MA2108S Tutorial 5 Solution

Prepared by: LuJingyi LuoYusheng

March 2011

Section 3.1

Question 7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.

(a). Use the difinition of limit to show that $\lim(x_n) = 0$.

Proof. Given any $\epsilon > 0$, since $\frac{1}{\epsilon} > 0$, $e^{\frac{1}{\epsilon}} > 1$. By the Archimedean Property, $\exists K \in \mathbb{N}$ such that $K > e^{\frac{1}{\epsilon}} - 1$. i.e. $\ln(K+1) > \frac{1}{\epsilon}$. i.e. $\frac{1}{\ln(K+1)} < \epsilon$. Hence $\left|\frac{1}{\ln(1+n)} - 0\right| = \frac{1}{\ln(1+n)} \leq \frac{1}{\ln(K+1)} < \epsilon \quad \forall n \geq K$.
Hence $\lim(x_n) = 0$. □

7(b). Find a specific value of $K(\epsilon)$ as required in the definition of limit for each of (i) $\epsilon = 1/2$, and (ii) $\epsilon = 1/10$.

(i) For $\epsilon = \frac{1}{2}$, take $K(\epsilon) = 7$. Then $\frac{1}{\ln(K(\epsilon)+1)} < \epsilon$. Hence by part(a)
$\left|\frac{1}{\ln(n+1)} - 0\right| < \frac{1}{\ln(K(\epsilon)+1)} < \epsilon \quad \forall n \geq K(\epsilon)$.

(ii) For $\epsilon = \frac{1}{10}$, take $K(\epsilon) = 22026$. Then $\frac{1}{\ln(K(\epsilon)+1)} < \epsilon$. Hence by part(a)
$\left|\frac{1}{\ln(n+1)} - 0\right| < \frac{1}{\ln(K(\epsilon)+1)} < \epsilon \quad \forall n \geq K(\epsilon)$.

Question 8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the comvergence of (x_n).

Proof. We can see that $\lim(x_n) = 0 \iff \forall \epsilon > 0, \exists K \in \mathbb{N}$ such that $|x_n - 0| = |x_n| = ||x_n| - 0| < \epsilon \quad \forall n \geq K \iff lim(|x_n|) = 0$. Hence $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$ $\square$

The convergence of $(|x_n|)$ need not imply the convergence of (x_n). Example: $x_n := (-1)^n$. Then $\lim(|x_n|) = 1$. But (x_n) does not converge.

Question 9. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$.

Proof. For all $\epsilon > 0$, since $\epsilon^2 > 0$ and $\lim(x_n) = 0, \exists K \in \mathbb{N}$ such that $|x_n| < \epsilon^2 \quad \forall n \geq K$. Since $x_n > 0$ for all $n \in \mathbb{N}$, $|x_n| = x_n < \epsilon^2 \quad \forall n \geq K$. This implies that $\sqrt{x_n} < \epsilon \quad \forall n \geq K$. Hence by definition, $\lim(\sqrt{x_n}) = 0$. $\square$

Question 10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

Proof. Since $\lim(x_n) = x > 0$, take $\epsilon = x > 0$, then $\exists M \in \mathbb{N}$ such that $|x_n - x| < x \quad \forall n \geq M$. Hence $x - x < x_n < x + x \quad \forall n \geq M$. Hence $x_n > 0 \quad \forall n \geq M$. $\square$

Question 11. Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Proof. $\forall n \in \mathbb{N}$, observe that $\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| = \left|\frac{1}{n(n+1)}\right| = \frac{1}{n^2+n} < \frac{1}{n}$. $\forall \epsilon > 0$, by the Archimedean Property, $\exists K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$. Hence $\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| < \frac{1}{n} < \epsilon \quad \forall n \geq K$. Hence by definition, $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$. $\square$

Question 12. Show that $\lim(\frac{1}{3^n}) = 0$.

Proof. Observe that for all n, $\left|\frac{1}{3^n} - 0\right| = \frac{1}{3^n} = \frac{1}{(1+2)^n} \leq \frac{1}{1+2n}$(by the Bernoulli's Inequality)$<$ $\frac{1}{2n}$. Given any $\epsilon > 0$, since $2\epsilon > 0$, by the Archimedean Property, $\exists K \in \mathbb{N}$ such that $\frac{1}{K} < 2\epsilon$, which implies that $\frac{1}{2K} < \epsilon$. Hence $\left|\frac{1}{3^n} - 0\right| < \frac{1}{2n} \leq \frac{1}{2K} < \epsilon \quad \forall n \geq K$. Hence by definition, $\lim(\frac{1}{3^n}) = 0$. $\square$

Question 15. Show that $\lim(\frac{n^2}{n!}) = 0$.

Proof. Observe that $\left|\frac{n^2}{n!} - 0\right| = \frac{n^2}{n!} < \frac{n^2}{n(n-1)(n-2)} = \frac{n^2}{n^3-3n^2+2n} < \frac{n^2}{n^3-3n^2} = \frac{1}{n-3} \quad \forall n > 3$. $\forall \epsilon > 0$, by the Archimedean Property, $\exists K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$. Then $\left|\frac{n^2}{n!} - 0\right| < \frac{1}{n-3} \leq \frac{1}{K} < \epsilon \quad \forall n \geq K+3$. Hence by definition, $\lim(\frac{n^2}{n!}) = 0$. □

Question 16. Show that $\lim(\frac{2^n}{n!}) = 0$.

Proof. For all $n \geq 3$, observe that $\left|\frac{2^n}{n!} - 0\right| = \frac{2^n}{n(n-1)\cdots 3\times 2\times 1} \leq \frac{2^{n-1}}{3^{n-2}} = 2\left(\frac{2}{3}\right)^{n-2}$.
Since $\frac{2}{3} < 1$, let $\frac{2}{3} = \frac{1}{1+h}$ where $h > 0$. Hence by the Bernoulli's inequality, $2\left(\frac{2}{3}\right)^{n-2} = 2\left(\frac{1}{1+h}\right)^{n-2} \leq 2\left(\frac{1}{1+(n-2)h}\right) < \frac{2}{(n-2)h}$ if $n \geq 3$.
Given any $\epsilon > 0$, since $\frac{h\epsilon}{2} > 0$, by the Archimedean Property, $\exists K \in \mathbb{N}$ such that $\frac{1}{K} < \frac{h\epsilon}{2}$, which means $\frac{2}{Kh} < \epsilon$. Hence $\forall n \geq K+2$, $\left|\frac{2^n}{n!} - 0\right| < \frac{2}{(n-2)h} \leq \frac{2}{kh} < \epsilon$. Hence $\lim(\frac{2^n}{n!}) = 0$. □

Question 17. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Proof. Since $\lim(x_n) = x > 0$, take $\epsilon = \frac{x}{2} > 0$. Then by definition, $\exists K \in \mathbb{N}$ such that $|x_n - x| < \epsilon = \frac{x}{2} \quad \forall n \geq K$. Hence $\frac{x}{2} < x_n < \frac{3}{2}x < 2x \quad$ if $n \geq K$. □

Section 3.2

Question 1(b). Establish either the convergence or the divergence of the sequence $X = (x_n)$. $x_n := \frac{(-1)^n n}{n+1}$.

Answer. X is divergent.

Proof. Suppose x_n is convergent and $\lim(x_n) = l$. Take $\epsilon = \frac{1}{2}$. By definition, $\exists K \in \mathbb{N}$ such that $\left|\frac{(-1)^n n}{n+1} - l\right| < \frac{1}{2} \quad \forall n \geq K$.
In particular, for all $n \geq K$, $\left|\frac{(-1)^{2n}2n}{2n+1} - l\right| < \frac{1}{2}$ and $\left|\frac{(-1)^{2n+1}(2n+1)}{(2n+1)+1} - l\right| < \frac{1}{2}$. These imply that $\left|\frac{2n}{2n+1} - l\right| < \frac{1}{2}$ and $\left|\frac{2n+1}{2n+2} + l\right| < \frac{1}{2}$. Hence by the triangle inequality, $\frac{1}{2} + \frac{1}{2} = 1 > \left|\frac{2n}{2n+1} - l\right| + \left|\frac{2n+1}{2n+2} + l\right| \geq \left|\frac{2n}{2n+1} + \frac{2n+1}{2n+2}\right| > \frac{2n}{2n+2} + \frac{2}{2n+2} = 1$ if $n \geq K$. This implies that $1 > 1$, which is a contradiction.
Hence the sequence is divergent. □

Question 2. Give an example of two divergent sequences X and Y such that:

(a). their sum $X + Y$ converges.

Answer. Let $X := ((-1)^n)$ and $Y := ((-1)^{n+1})$. Then X and Y diverge but $X + Y = 0$ converges.

(b). their product XY converges.

Answer. Let $X := (0, 1, 0, 1 \cdots)$ and $Y := (1, 0, 1, 0 \cdots)$. Then X and Y diverge but $XY = 0$ converges.

Question 3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.

Proof. Since X and $X+Y$ are convergent sequences. By limit theorem, $Y = (X+Y) - X$ also converges. $\square$

Question 4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.

Proof. Claim: there exists a K such that for all $n \geq K$, $x_n \neq 0$.

Proof of Claim:

Case 1: $x > 0$. Then take $\epsilon = x > 0$. Since $\lim(X) = x$, by definition, $\exists K \in \mathbb{N}$ such that $|x_n - x| < x \quad \forall n \geq K$. Hence $x_n > x - x = 0 \quad \forall n \geq K$.

Case 2: $x < 0$. Then take $\epsilon = -x > 0$. Since $\lim(X) = x$, by definition, $\exists K \in \mathbb{N}$ such that $|x_n - x| < -x \quad \forall n \geq K$. Hence $x_n < x - x = 0 \quad \forall n \geq K$.

Hence $\exists K \in \mathbb{N}$ such that $x_n \neq 0 \quad \forall n \geq K$. Since XY converges, let $\lim(XY) = z$. Consider the K-tail of X, Y, XY, the K-tail of X converges to x and the K-tail of XY converges to z. Since $Y = \frac{XY}{X} \quad if X \neq 0$, by limit theorem, the K-tail of Y converges to $\frac{z}{x}$. Hence Y converges. $\square$

Question 5. Show that the following sequences are not convergent.

(a). (2^n)

Proof. According to Ex1.13, $2^n > n \quad \forall n \in \mathbb{N}$.

Suppose to the contrary, $\lim(2^n) = l$ exists. Then take $\epsilon = 1$, by definition, $\exists K \in \mathbb{N}$ such that $|2^n - l| < 1 \; \forall n \geq K$. Hence $n < 2^n < l + 1 \quad n \geq K$.

Since $l + 1 \in \mathbb{R}$, by the Archimedean Property, $\exists H \in \mathbb{N}$ such that $H > l + 1$. Hence $\forall n \geq \max\{H, K\}$, $2^n > n > l + 1$. This is a contradiction. Hence (2^n) is not convergent. □

(b). $((-1)^n n^2)$

Proof. Suppose to the countrary, $\lim((-1)^n n^2) = l$ exists. Take $\epsilon = \frac{1}{2}$, by definition, $\exists K \in \mathbb{N}$ such that $|(-1)^n n^2 - l| < \frac{1}{2} \quad \forall n \geq K$.

In particular, $|(-1)^{2n} 4n^2 - l| < \frac{1}{2}$ and $|(-1)^{2n+1}(2n+1)^2 - l| < \frac{1}{2} \quad \forall n \geq K$. These imply that $|4n^2 - l| < \frac{1}{2}$ and $|4k^2 + 4k + 1 + l| < \frac{1}{2} \quad \forall n \geq K$. Hence by the triangle inequality, $1 > |4n^2 - l| + |4k^2 + 4k + 1 + l| \geq |8k^2 + 4k + 1| > 1 \quad \forall n \geq K$. This implies that $1 > 1$, which is a contradiction.

Hence $((-1)^n n^2)$ is divergent. □

Question 6. Find the limits of the following sequences.

(b). $\lim \left(\frac{(-1)^n}{n+2} \right)$

Answer. Since $-\frac{1}{n+2} \leq \frac{(-1)^n}{n+2} \leq \frac{1}{n+2}$. Also, $\lim\left(-\frac{1}{n+2}\right) = \lim\left(\frac{1}{n+2}\right) = 0$.

Hence by the Squeeze Theorem, $\lim\left(\frac{(-1)^n}{n+2}\right) = 0$.

(d). $\lim \left(\frac{n+1}{n\sqrt{n}} \right)$

Answer. $\lim\left(\frac{n+1}{n\sqrt{n}}\right) = \lim\left(\frac{\frac{n}{n\sqrt{n}} + \frac{1}{n\sqrt{n}}}{1}\right) = \lim\left(\frac{1}{\sqrt{n}}\right) + \lim\left(\frac{1}{n}\right)\lim\left(\frac{1}{\sqrt{n}}\right) = 0.$

Question 7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 cannot be used.

Proof. Since (b_n) is bounded, there exists $M > 0$ such that $|b_n| \leq M \quad \forall n \in \mathbb{N}$. Given any $\epsilon > 0$, since $\lim(a_n) = 0$, and $\frac{\epsilon}{M} > 0$, by definition, $\exists K \in \mathbb{N}$ such that $|a_n| < \frac{\epsilon}{M} \quad \forall n \geq K$. Hence $|a_n b_n| < |a_n| M < \epsilon \quad \forall n \geq K$.

Hence $\lim(a_n b_n) = 0$. Alternatively, it also follows from the squeeze theorem since $-M|a_n| \leq |b_n| \leq M|a_n|$ for all n. $\square$

Theorem 3.2.3 cannot be used because (b_n) may not be convergent.

Question 9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that (y_n) and $(\sqrt{n} y_n)$ converge. Find their limits.

Answer. Observe that $y_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. Hence $0 \leq \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}}$.
Since $\lim(0) = \lim\left(\frac{1}{\sqrt{n}}\right) = 0$, by the Squeeze Theorem, $\lim(y_n) = 0$.
Observe that $\sqrt{n} y_n = \sqrt{n(n+1)} - n = \frac{n}{\sqrt{n(n+1)}+n} = \frac{1}{\frac{\sqrt{n(n+1)}}{n}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$. Hence $\lim(\sqrt{n} y_n) = \frac{1}{\sqrt{1+\lim\left(\frac{1}{n}\right)}+1} = \frac{1}{2}$.

Question 10. Determine the following limits.

(a). $\lim((3\sqrt{n})^{1/2n})$.

Answer. Note that by Example 3.1.11(c), if $c > 0$, then $\lim(c^{1/n}) = 1$. By Example 3.1.11(d), $\lim(n^{1/n}) = 1$.

Hence, $\lim((3\sqrt{n})^{1/2n}) = \lim(3^{1/2n}) \lim(n^{1/4n}) = \lim(3^{1/2n}) \lim\left(\frac{1}{4}\right)^{1/4n} \lim((4n)^{1/4n}) = 1 \times 1 \times 1 = 1$.

(b). $\lim((n+1)^{1/\ln(n+1)})$.

Answer. Let $(n+1)^{1/\ln(n+1)} = a$, then $n+1 = a^{\ln(n+1)}$. Hence $a = e$.

Hence $\lim((n+1)^{1/\ln(n+1)}) = \lim(e) = e$.

Question 12 If $a > 0, b > 0,$ show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = \frac{a+b}{2}$.

Proof.

$$\lim\left(\sqrt{(n+a)(n+b)}-n\right)=\lim\left(\frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n}\right)=\lim\left(\frac{(a+b)n+ab}{\sqrt{(n+a)(n+b)}+n}\right)$$
$$=\lim\left(\frac{a+b+\frac{ab}{n}}{\sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1}\right)=\frac{\lim(a+b)+\lim\frac{ab}{n}}{\sqrt{\lim(1+\frac{a}{n})\cdot\lim(1+\frac{b}{n})}+1}=\frac{a+b}{2}.$$

Question 13 Use the Squeeze Theorem 3.2.7 to determine the limits of the following.

$(a)\ (n^{\frac{1}{n^2}})$ $(b)\ ((n!)^{\frac{1}{n^2}})$

Proof.

(a) Since $n \leq n^n$ for all n,

then $1 \leq (n)^{\frac{1}{n^2}} \leq (n^n)^{\frac{1}{n^2}} = n^{\frac{1}{n}}$.

But $\lim 1 = \lim(n^{\frac{1}{n}}) = 1$,

hence by the Squeeze Theorem, we have $\lim((n)^{\frac{1}{n^2}}) = 1$.

(b) Since $n! \leq n^n$ for all n,

then $1 \leq (n!)^{\frac{1}{n^2}} \leq (n^n)^{\frac{1}{n^2}} = n^{\frac{1}{n}}$.

But $\lim 1 = \lim(n^{\frac{1}{n}}) = 1$,

hence by the Squeeze Theorem, we have $\lim((n!)^{\frac{1}{n^2}}) = 1$.

Question 14 Show that if $z_n := (a^n + b^n)^{\frac{1}{n}}\ where\ 0 < a < b$, then $\lim(z_n) = b$.

Proof.

Since $0 < a < b$,

then $b^n \leq a^n + b^n \leq 2 \cdot b^n$,

then $(b^n)^{\frac{1}{n}} \leq (a^n + b^n)^{\frac{1}{n}} \leq (2 \cdot b^n)^{\frac{1}{n}}$.

But $\lim(b^n)^{\frac{1}{n}} = b = \lim 2^{\frac{1}{n}} \cdot \lim(b^n)^{\frac{1}{n}} = \lim(2 \cdot b^n)^{\frac{1}{n}}$,

hence by the Squeeze Theorem, we have $\lim(a^n + b^n)^{\frac{1}{n}} = b$,

i.e. $\lim(z_n) = b$.

Question 15 Apply theorem 3.2.11 to the following sequences, where a, b satisfy $0 < a < 1, b > 1$.

$(a)\ (a^n)$ $(b)\ (\frac{b^n}{2^n})$

$(c)\ (\frac{n}{b^n})$ $(d)\ (\frac{2^{3n}}{3^{2n}})$

Proof.

(a) Since $a^n > 0 \, \forall \, n, \; and \; \lim \frac{a^{n+1}}{a^n} = a < 1$,

hence by Theorem 3.2.11, we have $\lim (a^n) = 0$.

(b) Since $\frac{b^n}{2^n} > 0 \, \forall \, n, \; and \; \lim \frac{\frac{b^{n+1}}{2^{n+1}}}{\frac{b^n}{2^n}} = \frac{b}{2}$,

but $\frac{b}{2}$ can be either greater or smaller than 1,

hence Theorem 3.2.11 does not apply here.

(c) Since $\frac{n}{b^n} > 0 \, \forall \, n, \; and \; \lim \frac{\frac{n+1}{b^{n+1}}}{\frac{n}{b^n}} = \frac{1}{b} < 1$,

hence by Theorem 3.2.11, we have $\lim (\frac{n}{b^n}) = 0$.

(a) Since $\frac{2^{3n}}{3^{2n}} > 0 \, \forall \, n, \; and \; \lim \frac{\frac{2^{3(n+1)}}{3^{2(n+1)}}}{\frac{2^{3n}}{3^{2n}}} = \frac{8}{9} < 1$,

hence by Theorem 3.2.11, we have $\lim (\frac{2^{3n}}{3^{2n}}) = 0$.

Question 16

(a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim \frac{x_{n+1}}{x_n} = 1$.

(b) Give an example of a divergent sequence with this property.

Answer.

(a) Consider the convergent sequence $(x_n) = (\frac{1}{n})$.

We have $\lim (x_n) = 0$ and $\lim (\frac{x_{n+1}}{x_n}) = \lim \frac{n}{n+1} = 1$.

(b) Consider the divergent sequence $(x_n) = (n)$.

We have (x_n) is not bounded and hence diverge, and

$\lim (\frac{x_{n+1}}{x_n}) = \lim \frac{n+1}{n} = 1$.

Question 17 Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim (\frac{x_{n+1}}{x_n}) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.

Proof.

Since $\lim\left(\frac{x_{n+1}}{x_n}\right) > 1$,

then $\exists\, \rho > 1$ *and* $N_1 \in \mathbb{N}$, such that $\frac{x_{n+1}}{x_n} > \rho\ \forall\, n \geq N_1$.

Hence $x_{N_1+k} \geq \rho^k x_{N_1},\ \forall k \in \mathbb{N}$.

Let α be any given positive real number.

Since (ρ^k) is unbounded,

hence $\exists\, N_2 \in \mathbb{N}$, such that $\rho^k > \frac{\alpha}{x_{N_1}}\ \forall\, k \geq N_2$.

Then $\forall\, n \geq N_1 + N_2,\ x_n \geq \rho^{n-N_1} \cdot x_{N_1} > \frac{\alpha}{x_{N_1}} \cdot x_{N_1} = \alpha$.

Therefore, X is not a bounded sequence and hence not convergent.

Question 18 Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1, b > 1$.

$(a)\ (n^2 a^n)$ $\qquad$ $(b)\ \left(\frac{b^n}{n^2}\right)$

$(c)\ \left(\frac{b^n}{n!}\right)$ $\qquad$ $(d)\ \left(\frac{n!}{n^n}\right)$

Proof.

(a) Since $n^2 a^n > 0\ \forall\, n,$ *and* $\lim \frac{(n+1)^2 a^{n+1}}{n^2 a^n} = (\lim\,(1 + \frac{1}{n}))^2 \cdot a = a < 1$,

hence by Theorem 3.2.11, we have $\lim\,(a^n) = 0$.

(b) Since $\frac{b^n}{n^2} > 0\ \forall\, n,$ *and* $\lim \frac{\frac{b^{n+1}}{(n+1)^2}}{\frac{b^n}{n^2}} = (\lim\,(1 - \frac{1}{n+1}))^2 \cdot b = b > 1$,

hence by Question 17, we have $\left(\frac{b^n}{n^2}\right)$ diverges.

(c) Since $\frac{b^n}{n!} > 0\ \forall\, n,$ *and* $\lim \frac{\frac{b^{n+1}}{(n+1)!}}{\frac{b^n}{n!}} = \lim \frac{b}{n+1} = 0 < 1$,

hence by Theorem 3.2.11, we have $\lim\left(\frac{b^n}{n!}\right) = 0$.

(d) Since $\frac{n!}{n^n} > 0\ \forall\, n,$ *and* $\lim \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim\left(\frac{n}{n+1}\right)^n = \frac{1}{\lim\,(1+\frac{1}{n})^n} = e^{-1} < 1$,

hence by Theorem 3.2.11, we have $\lim\left(\frac{n!}{n^n}\right) = 0$.

Question 19 Let (x_n) be a sequence of positive real numbers such that $\lim\,(x_n{}^{\frac{1}{n}}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $\lim\,(x_n) = 0$.

Proof.

Since $\lim (x_n{}^{\frac{1}{n}}) = L < 1$,

then $\exists\, r < 1$ *and* $N \in \mathbb{N}$, such that $x_n{}^{\frac{1}{n}} < r \;\forall\, n \geq N$.

Hence $0 \leq x_n < r^n \;\forall\, n \geq N$,

but $\lim 0 = \lim (r^n : n \geq N) = 0$.

Hence, by squeeze theorem, we have $\lim (x_n : n \geq N) = 0$.

Hence $\lim (x_n) = \lim (x_n : n \geq N) = 0$.

Question 20

(a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim x_n{}^{\frac{1}{n}} = 1$.

(b) Give an example of a divergent sequence with this property.

Answer.

(a) Consider the convergent sequence $(x_n) = (\frac{1}{n})$.

We have $\lim (x_n) = 0$ and $\lim x_n{}^{\frac{1}{n}} = \lim (\frac{1}{n})^{\frac{1}{n}} = 1$.

(b) Consider the divergent sequence $(x_n) = (n)$.

We have (x_n) is not bounded and hence diverge, and

$\lim x_n{}^{\frac{1}{n}} = \lim n^{\frac{1}{n}} = 1$.

Question 21 Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$.

Does it follow that (y_n) is convergent?

Proof.

The answer is yes.

Let $x := \lim (x_n), \varepsilon > 0$,

then $\exists\, N_1 \in \mathbb{N}$, such that $|x_n - x| < \frac{\varepsilon}{2}, \;\forall\, n \geq N_1$.

By the assumption, $\exists\, N_2 \in \mathbb{N}$, such that $|y_n - x_n| < \frac{\varepsilon}{2}, \;\forall\, n \geq N_2$.

Hence, $|y_n - x| \leq |y_n - x_n| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ \forall\, n \geq max\{N_1, N_2\}$,

i.e. (y_n) converges (to x).

Question 22 Show that if (x_n) *and* (y_n) are convergent sequences, then the sequences (u_n) *and* (v_n) defined by $u_n := max\{x_n, y_n\}$ *and* $v_n := min\{x_n, y_n\}$ are also convergent.

Proof.

Notice that$u_n = \frac{1}{2}(x_n + y_n + |x_n - y_n|)$ *and* $y_n = \frac{1}{2}(x_n + y_n - |x_n - y_n|)$, $\forall\, n \in \mathbb{N}$ by Exercise 2.2.16.

Since (x_n) *and* (y_n) are convergent sequences,

$\lim{(u_n)} = \lim{(\frac{1}{2}(x_n + y_n + |x_n - y_n|))} = \frac{1}{2}(\lim{(x_n)} + \lim{(y_n)} + |\lim{(x_n)} - \lim{(y_n)}|)$,

and $\lim(v_n) = \lim(\frac{1}{2}(x_n + y_n - |x_n - y_n|)) = \frac{1}{2}(\lim(x_n) + \lim(y_n) - |\lim(x_n) - \lim(y_n)|)$.

Hence, (u_n) *and* (v_n) are also convergent.