AS/4602/WKC/1

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH4602 Scientific Computing
Assignment 1

Due Date: 12 Feb. 2021 (5:00pm)

1. [Basic matrix knowledge] (a) Let M_n be the set of all $n \times n$ real matrices. Prove or disprove the following statements. Let $S \in M_n$ such that $AS = SA$ for all $A \in M_n$ then we must have S being the **identity matrix**.
(b) Let $S \in M_n$ and all the eigenvalues of S are equal to zero. Then S must be the **zero matrix**.

2. [Computation of determinant] Consider the following $n \times n$ matrix

$$C_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

(a) Show that $\det(C_1) = 2$, $\det(C_2) = 3$ and

$$\det(C_n) = 2 \times \det(C_{n-1}) - \det(C_{n-2}).$$

(b)Use Mathematical Induction (M.I.) to prove that $\det(C_n) = n + 1$.

3. [Partition of a matrix] Let B and C be two $n \times n$ matrices such that $(B - I_n)$ is invertible where I_n is the $n \times n$ identity matrix. By using M.I. on k prove that

$$\begin{bmatrix} B & C \\ 0 & I_n \end{bmatrix}^k = \begin{bmatrix} B^k & (B^k - I_n)(B - I_n)^{-1}C \\ 0 & I_n \end{bmatrix}.$$

Here 0 is the $n \times n$ zero matrix.

4. [Inner product] Show that $\langle \mathbf{x}, A\mathbf{y} \rangle = \langle A^*\mathbf{x}, \mathbf{y} \rangle$ where $A^*_{ij} = \overline{A_{ji}}$.

5. [Computing the inverse of a matrix via row operations] Compute the **inverse** of the following matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

by using **elementary row operations** and write down all elementary matrices explicitly.

6. [Computation of eigenvectors and eigenvalues] Find the eigenvalues and associated eigenvectors of the following matrix A. Did you obtain a set of **three linearly independent** eigenvectors?

$$A = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix}$$

7. [Positive definite matrix] Determine the **values** of a suhc that the matrix A is **symmetric positive definite**.

$$A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}.$$

8. [Forward substitution for lower triangular matrix system] Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 & 0 \\ 1/8 & 1/4 & 1/2 & 1 & 0 \\ 1/16 & 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}.$$

Use **forward substitution** to solve the following system of linear equations:

$$L\mathbf{x} = [1\ 1\ 1\ 1\ 1]^T.$$

9. [Breakdown in the LU factorization] (a) Conduct the Doolittle's LU factorization to the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

What did you find?
(b) Interchange the second and the third row of A and re-do the Doolittle's LU factorization again.

10. [LU factorization and computing inverse] (a) Find the **Doolittle's LU factorization** of

$$B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

(b) From (a) obtain the **Cholesky factorization** of B.
(c) Find the inverse of A by solving the following linear systems:

$$B\mathbf{x}_1 = [1\ 0\ 0]^T, \quad B\mathbf{x}_2 = [0\ 1\ 0]^T \quad \text{and} \quad B\mathbf{x}_3 = [0\ 0\ 1]^T.$$

11. [For-loop and matrix-vector multiplication in MATLAB] To compute the inner product of two $n \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}^T\mathbf{y}$, there are two methods. Try $n = 10^5, 10^6, 10^7, 10^8$ and comment on the time for the following two MATLAB programs. Here Tic and Toc are used to measure the time elapsed.

Program 1 (For-Loop)

```
x=ones(n,1); y=ones(n,1); w=zeros(n,1);
tic
for i=1:n,
  w(i,1)=w(i,1)+x(i,1)*y(i,1);
end;
Time=toc
```

Program 2 (Direct Vector-Vector Multiplication)

```
x=ones(n,1); y=ones(n,1); w=zeros(n,1);
tic
w=x'*y;
Time=toc
```

12. [Sherman-Morrison-Woodbury formula] We are given an easy-to-solve linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 14 & 0 & 0 & 0 \\ 2 & 14 & 0 & 0 \\ 1 & 2 & 14 & 0 \\ 0 & 1 & 2 & 14 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = [1\ 0\ 0\ 0]^T.$$

The solution is given by $\mathbf{x} = [0.0714 \ \ -0.0102 \ \ -0.0036 \ \ 0.0012]^T$. Suppose A has perturbed to $\tilde{A}$, where

$$\tilde{A} = \begin{bmatrix} 15 & 1 & 1 & 1 \\ 3 & 15 & 1 & 1 \\ 2 & 3 & 15 & 1 \\ 1 & 2 & 3 & 15 \end{bmatrix}.$$

(a) Apply the Sherman-Morrison-Woodbury Formula to obtain the new solution.
(b) Try (a) again if

$$\tilde{A} = \begin{bmatrix} 13 & 0 & 0 & 1 \\ 2 & 14 & 0 & 0 \\ 1 & 2 & 14 & 0 \\ -1 & 1 & 2 & 15 \end{bmatrix}.$$