Vectors

Introduction:

- Need of vectors in physics:

Physics deals with many quantities that have both size and direction. To describe those quantities we use language of vectors.

Vectors:

- Definition:

In physics vector has magnitude as well as direction, and they follow certain rules of combination.

In computer programming, vector is a type of array that is one dimensional.

Scalar and Vector quantities

Vector quantity:

A physical quantity that has both magnitude and direction.

Example:

Displacement, velocity and acceleration

Scalar quantity:

Quantity that only involves magnitude but no direction.

They follow ordinary rules of algebra

Example:

Temperature, mass and work

Displacement:

- Change of position.
- It is straight line distance in a particular direction.
- A vector quantity
- The SI unit of displacement is meters.

Some important terminologies:

Point:

- Dimensionless.
- A notion that models exact location in space. It has no length, width or thickness.
- Element of some set called space.
- Represented by ordered pair.

Space:

Set with some added structure.

Consists of selected mathematical objects that are treated as points and selected relationship between these points.

Geometrical space is a boundless 3D extent in which objects and events have relative position and direction.

Some important terminologies:

- Coordinate system:

A system that uses 1 or more number or coordinates to uniquely determine the position of points or other geometric elements on a manifold.

- Axis

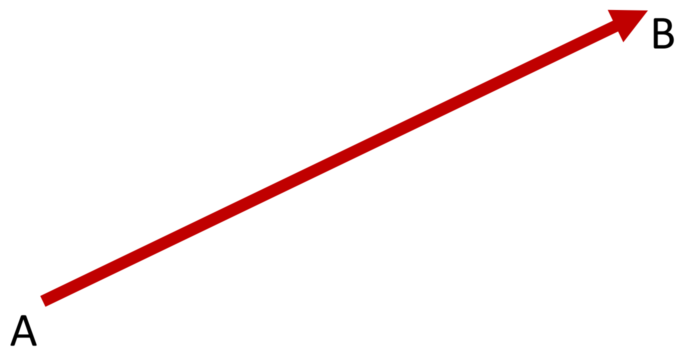
Fixed reference line for measurement of coordinates.

Representation of a vector:

- A vector is represented with an arrow pointing from A to B by a directed line segment whose length is magnitude of vector and with an arrow indicating the direction.
- A vector is represented using bold letter or arrow head.

Example:

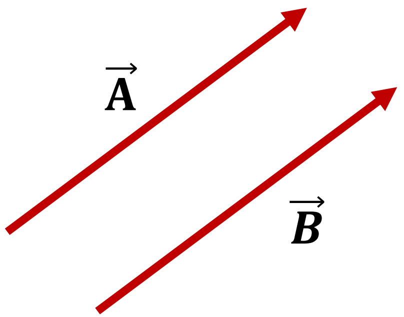
A, $\vec{a}$.



Equal vectors:

- Two vectors are said to be equal if they have same magnitude and direction.
- If we take a vector and translate it to a new position(without rotation) such that its magnitude and direction remains unchanged, then the vector we obtain at the end of this process is same as we had in beginning. Vectors are essentially independent of the position.

$$\vec{\mathbf{A}} = \vec{\mathbf{B}}$$



Distance vector

- Distance between two points is represented by a displacement/ distance vector.

Example:

Vector $\overrightarrow{\mathbf{P_{12}}}$ represents the distance between points 1 and 2

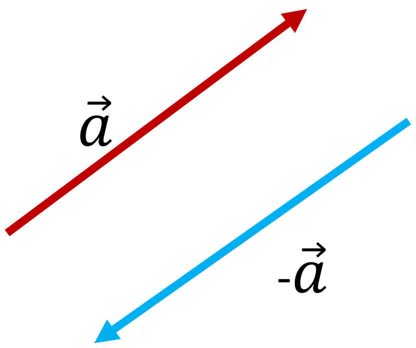
Zero vector

- A vector whose magnitude is 0 and does not point in any particular direction.

Negative of a vector

- A vector whose magnitude is same as the original vector but it is opposite in direction to the original vector. Adding these two vectors gives 0.

$$\vec{a} + (-\vec{a}) = 0$$



Distance and Displacement

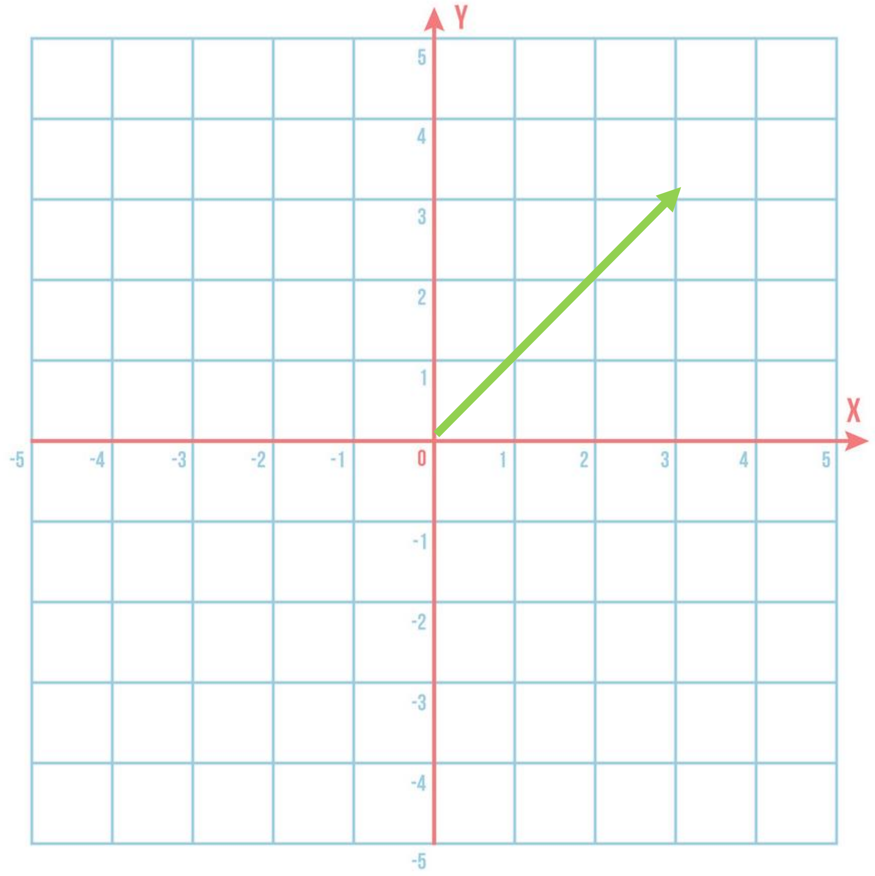
- Distance:

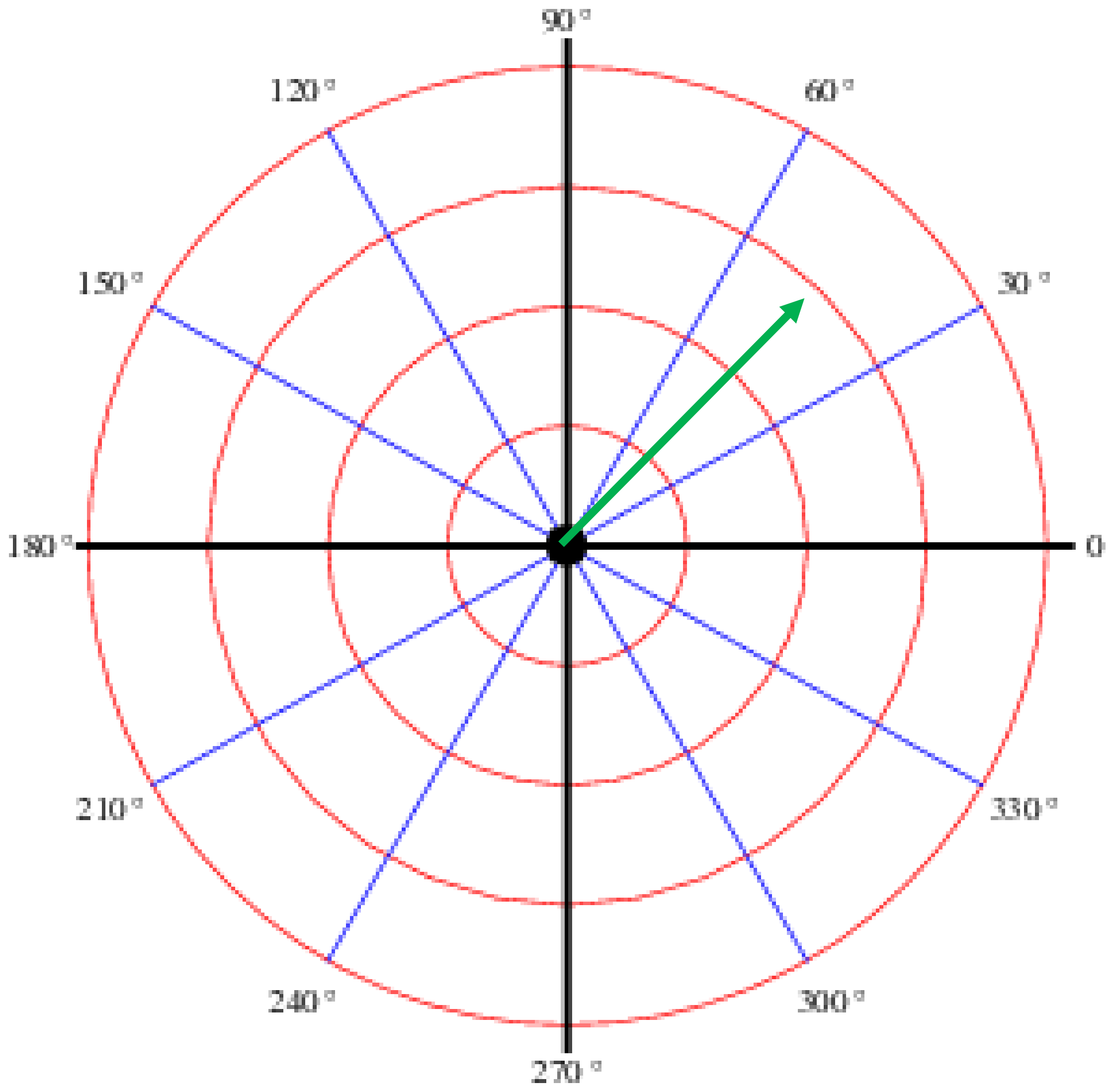
Length of the path travelled by a body while moving from initial to final position. It is a scalar quantity.

- Displacement:

Shortest distance between initial and final position of the body. It only represents the overall effect of the motion not motion itself. It is a vector quantity.

Vectors are drawn in coordinate system





Rectangular coordinate system(2D):

- The rectangular coordinate system consists of two real number lines that intersect at a right angle.

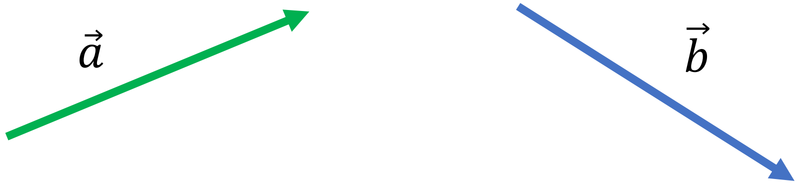
- The horizontal number line is called the x-axis, and the vertical number line is called the y-axis.

- These two number lines define a flat surface called a plane, and each point on this plane is associated with an ordered pair of real numbers (x,y).

- The intersection of the two axes is known as the origin which corresponds to the point (0,0).

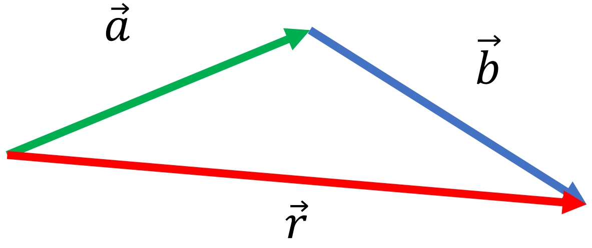
Adding vectors geometrically:

- Rules of vector addition are different. Consider two vectors $\vec{a}$ and $\vec{b}$ shown in figure:



- Their vector sum $\vec{r}$ is shown in figure:

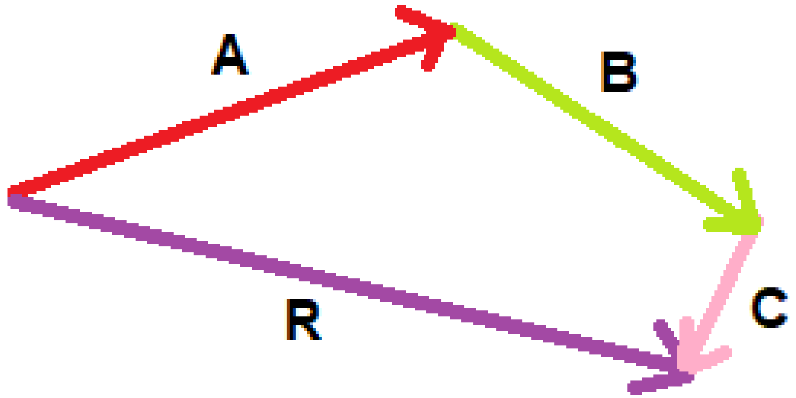
$$\vec{r} = \vec{a} + \vec{b}$$



Head to tail rule

- The head-to-tail method of adding vectors involves drawing the first vector on a graph and then placing the tail of each subsequent vector at the head of the previous vector. The resultant vector is then drawn from the tail of the first vector to the head of the final vector.

R = A + B + C



Important points to remember:

- On paper, sketch all vectors to some scale and at a proper angle.

Commutative law:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative Law:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

Subtraction/ difference between two vectors:

We find the difference between two vectors **A** and **B** by adding –**B** to the vector **A**.

$$\mathbf{A}-\mathbf{B} = \mathbf{A}+(-\mathbf{B})$$

Components of a vector:

A component of vector is the projection of the vector on an axis.

- X component:

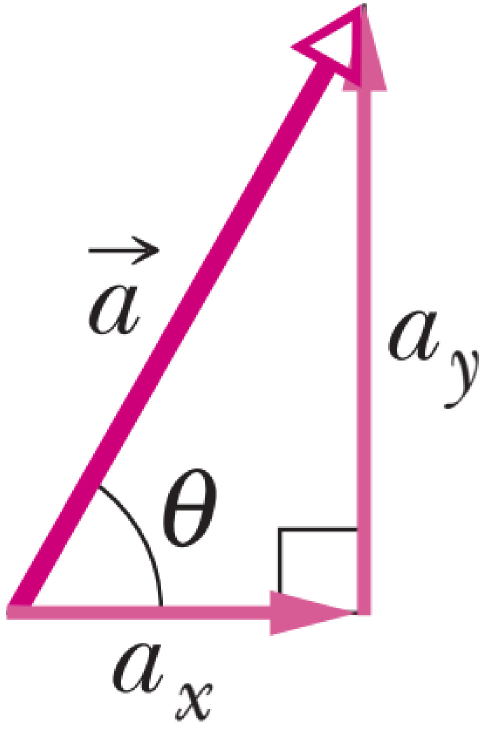
The projection of a vector on x-axis is its x component.

- Y component

The projection of a vector on y-axis is its y component.

- Resolving a vector:

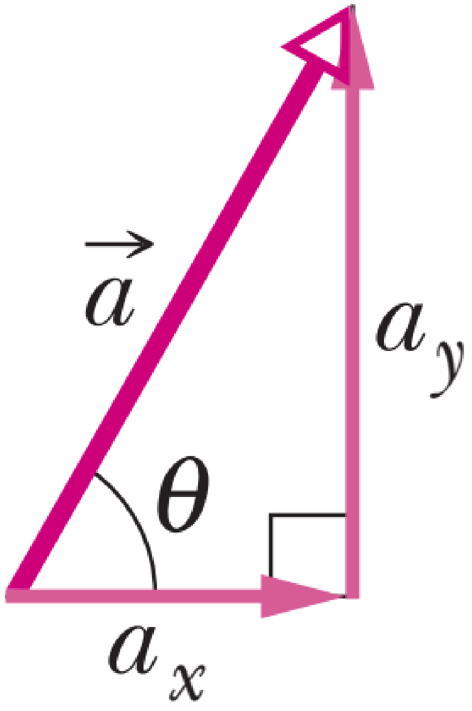
The process of finding the components of vector is called resolving the vector.



Components of a vector:

We can find components of vector from magnitude and angle using:

$$a_x = \text{a} \ \cos\theta \ and \quad a_y = a \ \sin\theta$$



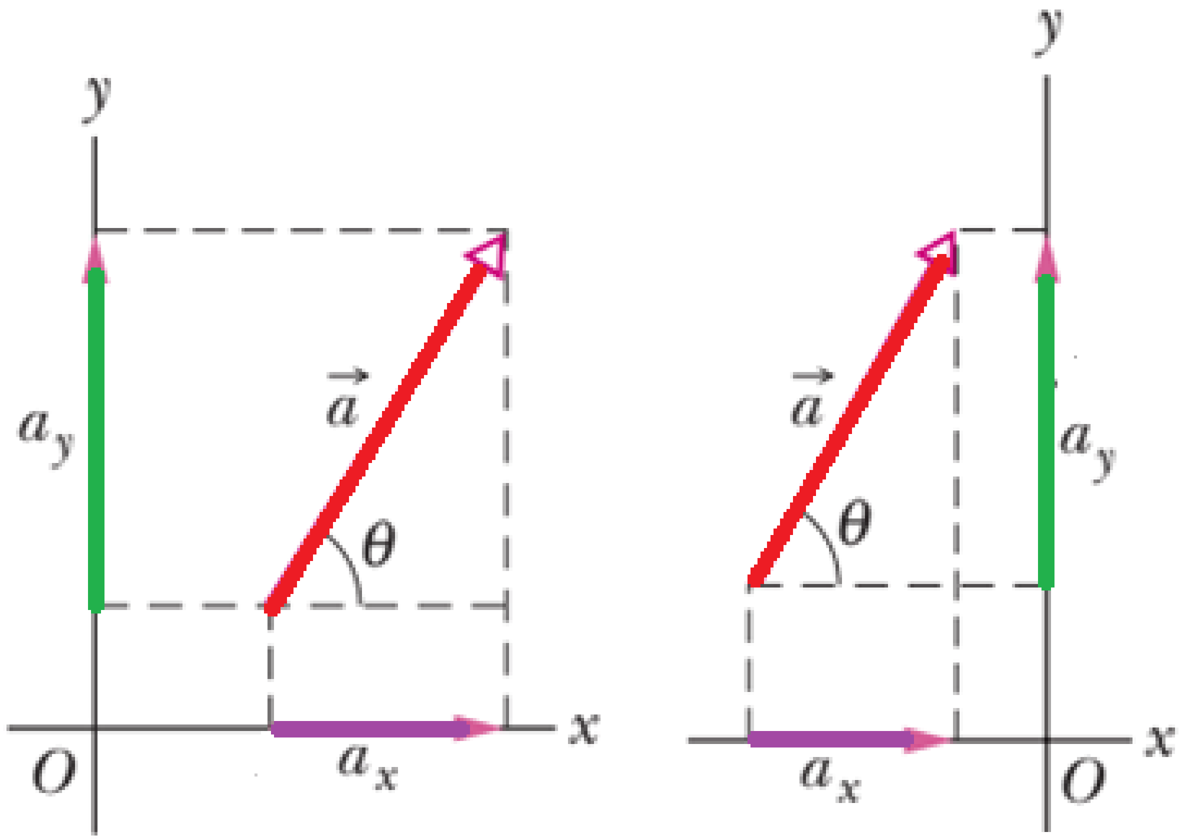
We can find magnitude and angle of vector from its components using:

$$a = \sqrt{a_x^2 + a_y^2} \ \ and \quad \tan\theta = \frac{a_y}{a_x}$$

ϴ is the angle that the vector $\vec{a}$ makes with the positive direction of x axis and a is the magnitude of vector $\vec{a}$. If angle is measured clockwise it is negative and if measured anticlockwise then it is taken positive.

Components of a vector:

- In the figure below a_x is the x component of vector $\vec{a}$ and a_y is the y component of vector $\vec{a}$.

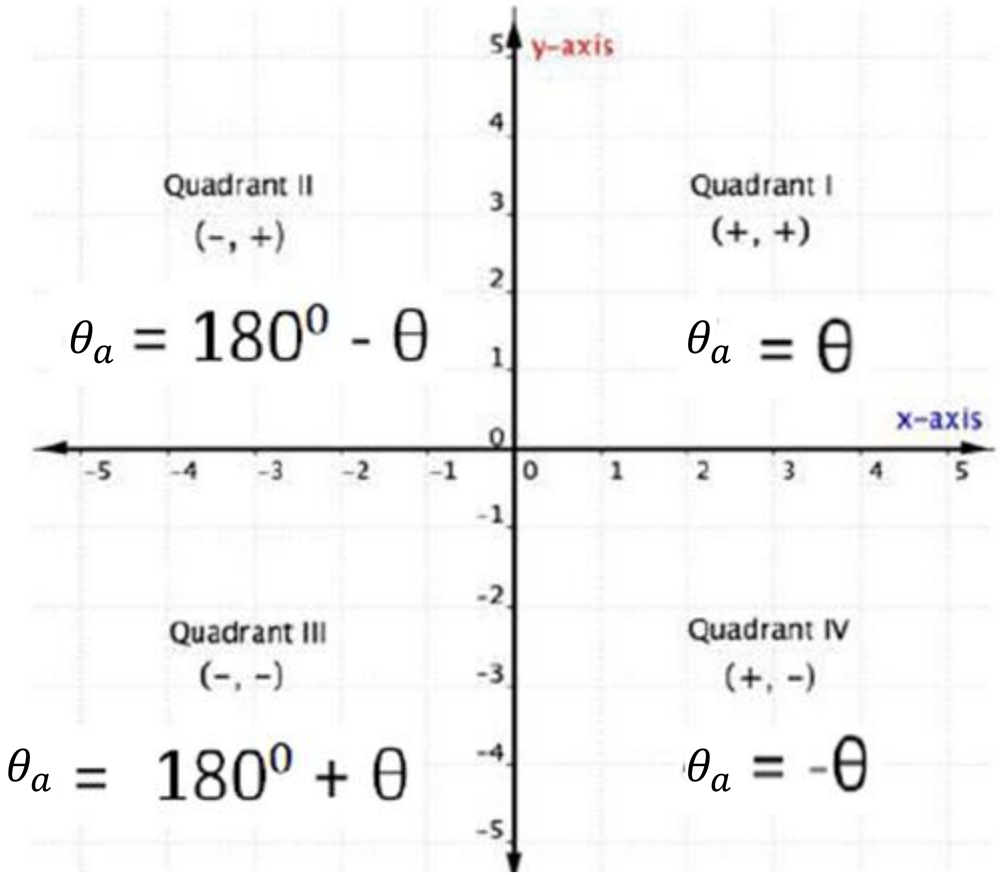


Calculations for ϴ

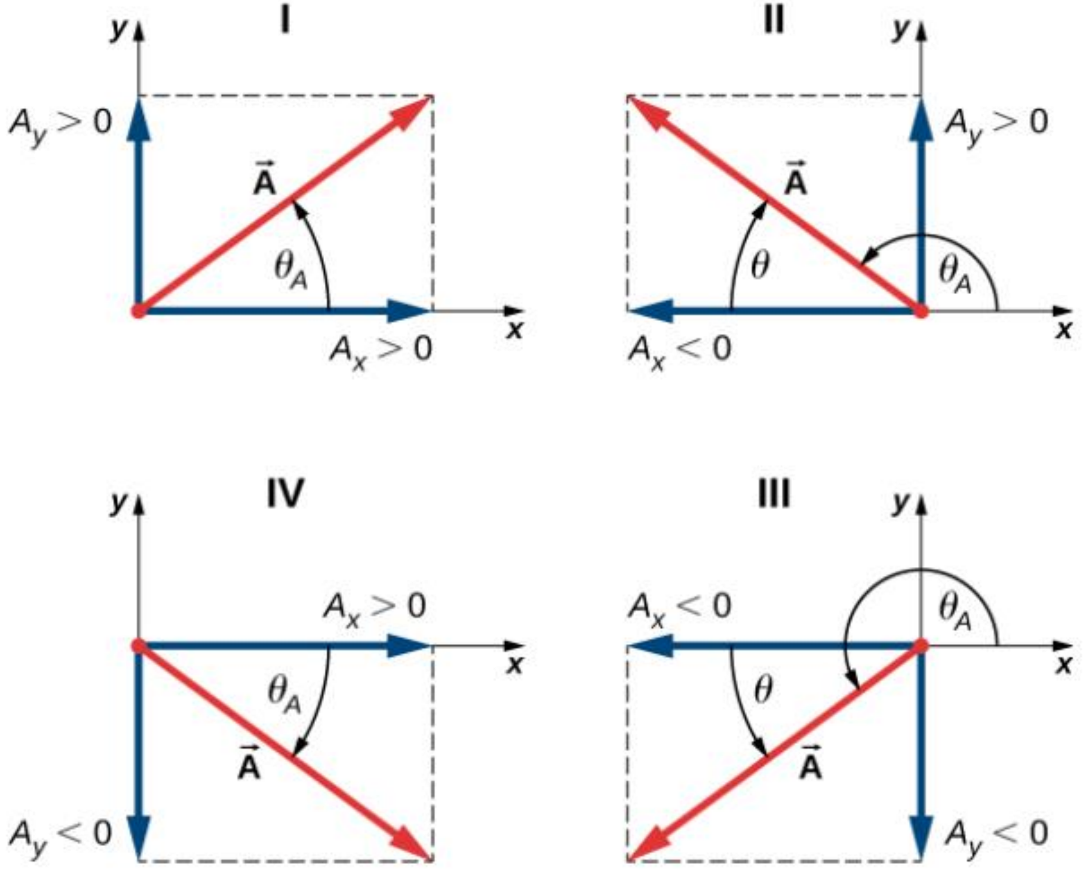
- ϴ is the angle that the vector makes with the positive direction of x axis. If angles are measured clockwise it is negative and if measured anticlockwise then they are taken positive.
- When range of ϴ is from 0^o to 360^o
- Feed the signed value of x and y in the calculator

Quadrant	**Value of $\tan^{-1}$**
I	Use the calculator value
II	Add 180° to the calculator value
III	Add 180° to the calculator value
IV	Add 360° to the calculator value

- When range of θ is from -180° to 180°. Let θ_a be the desired angle
- Feed the positive value of x and y in the calculator.



For finding components use angle θ_a,the clockwise angles are negative and anticlockwise are positive

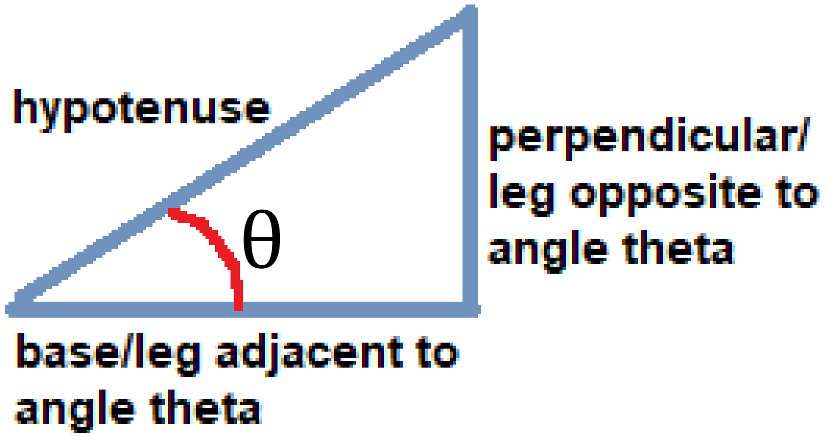


Important trigonometric formulas:

$$\sin\theta = \frac{perpendicular}{hypotenuse}$$

$$\cos\theta = \frac{base}{hypotenuse}$$

$$\tan\theta = \frac{perpendicular}{base}$$



Practice Problem 1:

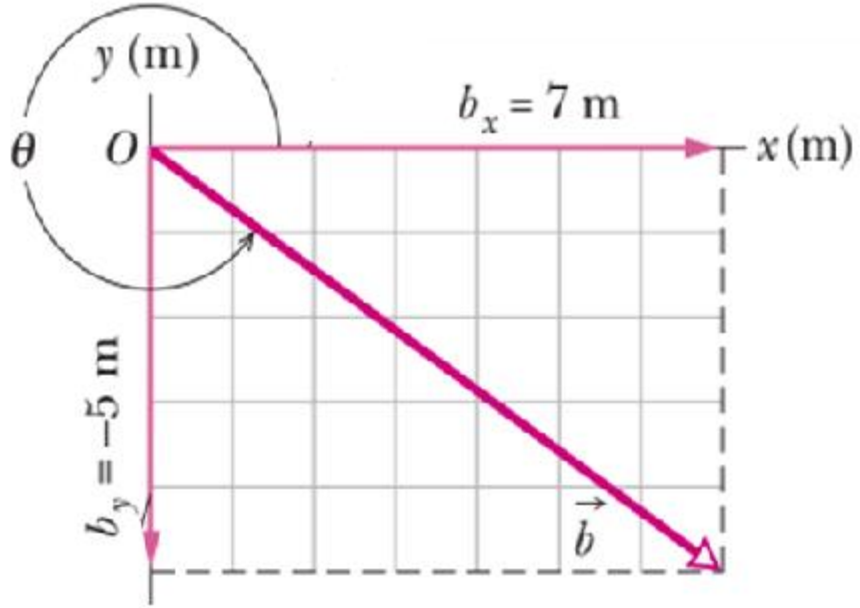
- Find length of the vector $\vec{b}$ and the unknown angle θ.

Answer:

Length of vector $\vec{b}$ = 8.60 m

And $\theta = 324.46°$

Here θ is marked and anti-clock wise, if Clock-wise angle was asked it would be -35.54°.



Practice Problem 2: (Sample problem 3.02)

A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. This means that the direction is not due north (directly toward the north) but is rotated 22° toward the east from due north. How far east and north is the airplane from the airport when sighted?

Answer:

The airplane is 81 km east and 200 km north of the airport.

Practice Problem 3: (Sample problem 3.04)

Concept of unit vectors:

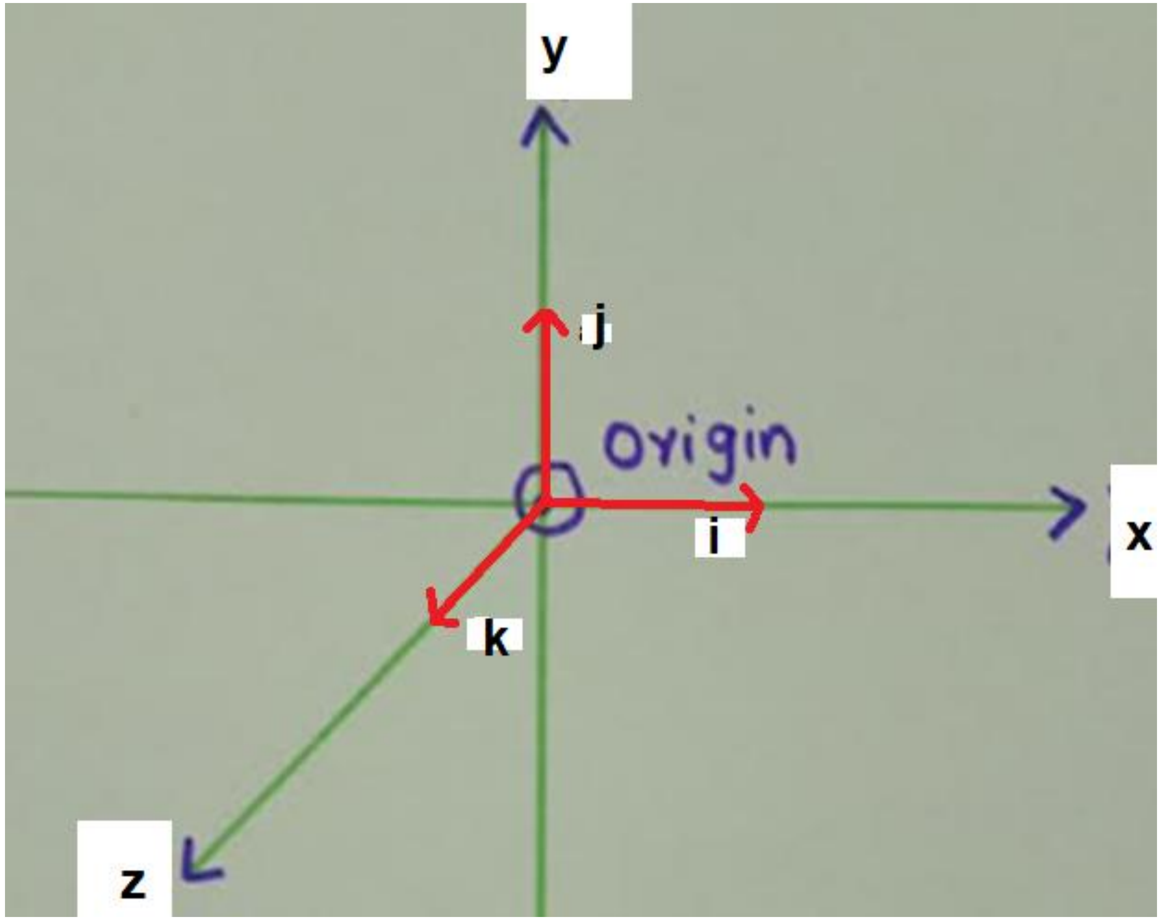
- A unit vector is a vector that has a magnitude of exactly 1 and points in a specific direction.

They are represented by hat instead of an overhead arrow.

For a vector $\overrightarrow{a}$ its unit vector $\widehat{a}$ or $\widehat{u_a}$ is given by:

$$\widehat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Unit vectors in right-handed coordinate system:



Vectors expressed using unit vectors:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

- The quantities $a_x\hat{i}$, $a_y\hat{j}$ and $a_z\hat{k}$ are called vector components of $\vec{a}$.
- The quantities a_x , a_y and a_z are called scalar components of $\vec{a}$(or simply components).

Adding vectors by components:

- Vectors can be added by combining their components axis by axis.
- Component wise Addition:(in 2 dimension)

$$\vec{r} = \vec{a} + \vec{b}$$
$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath}$$
$$r_x = a_x + b_x$$
$$r_y = a_y + b_y$$

Magnitude of $\vec{r} = \sqrt{r_x^2 + r_y^2}$

- Where,
- $a_x = \mathrm{a}\ \cos\theta$ *and* $a_y = a\ \sin\theta$
- $b_x = \mathrm{b}\ \cos\theta$ *and* $b_y = b\sin\theta$

Adding vectors by components:

- Vectors can be added by combining their components axis by axis.
- Component wise Addition:(in 3 dimension)

$$\vec{r} = \vec{a} + \vec{b}$$
$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$$
$$r_x = a_x + b_x$$
$$r_y = a_y + b_y$$
$$r_z = a_z + b_z$$

Magnitude of $\vec{r} = \sqrt{r_x^2 + r_y^2 + r_z^2}$

Vectors and Laws of Physics

The relation among the vectors do not depend on the location of origin or on the orientation of the axes. Rotating the axes changes the components but not the vector.

Multiplication and Division of a vector by scalar:

- (Scalar . Vector) = Vector

The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative To divide $\vec{v}$ by s, multiply $\vec{v}$ by $1/s$.

Dot Product/ Scalar Product

- The Projection of one vector on the other.
- If the result of the multiplication of two vectors is a scalar quantity then it is called dot product.
- (Vector . Vector) = Scalar

Examples:

work and electric flux

Formula:

$$\vec{\boldsymbol{a}}.\vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta_{AB}$$

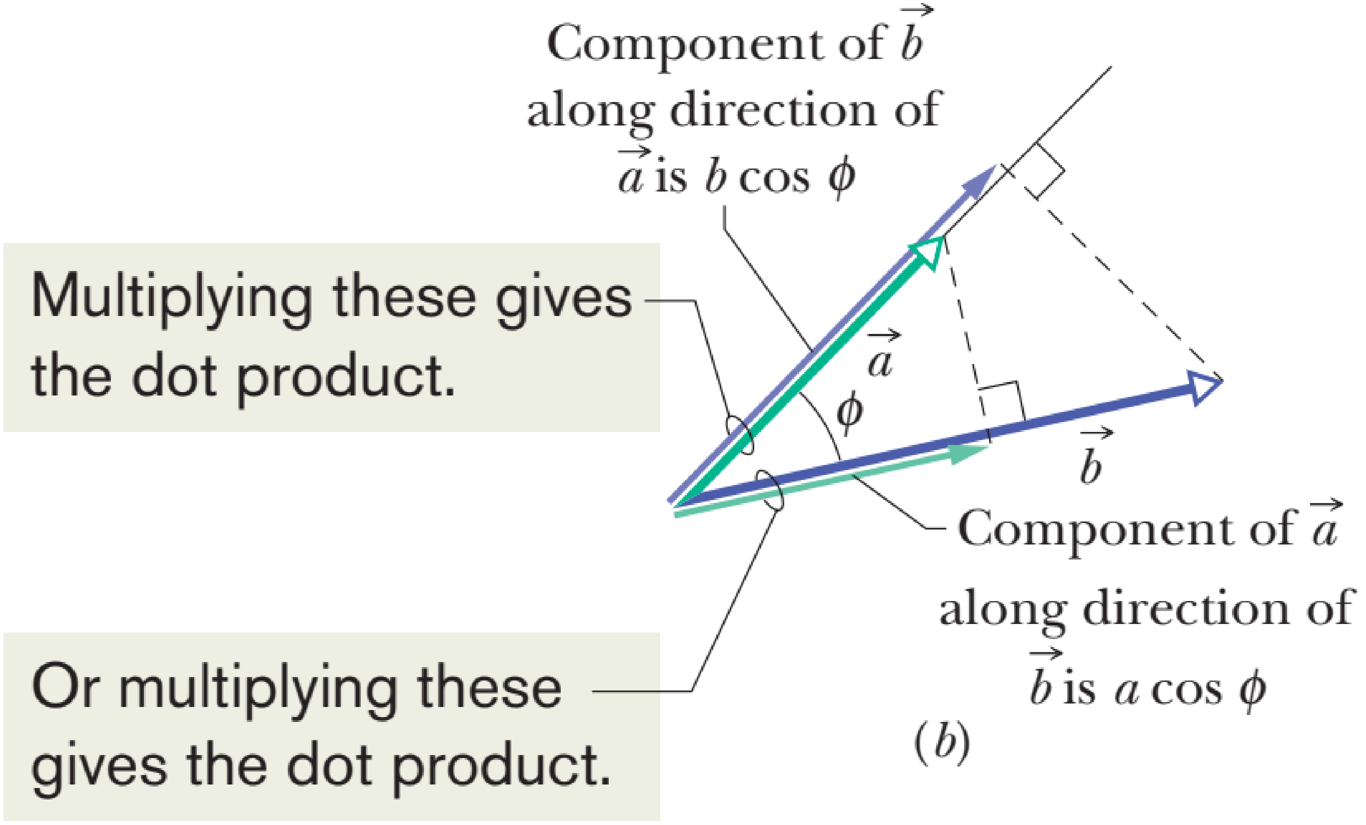
- In component form:

$$\mathbf{a.b} = a_x b_x + a_y b_y + a_z b_z$$

Component of Vector in the direction of another Vector/ The Projection of one vector on the other

- Component of **a** in the direction of **b** or projection of **a** in the direction of **b** = a cosϕ

a_b = $\vec{\boldsymbol{a}}.\hat{\boldsymbol{b}}$ = $|\vec{\boldsymbol{a}}|\,|\hat{\boldsymbol{b}}|$cosϕ$_{AB}$



In order to find PROJECTION

From the figure,

$$\cos\theta = \frac{ON}{B}$$

$$ON = B\cos\theta$$

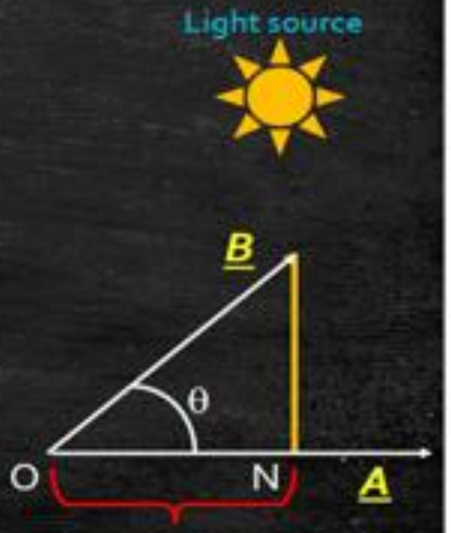
As we know,

$$\cos\theta = \frac{\underline{\boldsymbol{A}}.\underline{\boldsymbol{B}}}{AB}$$

$$B\cos\theta = \frac{\underline{\boldsymbol{A}}.\underline{\boldsymbol{B}}}{A}$$

So we reach to,

$$\boxed{ON = \frac{\underline{\boldsymbol{A}}.\underline{\boldsymbol{B}}}{A}}$$



Properties of Dot Product

- Commutative Law:
- **A.B** = **B.A**
- Distributive Law:
- **A.(B+C)= A.B + A.C**

Also:

- **A.A** = |**A**|^2 = A^2

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = (1)(1)\cos 0^\circ = 1$$

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = (1)(1)\cos 90^\circ = 0$$

Practice Problem 4: (sample problem 3.05)

What is the angle ϕ between $\vec{a} = 3.0\hat{\imath} - 4.0\hat{\jmath}$ and $\vec{b} = -2.0\hat{\imath} + 3.0\hat{k}$?

Answer:

$\phi = 110°$ (approximately)

Cross Product/ Vector Product

- If the result of the multiplication of two vectors is a vector quantity then it is called cross product.
- Rotational Information
- (Vector x Vector) = Vector

Examples:

Torque

Formula:

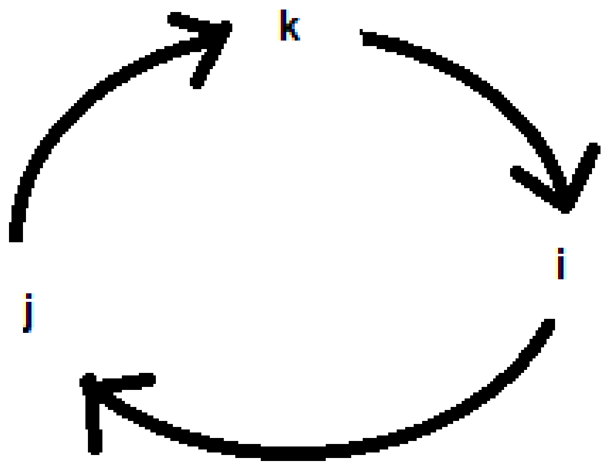
$$\vec{a} \times \vec{b} = absin\phi\hat{n} = \vec{c}$$

- The direction of the resultant vector is perpendicular to the plane containing both vectors **A** and **B**.
- The resultant vector is always perpendicular to the two vectors multiplied.
- To determine the correct direction we make use of right hand rule.

Properties of Cross Product

Non Commutative :

- **A x B = - B x A**



Not Associative:

- **A** x (**B** x **C**) ≠ **B** x (**A** x **C**)

Distributive Law:

- **A x** (**B** + **C**)= **A x B** + **A x C**

$$\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$$

$$\hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath}$$

$$\hat{k} \times \hat{\imath} = -\hat{\imath} \times \hat{k} = \hat{\jmath}$$

Also:

- **A x A** = 0
- **i x i = j x j= k x k** = 0

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{\mathrm{a}} \times \vec{\mathrm{b}}$ = 0. The magnitude of $\vec{\mathrm{a}} \times \vec{\mathrm{b}}$, which can be written as $\left|\vec{a} \times \vec{b}\right|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other

Cross product using components:

- $\vec{a} \times \vec{b} = det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}\right)$

- $\vec{a} \times \vec{b} = det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}\right)$

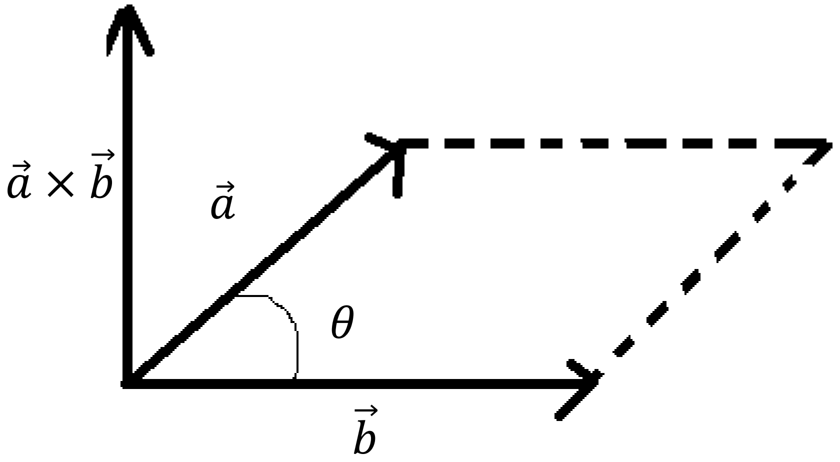
Area of a parallelogram

The length of the cross product of the two vectors is equal to the area of the parallelogram determined by the two vectors.

$$\left|\vec{a} \times \vec{b}\right| = ab\sin\theta$$

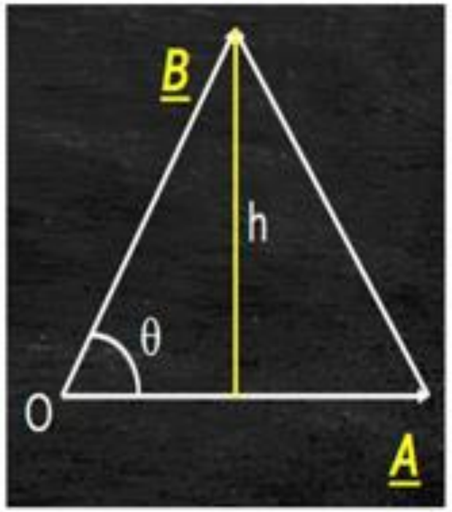
Length of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ of magnitude of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is the same as **area of parallelogram**.

$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ is perpendicular to the $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$



Area of triangle

the area of a triangle:



$$\text{Area of triangle} = \frac{1}{2} h \, |\underline{B}|$$

$$= \frac{1}{2} |\underline{A}| \sin\theta \, |\underline{B}|$$

$$= \frac{1}{2} |\underline{A} \times \underline{B}|$$

Triple Product

- Dot and cross products of three vectors ***A* , *B* *and C may*** produce meaningful products of the form ***A*.*(BxC)*** and ***Ax(BxC) then phenomenon is called triple product.***
- ***Two types of triple product***

1. ***Scalar Triple Product***
2. ***Vector Triple Product***

Scalar Triple Product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

which is known as the "bac-cab" rule.

Application of Triple product

Consider a parallelepiped. This is a six sided solid, the sides of which are parallelograms. Opposite parallelograms are identical. The volume, V, of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given by

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

This formula can be obtained by understanding that the volume is the product of the area of the base and the perpendicular height. Because the base is a parallelogram its area is $|\mathbf{b} \times \mathbf{c}|$. The perpendicular height is the component of $\mathbf{a}$ in the direction perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{a}$, and this is $h = \mathbf{a} \cdot \widehat{\mathbf{b} \times \mathbf{c}}$. So the volume is given by

$$\begin{aligned} V &= (\text{height})(\text{ area of base}) \\ &= \mathbf{a} \cdot \widehat{\mathbf{b} \times \mathbf{c}}\, |\mathbf{b} \times \mathbf{c}| \\ &= \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{|\mathbf{b} \times \mathbf{c}|} |\mathbf{b} \times \mathbf{c}| \\ &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

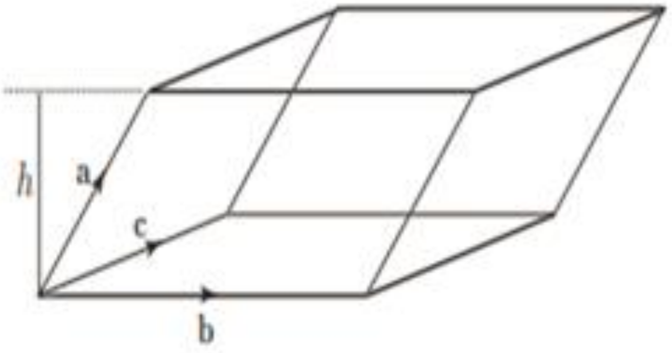


Figure . A parallelepiped with edges given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Practice Problem 5:

1. Find a unit vector which is perpendicular to both $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.
2. Find the area of the parallelogram with edges represented by the vectors $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} + \mathbf{j} + \mathbf{k}$.
3. Find the volume of the parallelepiped with edges represented by the vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
4. Calculate the **triple scalar product** $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ when $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Practice Problem 6:

1. Use the formula $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 9\mathbf{j}$.

(b) $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - 7\mathbf{j}$.

Practice Problem 7:

3. Use determinants to find the vector product $\mathbf{p} \times \mathbf{q}$ in each of the following cases.

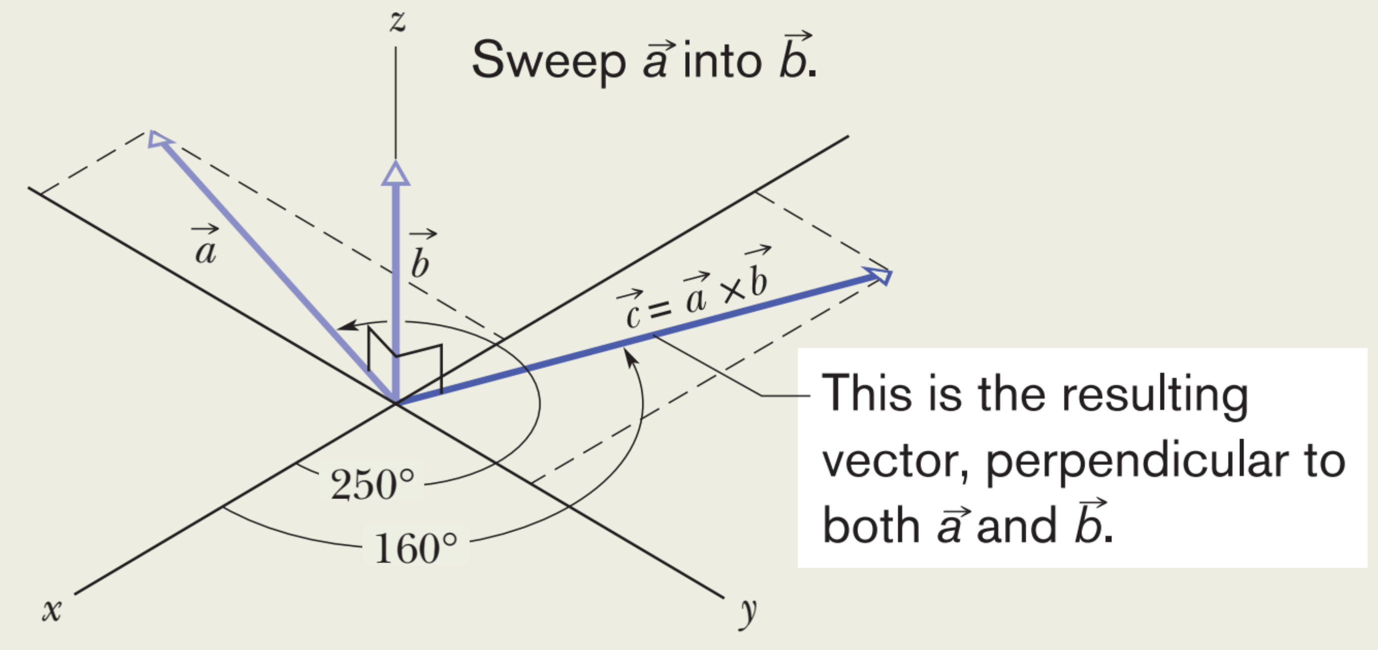
(a) $\mathbf{p} = \mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} - \mathbf{k}$.
(b) $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.

4. For the vectors $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ show that, in this special case, $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}$.
5. For the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, show that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

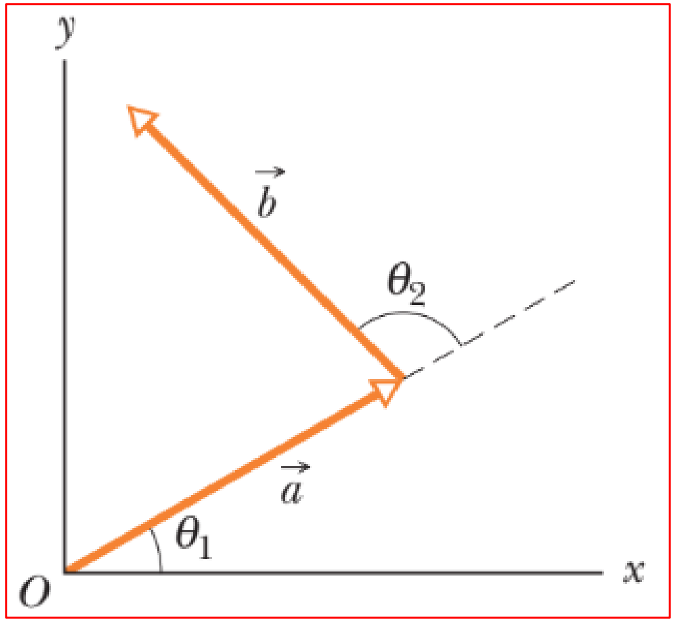
Practice Problem 8:(Sample problem 3.06)(homework)

In Fig. 3-20, vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units, and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?



Practice Problem 9:(End of chapter 3 problem 15)

•15 SSM ILW WWW The two vectors $\vec{a}$ and $\vec{b}$ in Fig. 3-28 have equal magnitudes of 10.0 m and the angles are $\theta_1 = 30°$ and $\theta_2 = 105°$. Find the (a) x and (b) y components of their vector sum $\vec{r}$, (c) the magnitude of $\vec{r}$, and (d) the angle $\vec{r}$ makes with the positive direction of the x axis.



Practice Problem 10:(Sample problem 3.07)

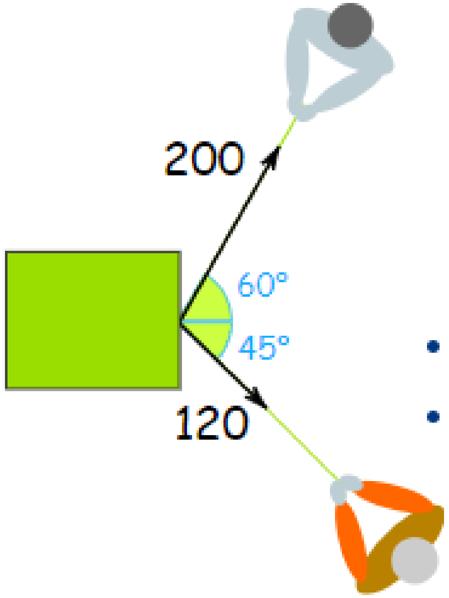
If $\vec{a} = 3.0\hat{\imath} - 4.0\hat{\jmath}$ and $\vec{b} = -2.0\hat{\imath} + 3.0\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

Answer:

$\vec{c} = -12\hat{\imath} - 9\hat{\jmath} - 8\hat{k}$

The vector c is perpendicular to both a and b vectors

Practice Problem 11:



An Example

Sam and Alex are pulling a box.

- Sam pulls with 200 Newtons of force at 60°
- Alex pulls with 120 Newtons of force at 45° as shown

What is the combined force, and its direction?

Answer:
Force = 205 N and the angle that it makes with positive x axis is 26°

Homework questions:

- Practice problems:
- End of chapter 3 textbook " Fundamentals of Physics"by Halliday & Resnick Jearl Walker 10th Edition"

2, 5, 13, 15, 20, 26

Answer of even problems:

2: $r_x = 13\ m\ and\ r_y = 7.5\ m$

20: Magnitude = 5 km and θ = 4.3° south of due west, In unit vector notation d= $(-5.01\ km\ )\hat{\imath}\ - (0.38\ km\ )\hat{\jmath}$

26. $\vec{R} = (-3.18\ \text{m})\hat{\text{i}} + (4.72\ \text{m})\hat{\text{j}}$. $(5.69 \angle 124°)$

References:

Book:

Fundamentals of Physics by Halliday Resnick & Jearl Walker 10th edition