PHYS2155 Methods in physics II

Tutorial 1

1. (a) If

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{pmatrix} \tag{1}$$

and

$$B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix} , \tag{2}$$

find $2A$, $-3B$, $A - 2B$, and $3A + 4B$.

(b) If

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} , \tag{3}$$

$$B = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \tag{4}$$

and

$$C = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{pmatrix} , \tag{5}$$

find the matrix D such that $2A + B - 3C + 2D = A + 4C$.

2. Compute the product AB if

$$A = \begin{pmatrix} 3 - 2i & i \\ -i & 1 \end{pmatrix} \tag{6}$$

and

$$B = \begin{pmatrix} -1 + i & 2 - i & 0 \\ 1 + 5i & 0 & 3 - 2i \end{pmatrix} , \tag{7}$$

where $i = \sqrt{-1}$.

3. Find a matrix

$$A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

such that

$$A^2 + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{9}$$

4. If A and B are both $n \times n$ matrices, prove the following properties of the trace: (i) $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ and (ii) $\mathrm{Tr}(\alpha A) = \alpha\, \mathrm{Tr}(A)$ where α is a scalar.

5. (a) If A and B are Hermitian matrices, show that $(AB + BA)$ and $i(AB - BA)$ are also Hermitian.
(b) Show that for any non-Hermitian matrix A, there exists two Hermitian matrices B and C such that $A = \alpha B + \beta C$ where α and β are complex numbers. (Hint: Show that $A + A^\dagger$ and $i(A - A^\dagger)$ are Hermitian.)

6. If A and B are $n \times n$ matrices, we define their commutator, denoted $[A, B]$, by
$$[A, B] = AB - BA \ . \tag{10}$$
Thus, $[A, B] = 0$ if and only if A and B commute, $AB = BA$. Indeed, we can define a commutator of two operators in the same manner and the commutators are very important in quantum mechanics. For instance, if two physical observables can be measured at the same time, then their associated operators must commute.

Verify that $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, and $[A_3, A_1] = A_2$ for
$$\begin{aligned} A_1 &= \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \\ A_2 &= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \\ A_3 &= \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \end{aligned} \tag{11}$$

7. Determine all minors and cofactors of the given matrices
$$A = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{pmatrix} . \tag{12}$$

8. Compute the determinant of the following matrix using cofactor expansion along the first row
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} . \tag{13}$$

9. Compute the determinant of the following matrx using cofactor expansion along any row or column that involves least computation,
$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{pmatrix} . \tag{14}$$

10. Find the determinants given below, assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7 \; . \tag{15}$$

(a) $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

(b) $\begin{vmatrix} 3a & -b & 2c \\ 3d & -e & 2f \\ 3g & -h & 2i \end{vmatrix}$

11. Recall that the inverse of an invertible matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{16}$$

is equal to

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{17}$$

Use this formula to find the inverses of the matrices if they exist.

(a)

$$A = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \tag{18}$$

(b)

$$B = \begin{pmatrix} 1 & -3 \\ 4 & -9 \end{pmatrix} \tag{19}$$

12. (a) Suppose A, B, and X are $n \times n$ invertible matrices. Solve the matrix equations (i) $(A^{-1}X)^{-1} = A(B^{-2}A)^{-1}$ and (ii) $ABXA^{-1}B^{-1} = I + A$ for X.

(b) Let A, B, and X be $n \times n$ matrices with A, X and $A - AX$ invertible. Suppose

$$(A - AX)^{-1} = X^{-1}B \; . \tag{20}$$

Explain why B is invertible. Hence, solve the above equation for X. If a matrix needs to be inverted, explain why that matrix is invertible.

13. In quantum mechanics, the state of any physical system is described by a state function that is specified by a vector in a complex vector space. Moreover, every physical observable in quantum mechanics corresponds to a linear Hermitian operator which in general maps one state function to another. In matrix formalism of quantum mechanics, each state function is represented by a column matrix while each operator is represented by a Hermitian square matrices.

The time evolution operator $\hat{T}(t+\epsilon,t)$ describes the change in the state function from time t to $t+\epsilon$. If ϵ is small enough such that the higher-order terms in ϵ can be neglected, the operator $\hat{T}$ is given by

$$\hat{T}(t+\epsilon,t) = 1 - \frac{2i\pi}{h}\epsilon\hat{H} \ , \tag{21}$$

where h is the Planck's constant and $\hat{H}$ is the Hamitonian operator. Let H and T be the matrices representing the operators $\hat{H}$ and $\hat{T}$ and so $T = I - (2i\pi\epsilon/h)H$.
(a) Show that H is Hermitian if T is unitary.
(b) Show that T is unitary if H is Hermitian.

14. (a) Show that $\mathbf{v}$ is an eigenvector of A and find the corresponding eigenvalue λ,

$$A = \begin{pmatrix} -1 & 1 \\ 6 & 0 \end{pmatrix} \ , \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \ . \tag{22}$$

(b) Show that λ is an eigenvalue of A and find the corresponding eigenvectors $\mathbf{v}$,

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{pmatrix} \ , \quad \lambda = -5 \ . \tag{23}$$

15. Find the eigenvalues and the corresponding eigenvectors.

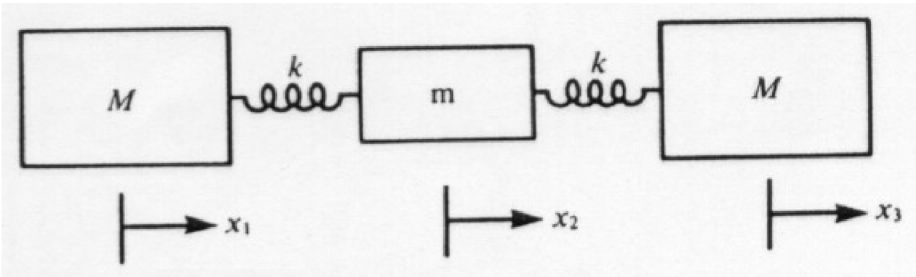
$$C = \begin{pmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{pmatrix} \ . \tag{24}$$

16. The moment of inertia matrix of a rigid body is defined by $\mathcal{I} = (\mathcal{I}_{ij})$, where its components are given by

$$\begin{aligned}
\mathcal{I}_{11} &= \sum_i m_i(r_i^2 - x_i^2) \ , \\
\mathcal{I}_{22} &= \sum_i m_i(r_i^2 - y_i^2) \ , \\
\mathcal{I}_{33} &= \sum_i m_i(r_i^2 - z_i^2) \ , \\
\mathcal{I}_{12} &= -\sum_i m_i x_i y_i \ ,
\end{aligned} \tag{25}$$

and so on.
(a) Find the moment of inertia matrix of a rigid body which consists of three point masses $m_1 = 1$ at $(1,1,-2)$, $m_2 = 2$ at $(-1,-1,0)$ and $m_3 = 1$ at $(1,1,2)$, with all quantities expressed in appropriate units.
(b) Find the eigenvalues and the corresponding normalized eigenvectors for the matrix found in (a).

17. Suppose three masses M, m, and M joined by identical springs of spring constant k are free to slide over a frictionless horizontal surface as shown below. Let $x_1(t)$, $x_2(t)$, and $x_3(t)$ be the displacements of the left, middle, and right masses where $x_1 = x_2 = x_3 = 0$ corresponds to the equilibrium configuration in which all the springs are unextended. Assume that the displacements are small so that the spring forces are governed by Hooke's law.



By applying Newton's second law for each block, it can be shown that the displacements of the masses satisfy the following system of linear differential equations,

$$\begin{aligned}
\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= -\frac{k}{M}(x_1 - x_2) \ , && (26)\\
\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= -\frac{k}{m}(2x_2 - x_1 - x_3) \ , && (27)\\
\frac{\mathrm{d}^2 x_3}{\mathrm{d}t^2} &= -\frac{k}{M}(x_3 - x_2) \ . && (28)
\end{aligned}$$

To find the common frequency ω such that all the masses vibrate at this same frequency, we can write their displacement as

$$\begin{aligned}
x_1(t) &= C_1 \exp(i\omega t) \ ,\\
x_2(t) &= C_2 \exp(i\omega t) \ ,\\
x_3(t) &= C_3 \exp(i\omega t) \ , && (29)
\end{aligned}$$

for some constants C_1, C_2, and C_3. Show that ω satisfy the following eigenvalue problem $A\mathbf{x} = \omega^2 \mathbf{x}$, where

$$A = \begin{pmatrix} k/M & -k/M & 0 \\ -k/m & 2k/m & -k/m \\ 0 & -k/M & k/M \end{pmatrix} \ . \qquad (30)$$

Hence, find all the possible values for ω.

18. Show that A and B are not similar matrices,

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . \qquad (31)$$

19. A matrix A is factored in the form $A = PDP^{-1}$ where D is a diagonal matrix. List the eigenvalues and the corresponding eigenvectors of A,

$$\begin{aligned} A &= \begin{pmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix} . \end{aligned}$$

20. Verify that the matrix A is diagonalizable. Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$,

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} . \tag{32}$$

21. Use the factorization $A = PDP^{-1}$ to compute A^{10}.

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix} . \tag{33}$$