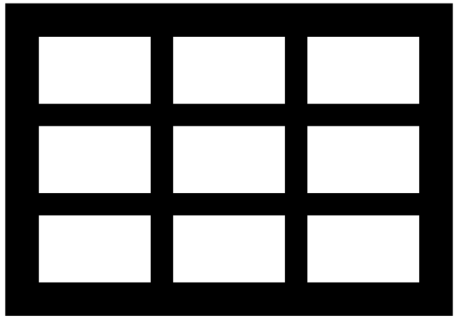
4COSC002W Mathematics for Computing

Lecture 7

Introduction to Matrices. Types of Matrices. Operations with Matrices

What is a Matrix?

A *matrix* is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. The individual items in a matrix are called its *elements* or entries.



Matrix Notation

A matrix is usually denoted by a capital letter (A, B, C, ...) and its elements by the corresponding lowercase letter with two subscript indices (a_{mn}), where:

- The first subscript (m) denotes the row number.
- The second (n) subscript denotes the column number.

For example:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Here, **A** is an **m×n** matrix with **m** rows and **n** columns.

Matrix Order

The ***order*** of a matrix is the number of rows and columns it has.

For Example:

$$M = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 7 & 5 \end{bmatrix}$$

A 3×2 matrix is a matrix that has 3 rows and 2 columns

In this matrix, M:

- The element in the first row, first column is 4 (M_{11}).
- The element in the first row, second column is -1 (M_{12}).
- The element in the second row, first column is 0 (M_{21}).
- The element in the second row, second column is 3 (M_{22}).
- The element in the third row, first column is 7 (M_{31}).
- The element in the third row, second column is 5 (M_{32}).

Types of Matrices

- **Row Matrix:** A matrix with only one row.
- **Column Matrix:** A matrix with only one column.
- **Square Matrix:** A matrix with the same number of rows and columns.
- **Zero Matrix:** A matrix with all elements equal to zero.
- **Identity Matrix:** A square matrix with 1's on the diagonal and 0's elsewhere.
- **Diagonal Matrix:** A square matrix where all elements off the main diagonal are zero.
- **Triangular Matrix:** A square matrix with zero elements above (upper triangular) or below (lower triangular) the main diagonal.

Square Matrix

A *square matrix* has the same number of rows and columns ($n \times n$). Matrices for which $m \neq n$ are called *non-square*.

Example:

$$S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

This is a 2×2 square matrix.

Square matrices have special properties (some explained in next week's lecture notes) and are particularly important in linear algebra.

Many operations, such as taking *determinants* and computing *eigenvalues*, are only defined for square matrices.

Identity Matrix (Unit Matrix)

An *identity matrix* is a square matrix where all elements on the main diagonal are 1, and all off-diagonal elements are 0.

Example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is a 2×2 identity matrix.

The subscript denotes its size.

Identity Matrix (Unit Matrix)

It is the multiplicative identity in matrix algebra, meaning any matrix multiplied by the identity matrix will result in the original matrix.

Important property of identity matrix

If A is an m x n matrix, then

$$I_m A = A \quad \text{and} \quad AI_n = A.$$

If A is a square matrix, then

$$IA = A = AI.$$

Diagonal Matrix

A *diagonal matrix* is a square matrix where all elements off the main diagonal are zero, but the diagonal can contain non-zero elements.

Example:

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

This is a 3×3 diagonal matrix.

Triangular Matrix

A *triangular matrix* is a type of square matrix where all elements above the main diagonal are zero in a lower triangular matrix or all elements below the main diagonal are zero in an upper triangular matrix.

Lower Triangular Matrix Example:

$$\mathrm{L} = \begin{bmatrix} \mathrm{l}_{11} & 0 & 0 \\ \mathrm{l}_{21} & \mathrm{l}_{22} & 0 \\ \mathrm{l}_{31} & \mathrm{l}_{32} & \mathrm{l}_{33} \end{bmatrix}$$

Upper Triangular Matrix Example:

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Row Matrix (Row Vector)

A *row matrix* has only one row and a certain number of columns (n). It is also referred to as a *row vector*.

Example:

$$R = \begin{bmatrix} 1 & -7 & 13 \end{bmatrix}$$

This is a 1×3 row matrix.

Column Matrix (Column Vector)

Conversely, a *column matrix* has only one column and a certain number of rows (m). It is also called a *column vector*.

Example:

$$C = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$$

This is a 3×1 column matrix.

Zero Matrix (Null Matrix)

A *zero matrix* is a matrix where all elements are **zero**. It can be of any dimension m×n.

Example:

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is a 2×2 zero matrix,

but zero matrices can be of any size.

ACTIVITY: Identify the Types of Matrices

Matrix A

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix B

$$\begin{bmatrix} 2 & 4 \\ -1 & 3 \\ 5 & 0 \end{bmatrix}$$

Matrix C

$$\begin{bmatrix} 0 & 4 \\ 7 & 8 \end{bmatrix}$$

Matrix D

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Matrix E

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Operations with Matrices

Similarly to addition, subtraction, multiplication, etc. on numbers, we can also perform these operations (and others) on matrices.

The operations we will now look at are:

- ✓addition, subtraction
- ✓multiplication by scalar
- ✓matrix transpose
- ✓multiplication of matrices

Addition / Subtraction

To add matrices, we add numbers in corresponding positions. However, to enable addition, matrices must be of the same order.

For example, you cannot add a 3x3 matrix and a 4x4 matrix because their numbers of rows and columns do not match

The addition of matrices **A** and **B** is written as **A + B**. If matrix **C** results from the addition of matrices **A** and **B**, then we write **C = A + B**.

If C is the sum of two m x n matrices, A and B, then C is an m x n matrix with elements:

$$\mathbf{c_{ij}} = \mathbf{a_{ij}} + \mathbf{b_{ij}}, \text{ where } i = 1,2,\ldots..m \text{ and } j = 1,2,\ldots...n$$

Example: Addition

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 6 \\ 5 & 2 \end{pmatrix}$ $\mathbf{C} = \mathbf{A} + \mathbf{B}$

$$\mathbf{C} = \begin{pmatrix} 2+0 & 3+6 \\ 4+5 & 5+2 \end{pmatrix} = \begin{pmatrix} 2 & 9 \\ 9 & 7 \end{pmatrix}$$

Multiplication By A Scalar

We can multiply a matrix **by a scalar** (scalar just means number)

This operation means that *each element* of the matrix is multiplied by the scalar

Example:

If matrix **A** = $\begin{pmatrix} 1 & 3 \\ 5 & 0 \end{pmatrix}$ Let's multiply it by **2.**

MULTIPLICATION BY A SCALAR

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 5 & 0 \end{pmatrix}$$

$$2\mathbf{A} = 2\begin{pmatrix} 1 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 10 & 0 \end{pmatrix}$$

Each element of the matrix **A** has been multiplied by **2.**

Matrix Transpose

The *transpose of a matrix* is a new matrix that simply has the rows and columns "exchanged".

We denote the transpose of matrix A as A^T. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

Example:

If matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ Then $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Transposition Process

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$A^T = \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \end{bmatrix}$$

A: 3x2

A^T = 2x3

Transpose

Row 1

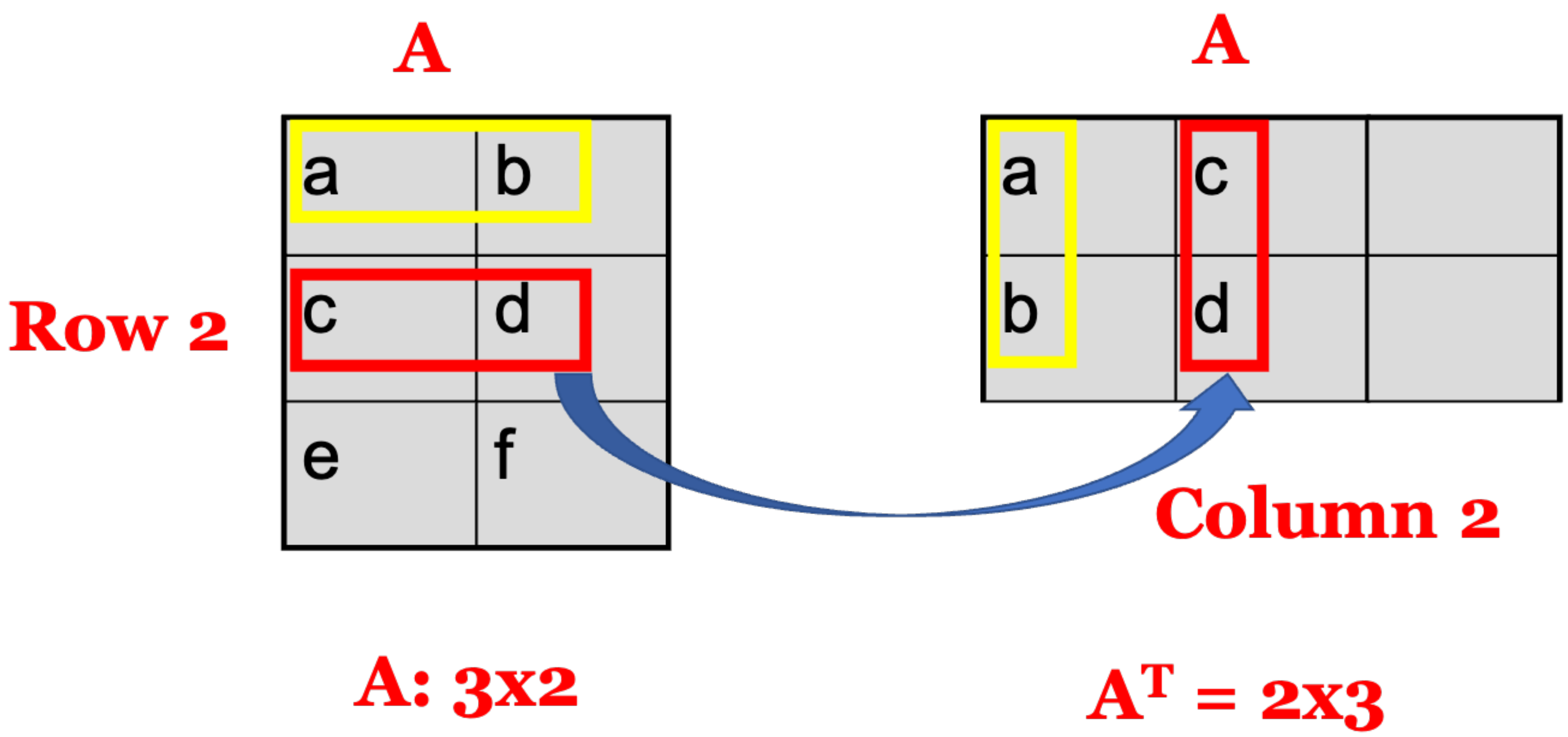
$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

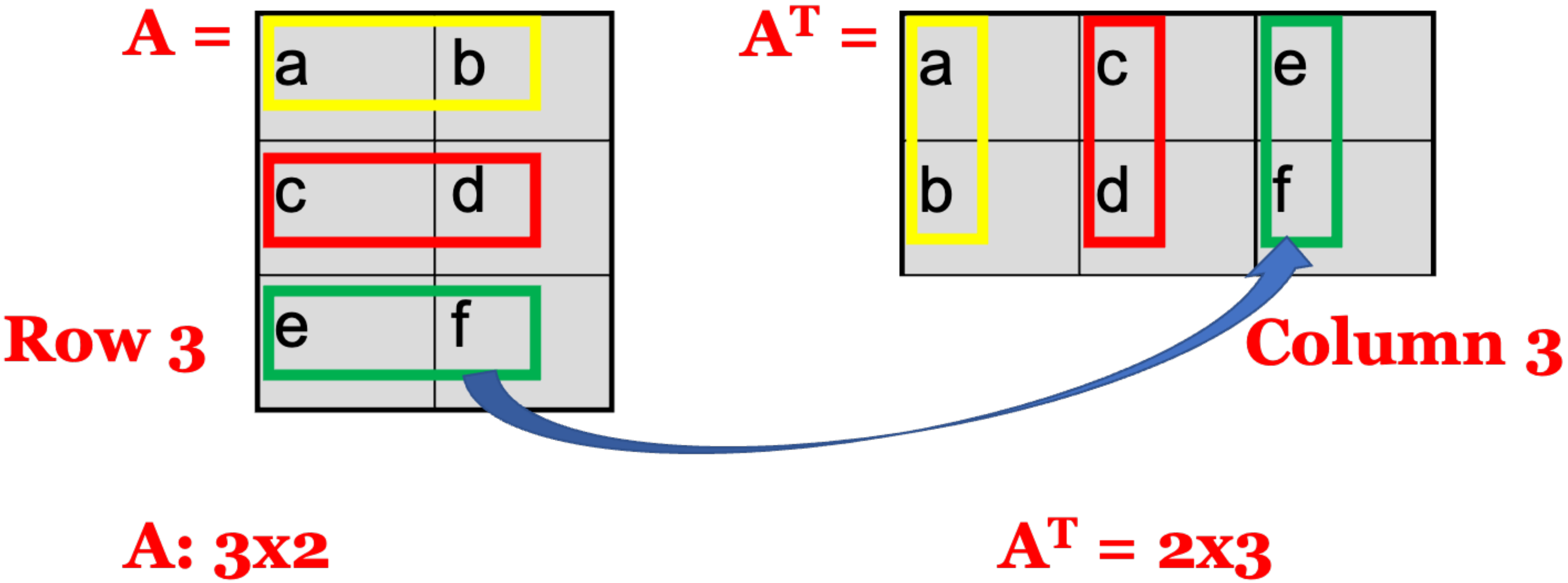
Column 1

$$A^T = \begin{bmatrix} a & & \\ b & & \end{bmatrix}$$

A: 3x2

A^T = 2x3


A
a
b
c
d
e
f
Row 2
A
a
c
b
d
Column 2
A: 3x2
A^T = 2x3



A =
a
b
c
d
Row 3
e
f
A^T =
a
c
e
b
d
f
Column 3
A: 3x2
A^T = 2x3

Matrix Transpose Properties

(1) $(A^T)^T = A$

(2) $(A + B)^T = A^T + B^T$

(3) For a scalar c, $(cA)^T = cA^T$

(4) $(AB)^T = B^T A^T$

Symmetric Matrix

A *symmetric matrix* is a square matrix that is equal to its transpose.

This means that the element in the i^{th} row and j^{th} column equals the element in the j^{th} row and i^{th} column.

Example:

$$Sym = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

For the matrix Sym to be symmetric,

it must hold that Sym = Sym^T.

Matrix Equality

Two matrices are considered *equal* if they have the same dimensions and their corresponding elements are equal.

For matrix equality to hold, every entry in one matrix must match the corresponding entry in the other matrix.

Example:

Matrix A:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix B:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

In contrast, if we have another

Matrix C:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

ACTIVITY: Matrix Equality

Given that the following matrices are equal, find the values of x, y, and z.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ \frac{2}{3} & -5 \\ 6 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} x & 1 \\ \frac{2}{3} & y-10 \\ \frac{z}{2} & 4 \end{bmatrix}$$

ACTIVITY: Answer

To have A = B, I must have all entries equal.

That is I must have $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$, and so forth.

In particular, I must have:

$$
\begin{aligned}
2 &= x \\
-5 &= y - 10 \\
6 &= z/2
\end{aligned}
$$

Solving these three equations, I get:

$x = 2$, $y = 5$, and $z = 12$.

Matrix Multiplication

Matrix multiplication is a binary operation that takes a pair of matrices and produces another matrix. This operation is not element-wise like addition or subtraction but rather involves a sequence of products and sums.

If we have two matrices **A** and **B** then we represent their product as **AB, or A×B.**

Two matrices can only be multiplied if their dimensions match in a specific way: we can multiply two matrices if the number of columns in the first matrix matches the number of rows in the second matrix.

Matrix multiplication is NOT commutative! That is, usually **AB ≠ BA**

Matrix Multiplication

If we have two matrices – one of order m x n and the other of order p x q then we can only multiply them if n = p

m x n p x q

m x q will be the size of the resulting matrix –sometimes called the 'outer dimensions'

Example: Matrix Multiplication

1. Let **A** be 2*x*2 and matrix **B** be 2*x*3 matrices. What is their product?

Answer: They can be multiplied, and the product will be matrix **AB** of the order 2*x*3.

2. Let **C** be 3*x*3, and **D** be 4*x*3. What is their product?

Answer: We cannot do this!

Matrix Multiplication

The steps for multiplying two matrices, A and B:

- The number of columns in the first matrix must equal the number of rows in the second matrix.
- The element at the position (i, j) in the resulting matrix C is calculated by taking the dot product of the i-th row of A and the j-th column of B.
- The dot product mentioned in step 2 is calculated by multiplying corresponding elements from the i-th row of A and the j-th column of B and then summing those products.

The resulting matrix has its elements defined as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Where c_{ij} is the element in the i-th row and j-th column of matrix C, a_{ik} is the element in the i-th row and k-th column of matrix A, and b_{kj} is the element in the k-th row and j-th column of matrix B.

Multiplication Process: C = AB

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ c_{20} & c_{21} \end{bmatrix}$$

E.g.

$$C_{00} = a_{00}*b_{00} + a_{01}*b_{10} + a_{02}*b_{20}$$

Example: Multiplication Process

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & -2 \\ 0 & 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 0 & 5 \\ 2 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & \\ -1 & \\ 4 & \end{bmatrix}$$

$$C_{00} = a_{00}*b_{00} + a_{01}*b_{10} + a_{02}*b_{20}$$

$$C_{00} = (2*3) + (0*0) + (1*2) = 8$$