not respect the vanishing slope at $\rho = 0$. Nonetheless, it gives $I[\Psi_1]_{\min} = -1.5136$, which is somewhat, but not greatly, worse than Ψ_2 (5.5% error). The insensitivity of $I[\Psi]$ to errors in the trial function illustrates both a strength and a weakness of the variational method. If the principle is used to estimate eigenvalues (related to the value of $I[\Psi]$), it does well. Used as a method of estimating a solution $\psi \approx \Psi$, it can fail badly, at least in parts of the configuration space.

The reader will recognize from (1.70) that a polynomial source density leads to an exact polynomial solution for ψ, but the idea here is to illustrate the variational method, not to demonstrate a class of explicit solutions. Further illustration is left to the problems at the end of this and later chapters.

1.13 *Relaxation Method for Two-Dimensional Electrostatic Problems*

The relaxation method is an iterative numerical scheme (sometimes called iterative finite difference method) for the solution of the Laplace or Poisson equation in two dimensions. Here we present only its basic ideas and its connection with the variational method. First we consider the Laplace equation with Dirichlet boundary conditions within a two-dimensional region S with a boundary contour C. We imagine the region S spanned by a square lattice with lattice spacing h (and the boundary contour C approximated by a step-like boundary linking lattice sites along C). The independent variables are the integers (i, j) specifying the sites; the dependent variables are the trial values of the potential $\psi(i, j)$ at each site. The potential values on the boundary sites are assumed given.

To establish the variational nature of the method and to specify the iterative scheme, we imagine the functional integral $I[\psi]$ over S as a sum over small domains of area h^2, as shown in Fig. 1.10*a*. We consider the neighboring trial values of the potential as fixed, while the value at the center of the subarea is a variational quantity to be optimized. The spacing is small enough to permit us to approximate the derivatives in, say, the northeast quarter of the subarea by

$$\left(\frac{\partial\psi}{\partial x}\right)_{\text{NE}} = \frac{1}{h}(\psi_{\text{E}} - \psi_0); \qquad \left(\frac{\partial\psi}{\partial y}\right)_{\text{NE}} = \frac{1}{h}(\psi_{\text{N}} - \psi_0)$$

and similarly for the other three quarters. The functional integral over the northeast quarter is

$$\begin{aligned} I_{\text{NE}} &= \frac{1}{2}\int_0^{h/2} dx \int_0^{h/2} dy \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \\ &\approx \frac{1}{8}\left[(\psi_0 - \psi_{\text{N}})^2 + (\psi_0 - \psi_{\text{E}})^2\right] \end{aligned} \tag{1.76}$$

The complete integral over the whole (shaded) subarea is evidently

$$I \approx \frac{1}{4}\left[(\psi_0 - \psi_{\text{N}})^2 + (\psi_0 - \psi_{\text{E}})^2 + (\psi_0 - \psi_{\text{S}})^2 + (\psi_0 - \psi_{\text{W}})^2\right] \tag{1.77}$$

Minimizing this integral with respect to ψ_0 gives the optimum value,

$$(\psi_0)_{\text{optimum}} = \frac{1}{4}(\psi_{\text{N}} + \psi_{\text{E}} + \psi_{\text{S}} + \psi_{\text{W}}) \tag{1.78}$$

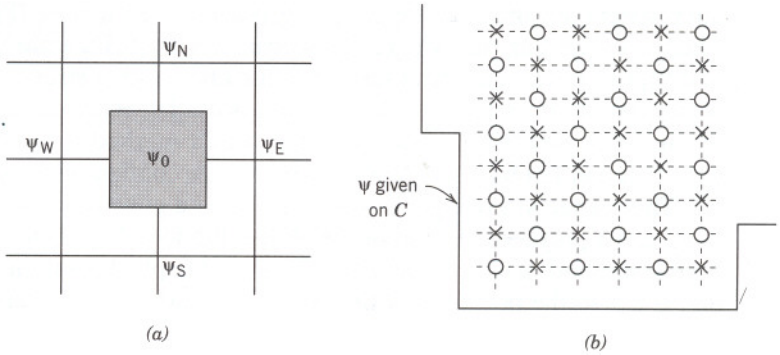

Figure 1.10 (*a*) Enlargement of one of the subareas in the functional integral (shaded). The trial values of the potential at the neighboring sites are labeled ψ_N, ψ_S, ψ_E, and ψ_W, while the value at the center of the subarea is ψ_0. (*b*) One possible iteration is to replace the trial values at the lattice sites (○) with the average of the values at the surrounding sites (×).

The integral is minimized if ψ_0 is equal to the average of the values at the "cross" points.

Now consider the whole functional integral, that is, the sum of the integrals over all the subareas. We guess a set of $\psi(i, j)$ initially and approximate the functional integral $I[\psi]$ by the sum of terms of the form of (1.77). Then we go over the lattice and replace half the values, indicated by the circles in Fig. 1.10*b*, by the average of the points (crosses) around them. The new set of trial values $\psi(i, j)$ will evidently minimize $I[\psi]$ more than the original set of values; the new set will be closer to the correct solution. Actually, there is no need to do the averaging for only half the points—that was just a replication for half of the subareas of the process for Fig. 1.10*a*.

There are many improvements that can be made. One significant one concerns the type of averaging. We could have taken the average of the values at the corners of the large square in Fig. 1.10*a* instead of the "cross" values. Or we could take some linear combination of the two. It can be shown (see Problem 1.21) by Taylor series expansion of any well-behaved function $F(x, y)$ that a particular weighted average,

$$\langle\langle F(x, y)\rangle\rangle \equiv \frac{4}{5}\langle F\rangle_c + \frac{1}{5}\langle F\rangle_s \tag{1.79}$$

where the "cross" and "square" averages are

$$\langle F(x, y)\rangle_c = \frac{1}{4}[F(x+h, y) + F(x, y+h) + F(x-h, y) + F(x, y-h)] \tag{1.80a}$$

$$\langle F(x, y)\rangle_s = \frac{1}{4}[F(x+h, y+h) + F(x+h, y-h) + F(x-h, y+h) + F(x-h, y-h)] \tag{1.80b}$$

yields

$$\langle\langle F(x, y)\rangle\rangle = F(x, y) + \frac{3}{10}h^2\,\nabla^2 F + \frac{1}{40}h^4\,\nabla^2(\nabla^2 F) + O(h^6) \tag{1.81}$$

In (1.81) the Laplacians of F are evaluated at (x, y). If $F(x, y)$ is a solution of the Laplace equation, the weighted averaging over the eight adjacent lattice sites in (1.79) gives F at the center with corrections only of order h^6. Instead of (1.78), which is the same as (1.80a), a better iteration scheme uses $\psi_{\text{new}}(i, j) = \langle\langle\psi(i, j)\rangle\rangle + O(h^6)$. With either the "cross" or "square" averaging separately, the error is $O(h^4)$. The increase in accuracy with $\langle\langle\psi\rangle\rangle$ is at the expense of twice as much computation for each lattice site, but for the same accuracy, far fewer lattice sites are needed: $\langle\langle N\rangle\rangle = O(\langle N\rangle^{2/3})$, where $\langle\langle N\rangle\rangle$ is the number of sites needed with $\langle\langle\psi\rangle\rangle$ and $\langle N\rangle$ is the corresponding number with the "cross" or "square" average.

Equation (1.81) has an added advantage in application to the Poisson equation, $\nabla^2\psi = -g$. The terms of order h^2 and h^4 can be expressed directly in terms of the specified charge density and the simplest approximation for its Laplacian. It is easy to show that the new value of the trial function at (i, j) is generated by

$$\psi_{\text{new}}(i, j) = \langle\langle\psi(i, j)\rangle\rangle + \frac{h^2}{5} g(i, j) + \frac{h^2}{10}\langle g(i, j)\rangle_{\text{c}} + O(h^6) \qquad (1.82)$$

where $\langle g\rangle_{\text{c}}$ is the "cross" average of g, according to (1.80a).

A basic procedure for the iterative numerical solution of the Laplace or Poisson equation in two dimensions with Dirichlet boundary conditions is as follows:

1. A square lattice spacing h is chosen and the lattice sites, including the sites on the boundary, are labeled in some manner [which we denote here as (i, j)].
2. The values of the potential at the boundary sites are entered in a table of the potential at all sites.
3. A guess is made for the values, called $\Phi_{\text{old}}(i, j)$, at all interior sites. A constant value everywhere is easiest. These are added to the table or array of "starting" values.
4. The first iteration cycle begins by systematically going over the lattice sites, one by one, and computing $\langle\langle\Phi(i, j)\rangle\rangle$ with (1.79) or one of the averages in (1.80). This quantity (or (1.82) for the Poisson equation) is entered as $\Phi_{\text{new}}(i, j)$ in a table of "new" values of the potential at each site. Note that the sites next to the boundary benefit from the known boundary values, and so their $\langle\langle\Phi\rangle\rangle$ values are likely initially to be closer to the ultimate values of the potential than those for sites deep in the interior. With each iteration, the accuracy works its way from the boundaries into the interior.
5. Once all interior sites have been processed, the set of $\Phi_{\text{old}}(i, j)$ is replaced by the set of $\Phi_{\text{new}}(i, j)$, and the iteration cycle begins again.
6. Iterations continue until some desired level of accuracy is achieved. For example, one might continue iterations until the absolute value of the difference of old and new values is less than some preassigned value at every interior site.

The scheme just outlined is called Jacobian iteration. It requires two arrays of values of the potential at the lattice sites during each iteration. A better scheme, called Gauss–Seidel iteration, employs a trivial change: one replaces $\Phi_{\text{old}}(i, j)$ with $\Phi_{\text{new}}(i, j)$ as soon as the latter is determined. This means that during an iteration one benefits immediately from the improved values. Typically, at any given site, $\langle\langle\Phi\rangle\rangle$ is made up half of old values and half of new ones, depending

on the path over the lattice. There are many other improvements possible—consult *Press et al.*, *Numerical Recipes*, or some of the references cited at the end of the chapter. The relaxation method is also applicable to magnetic field problems, as described briefly in Section 5.14.

References and Suggested Reading

On the mathematical side, the subject of delta functions is treated simply but rigorously by

Lighthill
Dennery and Kryzwicki

For a discussion of different types of partial differential equations and the appropriate boundary conditions for each type, see

Morse and Feshbach, Chapter 6
Sommerfeld, *Partial Differential Equations in Physics*, Chapter II
Courant and Hilbert, Vol. II, Chapters III–VI

The general theory of Green functions is treated in detail by

Friedman, Chapter 3
Morse and Feshbach, Chapter 7

The general theory of electrostatics is discussed extensively in many of the older books. Notable, in spite of some old-fashioned notation, are

Maxwell, Vol. 1, Chapters II and IV
Jeans, Chapters II, VI, VII
Kellogg

Of more recent books, mention may be made of the treatment of the general theory by Stratton, Chapter III, and parts of Chapter II.

Readers interested in variational methods applied to electromagnetic problems can consult

Cairo and Kahan
Collin, Chapter 4
Sadiku, Chapter 4

and

Pólya and Szegö

for elegant and powerful mathematical techniques.

The classic references to relaxation methods are the two books by R. V. Southwell:

Relaxation Methods in Engineering Science, Oxford University Press, Oxford (1940).
Relaxation Methods in Theoretical Physics, Oxford University Press, Oxford (1946).

Physicists will be more comfortable with the second volume, but much basic material is in the first. More modern references on relaxation and other numerical methods are

Sadiku
Zhou

Problems

1.1 Use Gauss's theorem [and (1.21) if necessary] to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)