4COSC002W Mathematics for Computing

Lecture 9

Probability Theory

UNIVERSITY OF WESTMINSTER

What is Probability?

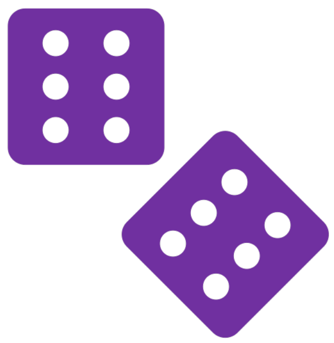
Probability measures the likelihood of an event occurring.

Output range: Between 0 (impossible) and 1 (certain)– [0,1]

For example:

The chance that a human lives for 1000 years is **0.**

The chance we all die someday is **1.**



Applications:

Critical in computing for decision-making, algorithms, and risk assessment.

Theoretical Probability

Probability of event A occurring:

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

Number of favourable outcomes: The count of outcomes that are considered as a success or the desired outcome for the event A.

Total number of possible outcomes (probability space): The total count of all possible outcomes for the event, assuming each outcome is equally likely.

NB! Theoretical probability assumes that every outcome has an equal chance of occurring.

Common Examples of Probability

Examples: Theoretical Probaility

Rolling a Die: The probability of rolling a 4 on a six-sided die. There is one favorable outcome (rolling a 4), and six possible outcomes.

$$\mathrm{P}(4) = \frac{1}{6}$$

Flipping a Coin: The probability of getting heads when flipping a fair coin. There is one favorable outcome (heads), and two possible outcomes (heads or tails).

$$\mathrm{P}(\text{Heads}) = \frac{1}{2}$$

Drawing a Card: The probability of drawing an Ace from a standard deck of 52 cards. There are four favorable outcomes (the four Aces), and 52 possible outcomes.

$$\mathrm{P}(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$$

Complementary Events

Events that together cover all possible outcomes are called *complementary*.

If A is an event, the complement of $\boldsymbol{A}$, denoted as $\boldsymbol{A}'$ or sometimes $\overline{\boldsymbol{A}}$, represents all outcomes that are not in A. The probability of the complementary event A′ is given by:

$$\boldsymbol{P(A') = 1 - P(A)}$$

$P(A)$ is the probability of event A occurring.

$P(A')$ is the probability of event A not occurring.

Example:

If the probability of raining is 0.3, the probability of not raining is 0.7.

Examples: Complementary Events

Coin Toss:

If $P(\text{Heads}) = 0.5$, then $P(\text{Tails}) = 1 - P(\text{Heads}) = 1 - 0.5 = 0.5$.

Drawing a Card:

If the probability of drawing a king from a standard deck of cards is $\frac{4}{52}$ or $\frac{1}{13}$, then the probability of not drawing a king is $1 - \frac{1}{13} = \frac{12}{13}$.

Dice Roll:

If the probability of rolling a 6 on a standard six-sided die is $\frac{1}{6}$, then the probability of not rolling a 6 is $1 - \frac{1}{6} = \frac{5}{6}$.

Independent Events

Independent events in probability theory are two or more events whose occurrence or non-occurrence does not affect the probability of the other events occurring.

If two events, A and B, are independent, the occurrence (or non-occurrence) of A does not change the probability of B occurring, and vice versa.

For two independent events, A and B, the probability of both events occurring is given by the product of their individual probabilities (*multiplication law*):

$$\mathbf{P(A\ and\ B) = P(A) \times P(B)}$$

Examples: Independent Events

Coin Tosses:

Tossing a coin twice. The outcome of the first toss (say, heads) does not affect the probability of the outcome of the second toss. If P(Heads) = 0.5 for each toss, then P(Heads and Heads) = $0.5 \times 0.5 = 0.25$.

Dice and Cards:

Rolling a die and drawing a card from a deck. The probability of rolling a 6 (1/6) and drawing an ace (4/52) are independent events. So, P(6 and Ace) = $\frac{1}{6} \times \frac{4}{52}$.

Traffic Lights:

The probability of hitting a green light on two different traffic signals might be independent if the lights are not synchronised. If the probability of green at each light is 0.3, then P(Green at both) = $0.3 \times 0.3 = 0.09$.

The probability of rolling certain totals

Total on Dice	Pairs of Dice	Probability
2	1+1	1/36 = 3%
3	1+2, 2+1	2/36 = 6%
4	1+3, 2+2, 3+1	3/36 = 8%
5	1+4, 2+3, 3+2, 4+1	4/36 = 11%
6	1+5, 2+4, 3+3, 4+2, 5+1	5/36 = 14%
7	1+6, 2+5, 3+4, 4+3, 5+2, 6+1	6/36 = 17%
8	2+6, 3+5, 4+4, 5+3, 6+2	5/36 = 14%
9	3+6, 4+5, 5+4, 6+3	4/36 = 11%
10	4+6, 5+5, 6+4	3/36 = 8%
11	5+6, 6+5	2/36 = 6%
12	6+6	1/36 = 3%

Probabilities of Outcomes in Three-Coin Toss

Specific Outcome	Probability of Specific Outcome	Total Heads in Outcome	Cumulative Probability of Total Heads (%)	Probability Calculation
TTT	1/8	0	12.5%	1/8 (1 outcome out of 8)
TTH	1/8	1	37.5%	3/8 (3 outcomes out of 8)
HTT	1/8	1	37.5%	3/8 (3 outcomes out of 8)
THT	1/8	1	37.5%	3/8 (3 outcomes out of 8)
THH	1/8	2	37.5%	3/8 (3 outcomes out of 8)
HHT	1/8	2	37.5%	3/8 (3 outcomes out of 8)
HTH	1/8	2	37.5%	3/8 (3 outcomes out of 8)
HHH	1/8	3	12.5%	1/8 (1 outcome out of 8)

Conditional Probability

Conditional probability measures the probability of an event occurring, given that another event has already occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ where } P(B) \text{ is not zero.}$$

Interpretation: It quantifies how the probability of one event changes when another related event is known to occur.

Example: Probability of drawing an ace after drawing a king is $\frac{4}{51}$.

Example: Conditional Probability

A fair die is rolled. Let **A** be the event that the outcome is an *odd number*, i.e., A={1,3,5}. Also let **B** be the event that the outcome is less than or equal to 3, i.e., B={1,2,3}.

- What is the Probability Space S?
- What is P(A)?
- What is P(B)?
- What is $P(A \cap B)$?
- What is the conditional probability $P(A|B)$?

Example: Conditional Probability

Solution:

Probability Space:

$$S=\{1,2,3,4,5,6\}$$

Probability of A:

$$P(A)=\frac{|A|}{|S|}=\frac{|\{1,3,5\}|}{6}=\frac{3}{6}=\frac{1}{2}$$

Example: Conditional Probability

Given sets: $A=\{1,3,5\}$ $B=\{1,2,3\}$

Let's find $P(A\cap B)$:

First, we need to define $A\cap B=\{1,3\}$.

$$P(A\cap B)=\frac{|A\cap B|}{|S|}=\frac{2}{6}=\frac{1}{3}.$$

Example: Conditional Probability

Now we can find the conditional probability of A given that B occurred as we know:

$$P(A \cap B) = \frac{1}{3} \text{ and } P(B) = \frac{1}{2}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3} \text{ x } \frac{2}{1} = \frac{2}{3}.$$

Bayes' Theorem

Bayes' Theorem is an extension of conditional probability:

$$\mathbf{P(A|B)} = \frac{\mathbf{P(B|A) \times P(A)}}{\mathbf{P(B)}}$$

where:

$P(A|B)$: The probability of event A occurring given that B is true.

$P(B|A)$: The probability of event B occurring given that A is true.

$P(A)$: The prior or initial probability of event A.

$P(B)$: The prior or initial probability of event B.

Example: Medical Test for a Disease

Imagine a medical test for a particular disease. Let's denote:

A as the event of actually having the disease.

B as the event of testing positive for the disease.

Bayes' Theorem can be used to calculate **P(A|B)**, the probability of having the disease given a positive test result.

In this context:

$P(A)$ would be the overall prevalence of the disease in the population.

$P(B|A)$ is the probability of testing positive if you have the disease (true positive rate).

$P(B)$ would be the overall probability of testing positive, which includes true positives and false positives.

Example: Medical Test for a Disease

Given Data:

1. **Prevalence of the Disease (P(A)):** Let's say 1% of the population has a certain disease. So, $P(A) = 0.01$.
2. **Sensitivity of the Test (P(B|A)):** The probability of the test correctly identifying the disease (true positive rate) is 95%. Hence, $P(B|A) = 0.95$.
3. **False Positive Rate:** Assume the test has a 5% false positive rate. This means that the test incorrectly identifies the disease in healthy people 5% of the time.

Example: Medical Test for a Disease

Calculating $\boldsymbol{P(B)}$**:**

$P(B)$ is the overall probability of testing positive. This includes both true positives (sick people correctly diagnosed) and false positives (healthy people incorrectly diagnosed).

Let's calculate it:

True Positives: $P(A) \times P(B|A) = 0.01 \times 0.95 = 0.0095$.

False Positives: $P(A') \times P(B|A')$ where $P(A') = 0.99$ (probability of not having the disease) and $P(B|A') = 0.05$ (false positive rate).

So, False Positives = $0.99 \times 0.05 = 0.0495$.

Therefore, $\mathbf{P(B)}$ = True Positives + False Positives = $0.0095 + 0.0495 = 0.059$.

Example: Medical Test for a Disease

Applying Bayes' Theorem:

We want to find **P(A|B):** the probability of having the disease given a positive test result.

Using the formula $P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$, we plug in the values:

$$P(A|B) = \frac{0.95 \times 0.01}{0.059}.$$

Let's calculate this.

The calculation yields that $P(A|B) \approx 0.161$, or approximately 16.1%.