5.2. 7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

Answer: all amounts of money greater than or equal 5.

Let $P(n)$: we can form n dollars using just 2 an 5-dollars bills. We want to prove that $P(n)$ is true for all $n \geq 5$.

Basis step: $P(5) = 5 \rightarrow$ true.

Inductive step: Assume the inductive hypothesis, That $P(j)$ is true for all j with $5 \leq j \leq k$, where k is a fixed integer greater than or equal to 6. We want to show that $P(k+1)$ is true. Because $k-1 \geq 5$, we know that $P(k-1)$ is true, that is, that we can form $k-1$ dollars. Add another 2-dollar bill, and we have formed $k+1$ dollars, as desired.

9. Use strong induction to prove that $\sqrt{2}$ is irrational.

Let $P(n)$ be the statement that $\sqrt{2} \neq \frac{n}{b}$ for any positive integer b.

Basis step: $P(1)$: $\sqrt{2} > 1 \geq \frac{1}{b}$ for all positive int b. is true.

Inductive step: Assume that $P(j)$ is true for all $j \leq k$, where k is an arbitrary positive integer; we must prove that $P(k+1)$ is true.

So assume the contrary, that $\sqrt{2} = \frac{k+1}{b}$ for some pos int b.

$(\sqrt{2})^2 = \frac{(k+1)^2}{b^2} \iff 2b^2 = (k+1)^2 \Rightarrow$

$(k+1)^2$ is even, and so $k+1$ is even. Therefore we can write $k+1 = 2t$ for some positive integer t. Subsituting: $2b^2 = 4t^2 \iff$

$b^2 = 2t^2 \Rightarrow$

b^2 is even, and so b is even, so $b = 2s$ for some int s. Then we have: $\sqrt{2} = \frac{(k+1)}{b} = \frac{2t}{2s} = \frac{t}{s}$. But $t \leq k$, so this contradicts the inductive hypothesis, and our proof of the inductive step is complete.

11. There are 4 base cases:

- If $n = 1 = 4 \cdot 0 + 1$, then clearly the first player is doomed, so the second player wins.
- If there are two, three or four matches ($n = 4 \cdot 0 + 2$, $n = 4 \cdot 0 + 3$, or $n = 4 \cdot 1$), then the first player can win by removing all but one match.

Inductive step: Assume that in games with k or fewer matches, the first player can win if $k \equiv 0, 2$ or $3 \pmod 4$ and the second player can win if $k \equiv 1 \pmod 4$. Suppose we have a game with $k+1$ matches, with $k \geq 4$. If $k+1 \equiv 0 \pmod 4$, then the first player can remove three matches, leaving $k-2$ matches for the other player. $k-2 \equiv 1 \pmod 4$. This is a game that the second player can win.

If $k+1 \equiv 2 \pmod 4$, then first player can remove one match, leaving k matches for the other player. Since $k \equiv 1 \pmod 4$, this is a game that the second player can win. And if $k+1 \equiv 3 \pmod 4$, then the first player can ~~win~~ remove two matches, leaving $k-1$ matches.
$k-1 \equiv 1 \pmod 4$, again the second p. can win.
If $k+1 \equiv 1 \pmod 4$, then the first p. must leave k, $k-1$ or $k-2$ matches for the other p.
Since $k \equiv 0 \pmod 4$, $k-1 \equiv 3 \pmod 4$ and $k-2 \equiv 2 \pmod 4$, this is game the first p. can win. Thus the first p. in our game is doomed, and the proof is complete

25. a) The inductive step allows us conclude that $P(3)$, $P(5)$, ... are true, but we can conclude nothing about $P(2)$, $P(4)$ ....
b) We can conclude $P(n)$ is true for all pos. int. n, using strong induction.
c) The inductive step allows us conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$, ... are all true, but we can conclude nothing about $P(n)$ when n is not a power of 2.
d) This is math. induction: we can conclude $P(n)$ is true for all pos. int. n.

35. Show that if $a_1, a_2, \dots a_n$ are n distinct real numbers, exactly $n-1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted into their product.

Basis step: If $n=1$, 0 multiplications are required.

Inductive step: for all $k<n$, no matter how parentheses are inserted into the product of k numbers, $k-1$ multiplications are required to compute the answer. $(a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_r) \cdot (a_{r+1} \cdot \dots \cdot a_n)$.

By the inductive hypothesis it requires $r-1$ multiplicat. to obtain the first product in parentheses and $n-r-1$ to obtain the second. 1 more m. is needed to multiply these two together.

$(r-1) + (n-r-1) + 1 = n-1$.