Mathematics in Computing 4COSC007C

Lecture 4: Relations, Functions, Graphs and Trees

September Intake 2021

Relations

- Given a Cartesian product of sets

$$A1 \times A2 \times .... X\ An$$

an n- ary relation R is a subset of the set of tuples

$$\{x,y,...,z : x \in A1,\ y \in A2,\ ...\ z \in An\}$$

- We will abbreviate it as R(x,y,...,z) read as

"R of x,y,..., z"

Relations- Example

- Consider a shop and the following three sets:
- S = {x: x is a sales assistant},
- P = {y: y is a product on sale},
- C = {z: z is a customer}.
- Then the following would be a 3-ary (ternary) relation R as a subset of the set of tuples

$$\{x, y, z: x \in S, y \in P, z \in C\}.$$

- Let's make a rule that there is a mapping which somehow associates elements of these sets.
- A particular tuple could be <John, Pen, Margaret> where

$$\text{John} \in S, \text{Pen} \in P, \text{Margaret} \in C.$$

Relations – Special Case

- When all the sets A1 x A2 x…An in the definition of the relation R are the same, for example, B, then R is simply called a relation on the set B

- When a relation is a set of pairs it is called a BINARY relation

- When a relation is a set of triples it is called a TERNARY relation

Relations – More Examples

- Let $A = \{1,2,3,4,5,6,7,8\}$
- Let's make the following association of the elements of A:
  $R = \{(1, 2), (2,4), (3, 6), (4, 8)\}$
- Then R is a BINARY relation over A. More specifically, with every number 1 to 4 the relation R associates the number which is twice greater than this number.

Relations – More Examples

- Let $A = \{1,2,3,4,5,6,7\}$
- Let's make the following association of the elements of A:
  $R = \{(1, 2, 3), (2,3,5), (3,4,7)\}$
- Then R is a TERNARY relation over A. More specifically, with every two consecutive numbers 1 to 4 the relation R associates the number which is their sum.

Properties of Relations

Symmetry: for any x, y: R(x, y) => R(y, x)

Transitivity: for any x, y, z: R(x, y) and R(y, z) => R(x, z)

Reflexivity: for any x R(x, x)

Examples?

Properties of Relations

- Consider the relation "greater or equal"
- It is reflexive, transitive but not symmetrical
- $x \le x$
- $x \le y$ and $y \le z$ then $x \le z$
- $x \le y$ does not imply $y \le x$

Properties of Relations

- Consider the relation "equal"
- It is reflexive, transitive and symmetrical
- $x = x$
- $x = y$ and $y = z$ then $x = z$
- $x = y$ does imply $y = x$
- A relation which is reflexive, transitive and symmetrical is called EQUIVALENCE

Some relations are special

- We call these special relations “functions” and you have probably met them before.
- A <u>function</u> is a way of transforming one set of objects (usually numbers) into another set of objects (also usually numbers).

Functions

- A ***function*** is a **rule** that assigns each input exactly one output.
- We call the output the (***range***)***I mage*** of the input.
- The set of all inputs for a function is called the ***domain***.
- The set of all allowable outputs is called the ***codomain***.
- We would write f: A→B to describe a function with name f, f, domain A and codomain B.
- This does not tell us *which* function f is though. To define the function, we must describe the rule. This is often done by giving a formula to compute the output for any input (although this is certainly not the only way to describe the rule).

Functions - Terminology

Terminology

$f : A \rightarrow B$

- This means that the function f maps elements in the Set A (the domain) to the elements in the Set B (the codomain)

Simple Rules

- We can deal with the rules of functions the same way we dealt with relations.

Domain: $\{1, 2, 3\}$

Codomain: $\{5, 6, 7, 8\}$

Rule: $\{(1, 5), (2, 6), (3, 8)\}$

More Advanced Rules

- Rules can also be written in the following style
  - f(a) = a + 4
  - g(b) = b x b + 2
  - h(c) = 5
- These would read
  - "f of a equals a plus 4"
  - "g of b equals b times b plus 2"
  - "h of c equals 5"

Rules continued

- When we see rules we often ask what their value might be when given concrete values.
- Take the formula from the previous page
  - $f(a) = a+4$
- What is its value when a equals 7?
  - Answer:

Rules continued

- When we see rules we often ask what their value might be when given concrete values.
- Take the formula from the previous page
  - $f(a) = a+4$
- What is its value when a equals 7?
  - Answer: 11

More Rules

- To abbreviate this question you might just ask what is f(7)?
- This means substitute the 7 for the a and solve the equation on the right hand side.
- Try out the following formulas and related questions on the next slide.

Formula Problems

- $f(x) = 2x + 3$
- $g(x) = 7$

What are

- $f(5)$?
- $f(8)$?
- $f(-4)$?
- $g(3)$?
- $g(7)$?
- $g(-10)$?

Formula Problems

- $f(x) = 2x + 3$
- $g(x) = 7$

What are

- $f(5) = 2x5 + 3 = 13$
- $f(8) = 2x8 + 3 = 19$
- $f(-4) = 2x(-4) + 3 = -8 +3 = -5$
- $g(3) = 7$
- $g(7) = 7$
- $g(-10) = 7$

So what is a function really?

- A function is a certain way that three of the components (domain, codomain and rule) are related.
- Something is a function if (and only if) you can take <u>every</u> value in the Domain, put the value in the formula, and get a single value that is in the Codomain

Explanation

- Let's look at an example.
  - Suppose I define my <u>Domain</u> to be $\{1, 2, 3\}$
  - And I define my <u>Codomain</u> to be $\{5, 6, 7, 8\}$
  - And my formula is $f(x) = x + 5$
- <u>Is this a function?</u>
  - To find out, we will go through each value of the domain

Explanation

Domain = {1, 2, 3} Codomain = {5, 6, 7, 8} f(x) = x + 5

- Let me try the value 1. f(1) = 1+ 5 = 6
  - 6 is in my Codomain. So far it's working.
- Let me try the value 2. f(2) = 2 + 5 = 7
  - 7 is in my Codomain. It is still working.
- Let me try the value 3. f(3) = 3 + 5 = 8
  - 8 is in my Codomain. It is still working.
- I have tried all values in my domain, and they all worked.
- Therefore, this <u>is a function.</u>

Explanation

Domain = {1, 2, 3} Codomain = {5, 6, 7} $f(x) = x + 5$

- Almost identical.
- Taking 1 from the domain works
- Taking 2 from the domain works
- Taking 3 from the domain fails.
  - Why? It produces the value 8. This value is no longer part of my Codomain. So therefore this example is <u>not a function</u>

Questions?

1. Is x^2 a function?

2. Is $\sqrt{x}$ a function?

Graphs

- A graph is a finite set of nodes with edges between nodes
- Formally, a graph G is a structure (V,E) where
  - V is a finite set of nodes, and
  - E that is a set of directed or undirected pairs of nodes x and y from V
- If a pair of nodes x and y is directed we abbreviate is as (x,y)
- If a pair of nodes x and y is undirected we abbreviate is as $\{x,y\}$

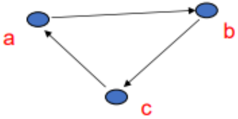
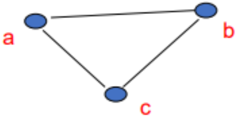
Directed vs. Undirected Graphs

- If the directions of the edges matter, then we show the edge directions, and the graph is called a *directed graph* (or a *digraph*)
- If the direction of the edges does not matter the graph is called an *undirected graph*.

Intuition Behind Graphs

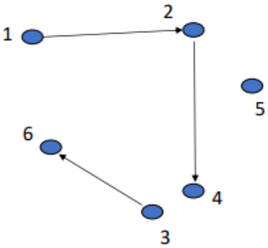
- The nodes represent entities (objects such as products, cars, numbers, words, etc.)
- Edges (x,y) represent relationships between entities x and y, such as:
  - "x < y"
  - "x larger than y"
  - "x is cheaper than y"
  - "x is a longer than y"
  - "x is bigger than y"
  - 'x is faster than y", ...

Directed vs. Undirected Graphs

DIRECTED GRAPHS	**UNDIRECTED GRAPHS**
A directed graph G is a structure (V,E) where V is a finite set of nodes, and E is a set of ordered pairs $\{(x,y): x \in V, y \in V\}$ 	*An undirected graph* G is a structure (V,E) where V is a finite set of nodes, and E is a set of unordered pairs $\{\{x,y\}: x \in V, y \in V\}$ 
EXAMPLE Consider a graph $G = (V,E)$ such that V = {a,b,c} and E = {(a,b), (b,c), (c,a)} this would be a ***directed*** graph with three nodes – a, b and c and three directed edges from a to b, b to c and c to a. 	EXAMPLE Consider a graph $G = (V,E)$ such that V = {a,b,c} and E = {{a,b}, {b,c}, {a,c}} this would be an ***undirected*** graph with three nodes – a, b and c and three undirected edges between a and b, b and c, a and c. 

Examples

- V={1,2,3,4,5,6}
- E={(1,2), (2,4), (3,6)}



This is a graph where only several vertices are connected.

Note the directions!

There is some pattern represented here - E is a set of all pairs (x,y) such that $y = 2x$

Question

- Choose the correct answer.

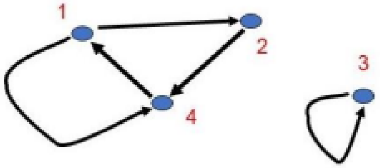


Figure: Graph G

- [a.] V={1,2,4} and E={{1,2}, {2,4}, {4,1},{1,4}}
- [b.] V={1,2,4} and E={(1,2), (2,4), (4,1),(1,4)}
- [a.] V={1,2,3,4} and E={{1,2}, {2,4}, {4,1},{1,4},{3,3}}
- [a.] V={1,2,3,4} and E={(1,2), (2,4), (4,1),(1,4),(3,3)}

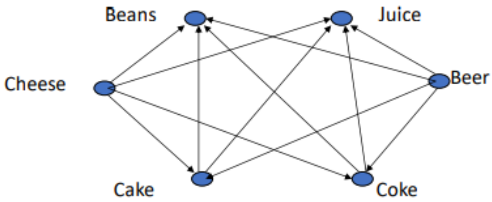
A "Corner Shop Sale" Example of a Graph

A "Corner Shop Sale" Example of a Graph

V is a set of 6 discounted products:

cheese	**£2.99**	**juice**	**£4.99**
beans	**£3.99**	**cake**	**£3.99**
beer	**£2.99**	**coke**	**£3.99**

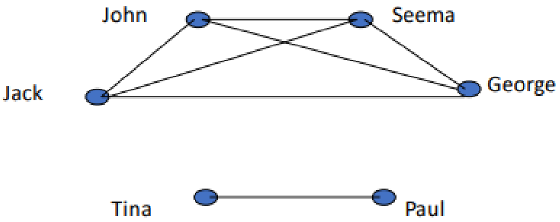
E is a set of all pairs (*x*,*y*) such that *x* is cheaper than *y*



Is this correct??

Examples of Undirected Graphs

- *V*=set of 6 students: John, Seema, Jack, George, Tina, and Paul, where the first 4 are friends, and the last two are friends
- *E* = is a set o all pairs {x,y} such that x and y are friends – note this is an undirected graph!

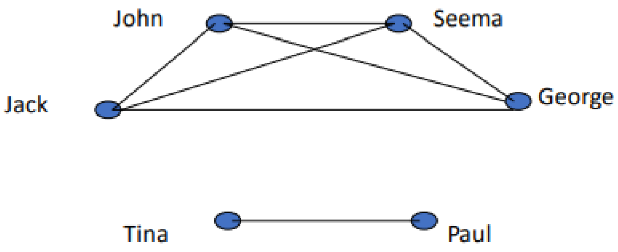


Paths

- A path in a graph G is a sequence of nodes x_1, x_2, …, x_k, such that for any node x_i $(1 \leq i \leq k-1)$ there is an edge from it to the next one in the sequence
- Consider the following examples and find paths through the graphs

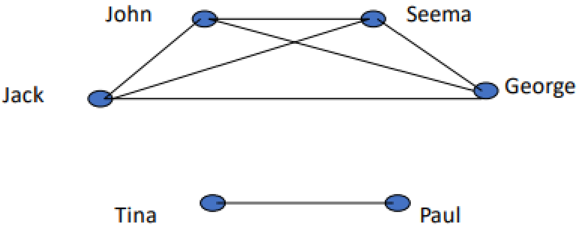
Examples of Undirected Graphs

- Are these sequences paths:
  - John, Seema, George, Jack ?
  - George, Paul ?
  - Jack, Seema, George ?
  - Jack, John, George, Jack ?



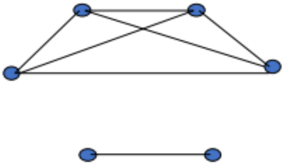
Cycle in a Graph

- A cycle in a graph G is a path where the last node is the same as the first node.
- In the "friends" graph, the following sequences are cycles:
- Jack, John, George, Jack is a cycle
- Jack, John, Seema, George, Jack is a cycle
- Tina, Paul, Tina is a cycle

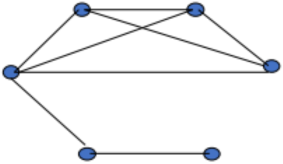


Connectivity

- A graph is *connected* if for every pair of nodes, there is a path between them. Otherwise, the graph is *disconnected.*



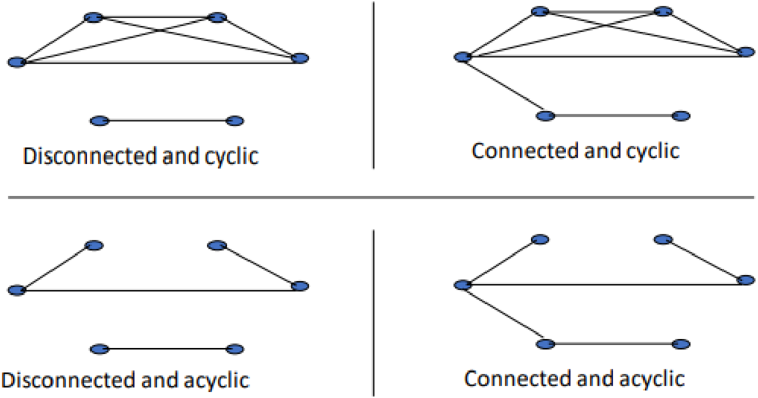
Disconnected



Connected

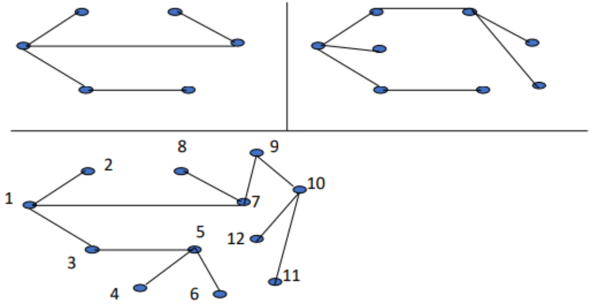
Graph Cyclicity

- A graph is *cyclic* if it has at least one cycle. Otherwise, it is *acyclic*



Trees

- A tree is a connected acyclic undirected graph.
- In a tree any two nodes are connected by *exactly one path*



Rooted Trees

- A rooted tree is a tree where one of the nodes is designated as the root node. (We cannot have two roots of a tree)

Tree-Related Concepts

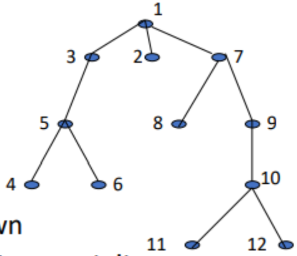
- The nodes adjacent to node x and below x are called the *children* of x,

  and x is called their parents
- A node that has no

  children is called a *leaf*
- The *descendents* of a node

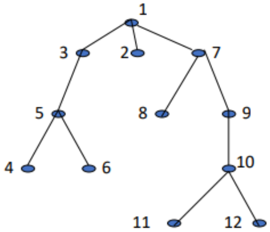
  are: its children,

  their children, all the way down
- The *ancestors* of a node are: its parent, its grandparent, all the way to the root
- WHAT ARE LEAFS OF THIS TREE?



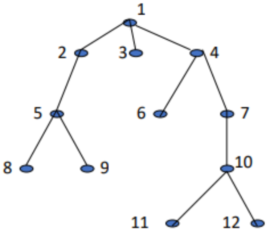
WHAT ARE LEAVES?

- A node that has no children is called a *leaf:*
  2, 4, 6, 8, 11,12



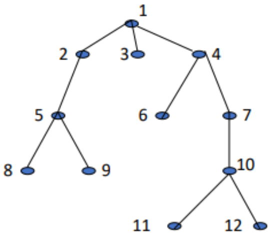
Tree-Related Concepts (Contd.)

- The *depth* of a node is the number of edges from the root to that node.
- The *depth (or height)* of a rooted tree is the depth of the lowest leaf
- Depth of node 10: 3
- Depth of this tree: 4

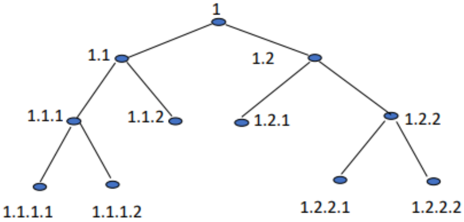


Binary Trees

- A tree is a binary tree if every node has at most two children



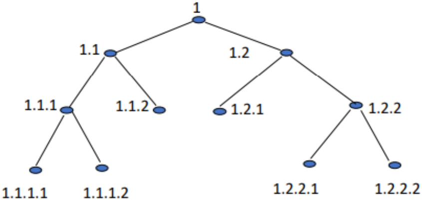
Nonbinary tree



Binary tree

Binary Trees

- The children of any node in a binary tree are ordered into a left child and a right child
- A node can have a left and a right child, a left child only, a right child only, or no children
- The tree made up of a left child (of a node x) and all its descendents is called the left subtree of x
- Right subtrees are defined similarly

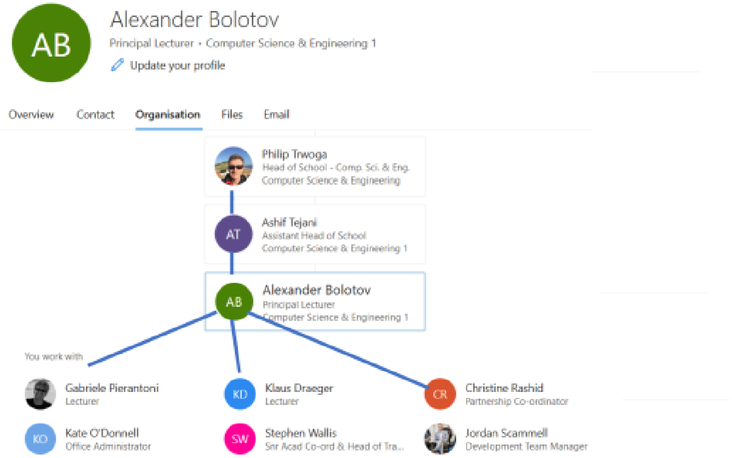


Binary tree

Applications of Graphs and Tress in Computing Disciplines

- Network
- Graph Databases
- Circuits
- Modelling – state diagrams, flow diagrams
- Search
- AI applications
- Logistics - optimisation

Applications of Graphs and Tress Organisational Structure



THANK YOU

Any Questions?