AS/4602/WKC/1

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH4602 Scientific Computing
Assignment 1

Due Date: 12 Feb. 2021 (5:00pm)

1. [Basic matrix knowledge] (a) Let $M_n$ be the set of all $n \times n$ real matrices. Prove or disprove the following statements. Let $S \in M_n$ such that $AS = SA$ for all $A \in M_n$ then we must have $S$ being the **identity matrix**.
(b) Let $S \in M_n$ and all the eigenvalues of $S$ are equal to zero. Then $S$ must be the **zero matrix**.

2. [Computation of determinant] Consider the following $n \times n$ matrix

$$C_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

(a) Show that $\det(C_1) = 2$, $\det(C_2) = 3$ and

$$\det(C_n) = 2 \times \det(C_{n-1}) - \det(C_{n-2}).$$

(b)Use Mathematical Induction (M.I.) to prove that $\det(C_n) = n + 1$.

3. [Partition of a matrix] Let $B$ and $C$ be two $n \times n$ matrices such that $(B - I_n)$ is invertible where $I_n$ is the $n \times n$ identity matrix. By using M.I. on $k$ prove that

$$\begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}^k = \begin{bmatrix} B^k & (B^k - I_n)(B - I_n)^{-1}C \\ 0 & I_n \end{bmatrix}.$$

Here 0 is the $n \times n$ zero matrix.

4. [Inner product] Show that $\langle \mathbf{x}, A\mathbf{y} \rangle = \langle A^*\mathbf{x}, \mathbf{y} \rangle$ where $A^*_{ij} = \overline{A_{ji}}$.

5. [Computing the inverse of a matrix via row operations] Compute the **inverse** of the following matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

by using **elementary row operations** and write down all elementary matrices explicitly.

6. [Computation of eigenvectors and eigenvalues] Find the eigenvalues and associated eigenvectors of the following matrix $A$. Did you obtain a set of **three linearly independent** eigenvectors?

$$A = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix}$$

7. [Positive definite matrix] Determine the **values** of $a$ such that the matrix $A$ is **symmetric positive definite** where

$$A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}.$$

8. [Forward substitution for lower triangular matrix system] Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 & 0 \\ 1/8 & 1/4 & 1/2 & 1 & 0 \\ 1/16 & 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}.$$

Use **forward substitution** to solve the following system of linear equations:

$$L\mathbf{x} = [1\ 1\ 1\ 1\ 1]^T.$$

9. [Breakdown in the LU factorization] (a) Conduct the Doolittle's LU factorization to the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

What did you find?
(b) Interchange the second and the third row of $A$ and re-do the Doolittle's LU factorization again.

10. [LU factorization and computing inverse] (a) Find the **Doolittle's LU factorization** of

$$B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

(b) From (a) obtain the **Cholesky factorization** of $B$.
(c) Find the inverse of $A$ by solving the following linear systems:

$$B\mathbf{x}_1 = [1\ 0\ 0]^T, \quad B\mathbf{x}_2 = [0\ 1\ 0]^T \quad \text{and} \quad B\mathbf{x}_3 = [0\ 0\ 1]^T.$$

11. [For-loop and matrix-vector multiplication in MATLAB] To compute the inner product of two $n \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}^T\mathbf{y}$, there are two methods. Try $n = 10^5, 10^6, 10^7, 10^8$ and comment on the time for the following two MATLAB programs. Here Tic and Toc are used to measure the time elapsed.

**Program 1 (For-Loop)**

```
x=ones(n,1); y=ones(n,1); w=zeros(n,1);
tic
for i=1:n,
  w(i,1)=w(i,1)+x(i,1)*y(i,1);
end;
Time=toc
```

**Program 2 (Direct Vector-Vector Multiplication)**

```
x=ones(n,1); y=ones(n,1); w=zeros(n,1);
tic
w=x'*y;
Time=toc
```

12. [Sherman-Morrison-Woodbury formula] We are given an easy-to-solve linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 14 & 0 & 0 & 0 \\ 2 & 14 & 0 & 0 \\ 1 & 2 & 14 & 0 \\ 0 & 1 & 2 & 14 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = [1\ 0\ 0\ 0]^T.$$

The solution is given by $\mathbf{x} = [0.0714 \quad -0.0102 \quad -0.0036 \quad 0.0012]^T$. Suppose $A$ has perturbed to $\tilde{A}$, where

$$\tilde{A} = \begin{bmatrix} 15 & 1 & 1 & 1 \\ 3 & 15 & 1 & 1 \\ 2 & 3 & 15 & 1 \\ 1 & 2 & 3 & 15 \end{bmatrix}.$$

(a) Apply the Sherman-Morrison-Woodbury Formula to obtain the new solution.
(b) Try (a) again if

$$\tilde{A} = \begin{bmatrix} 13 & 0 & 0 & 1 \\ 2 & 14 & 0 & 0 \\ 1 & 2 & 14 & 0 \\ -1 & 1 & 2 & 15 \end{bmatrix}.$$

1. (a) The statement is false. The matrix $S$ can also be the zero matrix.

(b) The statement is false. One can consider $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. All eigenvalues of $S$ are zero but $S \neq 0$.

2. It is easy to check that $\det(C_1) = 2$, $\det(C_2) = 2 \times 2 - 1 = 3$. Now

$$\det(C_k) = 2 \times \det(C_{k-1}) + \det(V)$$

where $V$ is an $(n-1) \times (n-1)$ matrix

$$V = \left[\begin{array}{c|ccccc} -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{array}\right].$$

We note that $\det(V) = -\det(C_{k-2})$. Hence we have

$$\det(C_n) = 2 \times \det(C_{n-1}) - \det(C_{n-2}).$$

(ii) We are going to prove that $\det(C_n) = n+1$. Clearly the result is true for $n = 1$ and 2. Suppose that $\det(C_n) = n+1$ then $\det(C_{n+1}) = 2(n+1) - n = n+2$. Hence we have proved the result by the principle of M.I. The result implies that $C_n$ is non-singular for all $n$.

3. Clearly when $k = 1$ the equality holds. We assume that

$$\begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}^k = \begin{bmatrix} B^k & (B^k - I_n)(B - I_n)^{-1}C \\ 0 & I_n \end{bmatrix}.$$

Then, we have

$$\begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}^{k+1} = \begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}^k \begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} B^k & (B^k - I_n)(B - I_n)^{-1}C \\ 0 & I_n \end{bmatrix} \begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}.$$

Since

$$B^kC + (B^k - I_n)(B - I_n)^{-1}C = (B^k(B - I_n) + B^k - I_n)(B - I_n)^{-1}C = (B^{k+1} - I_n)(B - I_n)^{-1}C,$$

we have

$$\begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}^{k+1} = \begin{bmatrix} B^{k+1} & (B^{k+1} - I_n)(B - I_n)^{-1}C \\ 0 & I_n \end{bmatrix}.$$

Therefore, by the principle of M.I. we proved the result.

4.

$$\begin{aligned} \langle \mathbf{x}, A\mathbf{y} \rangle &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \bar{A}_{ij} \bar{y}_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{A}_{ij} x_i \bar{y}_j \\ &= \sum_{i=j}^{n} \sum_{i=1}^{n} \bar{A}_{ij} x_i \bar{y}_j \\ &= \langle A^*\mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

5.
$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 30 \\ 0 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & \frac{1}{12} & 0 & -3 & 16 & -15 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 180 \end{bmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & \frac{1}{12} & 0 & -3 & 16 & -15 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

6. Let $\lambda$ be an eigenvalue of $A$, and $v = [v_1\ v_2\ v_3]^T$ be the corresponding eigenvector.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 & 6 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ \iff & \begin{bmatrix} 2-\lambda & -3 & 6 \\ 0 & 3-\lambda & -4 \\ 0 & 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \iff & \det \begin{bmatrix} 2-\lambda & -3 & 6 \\ 0 & 3-\lambda & -4 \\ 0 & 2 & -3-\lambda \end{bmatrix} = (\lambda+1)(\lambda-1)(\lambda-2) = 0 \\ \iff & \lambda = -1, 1, \texttt{ or } 2. \end{aligned}$$

When $\lambda = -1$,
$$\begin{bmatrix} 3 & -3 & 6 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Then, $\mathbf{v}^{(1)} = [-1\ 1\ 1]^T$.

When $\lambda = 1$,
$$\begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Then, $\mathbf{v}^{(2)} = [0\ 2\ 1]^T$.

When $\lambda = 2$,
$$\begin{bmatrix} 3 & -3 & 6 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Then, $\mathbf{v}^{(3)} = [1\ 0\ 0]^T$. Yes. the set of eigenvectors are linearly independent.

7. Let $\mathbf{x} = [x_1, x_2, x_3, x_4]$, then

$$\mathbf{x}A\mathbf{x}^t = (a-1)(x_1^2 + x_2^2 + x_3^2 + x_4^2) + (x_1 + x_2 + x_3 + x_4)^2.$$

Hence, $\mathbf{x}A\mathbf{x}^t > 0$ for all $\mathbf{x} \neq [0,0,0,0]$ if and only if $a > 1$.

8. We have $x_1 = 1$, and we proceed

$$\begin{cases} x_2 &= 1 - (1/2)x_1 = 1/2 \\ x_3 &= 1 - (1/4)x_1 - (1/2)x_2 = 1 - 1/2 = 1/2 \\ x_4 &= 1 - (1/8)x_1 - (1/4)x_2 - 1/2x_3 = 1 - 1/8 - 1/8 - 1/4 = 1/2 \\ x_5 &= 1 - (1/16)x_1 - (1/8)x_2 - (1/4)x_3 - (1/2)x_4 = 1 - 1/16 - 1/16 - 1/8 - 1/4 = 1/2 \end{cases}$$

9. (a)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & L_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

The process breaks when we are solving for $U_{22}$. The value of $U_{22} = 0$.

(b) After interchanging the two rows, we have

$$\tilde{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

10. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.50 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 1.5 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) We have

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0.50 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0.50 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $B = LL^T$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.50 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 05 & 0.5\sqrt{3} & \sqrt{3} \end{bmatrix}.$$

(c)

$$B^{-1} = [\mathbf{x}_1 \; \mathbf{x}_2 \; \mathbf{x}_3] = \frac{1}{36} \begin{bmatrix} 12 & -6 & 0 \\ -6 & 15 & -6 \\ 0 & -6 & 12 \end{bmatrix}$$

11.

| $n$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|---|---|---|---|---|
| Programme 1 time in second | $3.0 \times 10^{-3}$ | $5.3 \times 10^{-3}$ | 0.05 | 0.53 |
| Programme 2 time in second | $1.6 \times 10^{-4}$ | $9.1 \times 10^{-4}$ | 0.007 | 0.07 |

Program 2 is more efficient than Program 1. We remark that the recorded times will depend on the computing machine.

12. (a) We note that $\tilde{A} = A + [1\ 1\ 1\ 1]^T \cdot [1\ 1\ 1\ 1] = \mathbf{u}\mathbf{v}^T$. We first check

$$1 + \mathbf{v}^T A^{-1}\mathbf{u} = 1 + [1\ 1\ 1\ 1]A^{-1}[1\ 1\ 1\ 1]^T = 1.2491 \neq 0.$$

We can apply the formula as follows:

$$\begin{array}{lcl} \widetilde{A}^{-1}\mathbf{b} &=& A^{-1}\mathbf{b} - A^{-1}\mathbf{u}(I + \mathbf{v}^T A^{-1}\mathbf{u})^{-1}\mathbf{v}^T A^{-1}\mathbf{b} \\ \tilde{\mathbf{x}} &=& \mathbf{x} - (1.2491)^{-1} \cdot A^{-1}\mathbf{u}\mathbf{v}^T\mathbf{x} \\ &=& \mathbf{x} - 0.8006 \cdot A^{-1}\mathbf{u} \cdot 0.0588 \\ &=& \mathbf{x} - 0.8006 \cdot 0.0588 \cdot [0.0714\ \ 0.0612\ \ 0.0576\ \ 0.0588]^T \\ &=& [0.0681\ \ -0.0131\ \ -0.0064\ \ -0.0015]^T. \end{array}$$

(b) We note that $\tilde{A} = A + [1\ 0\ 0\ 1]^T \cdot [-1\ 0\ 0\ 1] = \mathbf{u}\mathbf{v}^T$. We first check

$$1 + \mathbf{v}^T A^{-1}\mathbf{u} = 1 + [-1\ 0\ 0\ 1]A^{-1}[1\ 0\ 0\ 1]^T = 1.0012 \neq 0.$$

We can apply the formula as follows:

$$\begin{array}{lcl} \widetilde{A}^{-1}\mathbf{b} &=& A^{-1}\mathbf{b} - A^{-1}\mathbf{u}(I + \mathbf{v}^T A^{-1}\mathbf{u})^{-1}\mathbf{v}^T A^{-1}\mathbf{b} \\ \tilde{\mathbf{x}} &=& \mathbf{x} - (1.0012)^{-1} \cdot A^{-1}\mathbf{u}\mathbf{v}^T\mathbf{x} \\ &=& [0.0764\ \ -0.0109\ \ -0.0039\ \ 0.0063]^T. \end{array}$$