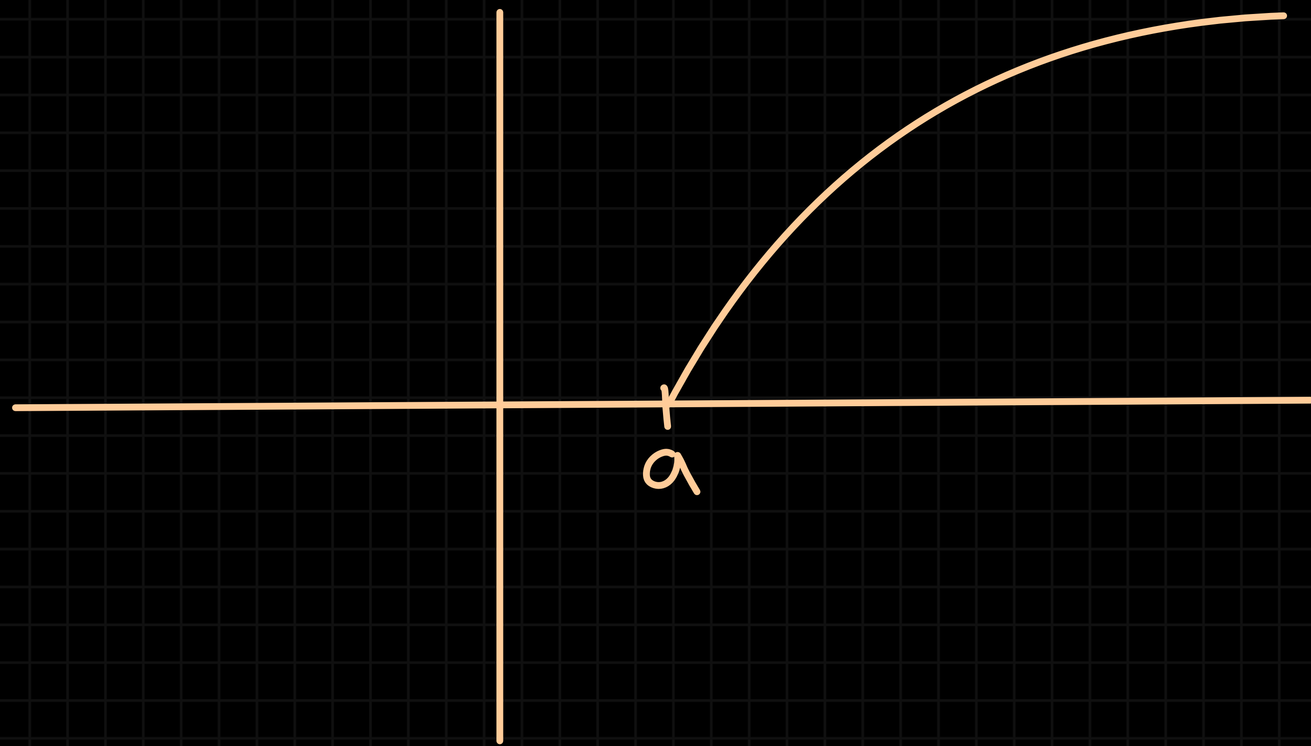
What is $f(t) \cdot u(t-a)$



$f(t)$:



what is $f(t-a) \cdot u(t-a)$



Second Translation Theorem

if $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number then,

$$\mathcal{L}\{f(t-a)\cdot u(t-a)\} = F(s)\, e^{-as}$$

rewritten as

$$\mathcal{L}\{f(t)\cdot u(t-a)\} = \mathcal{L}\{f(t+a)\}\cdot e^{-as}$$

if $\mathcal{L}\{f(t)\} = F(s)$ then,

$$\mathcal{L}\{f(t) \cdot u(t-a)\} = \mathcal{L}\{f(t+a)\} e^{-as}$$

Find the laplace transforms

i) $\mathcal{L}\{(t-1) \cdot u(t-1)\}$

ii) $\mathcal{L}\{e^{-2t} \cdot u(t-1)\}$

iii) $\mathcal{L}\{\cos 2t \cdot u(t-\pi)\}$

Find the laplace transforms

i) $\mathcal{L}\{(t-1)\cdot u(t-1)\}$

Solve:

$f(t) = t-1$

$\therefore a = 1$

$f(t+a) = f(t+1)$

$= t+1-1$

$= t$

$$\mathcal{L}\left\{(t-1)\,u(t-1)\right\}$$

$$= \mathcal{L}\left\{f(t+a)\right\} \cdot e^{-as}$$

$$= \mathcal{L}\{t\} \cdot e^{-s}$$

$$= \frac{1}{s^2} e^{-s}$$

Ans.

Find the laplace transforms

ii) $\mathcal{L}\{e^{-2t} \cdot u(t-1)\}$

<u>ans:</u>

$$f(t) = e^{-2t}$$

$$a = 1$$

$$\therefore f(t+a) = e^{-2(t+1)}$$

$$= e^{-2t} \cdot e^{-2}$$

$$\therefore \mathcal{L}\left\{ e^{-2t}\, u(t-1) \right\}$$

$$= \mathcal{L}\left\{ f(t) \right\} e^{-as}$$

$$= \mathcal{L}\left\{ e^{-2t} \cdot e^{-2} \right\} \cdot e^{-as}$$

$$= e^{-2}\, \mathcal{L}\left\{ e^{-2t} \right\} \cdot e^{-as}$$

$$= e^{-2} \times \frac{1}{s+2}\, e^{-s}$$

$$= \frac{e^{-2-s}}{s+2}$$

Find the laplace transforms

iii) $\mathcal{L}\{\cos 2t \cdot u(t-\pi)\}$

ans:

$f(t) = \cos 2t$

$a = \pi$

$f(t+a) = \cos 2(t+a)$

$= \cos 2(t+\pi)$

$= \cos(2t+2\pi)$

$= \cos 2t$

$$\mathcal{L}\left\{\cos 2t \cdot u(t-\pi)\right\}$$

$$= \mathcal{L}\left\{\cos 2t\right\} e^{-\pi s}$$

$$= \frac{s}{s^2+4} \times e^{-\pi s}$$

$$= \frac{s e^{-\pi s}}{s^2+4}$$

Converting Piecewise to Step function

$$f(t) = \begin{cases} \text{---} \\ \text{---} \\ g(t), & a \le t \le b \\ \text{---} \\ \text{---} \end{cases}$$

$$f(t) = \text{-- -- .} + g(t)\left[u(t-a) - u(t-b)\right]$$

$\hookrightarrow$ here, $u(t-0) = 1$

$u(t-\infty) = 0$

4) Find the laplace transform of

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ \cos(t), & t \geq \pi \end{cases}$$

$$f(t) = 0\left[u(t-0) - u(t-\pi)\right] +$$

$$\cos t\left[u(t-\pi) - u(t-\infty)\right]$$

$$= \cos t\left[u(t-\pi) - u(t-\infty)\right]$$

$$= \cos t \left[u(t-\pi) - 0 \right]$$

$$= \cos t \cdot u(t-\pi)$$

now,

$$\mathcal{L}\left\{ \cos t \cdot u(t-\pi) \right\}$$

here $f(t) = \cos t$

$a = \pi$

$f(t+a) = \cos(\pi + t)$

$= -\cos t$

$$\mathcal{L}\{\cos t \cdot u(t-\pi)\}$$

$$= \mathcal{L}\{f(t+a)\} e^{-as}$$

$$= \mathcal{L}\{-\cos t\} e^{-\pi s}$$

$$= -\frac{s}{s^2+1} e^{-\pi s}$$

1

$$\sin(2\pi+\theta)=\sin\theta$$

$$\cos(2\pi+\theta)=\cos\theta$$

$$\sin(\pi+\theta)=-\sin\theta$$

$$\cos(\pi+\theta)=-\cos\theta$$

☐ Find the laplace transform of

$$f(t) = \begin{cases} 5\sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

$$f(t) = 5\sin t\left[u(t-0) - u(t-\pi)\right]$$

$$+ 0 \cdot \left[u(t-\pi) - u(t-\infty)\right]$$

$$= 5\sin t\left[u(t-0) - u(t-\pi)\right]$$

$$= 5\sin t\left[1 - u(t-\pi)\right]$$

$$= 5\sin t - 5\sin t \cdot u(t-\pi)$$

$$\mathcal{L}\{5\sin t - 5\sin t \cdot u(t-\pi)\}$$

$$= \mathcal{L}\{5\sin t\} - 5\,\mathcal{L}\{\sin t \cdot u(t-\pi)\}$$

$$= 5 \times \frac{1}{s^2+1} - 5 \times$$

$$\mathcal{L}\{-\sin t\} \cdot e^{-\pi s}$$

$$\boxed{\begin{aligned} f(t) &= \sin t \\ a &= \pi \\ f(t+a) &= \sin(\pi+t) \\ &= -\sin t \end{aligned}}$$

$$= \frac{5}{s^2+1} + 5 \times \frac{1}{s^2+1} \times e^{-\pi s}$$

④ Find the laplace transform of

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ \sin t, & \pi \leq t < 2\pi \\ 0, & t > 2\pi \end{cases}$$

ans:

$$f(t) = 0 \cdot [\ldots] + \sin t \left[u(t-\pi) - u(t-2\pi) \right] + 0 [\ldots]$$

$$= \sin t \left[u(t-\pi) - u(t-2\pi) \right]$$

$$= \sin t \cdot u(t-\pi) - \sin t \cdot u(t-2\pi)$$

$$\mathcal{L}\left\{ \sin t \cdot u(t-\pi) - \sin t \cdot u(t-2\pi) \right\}$$

$$= \mathcal{L}\left\{ \sin t \cdot u(t-\pi) \right\} - \mathcal{L}\left\{ \sin t \cdot u(t-2\pi) \right\}$$

$f(t) = \sin t$

$a = \pi$

$f(t+a) = \sin(t+\pi)$

$= -\sin t$

$f(t) = \sin t$

$a = 2\pi$

$f(t+a) = \sin(2\pi+t)$

$= \sin t$

$$= \mathcal{L}\{-\sin t\} \cdot e^{-\pi s} - \mathcal{L}\{\sin t\} e^{-2\pi s}$$

$$= -\frac{1}{s^2+1} \cdot e^{-\pi s} - \frac{1}{s^2+1} e^{-2\pi s}$$

$$= \frac{-1}{s^2+1}\left(e^{-\pi s} + e^{-2\pi s}\right)$$