Date:

M T W T F S

PHYS 2155

Assignment 1

Q.1

a) A and B are upper triangular

i.e. $A_{ij} = 0$ for $i > j$

$B_{ij} = 0$ for $i > j$

WTS: AB is upper triangular

i.e $AB_{ij} = 0$ ~~for~~ if $i > j$

let $i > j$.

$$AB_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$= \sum_{k=1}^{i-1} A_{ik} B_{kj} + \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$= \sum_{k=1}^{i-1} (0) B_{kj} + \sum_{k=i}^{n} A_{ik} (0)$$

$\left(\begin{array}{l} k < i \\ \Rightarrow A_{ik} = 0 \end{array} \right)$ $\left(\begin{array}{l} k \geq i > j \\ \Rightarrow B_{kj} = 0 \end{array} \right)$

$$= 0 + 0$$

$$\boxed{= 0}$$ QED

Scanned with CamScanner

Date: ______
M T W T F S S

Q.1) b) A and B are symmetric

~~⇒ A and~~

$A = A^T$

~~$\Rightarrow A_{ij} = 0 \quad i \neq j$~~

$B = B^T$

~~$B_{ij} = 0 \quad i \neq j$~~

~~WTS $AB_{ij} = 0 \quad i \neq j$~~

WTS $(AB)^T = AB$

~~let $i \neq j$~~

~~AB.~~

~~$AB_{ij} =$~~

$(AB)^T = B^T A^T = BA$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 \\ -3 & \end{bmatrix}$$

counterexample

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{(Not symmetric)}$$

symmetric   symmetric

Date:

M T W T F S

A'

Q.1c) ~~AB~~ $(AB)^T = B^T A^T = (-B)(-A) = BA \neq -AB$

counter example

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

skew symmetric   skew symmetric   Not skew symmetric

Q.1d) $A^+ = A$ $B^+ = B$

$$(AB)^+ = B^+ A^+ = BA = AB$$

$\Rightarrow AB$ is hermitian

Scanned with CamScanner

Q2) a)

$$A(a,b)\,A(c,d) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

$$= \begin{pmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{pmatrix}$$

$$= \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}$$

$$= A(ac-bd,\ ad+bc)$$

equivalent to $(a+bi)(c+di) = ac-bd+(ad+bc)i$

b) $\det(A(a,b)) = a^2+b^2$

for $A(a,b)$ to be inverse, $\det(A(a,b))$ has to be non zero i.e either a or b has to be non zero.

$$A(a,b)^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{a}{a^2+b^2}\right) & \left(\frac{-b}{a^2+b^2}\right) \\ -\left(\frac{-b}{a^2+b^2}\right) & \left(\frac{a}{a^2+b^2}\right) \end{pmatrix}$$

$$= A\left(\frac{a}{a^2+b^2},\ \frac{-b}{a^2+b^2}\right)$$

The inverse is equivalent to $\frac{1}{(a+bi)}$.

Scanned with CamScanner

Q.3) a)

$$A = \begin{pmatrix} 3 & -8 \\ -2 & 5 \end{pmatrix}$$

$$\det \cancel{A - \lambda I}\ (\lambda I - A) = \begin{vmatrix} \lambda - 3 & 8 \\ 2 & \lambda - 5 \end{vmatrix} = 0$$

$$= (\lambda-3)(\lambda-5) - 16$$

$$= \lambda^2 - 8\lambda + 15 - 16$$

$$= \lambda^2 - 8\lambda - 1$$

$$\lambda = \frac{8 \pm \sqrt{64+4}}{2}$$

$$= \frac{8 \pm 2\sqrt{17}}{2} = \boxed{4 \pm \sqrt{17}}$$

eigenvectors :-

$\lambda = 4 + \sqrt{17}$:-

$$\lambda I - A = \begin{pmatrix} 1+\sqrt{17} & 8 \\ 2 & -1+\sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \cancel{\begin{pmatrix} \frac{1+\sqrt{17}}{1+\sqrt{17}} & 8 \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} 1 & 8/1+\sqrt{17} \\ 2 & -1+\sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow_{r_2 - 2r_1} \begin{pmatrix} 1 & 8/1+\sqrt{17} \\ 0 & \frac{-1-16}{-1+\sqrt{17}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Date: ______
M T W T F S S

$\Rightarrow$ $\cancel{y = s}$

x

eigenvectors:-

$\lambda_1 = 4 + \sqrt{17}$

$$(\lambda_1 I - A)\vec{x} = \begin{pmatrix} 1+\sqrt{17} & 8 \\ 2 & -1+\sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$r_1 = \frac{r_1}{1+\sqrt{17}}$

$r_2 = r_2 - 2r_1$

$$\Rightarrow \begin{pmatrix} 1 & \frac{8}{1+\sqrt{17}} \\ 0 & -1+\sqrt{17} - \frac{16}{1+\sqrt{17}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & \frac{8}{1+\sqrt{17}} \\ 0 & \frac{16-16}{1+\sqrt{17}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ let $y = s$, $\quad x + \frac{8}{1+\sqrt{17}} s = 0$

$$x = \frac{-8}{1+\sqrt{17}}$$

$$\text{eigenvector} = \begin{pmatrix} \frac{-8}{1+\sqrt{17}} \\ 1 \end{pmatrix}$$

$$\text{or } \boxed{c_1 \begin{pmatrix} 8 \\ -1-\sqrt{17} \end{pmatrix}}$$

Date:

M T W T F S

$\lambda_2 = 4 - \sqrt{17}$

$(\lambda_2 I - A)\vec{x} = \begin{pmatrix} 1-\sqrt{17} & 8 \\ 2 & -1-\sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & \frac{8}{1-\sqrt{17}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

~~eigen~~ let $x = \frac{-8}{1-\sqrt{17}} y$

$\boxed{\text{eigenvector} = c_2 \begin{pmatrix} 8 \\ -1+\sqrt{17} \end{pmatrix}}$

Q.3b) ~~Let~~

$$P = \begin{pmatrix} 8 & 8 \\ -1-\sqrt{17} & -1+\sqrt{17} \end{pmatrix}$$

$$P^{-1} = \frac{1}{16\sqrt{17}} \begin{pmatrix} -1+\sqrt{17} & -8 \\ 1+\sqrt{17} & 8 \end{pmatrix}$$

let $\vec{w} = P^{-1}\vec{y}$

then $\frac{d\vec{y}}{dx} = A\vec{y} + \vec{f}$

$$P\frac{d\vec{w}}{dx} = AP\vec{w} + \vec{f}$$

$$\frac{d\vec{w}}{dx} = P^{-1}AP\vec{w} + P^{-1}\vec{f}$$

$$\frac{d\vec{w}}{dx} = \begin{pmatrix} 4+\sqrt{17} & 0 \\ 0 & 4-\sqrt{17} \end{pmatrix}\vec{w} + P^{-1}f$$

$$\frac{dw_1}{dx} = (4+\sqrt{17})w_1 + \frac{(-1+\sqrt{17})f_1 - 8f_2}{16\sqrt{17}}$$

$$\frac{dw_2}{dx} = (4-\sqrt{17})w_2 + \frac{(1+\sqrt{17})f_1 + 8f_2}{16\sqrt{17}}$$

Q.3b (continued)

$$\Rightarrow w_1 = C_1 e^{(4+\sqrt{17})x} - \frac{[(-1+\sqrt{17})f_1 - 8f_2]}{16\sqrt{17}}$$

$$w_2 = C_2 e^{(4-\sqrt{17})x} - \frac{[(1+\sqrt{17})f_1 + 8f_2]}{16\sqrt{17}}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ -1-\sqrt{17} & -1+\sqrt{17} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$y_1 = 8w_1 + 8w_2$$
$$y_2 = (-1-\sqrt{17})w_1 + (-1+\sqrt{17})w_2$$

$$\Rightarrow y_1 = 8\left\{ C_1 e^{(4+\sqrt{17})x} + \frac{(1-\sqrt{17})f_1 + 8f_2}{16\sqrt{17}} + C_2 e^{(4-\sqrt{17})x} + \frac{(-1-\sqrt{17})f_1 - 8f_2}{16\sqrt{17}} \right\}$$

$$y_2 = (-1-\sqrt{17})\left[C_1 e^{(4+\sqrt{17})x} + \frac{(1-\sqrt{17})f_1 + 8f_2}{16\sqrt{17}} \right] + (-1+\sqrt{17})\left[C_1 e^{(4-\sqrt{17})x} + \frac{(-1-\sqrt{17})f_1 - 8f_2}{16\sqrt{17}} \right]$$

Date:

Q.4) a) $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

Base case:- $n = 1$

$$A^1 = A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (1)\lambda \\ 0 & 1 \end{pmatrix}$$

Base case valid.

Assume $A^k = \begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix}$

WTS:- $A^{k+1} = \begin{pmatrix} 1 & (k+1)\lambda \\ 0 & 1 \end{pmatrix}$

(want to show)

$$A^{k+1} = AA^k = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & k\lambda + \lambda \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)\lambda \\ 0 & 1 \end{pmatrix}$$

which is what we wanted to show

Since $A^k = \begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix}$ implies $A^{k+1} = \begin{pmatrix} 1 & (k+1)\lambda \\ 0 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix} \quad \forall \; n \geq 1$

No.

Date:
M T W T F S S

b) $B = \begin{pmatrix} -41 & 92 \\ -23 & 51 \end{pmatrix}$

$|\lambda I - B| = \begin{vmatrix} -41-\lambda & 92 \\ -23 & 51-\lambda \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -(51-\lambda)(41+\lambda) + 2116 = 0$

$\Rightarrow -(2091 - 41\lambda + 51\lambda - \lambda^2) + 2116 = 0$

$-(2091 + 10\lambda - \lambda^2) + 2116 = 0$

$\lambda^2 - 10\lambda - 2091 + 2116 = 0$

$\lambda^2 - 10\lambda + 25 = 0$

$\lambda^2 - 5\lambda - 5\lambda + 25 = 0$

$\lambda(\lambda-5) - 5(\lambda-5) = 0$

$\boxed{\lambda = 5}$

eigenvectors :-

$\begin{pmatrix} -41-5 & 92 \\ -23 & 51-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -46 & 92 \\ -23 & 46 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

eigenvectors $=$ $\cancel{c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$ $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Signature

Date:

Q.4b continued..

$$B = \begin{pmatrix} -41 & 92 \\ -23 & 51 \end{pmatrix}$$

eigenvalue = 5, eigenvectors = $c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Jordon Canonical form of B:-

let $N = B - 5I = \begin{pmatrix} -46 & 92 \\ -23 & 46 \end{pmatrix}$

~~Then~~ $\cancel{S = \begin{pmatrix} 0 & N\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \end{pmatrix}}$

$\cancel{= \begin{pmatrix} 0 & 92 \\ 1 & 46 \end{pmatrix}}$

Then $S = \begin{bmatrix} N\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{bmatrix} = \begin{bmatrix} 92 & 0 \\ 46 & 1 \end{bmatrix}$

$$S^{-1} = \frac{1}{92}\begin{bmatrix} 1 & 0 \\ -46 & 92 \end{bmatrix}$$

$\cancel{S[5}$ Let $J = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ (Jordon Block)

Date:
M T W T F S S

WTS :- $B = SJS^{-1}$:-

$$SJS^{-1} = \begin{bmatrix} 92 & 0 \\ 46 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \cdot \frac{1}{92} \begin{pmatrix} 1 & 0 \\ -46 & 92 \end{pmatrix}$$

$$= \frac{1}{92} \begin{bmatrix} 92 & 0 \\ 46 & 1 \end{bmatrix} \begin{bmatrix} -41 & 92 \\ -230 & 460 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1/2 & \frac{1}{92} \end{bmatrix} \begin{bmatrix} -41 & 92 \\ -230 & 460 \end{bmatrix}$$

$$= \begin{bmatrix} -41 & 92 \\ -\frac{41}{2} - \frac{\cancel{230}\,5}{\cancel{92}\,2} & 46 + \frac{460}{92} \end{bmatrix} = \begin{bmatrix} -41 & 92 \\ -23 & 51 \end{bmatrix} = B$$

Hence $B = SJS^{-1}$

$$B^n = (SJS^{-1})^n = \overbrace{(SJS^{-1})(SJS^{-1}) \ldots SJS^{-1}}^{n \text{ times}}$$
$$= SJ^nS^{-1}$$

$$J = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} = 5 \begin{pmatrix} 1 & 1/5 \\ 0 & 1 \end{pmatrix}$$

$$J^n = 5^n \begin{pmatrix} 1 & n/5 \\ 0 & 1 \end{pmatrix} \quad \text{(from part (a))}$$

$$\Rightarrow B^n = \frac{5^n}{92} \begin{pmatrix} 92 & 0 \\ 46 & 1 \end{pmatrix} \begin{pmatrix} 1 & n/5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -46 & 92 \end{pmatrix}$$

$$= \frac{5^n}{92} \begin{pmatrix} 92 & 0 \\ 46 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{46n}{5} & \frac{92n}{5} \\ -46 & 92 \end{pmatrix}$$

$$= 5^n \begin{pmatrix} 1 & 0 \\ 1/2 & 1/92 \end{pmatrix} \begin{pmatrix} 1 - 46n/5 & \frac{92n}{5} \\ -46 & 92 \end{pmatrix}$$

Signature

Page No.

Date:

$$= 5^n \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{92} \end{pmatrix} \begin{pmatrix} \frac{5-46n}{5} & \frac{92n}{5} \\ -46 & +92 \end{pmatrix}$$

$$= 5^n \begin{pmatrix} \frac{(5-46n)}{5} & \frac{92n}{5} \\ \frac{5-46n}{10} - \cancel{\frac{46}{92}}\frac{5}{10} & \frac{92n}{10}+1 \end{pmatrix}$$

$$= 5^n \begin{pmatrix} \frac{5-46n}{5} & 92n/5 \\ -\frac{23n}{5} & \frac{46n}{5}+1 \end{pmatrix}$$

Ans

$$\boxed{= 5^{n-1} \begin{pmatrix} 5-46n & 92n \\ -23n & 46n+5 \end{pmatrix}}$$

Q.5) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$|A-\lambda I| = \left|\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}\right| = 0$$

$$\Rightarrow \lambda^2 - 0 = 0$$

$\lambda = 0$ (multiplicity $= 2$)

eigenvector :-

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y = 0$$

eigenvectors $= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (Geometric multiplicity $= 1$)

Since the Algebraic multiplicity of $\lambda = 0$ is higher than Geometric Multiplicity, A is not diagonalizable.

Page No.

Signature