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PHYS 2155 Assignment #3

Q.1) a) $\int_C (1+x^2y^2)\,ds$

$C: r(t) = t\hat{i} + 2\sqrt{2}\,e^{t/2}\hat{j} + e^t\hat{k}, \quad 0\le t\le 2$

$= \int_{t=0}^{t=2} [1+t^2(8e^t)]\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

$= \int_0^2 [1+8t^2e^t]\left(1^2+(\sqrt{2}e^{t/2})^2+(e^t)^2\right)^{1/2} dt$

$= \int_0^2 [1+8t^2e^t](1+2e^t+e^{2t})^{1/2}\,dt$

$= \int_0^2 [1+8t^2e^t](1+e^t)^{2\cdot\frac{1}{2}}\,dt$

$= \int_0^2 (1+8t^2e^t+e^t+8t^2e^{2t})\,dt$

$\cancel{= \int_0^2 1+8t^2+1}$

$= [t]_0^2 + [e^t]_0^2 + \int_0^2 8t^2(e^t+e^{2t})\,dt$

$= 2+e^2-1+\left[8t^2\left(e^t+\frac{e^{2t}}{2}\right)\right]_0^2 - \int_0^2 16t\left(e^t+\frac{e^{2t}}{2}\right)dt$

$= 1+e^2+\left[32\left(e^2+\frac{e^4}{2}\right)\right] - \left[16t\left(e^t+\frac{e^{2t}}{4}\right)\right]_0^2 + \int_0^2 16\left(e^t+\frac{e^{2t}}{4}\right)$

$= 1+33e^2+16e^4-(32e^2+8e^4)+\left[16e^t+2e^{2t}\right]_0^2$

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$$= 1+e^2+8e^4+16e^2+2e^4-16-2$$

$$\boxed{= -17+17e^2+10e^4}$$

Ans

b) $\int_C (x^4+y^4)\,ds$

$$= \int_0^2 \left(t^4+64e^{2t}\right)\left(1+e^t\right)dt$$

$$= \int_0^2 t^4+64e^{2t}+t^4e^t+64e^{3t}\,dt$$

$$= \left[\frac{t^5}{5}\right]_0^2+\left[32e^{2t}\right]_0^2+\left[\frac{64e^{3t}}{3}\right]_0^2+\int_0^2 t^4e^t\,dt$$

$$= \frac{32}{5}+32e^4-32+\frac{64e^6}{3}-\frac{64}{3}+\left[t^4e^t\right]_0^2-\int_0^2 4t^3e^t\,dt$$

$$= \frac{96-480-320}{15}+32e^4+\frac{64e^6}{3}+16e^2-\left[4t^3e^t\right]_0^2+\int_0^2 12t^2e^t\,dt$$

$$= \frac{-704}{15}+32e^4+\frac{64e^6}{3}+16e^2-32e^2+\left[12t^2e^t\right]_0^2-\int_0^2 24te^t\,dt$$

$$= \frac{-704}{15}+32e^4+\frac{64e^6}{3}-16e^2+48e^2-\left[24te^t\right]_0^2+\int_0^2 24e^t\,dt$$

$$= \frac{-704}{15}+32e^4+\frac{64e^6}{3}+32e^2-48e^2+\left[24e^t\right]_0^2$$

$$= \frac{-704}{15}+32e^4+\frac{64e^6}{3}\cancel{-32e^2}-16e^2+24e^2-24$$

$$\boxed{= \frac{-1064}{15}+32e^4+\frac{64e^6}{3}+8e^2}$$

Ans.

$x = e^t \qquad \frac{dx^2}{dx}$

$x^2 = e^{2t}$

$\frac{dx^2}{dt} = 2x \cdot \frac{dx}{dt} = 2xe^{2t}$

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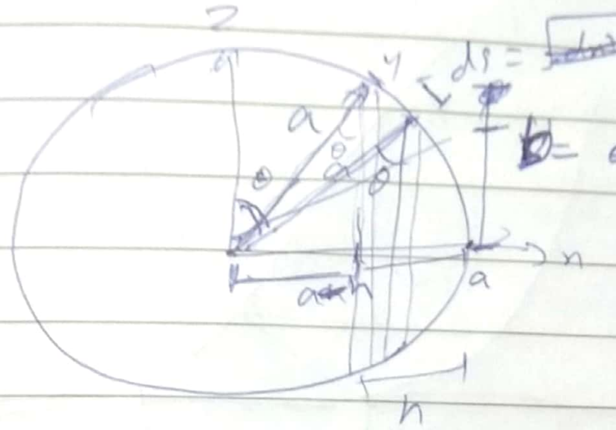
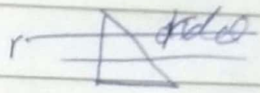
Q.2) a)

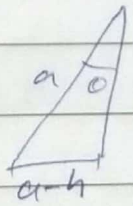




$dA = 2\pi b \cdot ds$, $ds = r\,d\theta$

$A = \int 2\pi a\cos\theta \cdot a\,d\theta$

$= 2\pi a^2 \int_{\sin\theta = 1-\frac{h}{a}}^{\sin\theta = 1} d(\sin\theta)$



$\sin\theta = \frac{a-h}{a}$

$= 2\pi a^2 \left[1 - 1 + \frac{h}{a}\right]$

Ans. $\boxed{Area = 2\pi a h}$

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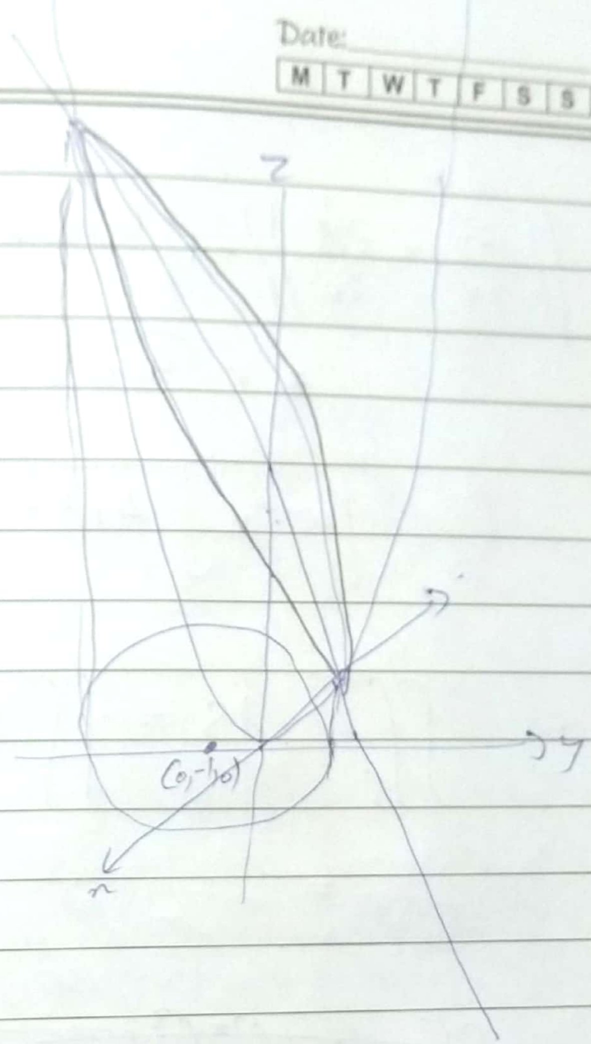
Q.2b)

$z = x^2 + y^2$

$z = 8 - 2y$



intersection:-

$x^2 + y^2 = 8 - 2y$

$x^2 + y^2 + 2y + 1 = 9$

$$\boxed{x^2 + (y+1)^2 = 9}$$

circle of radius 3
centered on (0,-1,0)

Parametrization:-

let $\boxed{x = r\cos\theta}$

$\boxed{y = -1 + r\sin\theta}$

$z = r^2\cos^2\theta + (-1 + r\sin\theta)^2$

$$\boxed{z = r^2 - 2r\sin\theta + 1}$$

let $\vec{r} = (x, y, z)$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 2r - 2\sin\theta \end{bmatrix} \qquad \frac{\partial \vec{r}}{\partial \theta} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ -2r\cos\theta \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{bmatrix} -2r\cos\theta\sin\theta - 2r^2\cos\theta + 2r\cos\theta\sin\theta \\ -2r^2\sin\theta + 2r\sin^2\theta + 2r\cos^2\theta \\ r\cos^2\theta + r\sin^2\theta \end{bmatrix}$$

$$= r\begin{bmatrix} -2r\cos\theta \\ 2 - 2r\sin\theta \\ 1 \end{bmatrix}$$

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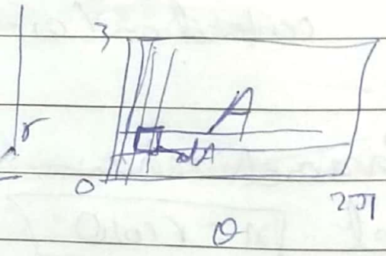
$$\left\|\frac{\partial \vec{r}}{\partial r}\times\frac{\partial \vec{r}}{\partial \theta}\right\| = r\sqrt{4r^2\cos^2\theta + (2-2r\sin\theta)^2 + 1^2}$$

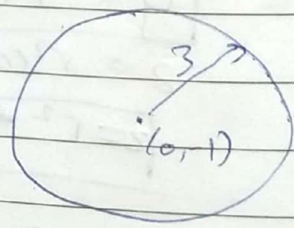
$$= r\sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta - 8r\sin\theta + 4 + 1}$$

$$= r\sqrt{4r^2 - 8r\sin\theta + 5}$$

$$\iint_S dS = \iint \left\|\frac{\partial \vec{r}}{\partial r}\times\frac{\partial \vec{r}}{\partial \theta}\right\| \, dA \; \cancel{dr\,d\theta}$$



$$\cancel{\Rightarrow \iint_S dS = \int_0^{2\pi}\int_0^3 \sqrt{4}}$$



$$\cancel{\Rightarrow \iint_S dS = \int_0^{2\pi}\int_0^3}$$

$$\Rightarrow \boxed{\iint_S dS = \int_0^{2\pi}\int_0^3 \left(\sqrt{4r^2 - 8r\sin\theta + 5}\right) r \, dr\, d\theta}$$

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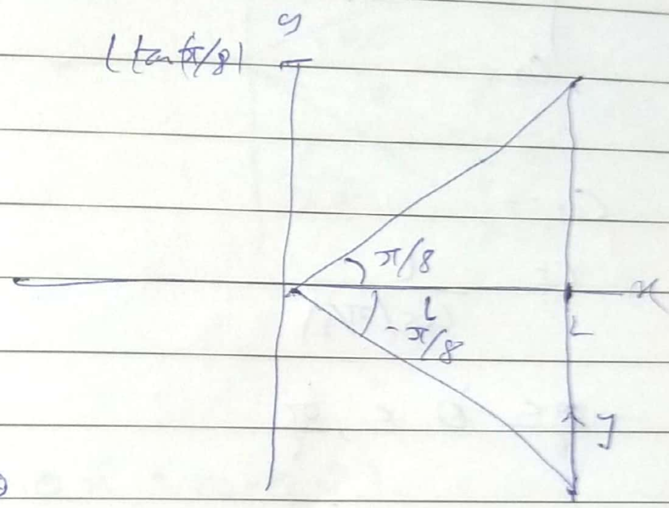
Q.3) a) $F = \frac{1}{r}\hat{\theta}$



$$\int_C F \cdot dl$$

$x = L$, $y = L\tan\theta$

$\frac{\partial x}{\partial \theta} = 0$ $\quad \frac{\partial y}{\partial \theta} = L\sec^2\theta$

$$\int_{-\pi/8}^{\pi/8} F \cdot \begin{bmatrix} 0 \\ L\sec^2\theta \end{bmatrix} d\theta$$

$$F = \frac{1}{\sqrt{x^2+y^2}} \left(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right)$$

$$\Rightarrow \int_{-\pi/8}^{\pi/8} \frac{1}{\sqrt{L^2 + L^2\tan^2\theta}} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ L\sec^2\theta \end{bmatrix} d\theta$$

$$= \int_{-\pi/8}^{\pi/8} \frac{1}{\sqrt{L^2\sec^2\theta}} (L\sec\theta)\, d\theta$$

$$= \int_{-\pi/8}^{\pi/8} \frac{1}{\cancel{L\sec\theta}} (\cancel{L\sec\theta})\, d\theta$$ ($\sec\theta$ is positive for $-\pi/8 \le \theta \le \frac{\pi}{8}$)

$$= \frac{\pi}{8} + \frac{\pi}{8} \boxed{= \frac{\pi}{4}}$$

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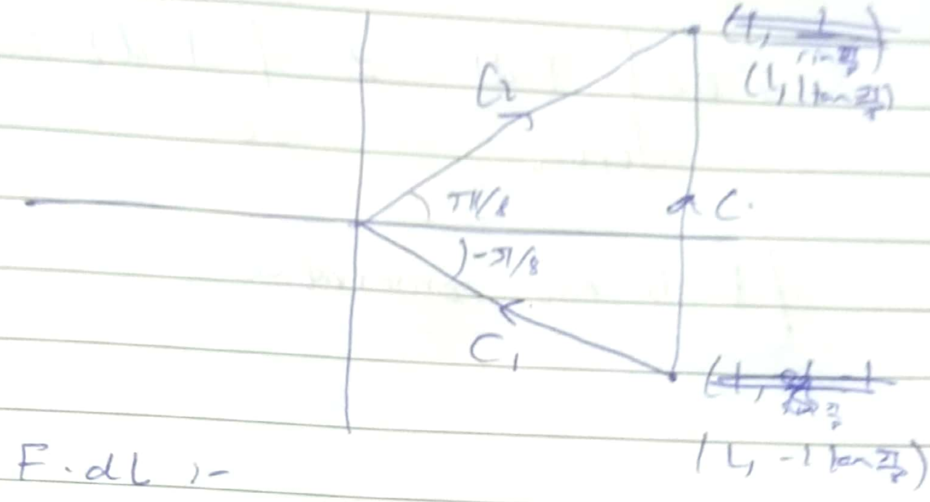
Q.3b)



$$\int_{C_1 \cup C_2} F \cdot dl :-$$

$$= \int_{C_1} F \cdot dl + \int_{C_2} F \cdot dl$$

$C_1 :-$ $\theta = -\pi/8$, $r = l\cos(\frac{\pi}{8}) - t$ $\quad 0 \le t \le l\cos(\frac{\pi}{8})$

$$\frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt}$$

~~$\int_{C_1} F \cdot dl =$~~

$$= r \cancel{\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}} + \hat{r}(-1)$$

$$\boxed{= -\hat{r}}$$

$$\int_{C_1} F \cdot dl = \int_{t=0}^{l\cos\pi/8} \frac{1}{r}\hat{\theta} \cdot (-\hat{r})\, dt \quad \boxed{= 0}$$

$C_2 :-$ $\theta = \frac{\pi}{8}$, $r = t$ $\quad 0 \le t \le \frac{l}{\cos(\pi/8)}$

$$\frac{d\vec{r}}{dt} = \hat{r}$$

$\Rightarrow$ ~~$\int_{C_2} F \cdot dl =$~~

$$\Rightarrow \int_{C_2} F \cdot dl = \int_0^{\frac{l}{\cos\pi/8}} \frac{1}{r}\hat{\theta} \cdot \hat{r}\, dt = 0$$

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Since $0 = \int_{C_1 \cup C_2} \vec{F} \cdot d\vec{l} \neq \int_C \vec{F} \cdot d\vec{l} = \frac{\pi}{4}$,

$\boxed{\vec{F} \text{ is not conservative}}$

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Q. 4/a) $F(x,y) = \left[\frac{\ln(\ln y)}{x} + \frac{2}{3}xy^3\right]\hat{i} + \left(\frac{\ln x}{y\ln y} + x^2y^2\right)\hat{j}$

$$\frac{\partial P}{\partial y} = \frac{1}{x}\cdot\frac{1}{\ln y}\cdot\frac{1}{y} + 2xy^2$$

$$\boxed{= \frac{1}{xy\ln y} + 2xy^2}$$

$$\boxed{\frac{\partial Q}{\partial x} = \frac{1}{xy\ln y} + 2xy^2}$$

$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \boxed{\checkmark \text{ conservative}}$

Note:- its conservative on the region where its defined (and differentiable) i.e. $x>0, y>0$. since $\ln x$ and $\ln y$ are not defined for ~~negative~~ non positive x and y ~~and $\frac{1}{x}$ and $\frac{1}{y}$ are not~~

$$\phi = \int \frac{\ln(\ln y)}{x} + \frac{2}{3}xy^3 \, dx$$

$$\boxed{\phi = \ln x \cdot \ln(\ln y) + \frac{x^2y^3}{3} + C}$$

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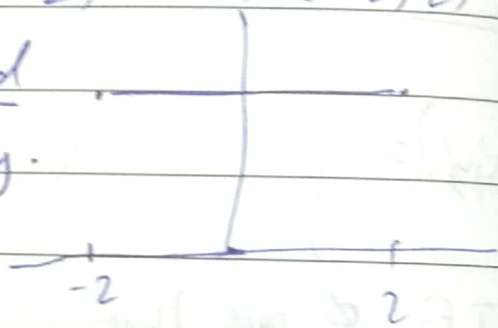
b) The line integral of F along the straight line from $(-2,2)$ to $(2,2)$ is <u>NOT</u> equal to $\phi(2,2) - \phi(-2,2)$ since F is <u>not defined</u> for nonpositive x and y.



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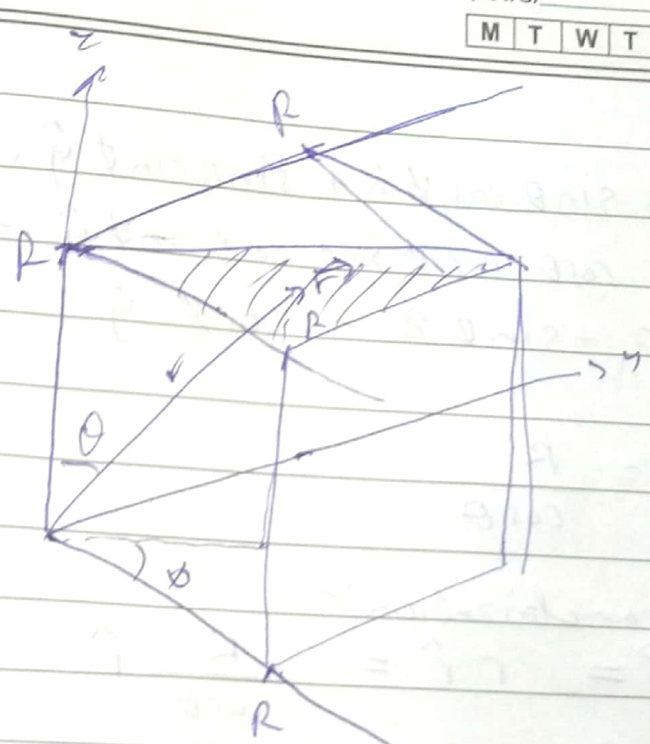
(Q.5)



$r\cos\theta = R$

$\Rightarrow \boxed{r = \dfrac{R}{\cos\theta}}$

$0 \le r\sin\theta\sin\phi \le r\sin\theta\cos\phi$

$\Rightarrow 0 \le \sin\phi \le \cos\phi \Rightarrow \boxed{0 \le \phi \le \pi/4}$

$0 \le r\sin\theta\cos\phi \le r\cos\theta$

$\Rightarrow \cancel{\tan\theta \le \cos\phi} \Rightarrow \cancel{\theta \le \tan^{-1}(\cos\phi)}$

$\Rightarrow \tan\theta \le \sec\phi \Rightarrow \boxed{\theta \le \tan^{-1}(\sec\phi)}$

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$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$$r = \frac{R}{\cos\theta}$$

Parametrization :-

$$\vec{r}(\theta,\phi) = r\hat{r} = \frac{R}{\cos\theta}\hat{r}$$

$$= \frac{R}{\cos\theta}\left(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\right)$$

$$\boxed{= R\tan\theta\cos\phi\,\hat{x} + R\tan\theta\sin\phi\,\hat{y} + R\hat{z}}$$

$$\frac{\partial\vec{r}}{\partial\theta} = R\sec^2\theta\cos\phi\,\hat{x} + R\sec^2\theta\sin\phi\,\hat{y}$$

$$= R\sec^2\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$

$$\frac{\partial\vec{r}}{\partial\phi} = -R\tan\theta\sin\phi\,\hat{x} + R\tan\theta\cos\phi\,\hat{y}$$

$$= R\tan\theta\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi} = R\sec^2\theta\begin{bmatrix}\cos\phi\\ \sin\phi\\ 0\end{bmatrix} \times R\tan\theta\begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\end{bmatrix}$$

$$= R^2\sec^2\theta\tan\theta\begin{bmatrix}0\\0\\1\end{bmatrix}$$

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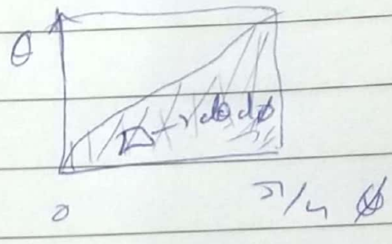
$$\Rightarrow \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right\| = R^2 \sec^2\theta \tan\theta$$

$$\iint_S \cancel{\sin\theta \, d\theta \, d\phi}$$

$$= \cancel{\int_0^{\pi/4} \int_0^{\tan^{-1}(\sec\phi)} \sin\theta \cdot R^2 \sec^2\theta \tan\theta \, d\theta \, d\phi}$$

$$= \int$$



$$\iint_S dS = \iint_S (\sin\theta \, d\theta \, d\phi)$$

$$\Rightarrow \iint_S dS = \iint \left\| \frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial \phi} \right\| dA$$

$$= \int_0^{\pi/4} \int_0^{\tan^{-1}(\sec\phi)} R^2 \sec^2\theta \tan\theta \, d\theta \, d\phi$$

$$= \int_0^{\pi/4} \int_0^{\tan^{-1}(\sec\phi)} R^2 \tan\theta \, d(\tan\theta) \, d\phi$$

$$= R^2 \int_0^{\pi/4} \left[\frac{\tan^2\theta}{2} \right]_0^{\sec\phi} d\phi$$

$$= R^2 \int_0^{\pi/4} \frac{\sec^2\phi}{2} \, d\phi = \frac{R^2}{2} \left[\tan\phi \right]_0^{\pi/4}$$

$$\boxed{= \frac{R^2}{2}}$$