5.3 Recursive definitions and Structural Induction

13. Prove that $f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$ when n is a positive integer.

Mathematical induction:

Basis step: $n=1$: $f_1 = f_{2\cdot 1} \rightarrow$ true, $f_1 = f_2 = 1$

Inductive hypothesis: $f_1 + f_3 + \ldots + f_{2n-1} + f_{2n+1} = f_{2n+2}$

$$\underbrace{f_1 + f_3 + \ldots + f_{2n-1}} + f_{2n+1} = f_{2n} + f_{2n+1}$$
$$= f_{2n+2}$$

25. Give a recursive definition of

a) the set of even integers

Basis step: $0 \in S$

Recursive step: If $x \in S$ then $x+2 \in S$ and $x-2 \in S$

b) the set of positive integers congruent to 2 modulo 3

Basis step: $2 \equiv 2 \pmod 3$; $2 \in S$

Recursive step: If $x \in S$ then $x+3 \in S$

c) the set of positive integers not divisible by 5

Basis step: 1, 2, 3, 4 modulo 5; $1 \in S, 2 \in S, 3 \in S, 4 \in S$

Recursive step: If $x \in S$ then $x+5 \in S$

33a) Give a recursive definition of the function $m(s)$, which equals the smallest digit in a nonempty string of decimal digit.

b) Use structural induction to prove that $m(st) = \min(m(s), m(t))$.

a) Basis step: string length $= 1$: If $x \in D$, then $m(x) = x$.
Recursive step: string $s = tx$, where $t \in D^*$ and $x \in D$, then $m(s) = \min(m(s), x)$.

b) Recall def. of concatenation. Let $t = wx$, where $w \in D^+$ and $x \in D$. If $w = \lambda$, then $m(st) = m(sx) = \min(m(s), x) = \min(m(s), m(x))$ by the recursive step and the basis step of the definition of m in par (a).

$m(st) = m((sw)x) = \min(m(sw), x)$ | part (a)

$m(sw) = \min(m(s), m(w))$ | by inductive hypothesis

$m(st) = \min(\min(m(s), m(w)), x) = \min(m(s), \min(m(w), x))$

$\min(m(w), x) = m(wx) = m(t)$

$m(st) = \min(m(s), m(t))$.

39. When does a string belong to the set A of bit strings defined recursively by

$$\lambda \in A$$
$$0x1 \in A \text{ if } x \in A$$

where λ is the empty string?

$A = \{0^n 1^n \mid n \geq 0\}$

43. Use structural induction to show that $n(T) \geq 2h(T)+1$, where T is a full ~~is a full~~ binary tree, $n(T)$ equals the number of vertices of T, and $h(T)$ is the height of T.

Basis step: only the root: $n(T)=1$, $h(T)=0$

$$1 \geq 2 \cdot 0 + 1$$

Inductive hypothesis: $n(T_1) \geq 2h(T_1)+1$ and
$n(T_2) \geq 2h(T_2)+1$,
where T_1 and T_2 are full binary trees.

$n(T) = 1 + n(T_1) + n(T_2)$ and $h(T) = 1 + \max(h(T_1), h(T_2))$

$n(T) = 1 + n(T_1) + n(T_2) \geq 1 + 2h(T_1) + 1 + 2h(T_2) + 1 \geq$
$\geq 1 + 2\cdot\max(h(T_1), h(T_2)) + 2 \geq 1 + 2(\max(h(T_1), h(T_2)) + 1) =$
$= 1 + 2h(T)$

61. Find the value of $\log^* n$ for these values of n.

a) 2 b) 4 c) 8 d) 16 e) 256 f) 65536 g) 2^{2048}

$\log^{(1)} n = \log n$; $\log^2 n = \log(\log n)$; $\log^{(3)} n = \log(\log(\log n))$

a) $\log^0 2 = 2$; $\log^1 2 = 1$ $\Rightarrow \log^* 2 = 1$

b) $\log^0 4 = 4$; $\log^1 4 = 2$; $\log^2 4 = 1$ $\Rightarrow \log^* 4 = 2$

c) $\log^0 8 = 8$; $\log^1 8 = 3$; $\log^2 8 \approx 1.585$; $\log^3 8 \approx 0.664 \Rightarrow \log^* 8 = 3$

d) $\log^0 16 = 16$; $\log^1 16 = 4$; $\log^2 16 = 2$; $\log^3 16 = 1 \Rightarrow \log^* 16 = 3$

e) $\log^0 256 = 256$; $\log^1 256 = 8$ (from part(c) +3) $\Rightarrow \log^* 256 = 4$

f) $\log^0 65536 = 65536$; $\log^1 65536 = 16$ (from part(d) +3) $\Rightarrow$

$\Rightarrow \log^* 65536 = 4$

g) $\log^0 2^{2048} = 2048$; $\log^1 2^{2048} = 2048$; $\log^2 2^{2048} = 11$;

$\log^3 2^{2048} \approx 3.46$; $\log^4 2^{2048} \approx 1.79$; $\log^5 2^{2048} \approx 0.84$ $\Rightarrow$

$\log^* 2^{2048} = 5$.