4COSCOO2W MATHEMATICS FOR COMPUTING

Week 9 Seminar Tasks

Matrices. Part 2

READING

Lecture 8 Notes (available on Blackboard)

Chapter 27. *Croft, T and Davison R (2016) Foundation maths, 6th ed. Harlow: Pearson.*

TASK 1. Singular & Non-Singular Matrices. Inverse of 2×2 Matrix.

See Lectures 8 Notes -Slides 5-11

Establish which of the following matrices have their inverse by any of the applicable techniques, and either explain why an inverse does not exist or calculate it.

A = $[0]$ **B** = $[-\ 0.5]$ **C** = $[2.5]$ **D** = $[1\ 2\ ]$ **E** = $[1\ \ -\ 2\ ]$

F = $[1\ 0\ 0\ 1\ ]$ **G** = $[1\ \ -\ 2\ \ -\ 2\ 4\ ]$ **H** = $[1\ \ -\ 2\ 2\ 4\ ]$ **J** = $[1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ]$

Solution:

Matrix A

A is a 0 matrix of the order 1, and Det(A) = 0; hence, A has no inverse.

Matrix B

B is a matrix of the order 1, and Det(B) = -0.5, i.e. it is the only value in B.

Hence B-1 =[$\frac{1}{-0.5}$] = [-2]

Matrix C

C is a matrix of the order 1, and Det(C) = 2.5, i.e. it is the only value in C.

Hence C-1 = [0.4]

Matrices D and E

D and E are not square matrices; hence, they do not have an inverse.

Matrix F

F is an identity matrix 0 matrix of the order 2, and Det(F) = 1, F-1 = F

Matrix G

G: Det(G) = 1x4 – (-2)2 = 0, hence G does not have an inverse

Matrix H

H: Det (H) = 1x 4 – (-4) = 8 hence H-1 = $\frac{1}{8} \times [4\ 2\ \ -\ 2\ 1\ ]\ =\ [0.5\ 0.25\ \ -\ 0.25\ 0.125\ ]$

Note we multiply the $\frac{1}{8}$ by $[4\ 2\ \ -\ 2\ 1\ ]$ not by H itself.

Matrix J

J is an identity matrix 0 matrix of the order 3, and Det(J) = 1, J-1 = J

TASK 2. The Inverse of 3×3 Matrix

See Lectures 8 Notes -Slides 12-25

For the following 3x3 matrices. We will use an enumeration of the elements starting with a11.

K = [0 1 1 1 0 1 1 1 0] **L** = [1 2 3 2 0 2 3 2 1] **N** = [1 − 2 0 3 1 5 − 1 2 3]

O = [1 3 − 1 − 2 1 2 0 5 3]

Find for each of these matrices:

a) Its Determinants using the diagonal method
b) Matrix of the minors of its elements, call this matrix M
c) Matrix cofactors of its elements, call this matrix C
d) The adjoint matrix called this matrix A
e) The inverse – K-1 ; L-1 ; N-1 and O-1
f) Check that the matrices you found are such that K x K-1 = I; L x L-1 = I; N x N-1 = I and O x O-1 = I where I is the identity matrix of the same size as K and L.
g) If you find any correlation between any of the given matrices, then try to find the reasoning path to determine how to use it to simplify the inverse calculation. Formulate this correlation as a general property and consider any restrictions.

Solution:

For Matrix K

a) Use the diagonal method to find **Det(K)**

Add columns [0 1 1 1 0 1 1 1 0] 0 1 1 1 0 1

Calculate: Det(K) = (0x0x0 + 1x1x1 + 1x1x1) – (1x0x1 + 0x1x1 + 1x1x0) = 2

The determinant of K is not 0 so K has an inverse.

b) **Matrix of all minors.** We have 9 elements in K hence 9 minors.

For the minor M_{11} we delete row 1 and column 1 and the remaining 2x2 matrix is

[0 1 1 0]

its determinant is -1, so M_{11} = -1

For M_{12} we delete row 1 and column 2 and the remaining 2x2 matrix is

[1 1 1 0]

its determinant is -1, so M_{12} = -1

For M_{13} we delete row 1 and column 3 and the remaining 2x2 matrix is

[1 0 1 1]

its determinant is 1, so M_{13} = 1

For M_{21} we delete row 2 and column 1 and the remaining 2x2 matrix is

[1 1 1 0]

its determinant is 0, so M_{21} = -1

For M_{22} we delete row 2 and column 2 and the remaining 2x2 matrix is 1

[0 1 1 0]

its determinant is -1, so M_{22} = -1

For M_{23} we delete row 2 and column 3 and the remaining 2x2 matrix is

$$[0\ 1\ 1\ 1]$$

its determinant is 0, so M_{23} = -1

For M_{31} we delete row 3 and column 1 and the remaining 2x2 matrix is

$$[1\ 1\ 0\ 1]$$

its determinant is 0, so M_{31} = 1

For M_{32} we delete row 3 and column 2 and the remaining 2x2 matrix is

$$[0\ 1\ 1\ 1]$$

its determinant is 0, so M_{32} = -1

For M_{33} we delete row 3 and column 3 and the remaining 2x2 matrix is

$$[0\ 1\ 1\ 0]$$

its determinant is -1, so M_{33} = -1

So, the matrix of minors

$$M = [-1\ -11\ -1\ -1\ -11\ -1\ -1]$$

c) To find for the **Matrix of Cofactors**,

$C_{11} = (-1)^2 \times M_{11} = 1 \times -1 = -1$

$C_{12} = (-1)^3 \times M_{12} = -1 \times -1 = 1$

$C_{13} = (-1)^4 \times M_{13} = 1 \times 1 = 1$

$C_{21} = (-1)^3 \times M_{21} = -1 \times -1 = 1$

$C_{22} = (-1)^4 \times M_{22} = 1 \times -1 = -1$

$C_{23} = (-1)^5 \times M_{23} = -1 \times -1 = 1$

$C_{31} = (-1)^4 \times M_{31} = 1 \times 1 = 1$

$C_{32} = (-1)^5 \times M_{32} = -1 \times -1 = 1$

$C_{33} = (-1)^6 \times M_{31} = 1 \times -1 = -1$

So, the matrix of co-factors is

$$C = [-\ 1\,1\,1\,1\ \ -\ 1\,1\,1\,1\ \ -\ 1\,]$$

d) The **Adjoint Matrix** is the transpose of the matrix of cofactors

Matrix of cofactors was C = $[-\ 1\,1\,1\,1\ \ -\ 1\,1\,1\,1\ \ -\ 1\,]$ so its transpose is C^T = $[-\ 1\,1\,1\,1\ \ -\ 1\,1\,1\,1\ \ -\ 1\,]$

e) The **Inverse** of the given matrix K is calculated as

$$0.5 \times [-\ 1\,1\,1\,1\ \ -\ 1\,1\,1\,1\ \ -\ 1\,] = [-\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,]$$

f) To check that K x K^{-1} = I

K x K^{-1} = $[0\,1\,1\,1\,0\,1\,1\,1\,0\,] \times [-\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,]$ and multiply following the rules

Call the resulting matrix I.

So, we get

To calculate a_{11}

$$[0\,1\,1\,1\,0\,1\,1\,1\,0\,]\ \mathrm{x}\ [-\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,]$$

a11 = 0x (-0.5) + 1 x 0.5 + 1 x 0.5 = 1

To calculate a_{12}

$$[0\,1\,1\,1\,0\,1\,1\,1\,0\,]\ \mathrm{x}\ [-\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,]$$

a12 = 0 x 0.5 + 1 x (-0.5) + 1 x 0.5 = 0

To calculate a_{13}

$$[0\,1\,1\,1\,0\,1\,1\,1\,0\,]\ \mathrm{x}\ [-\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,0.5\,0.5\,0.5\ \ -\ 0.5\,]$$

a13 = 1x 0.5 + 1 x 0.5 + 0 x (-0.5) = 0

To calculate a_{21}

$$[0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0]\ x\ [-\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ ]$$

a21 = 1x (-0.5) + 0 x 0.5 + 1 x 0.5 = 0

To calculate a_{22}

$$[0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0]\ x\ [-\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ ]$$

a22 = 1x 0.5 + 0 x (- 0.5) + 1 x 0.5 = 1

To calculate a_{23}

$$[0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0]\ x\ [-\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ ]$$

a_{23} = 1 x 0.5 + 0x0.5 + 1 x (-0.5) = 0

To calculate a_{31}

$$[0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0]\ x\ [-\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ ]$$

a31 = 1x (-0.5) + 1 x 0.5 + 0 x 0.5 = 0

To calculate a_{32}

$$[0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0]\ x\ [-\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ ]$$

a32 = 1x 0.5 + 1 x (- 0.5) + 0 x 0.5 = 0

To calculate a_{33}

$$[0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0]\ x\ [-\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ 0.5\ 0.5\ 0.5\ \ -\ 0.5\ ]$$

a33 = 1x 0.5 + 1 x 0.5 + 0 x (-0.5) = 1

So, we found the following values:

a_{11} = 0x (-0.5) + 1 x 0.5 + 1 x 0.5 = 1
a_{12} = 0 x 0.5 + 1 x (-0.5) + 1 x 0.5 = 0
a_{13} = 1x 0.5 + 1 x 0.5 + 0 x (-0.5) = 0
a_{21} = 1x (-0.5) + 0 x 0.5 + 1 x 0.5 = 0
a_{22} = 1x 0.5 + 0 x (- 0.5) + 1 x 0.5 = 1
a_{23} = 1 x 0.5 + 0x0.5 + 1 x (-0.5) = 0
a_{31} = 1x (-0.5) + 1 x 0.5 + 0 x 0.5 = 0
a_{32} = 1x 0.5 + 1 x (- 0.5) + 0 x 0.5 = 0
a_{33} = 1x 0.5 + 1 x 0.5 + 0 x (-0.5) = 1

This forms the following matrix

Row 1 would have the values a_{11} a_{12} a_{13} or 1 0 0
Row 2 would have the values a_{21} a_{22} a_{23} or 0 1 0
Row 3 would have the values a_{31} a_{32} a_{33} or 0 0 1

I = [1 0 0 0 1 0 0 0 1]

It is the identity matrix of the size 3.

For Matrix L

Use the diagonal method to find **Det(L)**

Add columns [1 2 3 2 0 2 3 2 1]1 2 2 0 3 2

Calculate: Det(L) (1x0x1 + 2x2x3 + 3x2x2) – (3x0x3 + 2x2x1 + 1x2x2) = 24 – 8 = 16

The determinant of L is not 0, so L has an inverse.

To find the Minors:

For M_{11} we delete row 1 and column 1, and the remaining 2x2 matrix is

[0 2 2 1]

its determinant is -4, so M_{11} = -4

For M_{12} we delete row 1 and column 2 and the remaining 2x2 matrix is

[2 2 3 1]

its determinant is -4, so M_{12} = -4

For M_{13} we delete row 1 and column 3 and the remaining 2x2 matrix is

[2 0 3 2]

its determinant is 4, so M_{13} = 4

For M_{21} we delete row 2 and column 1 and the remaining 2x2 matrix is

[2 3 2 1]

its determinant is -4, so M_{21} = -4

For M_{22} we delete row 2 and column 2 and the remaining 2x2 matrix is 1

[1 3 3 1]

its determinant is -8, so M_{22} = -8

For M_{23} we delete row 2 and column 3 and the remaining 2x2 matrix is

[1 2 3 2]

its determinant is -4, so M_{23} = -4

For M_{31} we delete row 3 and column 1 and the remaining 2x2 matrix is

$$[2\ 3\ 0\ 2\ ]$$

its determinant is 4, so $M_{31} = 4$

For M_{32} we delete row 3 and column 2 and the remaining 2x2 matrix is

$$[1\ 3\ 2\ 2]$$

its determinant is -4, so $M_{32} = -4$

For M_{33} we delete row 3 and column 3 and the remaining 2x2 matrix is

$$[0\ 2\ 2\ 0\ ]$$

its determinant is -4, so $M_{33} = -4$

So, the matrix of minors

$$M = [-4 \ -4\ 4 \ -4 \ -8 \ -4\ 4 \ -4 \ -4]$$

To find for the **Matrix of Cofactors**,

$C_{11} = (-1)^2 \times M_{11} = -4$

$C_{12} = (-1)^3 \times M_{12} = 4$

$C_{13} = (-1)^4 \times M_{13} = 4$

$C_{21} = (-1)^3 \times M_{21} = 4$

$C_{22} = (-1)^4 \times M_{22} = -8$

$C_{23} = (-1)^5 \times M_{23} = 4$

$C_{31} = (-1)^4 \times M_{31} = 4$

$C_{32} = (-1)^5 \times M_{32} = 4$

$C_{33} = (-1)^6 \times M_{33} = -4$

The matrix of Cofactors C is

$$C = [-\ 4\ 4\ 4\ 4\ \ -\ 8\ 4\ 4\ 4\ \ -\ 4\ ]$$

The adjoint Matrix A is the transpose of C

$$A = [-\ 4\ 4\ 4\ 4\ \ -\ 8\ 4\ 4\ 4\ \ -\ 4\ ]$$

Finally, the inverse of L, L^{-1}

$$\mathbf{L^{-1}} = \frac{1}{16} \times [-\ 4\ 4\ 4\ 4\ \ -\ 8\ 4\ 4\ 4\ \ -\ 4\ ] = [-\ 0.25\ 0.25\ 0.25\ 0.25\ \ -\ 0.5\ 0.25\ 0.25\ 0.25\ \ -\ 0.25\ ]$$

To check that L x L^{-1} gives the Identity matrix of the order 3, multiply these matrices.

For Matrix N:

First the determinant of this matrix **Det(N)** is 21.

Next, the following are matrix **<u>M of Minors</u>** and matrix **<u>C of Cofactors:</u>**

$$M = [-\ 7\ 14\ 7\ \ -\ 6\ 3\ 0\ \ -\ 10\ 5\ 7\ ]$$

$$C = [-\ 7\ \ -\ 14\ 7\ 6\ 3\ 0\ \ -\ 10\ \ -\ 5\ 7\ ]$$

Then obtain Adjoint Matrix A – the transpose of C and multiply by the $\frac{1}{21}$

$$A = C^T = [-\ 7\ 6\ \ -\ 10\ \ -\ 14\ 3\ \ -\ 5\ 7\ 0\ 7\ ]$$

$$N^{-1} = \frac{1}{21} \text{ x } A = [-\ 0.33\ 0.29\ \ -\ 0.48\ \ -\ 0.66\ 0.14\ \ -\ 0.24\ 0.33\ 0\ 0.33\ ]$$

For Matrix O:

Note that $N^T = O$, i.e. O is the transpose of matrix N, hence we can use the following rule: for any non-singular matrix A: $(A^T)^{-1} = (A^{-1})^T$

Hence $O^{-1} = (N^T)^{-1} = (N^{-1})^T$

And we obtain the result by transposing just found inverse N^{-1}

$$(N^{-1})^T = [-\ 0.33\ \ -\ 0.66\ 0.33\ 0.29\ 0.14\ 0\ \ -\ 0.48\ \ -\ 0.24\ 0.33\ ]$$