CSE330: Numerical Methods

Topic: Solving Non-Linear Equations

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Chapter - 4

Root finding of non-linear equations:

how we do it: $f(x) = x^2 - 2x = 0$

$$x(x-2) = 0$$

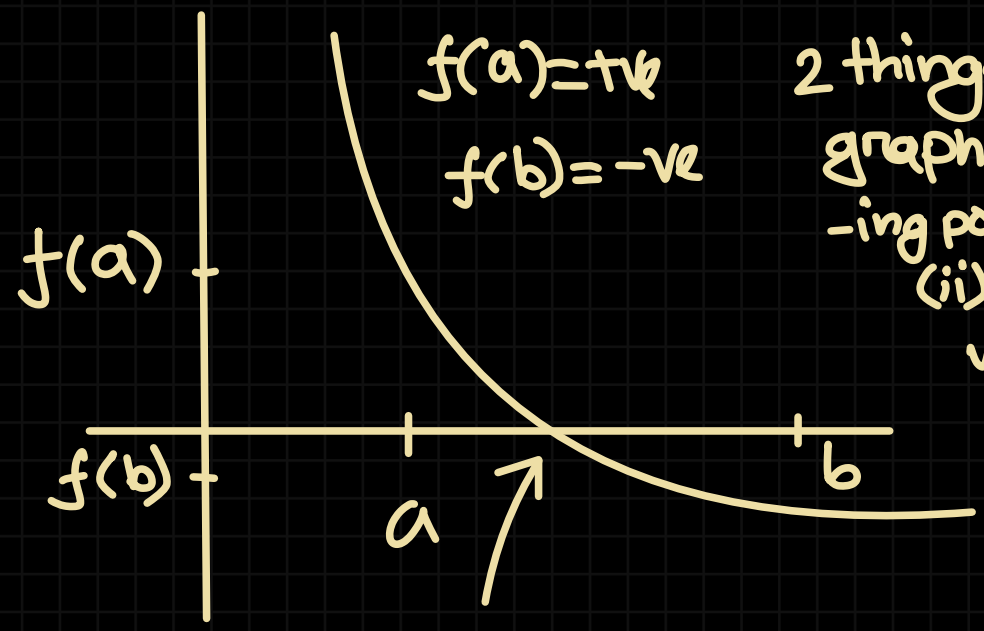
$$x = 0, \quad x = 2$$

$f(x) = 0 \Rightarrow$ root

How computers does it:

Bisection Root Finding Algorithm

bisection = equal two ↳ parts
↳ divide করা



root exists within [a,b]

$f(x)=0$

fig: graph 1

2 things to notice on the graph: (i) intersect -ing point ↳ $f(x)=0$
(ii) x ↳ specific value যে a এবং b এর মধ্যে

তাও mathemat -ical proof করা।

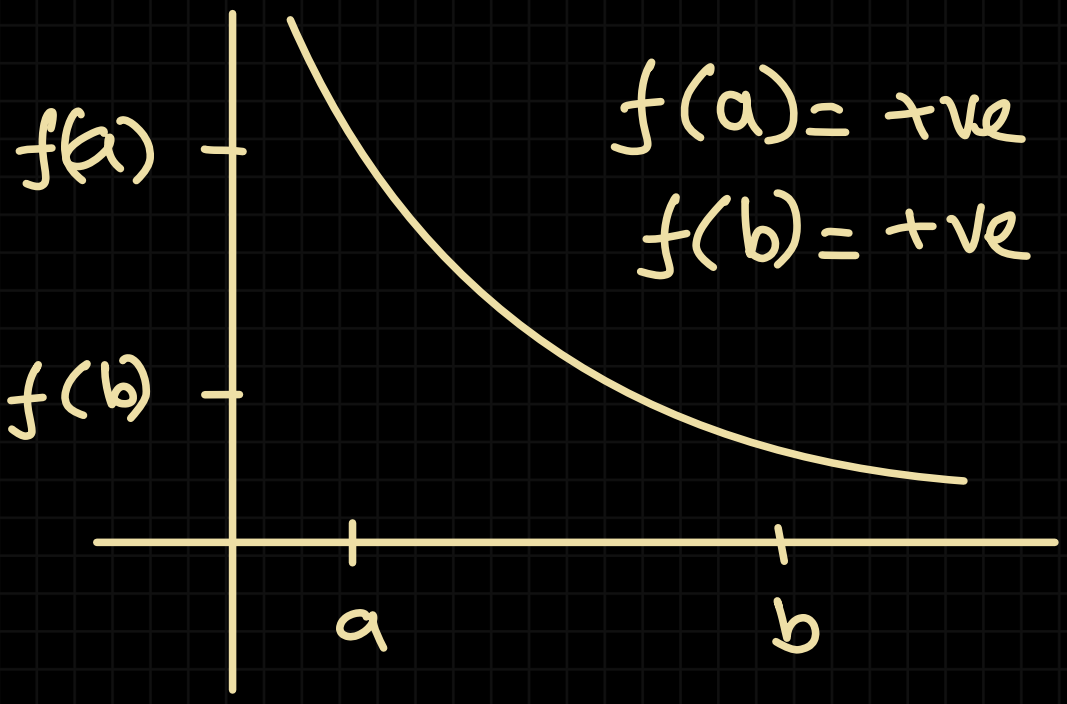


fig: Graph 2

no root within [a,b] range

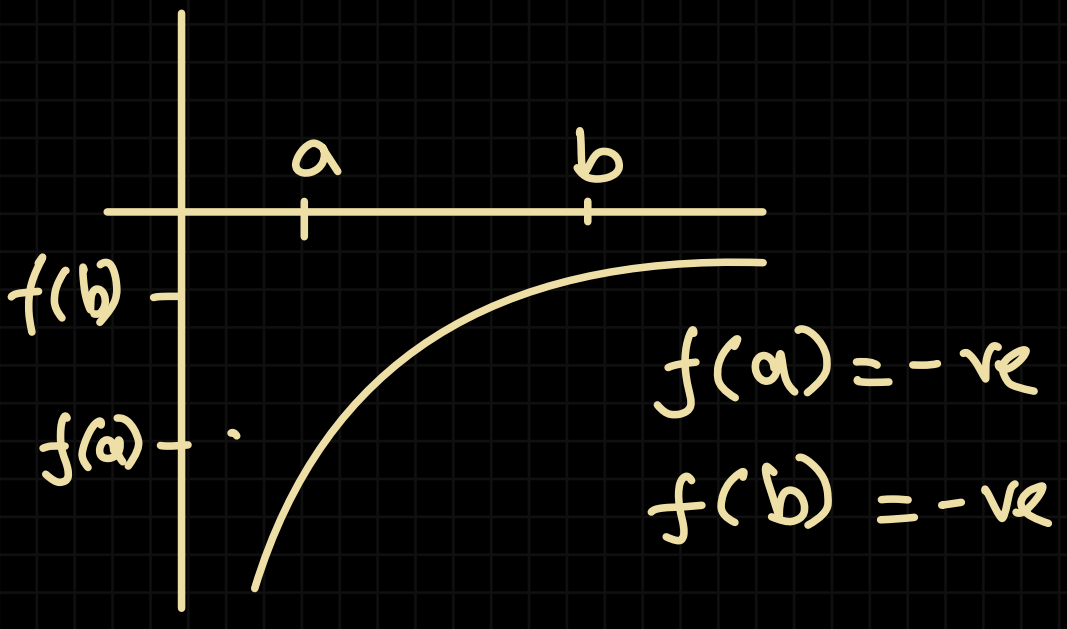


fig: graph 3

no root within [a,b] range

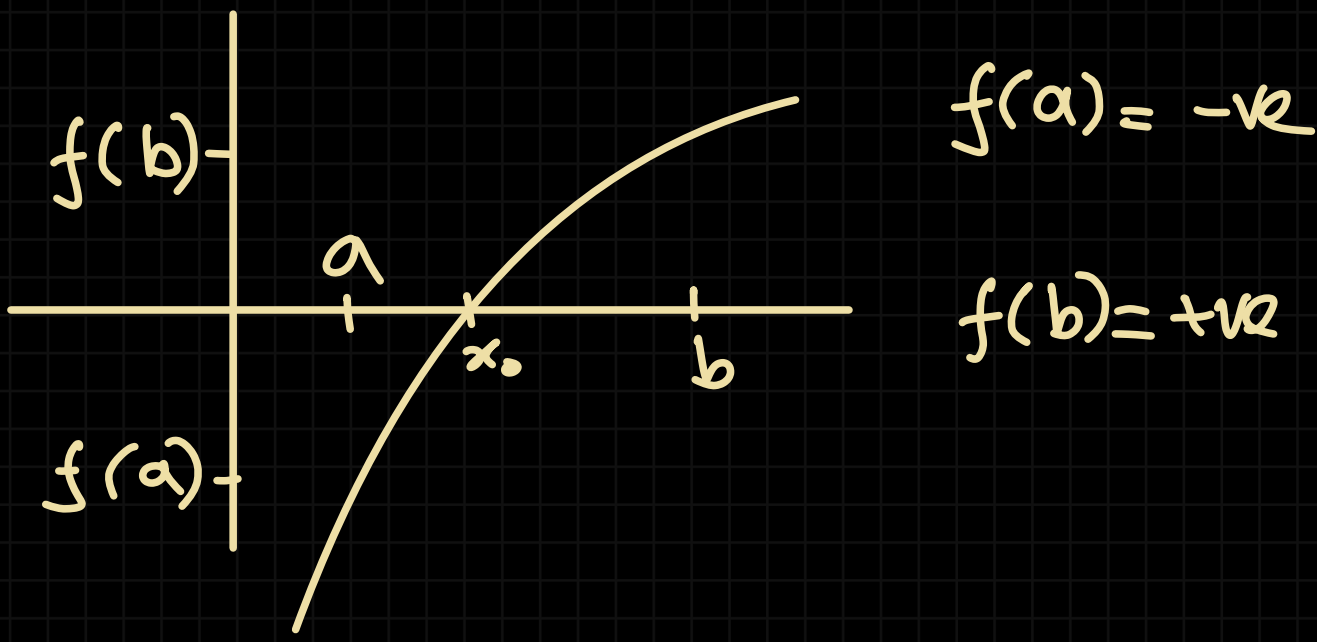


fig: Graph 4

conclusion:

if $f(a) \times f(b) < 0$,

root exists within [a,b] range

reason: either one has to be negative and other one has to be positive

$f(a) \times f(b) = 0$

↳ তাহলে f(b) root হলে
b is a root

here

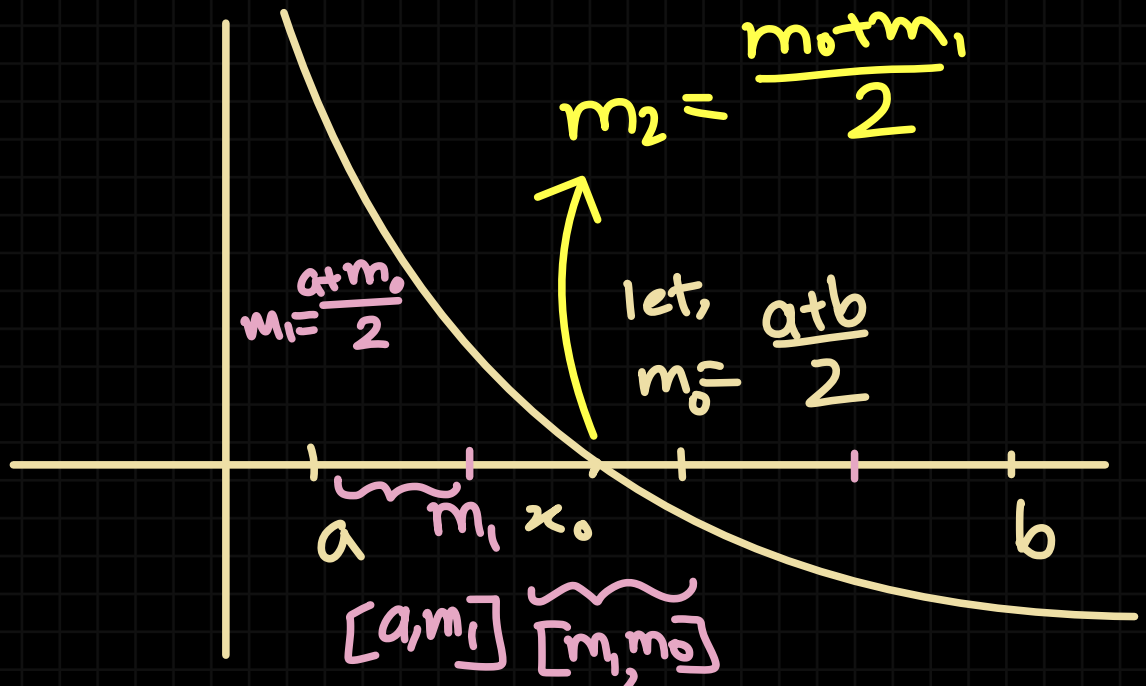
$f(a)=0$ হলে a is a root

first ↳ finding out if, within the range, the root exists at all.

Bisection Method

☆ range a,b এর value থাকতে হবে।
without it bisection cannot be done

☆ this method is similar to binary search



→ m এর জন্য 2 interval কে divide:
[a,m] এবং [b,m]

→ now we check on which side of m_1 we have x_0 (the root). so

if $f(a) \times f(m_1) < 0$

then root is on $[a, m_1]$ side

(otherwise its on the $[m_1, b]$ side)

→ and we keep diving into two parts (iteratively) until we get

$$\boxed{f(m) = 0}$$

there $f(m_2) = 0$, so m_2 is the root

solve can be found in two ways:

(1) with table (2) without table

better is with table

without using table =>

Que $f(x) = \frac{1}{x} - 0.5$ # [1.5, 3]

২টা iteration দেখাও।

question solve এর আগে,

notice if the question is asking for,

→ exact root

or

→ approximate root

or

→ n number of iteration
(root কে কত বলেছে)

soln:

$f(a_0) = 0.166$

$f(b_0) = -0.166$

$a_0 = 1.5$

$b_0 = 3$

since they have opposite signs, root exists within the range

$$m_0 = \frac{a_0 + b_0}{2}$$

$$= 2.25$$

$\therefore f(m_0) = -0.555$

$\because f(m_0) \neq 0$, m_0 is not root. so further iteration

$$f(a_0) \times f(m_0) < 0$$

$\therefore$ root lies between $[a_0, m_0]$

now, $a_1 = a_0 = 1.5$

$b_1 = m_0 = 2.25$

$$\therefore m_1 = \frac{a_1 + b_1}{2} = 1.875$$

$\therefore f(m_1) = 0.033$ ✓

$f(a_1) = 0.166$ ✗✗

$f(b_1) = -0.555$ ✓

$$f(a_1) \times f(m_1) > 0$$

$\therefore$ root is within $[m_1, b_1]$

আরেকটা same জিনিসটাকেই আমাদের বলবো,

$a_0 = 1.5$ $b_0 = 3$ $m_0 = \frac{a_0 + b_0}{2}$ $= 2.25$	$f(a_0) = 0.166$ $f(b_0) = -0.166$ $f(m_0) = -0.055$	$f(a_0) \times f(m_0) < 0$ ∴ root is within $[a_0, m_0]$
$a_1 = a_0 = 1.5$ $b_1 = m_0 = 2.25$ $m_1 = \frac{a_1 + b_1}{2}$ $= 1.875$	$f(a_1) = 0.166$ $f(b_1) = -0.055$ $f(m_1) = 0.033$	$f(a_1) \times f(m_1) > 0$ ∴ root is within $[m_1, b_0]$

[notice কর, $abs(f(m_1)) < abs(f(m_0))$, মানে যত divide, conquer করতে থাকলাম result ($f(m) = 0$) এর তত কাছে চলে যাচ্ছি।] so root এর দিকে এই আগানো is called **Convergence**
root এর বিপরীতে যাওয়া is called **Divergence**

track রাখার সুবিধার্থে we use tables instead (math process same tho)

Que $f(x) = x^3 - 7x^2 + 14x - 6 = 0$

$[1, 3.2]$ range ও accuracy 10^{-3} করে

accurate within 10^{-3}

same as saying → machine epsilon / error bound / accuracy

soln:

$\epsilon_m = 10^{-3}$

interval number / ลำดับ ของ iteration

k	a_k	m_k	b_k	$f(a_k)$	$f(m_k)$	$f(b_k)$	[ ]	$\lvert f(m)\rvert \leq 10^{-3}$
0	1	2.1	3.2	2	1.709	-0.11	[2.1, 3.2]	NO
1	2.1	2.65	3.2	1.709	0.55	-0.11	[2.65, 3.2]	NO
2	2.65	2.925	3.2	0.55	0.086	-0.11	[2.925, 3.0625]	NO
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9	2.999 8	3.000 105	3.002	1.96×10^{-3}	-1.95×10^{-4}	-2.3×10^{-3}		Yes

$x_* = 3.000\,105$

4 significant bit ~~ตัดทิ้ง~~, $x_* = 3.000$

Que type-2 => number of iteration বের করা।

যেগুলা থাকবে

→ ϵ_m

→ interval

number of iteration,

$$n \geq \frac{\log(|b-a|) - \log(\epsilon)}{\log 2} - 1$$

let $\epsilon_m = 1.1 \times 10^{-16}$, $a = 1.5$, $b = 3$

$$\therefore n \geq \frac{\log(|3-1.5|) - \log(1.1 \times 10^{-16})}{\log 2} - 1$$

$$n \geq 52.59$$

$$\Rightarrow n \geq 53$$

So, actual number of iteration can be 55, 57 or any number greater than 53.

n is the minimum number of iterations required