5.1.13. Prove that $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} n \frac{(n+1)}{2}$ whenever n is a positive integer.

Basis step: $n=1$ $\quad 1^2 = (-1)^{1-1} \cdot 1 \cdot \frac{(1+1)}{2}$

$1 = 1 \Rightarrow$ true

Inductive step: The inductive hypothesis is the statement that $P(k)$ is true, that is, $1^2 - 2^2 + 3^2 - \ldots + (-1)^{k-1} k^2 = (-1)^{k-1} \cdot k \frac{k+1}{2}$, where k is an arbitrary nonnegative integer.
We must show that if $P(k)$ is true, then $P(k+1)$ is also true. So assuming the inductive hypothesis, it follows that

$$P(k+1) = \underbrace{1^2 - 2^2 + 3^2 - \ldots + (-1)^{k-1} k^2}_{(-1)^{k-1} \cdot k \frac{k+1}{2}} + (-1)^k (k+1)^2 = (-1)^k \frac{(k+1)(k+2)}{2}$$

$$(-1)^{k-1} \cdot k \cdot \frac{k+1}{2} + (-1)^k (k+1)^2 = (-1)^k \cdot \left(\frac{-1}{1}\right) \cdot k \cdot \frac{(k+1)}{2} + (-1)^k (k+1)^2 =$$

$$= (-1)^k (k+1) \left(-\frac{k}{2} + k + 1\right) = (-1)^k (k+1) \left(\frac{k}{2} + 1\right) =$$

$$= \underbrace{(-1)^k \frac{(k+1)(k+2)}{2}}_{\text{the right side of } P(k+1)},$$

19. Let $P(n)$ be the statement that
$$1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{n^2} < 2-\frac{1}{n}$$, where n is an int. greater than 1.

a) What is the statement $P(2)$?

$$P(2) = 1+\frac{1}{4} < 2-\frac{1}{2}$$

b) Show that $P(2)$ is true, completing the basis step of the proof.

$$\frac{5}{4} < \frac{3}{2} \iff \frac{5}{4} < \frac{6}{4} \Rightarrow \text{true}$$

c) What is the inductive hypthesis?

The inductive hypothesis is the statement $P(k)$ is true, that is,

$$1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{k^2} < 2-\frac{1}{k}.$$

d) What do you need to prove in the inductive step?

We need to prove that if $P(k)$ is true, then $P(k+1)$ is also true for each $k \geq 2$.

So assuming the inductive hypothesis, it follows that

$$1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k+1}$$

e) Complete the inductive step.

$$1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

The right side of $P(k+1)$: $2-\frac{1}{k}+\frac{1}{(k+1)^2} = 2-\frac{(k+1)^2-k}{k(k+1)^2} =$

$$= 2 - \left(\frac{k^2+2k+1 - k}{k(k+1)^2} \right) = 2 - \left(\frac{k^2+k+1}{k(k+1)^2} \right) =$$

$$= 2 - \frac{k(k+1)}{k(k+1)^2} - \frac{1}{k(k+1)^2} = 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} < 2 - \frac{1}{k+1}$$

23. For which nonnegative integers n is $2n+3 \leq 2^n$? Prove your answer.

$n=1$: $5 \nleq 2$

$n=2$: $7 \nleq 4$

$n=3$: $9 \nleq 8$

$n=4$: $11 \leq 16$ → The basis step

Next assuming the inductive hypothesis that $2n+3 \leq 2^n$, $2(n+1)+3 = 2n+2+3 = \underbrace{2n+3}_{P(n)}+2$, which by the inductive hypothesis $\leq 2n+2$.

But since $n \geq 1 \Rightarrow 2^n + 2^n = 2^{n+1}$.

35. Prove that n^2-1 is divisible by 8 whenever n is an odd positive integer.

Proof:

The odd positive int can be written as $2n-1$.

$P(n) = (2n-1)^2 - 1$

Basis step: $P(1) = (2 \cdot 1 - 1)^2 - 1 = 0$ is divisible by 8 ✓.

Inductive step: The inductive hypothesis is the statement $P(n)$ is true. Then we want to prove $P(n+1)$,

$8 \mid ((2(n+1)-1)^2-1) = (2n+1)^2-1$

Lets look at the difference of these two expressions:

$(2n+1)^2-1-\underbrace{((2n-1)^2-1)}_{\text{true by inductive step}} = 4n^2+4n+1-1-4n^2+4n-1+1 =$

$=8n$ is divisible by 8. $\Rightarrow$ $P(n+1)$ is true.

45. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Proof:

Basis step: $n=2$: $2\frac{(2-1)}{2}=1$.

Inductive step: Assuming the inductive hypothesis, that a set with n elements has $\frac{n(n-1)}{2}$ subsets with exactly two elements. We want to prove that a set S with $n+1$ elements has $\frac{(n+1)n}{2}$ subsets with exactly two elements. Fix an element a is S, and let T be the set of elements of S other than a: Two varieties of subsets of S:

1. do not contain a: $\frac{n(n-1)}{2}$

2. contain a together with one element of T. Since T has n elements, there are exactly n subsets of this type. The total number of subsets:

$$\left(\frac{n(n-1)}{2}\right)+n=\frac{n(n-1)+2n}{2}=\frac{n^2-n+2n}{2}=\frac{n^2+n}{2}=$$

$$=\frac{n(n+1)}{2}, \text{ as desired.}$$

69. Find $G(1), G(2), G(3), G(4)$:

$G(1) = 0$

$G(2) = 1$

$G(3) = 3$ $(1 \to 2;\ 1 \to 3;\ 3 \to 2)$

$G(4) = 4$ $(1 \xrightarrow{S_1 S_2} 2;\ 3 \xrightarrow{S_3 S_4} 4;\ 1 \to 3,\ 2 \to 4)$.