Copying answers and steps are strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. The calculations and answers should be written neatly on paper which is attached as a single pdf. Box your answers where appropriate. Thanks!

Problem 1

Classify the following equations as ordinary or partial differential equations, state the order and degree of each equation and determine whether the equation is linear or nonlinear

(a) $\dfrac{\partial^3 u}{\partial x^3} + \dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial z}\right)^2 + ux^3 + uy^2 + uz = 0$

Solution:
order is 3, degree is 1, PDE, Non-Linear

(b) $\left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{d^2y}{dx^2} + y\right)^{\frac{3}{2}}$

Solution:
order is 2, degree is 3, ODE, Non-Linear

Problem 2

Verify that the indicated function is the solutions given differential equation and solve the following initial/boundary value problems.

(a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0,\ y(0) = -2, y^{'}(0) = 6$
where $y = c_1e^{4x} + c_2e^{-3x}$ is the general solution of the given differential equation.

Solution: Here $y = c_1e^{4x} + c_2e^{-3x}$ and $y^{'} = 4c_1e^{4x} - 3c_2e^{-3x}$.
Since $y(0) = -2$ and $y^{'}(0) = 6$, we have

$$-2 = c_1 + c_2 \tag{1}$$
$$6 = 4c_1 - 3c_2 \tag{2}$$

By solving we get $c_1 = 0$ and $c_2 = -2$.
Thus $y = -2e^{-3x}$ is the required solution.

(b) $x^3\dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6y = 0,\ y(2) = 2, y^{'}(2) = 2, y^{''}(2) = 6$
where $y = c_1x + c_2x^2 + c_3x^3$ is the general solution of the given differential equation.

Solution:
Here $y = c_1x + c_2x^2 + c_3x^3$, $y^{'} = c_1 + 2c_2x + 3c_3x^2$ and $y = c_2 + 6c_3x$.

Since $y(2) = 2$, $y^{'}(2) = 2$ and $y^{''}(2) = 6$, we have

$$c_1 + 2c_2 + 4c_3 = 0 \tag{3}$$
$$c_1 + 4c_2 + 12c_3 = 2 \tag{4}$$
$$c_2 + 6c_3 = 3 \tag{5}$$

By solving we get $c_1 = 2$,$c_2 = -3$ and $c_3 = 1$.
Thus $y = 2x - 3x^2 + x^3$ is the required solution.

(c) $\dfrac{d^2y}{dx^2} + y = 0,\ y(0) = 1, y^{'}(\frac{\pi}{2}) = -1$
whwere $y = c_1 \sin x + c_2 \cos x$ is the general solution of the given differential equation.

Solution:
Here $y = c_1 \sin x + c_2 \cos x$ and $y^{'} = c_1 \cos x - c_2 \sin x$
Applying boundary conditions, we get

$$1 = c_1 \sin 0 + c_2 \cos 0 \tag{6}$$
$$-1 = c_1 \cos \frac{\pi}{2} + c_2 \sin \frac{\pi}{2} \tag{7}$$

By solving we get $c_1 = 1$ and $c_2 = 1$.
Thus $y = \sin x + \cos x$ is the required solution.

Problem 3

Form the differential equation (eliminate the arbitrary constant)

(a) $x^3 + y^3 = 3cxy$

Solution:
Differentiating $(x^3 + y^3)/xy = 3c$ we obtain

$$\frac{xy(3x^2 + 3y^2y') - (x^3 + y^3)(xy' + y)}{x^2y^2} = 0$$
$$3x^3y + 3xy^3y' - x^4y' - x^3y - xy^3y' - y^4 = 0$$
$$(3xy^3 - x^4 - xy^3)y' = -3x^3y + x^3y + y^4$$
$$y' = \frac{y^4 - 2x^3y}{2xy^3 - x^4} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

(b) $3y = \dfrac{4x^3}{x^2 + 1} + \dfrac{3c}{x^2 + 1}$

Solution:
Rewrite the equation as $3(x^2 + 1)y = 4x^3 + 3c$ and differentiate with respect to x, we get

$6xy+3(x^2+1)\dfrac{dy}{dx}=12x^2$ or $3(x^2+1)\dfrac{dy}{dx}=6xy=12x^2$ is the required differential equation.

Bonus Problem 4

(a) Solve the DE by seperation of variable
$(xy+2x+y+2)dx+(x^2+2x)dy=0$

Solution:
Rewrite the DE as $(x+1)(y+2)dx+(x^2+2x)dy=0$ and seperate vaibales to get
$\dfrac{x+1}{x(x+2)}dx+\dfrac{1}{y+2}dy=0.$
By using prtial fraction we replace cofficient function of dx, that is $\dfrac{x+1}{x(x+2)}=\dfrac{\frac{1}{3}}{x}+\dfrac{\frac{1}{2}}{x+2}$.
Thus $\displaystyle\int(\frac{\frac{1}{3}}{x}+\frac{\frac{1}{2}}{x+2})dx+\int\frac{dy}{y+2}=0$
gives $\dfrac{1}{3}\ln x+\dfrac{1}{2}\ln(x+2)+\ln(y+2)=\ln c$, where c is a parameter.
That is $y+2=\dfrac{c}{x^{\frac{1}{3}}\sqrt{x+2}}$ is required solution

(b) Solve the linear DE
$\dfrac{dy}{dx}+\dfrac{y}{x\ln x}=\dfrac{3x^2}{\ln x}$

Solution:
Compairing with linear DE $\dfrac{dy}{dx}+P(x)y=Q(x)$.
Here $P(x)=\dfrac{1}{x\ln x}$ therefore I.F. is $e^{\int\frac{1}{x\ln x}dx}=\ln x$.
Multiplying given DE with I.F., we get $\dfrac{d}{dx}(\ln x\cdot y)=3x^2$.
On integration we get $\ln x\cdot y=x^3+c$ is requred solution.

(c) Solve the exact DE
$e^x\left[y-3(e^x+1)^2\right]dx+(e^x+1)dy=0, y(0)=4$

Solution:
Compairing with $M9x,y)dx+N(x,y)dy=0$.
Here $M=e^x\left[y-3(e^x+1)^2\right]$ and $N=e^x+1$ therefore $M_y=e^x=N_x$, so that DE is exact.
There exist a function f such that $f_x=M$ and $f_y=N$. That is

$$f_x=e^x\left[y-3(e^x+1)^2\right] \tag{8}$$
$$f_y=e^x+1 \tag{9}$$

from Equation 9, we get

$$f = e^x y + y + h(x) \tag{10}$$

Taking p.d. w.r.t. x, we get
$f_x = e^x y + h^{'}(x) \Rightarrow -3e^x(e^x + 1)^2 = h^{'}(x)$
integration gives $h(x) = (e^x + 1)^3$, substitute in Equation 10 $f = e^x y + y + (e^x + 1)^3$ so the implicit form of solution is $e^x y + y + (e^x + 1)^3 = c$