4COSC002W Mathematics for Computing

Lecture 8

Matrices – Part 2.

Power of a Matrix. Inverse of a Matrix

UNIVERSITY OF WESTMINSTER

Power of a Matrix

The *power of a matrix* is a concept similar to the power of numbers, but it applies to matrices.

When we say a matrix is raised to a power, it means the matrix is multiplied by itself a certain number of times.

It's important to note that matrix multiplication is not commutative (i.e., $AB \neq BA$ in general), so the power of matrices applies to square matrices only.

For a **square matrix A** (a matrix with the same number of rows and columns), **the power of the matrix is denoted as $\mathbf{A^n}$**, where n is a positive integer. It is defined as:

$$A^1 = A$$

$$A^2 = A \times A$$

$$A^3 = A \times A \times A$$

And so on, where A^n involves multiplying A by itself n times.

Example 1 of Power of a Matrix

Suppose we have the following **square matrix** $\boldsymbol{A}$:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Now, let's calculate $\mathbf{A^2}$, which means multiplying matrix A by itself: $A^2 = A \times A$

We can perform this multiplication to find A^2:

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

To calculate the product, we multiply the rows of the first matrix by the columns of the second matrix:

$$A^2 = \begin{bmatrix} (2\times2 + 1\times1) & (2\times1 + 1\times3) \\ (1\times2 + 3\times1) & (1\times1 + 3\times3) \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$$

Example 2 of Power of a Matrix

Let's use a 3x3 **diagonal matrix *D*** with diagonal elements of 3:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now, let's calculate $\boldsymbol{D^3}$:

$$D^3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

The Inverse of a Matrix

The *inverse of a matrix* is analogous to the *reciprocal* of a number. Just as multiplying a number by its reciprocal yields the identity value (1), multiplying a matrix by its inverse results in the identity matrix.

For a square matrix A, its inverse is denoted as A^{-1}, and it satisfies the following condition:

$$A \times A^{-1} = A^{-1} \times A = I$$

Here, I represents the identity matrix.

Identity Matrix (Unit Matrix)

An *identity matrix* is a square matrix where all elements on the main diagonal are 1, and all off-diagonal elements are 0.

Examples:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

These are 2×2, 3×3, and 4×4 identity matrices. The subscript denotes their sizes.

Identity Matrix (Unit Matrix)

It is the multiplicative identity in matrix algebra, meaning any matrix multiplied by the identity matrix will result in the original matrix.

Important property of identity matrix

If A is an m x n matrix, then

$$I_m A = A \quad \text{and} \quad A I_n = A.$$

If A is a square matrix, then

$$IA = A = AI.$$

Non-Singular Matrices

Not all matrices have inverses.

A matrix that has an inverse is called an *invertible or nonsingular matrix*.

If a matrix doesn't have an inverse, it's called a singular matrix.

The key requirement for a matrix to be invertible:

- square (having the same number of rows and columns)
- determinant must be non-zero.

The Determinant of a Square Matrix

The *determinant of a square matrix* is a scalar value that can be computed from the elements of the matrix. It is a fundamental concept in linear algebra and is denoted by "det(A)" or "|A|," where "A" is the matrix in question.

The determinant of a square matrix, such as an $n \times n$ matrix, is defined as follows:

- For a 1x1 matrix (a single number), the determinant is the number itself.
- For a 2x2 matrix:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

For larger square matrices ($n \times n$ where $n > 2$), the determinant can be computed using various methods, such as the expansion by minors, cofactor expansion, or using specialised algorithms like Gaussian elimination.

Inverse of a 2x2 Matrix

A 2x2 matrix A is invertible if its determinant (det(A)) is non-zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ det(A) = ad - bc}$$

Inverse Calculation:

If det(A) ≠ 0, the inverse of A is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse is obtained by multiplying the reciprocal of the determinant by the adjugate of A.

Example of Inverse of a 2x2 Matrix

Let A be a matrix where:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

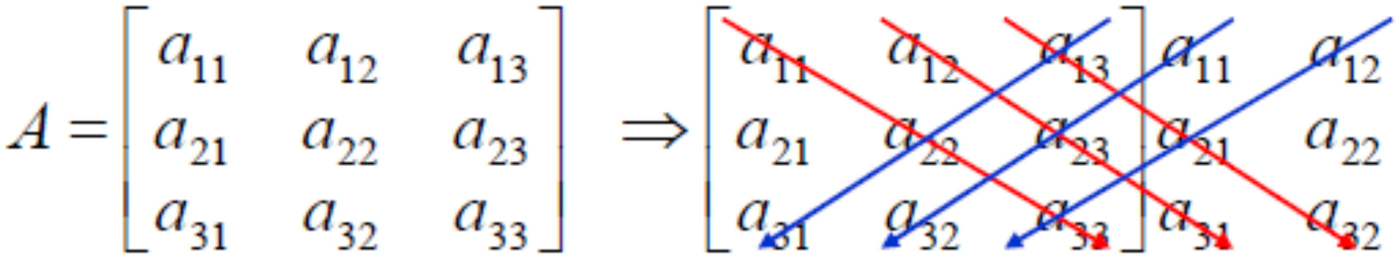
First, we find the determinant of A: $\det(A) = (4)(6) - (7)(2) = 24 - 14 = 10$

Since $\det(A) \neq 0$, matrix A is invertible. Now, we'll use the formula for the inverse of a 2x2 matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

When multiplied by the original matrix A, this inverse matrix will yield the identity matrix, confirming that it is indeed the correct inverse.

Determinant of a 3x3 Matrix: Diagonal Method



Determinant (|A|):

The sum of the products of the down diagonals minus the sum of the products of the up diagonals.

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -(a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

Example: Determinant of a 3x3 Matrix

$$\mathrm{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To use the diagonal method, we need to visualise the matrix with two additional columns that repeat the first two columns of the matrix to the right.

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{array}$$

$$\mathrm{Det(B)} = (1\times5\times9) + (2\times6\times7) + (3\times4\times8) - (3\times5\times7) - (1\times6\times8) - (2\times4\times9) = 0$$

So, matrix B is a *singular matrix*.

Determinant of a 3x3 Matrix: Minors and Cofactors

Square Matrix A:

Elements are denoted by a_{ij} where i and j represent the row and column indices.

Minors (Mij): The minor of element a_{ij} is the determinant of the 2x2 matrix formed by removing the i^{th} row and j^{th} column from A.

Cofactors (Cij): The cofactor of a_{ij} is calculated as

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

This considers the position of the element within the matrix by alternating signs.

Determinant Calculation: To find the determinant of A, multiply each element by its cofactor and sum the products for any row or column.

Determinant of a 3x3 Matrix: Minors and Cofactors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Element	Minor	Element	Minor
a_{11}	$M_{11} = \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$	a_{22}	$M_{22} = \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$
a_{21}	$M_{21} = \det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$	a_{23}	$M_{23} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$
a_{31}	$M_{31} = \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$	a_{33}	$M_{33} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Minors Calculation Formulae for 3 by 3 matrix

Determinant of a 3x3 Matrix:
Minors and Cofactors

Cofactor Definition: For element a_{ij} in an $n \times n$ matrix, the cofactor C_{ij} is found using $(-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}.

Determinant of 3x3 Matrix:

1. Pick any *row* or *column*.

2. Multiply the *minor of each element* by $+1$ or -1 based on the position (sum of indices even or odd).

3. Multiply each element by its *cofactor* and sum up these products to get the determinant.

Formula:

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots$$

Determinant of a 3x3 Matrix:
Minors and Cofactors

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix} \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \quad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$\left|C_{ij}\right| \equiv (-1)^{i+j}\left|M_{ij}\right|$$

The pattern of signs for cofactor minors

Example: Finding Minors and Cofactors for 3x3 matrix

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\text{Minor of } a_{11}: \; M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1\times 1) - (2\times(-2)) = 3$$

$$\text{Cofactor of } a_{11}: \; C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$$

$$\text{Minor of } a_{32}: \; M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1\times 2) - (3\times 4) = -10$$

$$\text{Cofactor of } a_{32}: \; C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$$

The Inverse of $n \times n$ Matrix

Normally, there are two ways to find the inverse of a n x n matrix.

Either using the *Augmented matrix method* or the *Adjoint method*.

We will only be looking at the *Adjoint method*.

The Inverse of $n \times n$ Matrix

Matrix A: An n×n square matrix.

Cofactor C_{ij}: The cofactor of element a_{ij} in matrix A.

Matrix of Cofactors: Constructed by replacing each element of A with its cofactor.

Adjoint of A (adj(A)): The transpose of the matrix of cofactors.

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{t}$$

adjoint matrix

The Inverse of $n \times n$ Matrix

The inverse of A, denoted as A^{-1}, is the adjoint of A divided by the determinant of A.

$$A^{-1} = \frac{1}{\det_{(A)}} \cdot \operatorname{adj}(A),\ \det(A) \neq 0$$

The Inverse of $n \times n$ Matrix

Calculating the Inverse of a *n x n* matrix requires more steps and calculations compared to a 2 x 2 matrix.

The following steps are needed:

Step 1: Make sure that the determinant of the matrix is not 0.

Step 2: Calculate the *Matrix of Minors*,

Step 3: Turn that into the *Matrix of Cofactors* by multiplying in the checkerboard manner

Step 4: Transpose; in other words, calculate the *Adjoint matrix*

Step 5: Multiply that by 1/Determinant.

Example: the Inverse of 3×3 Matrix

Let's consider a 3x3 **matrix *A***:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Calculate the **Matrix of Minors for *A***:

The matrix of minors is found by calculating the determinant for each 2x2 submatrix obtained by removing one row and one column from the original matrix at each element's position:

$$\text{Minors} = \begin{bmatrix} 24 & -5 & -4 \\ 12 & 3 & -2 \\ -2 & 5 & 4 \end{bmatrix}$$

Example: the Inverse of 3×3 Matrix

Calculate the **Matrix of Cofactors for A:**

The matrix of cofactors is obtained by applying $(-1)^{i+j}$ to each element of the matrix of minors:

$$\text{Cofactors} = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Calculate the **Adjoint of A:**

The adjoint matrix, denoted as adj(A), is the transpose of the matrix of cofactors:

$$\text{adj}(A) = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

Example: the Inverse of 3×3 Matrix

Calculate the **Determinant of A:**

The determinant of A is 22, calculated using the standard method for a 3x3 matrix.

Calculate the **Inverse of A:**

The inverse of matrix A, denoted as A^{-1}, is found by dividing the adjoint of A by the determinant of A:

$$A^{-1} = \frac{1}{22} \cdot \mathrm{adj}(A) = \begin{bmatrix} 1.09090909 & -0.54545455 & -0.09090909 \\ 0.22727273 & 0.13636364 & -0.22727273 \\ -0.18181818 & 0.09090909 & 0.18181818 \end{bmatrix}$$

How Matrices are Applied in Computer Science

1. **Computer Graphics:** Used for geometric transformations like translation, rotation, and scaling in 2D and 3D graphics.
2. **Machine Learning:** Represent and manipulate large datasets; crucial in neural networks for storing weights and activations.
3. **Cryptography**: Form the basis of certain encryption algorithms like the Hill Cipher, enhancing security through complex matrix operations.
4. **Network Analysis:** Employed in adjacency matrices for representing and analysing network connections in graph theory.
5. **Algorithm Optimisation:** Integral in optimising computational efficiency in various algorithms, especially in complex mathematical computations.