(a) Let A be an non-singular $n \times n$ matrix, u and v be two $n \times k$ $(k \leq n)$ matrices such that the matrix $(I_k + v^t A^{-1} u)$ is non-singular. Prove that

$$\left(A + uv^t\right)^{-1} = A^{-1} - A^{-1} u \left(I_k + v^t A^{-1} u\right)^{-1} v^t A^{-1}.$$

(b) Use the result in 1(a) or otherwise to obtain the solution of the following $n \times n$ linear system:

$$Bx \equiv \begin{bmatrix} n+1 & 1 & \cdots & 1 & 1 \\ 1 & n+2 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 2n-1 & 1 \\ 1 & \cdots & \cdots & 1 & 2n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

where $B_{ii} = n + i$ and $B_{ij} = 1$ when $i \neq j$.

(a)

$$
\begin{aligned}
& \left\{A^{-1} - A^{-1}u\left(I + v^tA^{-1}u\right)^{-1} v^tA^{-1}\right\}\{A + uv^t\} \\
= \quad & I_n + A^{-1}uv^t \quad -A^{-1}u(I + v^tA^{-1}u)^{-1}v^t - A^{-1}u(I + v^tA^{-1}u)^{-1}v^t \\
= \quad & I_n + A^{-1}(uv^t) - A^{-1}u(I + v^tA^{-1}u)^{-1}(I + v^tA^{-1}u)v^t \\
= \quad & I_n + A^{-1}(uv^t) - A^{-1}uI_kv^t \\
= \quad & I_n.
\end{aligned}
$$

(b) We note that

$$B = Diag(n, n+1, \ldots, 2n-1) + u^t \cdot u$$

where $u = (1, 1, \ldots, 1)$. Let

$$z = Diag(n, n+1, \ldots, 2n-1)^{-1} u = (\frac{1}{n}, \frac{1}{n+1}, \ldots, \frac{1}{2n-1})^t$$

and

$$y = Diag(n, n+1, \ldots, 2n-1)^{-1} (1, 0, \ldots, 0)^t = (\frac{1}{n}, 0, \ldots, 0)^t.$$

Since

$$(1+(1,1,\ldots,1)\cdot z) = 1+\sum_{i=1}^{n}\frac{1}{n+i-1} \neq 0$$

we can apply the result in 1(a). Let

$$\alpha = (1+\sum_{i=1}^{n}\frac{1}{n+i-1})^{-1}(n)^{-1}$$

then we have

$$x = B^{-1}(1,0,\ldots,0)^t = y-\alpha z = (\frac{1-\alpha}{n}, \frac{-\alpha}{n+1}, \ldots, \frac{-\alpha}{2n-1})^t.$$

(a) Consider the following $n \times n$ ($n \geq 3$) symmetric matrix:

$$A = \left[\begin{array}{ccccc|c} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & \ddots & \vdots & 0 \\ 0 & 1 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & \vdots \\ 0 & \cdots & 0 & 1 & 2 & 0 \\ \hline 1 & 0 & \cdots & \cdots & 0 & 2 \end{array}\right]_{n \times n} .$$

Prove that A is positive definite by showing that

$\mathbf{x}A\mathbf{x}^t > 0$ if and only if $\mathbf{x} \neq \mathbf{0}$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$.

(b) Suppose that the Cholesky factorization of A takes the following form:

$$\left[\begin{array}{cccccc|c} a_1 & 0 & \cdots & \cdots & 0 & 0 \\ l_1 & a_2 & \ddots & & \vdots & 0 \\ 0 & l_2 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & a_{n-2} & 0 & \vdots \\ 0 & \cdots & 0 & l_{n-2} & a_{n-1} & 0 \\ \hline d_1 & d_2 & \cdots & d_{n-2} & d_{n-1} & a_n \end{array}\right] \left[\begin{array}{ccccc|c} a_1 & l_1 & 0 & \cdots & 0 & d_1 \\ 0 & a_2 & l_2 & \ddots & \vdots & d_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & a_{n-2} & l_{n-2} & d_{n-2} \\ 0 & \cdots & \cdots & 0 & a_{n-1} & d_{n-1} \\ \hline 0 & 0 & \cdots & \cdots & 0 & a_n \end{array}\right].$$

show that

$$\begin{cases} a_1^2 = 2; & \\ a_{i+1}^2 + l_i^2 = 2 & \text{for} \quad i = 1, 2, \ldots, n-2; \\ a_i l_i = 1 & \text{for} \quad i = 1, 2, \ldots, n-2; \\ a_1 d_1 = 1; & \\ a_i d_i = -l_{i-1} d_{i-1} & \text{for} \quad i = 2, \ldots, n-1; \\ \sum_{i=1}^{n-1} d_i^2 + a_n^2 = 2. & \end{cases}$$

(c) Show that

$$a_{i+1}^2 = 2 - \frac{1}{a_i^2} \quad \text{for} \quad i = 1, 2, \ldots, n-2.$$

Hence prove that

$$a_i^2 = \frac{i+1}{i} \quad \text{for} \quad i = 1, 2, \ldots, n-1$$

and

$$l_i^2 = \frac{i}{i+1} \quad \text{for} \quad i = 1, 2, \ldots, n-2.$$

(d) Prove that

$$d_i^2 = \frac{1}{i(i+1)} \quad \text{for} \quad i = 1, 2, \ldots, n-1$$

and

$$a_n^2 = 1 + \frac{1}{n}.$$

(e) What is the operational cost of the Cholesky factorization in (b)?

(a) We note that

$$\mathbf{x}A\mathbf{x}^t = \sum_{i=1}^{n-2}(x_i + x_{i+1})^2 + (x_1 + x_n)^2 + x_{n-1}^2 + x_n^2 \geq 0.$$

Moreover if $\mathbf{x}A\mathbf{x}^t = 0$ then $x_n = x_{n-1} = \ldots = x_1 = 0$.

(b) We note that $LL^t =$

$$\left[\begin{array}{ccccc|c} a_1^2 & a_1 l_1 & \cdots & 0 & & a_1 d_1 \\ a_1 l_1 & l_1^2 + a_2^2 & a_2 l_2 & \vdots & & l_1 d_1 + a_2 d_2 \\ 0 & a_2 l_2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & a_{n-2} l_{n-2} & & \vdots \\ 0 & \cdots & a_{n-2} l_{n-2} & l_{n-2}^2 + a_{n-1}^2 & & l_{n-2} d_{n-1} + a_{n-1} d_{n-1} \\ \hline a_1 d_1 & l_1 d_1 + a_2 d_2 & \cdots & l_{n-2} d_{n-2} + a_{n-1} d_{n-1} & & \sum_{i=1}^{n-1} d_i^2 + a_n^2 \end{array}\right.$$

By comparing coefficients with A, we have

$$\begin{cases} a_1^2 = 2; & \\ a_{i+1}^2 + l_i^2 = 2 & \text{for} \quad i = 1, 2, \ldots, n-2; \\ a_i l_i = 1 & \text{for} \quad i = 1, 2, \ldots, n-2; \\ a_1 d_1 = 1; & \\ a_i d_i = -l_{i-1} d_{i-1} & \text{for} \quad i = 2, \ldots, n-1; \\ \sum\limits^{n-1} d_i^2 + a_n^2 = 2. & \end{cases}$$

(c) Since $a_{i+1}^2 + l_i^2 = 2$ and $a_i l_i = 1$ for $i = 1, 2, \ldots, n-2$, we have

$$a_{i+1}^2 = 2 - \frac{1}{a_i^2} \quad \text{for} \quad i = 1, 2, \ldots, n-2.$$

By using induction on i we have $a_1^2 = 2 = \frac{1+1}{1}$. Assume that $a_i^2 = (i+1)/i$ then

$$a_{i+1}^2 = 2 - \frac{i}{i+1} = \frac{i+2}{i+1}.$$

Hence by the principle of Mathematical Induction we have

$$a_i^2 = \frac{i+1}{i} \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Since $l_i^2 a_i^2 = 1$, we have

$$l_i^2 = \frac{i}{i+1} \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

(d) We have $a_1^2 d_1^2 = 1$ and $a_i^2 d_i^2 = l_{i-1}^2 d_{i-1}^2$ for $i = 2, \ldots, n-1$ and

$$d_1^2 = \frac{1}{a_i^2}, \quad \text{and} \quad d_i^2 = \frac{l_{i-1}^2}{a_i^2} d_{i-1}^2 = \frac{i-1}{i+1} d_{i-1}^2.$$

Therefore

$$d_i^2 = \frac{i-1}{i+1} \times \frac{i-2}{i} \times \frac{i-3}{i-1} \times \frac{i-4}{i-2} \times \cdots \times \frac{2}{4} \times \frac{1}{3} \times d_1^2 = \frac{2}{i(i+1)} \times d_1^2.$$

Since $d_1^2 = \frac{1}{2}$, the result follows.

Moreover,

$$a_n^2 = 2 - \sum_{i=1}^{n-1} d_i^2 = 2 - \sum_{i=1}^{n-1} \frac{1}{i(i+1)} = 2 - \sum_{i=1}^{n-1} (\frac{1}{i} - \frac{1}{i+1}) = 1 + \frac{1}{n}.$$

(e) The operational cost for the Cholesky factorization is $O(n)$. Because from 1(b)-1(d) the total cost for the obtaining a_i^2, l_i^2 and d_i^2 is $O(n)$. Hence the Cholesky factorization takes $O(n)$ in this case.

(a) Show that the inverse of a lower triangular Toeplitz matrix is still a lower triangular Toeplitz matrix.
(b) Let T be a lower triangular Toeplitz matrix given as follows:

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ p & 1 & 0 & \cdots & 0 \\ p^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & p & 1 & 0 \\ p^{n-1} & \cdots & p^2 & p & 1 \end{bmatrix}.$$

Find T^{-1}.

(a) For a lower triangular Toeplitz matrix

$$T = \begin{bmatrix} t_1 & 0 & \cdots & 0 & 0 \\ t_2 & t_1 & 0 & \cdots & 0 \\ t_3 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & t_2 & t_1 & 0 \\ t_n & \cdots & t_3 & t_2 & t_1 \end{bmatrix}.$$

We can obtain the i column of T^{-1} solving $T\mathbf{x}_i = \mathbf{e}_i$ where $\mathbf{e}_i$ is the unit column vector with the ith entry being equal to 1. The linear system can be solved by using forward substitution.

Suppose that $\mathbf{x}_1 = (x_1, x_2, \ldots, x_n)^T$, then one can see that $\mathbf{x}_2 = (0, x_1, x_2, \ldots, x_{n-1})^T$, $\mathbf{x}_3 = (0, 0, x_1, x_2, \ldots, x_{n-2})^T \cdots$, $\mathbf{x}_n = (0, 0, \ldots, 0, x_1)^T$. Eventually, T^{-1} is a lower triangular matrix.

(b) Using the method described in (a)(i) we can solve

$$T^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -p & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -p & 1 & 0 \\ 0 & \cdots & 0 & -p & 1 \end{bmatrix}.$$

(a) Show that the following symmetric matrix B is positive definite where

$$B = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

(b) Consider the following linear system of equations:

$$B\mathbf{x} = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 21 \end{bmatrix}.$$

Apply the Gaussian elimination method (show all your steps) to solve the above linear system.

(c) Using the results in (b) or otherwise, find the Doolittle's LU factorization of B and hence the Cholesky's factorization of B.

(a) We note that

$$[x_1, x_2, x_3, x_4]B[x_1, x_2, x_3, x_4]^T = 3\sum_{i=1}^{4} x_i^2 + (x_1 + x_2 + x_3 + x_4)^2 = 0$$

if and only if $x_1 = x_2 = x_3 = x_4 = 0$.

(b) With multiplier $1/4$ we have

$$B_1 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & 15/4 & 3/4 & 3/4 \\ 0 & 3/4 & 15/4 & 3/4 \\ 0 & 3/4 & 3/4 & 15/4 \end{pmatrix}.$$

Then with multiplier $1/5$, we have

$$B_2 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & 15/4 & 3/4 & 3/4 \\ 0 & 0 & 18/5 & 3/5 \\ 0 & 0 & 3/5 & 18/5 \end{pmatrix}.$$

Finally, with multiplier $1/6$, we have

$$B_3 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & 15/4 & 3/4 & 3/4 \\ 0 & 0 & 18/5 & 3/5 \\ 0 & 0 & 0 & 7/2 \end{pmatrix}.$$

Hence we can solve

$$B_3 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & 15/4 & 3/4 & 3/4 \\ 0 & 0 & 18/5 & 3/5 \\ 0 & 0 & 0 & 7/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 21 \end{pmatrix}$$

by backward substitution, $[x_1, x_2, x_3, x_4] = [-1, -1, -1, 6]$.

(c) Using the multiplier obtained in the Gaussian elimination process, we have $B = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & 1/5 & 1 & 0 \\ 1/4 & 1/5 & 1/6 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 0 & 15/4 & 3/4 & 3/4 \\ 0 & 0 & 18/5 & 3/5 \\ 0 & 0 & 0 & 7/2 \end{bmatrix}.$$

Now we have $B =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & 1/5 & 1 & 0 \\ 1/4 & 1/5 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \frac{15}{4} & 0 & 0 \\ 0 & 0 & \frac{18}{5} & 0 \\ 0 & 0 & 0 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} 1 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $B \equiv L \cdot D \cdot L^T = (L \cdot \sqrt{D})(L \cdot \sqrt{D})^T = P \cdot P^T$ where

$$P = L \cdot \sqrt{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1/2 & \sqrt{15}/2 & 0 & 0 \\ 1/2 & \sqrt{15}/10 & \sqrt{18/5} & 0 \\ 1/2 & \sqrt{15}/10 & \sqrt{1/10} & \sqrt{7/2} \end{bmatrix}.$$

A matrix $A \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$. Show that a strictly diagonally dominant matrix is nonsingular.

Proof:
(Method I.) It is sufficient to show that if $Ax = 0$, then $x = 0$.

Let

$$|x_k| = \max_{1 \leq j \leq n} |x_j|,$$

and we note that $Ax = 0$ can be written as

$$\sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, \ldots, n.$$

We thus have

$$
\begin{aligned}
& |a_{kk}x_k| \\
= \; & |a_{k1}x_1 + \ldots + a_{k,k-1}x_{k-1} + a_{k,k+1}x_{k+1} + \ldots + a_{kn}x_n| \\
\leq \; & |a_{k1}||x_1| + \ldots + |a_{k,k-1}||x_{k-1}| + |a_{k,k+1}||x_{k+1}| + \ldots + |a_{kn}||x_n| \\
\leq \; & (|a_{k1}| + \ldots + |a_{k,k-1}| + |a_{k,k+1}| + \ldots + |a_{kn}|)|x_k|,
\end{aligned}
$$

which implies that

$$|x_k|(|a_{kk}| - \sum_{j=1, j\neq k}^{n} |a_{kj}|) \leq 0.$$

Since the matrix A is strictly diagonally dominant, i.e. $|a_{kk}| > \sum_{j=1,j\neq k}^{n} |a_{kj}|$, we conclude that $|x_k| = 0$. It follows from the definition of $|x_k|$ that $x = 0$.

(Method II.) Let λ be an eigenvalue of A with an eigenvector $\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$ we have for some r ($1 \leq r \leq n$)

$$|x_r| = \max_i \{|x_i|\} > 0.$$

We then define $\mathbf{v} = \frac{1}{x_r}\mathbf{x}$, which is an eigenvector of A such that $A\mathbf{v} = \lambda\mathbf{v}$ and $|v_i| \leq 1$ and $v_r = 1$. Now we consider the rth row of the matrix equation and recall that $v_r = 1$,

$$\sum_{j=1}^{n} a_{rj} v_j = a_{rr} + \sum_{j=1, j \neq r}^{n} a_{rj} v_j = \lambda v_r = \lambda.$$

Thus we have

$$|a_{rr} - \lambda| = \left| \sum_{j=1, j \neq r}^{n} a_{rj} v_j \right| \leq \sum_{j=1, j \neq r}^{n} |a_{rj} v_j| \leq \sum_{j=1, j \neq r}^{n} |a_{rj}| < |a_{rr}|$$

and λ cannot be zero.