Discrete Structures

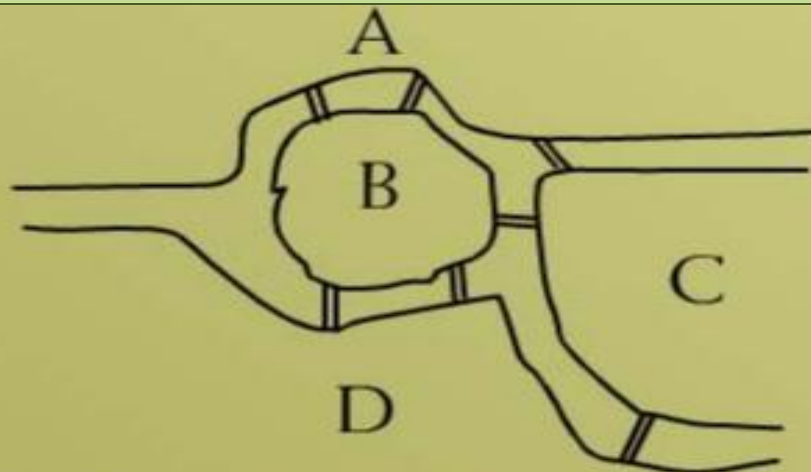
Lecture # 12

Dr. Muhammad Ahmad

Department of Computer Science

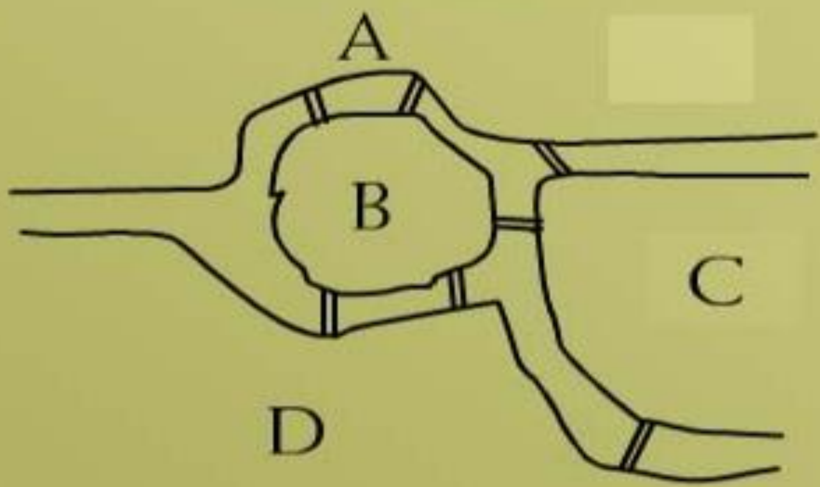
FAST -- National University of Computer and Emerging Sciences. CFD Campus

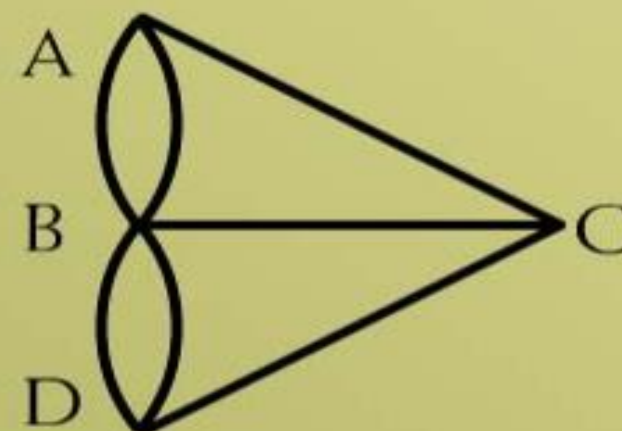
KONIGSBERG BRIDGES PROBLEM



Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?

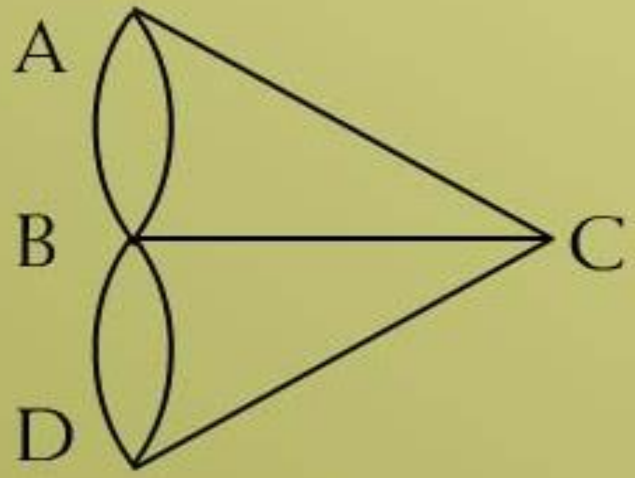
SOLUTION





Is it possible to find **a route** through the **graph** that **starts and ends** at some **vertex A, B, C** or **D** and **traverses** each edge exactly once?

EQUIVALENT FORM OF BRIDGE PROBLEM



Is it possible to trace this graph, starting and ending at the same point, without ever lifting your pencil from the paper?

TERMINOLOGY

Let G be a graph and let v and w be vertices in graph G.

1. WALK

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G.

TERMINOLOGY

Thus a walk has the form

$$v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$$

where the v's represent vertices, the e's represent edges $v_0 = v, v_n = w$, and for all $i = 1, 2 \ldots n$, v_{i-1} and v_i are endpoints of e_i.

The trivial walk from v to v consists of the single vertex v.

TERMINOLOGY

2. CLOSED WALK

A **closed walk** is a **walk** that **starts** and **ends** at the **same vertex**.

3. CIRCUIT

A **circuit** is a **closed walk** that does not contain a **repeated edge**.

Thus a **circuit** is a **walk** of the form

$$v_0 e_1 v_1 e_2 \ldots v_{n-1}\ e_n\ v_n$$

where $v_0 = v_n$ and all the e_i's are **distinct**

TERMINOLOGY

4. SIMPLE CIRCUIT

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

Thus a simple circuit is a walk of the form

$$v_0 e_1 v_1 e_2 \ldots v_{n-1}\, e_n\, v_n$$

where all the e_i's are distinct and all the v_j's are distinct except that $v_0 = v_n$

TERMINOLOGY

5. PATH

A **path** from **v** to **w** is a **walk** from **v** to **w** that does not contain a **repeated edge**.

Thus a path from **v** to **w** is a **walk** of the form

$$v = v_0 e_1 v_1 e_2 \ldots v_{n-1}\, e_n\, v_n = w$$

where all the e_i's are **distinct** (that is $e_i \neq e_k$ for any $i \neq k$).

TERMINOLOGY

6. SIMPLE PATH

A **simple path** from **v** to **w** is a path that does not contain a **repeated vertex**.

Thus a **simple path** is a **walk** of the form

$$v = v_0 e_1 v_1 e_2 \ldots v_{n-1}\ e_n\ v_n = w$$

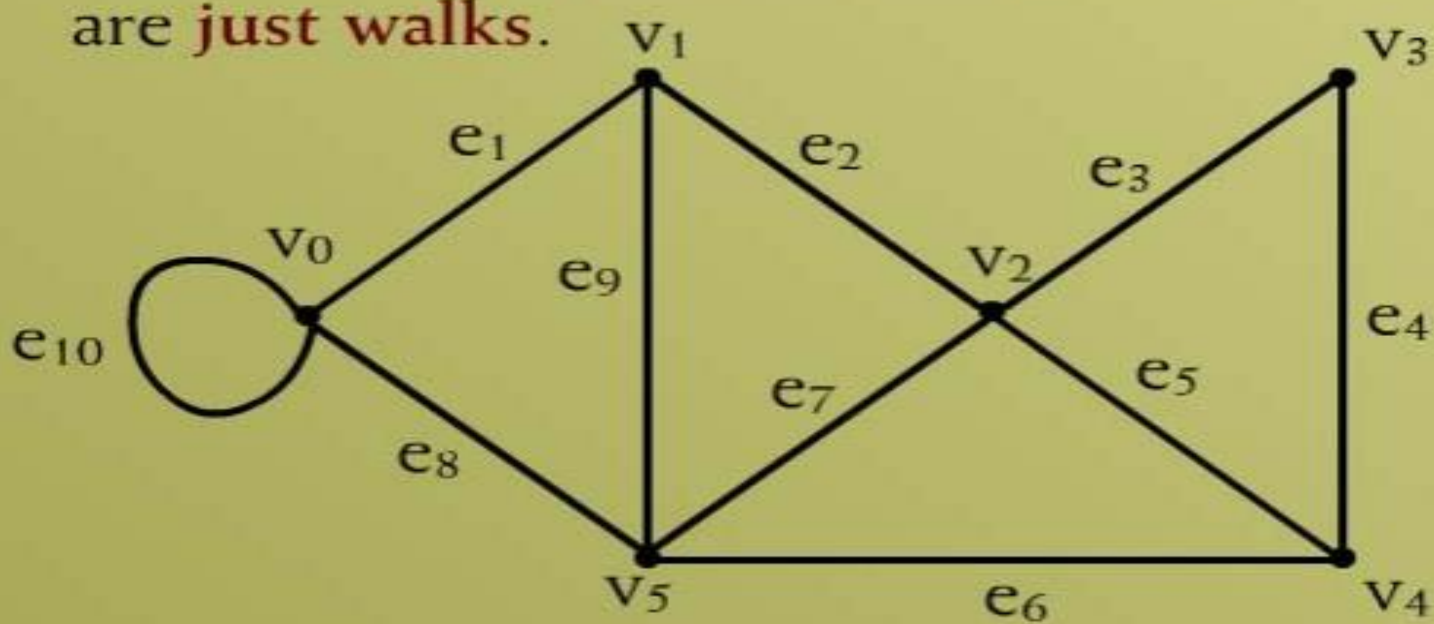
where all the e_i's are **distinct** and all the v_j's are also **distinct** (that is, $v_j \neq v_m$ for any $j \neq m$).

SUMMARY

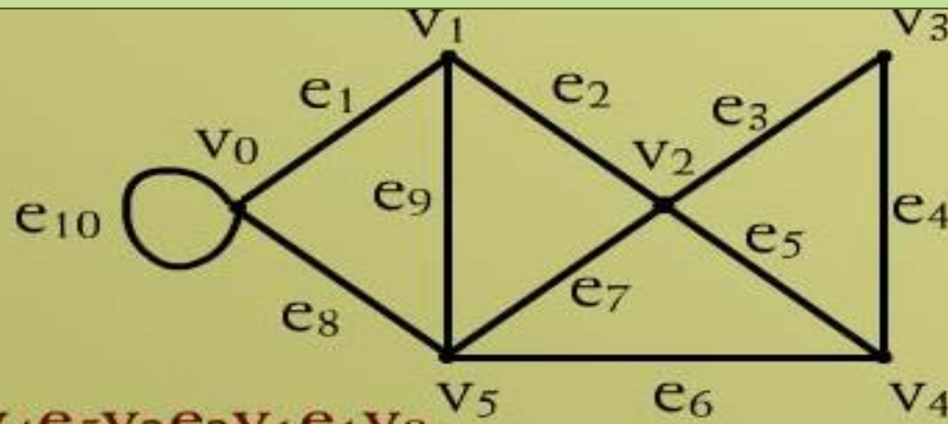
Criteria / Terms	Repeated Edge	Repeated Vertex	Starts and Ends at Same Point
walk	allowed	allowed	allowed
closed walk	allowed	allowed	yes
circuit	no	allowed	yes
simple circuit	no	first and last only	yes
path	no	allowed	no
simple path	no	no	no

PROBLEM

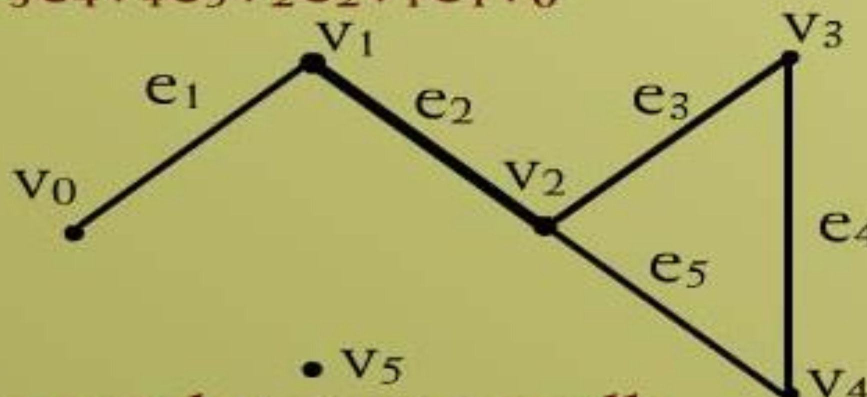
In the **graph** below, determine whether the following **walks** are **paths**, **simple paths**, **closed walks**, **circuits**, **simple circuits**, or are **just walks**.



SOLUTION

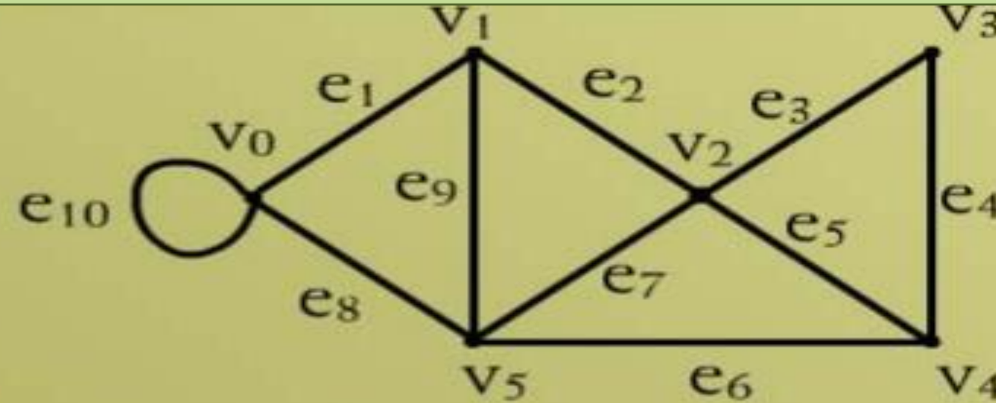


(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

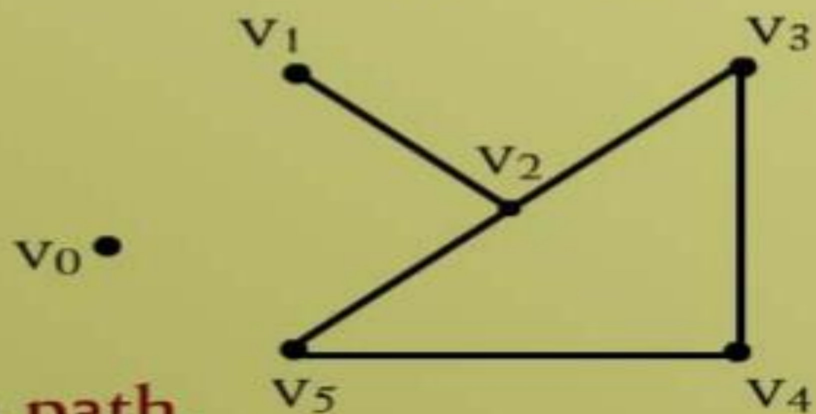


The **graph** is just a **walk**.

SOLUTION

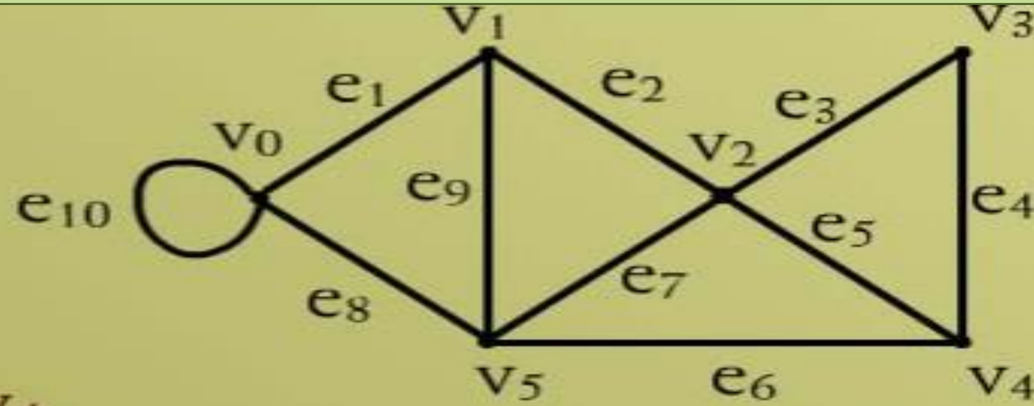


(b) $v_1v_2v_3v_4v_5v_2$

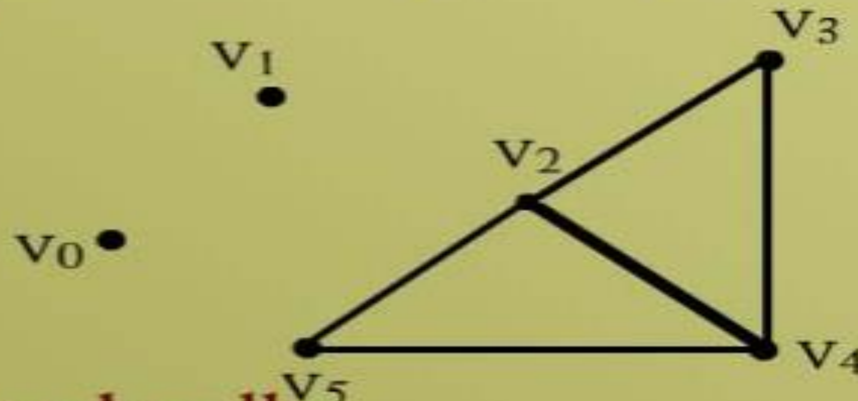


The graph is a path.

SOLUTION

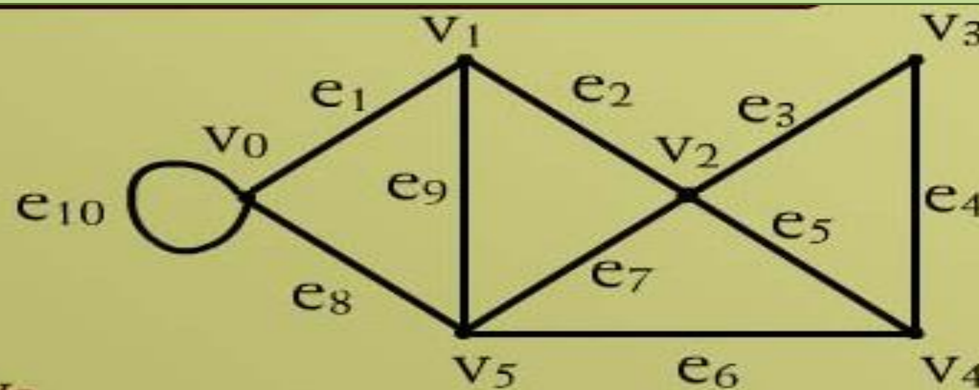


(c) $v_4v_2v_3v_4v_5v_2v_4$

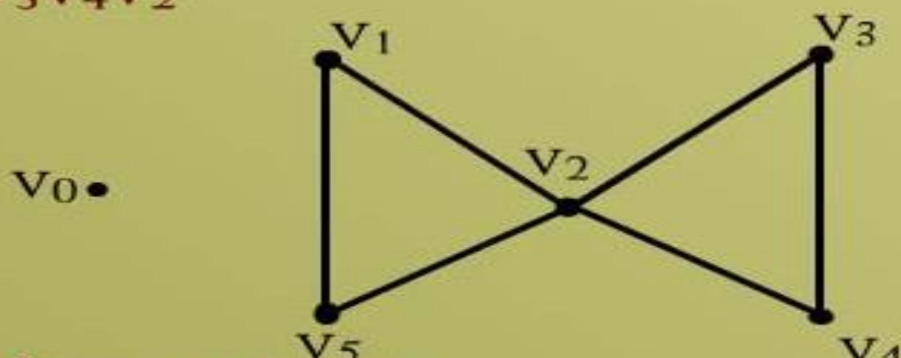


The graph is a closed walk.

SOLUTION

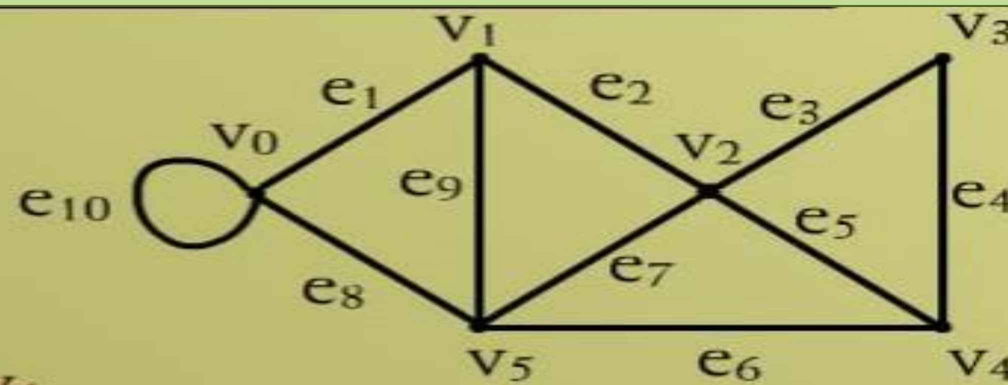


(d) $v_2v_1v_5v_2v_3v_4v_2$

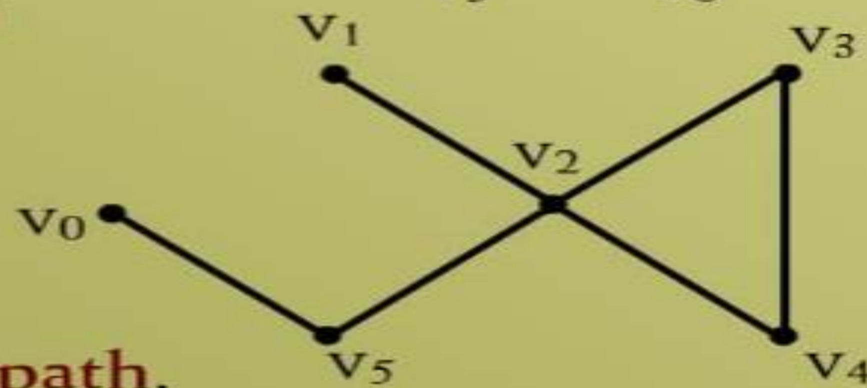


The graph is a circuit.

SOLUTION

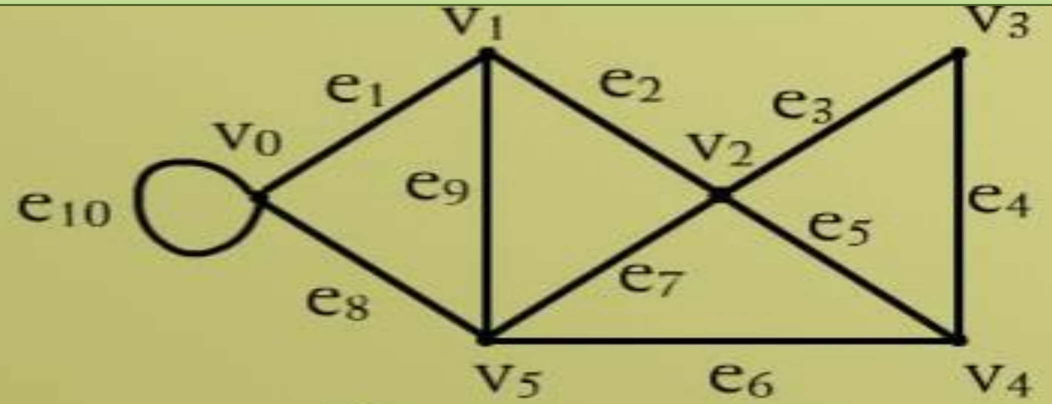


(e) $v_0v_5v_2v_3v_4v_2v_1$

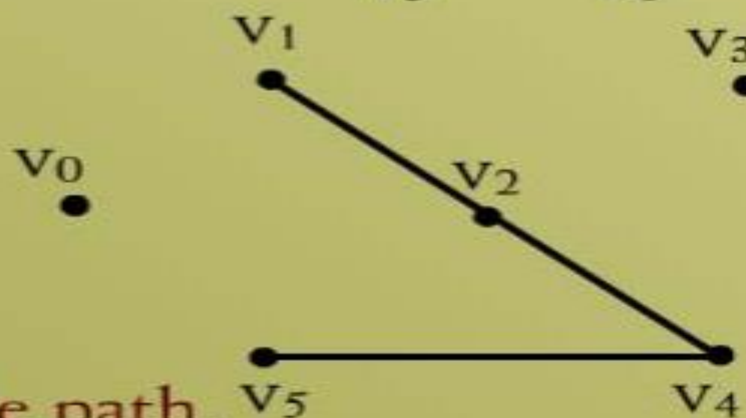


The graph is a path.

SOLUTION



(f) $v_5v_4v_2v_1$



The graph is a simple path.

CONNECTEDNESS

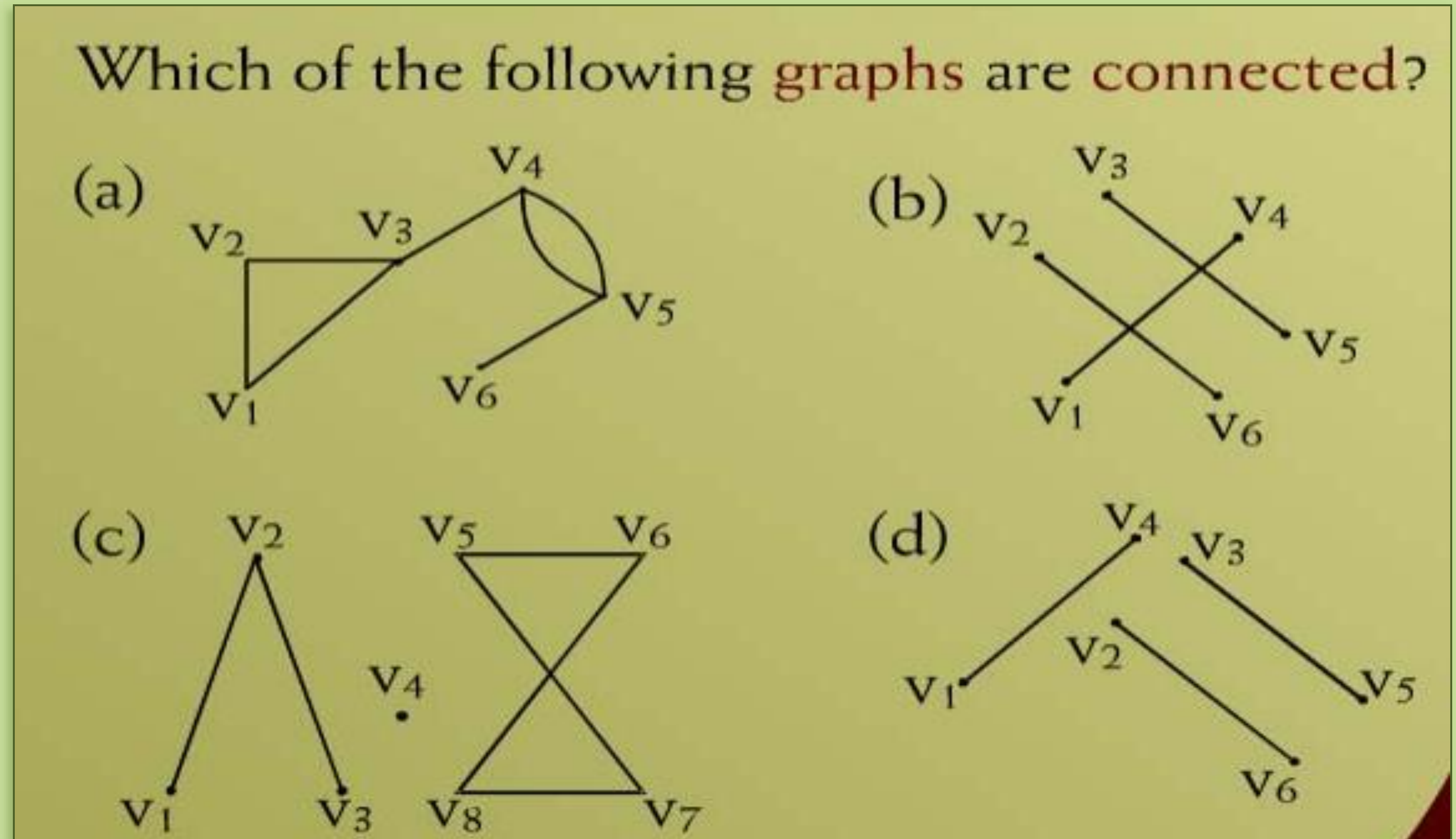
Let G be a graph. Two vertices v and w of G are connected if, and only if, there is a walk from v to w.

The graph G is connected if, and only if, given any two vertices v and w in G, there is a walk from v to w.

Symbolically:

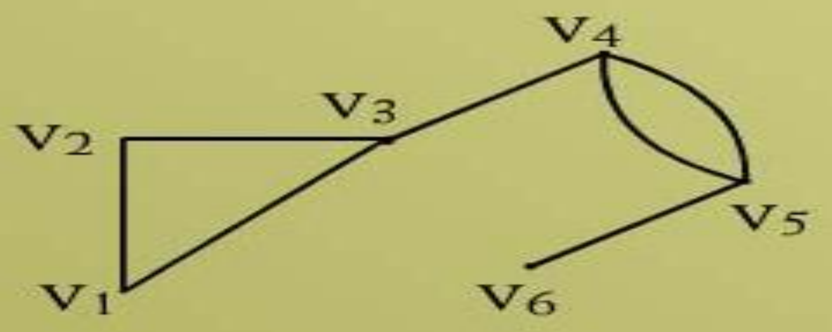
G is connected $\Leftrightarrow$ $\forall$ vertices $v, w \in V(G)$, $\exists$ a walk from v to w:

EXAMPLE


Which of the following graphs are connected?
(a)
v4
v2
v3
v5
v1
v6
(b)
v3
v2
v4
v5
v1
v6
(c)
v2
v5
v6
v4
v1
v3
v8
v7
(d)
v4
v3
v2
v1
v5
v6

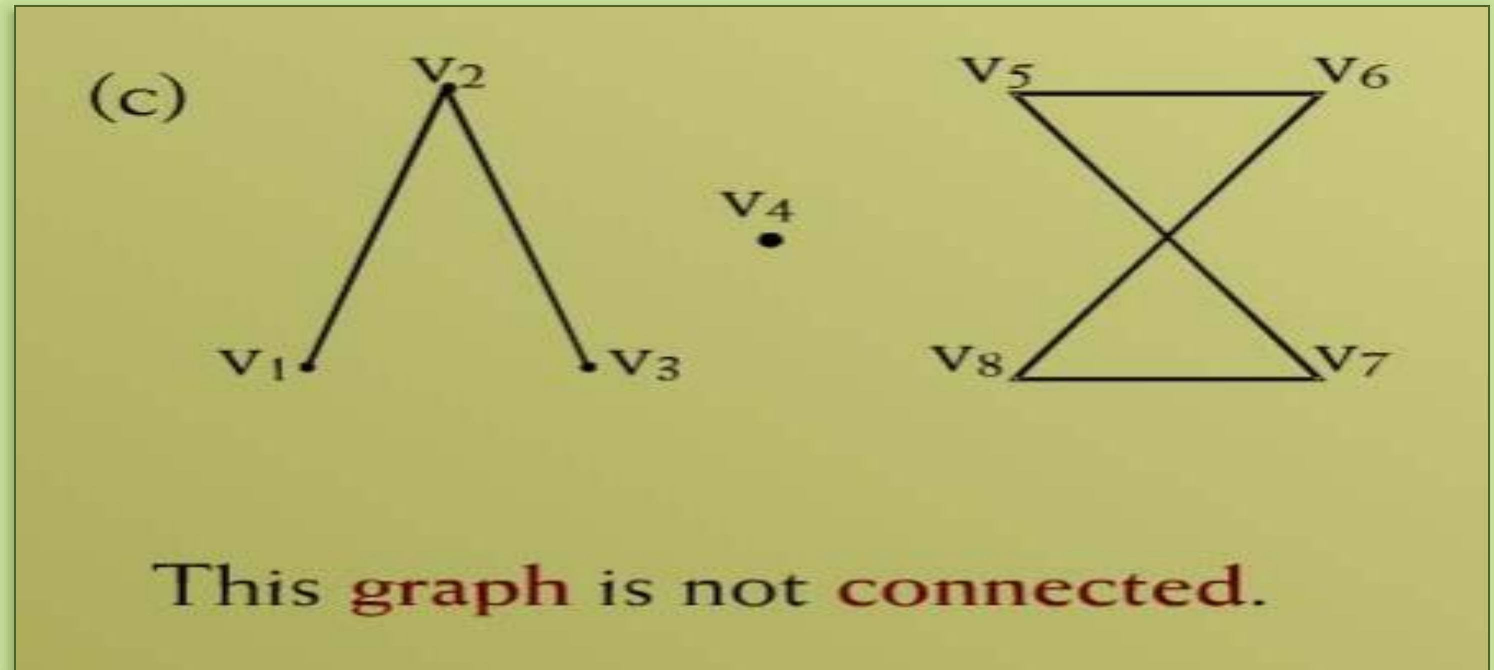
EXAMPLE

(a)

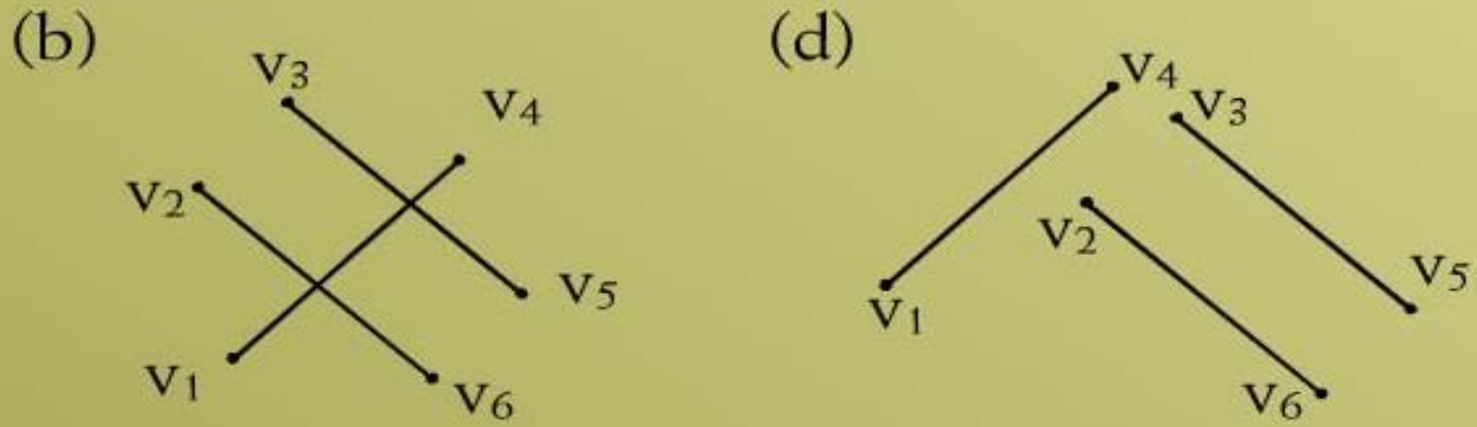


It has six vertices $v_1, v_2, \ldots, v_6$.
The graph is connected.

EXAMPLE



EXAMPLE



So **graph (b)** and **(d)**are not **connected**.

EULER CIRCUITS

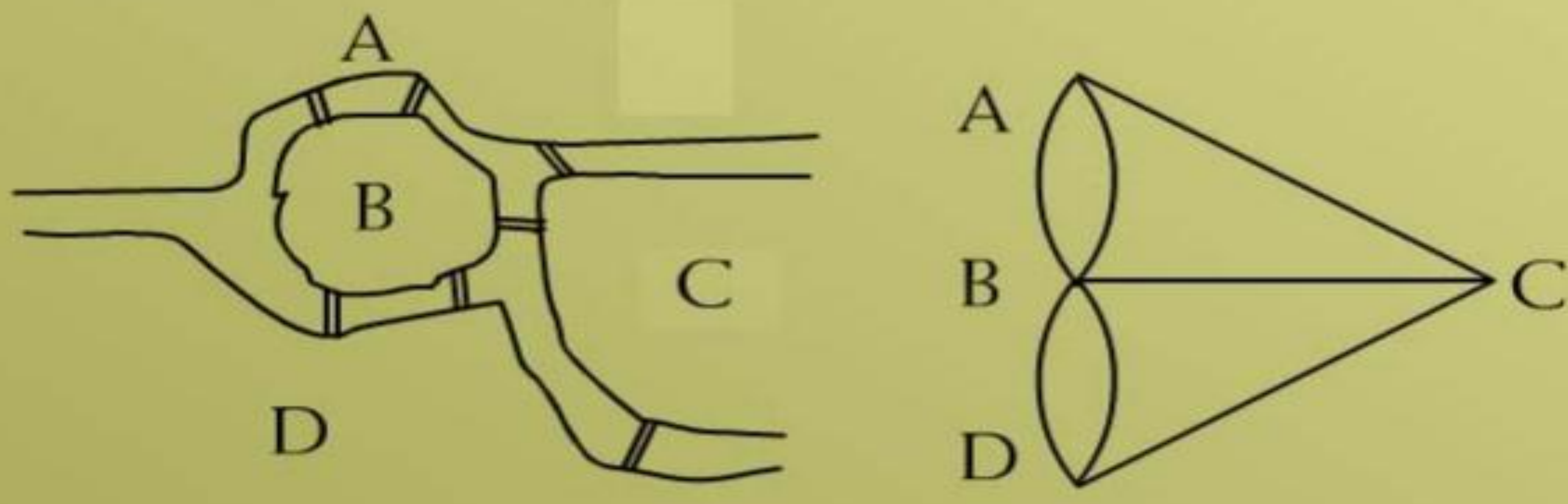
Let G be a graph. An Euler circuit of G is a circuit that contains every vertex and every edge of G.

That is, an Euler circuit of G is sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

EULER RESULT

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has an even degree.

KONIGSBERG BRIDGES PROBLEM



$$\deg(a) = 3 = \deg(c)$$

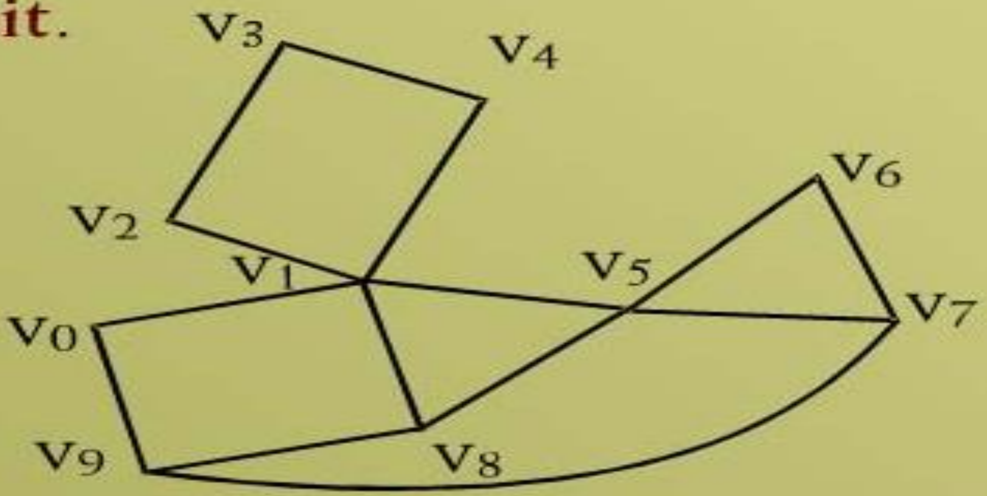
$$\deg(b) = 5, \deg(d) = 3$$

KONIGSBERG BRIDGES PROBLEM

No vertex has even degree so there is no possibility of an Euler circuit.

EXERCISE

Determine whether the following **graph** has **an Euler circuit**.

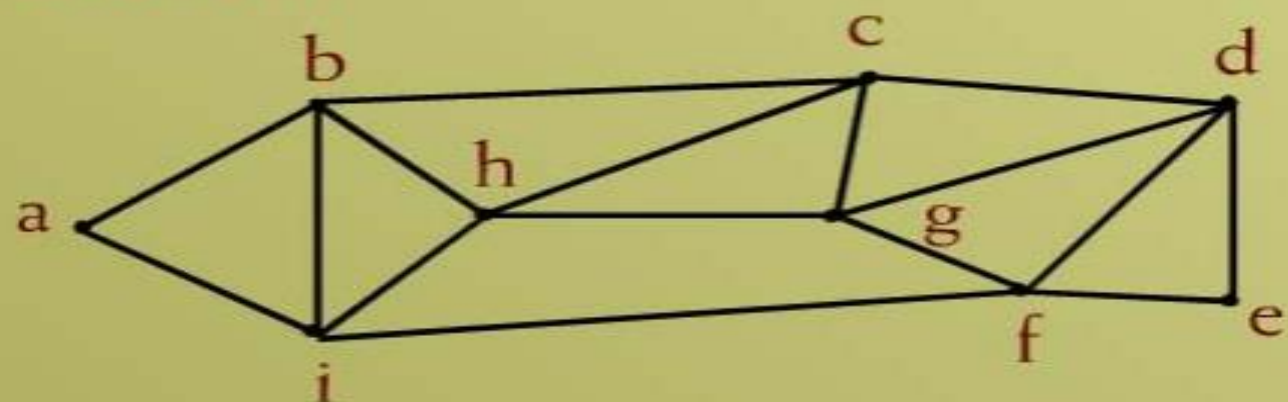


$deg(v_3) = 2 = deg(v_4) = deg(v_2)$, $deg(v_1) = 5$

As v_1 has **odd degree** so this **graph** can't have **an Euler circuit**.

EXERCISE

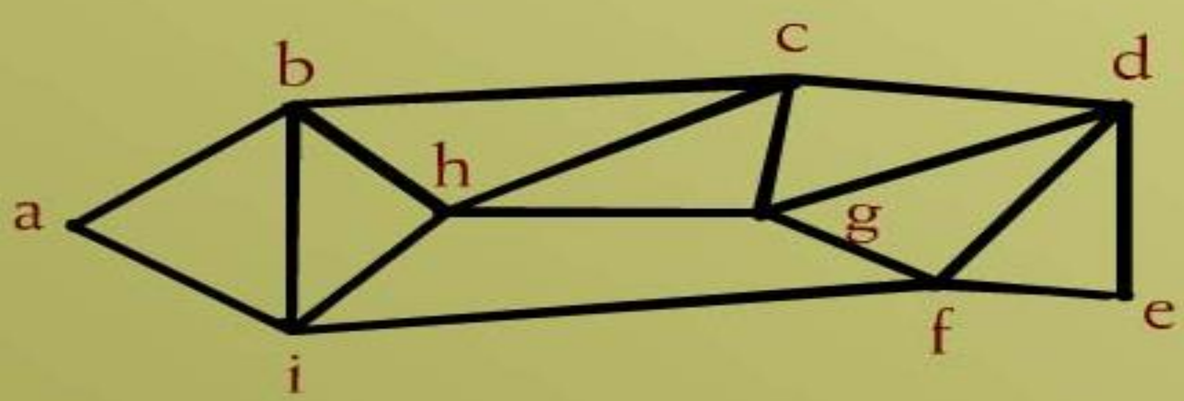
Determine whether the following **graph** has **Euler circuit**.



deg(a)=2, deg(b)=4, deg(c)=4, deg(d)=4, deg(e)=2, deg(f)=4, deg(g)=4, deg(h)=4, deg(i)=4

EXERCISE

So the **every vertex** is of **even degree**, clearly **Euler theorem** is applicable. We should be able to find **Euler circuit** here:



Euler circuit: {a, b, c, d, f, e, d, g, f, i, h, g, c, h, b, i, a}.

EULER PATH

Let G be a graph and let v and w be two vertices of G.

An Euler path from v to w is a sequence of adjacent edges and vertices that starts at v, end at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

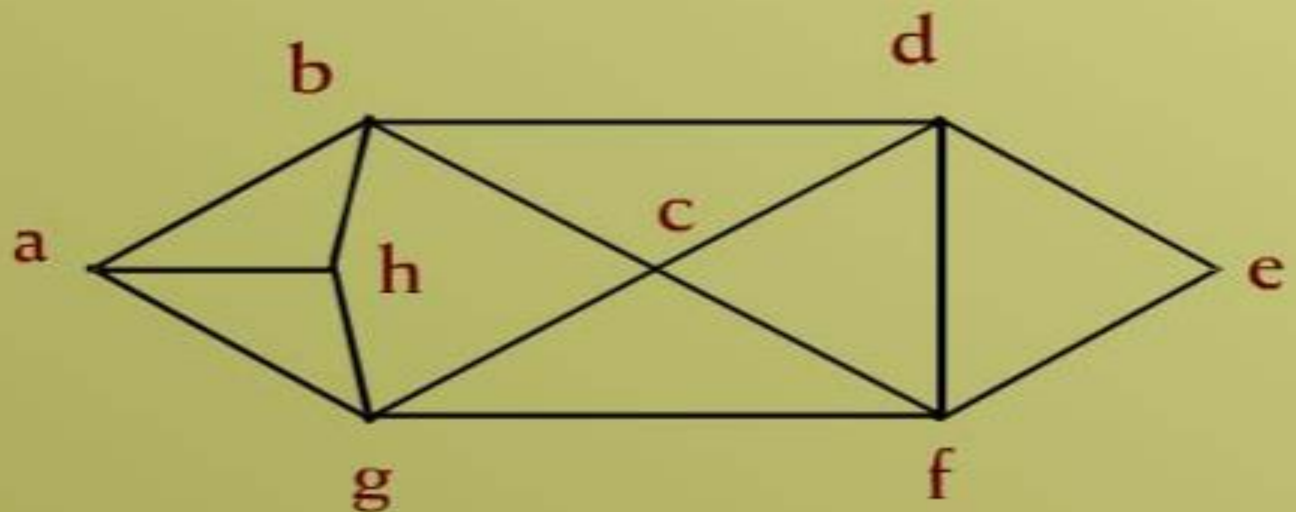
HAMILTONIAN CIRCUITS

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G.

That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

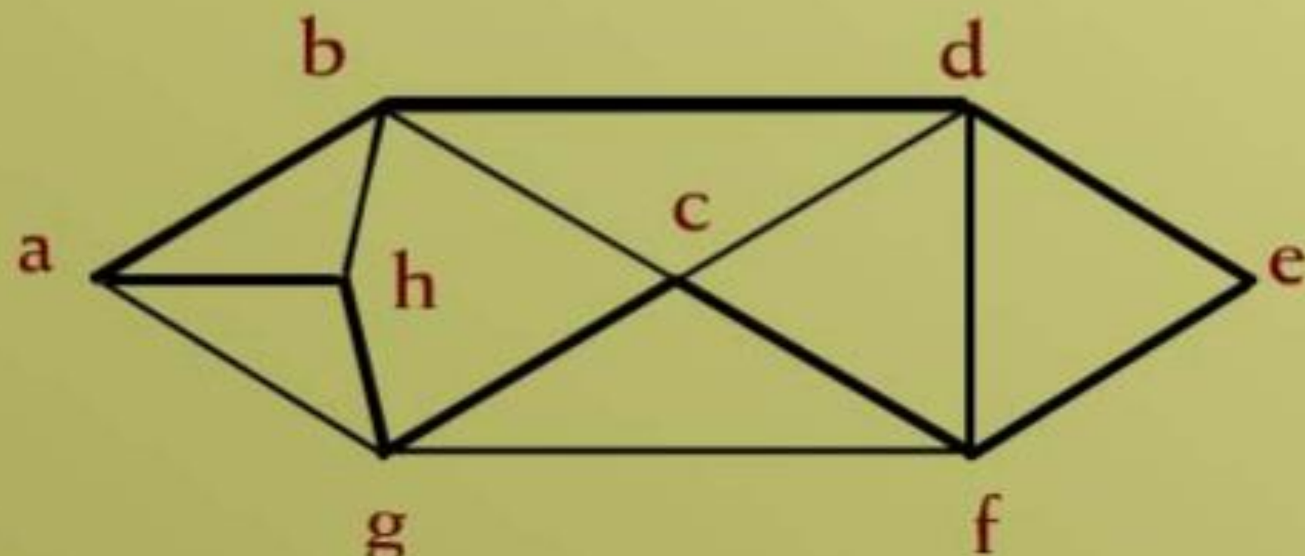
EXERCISE

Find Hamiltonian Circuit for the following graph.



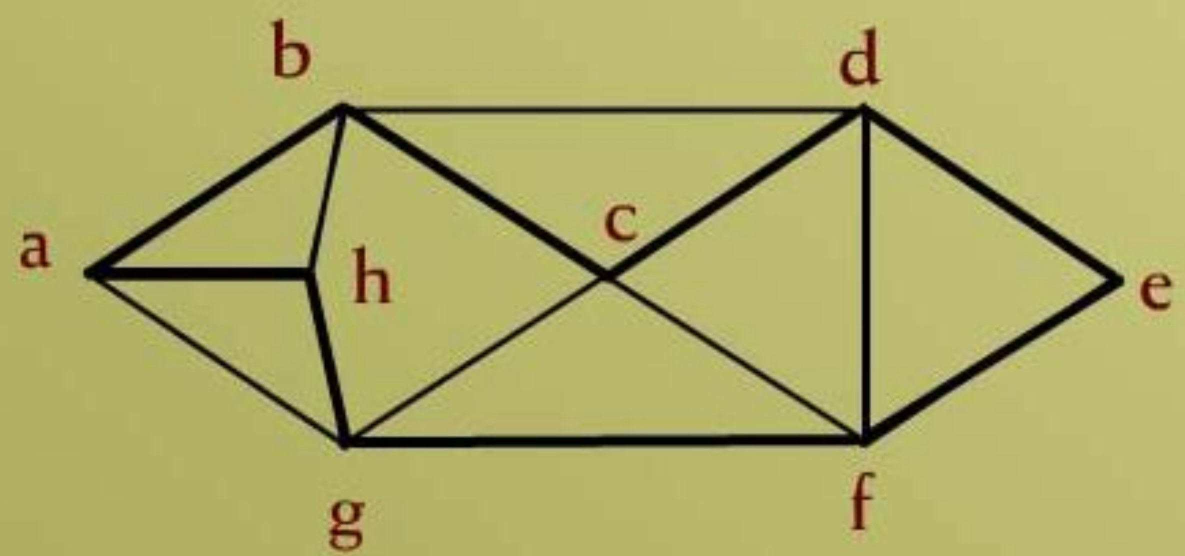
SOLUTION

The Hamiltonian Circuit is:



SOLUTION

Another Hamiltonian Circuit could be:



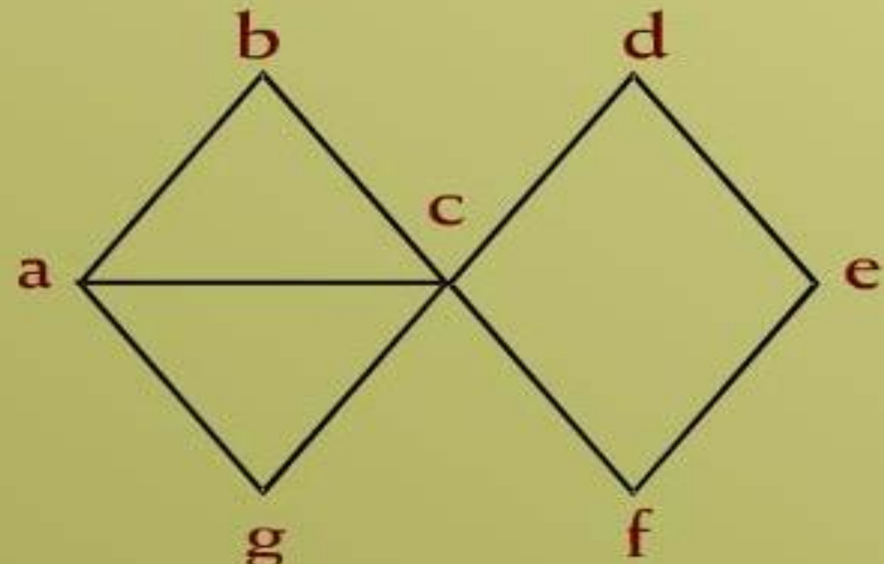
PROPERTIES

If a graph G has a Hamiltonian circuit then G has a sub-graph H with the following properties:

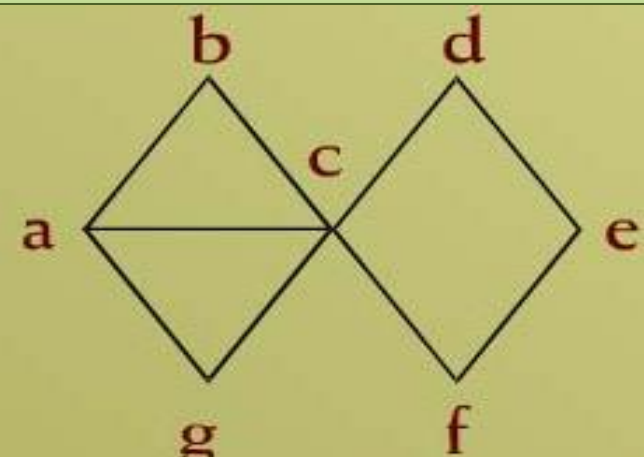
1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

EXAMPLE

Show that whether the Hamiltonian circuit is possible or not?



EXAMPLE



$\deg(c)=5$, if we remove 3 edges from vertex c then $\deg(a)<2, \deg(b)<2, \deg(g)<2, \deg(d)<2$

It means that this graph does not have a subgraph with the desired properties, so the Hamiltonian circuit is not possible.

Is the following graph a Hamiltonian graph? Give the explicit reason.