MAT215: Machine Learning & Signal Processing

Former Title: Complex variables & Laplace Transformations

Assignment 3

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BRAC University
Department of Mathematics and Natural Sciences
MAT 215: Mathematics for Machine Learning & Signal Processing

Assignment 3

Deadline : SPRING 2024 Total Marks: 60

Make a Front Page by yourself, mentioning your #name, #ID, and #section. (Compulsory)

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1. Evaluate $\oint_C (x+2y)dx + (y-2x)dy$ around the ellipse C defined by $x = 4\cos\theta, y = 3\sin\theta, \quad 0 \le \theta < 2\pi$ if C is described in a counterclockwise direction. **(5)**

2. Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0),(1,0),(1,1),(0,1)$. **(5)**

3. Evaluate $\oint_C \bar{z}^2 dz$ around the circles (a) $|z| = 1$, (b) $|z-1| = 1$. **(5)**

4. Evaluate $\int_C \left(z^2+1\right)^2 dz$ along the arc of the cycloid $x = a(\theta - \sin\theta), y = a(1-\cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$. **(5)**

5. Evaluate $\oint_C \left(5z^4 - z^3 + 2\right) dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at $(0,0),(1,0),(1,1)$, and $(0,1)$, (c) the curve consisting of the parabolas $y = x^2$ from $(0,0)$ to $(1,1)$ and $y^2 = x$ from $(1,1)$ to $(0,0)$. **(5)**

6. Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2| = 4$, (b) the circle $|z-1| = 5$, (c) the square with vertices at $3 \pm 3i, -3 \pm 3i$. **(5)**

7. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i} dz$ where C is the curve : (a) the circle $|z-1| = 4$, (b) the ellipse $|z-2| + |z+2| = 6$. **(5)**

8. Evaluate $\frac{1}{2\pi i} \oint_C \frac{\cos \pi z^2}{z^2-1} dz$ around a rectangle with vertices at: (a) $2 \pm i, -2 \pm i$; (b) $-i, 2-i, 2+i, i$. **(5)**

9. Show that $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2+1} dz = \sin t$ if $t > 0$ and C is the circle $|z| = 3$. **(5)**

10. Evaluate $\oint_C \frac{e^{iz}}{z^3} dz$ where C is the circle $|z| = 2$. **(5)**

11. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{\left(z^2+1\right)^2} dz$ when $t > 0$ and C is the circle $|z| = 3$ **(5)**

12. Evaluate $\oint_C \frac{e^z}{(z^2-1)^2} dz$ where C is the circle $|z-1| = 2$. **(5)**

Answer to the question no-1

1. Evaluate $\oint_C (x+2y)dx + (y-2x)dy$ around the ellipse C defined by $x = 4\cos\theta, y = 3\sin\theta, \quad 0 \leq \theta < 2\pi$ if C is described in a counterclockwise direction. **(5)**

$x = 4\cos\theta$

$dx = -4\sin\theta \, d\theta$

$y = 3\sin\theta$

$dy = 3\cos\theta \, d\theta$

$\oint_C (x+2y)\,dx + (y-2x)\,dy$

$= \int_0^{2\pi} (4\cos\theta + 6\sin\theta)(-4\sin\theta\, d\theta) +$

$$(3\sin\theta - 8\cos\theta)(3\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} \Big(-16\cos\theta\sin\theta - 24\sin^2\theta + 9\sin\theta\cos\theta - 24\cos^2\theta\Big)\, d\theta$$

$$= \int_0^{2\pi} \left\{ -\frac{7}{2} \times 2\sin\theta\cos\theta - 24(\sin^2\theta + \cos^2\theta) \right\} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{7}{2}\sin 2\theta - 24\right) d\theta$$

$$= \left[\frac{7}{2} \times \frac{1}{2}\cos 2\theta - 24\theta \right]_0^{2\pi}$$

$$= \frac{7}{4}\cos(4\pi) - 48\pi - \frac{7}{4}\cos 0 - 0$$

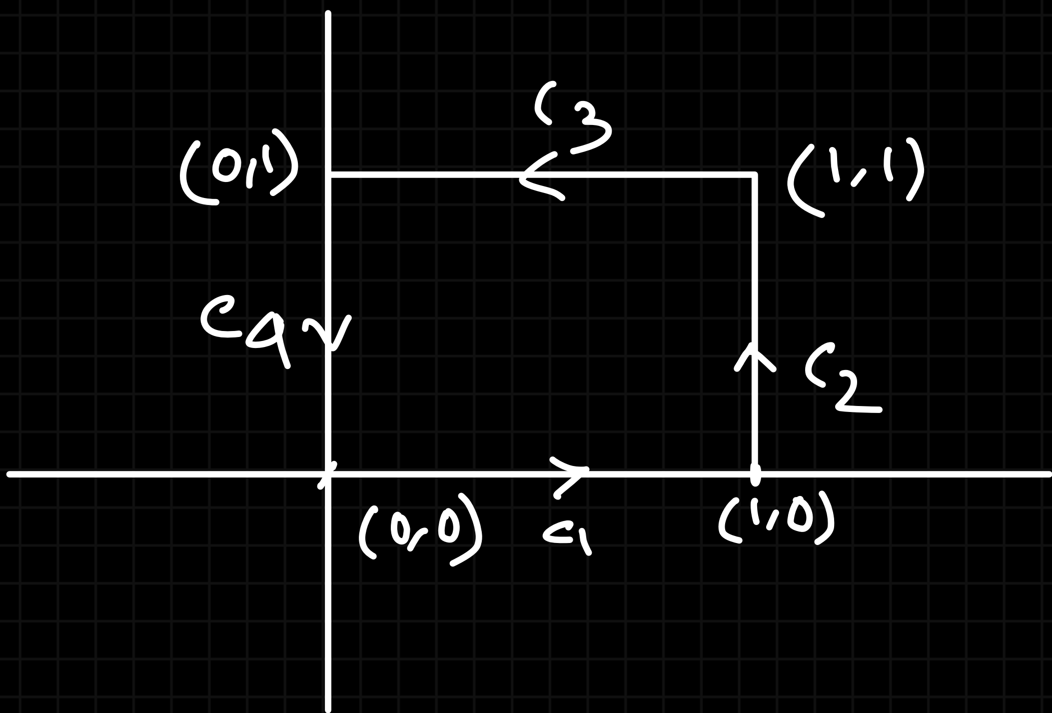
$$= \frac{7}{4} - 48\pi - \frac{7}{4}$$

$$= -48\pi$$

(Ans)

Answer to the question no-2

2. Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0),(1,0),(1,1),(0,1)$.



$C \equiv C_1 \to C_2 \to C_3 \to C_4 \to C_1$

$$\therefore \oint_C |z|^2 dz = \int_{c_1} |z|^2 dz + \int_{c_2} |z|^2 dz +$$

$$\int_{c_3} |z|^2 dz + \int_{C_4} |z|^2 dz$$

<u>for c_1:</u>

$$z(t) = z_1 + (z_2 - z_1)t$$

$$= t$$

$$dz = dt$$

$x = t$ | $t \to 0$

$y = 0$ | $t \to 1$

$$\int_{c_1} |z|^2 \, dz = \int_0^1 |t|^2 \, dt$$

$$= \left[\frac{t^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

for c_2:

$$z = z_1 + (z_2 - z_1)t$$

$$= 1 + (1 + i - 1)t$$

$$= 1 + it$$

$x = 1$ | $dz = i\,dt$

$y = t$ | $t \to 0$, $t \to 1$

$$\int_{C_2} (x^2 + y^2)\, dz$$

$$= \int_0^1 (1 + t^2)\, i\, dt$$

$$= i \int_0^1 (1 + t^2)\, dt$$

$$= i \left[t + \frac{t^3}{3} \right]_0^1$$

$$= i \left[1 + \frac{1}{3} \right]$$

$$= \frac{4i}{3}$$

<u>for c_3:</u>

$$z(t) = 1 + i + (i - 1 - i)t$$

$$= 1 + i - t$$

$$= (1 - t) + i$$

$x = 1 - t$ | $dz = -dt$

$y = 1$ | $t \to 0$, $t \to 1$

$$\int_{C_3} (x^2+y^2)\, dz$$

$$= \int_0^1 \left\{ (1-t)^2 + 1^2 \right\} - dt$$

$$= \int_1^0 (1-2t+t^2+1)\, dt$$

$$= \int_1^0 (2-2t+t^2)\, dt$$

$$= \left[2t - 2\times\frac{t^2}{2} + \frac{t^3}{3} \right]_1^0$$

$$= -2(1) + 1 - \frac{1}{3} = -\frac{4}{3}$$

for c_4:

$$z(t) = i + (-i)t$$
$$= i(1-t)$$

$x = 0$ | $dz = -i\,dt$

$y = 1-t$ | $t \to 0$ to 1

$$\int_{c_4} (x^2+y^2)\,dz$$

$$= \int_0^1 (1-t)^2 (-i\,dt)$$

$$= \int_1^0 (1 - 2t + t^2)\, i \, dt$$

$$= \int_1^0 (i - 2ti + it^2)\, dt$$

$$= \left[it - \cancel{2} \times \frac{t^2}{\cancel{2}} i + i \frac{t^3}{3} \right]_1^0$$

$$= i \left[t - t^2 + \frac{t^3}{3} \right]_1^0$$

$$= i \left(-1 + 1 - \frac{1}{3} \right) = {}^{-i}/_{3}$$

$$\therefore \oint_C |z|^2 \, dz$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= -1 + i$$

(Ans)

Answerr to the question no-3

3. Evaluate $\oint_C \bar{z}^2 dz$ around the circles (a) $|z| = 1$, (b) $|z-1| = 1$.

$\underline{3(a)}$

$|z| = 1$

$z = e^{i\theta}$

$\therefore \bar{z} = e^{-i\theta}$

$dz = i e^{i\theta} d\theta$

$$\therefore \oint_C \bar{z}^2 \, dz$$

$$= \int_0^{2\pi} (e^{-i\theta})^2 \, i e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} e^{-i\theta} \, d\theta$$

$$= i \left[\frac{1}{-i} \times e^{i\theta} \right]_0^{2\pi}$$

$$= \left[- e^{i\theta} \right]_0^{2\pi}$$

$$= - e^{i(2\pi)} + e^0$$

$$= -\cos(2\pi) - i\sin(2\pi) + 1$$

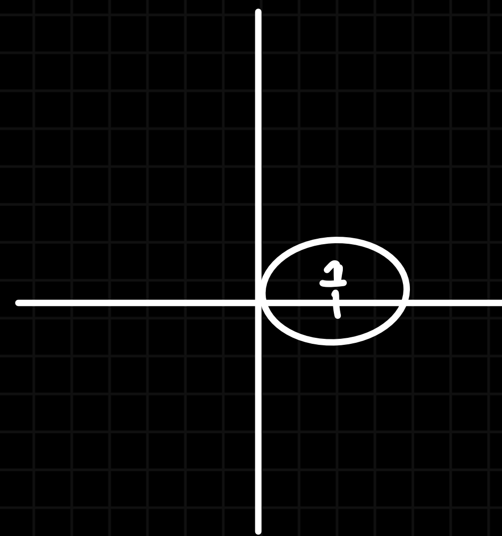
$$= -1 + 1$$

$$= 0$$

(Ans)

3(b)

$$|z-1| = 1$$



$$z-1 = e^{i\theta}$$

$$z = 1 + e^{i\theta}$$

$$dz = i e^{i\theta}\, d\theta$$

$$\bar{z} = 1 + e^{-i\theta}$$

$$\therefore \oint_C \bar{z}^2\, dz = \int_0^{2\pi} (1+e^{-i\theta})^2 \times (ie^{i\theta} d\theta)$$

$$= \int_0^{2\pi} \left(1 + 2e^{-i\theta} + e^{-2i\theta}\right) \times ie^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} \left(e^{i\theta} + 2 + e^{-i\theta}\right) d\theta$$

$$= i \left[\frac{1}{i} e^{i\theta} + 2\theta + \frac{1}{-i} e^{-i\theta}\right]_0^{2\pi}$$

$$= \left[e^{i\theta} + 2\theta i - e^{-i\theta}\right]_0^{2\pi}$$

$$= \left[e^{i(2\pi)} + 2(2\pi)i - e^{i(-2\pi)} - \right.$$

$$\left(e^{0} + 0 - e^{0}\right)\Big]$$

$$= \cos(2\pi) + i\sin(2\pi) + 4\pi i - \cos(-2\pi) - i\sin(-2\pi)$$

$$= 1 + 4\pi i - 1$$

$$= 4\pi i$$

(Ans)

Answer to the question no-4

Evaluate $\int_C (z^2+1)^2 \, dz$ along the arc of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$. **(5)**

Given the integral

$$\int_C (z^2+1)^2 \, dz$$

along the arc of the cycloid,

$$x = a(\theta - \sin\theta)$$

$$y = a(1 - \cos\theta)$$

where θ ranges from

$$\theta = 0 \quad \text{to} \quad \theta = 2\pi$$

First we need to parameterize z in terms of θ. since $z = x + iy$ we get,

$$z = a(\theta - \sin\theta) + ia(1 - \cos\theta)$$

$$dz = \{a(1 - \cos\theta) + i\, a\sin\theta\}\, d\theta$$

now the integral becomes,

$$\int_0^{2\pi} \left[\left\{ a(\theta - \sin\theta) + i\,a(1-\cos\theta) \right\}^2 + 1 \right]^2 .$$

$$\left\{ a(1-\cos\theta) + i\,a\sin\theta \right\} d\theta$$

$$= \int_0^{2\pi} \left[a^2(\theta - \sin\theta)^2 + i^2 a^2 (1-\cos\theta)^2 \right.$$

$$\left. + 2a(\theta - \sin\theta) . i\,a(1-\cos\theta) + 1 \right]^2 .$$

$$(a - a\cos\theta + ai\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} \Big\{ (1 - 2a^2 + a^4 + 4ia^2\theta - 4a^4 i\theta$$

$$+ 2a^2\theta^2 - 6a^4\theta^2 + 4ia^4\theta^3 + a^4\theta^4 +$$

$$4a^2\cos\theta - 4a^4\cos\theta - 4ia^2\theta\cos\theta$$

$$+ 12ia^4\theta\cos\theta + 12a^4\theta^2\cos\theta -$$

$$4ia^4[\theta]^3\cos\theta - 2a^2\cos\theta^2 +$$

$$6a^4\cos[\theta]^2 - 12ia^4\theta\cos^2\theta - 6a^4\theta^2$$

$$\cos[\theta]^2 - 4a^4\cos[\theta]^3 + 4ia^4\theta\cos\theta^3$$

$$+ a^4\cos(\theta)^4 - 4ia^2\sin(\theta) + 4ia^4\sin\theta -$$

$$4a^2\theta\sin[\theta] + 12a^4\theta\sin[\theta] - 12ia^4 \times$$

$$\theta^2\sin\theta - 4a^4\theta^3\sin[\theta] + 4ia^2\cos[\theta] \times$$

$\sin\theta - 12 i a^4 \cos[\theta] \sin[\theta] - 24 a^4 \times$

$\cos\theta \sin\theta + 12 i a^4 \theta^2 \cos\theta \sin\theta +$

$12 i a^4 \cos[\theta]^2 \sin[\theta] + 12 a^4 \theta \cos[\theta] \times$

$\sin[\theta] - 4 i a^4 \cos[\theta]^3 \sin\theta + 2a^2 \sin[\theta]^2$

$- 6 a^4 \sin[\theta]^2 + 12 i a^4 \theta \sin[\theta]^2 +$

$6 a^4 \theta^2 \sin[\theta]^2 + 12 a^4 \cos[\theta] \sin[\theta]^2$

$- 12 i a^4 \theta \cos[\theta] \sin[\theta]^2 - 6 a^4 \times$

$\cos[\theta]^2 \sin[\theta]^2 - 4 i a^4 \sin[\theta]^3$

$- 4 a^4 \sin[\theta]^3 + 4 i a^4 \cos[\theta] \sin[\theta]^3 +$

$a^4 \sin[\theta]^4) \{a(1-\cos\theta) + i a \sin\theta\} \, d\theta$

$$= -\frac{2}{15} a\pi \left\{ 15 + a^2(3 + 6i\pi + 4\pi^2) + 2a^4(-70 - 75i\pi + 60\pi^2 + 60i\pi^3 + 24\pi^4) \right\} \left\{ -1 + \cos\theta - i\sin\theta \right\}$$

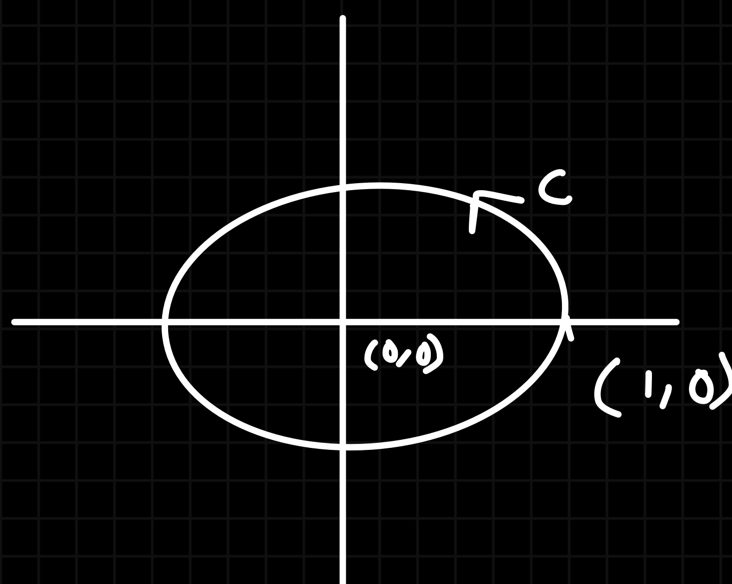
Ans.

Answer to the question no-5

5. Evaluate $\oint_C (5z^4 - z^3 + 2)\, dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at $(0,0), (1,0), (1,1)$, and $(0,1)$, (c) the curve consisting of the parabolas $y = x^2$ from $(0,0)$ to $(1,1)$ and $y^2 = x$ from $(1,1)$ to $(0,0)$. **(5)**

5(a)

$|z| = 1$



The function $5z^4 - z^3 + 2$ has no singularity inside or on the region C.

so according to Cauchy - Goursat theorem

$$\oint_C (5z^4 - z^3 + 2)\, dz = 0$$

also we can evaluate the value as such:

$$|z| = 1$$

$$z = e^{i\theta}$$

$$dz = i e^{i\theta}\, d\theta$$

$$\therefore \oint_C (5z^4 - z^3 + 2)\, dz$$

$$= \int_0^{2\pi} \left(5e^{(4i)\theta} - e^{(3i)\theta} + 2\right) ie^{i\theta} d\theta$$

$$= i\int_0^{2\pi} \left(5e^{(5i)\theta} - e^{(4i)\theta} + 2e^{i\theta}\right) d\theta$$

$$= i\left[5\times\frac{1}{5i}e^{(5\theta)i} - \frac{1}{4i}e^{i(4\theta)} + \frac{2}{i}e^{i\theta}\right]_0^{2\pi}$$

$$= \left[e^{i(5\theta)} - \frac{e^{i(4\theta)}}{4} + 2e^{i\theta}\right]_0^{2\pi}$$

$$= \left[e^{i(10\pi)} - \frac{e^{i(8\pi)}}{4} + 2e^{i(2\pi)} \right.$$

$$\left. - e^{0} + \frac{e^{0}}{4} - 2e^{0} \right]$$

$$= \cos(10\pi) + i\sin(10\pi) - \frac{1}{4}\cos(8\pi)$$

$$- i \cdot \frac{1}{4} \cdot \sin(8\pi) + 2\cos(2\pi) + i \cdot 2\sin(2\pi)$$

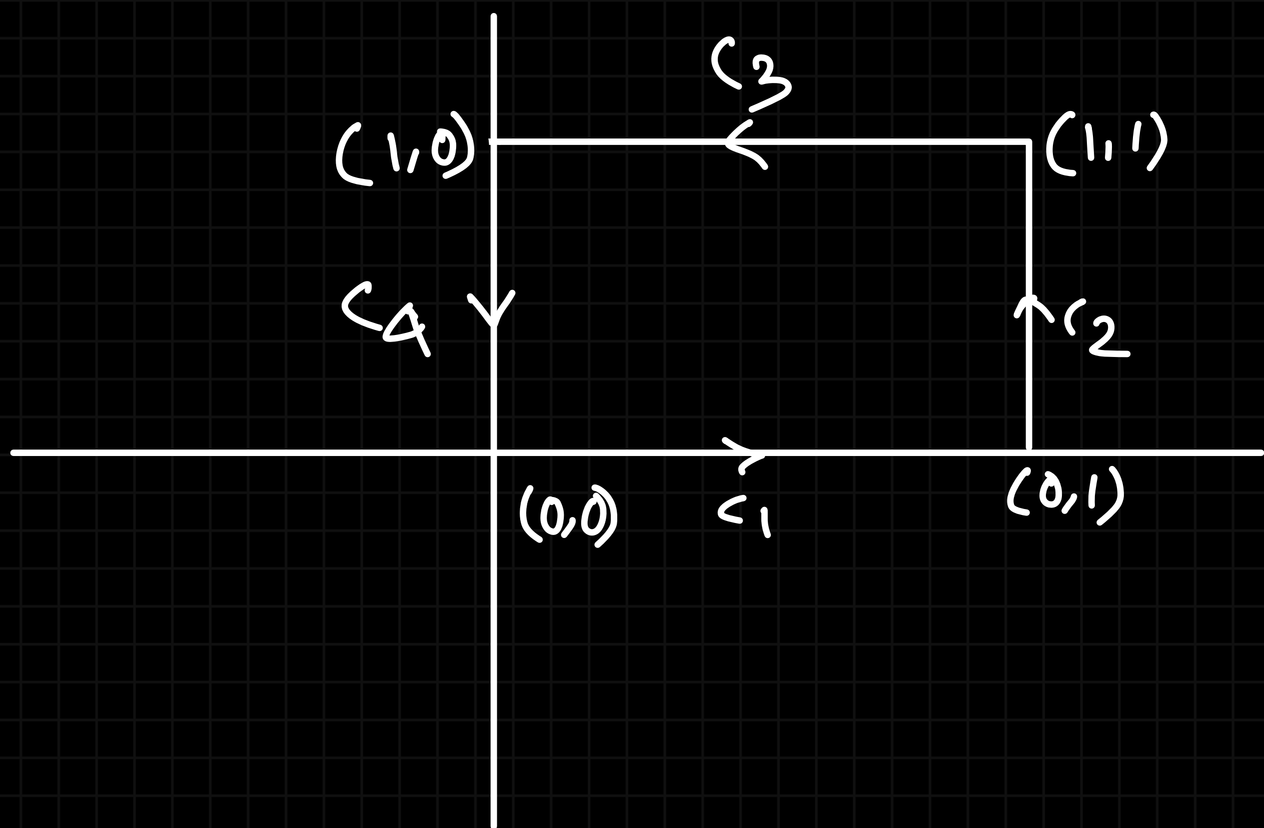
$$-1 + \frac{1}{4} - 2$$

$$= \cancel{1} + 0 - \cancel{\frac{1}{4}} - 0 + \cancel{2} + 0 - \cancel{1} + \cancel{\frac{1}{4}} - \cancel{2}$$

$$= 0$$

(Ans)

5 (b)



C is the square with the vertices

at (0,0), (0,1), (1,1), (1,0)

the function $(5z^4 - z^3 + 2)$ has no singularity whatsoever.

so, the function is analytic inside and on the boundary of C.

using cauchy goursat theorem,

$$\oint_C (5z^4 - z^3 + 2)\, dz$$

$$= 0$$

(Ans)

$\underline{5(c)}$

c is the region of the parabolas

$y = x^2$ from $(0,0)$ to $(1,1)$ and

$y^2 = x$ from $(1,1)$ to $(0,0)$

the function $5z^4 - z^3 + 2$ has no singularity and therefore is analytic inside the parabolas and on their boundaries. So using cauchy goursat thm,

$$\int (5z^4 - z^3 + 2)\, dz = 0$$

(Ans)

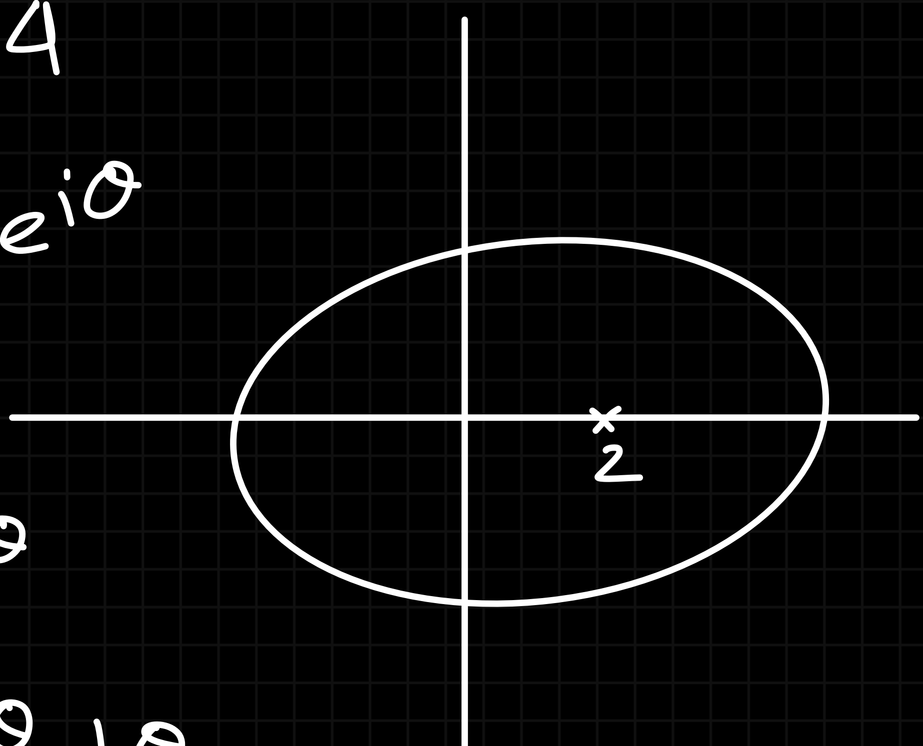
Answer to the question no. 6

6. Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2| = 4$, (b) the circle $|z-1| = 5$, (c) the square with vertices at $3 \pm 3i, -3 \pm 3i$. **(5)**

6(a)

$|z-2| = 4$

$z-2 = 4e^{i\theta}$



$z = 2 + 4e^{i\theta}$

$dz = 4ie^{i\theta}\, d\theta$

$$\oint_C \frac{dz}{z-2}$$

$$= \int_0^{2\pi} \frac{4ie^{i\theta}\, d\theta}{4e^{i\theta}}$$

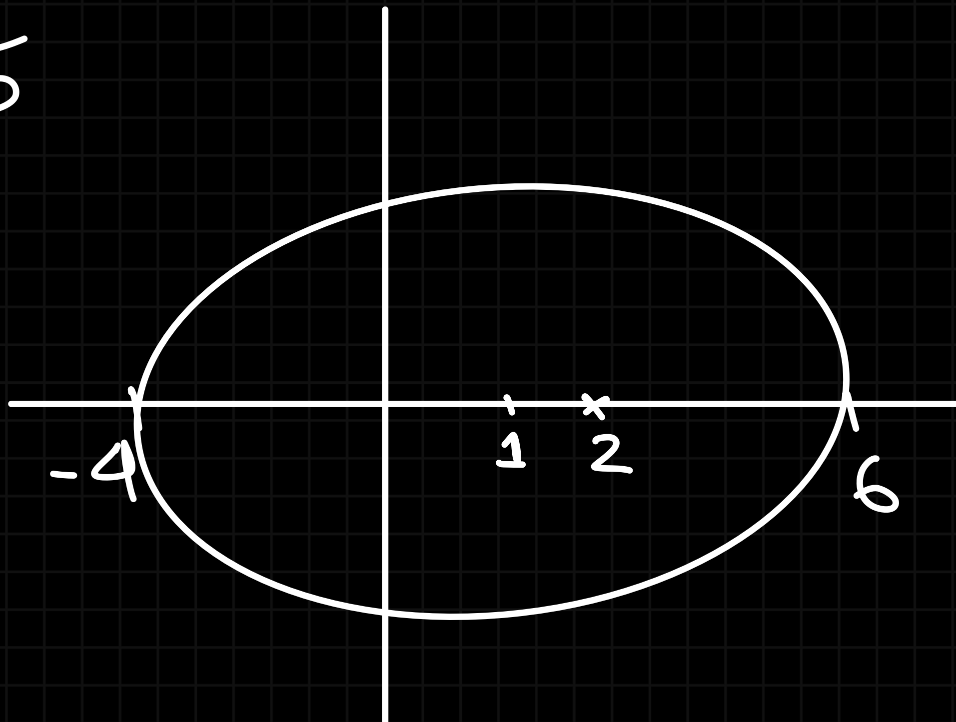
$$= i\int_0^{2\pi} d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$ (Ans)

6(b)

$|z-1| = 5$



$|z-2| = 2$

$z-2 = 2e^{i\theta}$

$z = 2+2e^{i\theta}$

$dz = 2ie^{i\theta}\, d\theta$

$$\int_C \frac{dz}{z-2}$$

$$= \int_0^{2\pi} \frac{2ie^{i\theta}\,d\theta}{2e^{i\theta}}$$

$$= i\int_0^{2\pi} d\theta$$

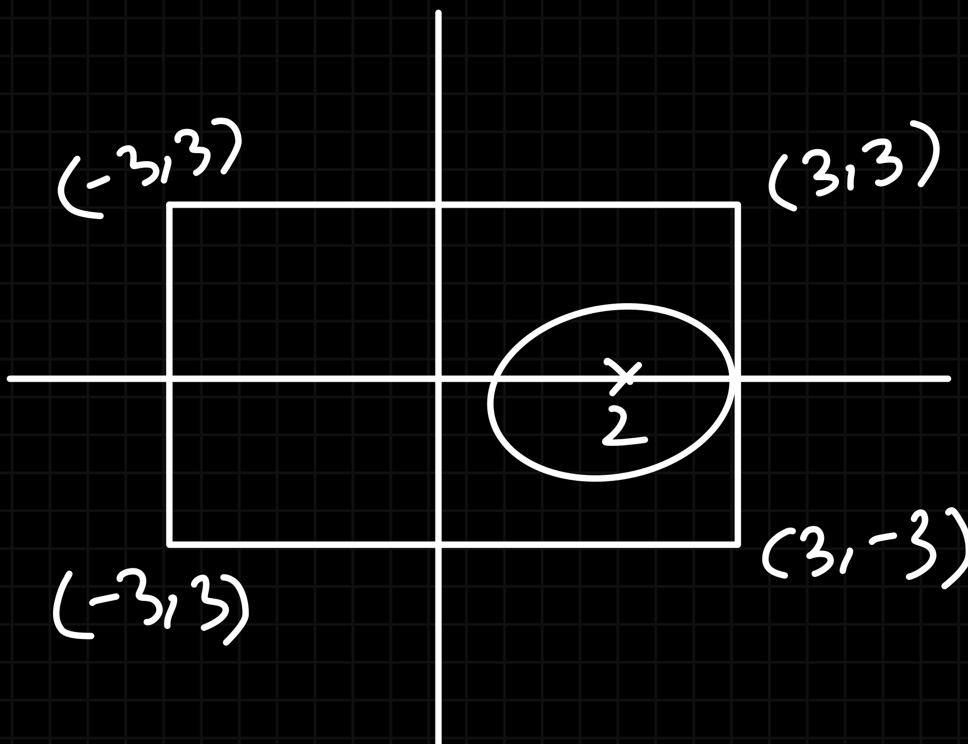
$$= i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$

(Ans)

6(c)

the vertices: $3+3i, -3+3i, 3-3i, -3-3i$



let $|z-2|=1$

$\Rightarrow z-2=e^{i\theta}$

$z=2+e^{i\theta}$

$$dz = i e^{i\theta} \, d\theta$$

$$\oint_C \frac{dz}{z2} = \int_0^{2\pi} \frac{i e^{i\theta} \, d\theta}{e^{i\theta}}$$

$$= i \int_0^{2\pi} d\theta$$

$$= i \left[\theta \right]_0^{2\pi}$$

$$= 2\pi i$$

(Ans)

Answerr to the question no- 7

7. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i} dz$ where C is the curve : (a) the circle $|z-1| = 4$, (b) the ellipse $|z-2| + |z+2| = 6$. **(5)**

7(a)

$$\oint \frac{e^{3z}}{z-\pi i} dz$$

$c: |z-1| = 4$

let $f(z) = e^{3z}$ and $a = \pi i$

now, $f(z)$ is analytic inside and on the boundary of C. Also

$z = \pi i$ is inside C.

so applying cauchy integral formula,

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z-a} \, dz = f(a)$$

$$= f(\pi i) = e^{(3\pi) i}$$

$$= \cos(3\pi) + i \sin(3\pi)$$

$$= -1$$

$$\therefore \oint \frac{e^{3z}}{z-\pi i}\,dz = -2\pi i$$

(Ans)

7(b)

$$|z-2| + |z+2| = 6$$

now, $|\pi i - 2| + |\pi i + 2|$

$$= \sqrt{\pi^2 + (-2)^2} + \sqrt{\pi^2 + 2^2}$$

$$= 2\sqrt{\pi^2+4}$$

$$= 7.448 > 6$$

$\therefore$ its outside the eclipse.

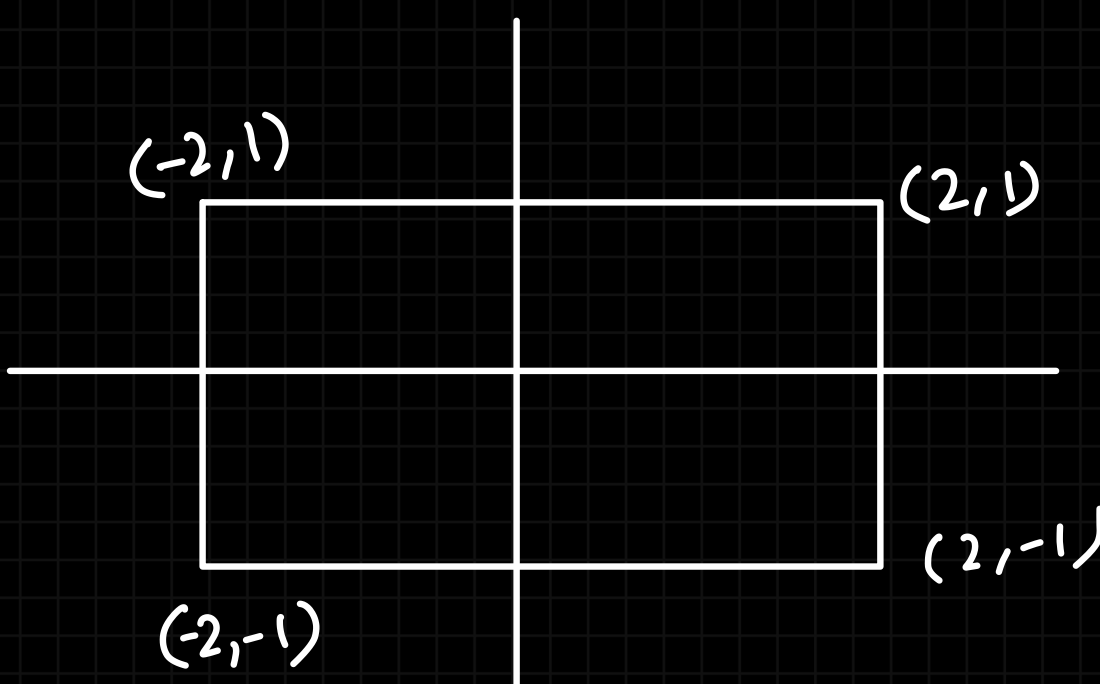
so $\oint_C \frac{e^{3z}}{z - \pi i} \, dz = 0$ (Ans)

Answer to the question no-8

8(a)

a) $\frac{1}{2\pi i}\oint_{C} \frac{\cos \pi z^2}{z^2-1}\, dz$

the vertices: $2+i,\ 2-i,\ -2+i,\ -2-i$



now,

$$\oint \frac{\cos \pi z^2}{(z+1)(z-1)} \, dz$$

$$\operatorname{Res}(f, z=-1) = \lim_{z \to -1} \left\{ (z+1) \frac{\cos \pi z^2}{(z+1)(z-1)} \right\}$$

$$= \frac{\cos \pi}{-2}$$

$$= \frac{-1}{-2}$$

$$= \frac{1}{2}$$

$$\operatorname{Res}(f, z=1) = \lim_{z \to 1} \left\{ \frac{\cos \pi z^2}{(z+1)} \right\}$$

$$= \frac{\cos \pi}{2}$$

$$= \frac{-1}{2}$$

$$\therefore \oint_C \frac{\cos \pi z^2}{z^2-1} dz = 2\pi i \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

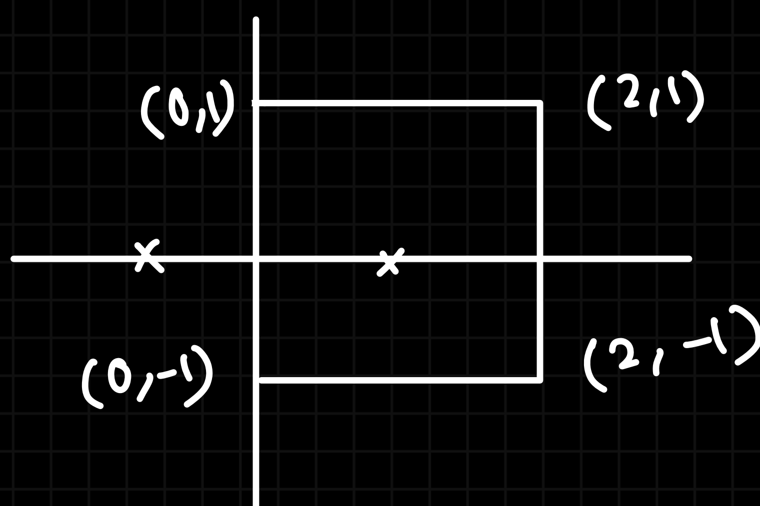
$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{\cos \pi z^2}{z^2-1} dz = \frac{1}{2\pi i} \times 0$$

$$= 0 \quad \text{(Ans)}$$

8(b)

$$\frac{1}{2\pi i} \oint \frac{\cos \pi z^2}{z^2 - 1} \, dz$$

$\pm i, 2 \pm i$



now,

$$\oint_C \frac{\cos \pi z^2}{(z+1)(z-1)}\, dz$$

$$= \oint_C \frac{\frac{\cos \pi z^2}{(z+1)}}{(z-1)}\, dz$$

$$\text{Res}(f, z=1) = \lim_{z \to 1} \left\{ (z-1) \frac{\frac{\cos \pi z^2}{(z+1)}}{(z-1)} \right\}$$

$$= \frac{\cos \pi}{2}$$

$$= \frac{-1}{2}$$

$$\therefore \oint_C \frac{\cos \pi z^2}{(z+1)(z-1)}\, dz = 2\pi i \times \frac{-1}{2}$$

$$= -\pi i$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{\cos \pi z^2}{(z+1)(z-1)}\, dz = \frac{1}{2\pi i} \times (-\pi i)$$

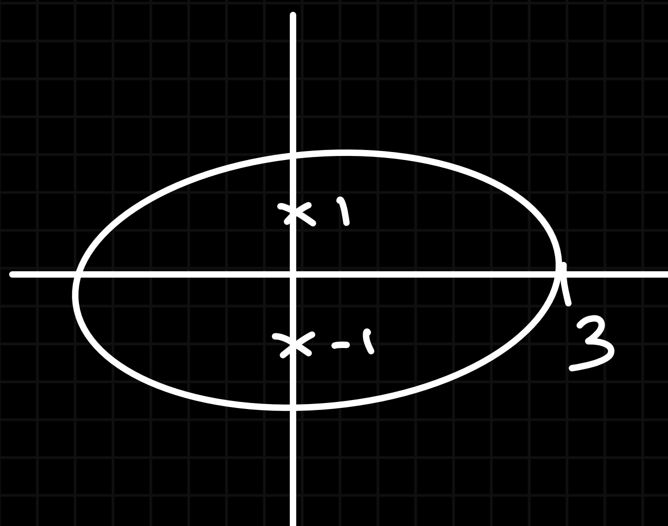
$$= -\frac{1}{2}$$

(Ans)

Answer to the question no. 9

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz$$

$c: |z| = 3$



now,

$$\oint_C \frac{e^{zt}}{z^2-i^2}\,dz$$

$$= \oint_C \frac{e^{zt}}{(z+i)(z-i)}\, dz$$

now,

$$\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$\Rightarrow 1 = A(z-i) + B(z+i)$$

let $z = i$

$$\therefore \; 1 = B \times 2i$$

$$\therefore B = \frac{1}{2i}$$

let $z = -i$

$$\therefore \; 1 = A(-2i)$$

$$A = \frac{-1}{2i}$$

$$\therefore \; \frac{1}{(z+i)(z-i)} = \frac{-1/2i}{(z+i)} + \frac{1/2i}{(z-i)}$$

$$\Rightarrow \frac{1}{2\pi i} \oint \frac{e^{zt}\,dz}{(z+i)(z-i)}$$

$$= \frac{1}{2\pi i}\left[-\frac{1}{2i}\oint \frac{e^{zt}}{z+i}\,dz + \frac{1}{2i}\oint \frac{e^{zt}\,dz}{z-i}\right]$$

now,

$$\frac{1}{2\pi i}\oint \frac{e^{zt}}{z+i}\,dz$$

$$= f(-i) = e^{-it}$$

$$= \cos t - i\sin t$$

$$\therefore \frac{1}{2\pi i}\oint \frac{e^{zt}}{(z+i)(z-i)}\,dz$$

$$= \frac{1}{2i}\left[-\frac{1}{2\pi i}\oint \frac{e^{zt}\,dz}{z+i} + \frac{1}{2\pi i}\oint \frac{e^{zt}\,dz}{z-i}\right]$$

$$= \frac{1}{2i}\left(-\cos t + i\sin t + \cos t + i\sin t\right)$$

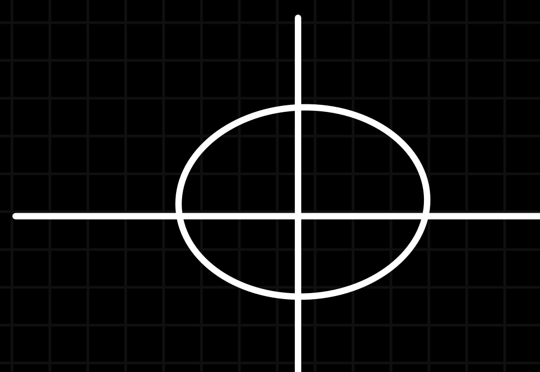
$$= \frac{1}{2i} \times 2i\sin t$$

$$= \sin(t)$$

[shown]

Answer to the question no-10

$$\oint_C \frac{e^{iz}}{z^3}\,dz \qquad c: |z| = 2$$



$$= \oint_C \frac{e^{iz}}{(z-0)^3}\,dz$$

using cauchy integral general formula,

$$\frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}\,dz = f^{n}(a)$$

$$\Rightarrow \frac{2!}{2\pi i} \oint \frac{e^{iz}}{z^3}\, dz$$

$$= f^2(0)$$

$$= \frac{d^2}{dz^2}\left(e^{iz}\right)\Big|_{z=0}$$

$$= \frac{d}{dz}\left(ie^{iz}\right)$$

$$= i^2 e^{iz}$$

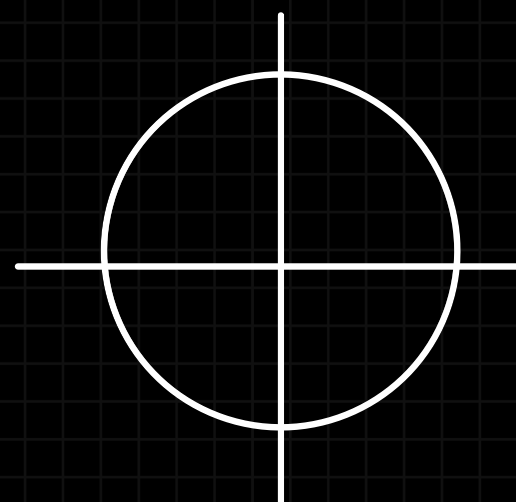
$$= i^2 = -1$$

$$\therefore \oint \frac{e^{iz}}{z^3}\, dz = \pi i \cdot (-1) = -\pi i$$

(Ans)

Answer to the question no-11

$$\oint_c \frac{e^{zt}}{(z^2+1)^2}\, dz \qquad c: |z|=3$$



$$= \oint_c \frac{e^{zt}}{(z^2-i^2)^2}\, dz$$

$$= \oint_c \frac{e^{zt}}{(z+i)^2 (z-i)^2}\, dz$$

$$= \oint_c f(z)\, dz$$

for $z = -i$:

$$\text{Res}(f, z=-i) = \frac{1}{1!} \lim_{z \to -i} \left\{ \frac{d'}{dz} (z+i)^2 f(z) \right\}$$

$$= \lim_{z \to -i} \frac{d}{dz} \left\{ (z+i)^2 \frac{e^{zt}}{(z+i)^2 (z-i)^2} \right\}$$

$$= \lim_{z \to -i} \frac{d}{dz} \left\{ \frac{e^{zt}}{(z-i)^2} \right\}$$

$$= \lim_{z \to -i} \frac{(z-i)^2 \cdot t \cdot e^{zt} - e^{zt} \cdot 2(z-i)}{(z-i)^4}$$

$$= \frac{-4te^{-it} + 4ie^{-it}}{16}$$

$$= \frac{-te^{-it} + ie^{-it}}{4}$$

$$= \frac{e^{-it}(i-t)}{4}$$

$$= \frac{(\cos t - \sin t)(i-t)}{4}$$

for $z = i$:

$$\operatorname{Res}(f, z=i) = \frac{1}{1!} \lim_{z \to i} \left\{ \frac{d}{dz} (z-i)^2 f(z) \right\}$$

$$= \lim_{z \to i} \left\{ \frac{d}{dz} \left(\frac{e^{zt}}{(z+i)^2} \right) \right\}$$

$$= \lim_{z \to i} \frac{(z+i)^2 t e^{zt} - e^{zt} 2 \cdot (z+i)}{(z+i)^4}$$

$$= \frac{-4 t e^{it} - 4 i e^{it}}{16}$$

$$= \frac{-t e^{it} - i e^{it}}{4}$$

$$= \frac{e^{it}(-i-t)}{4}$$

$$= \frac{(\cos t + i \sin t)(-i-t)}{4}$$

$$\therefore \oint_C f(z)\, dz = 2\pi i \left\{ \frac{(\cos t - i \sin t)(i-t)}{4} + \frac{(\cos t + i \sin t)(-i-t)}{4} \right\}$$

$$= 2\pi i \left\{ \frac{\begin{array}{r} i\cos t - t\cos t + \sin t + it\sin t - \\ - i\cos t - t\cos t + \sin t - it\sin t \end{array}}{4} \right\}$$

$$= \frac{2\pi i}{4} \times (2\sin t - 2t\cos t)$$

$$= \pi i (\sin t - t\cos t)$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{e^{zt}\,dz}{(z^2+1)^2} = \frac{1}{2\pi i} \times \pi i (\sin t - t\cos t)$$

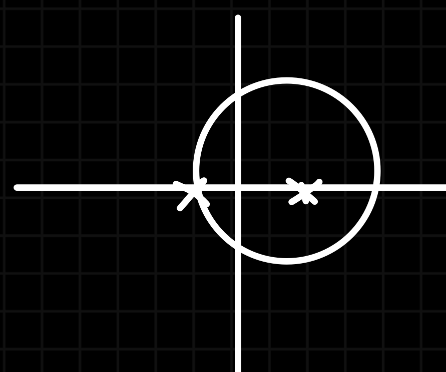
$$= \frac{1}{2}(\sin t - t\cos t)$$

(Ans)

Answer to the question no-12

$$\oint_C \frac{e^z}{(z^2-1)^2}\,dz$$

$C : |z-1| = 2$



$$= \oint_C \frac{e^z}{(z+1)^2(z-1)^2}\,dz$$

$$\therefore f(z) = \frac{e^z}{(z+1)^2(z-1)^2}$$

now

for $z=-1$,

$$\operatorname{Res}(f, z=-1) = \frac{1}{1!} \lim_{z\to -1} \frac{d}{dz}\left\{(z+1)^2 f(z)\right\}$$

$$= \lim_{z\to -1} \frac{d}{dz}\left\{\frac{e^z}{(z-1)^2}\right\}$$

$$= \lim_{z\to -1} \frac{(z-1)^2 e^z - e^z \cdot 2(z-1)}{(z-1)^4}$$

$$= \frac{4e^{-1} + 4e^{-1}}{16}$$

$$= \frac{e^{-1}}{2}$$

<u>for $z=1$:</u>

$$\text{Res}(f, z=1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left\{ \frac{e^z}{(z+1)^2} \right\}$$

$$= \lim_{z \to 1} \frac{(z+1)^2 e^z - e^z \cdot 2(z+1)}{(z+1)^4}$$

$$= \frac{4e^{-1} - 4e^{-1}}{16}$$

$= 0$

$$\therefore \oint_C \frac{e^z}{(z^2-1)^2}\,dz = 2\pi i\left(\frac{e^{-1}}{2}+0\right)$$

$$= e^{-1}\pi i$$

$$= \frac{\pi i}{e}$$

(Ans)