Oscillations

Simple Harmonic Motion

Any motion that repeats at regular intervals is called periodic motion or harmonic motion.

However, here we are interested in a particular type of periodic motion called simple harmonic motion (SHM). Such motion is a sinusoidal function of time t. That is, it can be written as a sine or a cosine of time t.

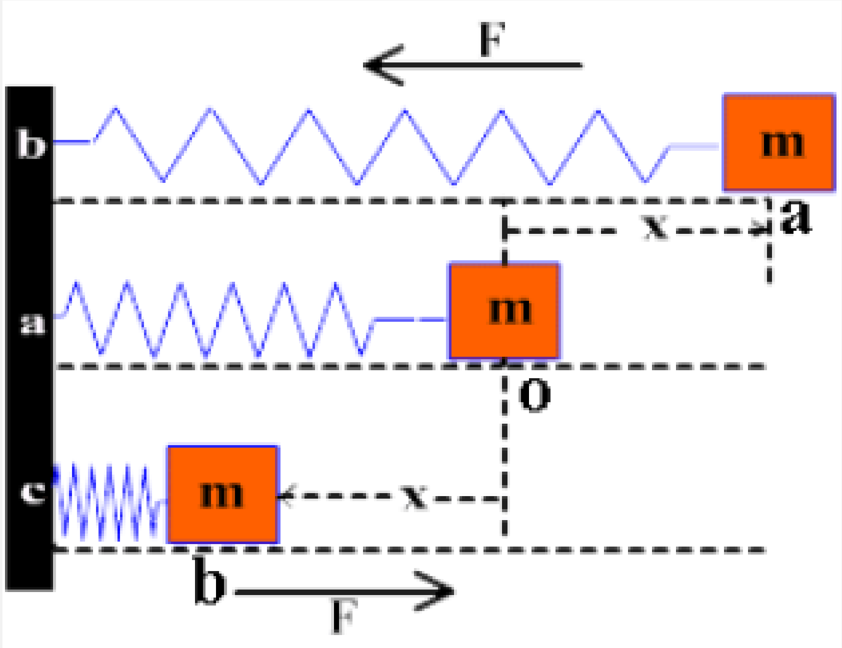
Simple Harmonic Motion or SHM is defined as a motion in which the restoring force is directly proportional to the displacement of the body from its mean position. The direction of this restoring force is always towards the mean position. The acceleration of a particle executing simple harmonic motion is given by,

$$a(t) = -\omega^2 x(t)$$

Here, ω is the angular velocity of the particle.

Definition:

- Such a motion in which acceleration is directly proportional to the displacement and is directed towards the mean position is called simple harmonic motion(SHM).



Condition FOR SHM:

- The system should have restoring force.
- The system should have inertia.
- The system should be frictionless.

General Equation

$$x(t) = A\cos(2\pi ft + \phi)$$

where,

x → Displacement

A → Amplitude of the oscillation

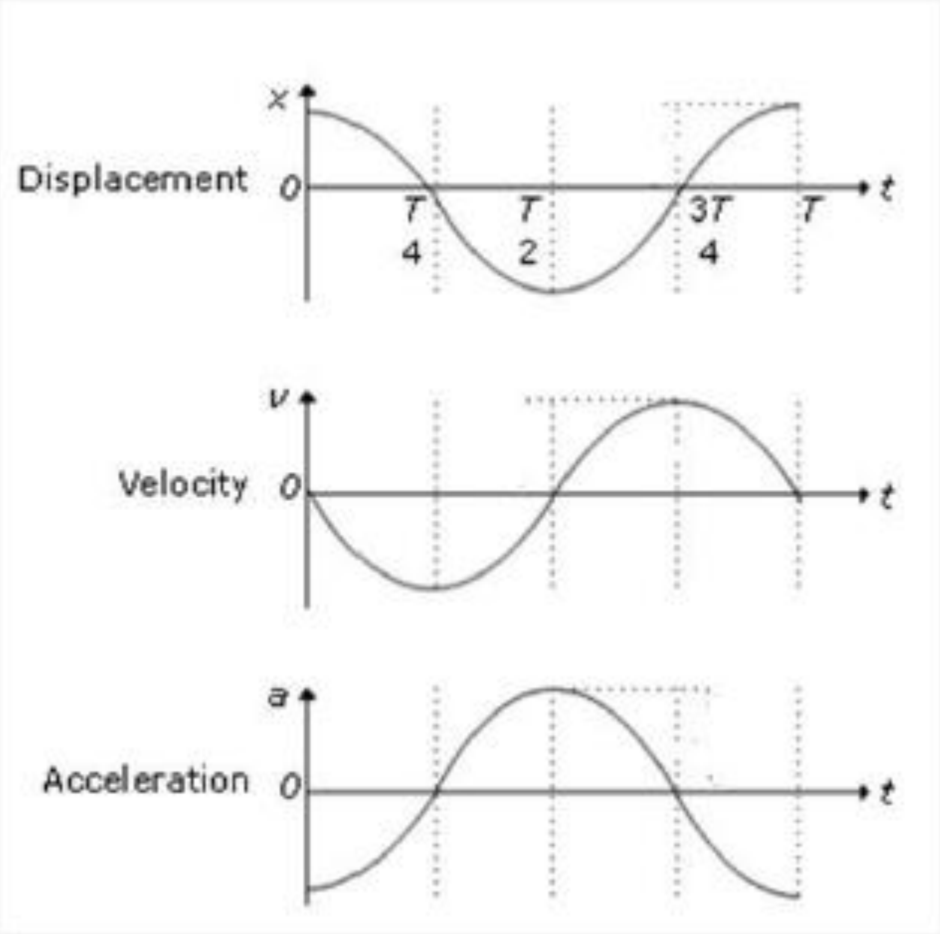
f → Frequency

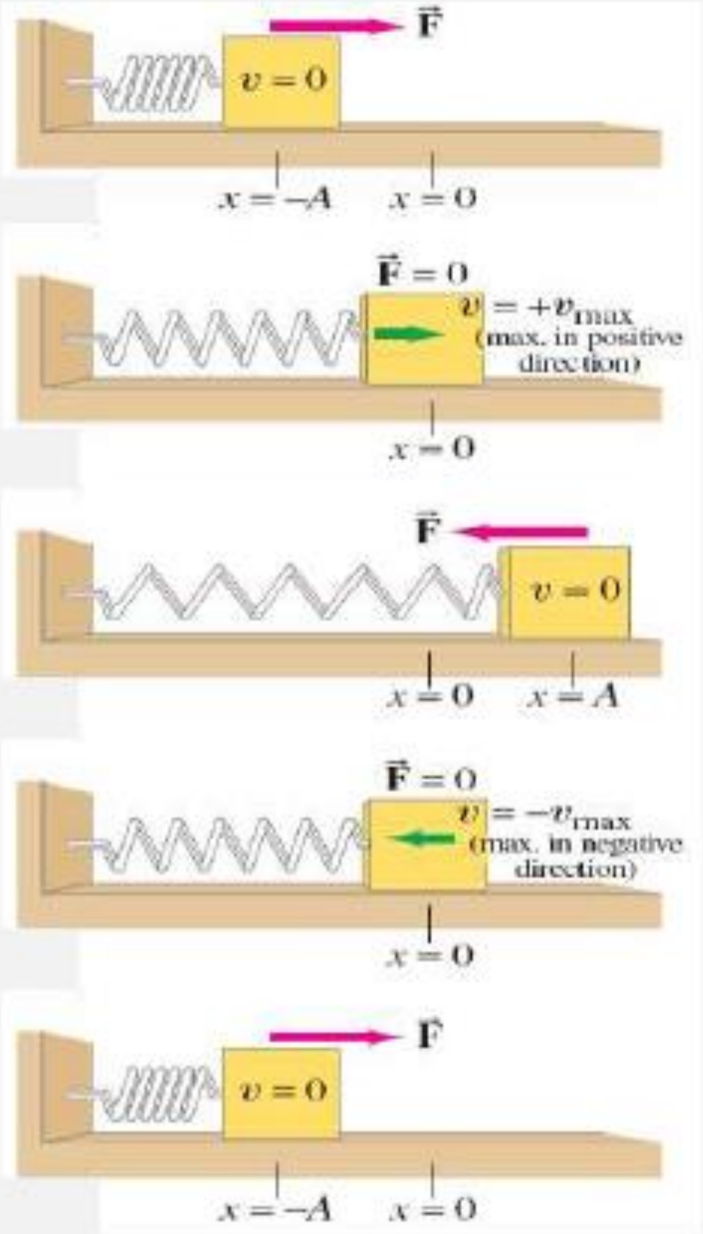
t → time

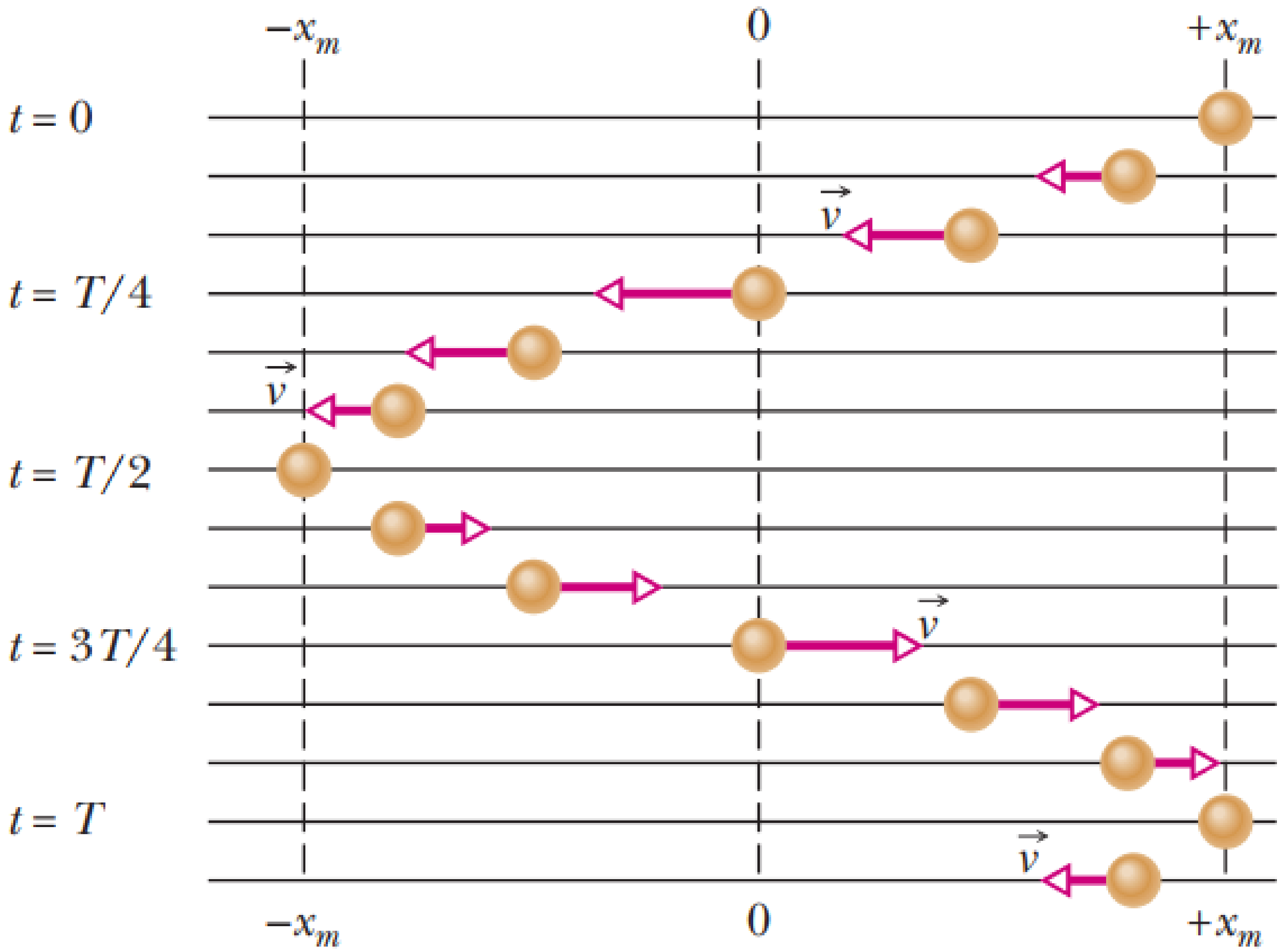
ϕ → Phase of oscillation

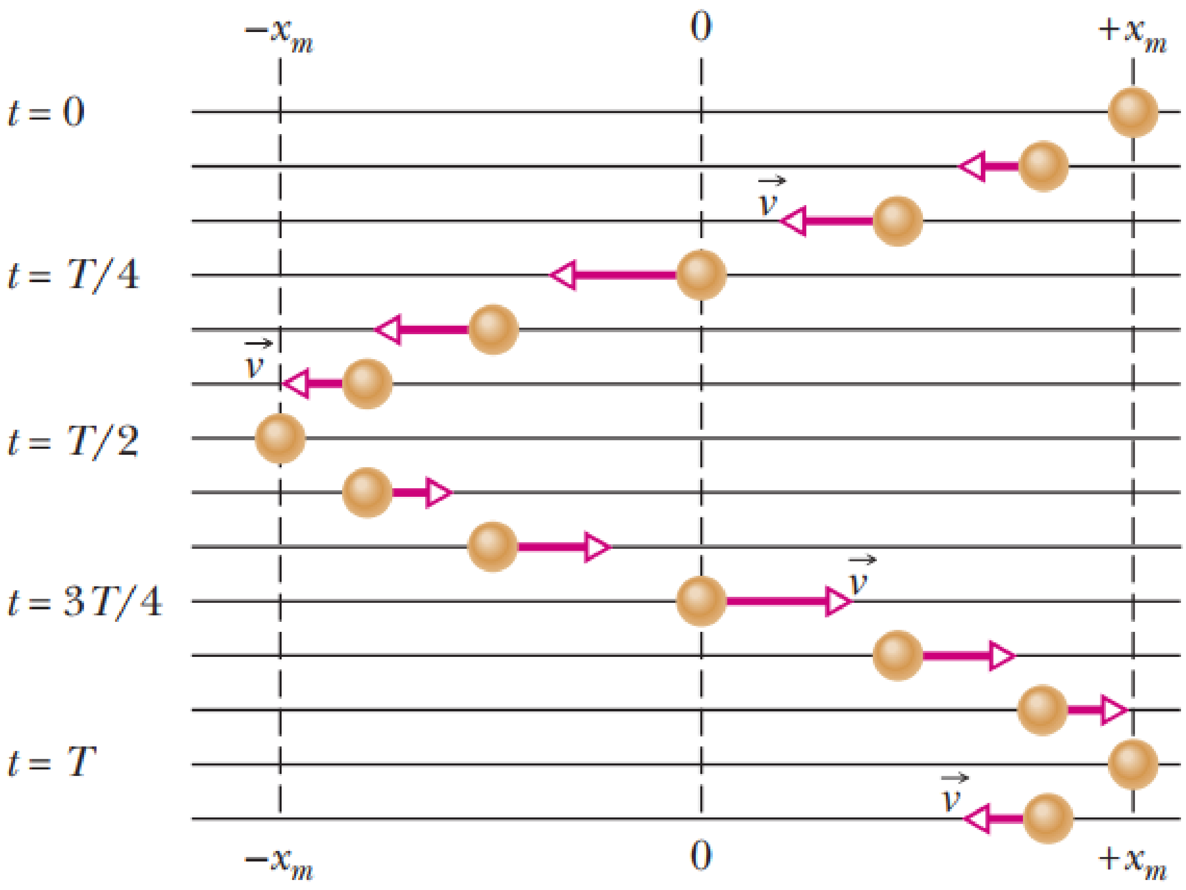
If there is no displacement at time t = 0, the phase is $\phi = \pi/2$

Graphical representation

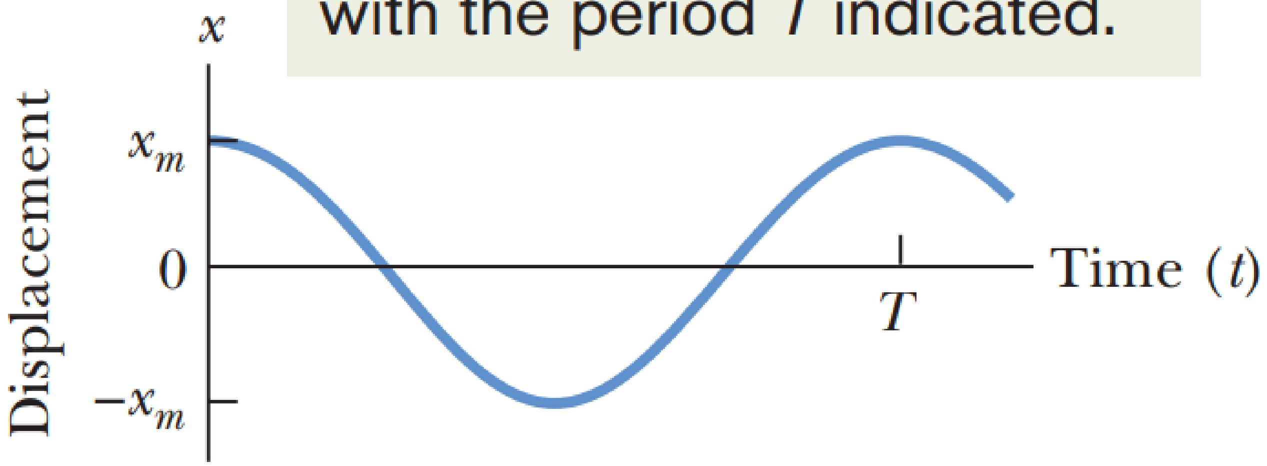





$-x_m$
0
$+x_m$
$t = 0$
$\vec{v}$
$t = T/4$
$\vec{v}$
$t = T/2$
$t = 3T/4$
$\vec{v}$
$t = T$
$\vec{v}$
$-x_m$
0
$+x_m$



This is a graph of the motion, with the period T indicated.

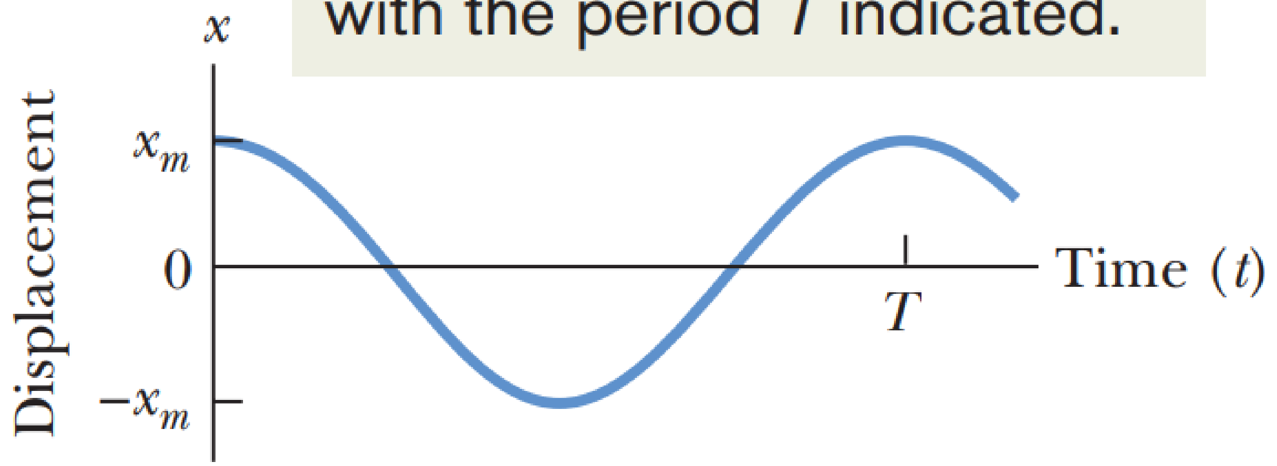


$$T = \frac{1}{f}.$$

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1}.$$

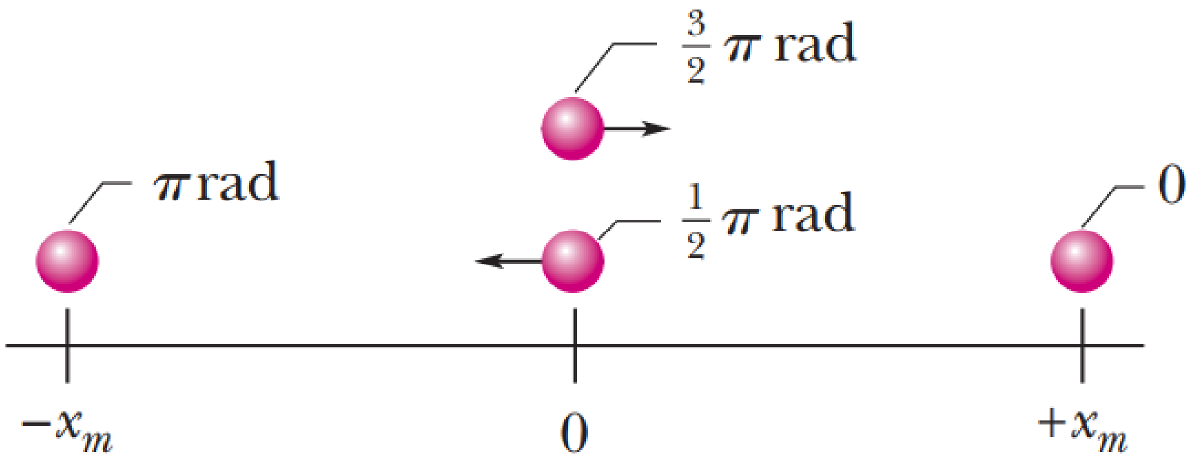
$$\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos(\overbrace{\underset{\text{Angular frequency}}{\omega}\, \underset{\text{Time}}{t} + \underset{\text{Phase constant or phase angle}}{\phi}}^{\text{Phase}})$$

This is a graph of the motion, with the period T indicated.

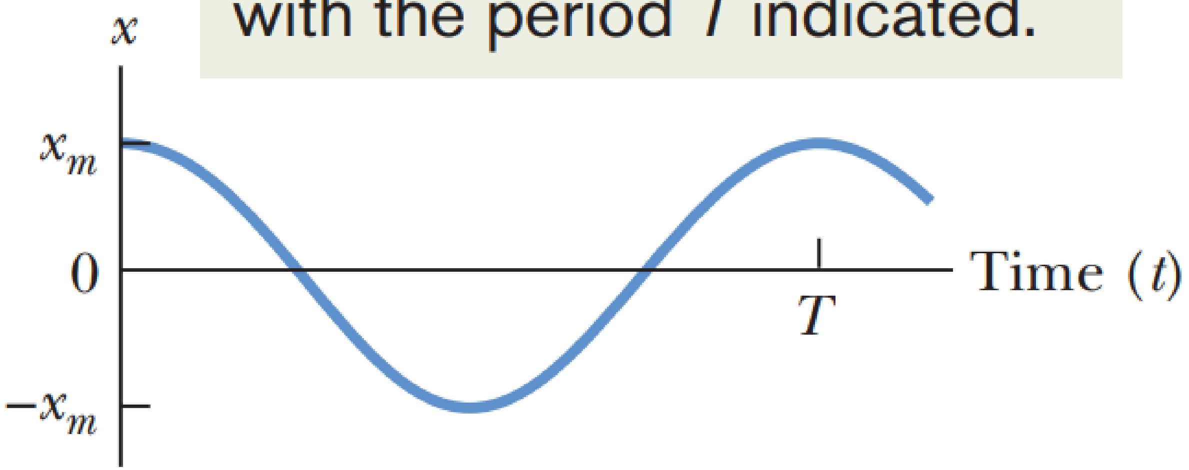


The value of x_m determines how far the particle moves in its oscillations and is called the amplitude of the oscillations.

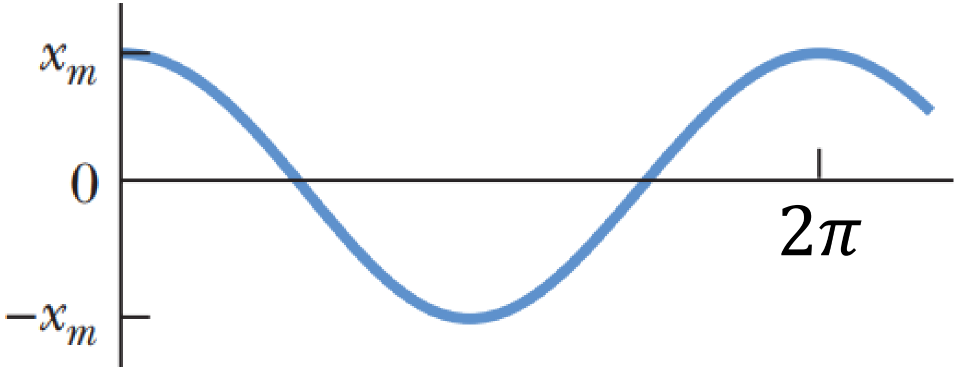
The argument of the cosine function is called the phase of the motion. As it varies with time, the value of the cosine function varies. The constant ϕ is called the phase angle or phase constant.



This is a graph of the motion, with the period T indicated.

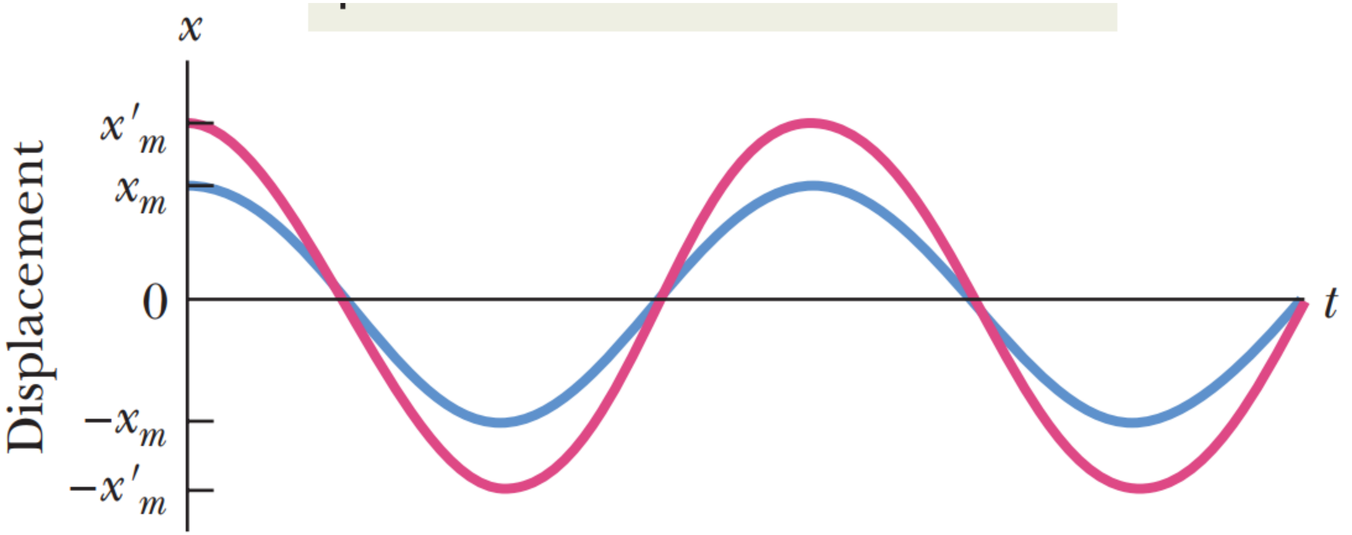


$$x_m \cos \omega t$$

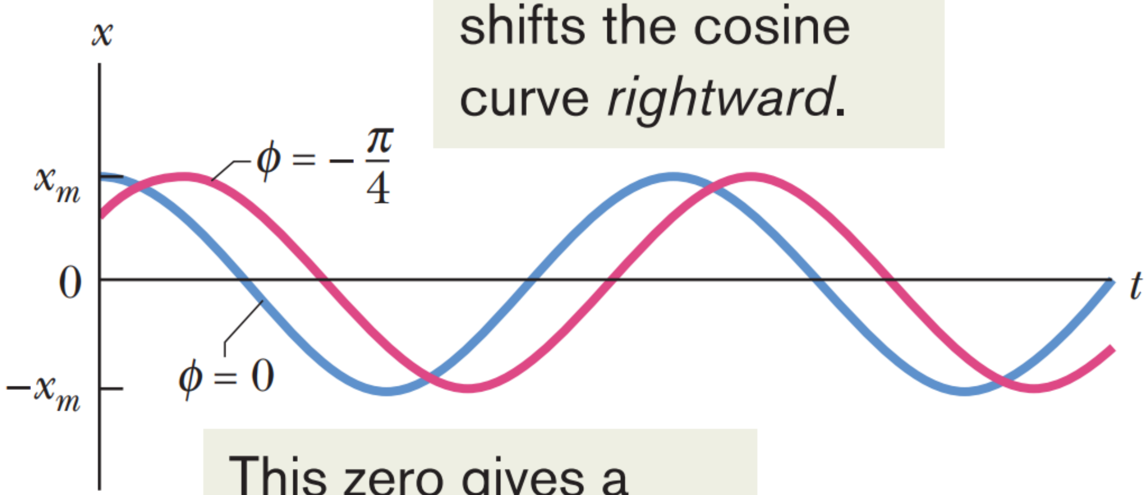


$$\omega = \frac{2\pi}{T} = 2\pi f.$$

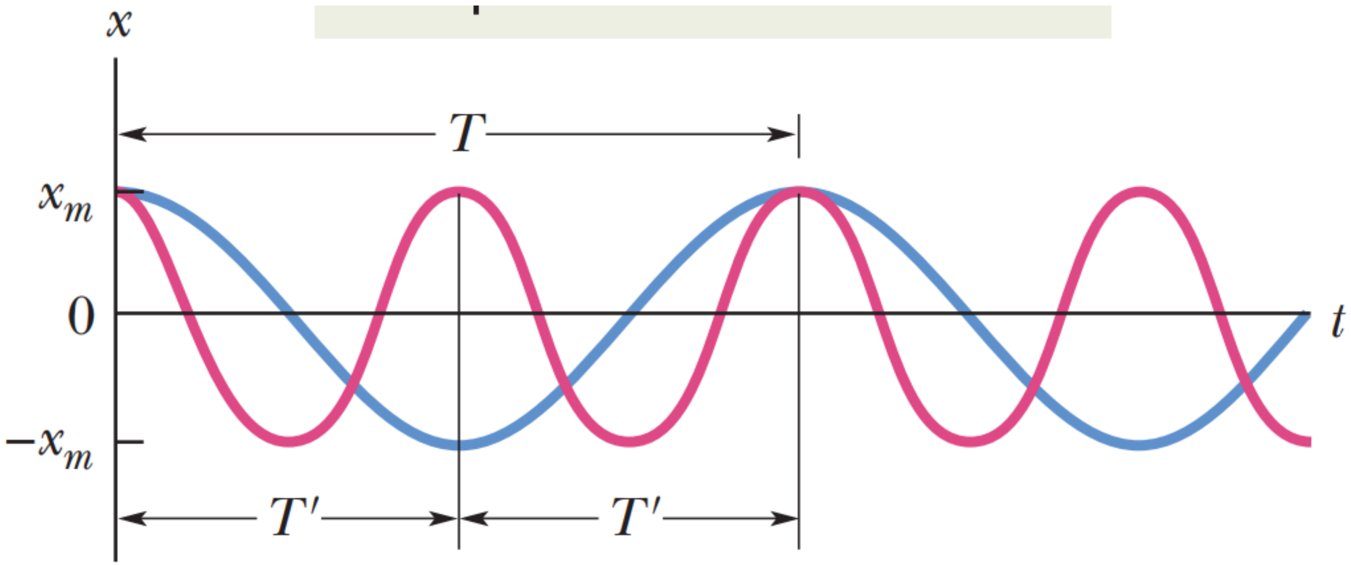
$$x(t) = x_m \cos(\omega t + \phi)$$

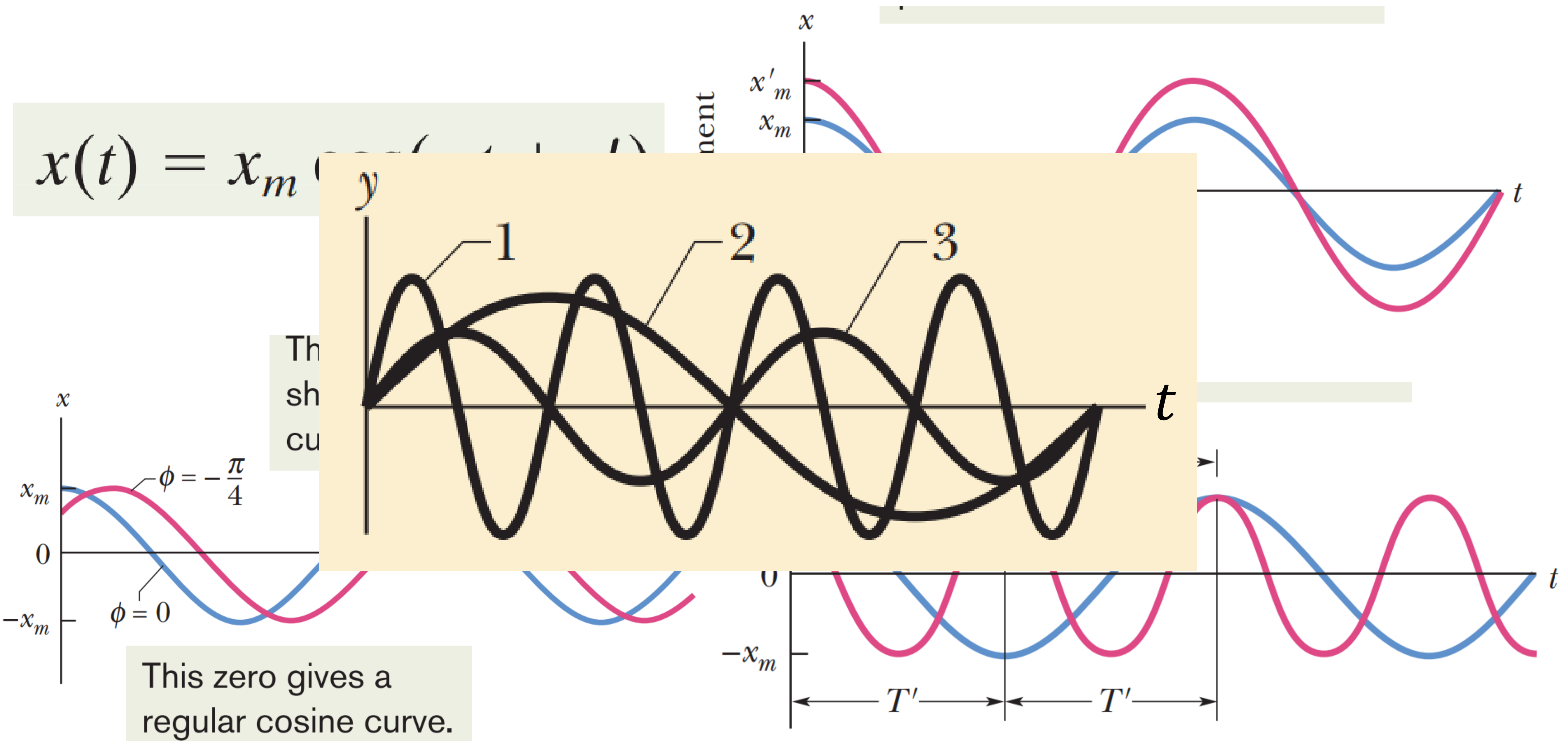


This *negative* value shifts the cosine curve *rightward*.



This zero gives a regular cosine curve.





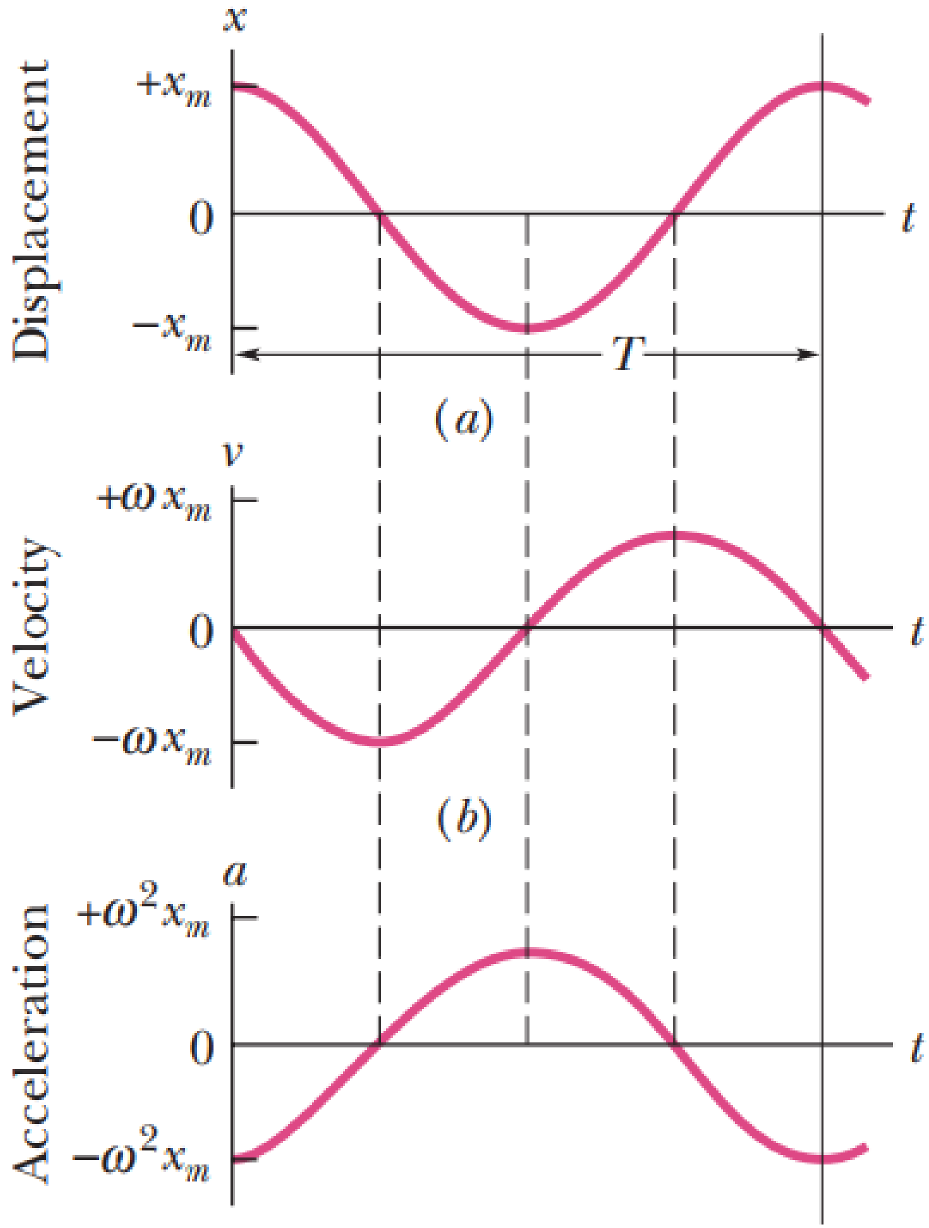
$x(t) = x_m$
Th
sh
cu
This zero gives a
regular cosine curve.
y
1
2
3
t
x
x'_m
x_m
0
$-x_m$
$\phi = -\frac{\pi}{4}$
$\phi = 0$
T'
T'
ment

$$x(t) = x_m \cos(\omega t + \phi)$$

$$v(t) = -\omega x_m \sin(\omega t + \phi)$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

$$a(t) = -\omega^2 x(t)$$

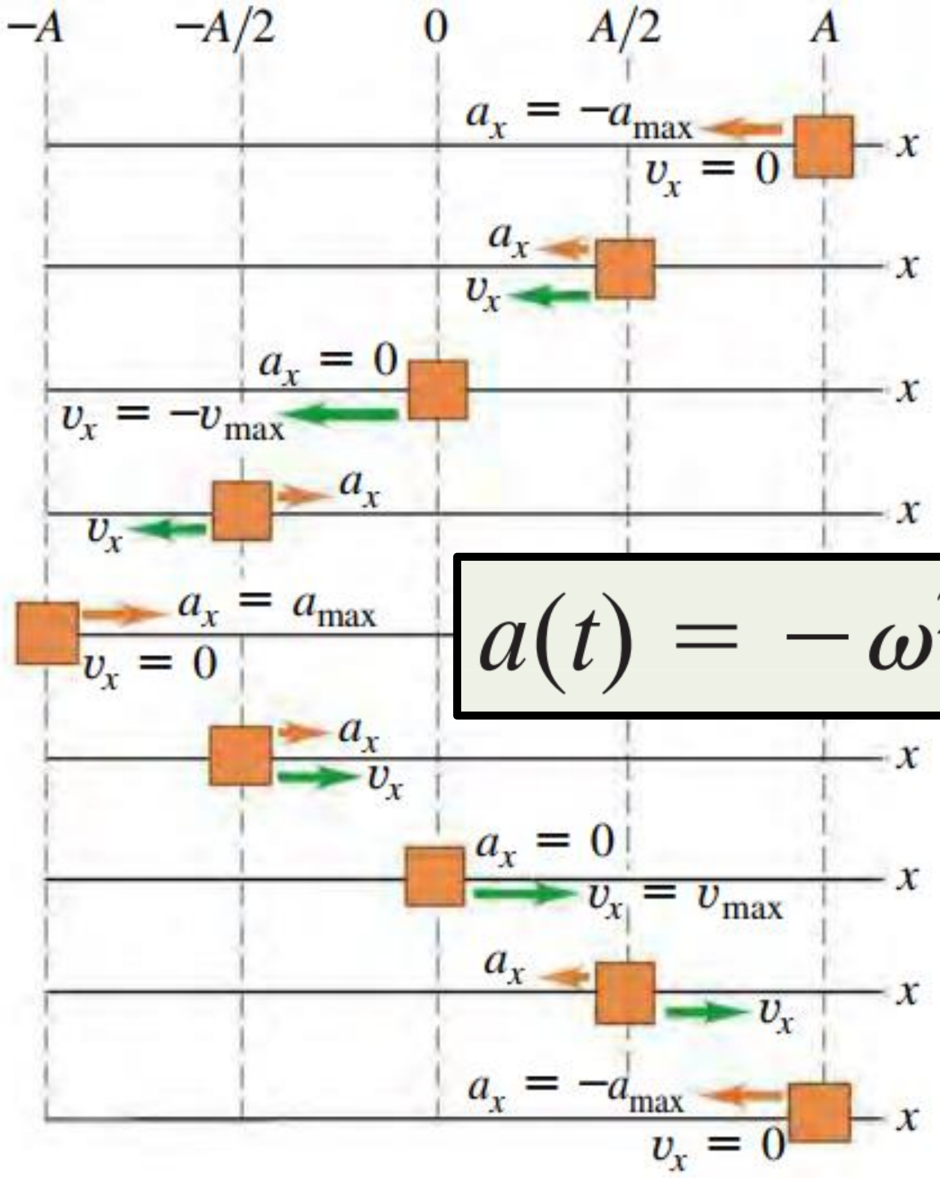


Extreme values here mean ...

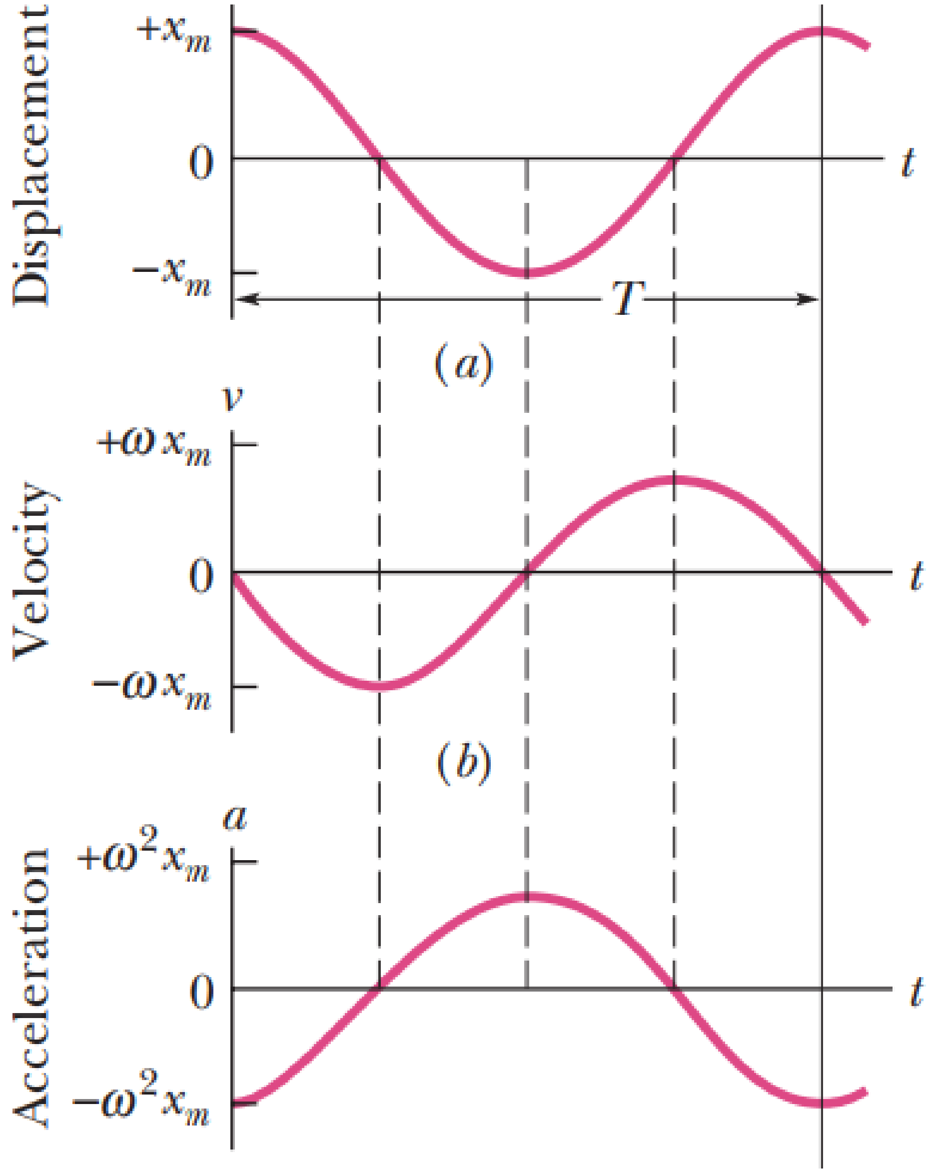
zero values here and ...

extreme values here.

In SHM, the acceleration a is proportional to the displacement x but opposite in sign, and the two quantities are related by the square of the angular frequency ω.



$$a(t) = -\omega^2 x(t)$$



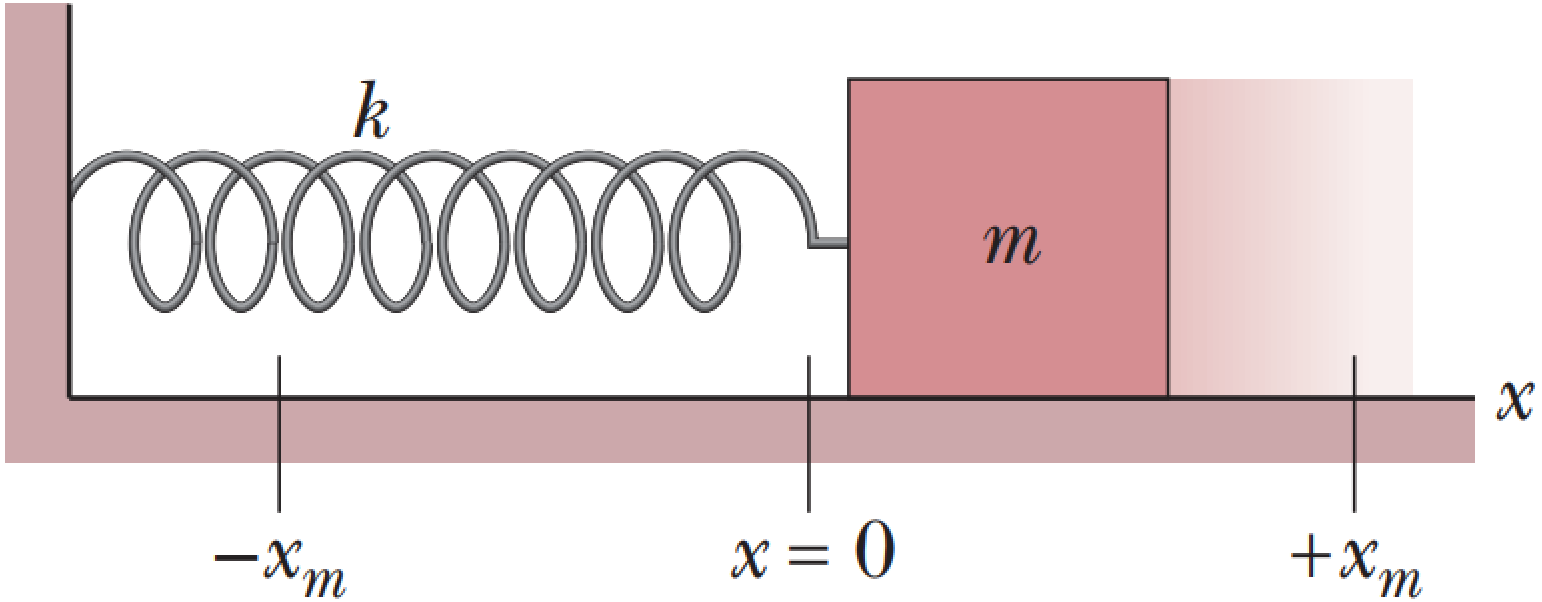
Extreme values here mean ...

zero values here and ...

extreme values here.

In SHM, the acceleration a is proportional to the displacement x but opposite in sign, and the two quantities are related by the square of the angular frequency ω.

Simple harmonic motion is the motion of a particle when the force acting on it is proportional to the particle's displacement but in the opposite direction.



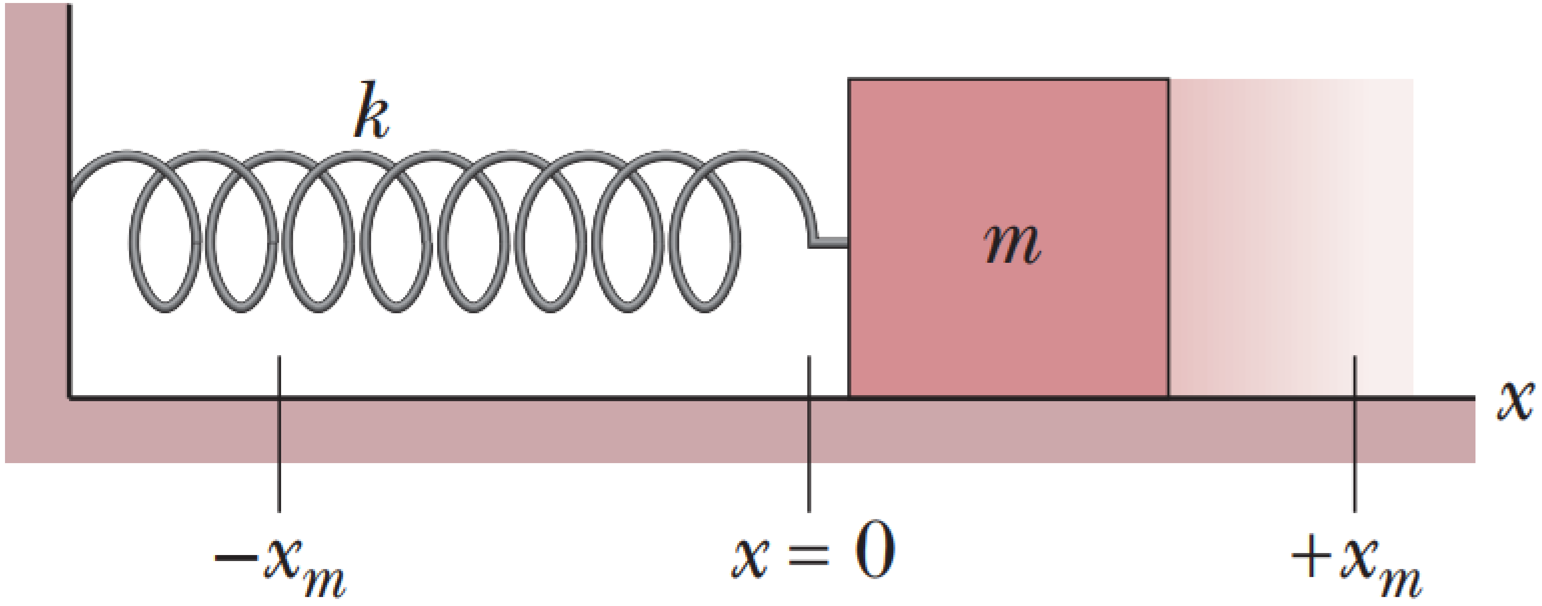
Force Law for Simple Harmonic Motion

$$F = ma = m(-\omega^2 x) = -(m\omega^2)x. \quad\text{- - - - - - - - - -}\quad F = -kx,$$

The minus sign means that the direction of the force on the particle is opposite the direction of the displacement of the particle. That is, in SHM the force is a restoring force in the sense that it fights against the displacement, attempting to restore the particle to the center point at x = 0.

Block-Spring System

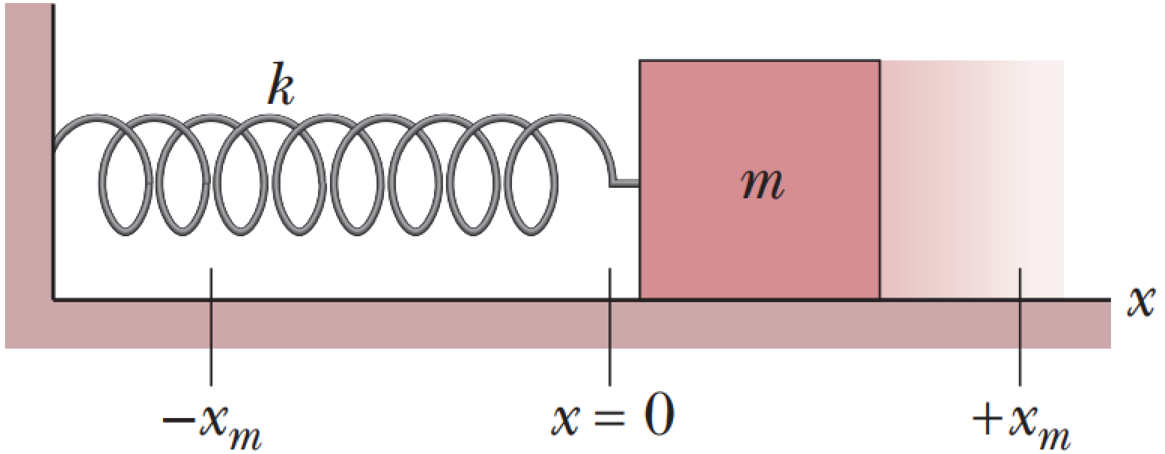
- The block–spring system of Fig. is called a linear simple harmonic oscillator (linear oscillator, for short), where linear indicates that F is proportional to x to the first power (and not to some other power).



$$F = ma = m(-\omega^2 x) = -(m\omega^2)x. \quad F = -kx,$$

$$k = m\omega^2.$$

Spring Constant



$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency).}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period).}$$

Important points to remember

Every oscillating system, be it a diving board or a violin string, has some element of “springiness” and some element of “inertia” or mass.

In block-spring system, these elements are separated: The springiness is entirely in the spring, which we assume to be massless, and the inertia is entirely in the block, which we assume to be rigid. In a violin string, however, the two elements are both within the string.

Hooke's Law

- force that is applied to spring is **directly proportional** to the displacement.

- if the mass is displaced from equilibrium, the spring will exert a restoring force, which is a force that tends to restore it to the equilibrium position.

$$\boxed{F = -kx}$$

Expression for acceleration of the body executing SHM:

- Consider a mass ‘m’ attached to one end of elastic spring which can move freely on a frictionless horizontal surface.

- When the mass is released, it begins to vibrate about its mean or equilibrium position.

- But due to elasticity, spring opposes the applied force which produces the displacement. This opposing force is called restoring force.

Expression for acceleration of the body executing SHM:

The restoring force is written by;

$$F_r = -kx \text{.......................(i)}$$

If ‘a’ is the acceleration produced by force ‘F’ in mass-spring system at any instant, then according to Newton’s law of motion.

$$F = ma$$

$$F = m\ddot{x} \text{........................(ii)}$$

Comparing (i) and (ii)

$$m\ddot{x} = -kx \qquad \Rightarrow \ddot{x} = \left(\frac{-k}{m}\right)x$$

$$\because \frac{k}{m} = constant$$

a α -x

From above equation we can write it as

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0$$

Solution of the form

$$x = A\cos(\omega t + \phi).$$

$$\therefore \omega = \sqrt{\frac{k}{m}}$$

- Velocity can found by differentiating displacement

$$v = -A\omega \sin(\omega t + \emptyset)$$

- Acceleration can found by differentiating velocity

$$a = -A\omega^2 \cos(\omega t + \emptyset)$$

Simplifying acceleration in terms of displacement:

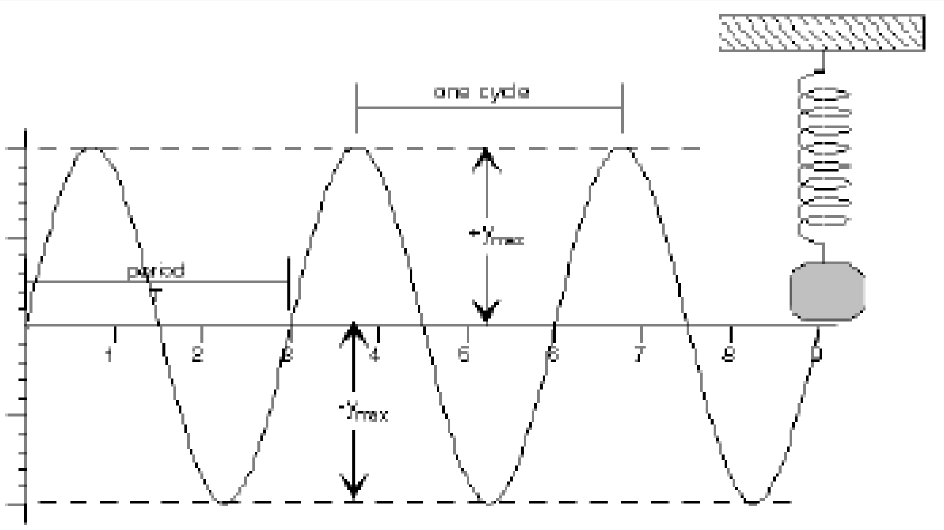
$$a = \frac{d^2x}{dt^2} = -\omega^2 x,$$

Acceleration can also be expressed as:

$$a(t) = -(2\pi f)^2 x(t)$$

Characteristic of Mass-spring system executing SHM:

When a body is vibrating, its displacement from the mean position changes with time. The value of its distance from the mean position at any time is known as its instantaneous displacement.



- The maximum value of displacement is known as its amplitude.

- A vibration means one complete round trip of the body in motion.

- The time required to complete one vibration is called time period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- The number of cycles per second. A cycle is a complete round trip is called Frequency

$$f = \frac{1}{T}$$ $$f = \frac{1}{2\pi\sqrt{\frac{m}{k}}}$$ $$f = 2\pi\sqrt{\frac{k}{m}}$$

- If T is time period of a body executing SHM, its angular frequency can be written as;

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Checkpoint 1

A particle undergoing simple harmonic oscillation of period T ( ) is at $-x_m$ at time $t = 0$. Is it at $-x_m$, at $+x_m$, at 0, between $-x_m$ and 0, or between 0 and $+x_m$ when (a) $t = 2.00T$, (b) $t = 3.50T$, and (c) $t = 5.25T$?

Which of the following relationships between a particle's acceleration a and its position x indicates simple harmonic oscillation: (a) $a = 3x^2$, (b) $a = 5x$, (c) $a = -4x$, (d) $a = -2/x$? For the SHM, what is the angular frequency (assume the unit of rad/s)?

Checkpoint 3

Which of the following relationships between the force F on a particle and the particle's position x gives SHM: (a) $F = -5x$, (b) $F = -400x^2$, (c) $F = 10x$, (d) $F = 3x^2$?

Practice Problem 1: Sample Problem 15.01

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-12:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ \text{N/m}}{0.68\ \text{kg}}} = 9.78\ \text{rad/s}$$

$$\approx 9.8\ \text{rad/s}. \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-5, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78\ \text{rad/s}}{2\pi\ \text{rad}} = 1.56\ \text{Hz} \approx 1.6\ \text{Hz}. \qquad \text{(Answer)}$$

The period follows from Eq. 15-2, which yields

$$T = \frac{1}{f} = \frac{1}{1.56\ \text{Hz}} = 0.64\ \text{s} = 640\ \text{ms}. \qquad \text{(Answer)}$$

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(b) What is the amplitude of the oscillation?

KEY IDEA With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm}. \qquad \text{(Answer)}$$

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

KEY IDEA The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-6.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m})$$
$$= 1.1 \text{ m/s}. \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-7.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m})$$
$$= 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path. At those points, the force acting on the block has its maximum magnitude

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(e) What is the phase constant ϕ for the motion?

Solution:

$$x(t) = x_m \cos(\omega t + \phi)$$

Calculations: Equation 15-3 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-3 and canceling x_m give us

$$1 = \cos \phi. \qquad (15\text{-}14)$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad}. \qquad \text{(Answer)}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(f) What is the displacement function $x(t)$ for the spring–block system?

Calculation: The function $x(t)$ is given in general form by Eq. 15-3. Substituting known quantities into that equation gives us

$$\begin{aligned} x(t) &= x_m \cos(\omega t + \phi) \\ &= (0.11 \text{ m}) \cos[(9.8 \text{ rad/s})t + 0] \\ &= 0.11 \cos(9.8t), \qquad \text{(Answer)} \end{aligned}$$

where x is in meters and t is in seconds.

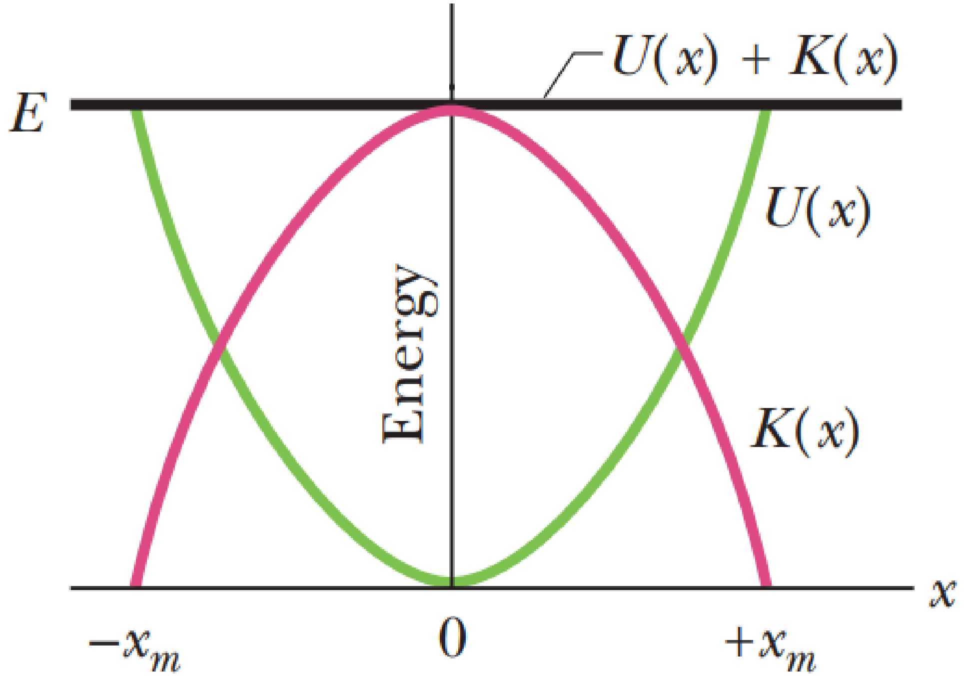
Energy in Simple Harmonic Motion

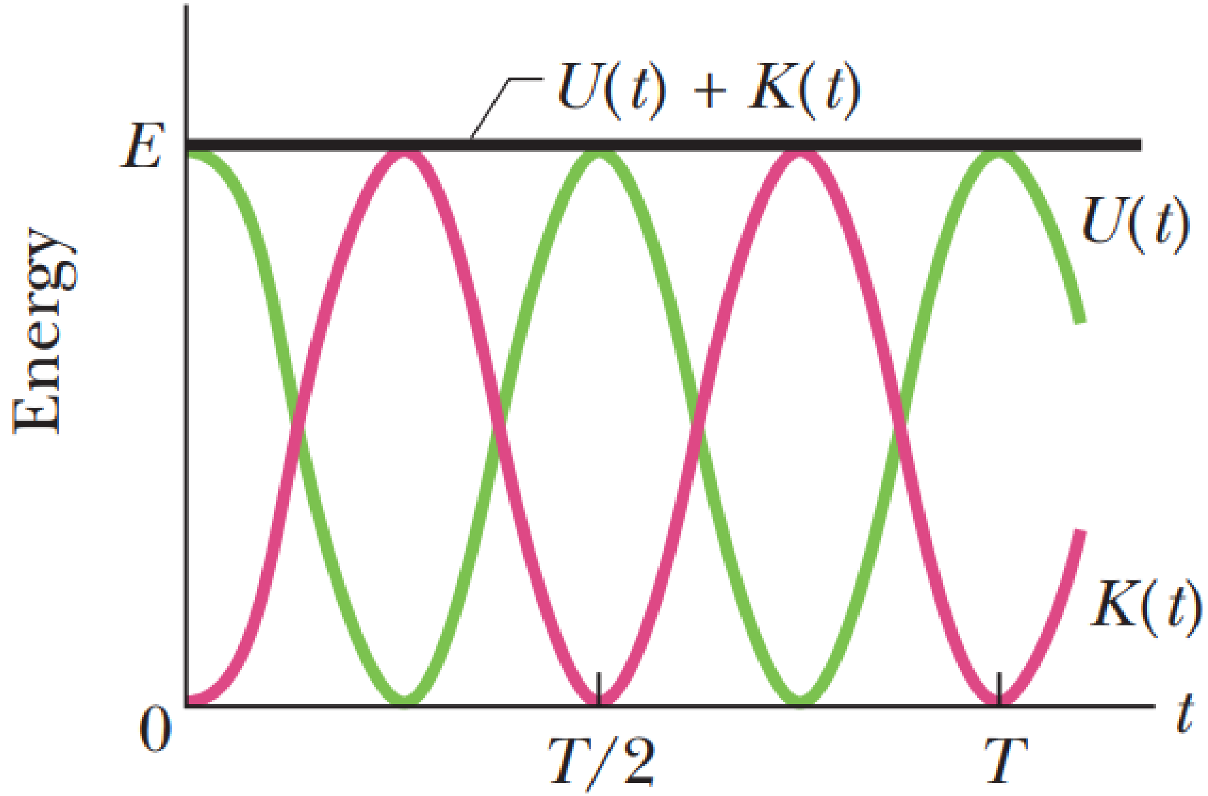
Energy conservation in Simple harmonic motion:

- if the friction effect are neglected, total mechanical energy of vibrating mass spring system remains constant
- The velocity and position of the vibrating body are continually changing
- The kinetic and potential energies also change, but their sum must have the same values at any instant.
- By hook's law
- F=-kx
- $W = \int f\,dx$
- U=-W
- U=$-\int f\,dx$
- $U = -\int -kx\,dx$
- $U = k\int x\,dx$
- $U = k\frac{x^2}{2}$

$$U(t) = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi).$$

$$K(t) = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

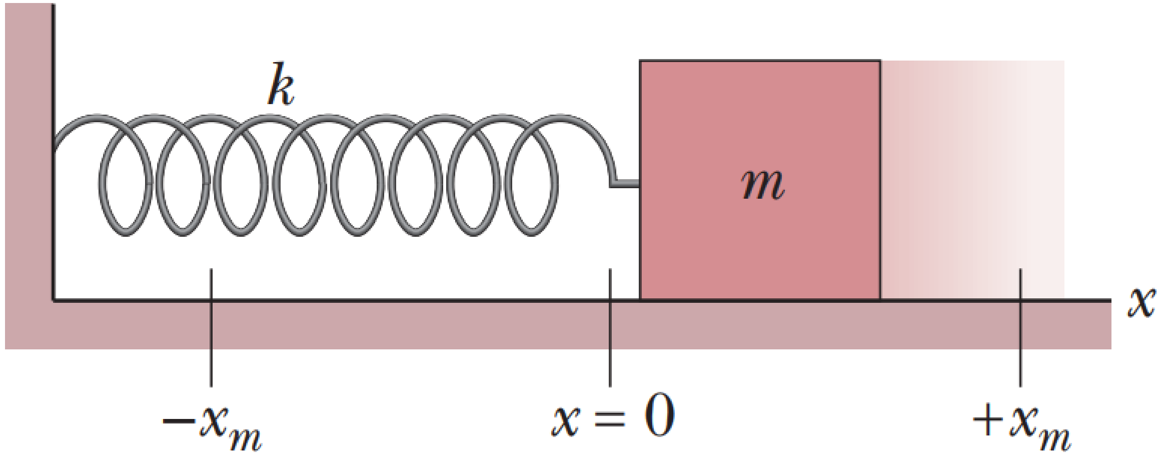




$$E = U + K = \tfrac{1}{2}kx_m^2.$$

Checkpoint 4

In Fig. the block has a kinetic energy of 3 J and the spring has an elastic potential energy of 2 J when the block is at $x = +2.0$ cm. (a) What is the kinetic energy when the block is at $x = 0$? What is the elastic potential energy when the block is at (b) $x = -2.0$ cm and (c) $x = -x_m$?



Practice Problem 2: (Sample Problem 15.03)

Many tall building have mass dampers, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building. Suppose that the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring-block system?

KEY IDEA The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$\begin{aligned} k &= m\omega^2 = m(2\pi f)^2 \\ &= (2.72 \times 10^5\ \text{kg})(2\pi)^2(10.0\ \text{Hz})^2 \\ &= 1.073 \times 10^9\ \text{N/m}. \end{aligned}$$

We can now evaluate E as

$$\begin{aligned} E &= K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \\ &= 0 + \tfrac{1}{2}(1.073 \times 10^9\ \text{N/m})(0.20\ \text{m})^2 \\ &= 2.147 \times 10^7\ \text{J} \approx 2.1 \times 10^7\ \text{J}. \qquad \text{(Answer)} \end{aligned}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

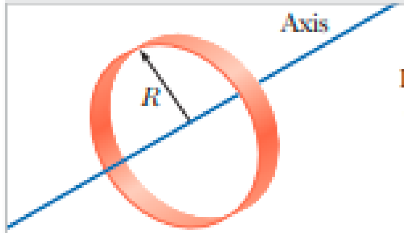
$$2.147 \times 10^7\ \text{J} = \tfrac{1}{2}(2.72 \times 10^5\ \text{kg})v^2 + 0,$$

or

$$v = 12.6\ \text{m/s}. \qquad \text{(Answer)}$$

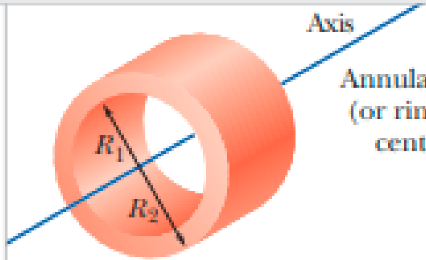
Because E is entirely kinetic energy, this is the maximum speed v_m.

Table 10-2 Some Rotational Inertias



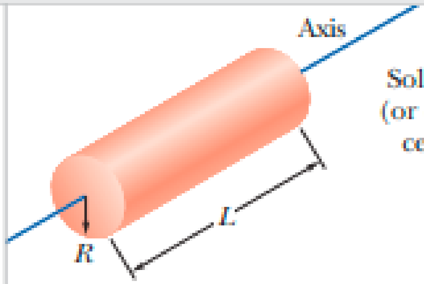
Hoop about central axis

$I = MR^2$ (a)



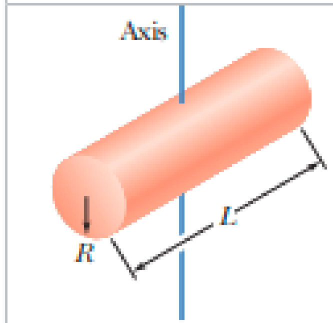
Annular cylinder (or ring) about central axis

$I = \frac{1}{2}M(R_1^2 + R_2^2)$ (b)



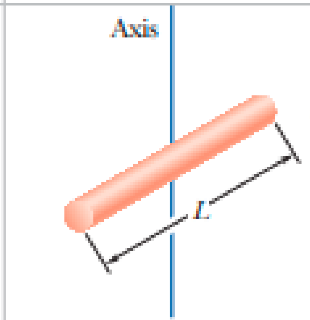
Solid cylinder (or disk) about central axis

$I = \frac{1}{2}MR^2$ (c)



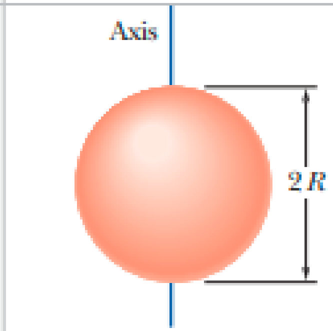
Solid cylinder (or disk) about central diameter

$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ (d)



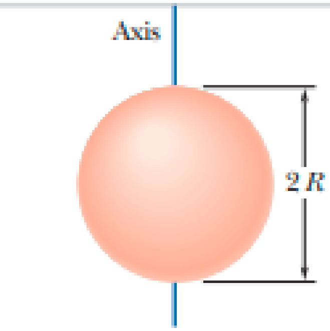
Thin rod about axis through center perpendicular to length

$I = \frac{1}{12}ML^2$ (e)



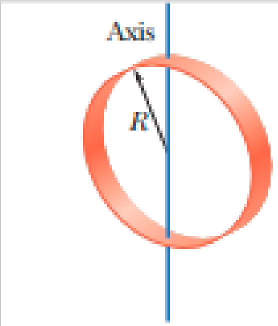
Solid sphere about any diameter

$I = \frac{2}{5}MR^2$ (f)



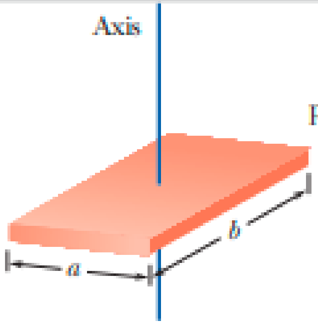
Thin spherical shell about any diameter

$I = \frac{2}{3}MR^2$ (g)



Hoop about any diameter

$I = \frac{1}{2}MR^2$ (h)



Slab about perpendicular axis through center

$I = \frac{1}{12}M(a^2 + b^2)$ (i)

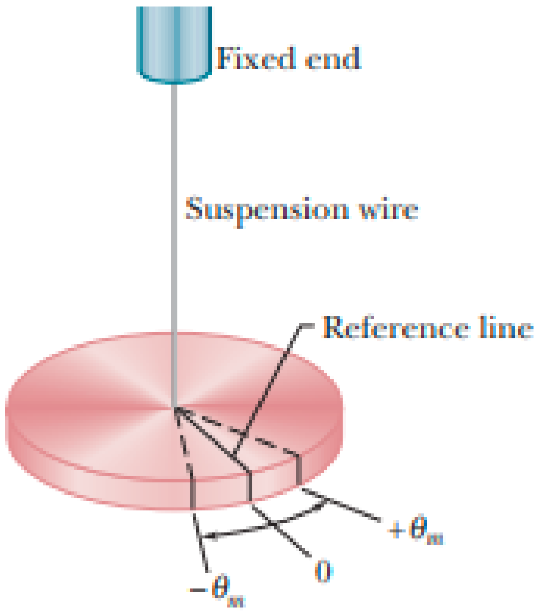
Angular Simple Harmonic Oscillator

- Figure shows an angular version of a simple harmonic oscillator; the element of springiness or elasticity is associated with the twisting of a suspension wire rather than the extension and compression of a spring. The device is called a torsion pendulum, with torsion referring to the twisting.

- Rotating the disk through an angle θ in either direction introduces a restoring torque given by:

$$\tau = \kappa\theta$$

where, κ = torsion constant, that depends on the length, diameter, and material of the suspension wire.



Time period of torsion pendulum

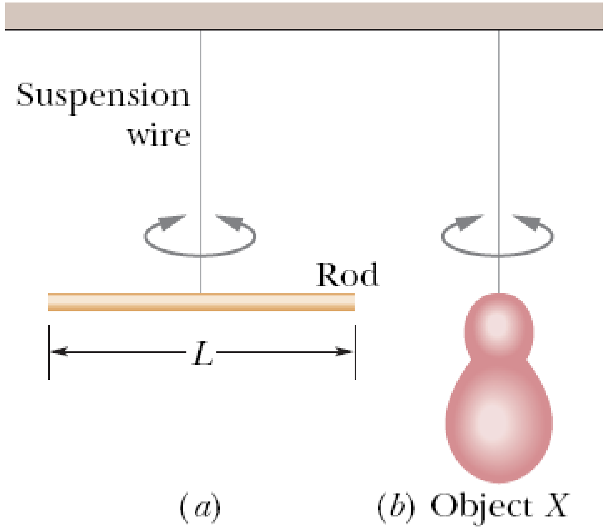
$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad \text{(torsion pendulum).}$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

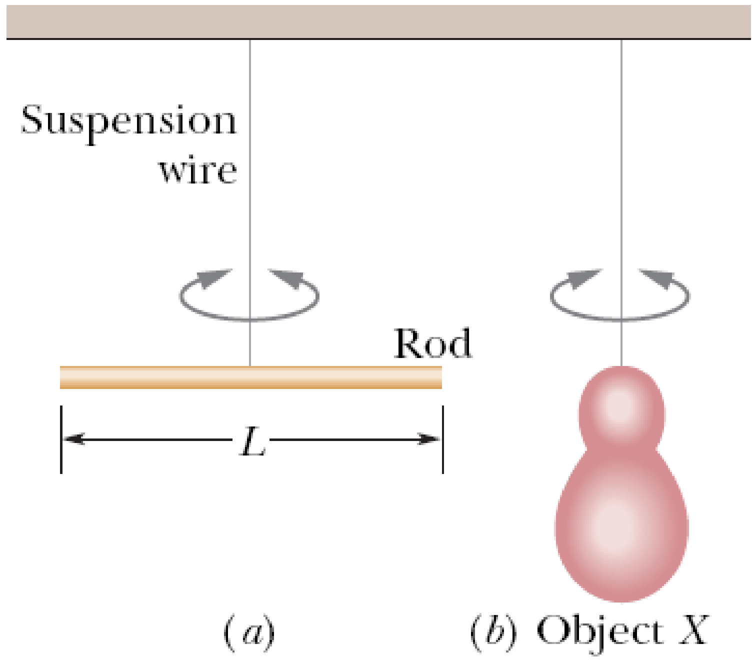
Where,
$\kappa =$ torsion constant, that depends on the length, diameter, and material of the suspension wire.
I = rotational inertia of the oscillating disk.

Example, angular SHM (sample Problem 15.04):

Figure a shows a thin rod whose length L is 12.4 cm and whose mass m is 135 g, suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call object X, is then hung from the same wire, as in Fig. b, and its period T_b is found to be 4.76 s. What is the rotational inertia of object X about its suspension axis?



Solution



The rotational inertia of either the rod or object X is related to the measured period. The rotational inertia of a thin rod about a perpendicular axis through its midpoint is given as 1/12 mL^2.Thus, we have, for the rod in Fig. a,

$$I_a = \tfrac{1}{12}mL^2 = (\tfrac{1}{12})(0.135\text{ kg})(0.124\text{ m})^2$$
$$= 1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2.$$

Now let us write the periods, once for the rod and once for object *X:*

Alternate option, find κ of rod L from Equation 1 . Then put this κ in equation 2 to find Ib

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

The constant κ, which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

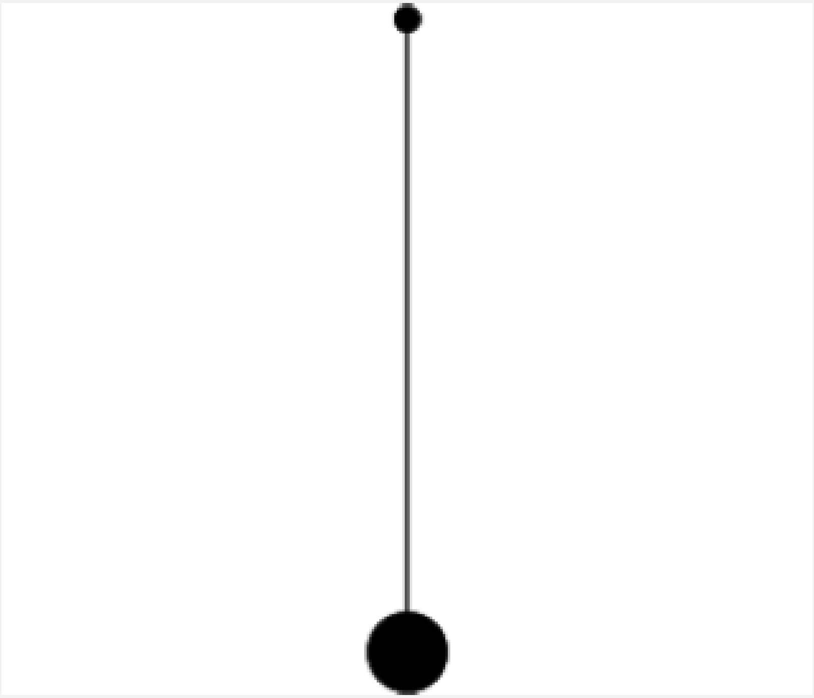
Let us square each of these equations, divide the second by the first, and solve the resulting equation for I_b. *The result is*

$$I_b = I_a\frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2)\frac{(4.76\text{ s})^2}{(2.53\text{ s})^2}$$
$$= 6.12 \times 10^{-4}\text{ kg}\cdot\text{m}^2. \qquad \text{(Answer)}$$

Pendulums:

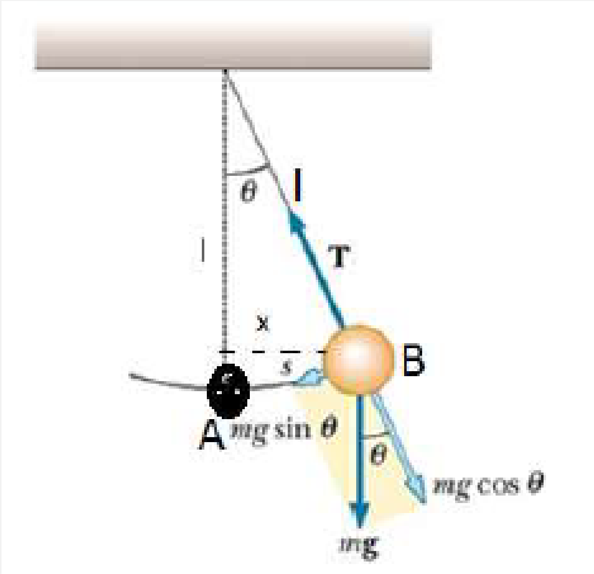
Pendulums belong to a class of simple harmonic oscillators in which the springiness is associated with the gravitational force rather than with the elastic properties of a twisted wire or a compressed or stretched spring.

Simple pendulum:



Simple pendulum:

- A simple pendulum is idealized model consist of a point mass suspended by an inextensible string of length *l* fixed
- When pulled to one side of its equilibrium position A to the position B through a small angle θ and released, it starts oscillating to and fro over the small.



Simple pendulum performs SHM:

- Condition for SHM is that restoring force F should be directly proportional to the displacement oppositely directed. The path of the bob is not straight line, but the arc of the circle of radius *l*

- Let T is the tension in the spring. When the particle is at point B two forces are acting on it:

1. mg, the weight of the point bob acting vertically downward.
2. T, the tension along the string.

- The weight mg can be resolved into two rectangular components.

1. Component of weight mg along the spring=$mg\cos\theta$
2. Component of weight mg perpendicular to the string=$mg\sin\theta$

- Since there is no motion of bob along the string, so the component mg cosθ must be equal to tension in the string T

$$T = mg\cos\theta$$

- Component mg sinθ is responsible for the motion of the bob towards the mean position. Thus, the restoring force F is;

$$F = -mg\sin\theta \qquad \ldots\ldots\ldots(iv)$$

$$\text{So } \sin\theta = \frac{x}{l}$$

$$F = -mg\frac{x}{l}$$

- By 2nd law of motion

F=ma ……………(v)

- Comparing (iv) and (v)

$$ma = -\,mg\frac{x}{l}$$

$$a = -\,\frac{gx}{l}$$

$$a = -(\text{constant})x$$

$$a \propto -x$$

Expression for time period:

We know that

$a = -x\omega^2$ $\Rightarrow$ $a = -\frac{gx}{l}$

$-x\omega^2 = -\frac{gx}{l}$ $\Rightarrow \omega = \sqrt{g/l}$

$T = \frac{2\pi}{\omega}$ $\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$

Frequency:

$F = {}^{1}/_{T}$

$F = {}^{1}/_{2\pi} \times \sqrt{g/l}$

Simple Pendulum

In a simple pendulum, a particle of mass m is suspended from one end of an unstretchable massless string of length L that is fixed at the other end.
The restoring torque acting on the mass when its angular displacement is θ, is:

$$\boxed{\tau = -L(F_g \sin\theta) = I\alpha}$$

α is the angular acceleration of the mass.
Finally,

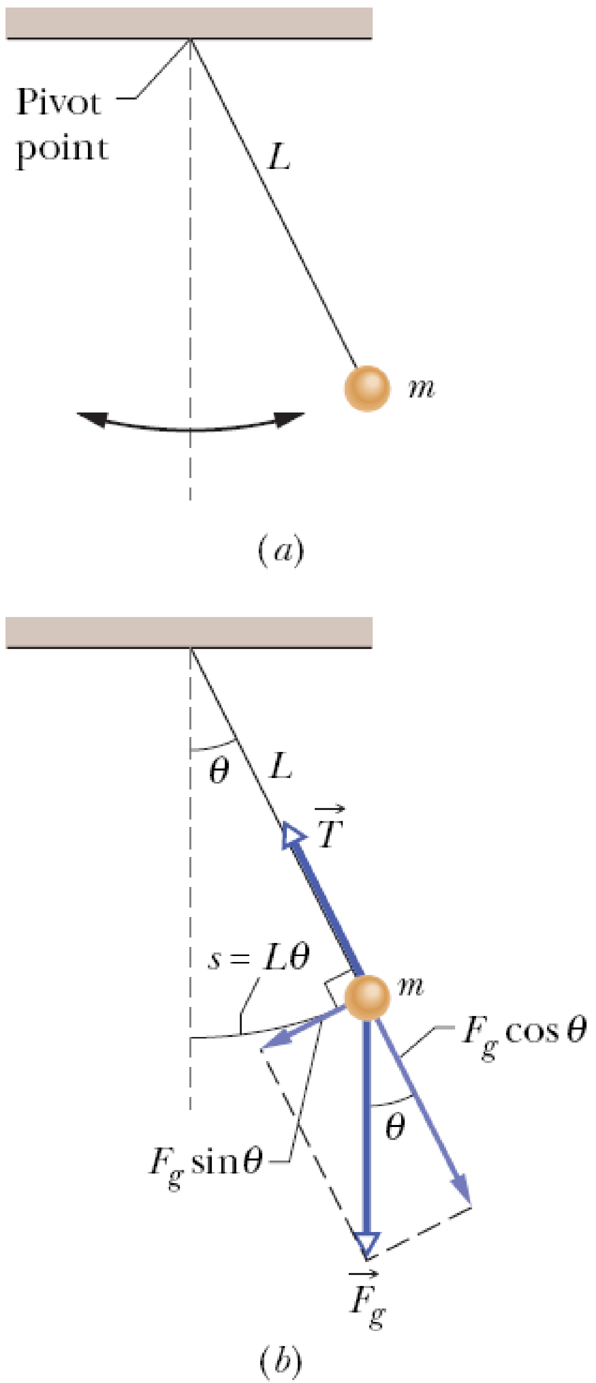
$$\alpha = -\frac{mgL}{I}\theta, and$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$T = 2\pi\sqrt{\frac{I}{mgL}}.$$

Also, $I = ml^2$

This is true for small angular displacements, θ.



Pendulums

In the **small-angle approximation** we can assume that θ << 1 and use the approximation sin $\theta \cong \theta$. Let us investigate up to what angle θ is the approximation reasonably accurate?

θ (degrees)	θ (radians)	sin θ
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259 (1% off)
20	0.349	0.342 (2% off)

Conclusion: If we keep $\theta < 10\,°$ we make less than 1 % error.

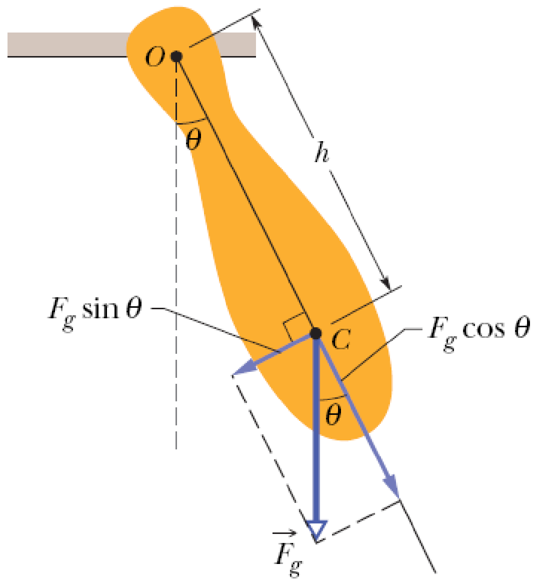
Physical Pendulum

A real pendulum, usually called a physical pendulum, can have a complicated distribution of mass.

Figure shows an arbitrary physical pendulum displaced to one side by angle θ. The gravitational force g acts at its center of mass C, at a distance h from the pivot point O. Comparison reveals only one important difference between an arbitrary physical pendulum and a simple pendulum. For a physical pendulum the restoring component $F_g \sin\theta$ of the gravitational force has a moment arm of distance h about the pivot point, rather than of string length L. Again (for small θ_m), we would find that the motion is approximately SHM. If we replace L with h in eq of time period of simple pendulum, we can write the period as:

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

Here, I is the rotational inertia of the pendulum about O.

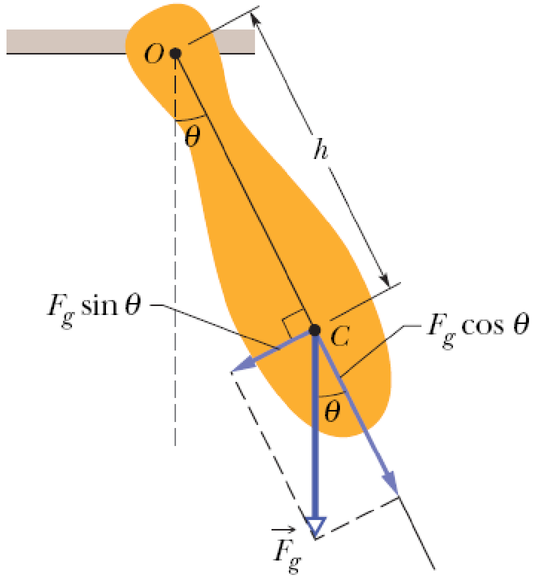


Physical Pendulum

A physical pendulum can have a complicated distribution of mass. If the center of mass, C, is at a distance of h from the pivot point (figure), then for small angular amplitudes, the motion is simple harmonic.

The period, T, is:

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgh}}}$$



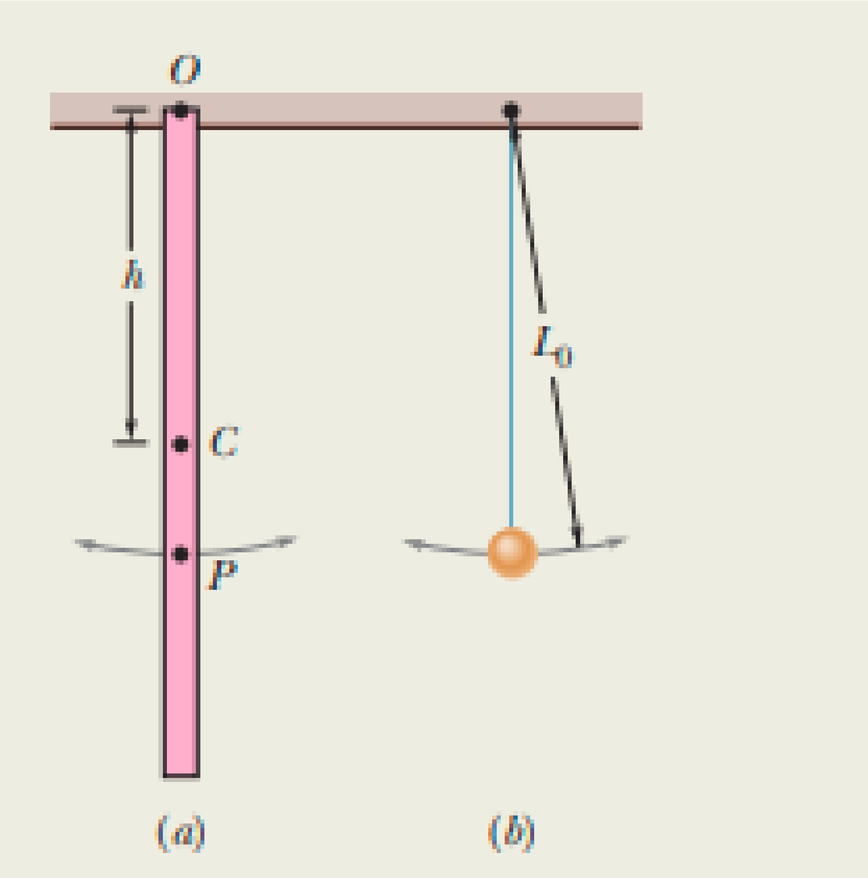
Here,
I is the rotational inertia of the pendulum about O.

Special Case of Physical Pendulum:

Take the pendulum to be a uniform rod of length L, suspended from one end. For such a pendulum, h, the distance between the pivot point and the center of mass, is $\frac{1}{2}L^2$

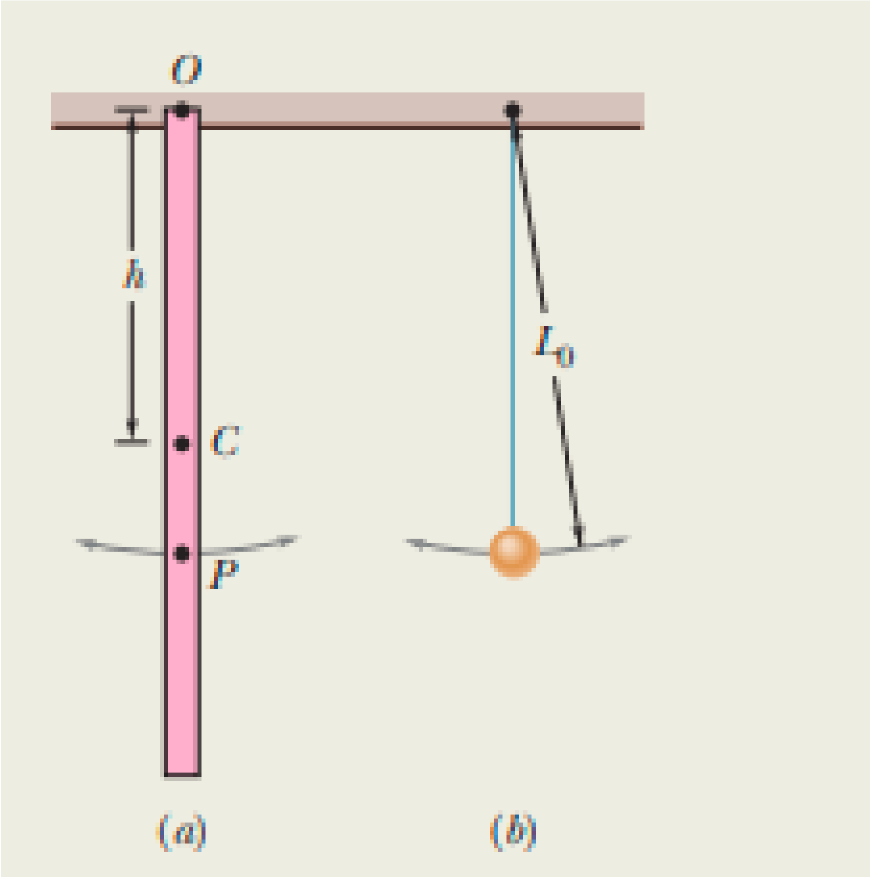
$$h = \frac{1}{2}L^2$$

$$I = \frac{1}{3}\mathrm{m}L^2$$



The center of Oscillation

Corresponding to any physical pendulum that oscillates about a given pivot point O with period T is a simple pendulum of length L_0 with the same period T. We can find L_0 with Eq. of the simple pendulum. The point along the physical pendulum at distance L_0 from point O is called the center of oscillation of the physical pendulum for the given suspension point.



Measuring g

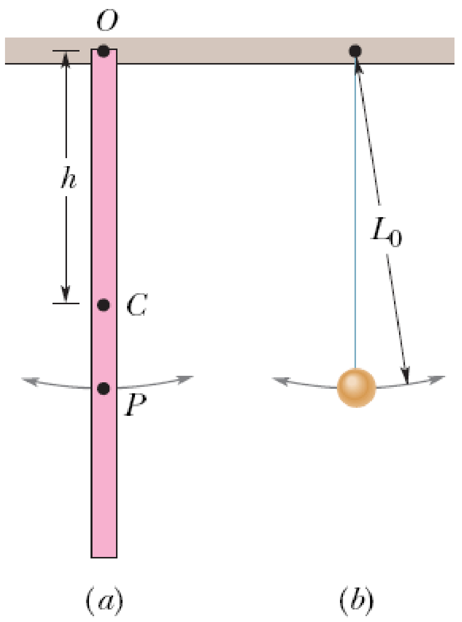
$$g = \frac{8\pi^2 L}{3T^2}$$

Sample Problem 15.05

In Fig. a, a meter stick swings about a pivot point at one end, at distance h from the stick's center of mass.
(a)What is the period of oscillation T?

KEY IDEA: The stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite the pivot point—so the stick is a physical pendulum.

Calculations: The period for a physical pendulum depends on the rotational inertia, I, of the stick about the pivot point. We can treat the stick as a uniform rod of length L and mass m. Then I =1/3 mL^2, where the distance h is L.
Therefore,



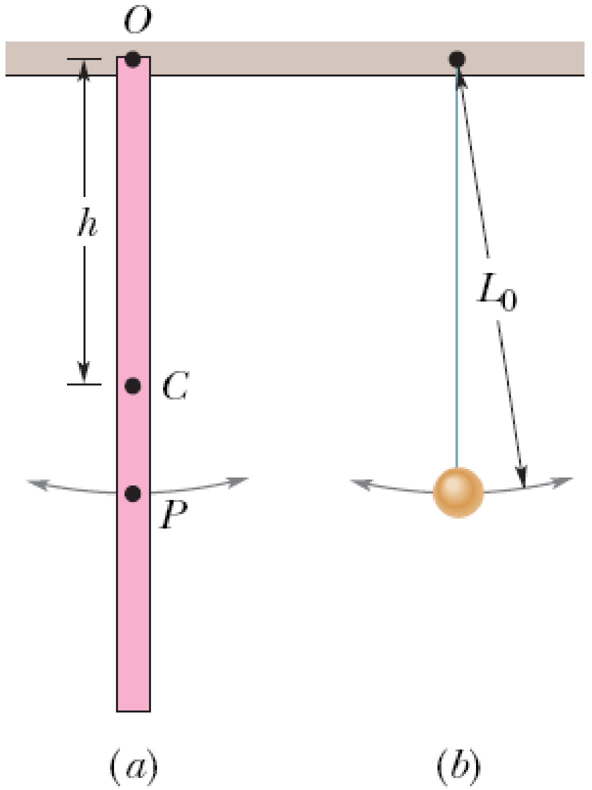
$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg(\frac{1}{2}L)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$= 2\pi\sqrt{\frac{(2)(1.00\text{ m})}{(3)(9.8\text{ m/s}^2)}} = 1.64\text{ s.} \qquad \text{(Answer)}$$

Note the result is independent of the pendulum's mass *m*.

(b) What is the distance L_0 *between the pivot point O* of the stick and the center of oscillation of the stick?

Calculations: We want the length L_0 of the simple pendulum (drawn in Fig. *b)* that has the same period as the physical pendulum (the stick) of Fig. *a*.



Corresponding to any physical pendulum that oscillates about a given pivot point O with period T is a simple pendulum of length L_0 with the same period T. We can find L_0 with Eq. of the simple pendulum. The point along the physical pendulum at distance L_0 from point O is called the center of oscillation of the physical pendulum for the given suspension point.

Keep T = 1.64 sec , 9.8m/s^2

Solving the equation we get:

L_0 = 0.6676 m or 66.76 cm

$$T = 2\pi\sqrt{\frac{L_0}{g}}$$

Time period of different Pendulums:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad (torsion\ Pendulum)$$

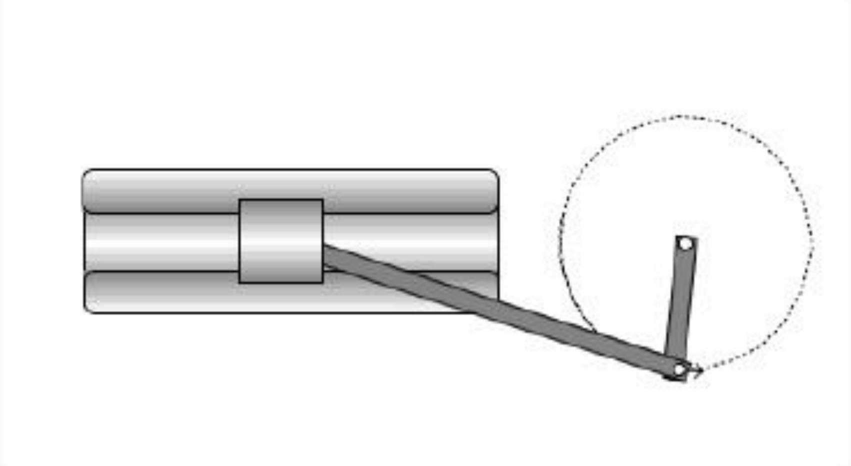
$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{(simple pendulum)}$$

$$T = 2\pi\sqrt{\frac{I}{mgh}} \quad \text{(physical pendulum)}$$

Checkpoint 5

Three physical pendulums, of masses m_0, $2m_0$, and $3m_0$, have the same shape and size and are suspended at the same point. Rank the masses according to the periods of the pendulums, greatest first.

UNIFORM CIRCULAR MOTION



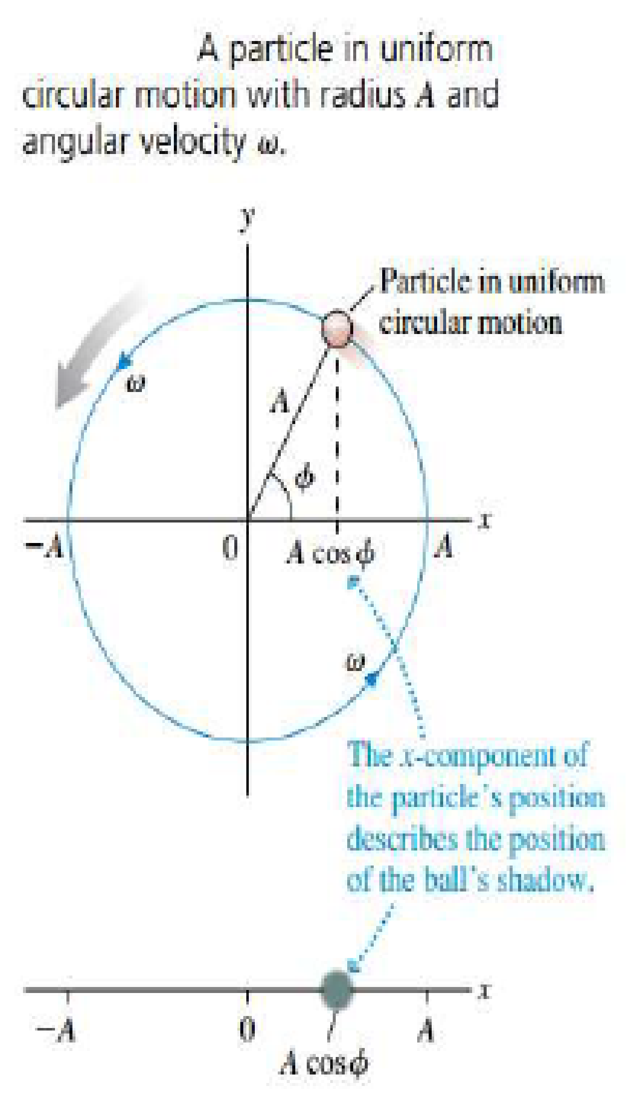
Relation between uniform circular motion and SHM

An object in simple harmonic motion has the same motion as of an object in uniform circular motion:

- Consider the particle in uniform circular motion with radius A and angle ϕ
- x= A cos ϕ
- Particle's angular velocity, in rad/s, is
- $\frac{d\varphi}{dt}$=ω
- This is the rate at which the angle ϕ is increasing.

If the particle starts from $\emptyset_0$= 0 at t = 0, its angle at a later time t is simply

- φ =ωt



- As φ increases, the particle's x-component is

$x(t) = A \cos \omega t$

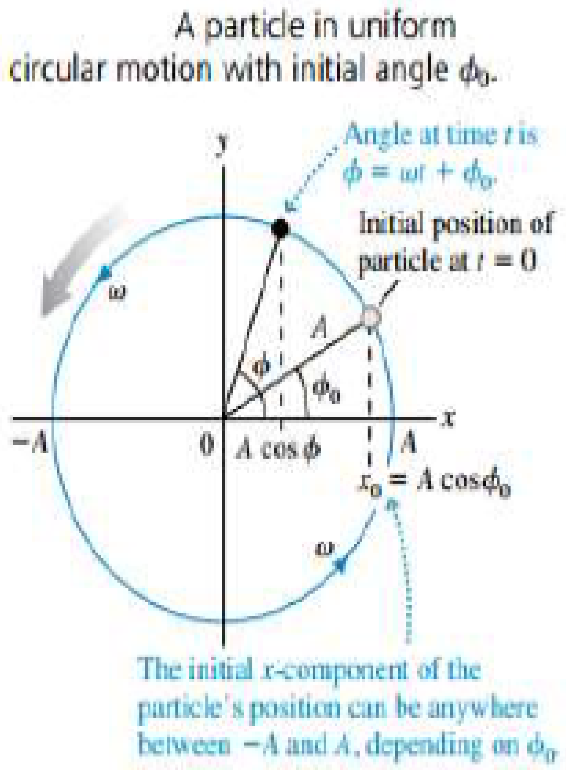
- The particle in started at $\varnothing_0 = 0$. fig shows a more general situation in which the initial angle

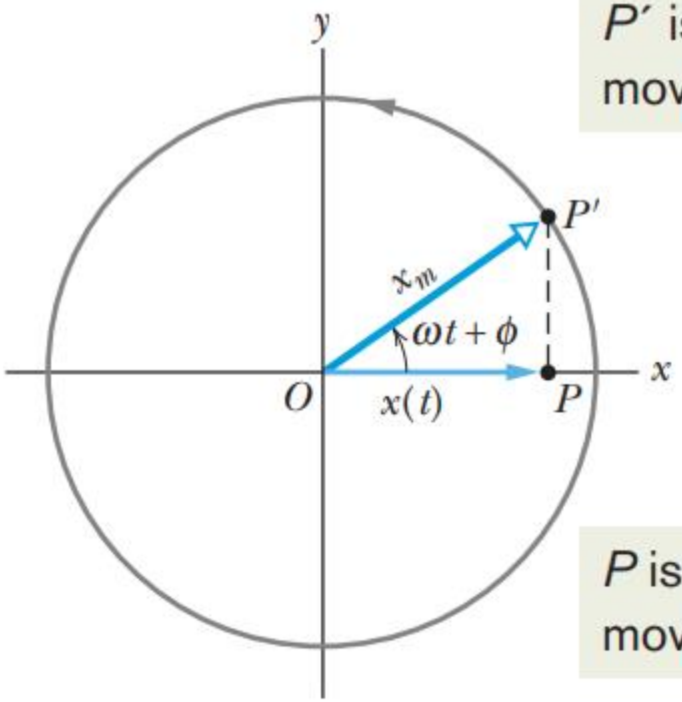
$\varnothing_0$ can have any value. The angle at a later time t is then

$$\varphi = \omega t + \varnothing_0$$

- $v(t) = -\omega A \sin(\omega t + \varnothing_0) =$

$v(t) = -v_{max} \sin(\omega t + \varnothing_0)$

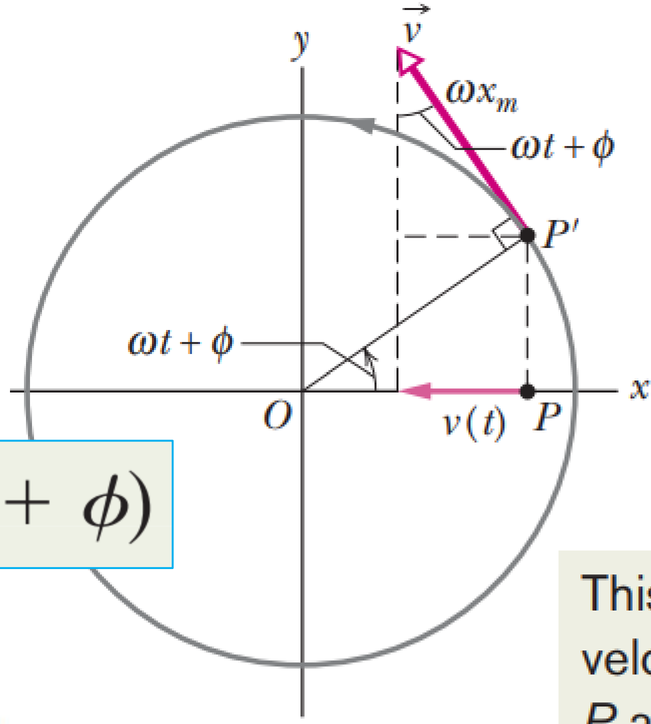




P' is a particle moving in a circle.

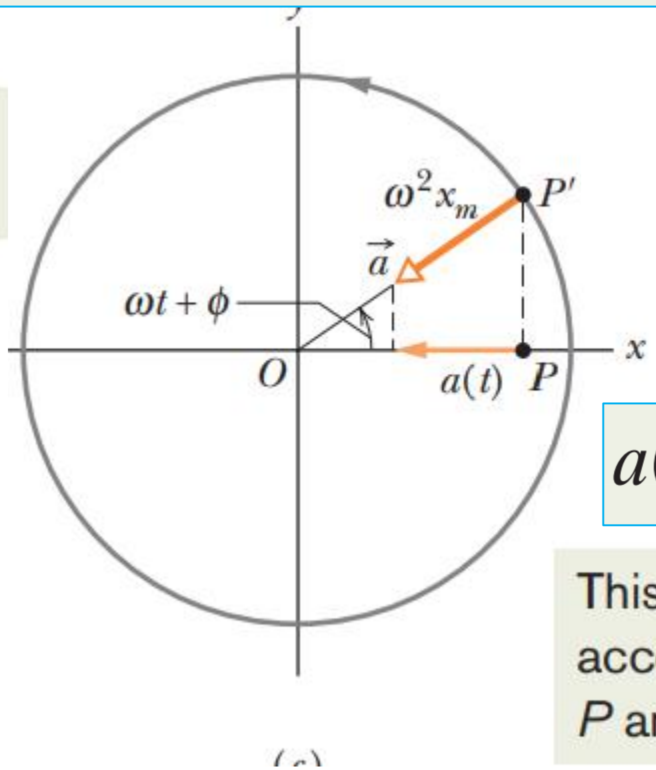
P is a projection moving in SHM.

$$x(t) = x_m \cos(\omega t + \phi)$$



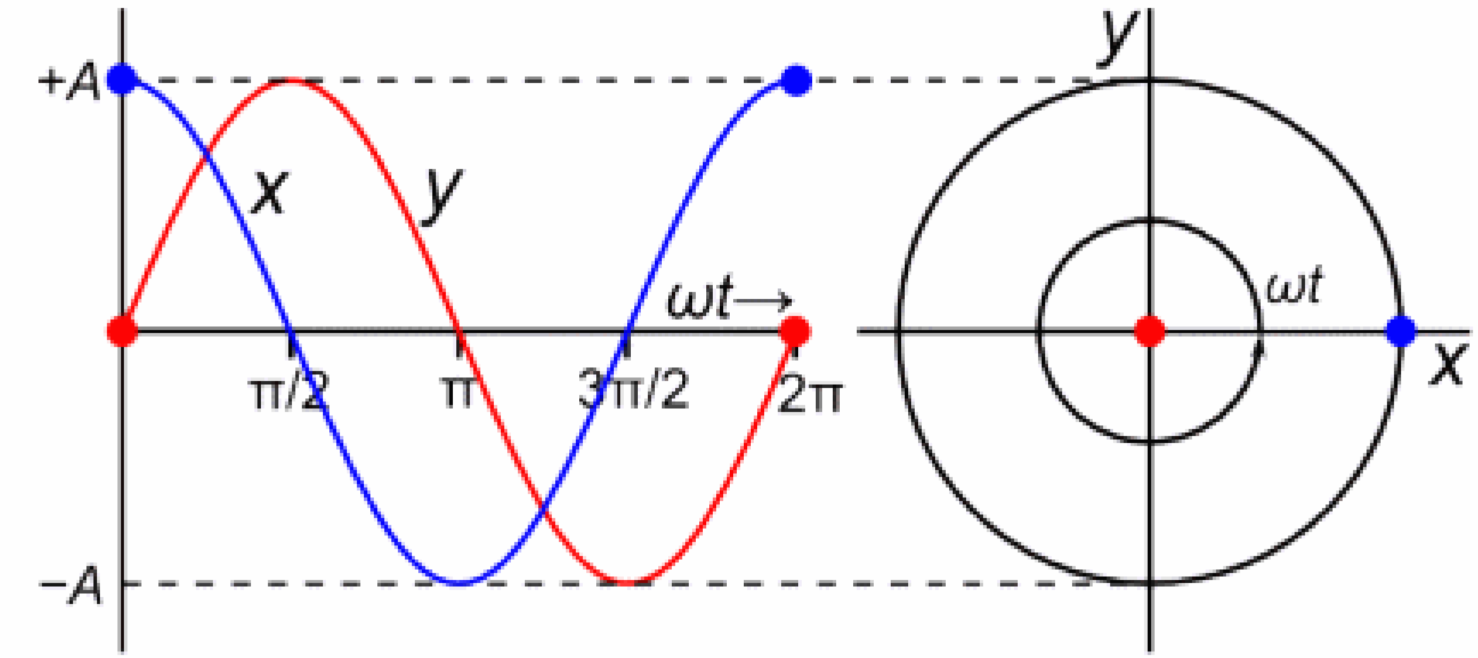
$$v(t) = -\omega x_m \sin(\omega t + \phi)$$

This relates the velocities of P and P'.

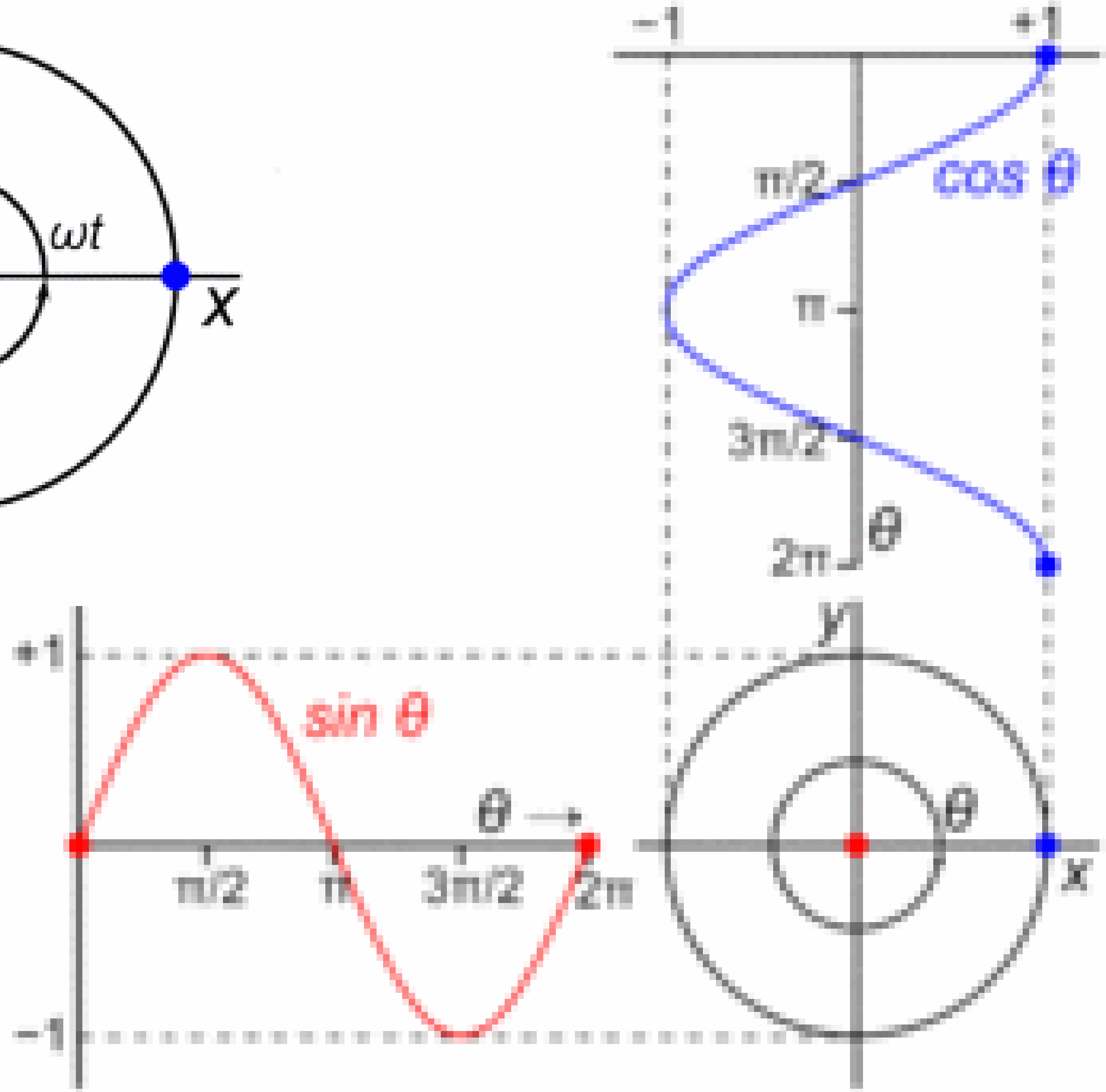


$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

This relates the accelerations of P and P'.



Sine and Cosine Functions were essentially made to represent the points on circle (and triangle)



Path of a Pendulum also contains the part of circle

Practice Problem:

A particle rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. As the particle has an *x* coordinate of 2.00 m at t =0 s and is moving to the right. (a) Determine the *x* coordinate as a function of time.

Solution Because the amplitude of the particle's motion equals the radius of the circle and $\omega = 8.00$ rad/s, we have

$$x = A\cos(\omega t + \phi) = (3.00 \text{ m})\cos(8.00t + \phi)$$

We can evaluate ϕ by using the initial condition that $x = 2.00$ m at $t = 0$:

$$2.00 \text{ m} = (3.00 \text{ m})\cos(0 + \phi)$$

$$\phi = \cos^{-1}\left(\frac{2.00 \text{ m}}{3.00 \text{ m}}\right)$$

If we were to take our answer as $\phi = 48.2°$, then the coordinate $x = (3.00 \text{ m})\cos(8.00t + 48.2°)$ would be decreasing at time $t = 0$ (that is, moving to the left). Because our particle is first moving to the right, we must choose $\phi = -48.2° = -0.841$ rad. The *x* coordinate as a function of time is then

$$x = \boxed{(3.00 \text{ m})\cos(8.00t - 0.841)}$$

Note that ϕ in the cosine function must be in radians.

(b) Find the *x* components of the particle's velocity and acceleration at any time *t*.

Solution

$$v_x = \frac{dx}{dt} = (-3.00 \text{ m})(8.00 \text{ rad/s})\sin(8.00t - 0.841)$$

$$= \boxed{-(24.0 \text{ m/s})\sin(8.00t - 0.841)}$$

$$a_x = \frac{dv_x}{dt} = (-24.0 \text{ m/s})(8.00 \text{ rad/s})\cos(8.00t - 0.841)$$

$$= \boxed{-(192 \text{ m/s}^2)\cos(8.00t - 0.841)}$$

From these results, we conclude that $v_{max} = 24.0$ m/s and that $a_{max} = 192$ m/s^2. Note that these values also equal the tangential speed ωA and the centripetal acceleration $\omega^2 A$.

Undamped vs Damped Oscillations

The difference between damped and undamped oscillations is that the amplitude of the waves that are being generated keeps on decreasing gradually in damped oscillations, while in undamped oscillations, the amplitude of the waves that are being generated remained unchanged and constant over time.

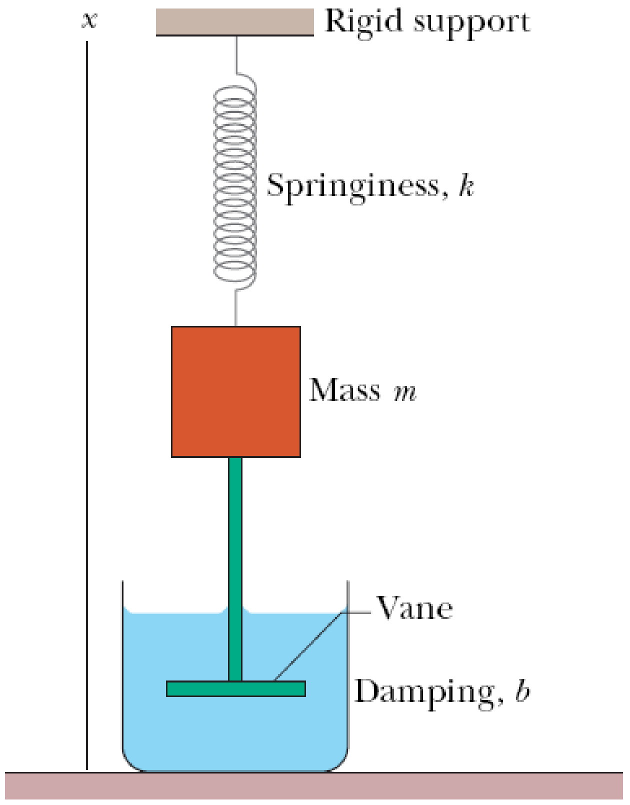
Damped Oscillations/ Damped SHM

In a damped oscillation, the motion of the oscillator is reduced by an external force. The oscillator and its motion are said to be damped.

Example:

A block of mass m oscillates vertically on a spring with spring constant k.

From the block a rod extends to a vane which is submerged in a liquid. The liquid provides the external damping force, F_d.



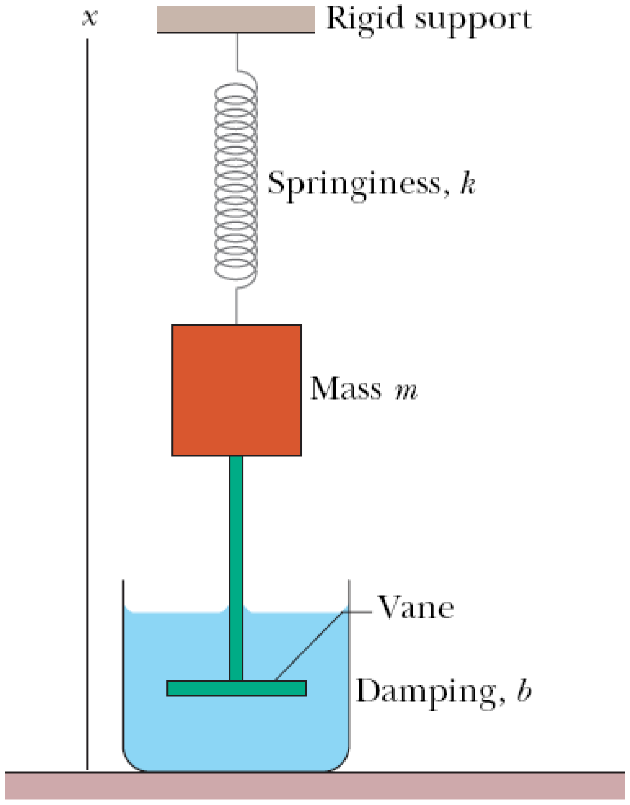
Damped SHM

Often the damping force, F_d, is proportional to the 1st power of the velocity v. That is,

$$F_d = -bv$$

Fs = -kx

-bv –kx = ma



From Newton's 2nd law, the following DE results:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The solution is:

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)$$

Here ω' is the angular frequency, and is given by:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Damping Constant

where b is a damping constant that depends on the characteristics of both the vane and the liquid and has the SI unit of kilogram per second. The minus sign indicates that opposes the motion.

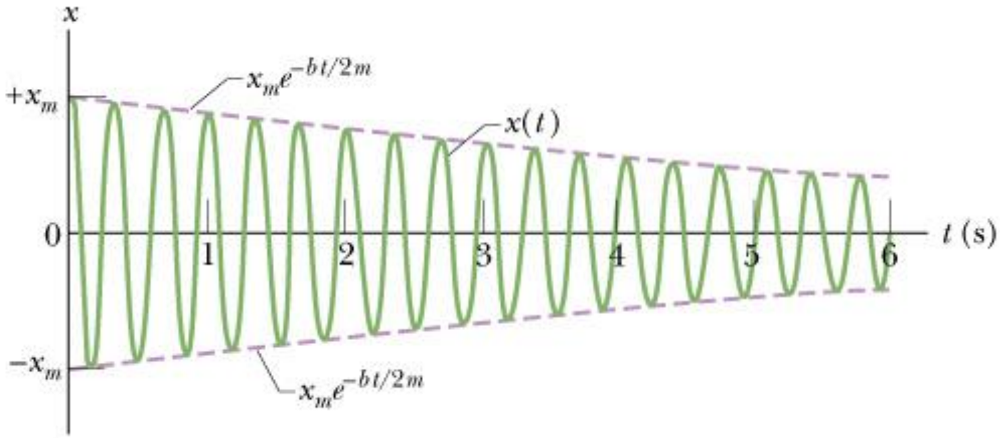
Undamped Oscillator:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

If $b = 0$ (there is no damping), then Eq. reduces to ($\omega = \sqrt{k/m}$) for the angular frequency of an undamped oscillator, If the damping constant is small but not zero (so that $b \ll \sqrt{km}$), then $\omega' \approx \omega$.

Damped Oscillations

$$\boxed{x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)}$$

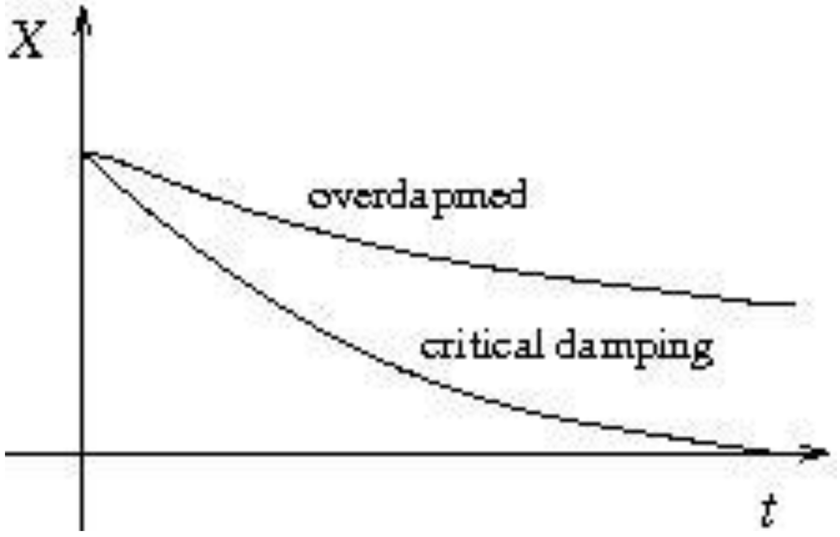


The above figure shows the displacement function *x(t)* for the damped oscillator described before.

The amplitude decreases as $x_0 \exp(-bt / 2m)$ with time.

The above is for b < $2m\omega_0$ (underdapmed).

For b > $2m\omega_0$ (overdapmed)
and b = $2m\omega_0$ (critical damping),



Damped Energy:

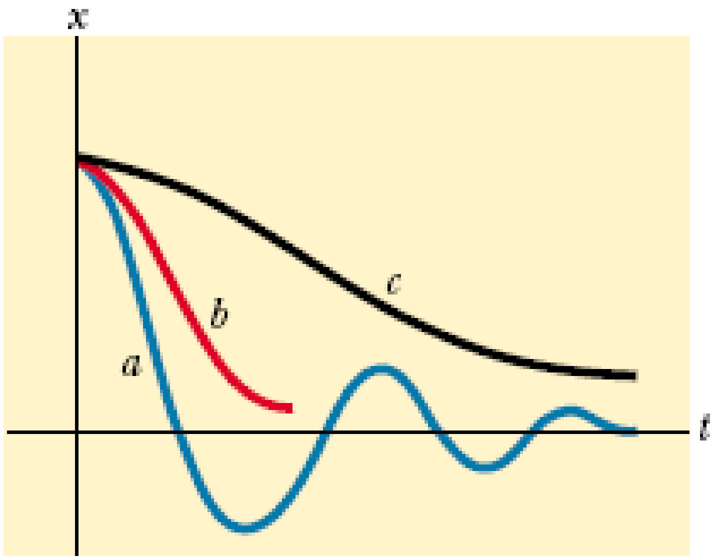
$$E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m},$$

$$E(t) \approx \frac{1}{2}kx_m^2 e^{-\frac{bt}{m}}$$

DAMPED OSCILLATIONS

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

where $\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency**



Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

- critical damping: the condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

- over damping: the condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system

- under damping**:** the condition in which damping of an oscillator causes it to return to equilibrium with the amplitude gradually decreasing to zero; system returns to equilibrium faster but overshoots and crosses the equilibrium position one or more times that means an underdamped system will oscillate through the equilibrium position.

Practice Problem (Sample Problem 15.06)

For the damped oscillator of Fig. 15-16, $m = 250$ g, $k = 85$ N/m, and $b = 70$ g/s.

(a) What is the period of the motion?

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

- Definition of Free Vs. Forced Oscillations
- Free oscillations are oscillations execute by a subject without being acted upon by an external force. They occur due to the elastic forces and inertia of the system.

- Forced oscillations can be defined as the oscillations in which the body oscillates with a frequency other than its natural frequency under the influence of an external periodic force. The external force here is called the driving force.

- Example of Free vs. Forced Oscillations

Let’s take an example of a playground swing. When you push the swing just once, it oscillates at its own natural frequency without any interference from any external force, so it acts as a free oscillator. But if you push the swing each time it begins to slow down or reaches a certain point, it will continue to swing because it’s now being subjected to an external force, so it acts as a forced oscillator. Other examples of free oscillations include a tuning fork and a pendulum.

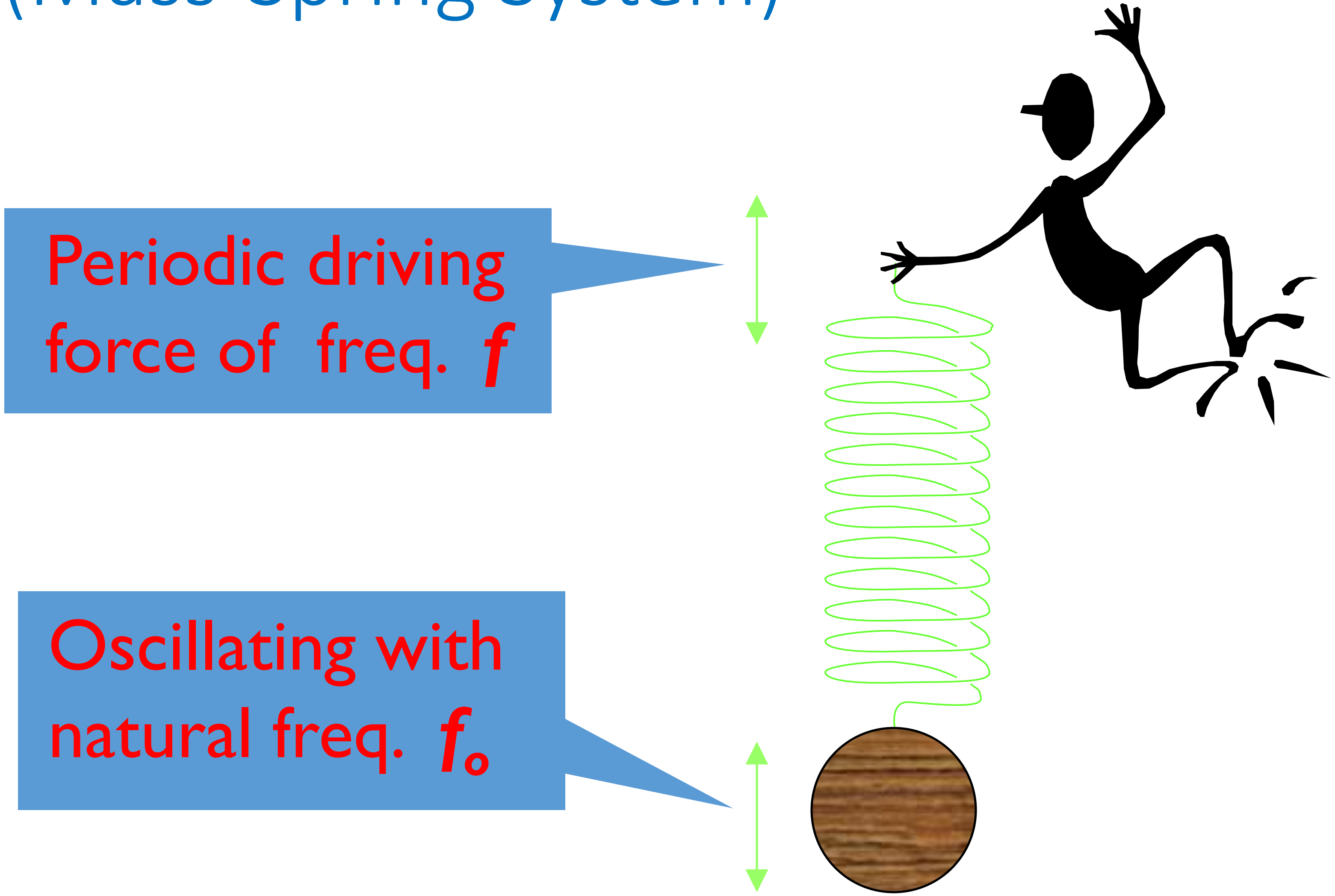
Forced Oscillations and Resonance

A person swinging in a swing without anyone pushing it is an example of free oscillation. However, if someone pushes the swing periodically, the swing has forced, or driven, oscillations. Two angular frequencies are associated with a system undergoing driven oscillations:

(1) the natural angular frequency ω of the system, which is the angular frequency at which it would oscillate if it were suddenly disturbed and then left to oscillate freely, and

(2) the angular frequency ω_d of the external driving force causing the driven oscillations.

Example (Mass-Spring System)



Resonance

When a system is disturbed by a periodic driving force whose frequency(angular frequency) is equal to the natural frequency (fo) of the system, the system will oscillate with LARGE amplitude. Resonance is said to occur.

$$\omega_d = \omega$$

Resonance

- Resonance is a particular case of forced oscillation. When the frequency difference between the system and that of the external force is minimal, the resultant amplitude of the forced oscillations will be enormous. However, when the two frequencies match or become the same, resonance occurs. Thus, at resonance, the amplitude of forced oscillation is maximal, and the natural oscillating frequency of the system is equal to the frequency of the periodic driving force.

Example 1

Breaking Glass

System : ***glass***

Driving Force :
sound wave

Example 2

Collapse of the Tacoma Narrows suspension bridge in America in 1940

System : ***bridge***

Driving Force : ***strong wind***

Homework questions:

- Practice problems:
- End of chapter 15 textbook “ Fundamentals of Physics” by Halliday & Resnick Jearl Walker 10th Edition”
- Questions:

1, 3, 5, 7, 8, 12

- Problems:
- 1, 2, 3, 6, 8, 9, 11, 12, 13, 27, 28, 32, 58, 59, 62, 77, 94, 106

Answers of even problems:

Answer of even problems:

2: a. $F_{max} = 10\ N$ $b.\ \omega = 1.2 \times 10^2$ N/m

6: a. $\omega = 6.28 \times 10^5$ rad/s b. $x_m = 1.59 \times 10^{-3}$ m

8. ϕ = +1.91 rad or – 4.37 rad. The other "root" (+4.37 rad) can be rejected on the grounds that it would lead to a positive slope at t = 0

12. ϕ = – 0.927 rad or +5.36 rad. The other "root" (+4.07 rad) can be rejected on the grounds that it would lead to a positive slope at t = 0.

28: T.E. = 0.72 J answers a. the mass does turn back before reaching x = 15 cm. b. It turns back at x = 12 cm.

Answers of even problems:

Answer of even problems:

32: $k = 8.3 \times 102$ N/m

58. 0.39

62. For the given range of $2.00 < \omega < 4.00$ (in rad/s), we find only two of the given pendulums have appropriate values of ω. pendulum (d) with length of 0.80 m (for which $\omega = 3.5$ rad/s) and pendulum (e) with length of 1.2 m (for which $\omega = 2.86$ rad/s).

94. $\phi = +1.82$ rad or – 4.46 rad. The other “root” (+4.46 rad) can be rejected on the grounds that it would lead to a negative slope at $t = 0$.

- Answer of 106:

(a) The graph makes it clear that the period is T = 0.20 s.

(b) m = 0.203 = 0.20 kg.

(c) The graph indicates that the speed is (momentarily) zero at t = 0, which implies that the block is at $x_0 = \pm x_m$. From the graph we also note that the slope of the velocity curve (hence, the acceleration) is positive at t = 0, which implies (from ma = –kx) that the value of x is negative. Therefore, with x_m = 0.20 m, we obtain x_0 = –0.20 m.

(d) We note from the graph that v = 0 at t = 0.10 s, which implied $a = \pm a_m = \pm\omega^2 x_m$. Since acceleration is the instantaneous slope of the velocity graph, then (looking again at the graph) we choose the negative sign. Recalling ω^2= k/m we obtain a = –197 ≈ –2.0 ×10^ 2 m/s^2 .

(e) The graph shows v_m = 6.28 m/s, so $K_m = \frac{1}{2}\mathrm{m}v_m^2 = \frac{1}{2}$(0.20 kg)(6.28 m/s) = 4.0 J

Adapted from:

Book:

Fundamentals of Physics by Halliday Resnick & Jearl Walker 10th edition