Copying answers and steps are strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. The calculations and answers should be written neatly on paper which is attached as a single pdf. Box your answers where appropriate. Thanks!

Problem 1

Solve the following `homogeneous linear differential equations with constant coefficient`.

(a) $(D^4 + 6D^3 + 15D^2 + 20D + 12)y = 0$

Solution:
The auxiliary equation of the differential equation is $m^4 + 6m^3 + 15m^2 + 20m + 12 = 0$.
Its roots are $-2, -2, 1 \pm i\sqrt{2}$.
So the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^x \cos\sqrt{2}x + c_4 e^x \sin\sqrt{2}x$

(b) $(D^3 - 27)y = 0$

Solution:
The auxiliary equation of the differential equation is $m^3 - 27 = 0$.
Its roots are $3, -\frac{3}{2} \pm i\frac{3\sqrt{3}}{2}$.
So the general solution is $y = c_1 e^{3x} + c_2 e^{-3x/2} \cos\frac{3\sqrt{3}}{2} + c_3 e^{-3x/2} \sin\frac{3\sqrt{3}}{2}$

Problem 2

Solve the differential equations by method of `Undetermine Coefficient-Superposition Approach`.

(a) $(D^2 - 7D + 12)y = e^{2x}(x^3 - 5x^2)$

Solution:
Let's compose the characteristic equation $\mathbf{a_0\,\lambda^n + a_1\,\lambda^{n-1} + \ldots + a_{n-1}\,\lambda + a_n = 0}$:

$$\boldsymbol{\lambda^2 - 7\,\lambda + 12 = 0 \rightarrow (\lambda - 4)\,(\lambda - 3) = 0}$$
$$\overline{y} = C\,e^{4\,x} + C_1\,e^{3\,x}$$

Method of undefined coefficients search for a particular solution

Particular solution for $(x^3 - 5x^2)\,e^{2x}$:

$$\alpha + \beta i = 2 \rightarrow s = 0$$

$$y_0 = (A x^3 + B x^2 + C x + D)\,e^{2x} \{1\}$$

Calculate derivatives:

$$y_0' = (2 A x^3 + (2B + 3A)\,x^2 + (2C + 2B)\,x + 2D + C)\,e^{2x}$$

$$y_0'' = (4 A x^3 + (4B + 12A)\,x^2 + (4C + 8B + 6A)\,x + 4D + 4C + 2B)\,e^{2x}$$

Substitute in original equation:

$$2 A x^3 e^{2x} + (2B - 9A)\,x^2 e^{2x} + (2C - 6B + 6A)\,x e^{2x} + (2D - 3C + 2B)\,e^{2x} = (x^3 - 5x^2)\,e^{2x}$$

Find coefficients:

$$\begin{cases} 2B - 9A = -5 \\ 2A = 1 \\ 2C - 6B + 6A = 0 \\ 2D - 3C + 2B = 0 \end{cases} = \begin{cases} A = \dfrac{1}{2} \\ B = -\dfrac{1}{4} \\ C = -\dfrac{9}{4} \\ D = -\dfrac{25}{8} \end{cases}$$

Substitute in $\{1\}$:

$$y_0 = \left(\frac{x^3}{2} - \frac{x^2}{4} - \frac{9x}{4} - \frac{25}{8}\right) e^{2x}$$

$$y = C e^{4x} + C_1 e^{3x} + \left(\frac{x^3}{2} - \frac{x^2}{4} - \frac{9x}{4} - \frac{25}{8}\right) e^{2x}$$

(b) $y'' + y' - 2y = x^2 + 2\sin x - e^{3x}$

Solution:

Let's compose the characteristic equation $a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$:

$$\lambda^2 + \lambda - 2 = 0 \rightarrow (\lambda - 1)\,(\lambda + 2) = 0$$

$$\overline{y} = C e^{x} + \frac{C_1}{e^{2x}}$$

Method of undefined coefficients search for a particular solution

Particular solution for x^2 :

$$\alpha + \beta i = 0 \rightarrow s = 0$$

$$y_0 = A x^2 + B x + C \; \{1\}$$

Calculate derivatives:

$$y_0' = 2 A x + B$$

$$y_0'' = 2 A$$

Substitute in original equation:

$$-2 A x^2 + (2 A - 2 B) x - 2 C + B + 2 A = x^2$$

Find coefficients:

$$\begin{cases} -2 A = 1 \\ 2 A - 2 B = 0 \\ -2 C + B + 2 A = 0 \end{cases} = \begin{cases} A = -\frac{1}{2} \\ B = -\frac{1}{2} \\ C = -\frac{3}{4} \end{cases}$$

Substitute in $\{1\}$:

$$y_0 = -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$$

Particular solution for $-e^{3x}$:

$$\alpha + \beta i = 3 \rightarrow s = 0$$

$$y_1 = A e^{3x} \downarrow ? \{2\}$$

Calculate derivatives:

$$y_1' = 3 A e^{3x}$$

$$y_1'' = 9 A e^{3x}$$

Substitute in original equation:

$$10 A e^{3x} = -e^{3x}$$

Find coefficients:

$$10 A = -1 \rightarrow A = -\frac{1}{10}$$

Substitute in $\{2\}$:

$$y_1 = -\frac{e^{3x}}{10}$$

Particular solution for $2 \sin(x)$:

$$\alpha + \beta i = i \rightarrow s = 0$$

$$y_2 = B \sin(x) + A \cos(x) \downarrow ? \{3\}$$

Calculate derivatives:

$$y_2' = B \cos(x) - A \sin(x)$$

$$y_2'' = -B \sin(x) - A \cos(x)$$

Substitute in original equation:

$$(-3B - A) \sin(x) + (B - 3A) \cos(x) = 2 \sin(x)$$

Find coefficients:

$$\begin{cases} -3B - A = 2 \\ B - 3A = 0 \end{cases} = \begin{cases} A = -\frac{1}{5} \\ B = -\frac{3}{5} \end{cases}$$

Substitute in $\{3\}$:

$$y_2 = -\frac{3 \sin(x)}{5} - \frac{\cos(x)}{5}$$

$$y = -\frac{3 \sin(x)}{5} - \frac{\cos(x)}{5} - \frac{e^{3x}}{10} + C e^{x} + \frac{C_1}{e^{2x}} - \frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$$

Problem 3

Solve differential using `Variation of Parameters`.

(a) $(D^2 + 1)y = \csc x$

Solution:

$$(D^2+1)\,y = \operatorname{cosec} x$$
$$D^2+1 = 0 \text{ or } D = \pm i$$
$$\text{C.F.} = A\cos x + B\sin x$$
$$y_1 = \cos x, \quad y_2 = \sin x$$
$$\text{P.I.} = y_1 u + y_2 v$$
$$u = \int \frac{-y_2 \cdot \operatorname{cosec} x\, dx}{y_1 \cdot y'_2 - y'_1 \cdot y_2} = \int \frac{-\sin x \cdot \operatorname{cosec} x\, dx}{\cos x(\cos x) - (-\sin x)(\sin x)}$$
$$= \int \frac{-\sin x \cdot \frac{1}{\sin x}\, dx}{\cos^2 x + \sin^2 x} = -\int dx = -x$$
$$v = \int \frac{y_1 \cdot X\, dx}{y_1 \cdot y'_2 - y'_1 \cdot y_2} = \int \frac{\cos x \cdot \operatorname{cosec} x\, dx}{\cos x(\cos x) - (-\sin x)(\sin x)}$$
$$= \int \frac{\cos x \cdot \frac{1}{\sin x}}{\cos^2 x + \sin^2 x}\, dx = \int \frac{\cot x\, dx}{1} = \log \sin x$$
$$\text{P.I.} = \cos x \cdot (-x) + \sin x \cdot (\log \sin x)$$
$$\text{General solution} = \text{C.F.} + \text{P.I.}$$
$$y = A\cos x + B\sin x - x\cos x + \sin x \cdot \log \sin x \quad \textbf{Ans.}$$

(b) $(D^2-1)y = \dfrac{2}{1+e^x}$

Solution:

1. We have, $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$
$$(D^2-1) = 0$$
$$D^2 = 1, \quad D = \pm 1$$
$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$
$$\therefore \text{P.I.} = uy_1 + vy_2$$
Here, $y_1 = e^x, \quad y_2 = e^{-x}$
$$y_1 \cdot y'_2 - y'_1 \cdot y_2 = -e^x \cdot e^{-x} - e^x \cdot e^{-x} = -2$$
$$u = \int \frac{-y_2 X}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\, dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x}\, dx$$
$$= \int \frac{e^{-x}}{1+e^x}\, dx$$
$$= \int \frac{dx}{e^x(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$
$$= \int e^{-x}\, dx - \int \frac{e^{-x}}{e^{-x}+1}\, dx = -e^{-x} + \log(e^{-x}+1)$$
$$v = \int \frac{y_1 X}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\, dx = \int \frac{e^x}{-2}\, \frac{2}{1+e^x}\, dx$$
$$= -\int \frac{e^x}{1+e^x}\, dx = -\log(1+e^x)$$
$$\text{P.I.} = u \cdot y_1 + v \cdot y_2 = [-e^{-x} + \log(e^{-x}+1)]\, e^x - e^{-x} \log(1+e^x)$$
$$= -1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$$
$$y = c_1 e^x + c_2 e^{-x} - 1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$$

Problem 4

Solve the differential equations by `method of Undetermine Coefficient-Annihilator Approach`.

(a) $y'' + y' + \frac{1}{4}y = e^x(\sin 3x + \cos 3x)$

Solution:

Applying $D^2 - 2D + 10$ to the differential equation we obtain

$$(D^2 - 2D + 10)\left(D^2 + D + \frac{1}{4}\right)y = (D^2 - 2D + 10)\left(D + \frac{1}{2}\right)^2 y = 0.$$

Then

$$y = \underbrace{c_1e^{-x/2} + c_2xe^{-x/2}}_{y_c} + c_3e^x \cos 3x + c_4e^x \sin 3x$$

and $y_p = Ae^x \cos 3x + Be^x \sin 3x$. Substituting y_p into the differential equation yields

$$(9B - 27A/4)e^x \cos 3x - (9A + 27B/4)e^x \sin 3x = -e^x \cos 3x + e^x \sin 3x.$$

Equating coefficients gives

$$-\frac{27}{4}A + 9B = -1$$

$$-9A - \frac{27}{4}B = 1.$$

Then $A = -4/225$, $B = -28/225$, and the general solution is

$$y = c_1e^{-x/2} + c_2xe^{-x/2} - \frac{4}{225}e^x \cos 3x - \frac{28}{225}e^x \sin 3x.$$

(b) $y'' + 2y' + y = x^2e^{-x}$

Solution:

Applying $(D + 1)^3$ to the differential equation we obtain

$$(D + 1)^3(D^2 + 2D + 1)y = (D + 1)^5y = 0.$$

Then

$$y = \underbrace{c_1e^{-x} + c_2xe^{-x}}_{y_c} + c_3x^4e^{-x} + c_4x^3e^{-x} + c_5x^2e^{-x}$$

and $y_p = Ax^4e^{-x} + Bx^3e^{-x} + Cx^2e^{-x}$. Substituting y_p into the differential equation yields

$$12Ax^2e^{-x} + 6Bxe^{-x} + 2Ce^{-x} = x^2e^{-x}.$$

Equating coefficients gives $A = \frac{1}{12}$, $B = 0$, and $C = 0$. The general solution is

$$y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{12}x^4e^{-x}.$$

Problem 5

Solve Cauchy Euler equation $x^2y'' + xy' - y = x^3e^x$

Solution:

Solution. The given differential equation is

$$x^2 y'' + x y' - y = x^3 e^x$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^3 e^x$$

Put $x = e^z \Rightarrow D = \frac{d}{dz}$, $x \frac{dy}{dx} = Dy$, $\quad x^2 \frac{d^2y}{dx^2} = D(D-1)y$ in (1), we get

$$D(D-1)y + Dy - y = e^{3z} e^{e^z}$$

$$\Rightarrow (D^2 - 1)\, y = e^{3z} e^{e^z}$$

A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$$\therefore \ C.F. = c_1 e^z + c_2 e^{-z}$$

$$\Rightarrow \qquad \text{C.F.} = c_1 x + c_2 x^{-1} = c_1 x + \frac{c_2}{x}$$

$\therefore$ P.I. $= uy_1 + vy_2$ Here, $y_1 = x$ and $y_2 = \frac{1}{x}$

$$\text{Also, } u = \int \frac{-y_2 X\, dx}{y_1 y_2' - y_2 y_1'} = \int \frac{-\frac{1}{x} x^3 e^x\, dx}{x\left(-\frac{1}{x^2}\right) - \frac{1}{x}(1)} = \int \frac{-x^2 e^x}{-\frac{2}{x}}\, dx$$

$$= \frac{1}{2} \int x^3 e^x\, dx$$

$$u = \frac{1}{2}\left[x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x\right]$$

$$= \frac{1}{2}\left[x^3 - 3x^2 + 6x - 6\right] e^x$$

$$v = \int \frac{y_1 X\, dx}{y_1 y_2' - y_2 y_1'} = \int \frac{x \,.\, x^3 e^x\, dx}{x\left(-\frac{1}{x^2}\right) - \frac{1}{x}(1)} = \int \frac{x^4 e^x}{-\frac{2}{x}}\, dx = -\frac{1}{2} \int x^5 e^x\, dx$$

$$v = -\frac{1}{2}\left[x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120 x e^x - 120 e^x\right]$$

$$\text{Now, P.I.} = u y_1 + v y_2 = \frac{1}{2}(x^3 - 3x^2 + 6x - 6)\, e^x \,.\, x - \frac{1}{2}(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)\, e^x \,.\, \frac{1}{x}$$

$$= \frac{e^x}{2}\left[x^4 - 3x^3 + 6x^2 - 6x - x^4 + 5x^3 - 20x^2 + 60x - 120 + \frac{120}{x}\right]$$

$$= \frac{e^x}{2}\left[2x^3 - 14x^2 + 54x - 120 + \frac{120}{x}\right]$$

$$y = \text{C.F.} + P.I.$$

$$= C_1 x + \frac{C_2}{x} + \left(x^3 - 7x^2 + 27x - 60 + \frac{60}{x}\right) e^x \qquad \textbf{Ans.}$$

Bonus Problem 6

(a) Determine whether the the given set of fucntion is `linearly independent or lineearly dependent` on $(-\infty, \infty)$.

i. $y_1 = \cos 2x, y_2 = 1, y_3 = \cos^2 x$

Solution:
Since $W(y_1, y_2, y_3) = 0$, hence the set of function is linearly dependent.

ii. $y_1 = x, y_2 = x^{-2}, y_3 = x^2 \ln x$

Solution:
Since $W(y_1, y_2, y_3) = 9x^{-6} \neq 0$, hence the set of function is linearly independent.

(b) Solve differential equation using `reduction of order` , then varify by formulation $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$.

i. $9y^{''} - 12y^{'} + 4y = 0, y_1 = e^{2x/3}$

Solution:

Define $y = u(x)e^{2x/3}$ so

$$y' = \frac{2}{3}e^{2x/3}u + e^{2x/3}u', \quad y'' = e^{2x/3}u'' + \frac{4}{3}e^{2x/3}u' + \frac{4}{9}e^{2x/3}u$$

and

$$9y'' - 12y' + 4y = 9e^{2x/3}u'' = 0.$$

Therefore $u'' = 0$ and $u = c_1x + c_2$. Taking $c_1 = 1$ and $c_2 = 0$ we see that a second solution is $y_2 = xe^{2x/3}$.

ii. $y^{''} - 3y^{'} + 2y = 5e^{3x}, y_1 = e^x$

Solution:

Define $y = u(x)e^x$ so

$$y' = ue^x + u'e^x, \quad y'' = u''e^x + 2u'e^x + ue^x$$

and

$$y'' - 3y' + 2y = e^x u'' - e^x u' = 5e^{3x}.$$

If $w = u'$ we obtain the linear first-order equation $w' - w = 5e^{2x}$ which has the integrating factor $e^{-\int dx} = e^{-x}$. Now

$$\frac{d}{dx}[e^{-x}w] = 5e^x \quad \text{gives} \quad e^{-x}w = 5e^x + c_1.$$

Therefore $w = u' = 5e^{2x} + c_1e^x$ and

$$u = \frac{5}{2}e^{2x} + c_1e^x + c_2$$
$$y = \frac{5}{2}e^{3x} + c_1e^{2x} + c_2e^x$$

From the last equation we see that a second solution is $y_2 = e^{2x}$ and $y_p = \frac{5}{2}e^{3x}$.

Bonus Problem 7

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

Solution:

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as follows.

$$40 = \frac{4000}{1 + be^{-k(0)}}$$

$$40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

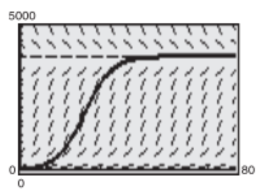
$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \dfrac{4000}{1 + 99e^{-0.194t}}$.

b. Using a graphing utility, you can graph the slope field for

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure



Slope field for $\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$ and the solution passing through $(0, 40)$

c. To estimate the elk population after 15 years, substitute 15 for t in the model.

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \quad \text{Substitute 15 for } t.$$

$$= \frac{4000}{1 + 99e^{-2.91}} \approx 626 \quad \text{Simplify.}$$

d. As t increases without bound, the denominator of $\dfrac{4000}{1 + 99e^{-0.194t}}$ gets closer and closer to 1.

So, $\displaystyle\lim_{t\to\infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000.$ ■