4.2 31. Show that a positive integer is divisible by 3 if only if the sum of its decimal digits is divisible by 3.

(1) $a = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$

$$(2)\begin{cases} 10 = 9+1 = 3\cdot 3+1 \\ 100 = 99+1 = 33\cdot 3+1 \\ 1000 = 999+1 = 333\cdot 3+1 \quad \ldots \end{cases}$$

(2) in (1)

$$a = a_n \cdot 10^n + \ldots + a_2 \cdot 100 + a_1 \cdot 10 + a_0 =$$

$$= a_n \cdot (33\ldots \cdot 3\cdot 3+1) + \ldots + a_2 \cdot (33\cdot 3+1) + a_1 \cdot (3\cdot 3+1) + a_0 =$$

$$= 33\ldots 3 \cdot a_n + a_n + \ldots + 33\cdot 3 \cdot a_2 + a_2 + 3\cdot 3\cdot a_1 + a_1 + a_0 =$$

$$= 33\ldots 3 \cdot a_n + \ldots + 33\cdot 3\cdot a_2 + 3\cdot 3\cdot a_1 + (a_n + \ldots + a_2 + a_1 + a_0) =$$

$$= 3\cdot(33\ldots a_n + \ldots + 33\cdot a_2 + 3a_1) +$$

$$+ (a_n + \ldots + a_2 + a_1 + a_0)$$

$$A = (a_n + \ldots + a_2 + a_1 + a_0$$

$$a = \underline{3} \cdot (33\ldots \cdot a_n + \ldots + 33\cdot a_2 + 3a_1) + A$$

Here we clearly see that divisibility by 3 depends on A.

33. Show that a positive integer is divisible by 3 if and only if the difference of the sum of its binary digits is even-numbered positions and the sum of its binary digits in odd-numbered positions is divisible by 3.

$$a = (a_{n-1} a_{n-2} \ldots a_1 a_0)_2$$
$$a = a_0 + 2a_1 + 2^2 a_2 + \ldots + 2^{n-1} a_{n-1}$$

$2^k = 1 \pmod 3$, when k is even
$2^k = -1 \pmod 3$, when k is odd

$$a = a_0 - a_1 + a_2 - a_3 + \ldots \pm a_{n-1} \pmod 3$$

$a \equiv 0 \pmod 3$ if and only if the sum even-numbered digits minus odd-numbered digits congruent to 0 modulo 3.

37. How is the one's complement representation of the sum of two integers obtained from the one's complement representations of these integers?

Assume that n bits are being used, so the range of that numbers is between -2^{n-1} and 2^{n-1}.

To obtain the one's complement representation of the sum of two numbers, we add the two strings representing these numbers using algorithm. After perfoming this operation, there may be a carry out of the left-most column, in such case we then add 1 more to the answer.

43. Answer Ex. 37 for two's complement expansions

To obtain the two's complement representation of the sum of two integers given in two's complement representation, add them as if they were binary integers, and ignore any carry out of the left-most column. However, if the left-most digits of the two addends agree and the left-most digit of the answer is different from their common value, then an overflow has occured, and the answer is not valid.

55. Devise an algorithm that, given the binary expasions of the integers a and b, determines whether $a>b$, $a=b$, or $a<b$.

$a=(a_{n-1}a_{n-2}\ldots a_2a_1a_0)$; $b=(b_{n-1}b_{n-2}\ldots b_2b_1b_0)$

procedure compare (a, b: nonnegative integers)
$i = n-1$
while $i>0$ and $a_i = b_i$
    i --;
if $a_i > b_i$ then answer = "$a>b$"
else if $a_i < b_i$ then aswer = "$a<b$"
else answer = "$a=b$"
return aswer