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Graph Dynamical Systems for Opinion Formation

Group Scotland Hippos Project Report for 851-0101-86L Complex Social Systems: Modeling Agents, Learning, and Games HS2021

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Abstract

It has been found that individuals adapt and revise their opinions based on the influence of close connections (mass media, friends, coworkers) [1]. The goal of this paper is to reveal insights about the effect of different network structures, skill levels and their interaction on the (collective) opinion that gets formed during discussion. The model we present is properly discussed in 2.2. It is a graph dynamical systems which uses an opinion revision equation based on the work on [2]. The network toplologies were selected in order to investigate the opinion formation dynamics in the presence of an pyramidal hierarchy (Teacher), hierarchical decision making with consultancy (Diplomat), symmetrical recurrence (Circle) in a small social network. The emerged patterns and internal dynamics are properly discussed in 4.1. Overall, these simulations demonstrated that independently of the network topologies, a group formed of heterogeneous individuals, the high skilled individuals tend to stick to each other and ignore the opinions from poor skilled individuals. On the other hand, poor skilled individuals will always be strongly influenced by their connections regardless of the skill set.

Contents

1. Introduction

Sociologists and psychologists strongly believe that a crucial component of opinion formation relies in the interactions between individuals in a social network [3]. It has been found that individuals adapt and revise their opinions based on the influence of close connections (mass media, friends, coworkers) [1]. Contrarily to decision-making task, which is a subclass of opinion formation mainly based on the time localization of episodes, opinion formation is a time evolution process [4]. Therefore, it appears that investigating the dynamics within a small social network is a key task for computational sociologists in understanding the dynamics behind opinion formation. This can result in understanding the mechanisms behind phenomena such as political voting [5], spread of misinformation [6], polarization of belief [7]. In the last years, there have been a multitude of opinion formation framework proposed in the literature [8]. In the following section, few studies will be outlined.

1.1 Related Work

The progress of opinion dynamics up to now has been uncoordinated and relied on individual attempts. Social mechanisms considered reasonable turned into mathematical rules, without a general shared framework and often with no reference to real sociological studies. The first opinion dynamics designed by a physicist was a model proposed by [9]. The model is based on the probabilistic framework of sociodynamics. Later on, the Ising model made its first appearance in opinion dynamics [10], [11]. The spin-spin coupling represents the pairwise interaction between agents, the magnetic field represents the cultural majority, or propaganda. Moreover, individual fields are introduced that determine personal preferences toward either orientation. Depending on the strength of the individual fields, the system may reach total consensus toward one of the two possible opinions, or a state in which both opinions coexist.

On the other hand, recently there have been plenty of works on investigating social influence networks and their interaction. For example, Luo et al. proposes a temporal network based on the Voter model [5] with the presence of zealots in order to investigate opinion formation. The study is split into two scenarios of opinion formation, innovation eruption and election [12].Each individual in the Voter model, either sticks with his opinion with probability α, change to opposite opinion with probability γ, or follow a neighbour's opinion with probability β. For the first scenario, it was found that the presence of zealots will maintain the negative opinion, which zealots hold, alive among the population. For the other scenario, temporal network revealed to undermine the effect of the zealots compared with the case of static network.

The magnitude of the effect primarily depends on the connectivity of temporal networks and partly depends on the temporality.

Furthermore, Moussaïd et al. investigate how different participants update their initial answer to factual questions in response to the peer-opinion (and the confidence in their opinion) by conducting two controlled experiments (one without and one with allowing for social influence) [13]. The authors carry out a simulation study for collective dynamics based on an individual-based model of opinion adaptation that they derived from the experiments utilizing methodologies stemming from the social psychology research combined with statistical physics. than the social information they have received [see 14, 15, for previous studies that have demonstrated this effect]. On the other hand, a confirmation bias, the tendency to give more weight to confirming information than to contradicting information [16], can be observed [see also 17, for an earlier study that provided evidence for this effect in collective dynamics on opinion formation]. This finding is also in line with so-called "bounded-confidence" models and the corresponding results from those [18, 19]. A central idea of these models is that the influence of an individual with a completely different opinion than another individual is very small. Moussaïd et al. also show that through repeated collusion with other individuals, confidence in one's own adjusted opinion increases and thus the probability of changing one's opinion on the basis of social influence decreases. However, the confidence in the opinion soon stops to be a good indicator of opinion accuracy after social influence has occurred [see 20, for a more thorough discussion of the influence of the confidence in an opinion on the collective dynamics]. In summary, it can be concluded that the combination of different effects influences group opinion and thus makes the prediction of group opinion difficult. However, there would be an expert effect on the one hand and a majority effect on the other, which can be seen as significant factors influencing group opinion.

The first point relies in the existence of a critical mass of uncertain individuals who happen to share a similar opinion. Indeed, when such a bundle of individuals is initially present in the system, either randomly or because individuals share a common bias, the rest of the crowd tends to converge toward it. This majority effect is typical of conformity experiments that have been conducted in the past [21], where a large number of people sharing the same opinion have a strong social influence on others. The second point is the existence of one or a few highly confident individuals. The root of this expert effect is twofold: First, very confident individuals exert strong persuasive power. Second, unconfident people tend to increase their own confidence after interacting with a very confident person, creating a basin of attraction around that person's opinion [22], [23].

2. Methodology

Now that we have outlined the research field we are dealing with and provided an overview of the relevant research in Chapter 1, in this Chapter we will describe the design of the study conducted in this work. We intend to provide insights into the formation of opinions in small groups considering various group compositions. The formation of opinion refers to how individual opinions or assessments change in the course of a social interaction. On the basis of this observation, we can then draw conclusions about how the collective opinion or assessment develops during a social interaction. Group composition refers on the one hand to the structure of communication possibilities between individual actors (network structure) and on the other hand to the initial skill levels of the individual group members (skill levels).
The presence of a collective intelligence factor within a group discussion was already investigated by Woolley et al. [2]. This factor indicates the social sensitivity (listening,integrating others, revising own ideas) of each individual for reaching a common consensus, rather than the individual persisting on their individual opinion based on the individual skill level. Thus, we would suspect that there is an unknown mechanism within a social network that influences the formation of a common final assessment, idea or opinion (used interchangeably). In our opinion, the design of this mechanism and thus its effectiveness heavily depends on the network structure and the initial skill level of the individuals involved. Furthermore, the motivation of the individuals (benevolent/communicative vs. uncommunicative vs. malicious) can influence the collective formation of opinions.
Therefore, the goal of this paper is to reveal insights about the effect of different network structures, skill levels and their interaction on the (collective) opinion that gets formed during discussion. This involves the identification of optimal social network influence weights for each group composition. Those weights determine the effect of each individual on the opinion formation of another individual and the way the individual opinion is adapted during discussion. Thereupon, we investigate if the groups are capable of finding the optimal social influence network their selves when they are allowed to conduct multiple consecutive discussion rounds and to adapt their initial weights after each discussion round With this purpose in mind, we consider our work as a descriptive study and deliberately limit the various possible network structure design specifications and available skill level definitions as well as number of actors. In the following Sections we will outline the experimental design. This will include a description of the group compositions we consider in this work and a high-level overview of the experimental stages and research goals (see Section 2.1). Thereupon, in Section 2.2 we will describe the methodology for simulating a group discussion and corresponding formation of opinions within a group. In Section 2.3 we define the procedure for identifying optimal the optimal social influence network for different group compositions. Finally, we elaborate in

Section 2.4 on the employed methodology when allowing for the learning of the optimal social influence weight configurations to solve an estimation task. All of the basic methodology we employ in this work is based on the research paper [2] by Woolley et al..

2.1 Experimental setup

In this study we are focusing on the opinion formation in small groups. Thus, all our simulations are run with groups of $N_I = 5$ individuals. We simulate a situation in which these five individuals have to give an initial opinion on an estimation task (without loss of generalization, correct answer is zero). Then there are multiple successive speaking turns ($N_R = 15$), i.e. opportunities for one of the five individuals to express his or her opinion and for each of the other individuals to adjust his or her opinion. The selection of the speaking individual is random and the formation or adaption of the opinion is defined by the model explained in Section 2.2. This procedure is repeated for several times to account for the randomness induced from the selection of the speaking individual. Each of the five individuals will be equipped with predefined skill level, which determines the initial estimate or opinion of the individual. In fact, for each individual at the beginning of each simulation iteration, a random initial estimate is generated based on a normal distribution with mean zero and standard deviation defined by the skill level of the individual ($\mathcal{N}(0, \sigma_i)$, with $\sigma_i = \text{skill\_level}_{p_i}$ and p_i representing the individual with $i \in [1, 5]$). We use two different skill levels, i.e. poor-skilled and high-skilled, which take the values 5 and 1. This means the initial estimate of the poor-skilled individual is likely to deviate far more from the true value than the estimate of the high-skilled individual. We are considering four different skill level compositions:

- Skill set 1: 2-high-2-poor-skilled individuals + 1-high-skilled focus individual (Teacher, Diplomat or arbitrary individual).
- Skill set 2: 2-high-2-poor-skilled individuals + 1-poor-skilled focus individual (Teacher, Diplomat or arbitrary individual).
- Skill set 3: 5-high-skilled individuals.
- Skill set 4: 5-poor-skilled individuals.

While in the work of Woolley et al. only fully-connected weighted directed social networks were considered, we will look at network structures that contain a limitation in the way the group individuals can communicate with each other. Such social networks are particularly interesting because in reality it is often not possible for all group individuals to communicate with each other. There are often intermediaries or social barriers or structures that determine the flow of communication. All of the considered social networks are represented as a directed weighted graph where each nodes represents the individuals within the group discussion and the edges are the weighted influence each nodes exert on each other. Figure 2.1 depicts the three different network structures that we investigate in this work, while the colors indicate the skill level of each individual considering skill set 1. We refer to the three network structures as Teacher, Diplomat and Circle. While the aforementioned Figure only visualizes the network structures under skill set 1, the others can be found in the appendix in Figure 5.1, 5.2 and 5.3.

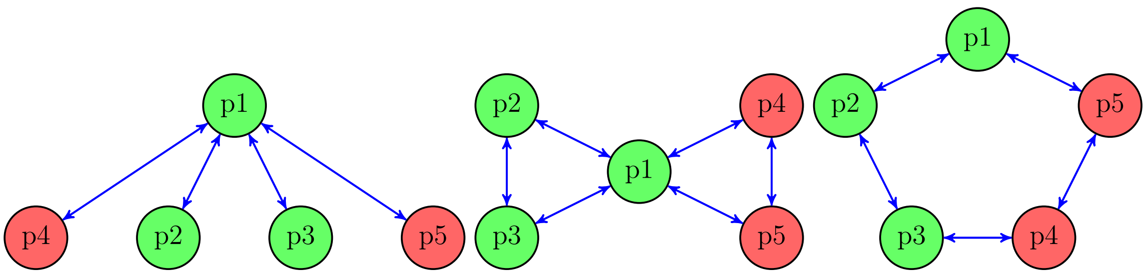

Figure 2.1: From left to right: Teacher, Diplomat and Circle. Skill set 1: 2-high-2-poor-skilled individuals + 1-high-skilled focus individual (Teacher, Diplomat or arbitrary individual).

The network structures in Figure 5.2 allow us to investigate the opinion formation dynamics in the presence of an pyramidal hierarchy (Teacher), hierarchical decision making with consultancy (Diplomat), symmetrical recurrence (Circle) in a small social network.
The first stage of our experiment is to identify the most efficient social influence network for each of the available group compositions, i.e. the one with the lowest group error on the estimation task. The methodology for obtaining those optimal influence networks is described in Section 2.3. The optimal social influence networks and achieved social errors for each group composition will be analyzed and compared in Section 3.1. We will explore the effect of different skill level configurations on the collective opinion formation under certain network structures. Furthermore, we will investigate whether there are certain network structures that are particularly suitable for a certain skill level composition. This would be the case, for example, if despite the same skill level composition, a certain network structure would result in a better collective assessment than another (macro effect). At the same time we can examine the effect of a certain network structure on the opinion formation of individuals with a certain skill level (micro effect). For example, a certain network structure could help individuals with a low skill level to reach the correct assessment more quickly.
The second stage of our experiment is to investigate whether the groups are able to learn the optimal social influence network configuration when they are able to adapt their social influence network weights over multiple estimation tasks. We refer to this as social influence learning and describe the methodology for the weight learning process in Section 2.4 while the results will be presented in Section 3.2. First, we investigate whether the estimation accuracy achieved in the previous stage, i.e. the search for the optimal network configuration, can be achieved or even exceeded. On the other hand, we investigate the effect of the network structure and the skill level as well as the interaction of these on the learning process.

2.2 Model for opinion formation during a simple discussion

As described in Section 2.1 one discussion lasts for multiple successive speaking turns ($N_S = 15$). In each of those N_s speaking turns a randomly selected individual p_i acts as the speaker and expresses her or his estimate x_i. Thereupon, the each of the other individuals adapts her or his previous estimate. The equation that governs the revision of the estimate (opinion) x_j of the individual p_j to an estimate x_i of the speaker p_i during each speaking turn r is given by:

$$x_j^r = x_j^{r-1} + W_{ij}(x_i^r - x_j^{r-1}), \tag{2.1}$$

where W_{ij} corresponds to the social influence that p_i has on p_j. Thus, in each speaking round the estimates of the individuals will only be influenced by the last estimate of the speaker. The estimates of the other individuals other than p_i and p_j will not have an effect on the opinion formation of p_j. This prevents the final opinion to be polarized to the beliefs of the crowd.

2.3 Identification of optimal influence network

In order to find the optimal weights for a social influence network to collectively achieve good performance on an estimation task we have to solve an optimization problem. The goal is to minimize the social error ε by finding the optimal weights $W_{ij} \forall i, j \in [1, 5]$ of the social influence network. The social error is defined by the average estimation error over all individual errors at the end of the discussion. For optimization we employ an exhaustive search over a grid of initial influence network weights, with $W_{ij} \in 0, 0.33, 0.66, 1 \forall i, j \in [1, 5]$. In fact, for each network weight configuration we repeat the simulation of the discussion for multiple times ($N_E = 50$) and average the resulting social error to account for the randomness induced by the selection of the speaking individual in each speaking round. In order to achieve a higher generalizability we then calculate the average over the distinct weight matrices of the social influence networks that yielded the best N_B ($N_B = 30$) social errors. The resulting averaged weight matrix will define the optimal social influence network. This can then be used to simulate another discussion and investigate the estimation results or to compare the optimal weight configurations across various group compositions.

2.4 Model for opinion formation under social influence learning

When simulating a simple discussion the social influence weights stay constant over the speaking turns and only the estimates of the individuals change (see Section 2.2). In order to learn the optimal weight configurations we simulate multiple estimation tasks ($N_T = 50$) and corresponding discussions, each lasting over multiple successive speaking turns ($N_S = 15$). Thereby, we allow the individuals to update their social influence weight after each estimation task. This is an adaptation task where the weights W_{ij} get updated based on the relative performance of the speaker and the social discounting bias α. The social discounting bias is a factor that indicates that individuals tend to downgrade the performance of others compared to their own [see e.g. 24, 25, 26, for research work that provide evidence for the relevance of this factor in group interactions]. The following equation defines the governing process for the opinion formation over multiple speaking rounds within one discussion, which relies on the definition given in Equation (2.1).

$$x_j^{t,r} = x_j^{t,r-1} + W_{ij}^{t-1}(x_i^{t,r} - x_j^{t,r-1}), \tag{2.2}$$

where $x_j^{t,r}$ / $x_i^{t,r}$ defines the estimate of individual p_j / p_i during the estimation task t and the speaking round r. Furthermore, W_{ij}^{t-1} defines the social influence that p_i has on p_j after the estimation task $t-1$ was completed. Equation 2.3 defines governing process of the adaption of the weights of the social influence network after each estimation task and shows how the social influence W_{ij}^{t-1} that p_i has on p_j changes depending on the estimation performance of the preceding estimation task.

$$W_{ij}^{(t)} = \begin{cases} W_{ij}^{t-1} + W^* & \text{if } \varepsilon_j^{t,*} + \alpha < \varepsilon_i^{t,0} \\ W_{ij}^{t-1} - W^* & \text{if } \varepsilon_j^{t,*} + \alpha > \varepsilon_i^{t,0}, \end{cases} \tag{2.3}$$

where W^* is a predefined offset or so-called learning rate (lr = 0.1) that determines by how much the weights are increased or decreased after each discussion. The parameter α defines the social discounting bias ($\alpha \in [0, 2]$) and $\varepsilon_i^{t,0}$ is the error of the individual p_i during the previous estimation task on her or his initial estimate. On the other hand $\varepsilon_i^{t,*}$ is defined as the error on the first communicated estimate of individual p_j during the previous estimation task. That means that when the error of individual p_i on the initial estimate is lower than the sum of the first communicated estimate of individual p_j and the social discounting bias the influence of individual p_i on individual p_j is increased, and decreased otherwise. This could be also interpreted from the perspective of the listening individual p_j, which updates the social influence that others can have on the own opinion formation after each discussion. Thereby, it becomes evident why we are using $\varepsilon_i^{t,*}$ as the individual p_j does not know the initial estimate of the individual p_i but only the first communicated one. The preceding process of adapting the social influence weights is done for every weight in the social influence network and after each estimation task. We are repeating this simulation of the multiple estimation tasks for multiple times ($N_E = 50$) and average the learned optimal weight configurations to account for the randomness induced by the selection of the speaking individual in each speaking turn of the discussions.

3. Results

3.1 Optimal social influence networks

The results obtained in the optimal weight configuration can be visualized in the figures(3.1, 3.2 and 3.3). Each cell W_{ij} of the weight matrix can take values between 0 and 1, and in these plots are represented by a colour range. White indicates not a number (NaN, individuals can't influence themselves), dark colours indicate poor connection (small influence of j on i) and light colours indicate strong connections (high influence of j on i).
Note : all squares in dark purple indicate a value of 0, meaning no connection at all. As expected, that was part of our design of the structures and some individuals are not connected to respect the network structures. In particular : non-neighbours have zero weight in the Circle model.

3.1.1 Network structure - Teacher

We start with the results from the 'Teacher' network structure.

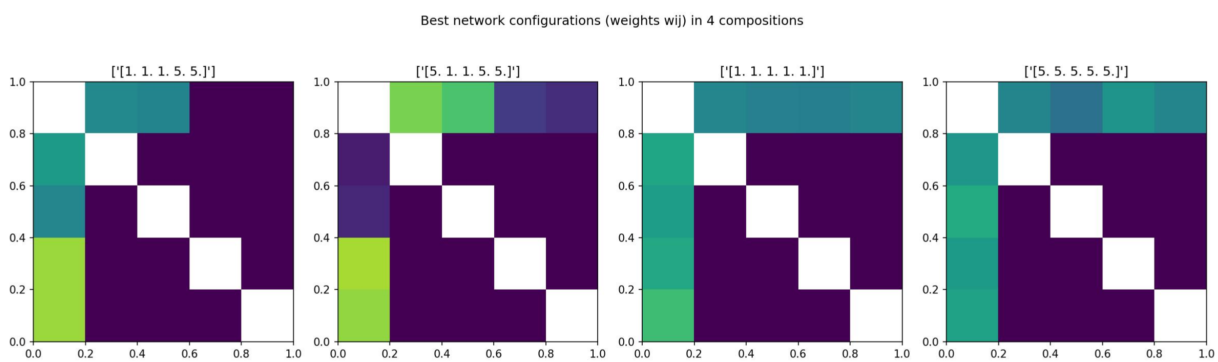

Figure 3.1: Weight matrix W_{ij} for the 4 different skill sets in the Teacher network. The titles correspond to the skillsets σ, where 1 is high-skilled and 5 is poor-skilled, for each of the 5 individuals.

For reference, for interaction and connection is meant how much individual j influences the individual i.
From the first figure on the left, looking at the first row, we can observe that high-skilled

teacher (p1) tends to totally disconnect to poor-skilled individuals (p4 and p5). On the other hand, the poor-skilled individuals want to be establish strong connections with highly skilled individual (p1).

It is interesting to observe, in the second figure on the left, that even if the teacher (p1) continues to reject the opinions coming from (or in this case build a small influence) the poor skilled individuals, these create a strong bond with a poor-skilled teacher. From these simulations, it shows that in a hierarchical configuration, the poor-skilled individuals will always tend to be strongly influenced by the head individual (teacher), no matter his skills. Contrarily, the high skilled individuals will tend to reject the teacher opinion, if the teacher is poorly-skilled. Also, the teacher, being connected to everyone, has a clearer view of each individual performance so it's optimal weights are correct : trust high-skilled students and ignore poor-skilled ones.

When the network has all individuals that share the same skill level, all the connections tend to have an average of strength. (We assume that this scenario will appear in the following network structures, indicating that individuals who all share the same skills tend to average their influence on each other). Note that if we evaluate the exact value of those weights, it seems that the students systematically give more credit to what the teacher says (approx. 0.6 weight in the first column) than the teacher does for the students (approx 0.45 weight in the first row). This means that in the discussion process, the teacher is often in average a little bit more right than each individual student. We can also see that in fig 3.1 by observing that first row is darker for the last two figures.

3.1.2 Network structure - Diplomat

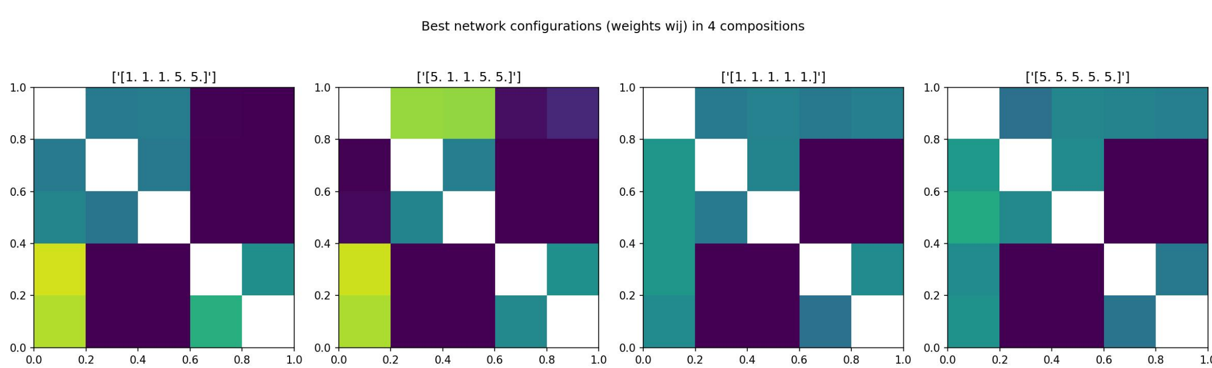

Figure 3.2: Weight matrix W_{ij} for the 4 different skill sets in the Diplomat network. The titles correspond to the skillsets σ, where 1 is high-skilled and 5 is poor-skilled, for each of the 5 individuals.

The strength of interactions of the diplomat (p1) can be seen in the first row. In the case the diplomat is highly-skilled, interactions with all individuals are weak or very weak (first figure from left). It appears that individuals will poorly influence the high skilled diplomat regardless of their skill sets.
A poorly-skilled diplomat however is much strongly influenced by high-skilled individuals (p2 and p3) than with poor-skilled individuals (p4 and p5). Consequently, the poor-skilled individuals are strongly influenced by the high-skilled diplomat. A trend which appears to repeat even in this network is that the poor skilled individuals will always be strongly influenced by the diplomat (p1) regardless of his skillsets and the diplomat is capable of spotting high skilled consultants.
However, in the second figure on the left, it is interesting to notice that high-skilled individuals who rejects the poor-skilled diplomat, will establish strong connections between each other.

3.1.3 Network structure - Circle

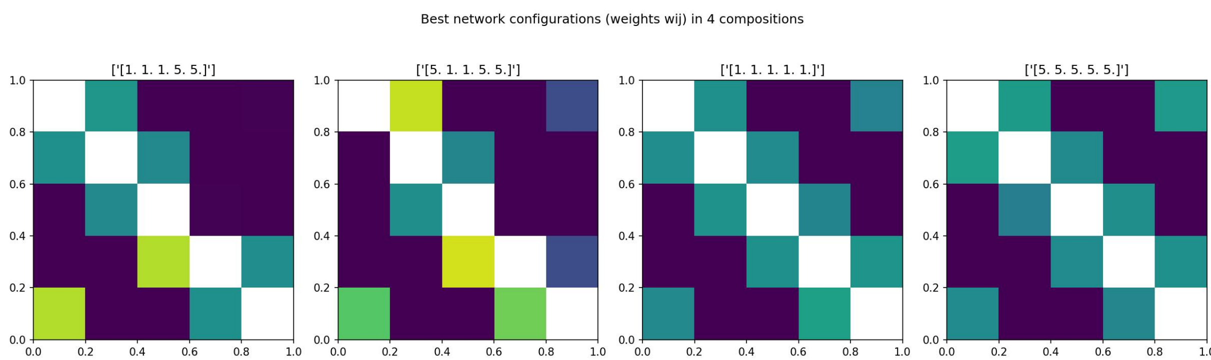

Figure 3.3: Weight matrix W_{ij} for the 4 different skill sets in the Circle network. The titles correspond to the skillsets σ, where 1 is high-skilled and 5 is poor-skilled, for each of the 5 individuals.

Similar results occurred in a circle configuration with high-skilled and poor-skilled individuals connections. As we see in first figure from the left, to form an optimal network, the poor-skilled individuals need to form a strong connection with the high-skilled individuals. This observation can also be made in the second figure from the left, where p1 is in this case poor-skilled and forms an additional strong connection with p2 compared to the first figure. Interestingly, the strength of the bonds p3→p4 strengthen and p5→p1 strengthen even if their skillsets didn't change. This means that the loss of a one high-skilled individual within the newtwork, strengthen the influence of a high-skilled to a poor-skilled individual. If all the individuals have the same skillset, the connection to both neighbours are of the same strength. This observation does not dependent on the actual skillset, meaning if the individuals are poor or high-skilled.

More on the effects on the optimized weights for the three network topologies on speaking discussions can be seen in the appendix 5.2.

3.2 Social influence learning

In this section, we evaluate the adaptation task we mentioned : individuals perform 30 estimation tasks one after another and after each one they update their weights based on the performance of their (connected) peers.

The following figure indicates the impact of the social discounting bias for different skillsets. On the figures the red line indicates the average error of the group as a function of the social discounting bias. The blue dotted line indicates the average error with an optimal weight configuration.

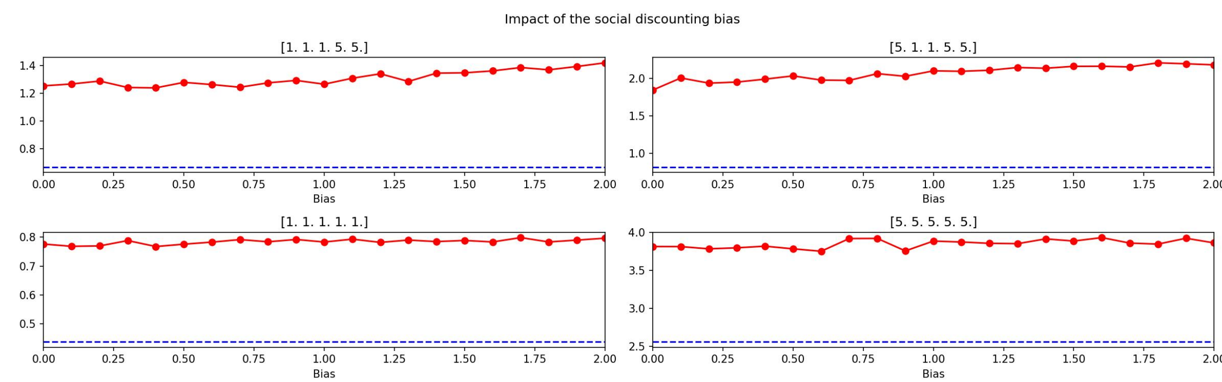

Figure 3.4: Social discounting effect pattern emerged in the teacher network after 30 estimation tasks.

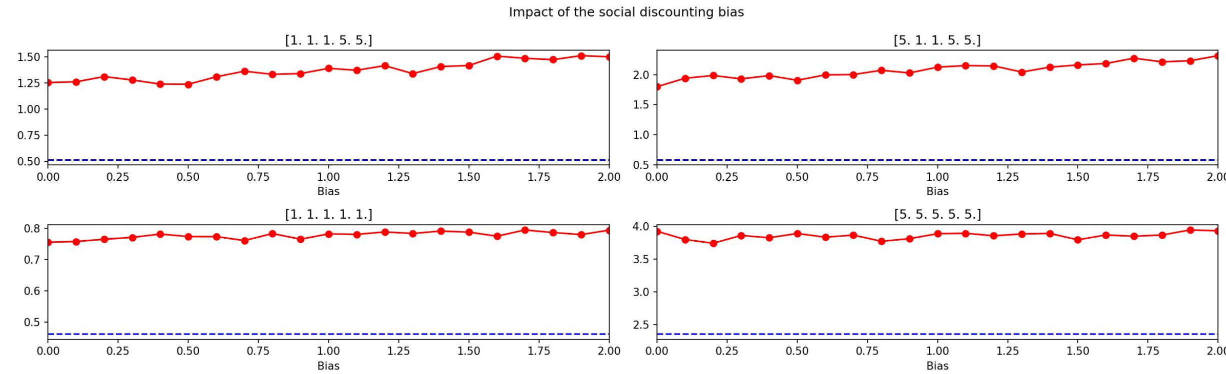

Figure 3.5: Social discounting effect pattern emerged in the diplomat network after 30 estimation tasks.

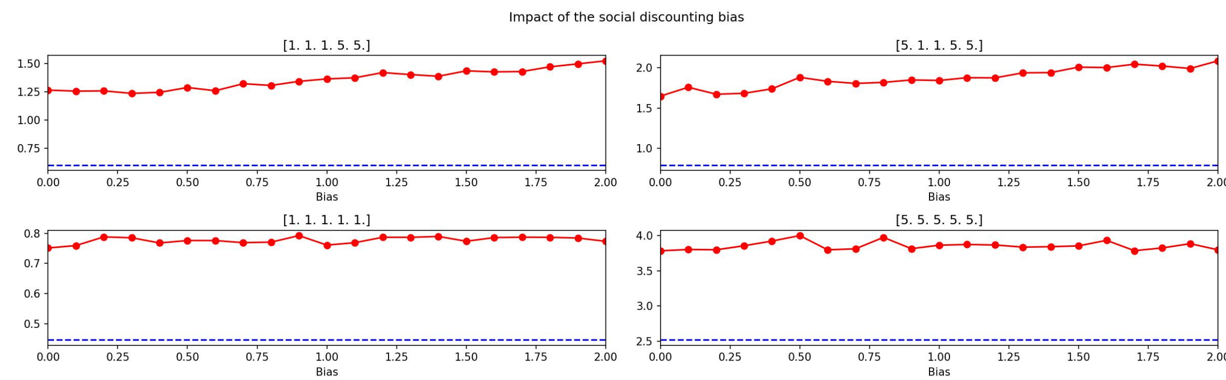

Figure 3.6: Social discounting effect pattern emerged in the circle network after 30 estimation tasks.

From the results, we notice that for every skill set, network configuration or bias value, the average error converges to a value higher than the optimal error found earlier. However, each convergence value is different and are analyzed in section 4.1.

4. Discussion

As mentioned in chapter 2, our work is a continuation of a [2]. We wanted to investigate opinion dynamics in a small influence network with 3 particular networks topologies (Teacher, Diplomat, Circle). This section is dedicated for the comparison of these structures.

4.1 Comparison of network structures

In section 3.1, we obtained the optimal weights that arise from each network structure. We simulated discussions for each skill set in each network configuration. Each time, we kept 2 interesting discussions. The 24 discussions are in section 5.2 in the Appendix. To keep it concise, we'll only compare results across network configurations (for a single skill level each time) as results within a single network configuration have been mentioned and the simulations are only a way to visualize them. Note that at each speaking turn, the speaker is showed by a black circle.

Skill set 1 : [1,1,1,5,5] We notice in fig 5.4 that the Teacher (p1) only updates it's weights based on good students (p1 and p2) whereas all the students trust the teacher : all opinions shrink to the bold blue one whenever he speaks. A similar behavior is visible for the Diplomat in fig 5.8 : good individuals trust the diplomat a little (because he is good) and poor individuals blindly follow his opinion. We observe that both these networks converge to a consensus. On the other hand, in the Circle configuration it is more difficult to converge and some differences are still here after 15 speaking turns. Moreover, the convergence in Circle is less monotonic : some individuals go back and forth on their opinion (fig 5.12) depending on which neighbour they're talking too. This comes from the speaker randomness. Trying to see if the Circle network ever converges, we noticed that it actually does but aroun 20 to 25 speaking turns.

Skill set 2 : [5,1,1,5,5]. Now p1 is poor-skilled and we see a difference between the Teacher and Diplomat networks : in fig 5.5 we notice that the good students just prefer to keep their belief stable : we therefore have 2 "truths" in orange or green and the teacher will converge to one of them (and then the bad students follow). In fig 5.9, this bissection isn't there anymore : the good individuals discuss a consensus and it is spreaded through the DIPLOMAT as soon as he speaks : the convergence is easier. Finally for Circle (fig 5.13), the opinions are again non-monotonic but still end up converging to the two good individuals as they don't update their belief much.

Skill set 3 : [1,1,1,1,1]. In teacher (fig 5.6), it's 4 parallel path that converges to some kind of consensus : each time a student talks, the teacher listens and updates his belief; each time the teacher talks, the students listen and all update their estimate. Overall, in this situation where every individual is good, we just converge to the mean of their initial estimate. A similar observation can be done for Diplomat (fig 5.10) but we can see 2 parallel paths emerge (depending on the initial estimates and speaking order) : this is expected as it simulates two groups of discussion (left and right, diplomat being the link).

Skill set 4 : [5,5,5,5,5]. As the weights are very similar to the one for Skills set 3, the conclusions we draw are also the same. One interesting points though is that now the initial estimates are more spread and therefore the Diplomat network (fig 5.11) is more efficient to find a consensus than the teacher and circle networks (figs 5.7 and 5.15). We believe it is inherently due to the fact that people in the Diplomat network are more interconnected (6 links) vs 4 and 5 links in respectively Teacher and Circle.

Best performing network. In the Social Learning section (3.2) figures, the horizontal blue line gives us the average error when simulating with optimal weights (for each skill set in each network configuration) and the red line gives us the socially learned one for each bias value. As noticed earlier, the red line is approximately constant, we decide to document the error values here :

skillset	1	2	3	4	1	2	3	4
TEACHER	0.65	0.8	0.45	2.6	1.3	2.0	0.78	3.8
DIPLOMAT	0.5	0.6	0.47	2.4	1.35	2.0	0.75	3.8
CIRCLE	0.6	0.8	0.43	2.5	1.4	1.8	0.75	3.75

Average error using : optimal weights (left) or socially learned weights after 30 rounds (right)

Assuming optimal weights, it seems that the Diplomat network outperforms the two others by a small margin. Considering that Circle takes more time to find a consensus, Teacher seems to be a better network too.

However, in real life, people are not aware of the optimal weights and encounter more often the social learning setup. This time, no significant winner emerges from the three network configurations.

4.2 Limitations

Throughout the experiments some limitations were identified, which will be briefly mentioned here.

- The optimization technique (grid search) limited discrete weight parameters. Using grid search, we couldn't analyze more than 4 values for each parameters as the optimization was already taking 24 hours for Diplomat and we had to split it into 10 different nodes (and wait 3 hours) to do the computation. About the technique, we suppose that we can do a better optimization using Gradient descent in a continuous space but we didn't have time to dive into it.

- The number of experimental repetitions.
- Extrapolation to larger networks is difficult to predict and the current networks might not ideally represent what happens in real life network structures. For example, in a classroom, students are actually also influenced by the other students.
- Space of possible skill levels. We only investigated highly and poorly skilled individuals in arbitrarily chosen configurations. Having more than 4 skill sets would have been confusing in term of reporting the results so we tried to choose interesting cases based on these limitations we gave ourselves.

4.3 Future Works

Based on our proposed network, we want to investigate further the presence of a disruptor within the network. This disruptor is simply an individual that keeps his belief constant to an outlier estimate. This indicates that his ground truth is different from the other individuals and he is trying to deviate the individuals to a different common consensus. From this it would be interesting to compare opinion formation with works on the presence of stubborn [27], zealots [12] and extremist persuaders [28]. Some ideas on this question are shared in the appendix, section 5.3.
Furthermore, another aspect that we would like to investigate is speaking ratio. Contrarily to the work done [2] by Woolley et al., who focuses on the effect of speaking order in the final judgment, our interests is in analysing the effect of individuals with more or less speaking rounds in the final opinion formation.

4.4 Conclusion

In conclusion, the goal of this paper is to reveal insights about the effect of different network structures, skill levels and their interaction on the (collective) opinion that gets formed during discussion. The model we present is properly discussed in 2.2. It is a graph dynamical systems which uses an opinion revision equation based on the work on [2]. The emerged patterns and internal dynamics were discussed in 4.1. Overall, these simulations demonstrated that independently of the network topologies, a group formed of heterogeneous individuals, the high skilled individuals tend to stick to each other and ignore the opinions from poorly skilled individuals. On the other hand, poorly skilled individuals will always be strongly influenced by their connections regardless of their connection's skillsets. Also, we showed that in these particular topologies, social learning is sub-optimal and that the topology itself does not affect the final error you obtain, only the skill of each individual does.

5. Appendix

5.1 Network Configurations

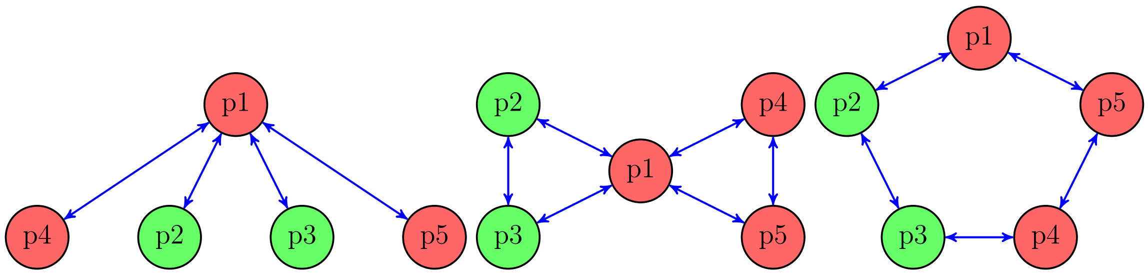

Figure 5.1: From left to right: Teacher, Diplomat and Circle. Skill set 2: 2-high-2-poor-skilled individuals + 1-poor-skilled focus individual (Teacher, Diplomat or arbitrary individual).

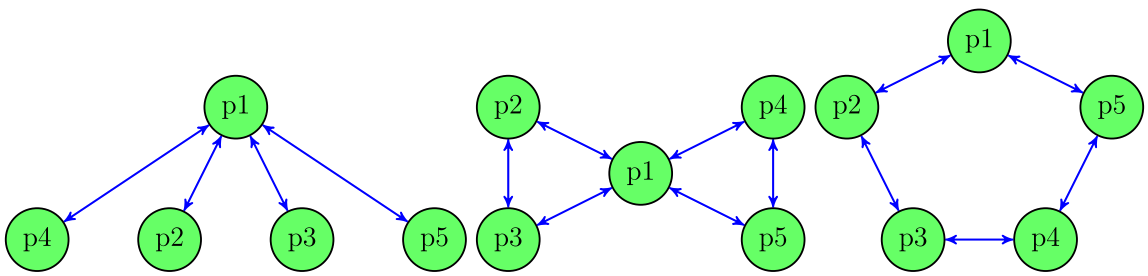

Figure 5.2: From left to right: Teacher, Diplomat and Circle. Skill set 3: 5-high-skilled individuals.

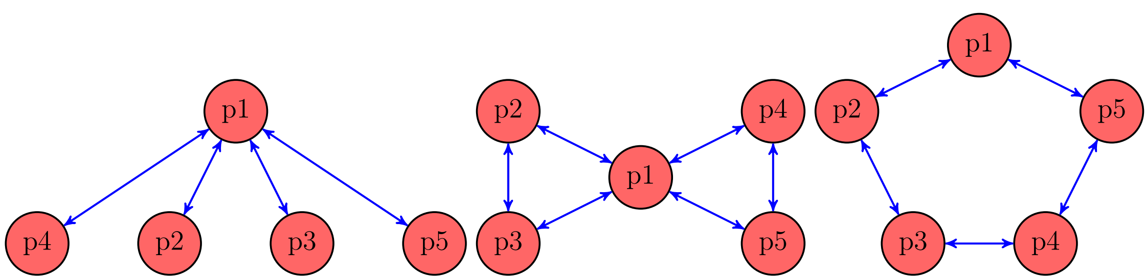

Figure 5.3: From left to right: Teacher, Diplomat and Circle. Skill set 4: 5-poor-skilled individuals.

5.2 Simulated discussions with optimal weights

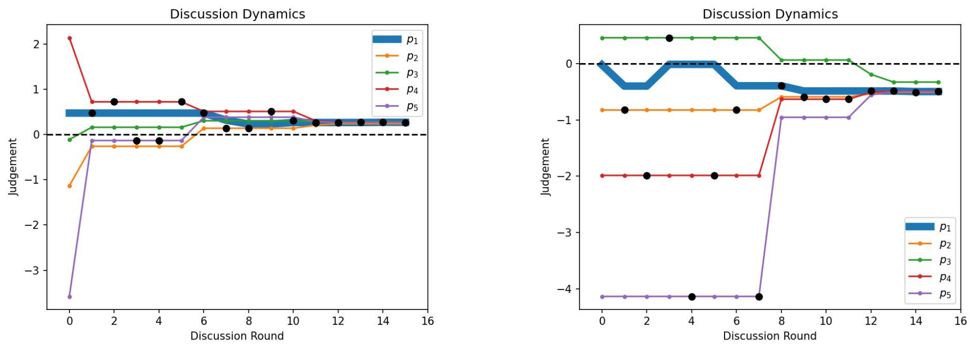

Figure 5.4: TEACHER with skill set 1 : [1,1,1,5,5]

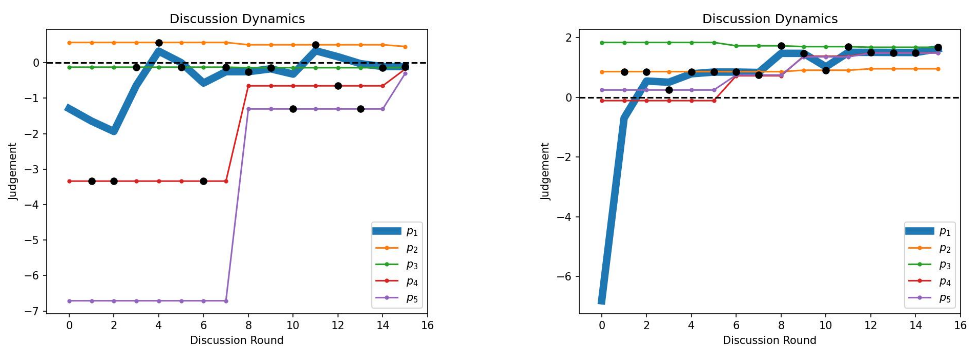

Figure 5.5: TEACHER with skill set 2 : [5,1,1,5,5]

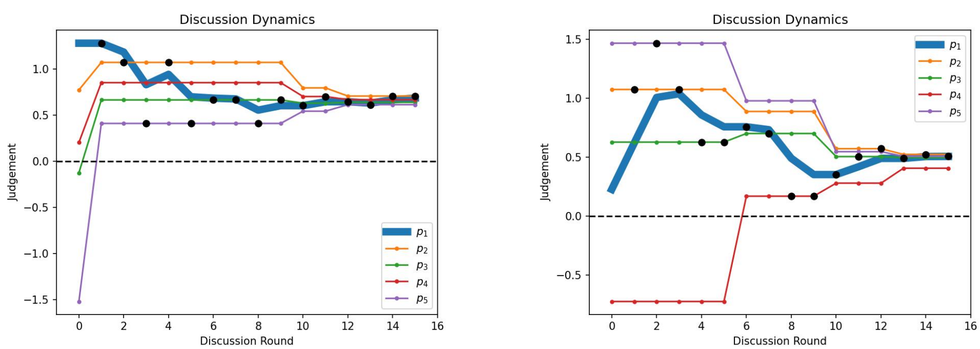

Figure 5.6: TEACHER with skill set 3 : [1,1,1,1,1]

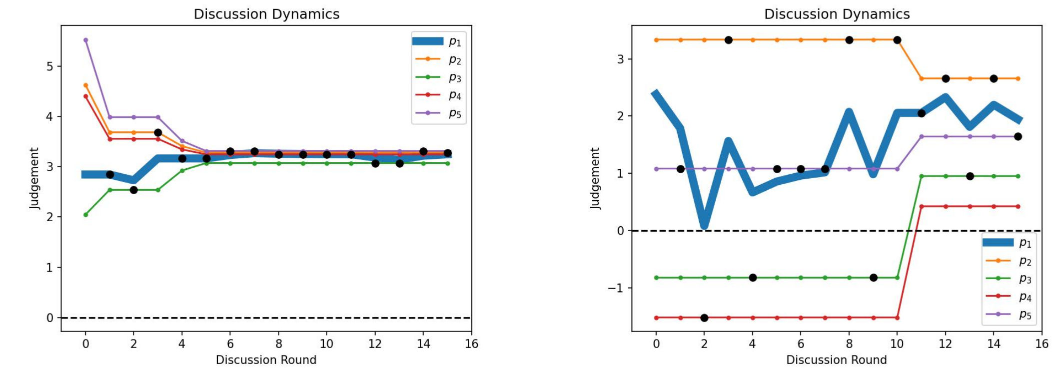

Figure 5.7: TEACHER with skill set 4 : [5,5,5,5,5]

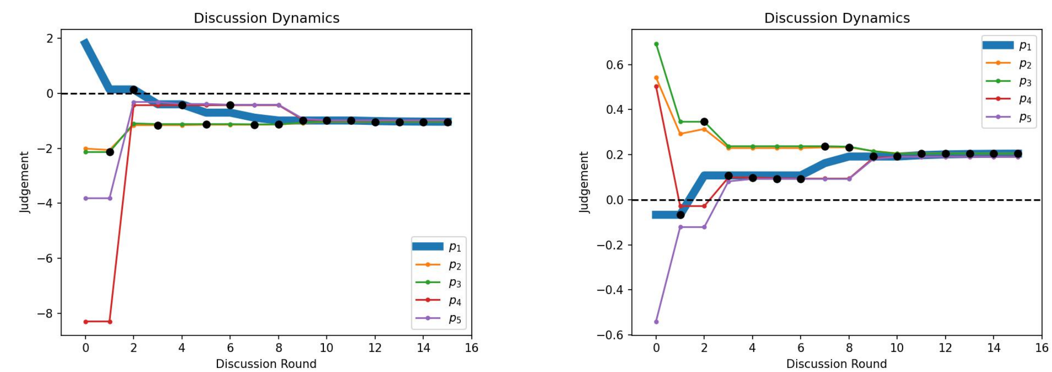

Figure 5.8: DIPLOMAT with skill set 1 : [1,1,1,5,5]

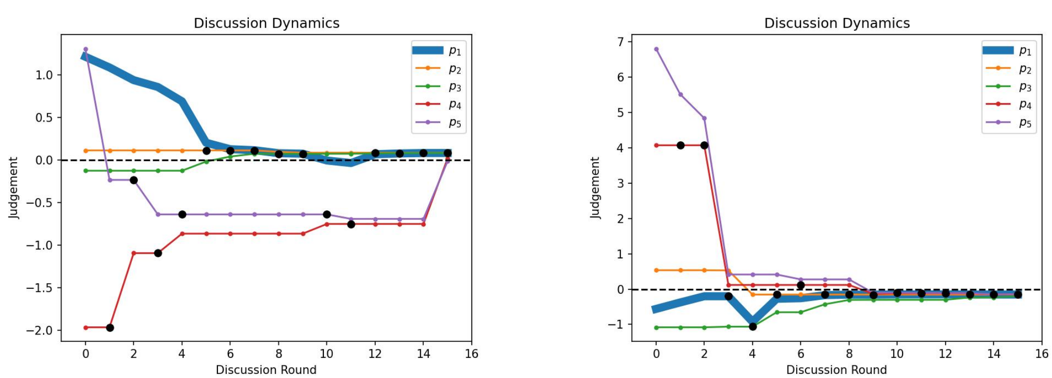

Figure 5.9: DIPLOMAT with skill set 2 : [5,1,1,5,5]

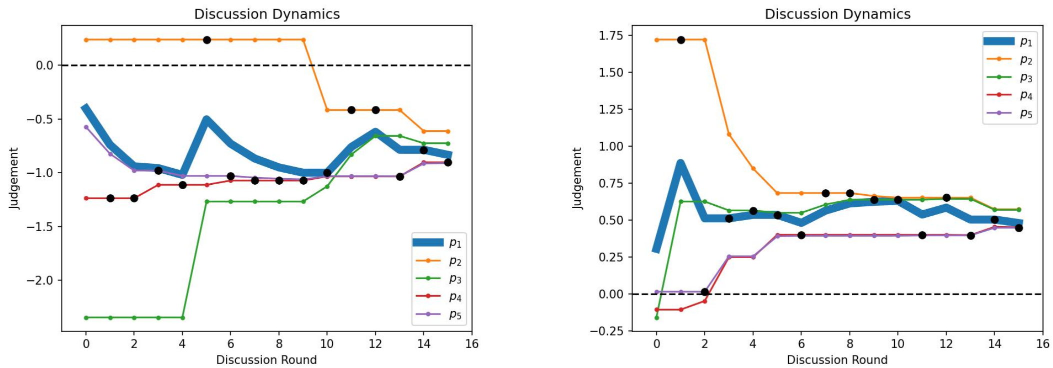

Figure 5.10: DIPLOMAT with skill set 3 : [1,1,1,1,1]

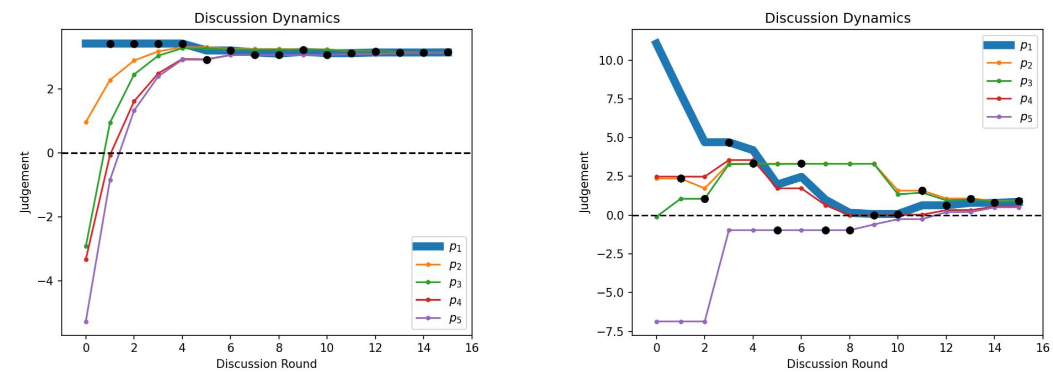

Figure 5.11: DIPLOMAT with skill set 4 : [5,5,5,5,5]

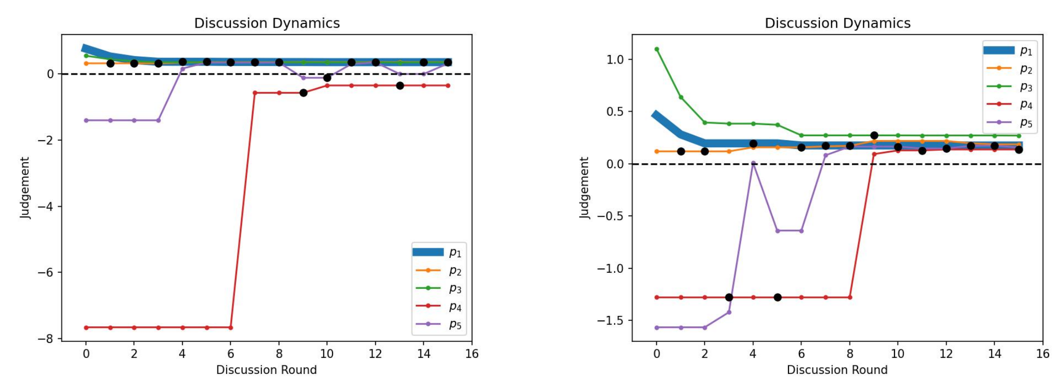

Figure 5.12: CIRCLE with skill set 1 : [1,1,1,5,5]

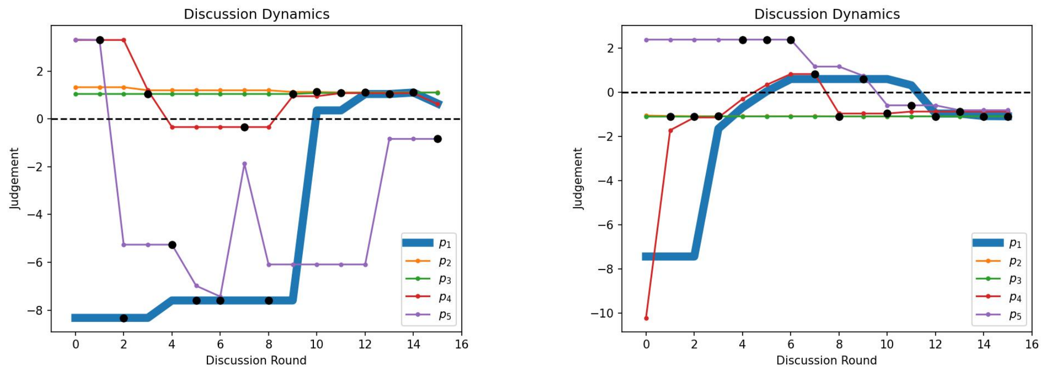

Figure 5.13: CIRCLE with skill set 2 : [5,1,1,5,5]

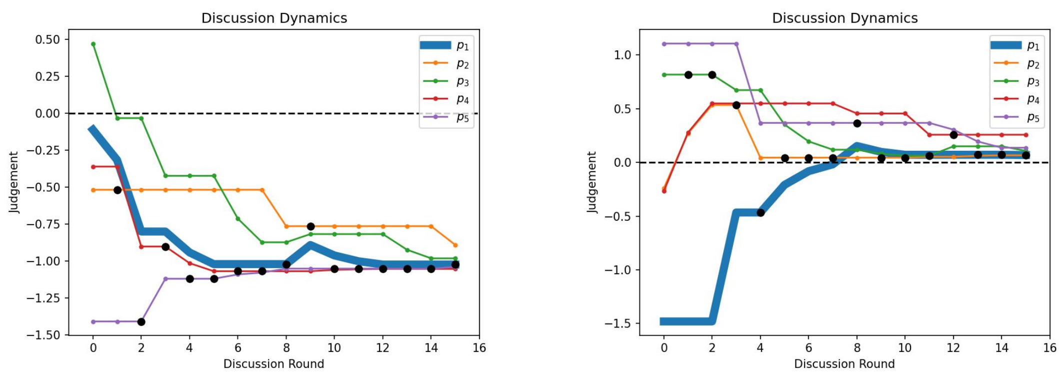

Figure 5.14: CIRCLE with skill set 3 : [1,1,1,1,1]

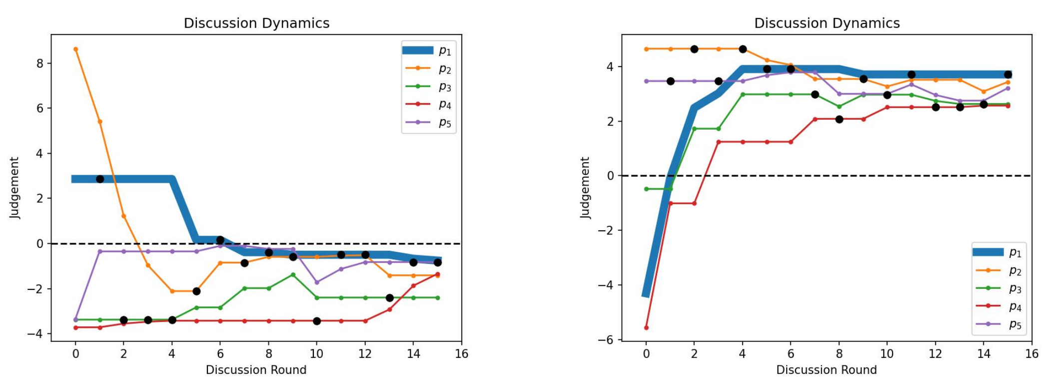

Figure 5.15: CIRCLE with skill set 4 : [5,5,5,5,5]

5.3 Simulated discussions with a disruptor

In this paper, when building and analyzing the model, we assumed that every individual is cooperative and willing to find the true value (0). This is not always true and have been studied in some references mentioned in and . We simulated discussions with optimal weights in presence of a disruptor and found out some interesting insights:

- If the disruptor is a good student or the teacher, he succeeds in his goal and the consensus converges to his opinion (fig 5.16 and 5.17) If the disruptor is a bad student, it is ignored by the other individuals (fig 5.18) similarly to what happens in the famous "Peter and Wolf" story.
- In the DIPLOMAT and CIRCLE network configuration, similar conclusions are drawn

Note that the disruptor is simply modeled by a stubborn individual (constant belief) with initial estimate -10 (to be far enough from the ground truth).

This observation shows some of the limit of our model and keeps the way open to find a general model of opinion formation.

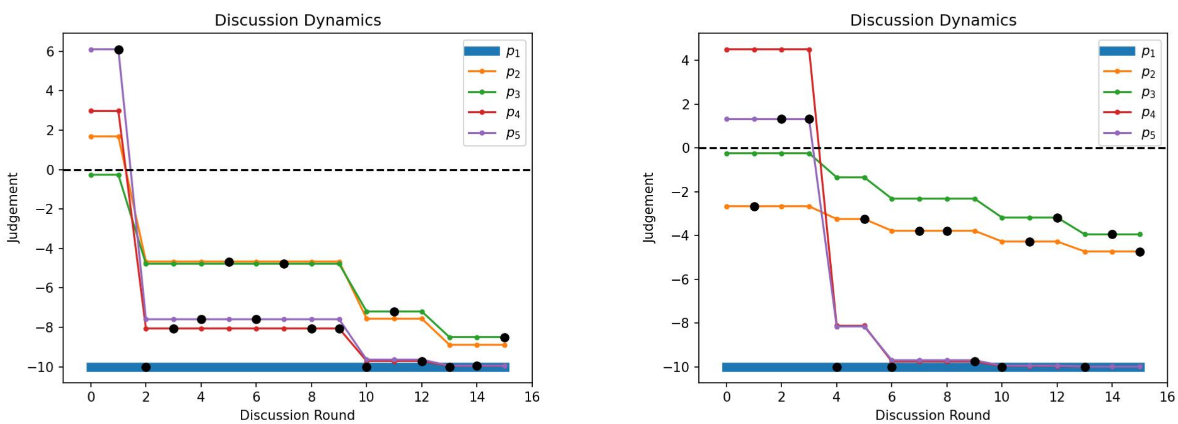

Figure 5.16: Discussion simulation with an impostor TEACHER (skill sets 1 and 2)

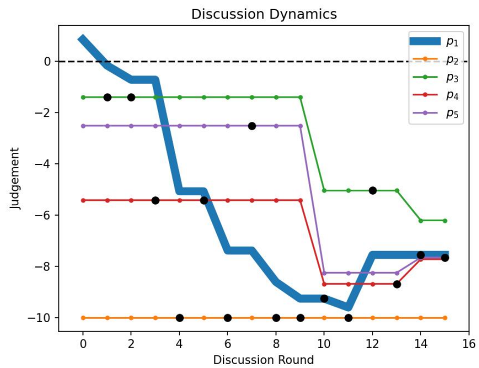

Figure 5.17: Discussion simulation in TEACHER with a good student impostor (skill set 1)

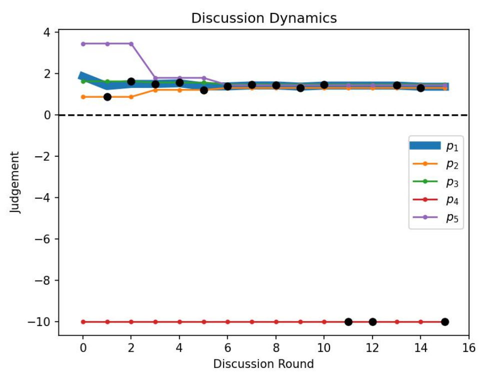

Figure 5.18: Discussion simulation in TEACHER with a bad student impostor (skill set 1)

5.4 Contributions

Leonardo Di Felice: Writing and Editing Report, implementation of code for simulation
Saad Himmi: Writing and Editing Report, implementation of code for simulation
Marcel Müller: Writing and Editing Report, implementation of code for simulation
Robin Bölsterli: Writing and Editing Report, implementation of code for simulation

5.5 Supplementary material

For more information on the numerical procedure can be found on Github:
https://github.com/saadhimmi/graph-dynamical-systems-for-opinion-formation.git.

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