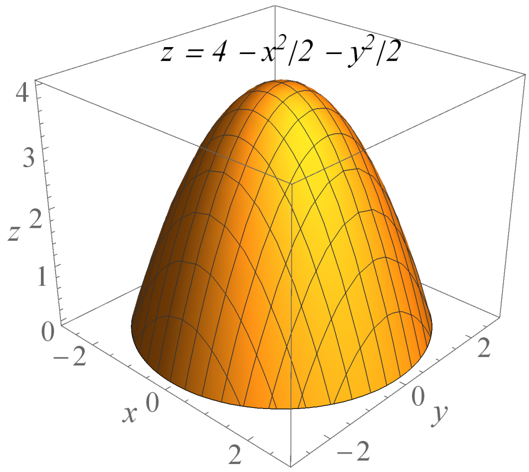
PHYS2155 Methods in physics II

Tutorial 4

1. (a) A conical shell S is given by $z = 4 - 2\sqrt{x^2 + y^2}$, $0 \leq z \leq 4$. At the point (x, y, z) on S, the area mass density $\sigma(x, y, z)$ is proportional to the distance between the point and the z-axis. Find the mass of the shell.

   (b) A surface lamina S is made up of the part of the paraboloid $z = 4 - x^2/2 - y^2/2$, $z \geq 0$. Find the moment of inertia about the z-axis of the lamina if its area mass density at the point (x, y, z) is $\sigma(x, y, z) = k$ where k is a constant.

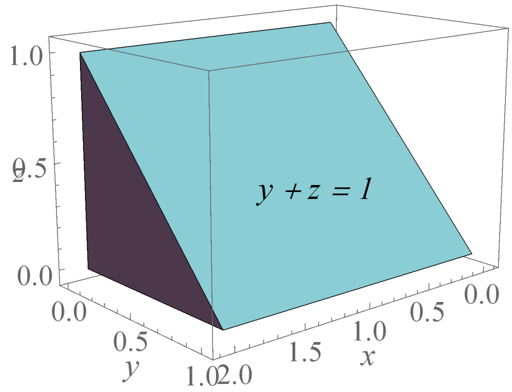


   (c) A thin sheet S is in the shape of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its area mass density at the point (x, y, z) is $\sigma(x, y, z) = 1 + x$. Find the mass and location of center of mass of the sheet.

2. Evaluate the surface integrals over the surfaces S.

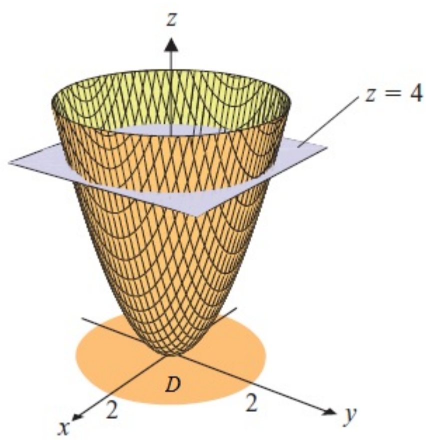
   (a) $\displaystyle\iint_S xyz\,\mathrm{d}S$, S is the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

   (b) $\displaystyle\iint_S (y + z)\,\mathrm{d}S$, S is the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.



(c) $\displaystyle\iint_S (x^2 + y^2 + z^2)\,\mathrm{d}S$, S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$ together with its top and bottom disks.

3. Find the flux of the vector fields $\mathbf{F}$ across the surfaces S. If S is closed, use the positive orientation.
(a) $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2}\,\hat{k}$, S is the portion of the cone $\mathbf{r}(u, v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + 2u\,\hat{k}$, $0 \le u \le \sin v$, $0 \le v \le \pi$, oriented by an upward unit normal vector.
(b) $\mathbf{F}(x, y, z) = x\,\hat{\imath} + y\,\hat{\jmath}$, S is the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$, oriented by an upward unit normal vector



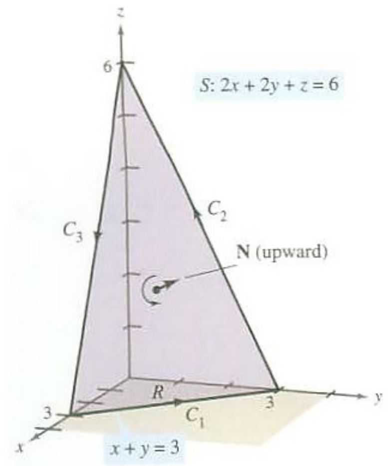
(c) $\mathbf{F}(x, y, z) = x^2\,\hat{\imath} + y^2\,\hat{\jmath} + z^2\,\hat{k}$, S is the surface of the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

4. If the electric field is $\mathbf{E}(r, \theta, z) = (\lambda/2\pi\epsilon_0 r)\,\hat{\boldsymbol{r}}$ in cylindrical coordinates, find the electric flux $\Phi = \iint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S}$ in the outward direction across the cylindrical side wall S of a cylinder defined by $-h \le z \le h$ and $r = R$ where $R > 0$.

5. (a) A fluid has a mass density $870\,\mathrm{kg/m^3}$ and flows in a velocity field $\mathbf{v}(x, y, z) = 3x\,\hat{\imath} + 4ye^{\sin x}\,\hat{\jmath} + z^x\,\hat{k}$ in meter per seconds, where x, y, and

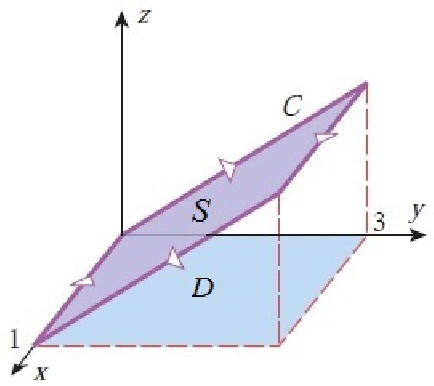
z are measured in meters. Find the rate of flow of the fluid that passes in the $+y$-direction through the surface S defined by $2 \leq x \leq 5$, $y = 0$, and $4 \leq z \leq 6$.
(b) A fluid has a mass density $1060\,\mathrm{kg/m^3}$ and flows with velocity $\mathbf{v}(x, y, z) = \ln y\,\hat{\imath} + \ln z\,\hat{\jmath} + \ln x\,\hat{k}$ in meter per seconds, where x, y, and z are measured in meters. Find the rate of flow of the fluid that passes in $+z$-direction through the surface S lying on the $z = 1$ plane bounded by the lines $2x + 2y = 5$ and the hyperbola $xy = 1$ with $x \geq 0$ and $y \geq 0$.

6. Use Stoke's Theorem to evaluate the following integrals.
(a) The line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (z+\sin x)\,\hat{\imath}+(x+y^2)\,\hat{\jmath}+(y+e^z)\,\hat{k}$ and C is the curve of intersection of the sphere $x^2+y^2+z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$ oriented in a counterclockwise direction when viewed from the positive z-axis.
(b) The line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = -y^2\,\hat{\imath}+z\,\hat{\jmath}+x\,\hat{k}$ and C is the triangle formed by the intersection of the plane $2x+2y+z = 6$ with the coordinate planes oriented in a counterclockwise direction when viewed from the positive z-axis.



(c) The surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = e^{z^2}\,\hat{\imath} + (4z - y)\,\hat{\jmath} + 8x\sin y\,\hat{k}$ and S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ lying above the xy-plane with upward orientation.
(d) The surface integral $\iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xyz\,\hat{\imath}+xy\,\hat{\jmath}+x^2yz\,\hat{k}$ and S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$ with outward orientation.

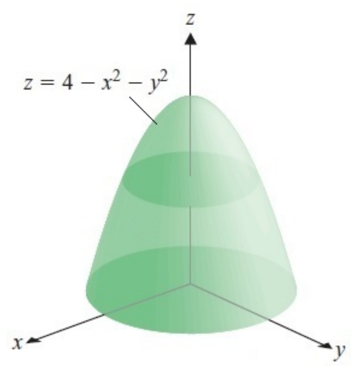
7. (a) Use Stoke's Theorem to find the work done by the force field $\mathbf{F}(x, y, z) = x^2\,\hat{\imath} + 4xy^3\,\hat{\jmath} + y^2x\,\hat{k}$ on a particle that transverses the rectangle C in the plane $z = y$ with vertices $(0, 0, 0)$, $(0, 3, 3)$, $(1, 3, 3)$ and $(1, 0, 0)$ oriented in a clockwise direction when viewed from the positive z-axis.



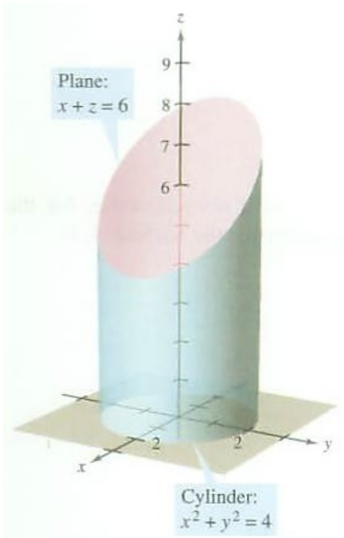
(b) Use Stoke's Theorem and Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ to show that the flux of a magnetic field $\mathbf{B}$ across a surface S with boundary C satisfying the hypotheses of Stoke's Theorem equals the circulation of the vector potential $\mathbf{A}$ around C where $\mathbf{B} = \nabla \times \mathbf{A}$.

8. Evaluate the volume integral of the vector field $\mathbf{F}$ over the region E.
(a) $\mathbf{F}(x, y, z) = x^2\,\hat{\imath} + 2\,\hat{\jmath}$, E is the region defined by $0 \le x \le 1$, $0 \le y \le 2$ and $0 \le z \le (1 - x)$.
(b) $\mathbf{F}(x, y, z) = x\cos y\,\hat{\imath} + z^3\,\hat{\jmath} + xy^2\,\hat{k}$, E is the region defined by $0 \le x \le \sqrt{4 - z^2}$, $0 \le y \le \pi/2$ and $0 \le z \le 2$.
(c) $\mathbf{F}(x, y, z) = e^y\,\hat{\imath} + x\ln z\,\hat{\jmath} + \dfrac{z}{y}\,\hat{k}$, E is the region defined by $0 \le x \le 2y^2$, $z \le y \le 1$ and $0 \le z \le 1$.

9. Use the Divergence theorem to find the outward flux of the vector fields $\mathbf{F}$ across the surfaces S.
(a) $\mathbf{F}(x, y, z) = 2x\,\hat{\imath} + 3y\,\hat{\jmath} + z^2\,\hat{k}$, S is the surface of the cubical region E cut from the first octant by the planes $x = 1$, $y = 1$ and $z = 1$.
(b) $\mathbf{F}(x, y, z) = x^3\,\hat{\imath} + y^3\,\hat{\jmath} + z^3\,\hat{k}$, S is the surface of the solid region E bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.



(c) $\mathbf{F}(x, y, z) = (x^2 - \sin z)\,\hat{\imath} + (xy + \cos z)\,\hat{\jmath} + e^y\,\hat{k}$, S is the surface of the solid region E bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$ and the xy-plane.



10. (a) Use the Divergence Theorem and Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ to show that the flux of a magnetic field $\mathbf{B}$ is equal to zero, i. e. $\iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$, for any closed surface S.
(b) If the temperature function T has a continuous partial derivative with respect to time t, then the heat flow out of the solid with constant mass density ρ occupying the region E bounded by the surface S is given by

$$\iint_S (-k\nabla T) \cdot \mathrm{d}\mathbf{S} = -\iiint_E \rho\sigma \frac{\partial T}{\partial t} \, \mathrm{d}V \ , \tag{1}$$

where k is the conductivity and σ is the specific heat of the solid. Use the Divergence theorem and the above equation to derive the heat equation

$$\frac{\partial T}{\partial t} = \alpha^2 \, \nabla^2 T \ , \tag{2}$$

where $\alpha^2 = k/(\rho\sigma)$.

11. Find the Fourier series for the function $f(x) = x(L - x)$ for $0 \le x \le L$.

12. Find the Fourier series for the function $f(x) = x^3$ for $-1 \le x \le 1$.

13. Find the Fourier transform of the function $f(x) = xe^{-x^2}$.

14. Assume well-behaved, how does the Fourier transform of the derivative of a function relate to the Fourier transform of the function? And higher derivatives?

15. Solve the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \delta(x) \ . \tag{3}$$