Discrete Structures

Lecture # 07

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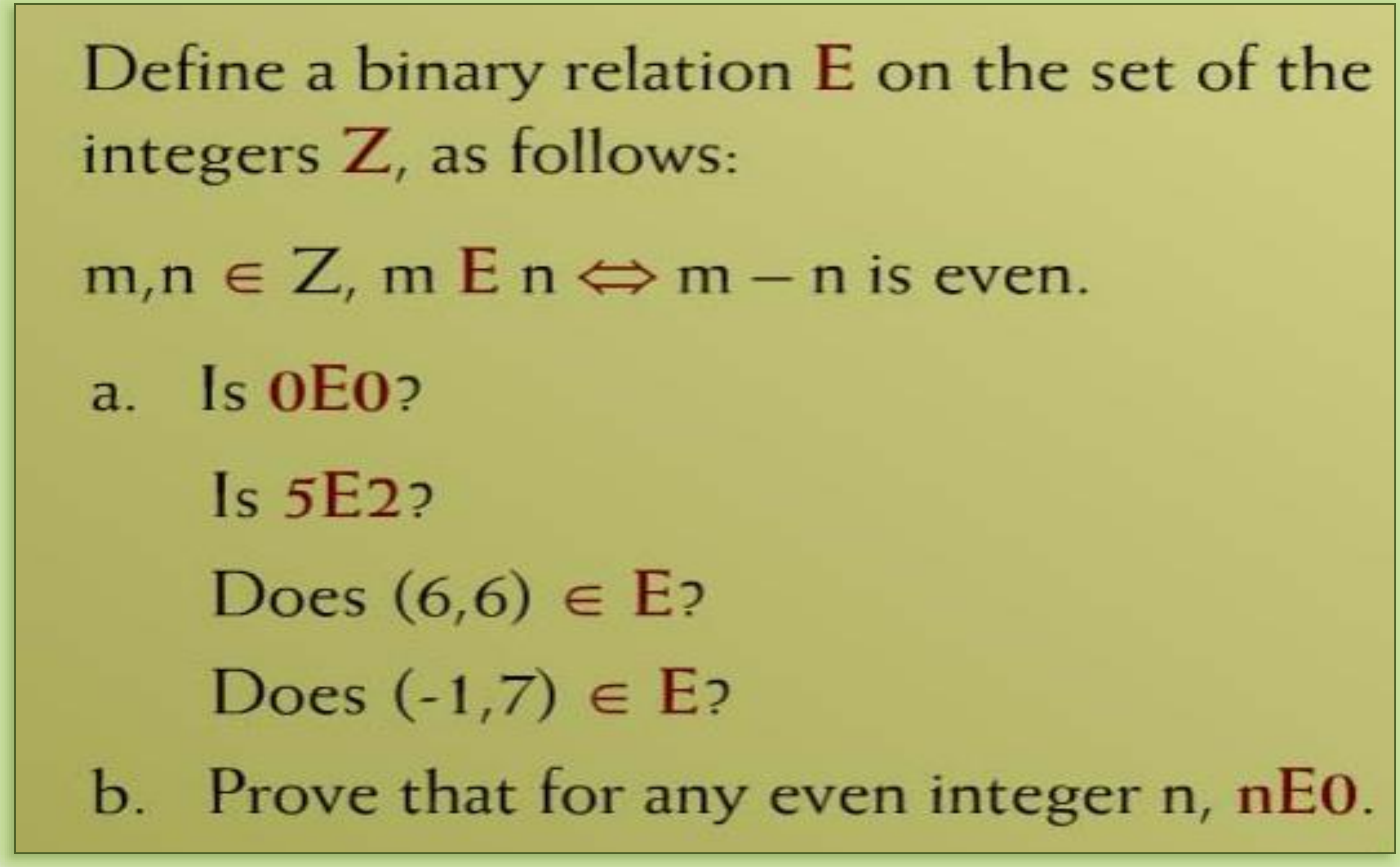
EXERCISE



Define a binary relation E on the set of the integers Z, as follows:

$m, n \in Z$, $m \; E \; n \Leftrightarrow m - n$ is even.

a. Is **0E0**?

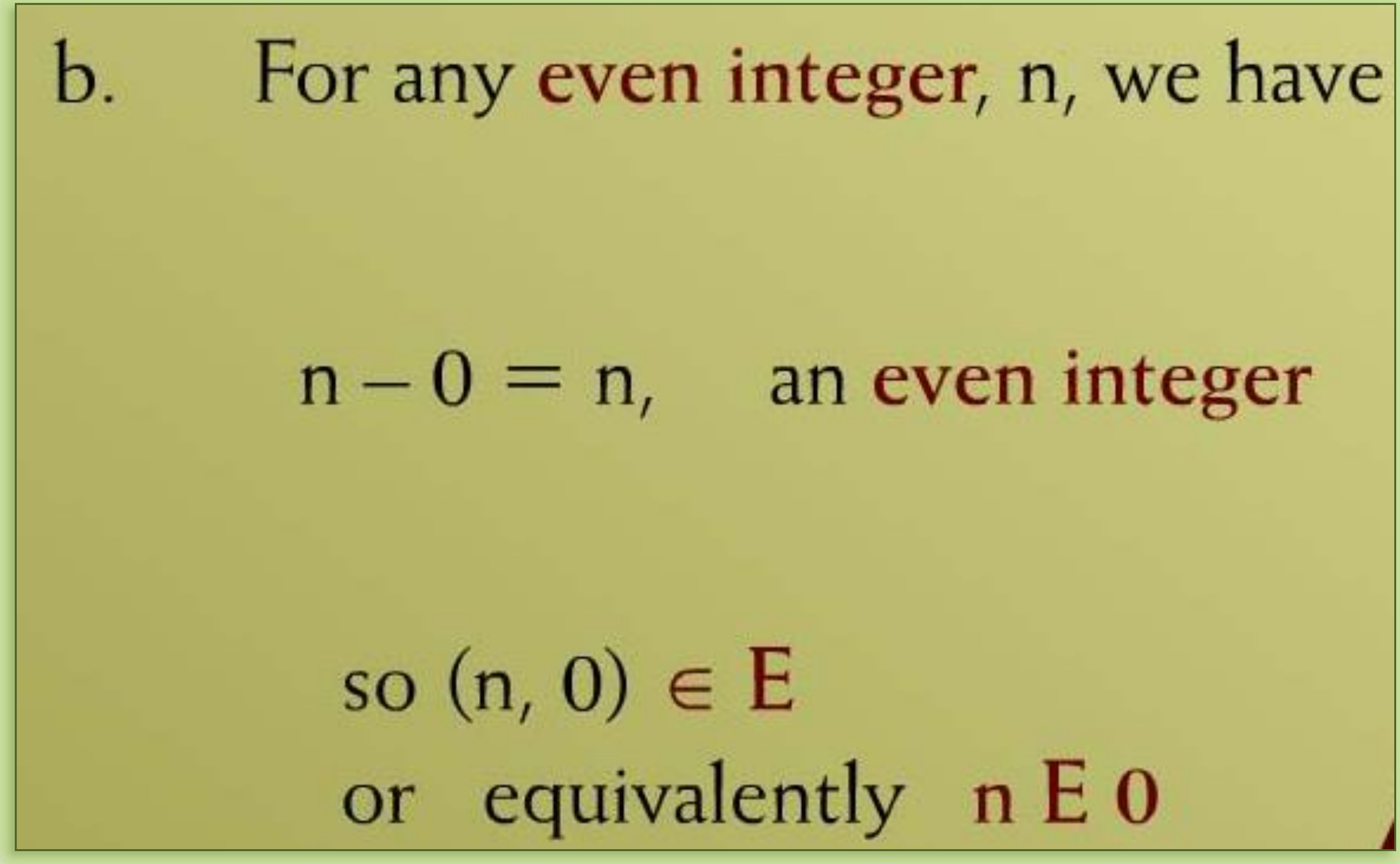
Is **5E2**?

Does $(6,6) \in E$?

Does $(-1,7) \in E$?

b. Prove that for any even integer n, **nE0**.

SOLUTION



b. For any **even integer**, n, we have

$n - 0 = n$, an **even integer**

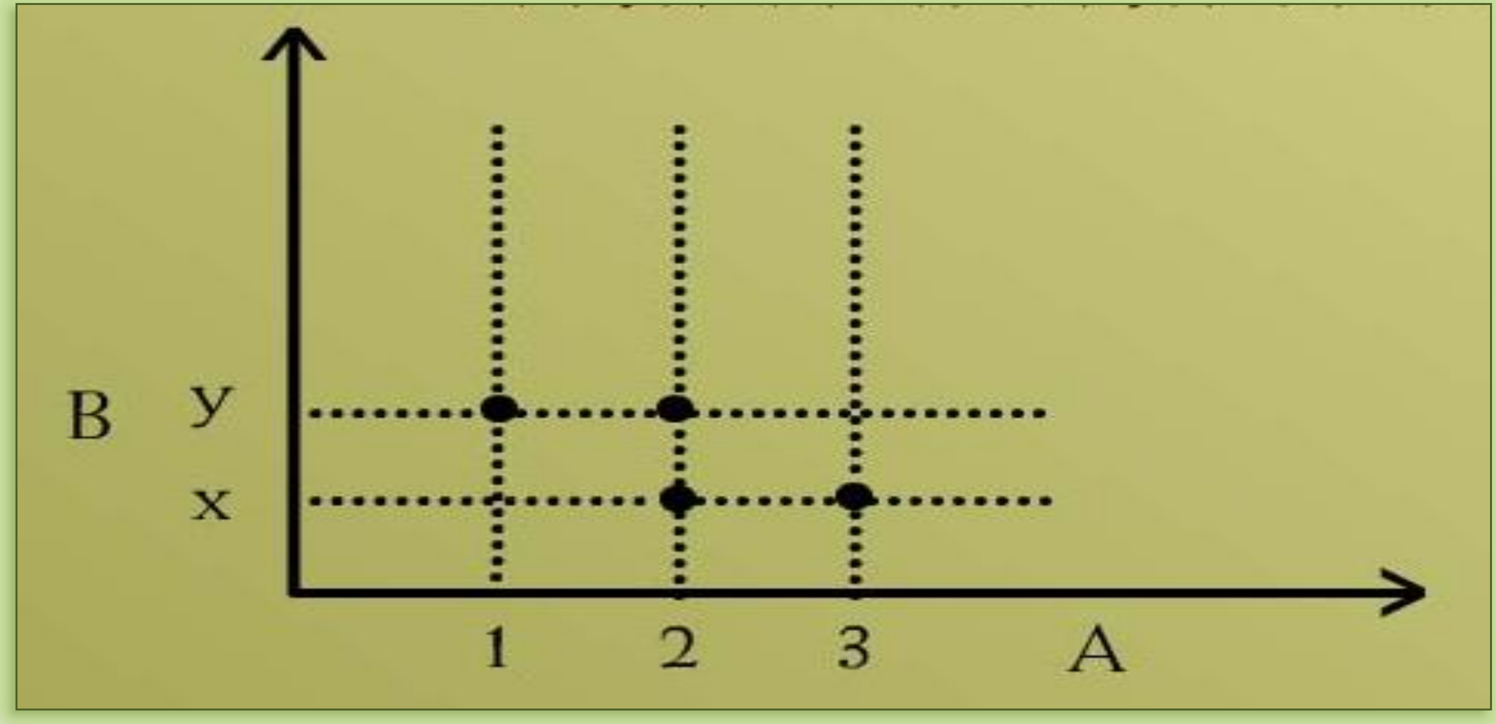
so $(n, 0) \in E$

or equivalently $n \, E \, 0$

CARTESIAN DIAGRAM OF A RELATION

Let A = {1, 2, 3}   and   B = {x, y}

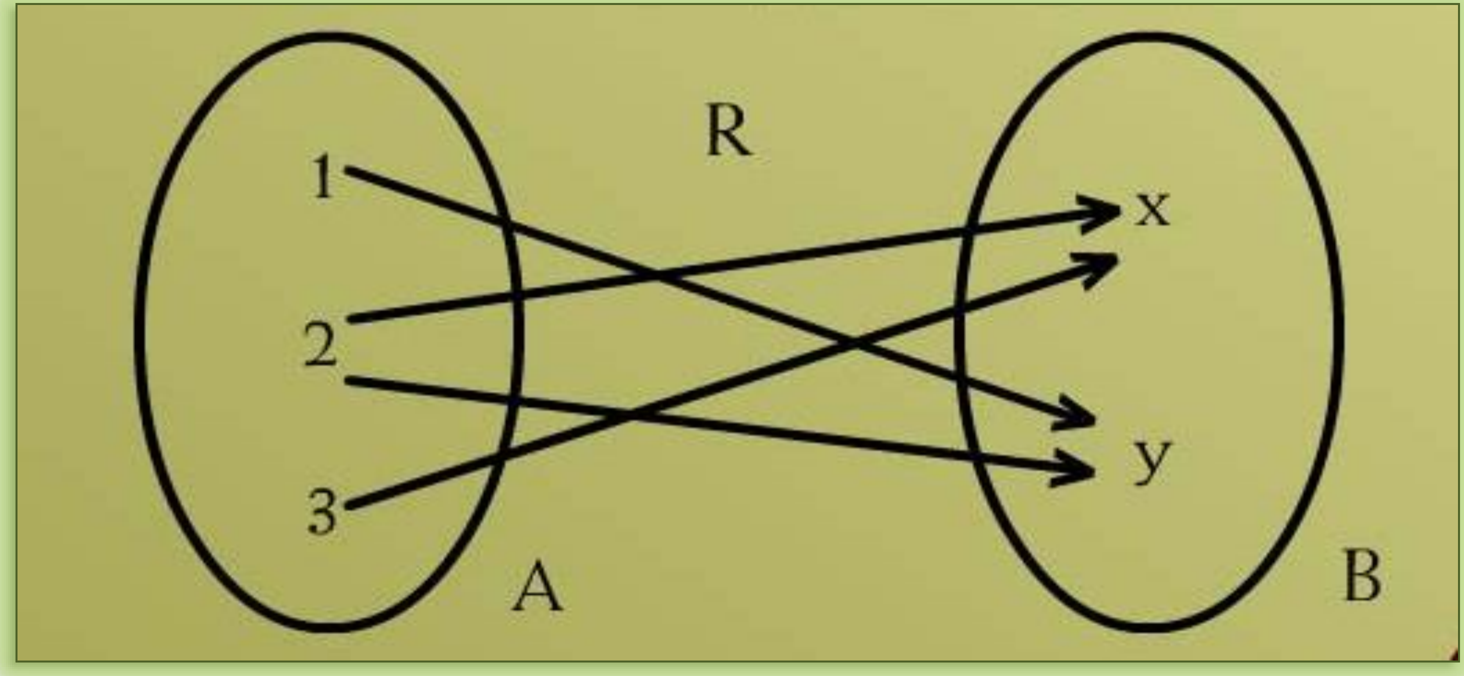
R = { (1, y) , (2, x) , (2, y) , (3, x) }



ARROW DIAGRAM OF A RELATION

Let A = {1, 2, 3} and B = {x, y}

R = { (1, y) , (2, x) , (2, y) , (3, x) }

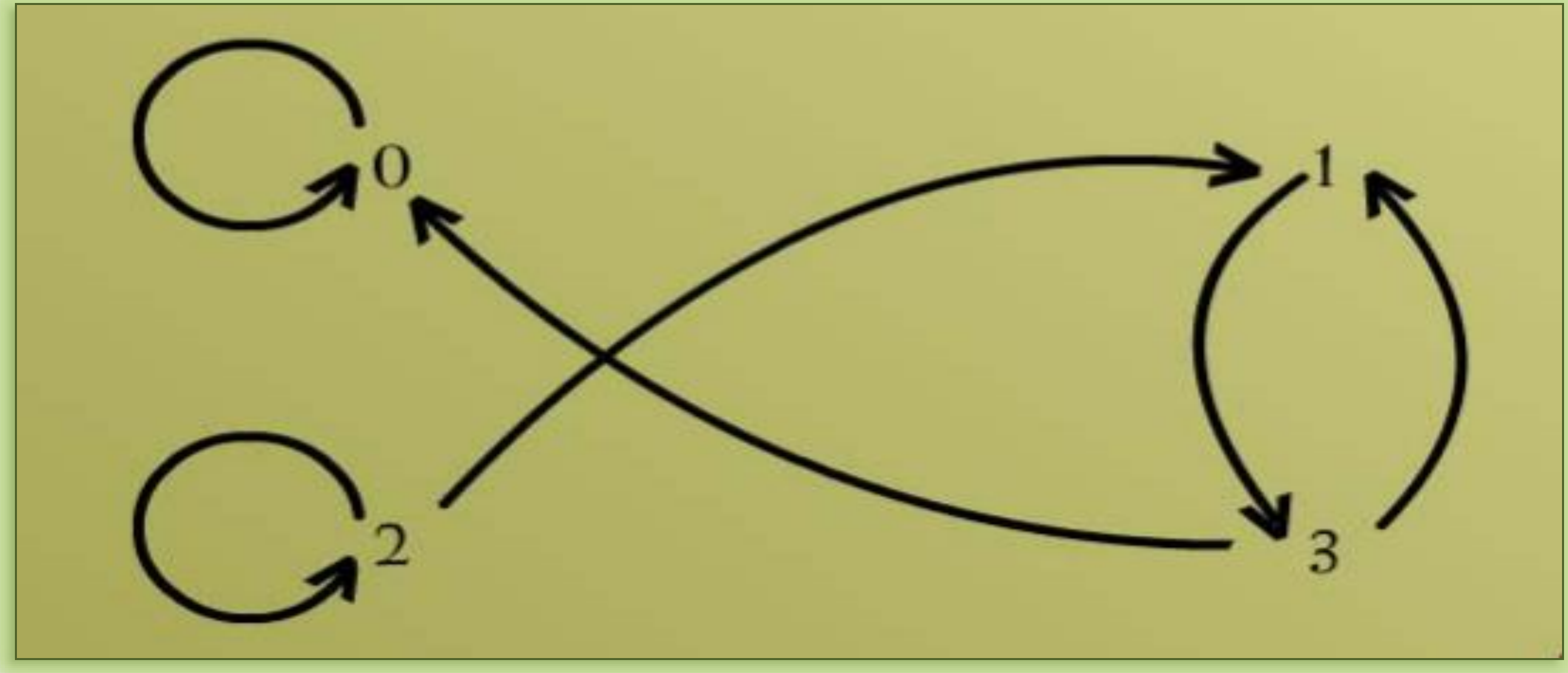


DIRECTED GRAPH OF A RELATION

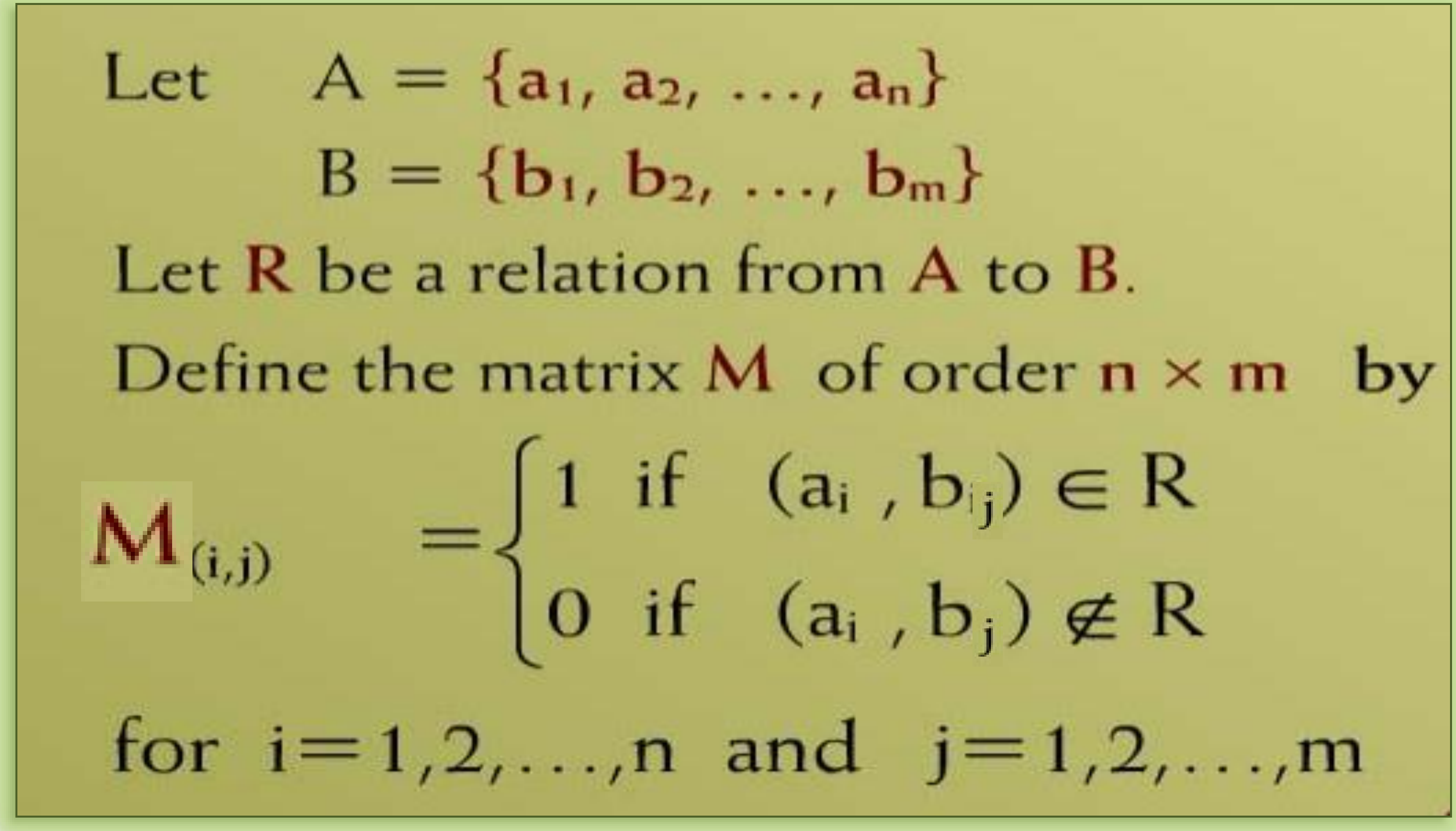
Let A = {0, 1, 2, 3}

A x A

R = { (0,0) , (1, 3) , (2, 1) , (2, 2) , (3, 0) , (3, 1) }



MATRIX REPRESENTATION OF A RELATION



Let $A = \{a_1, a_2, \ldots, a_n\}$

$B = \{b_1, b_2, \ldots, b_m\}$

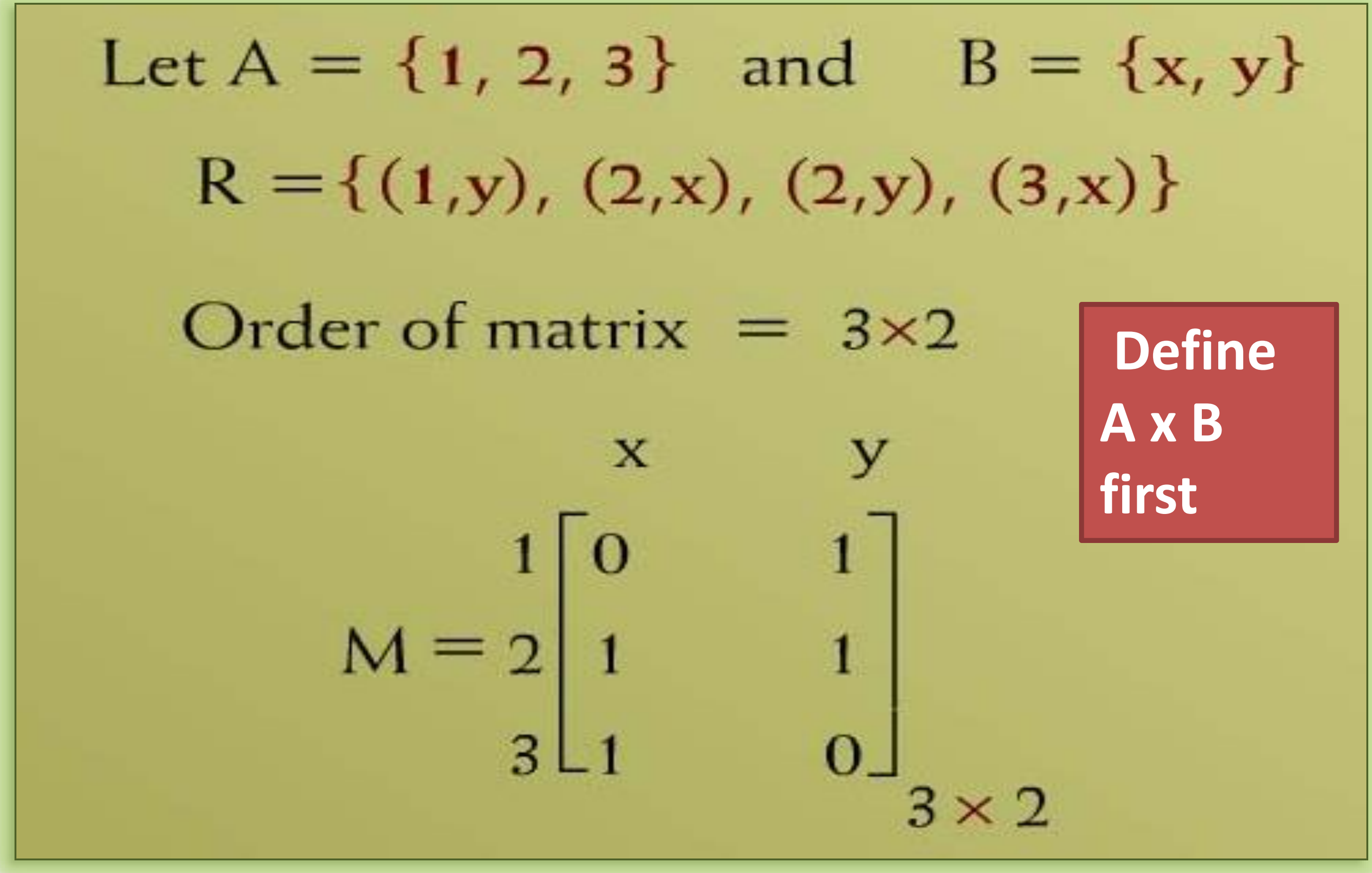
Let R be a relation from A to B.

Define the matrix M of order $n \times m$ by

$$M_{(i,j)} = \begin{cases} 1 \text{ if } (a_i, b_{ij}) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$

EXAMPLE



Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$

$R = \{(1,y), (2,x), (2,y), (3,x)\}$

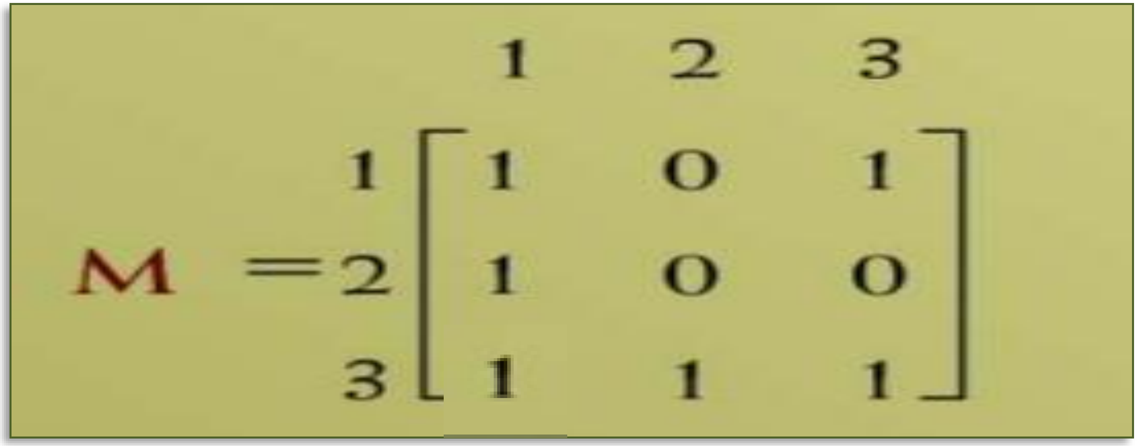
Order of matrix $= 3\times2$

$$M = \begin{array}{c|cc} & x & y \\ \hline 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{array}_{3 \times 2}$$

Define A x B first

EXAMPLE

For the relation matrix



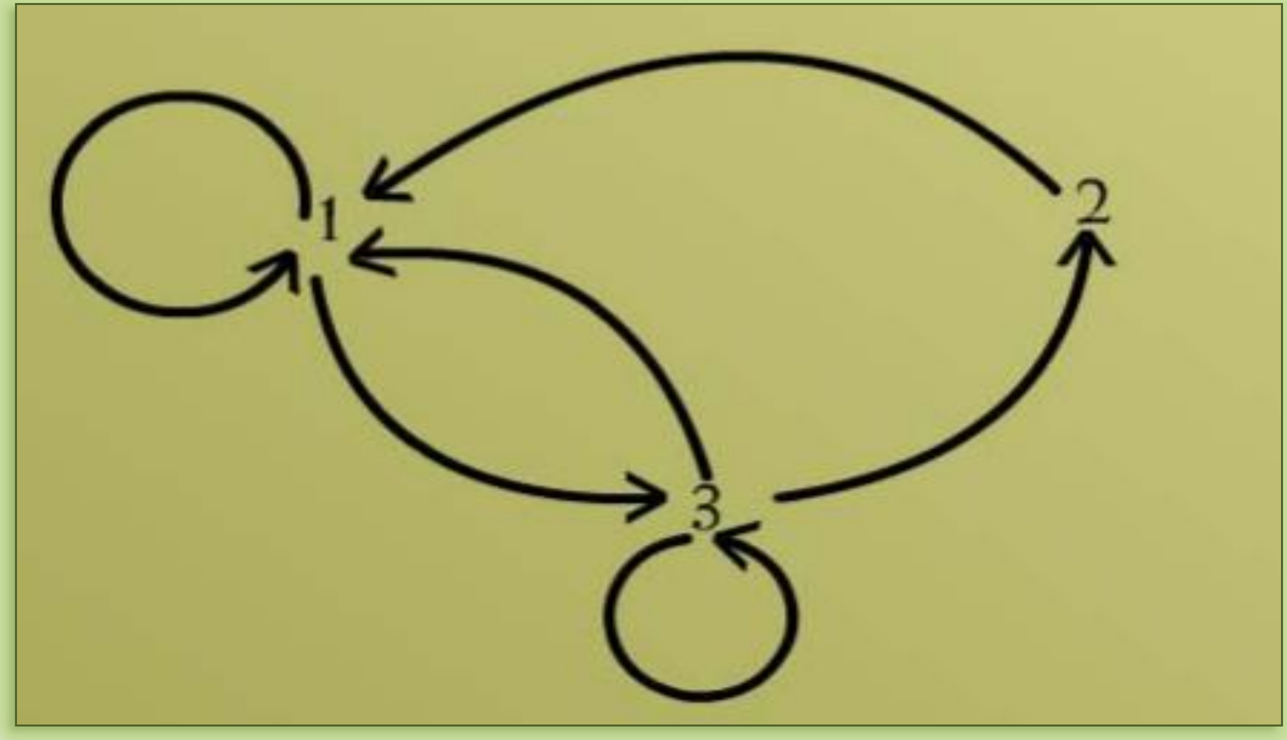
$$M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

1. List set of ordered pairs represented by M.
2. Draw the directed graph of the relation.

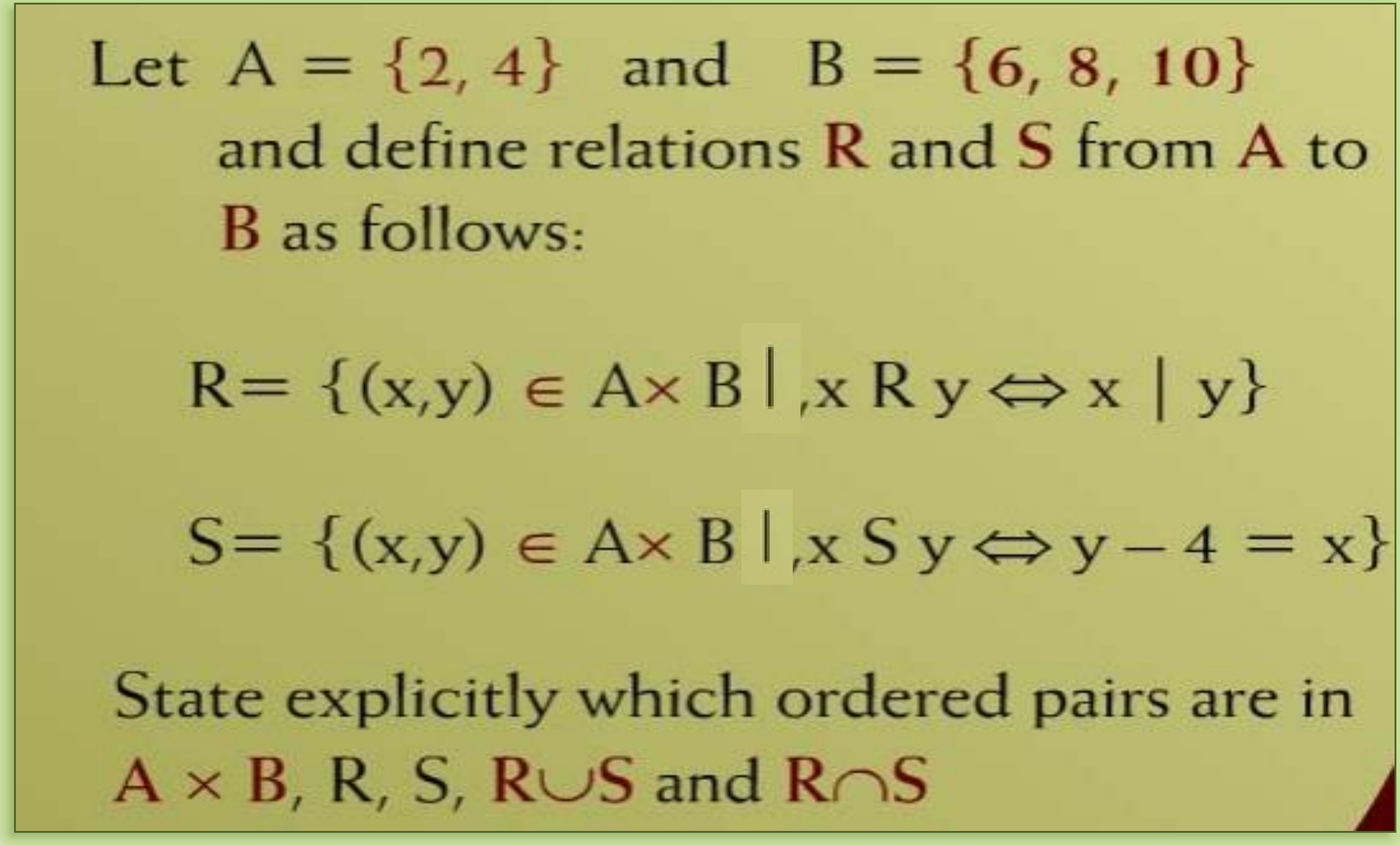
Solution contd…

$$M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

1. R = { (1, 1) , (1, 3) , (2, 1) , (3, 1) , (3, 2) , (3, 3) }



EXERCISE



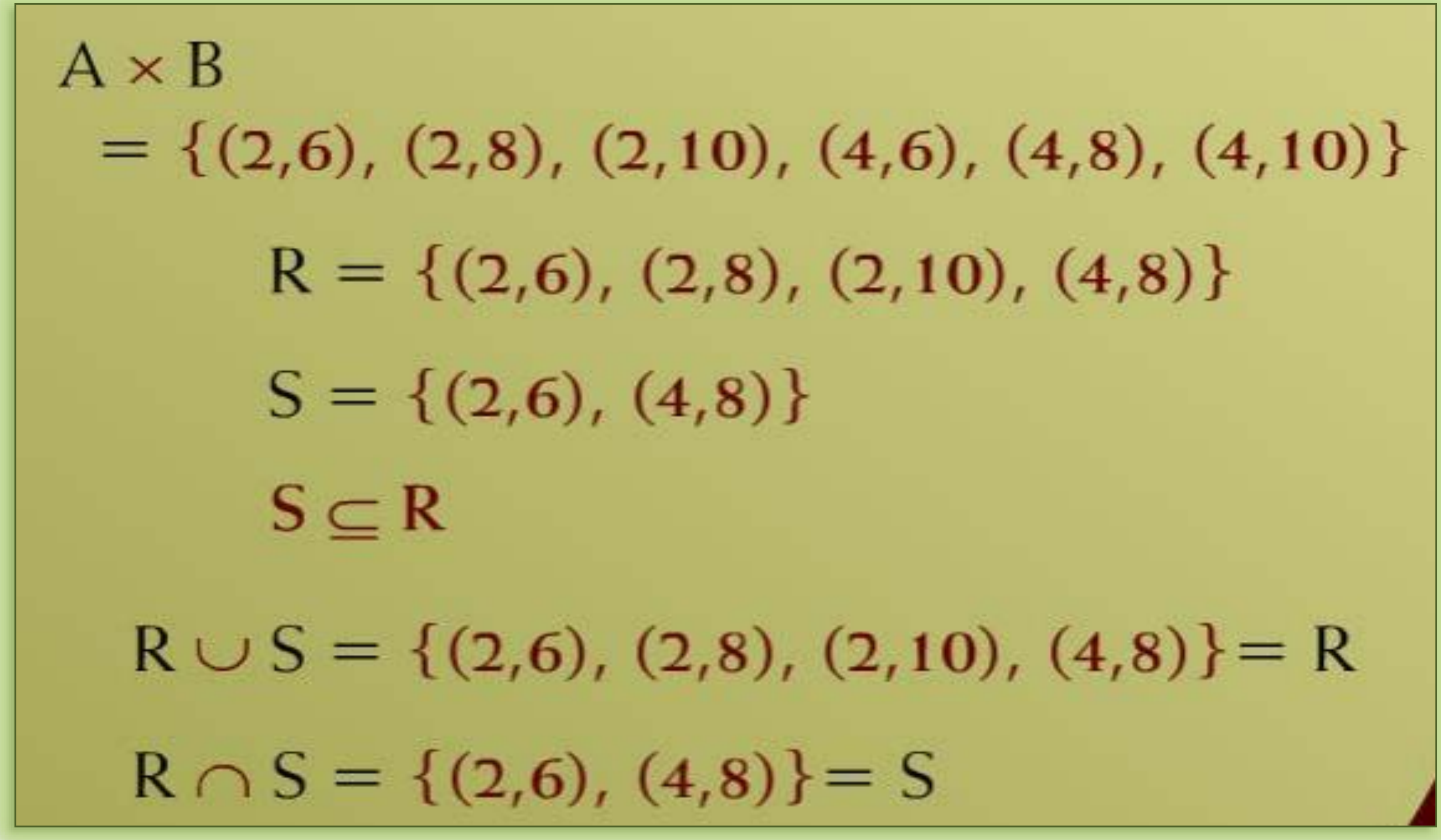
Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define relations R and S from A to B as follows:

$$R = \{(x,y) \in A \times B \mid ,x\,R\,y \Leftrightarrow x \mid y\}$$

$$S = \{(x,y) \in A \times B \mid ,x\,S\,y \Leftrightarrow y - 4 = x\}$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$ and $R \cap S$

SOLUTION



$$A \times B$$

$$= \{(2,6),\ (2,8),\ (2,10),\ (4,6),\ (4,8),\ (4,10)\}$$

$$R = \{(2,6),\ (2,8),\ (2,10),\ (4,8)\}$$

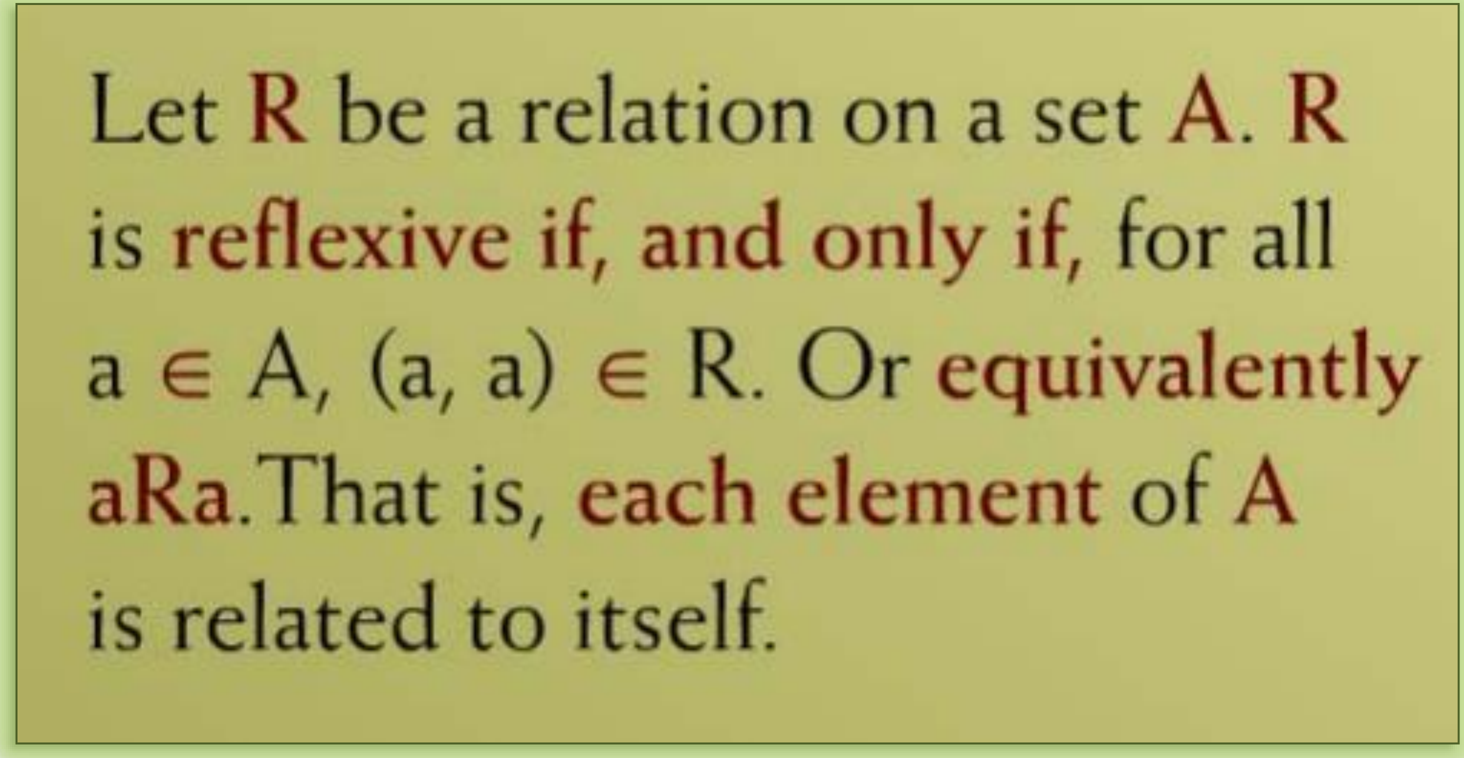
$$S = \{(2,6),\ (4,8)\}$$

$$S \subseteq R$$

$$R \cup S = \{(2,6),\ (2,8),\ (2,10),\ (4,8)\} = R$$

$$R \cap S = \{(2,6),\ (4,8)\} = S$$

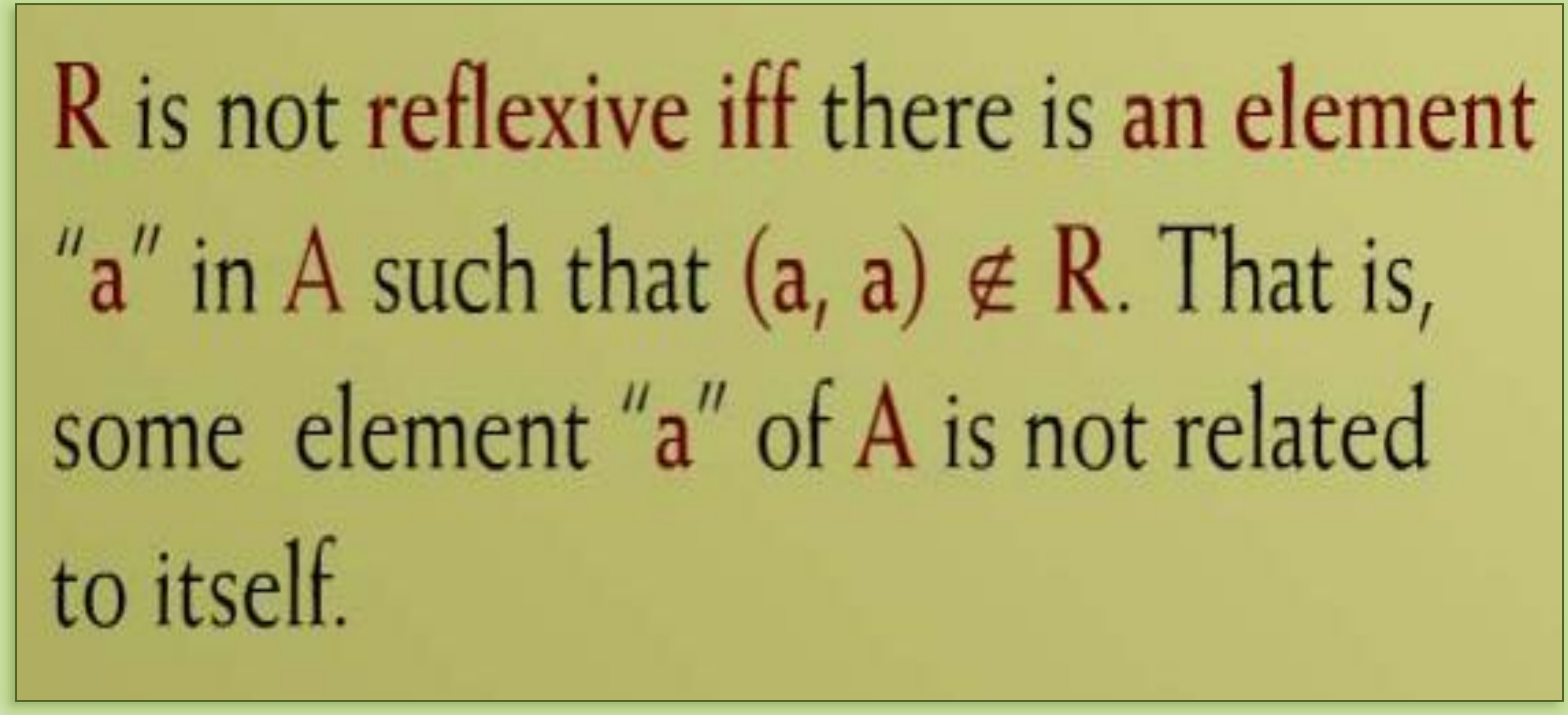
REFLEXIVE RELATION



Let R be a relation on a set A. R is **reflexive if, and only if,** for all a ∈ A, (a, a) ∈ R. Or **equivalently** **aRa**. That is, **each element** of A is related to itself.

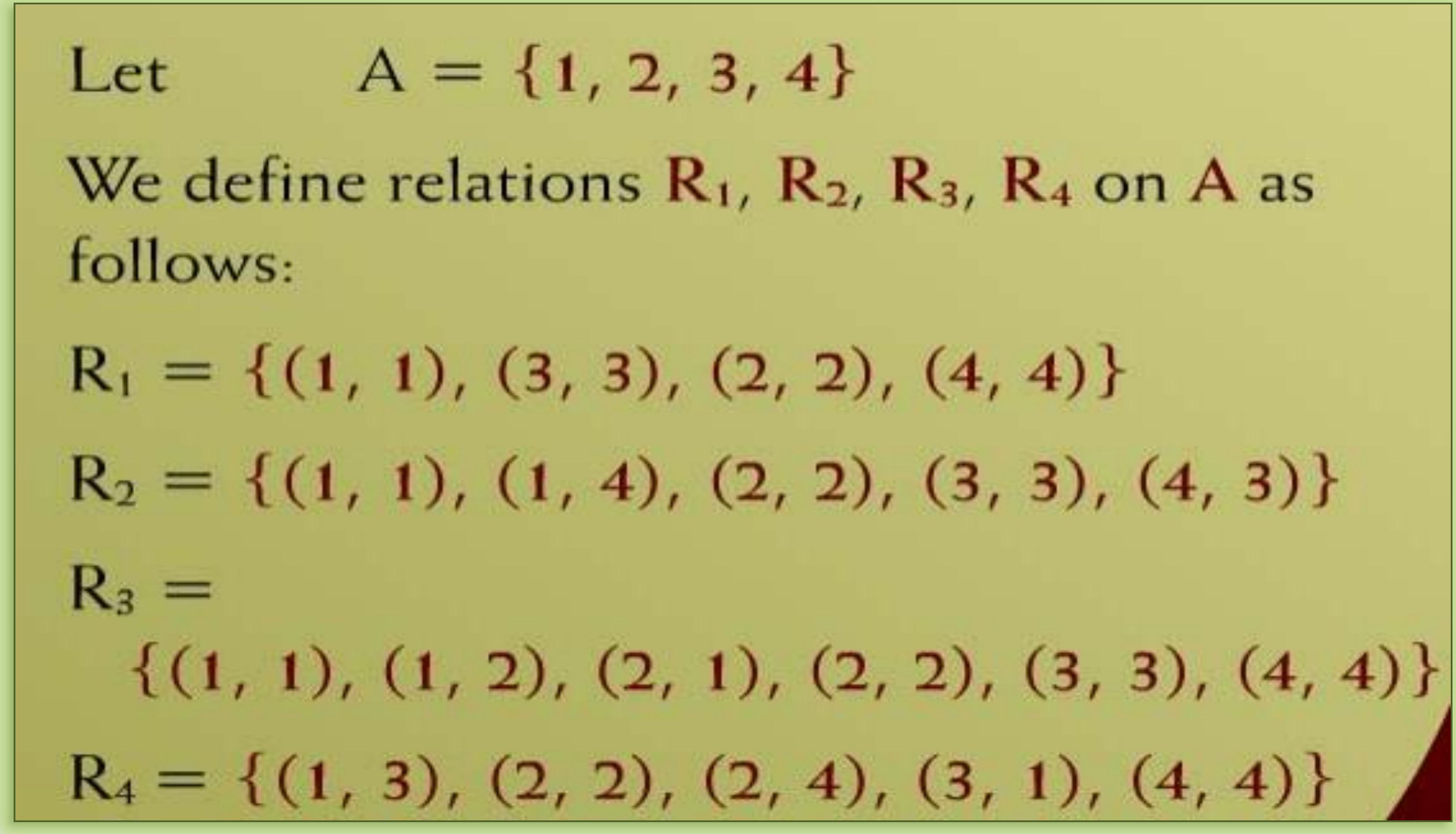
REFLEXIVE RELATION

REMARK:



R is not reflexive iff there is an element “a” in A such that $(a, a) \notin R$. That is, some element “a” of A is not related to itself.

EXAMPLE



Let $A = \{1, 2, 3, 4\}$

We define relations R_1, R_2, R_3, R_4 on A as follows:

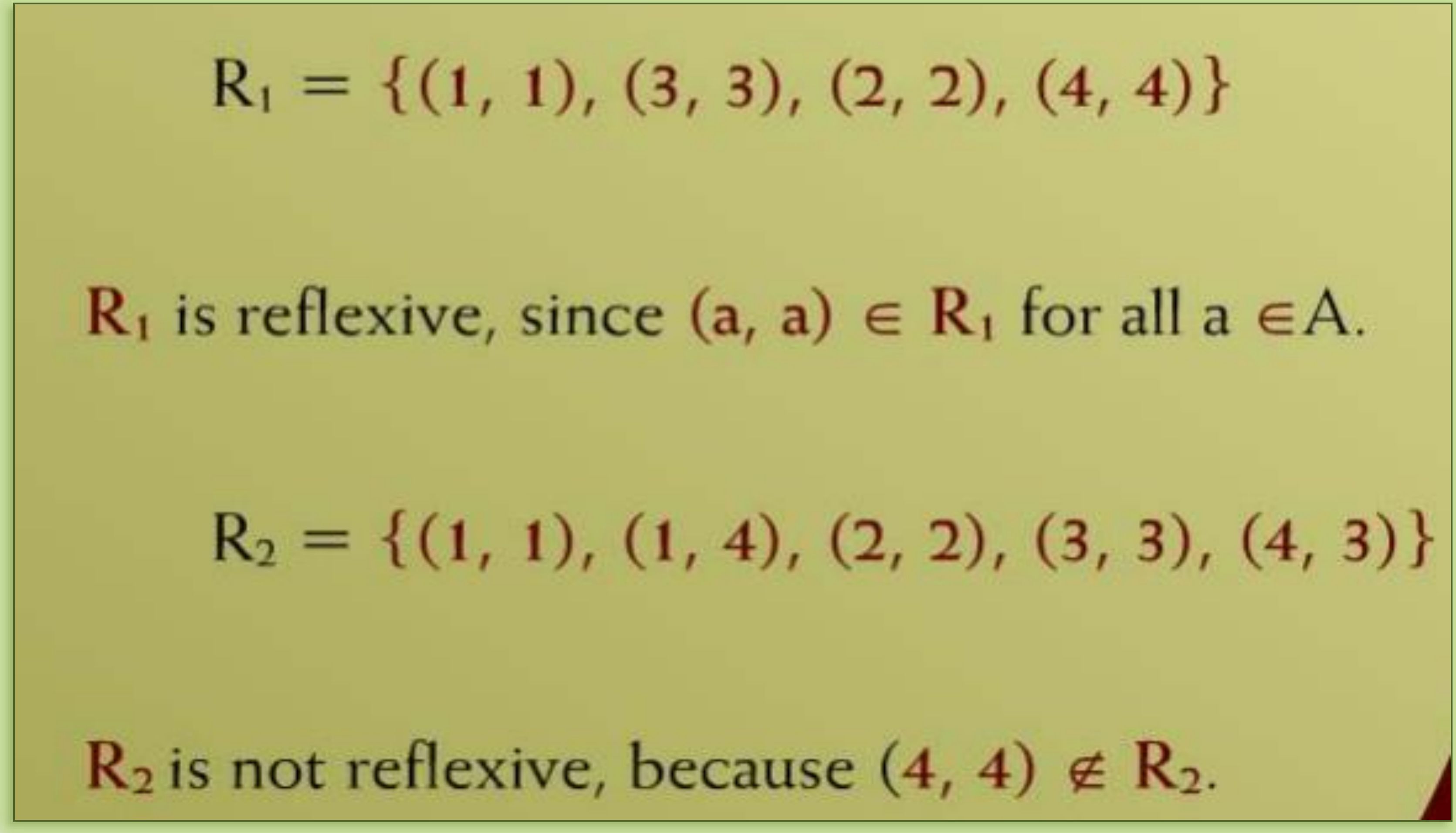
$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

EXAMPLE



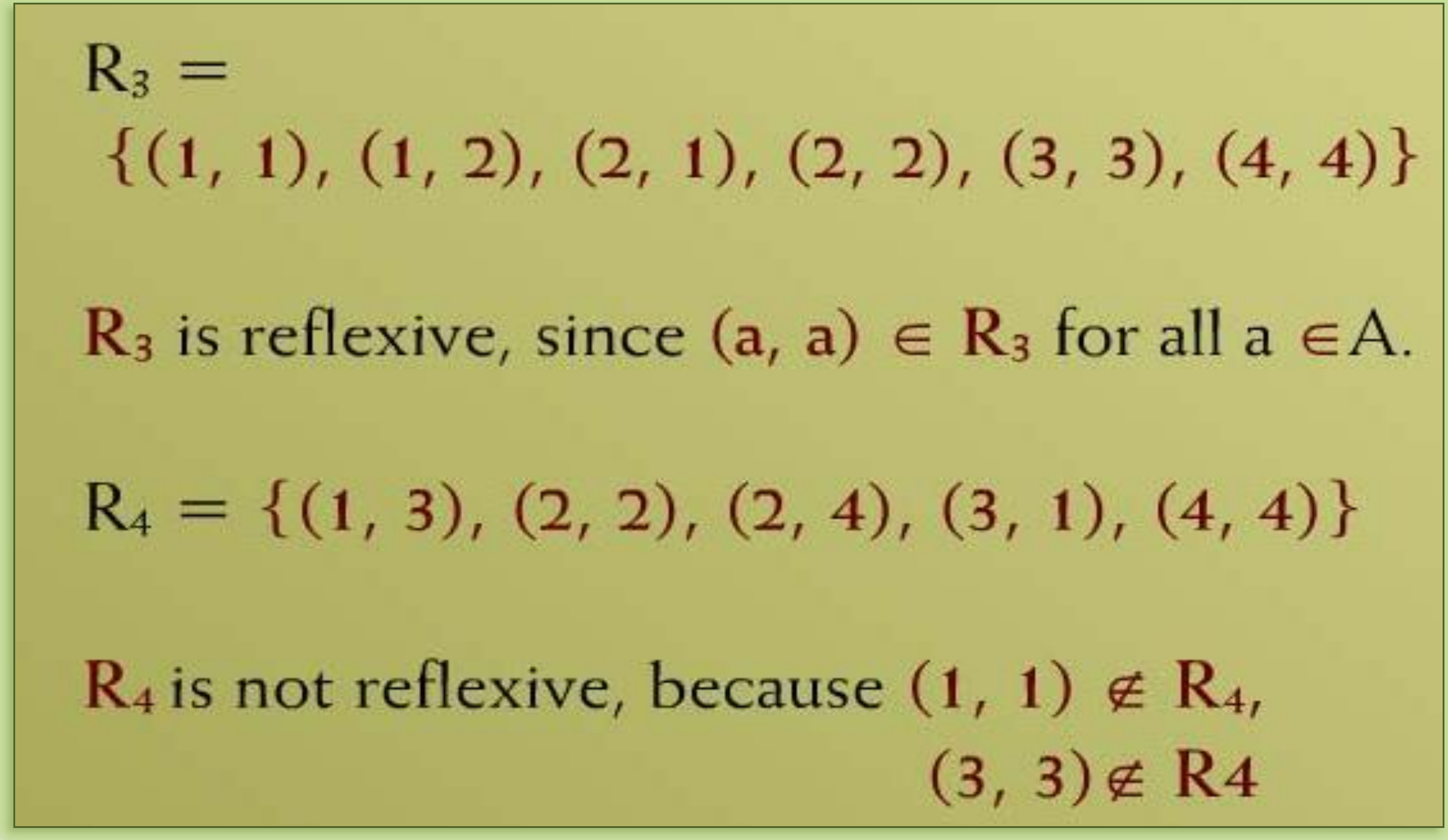
$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

R_2 is not reflexive, because $(4, 4) \notin R_2$.

EXAMPLE



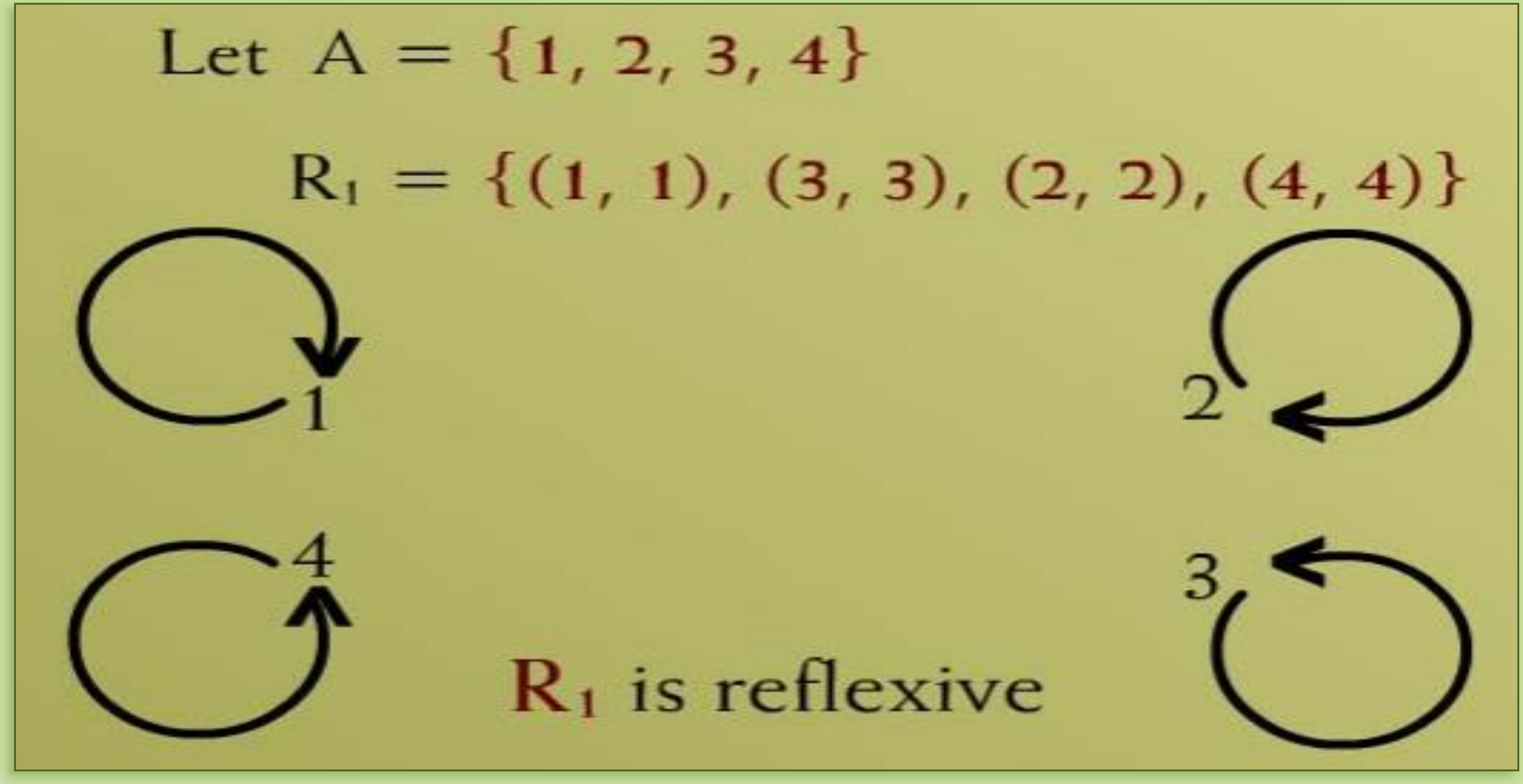
$R_3 =$
$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

R_4 is not reflexive, because $(1, 1) \notin R_4$,
$(3, 3) \notin R4$

DIRECTED GRAPH OF A REFLEXIVE RELATION

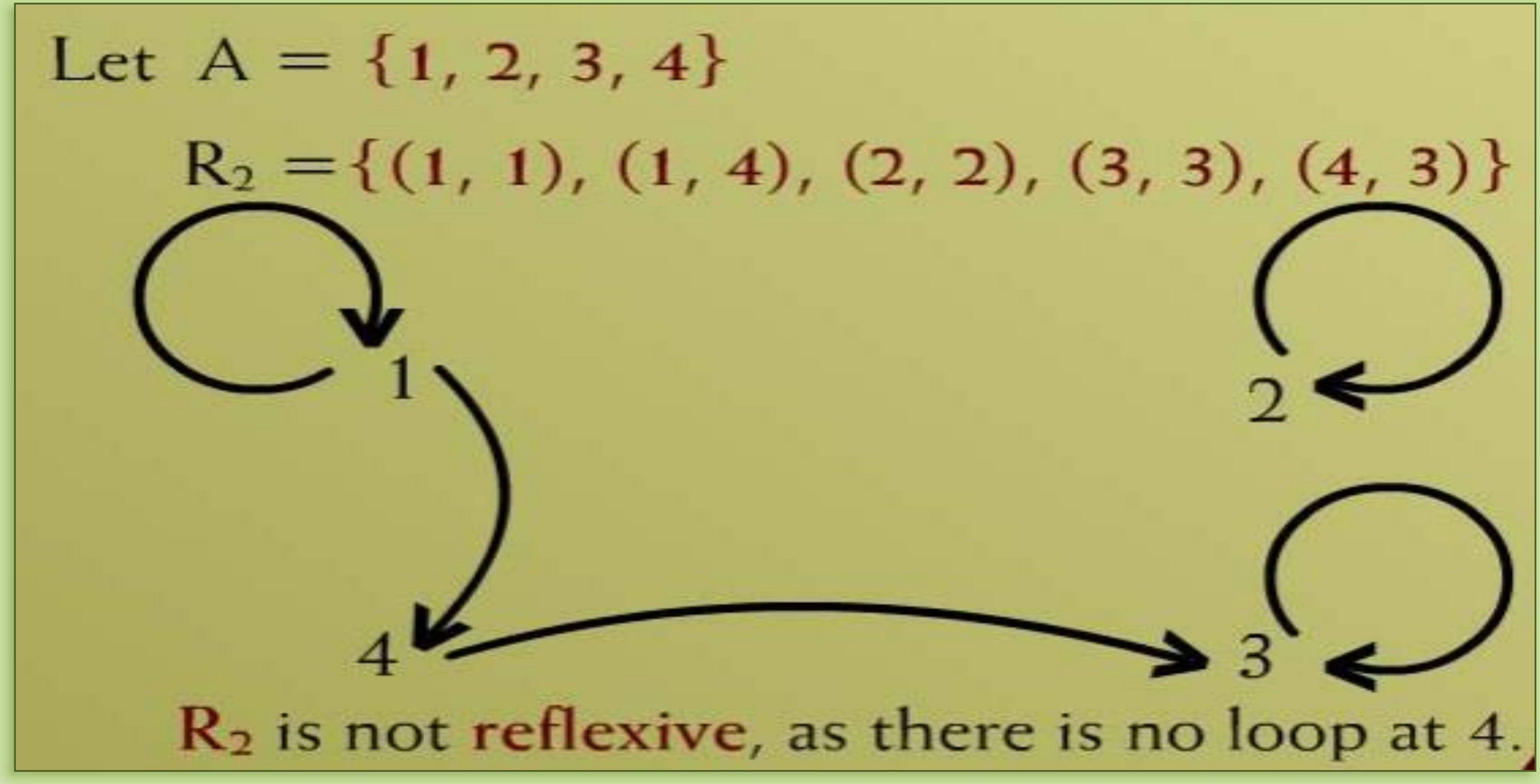


Let A = $\{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

R_1 is reflexive

DIRECTED GRAPH OF A REFLEXIVE RELATION

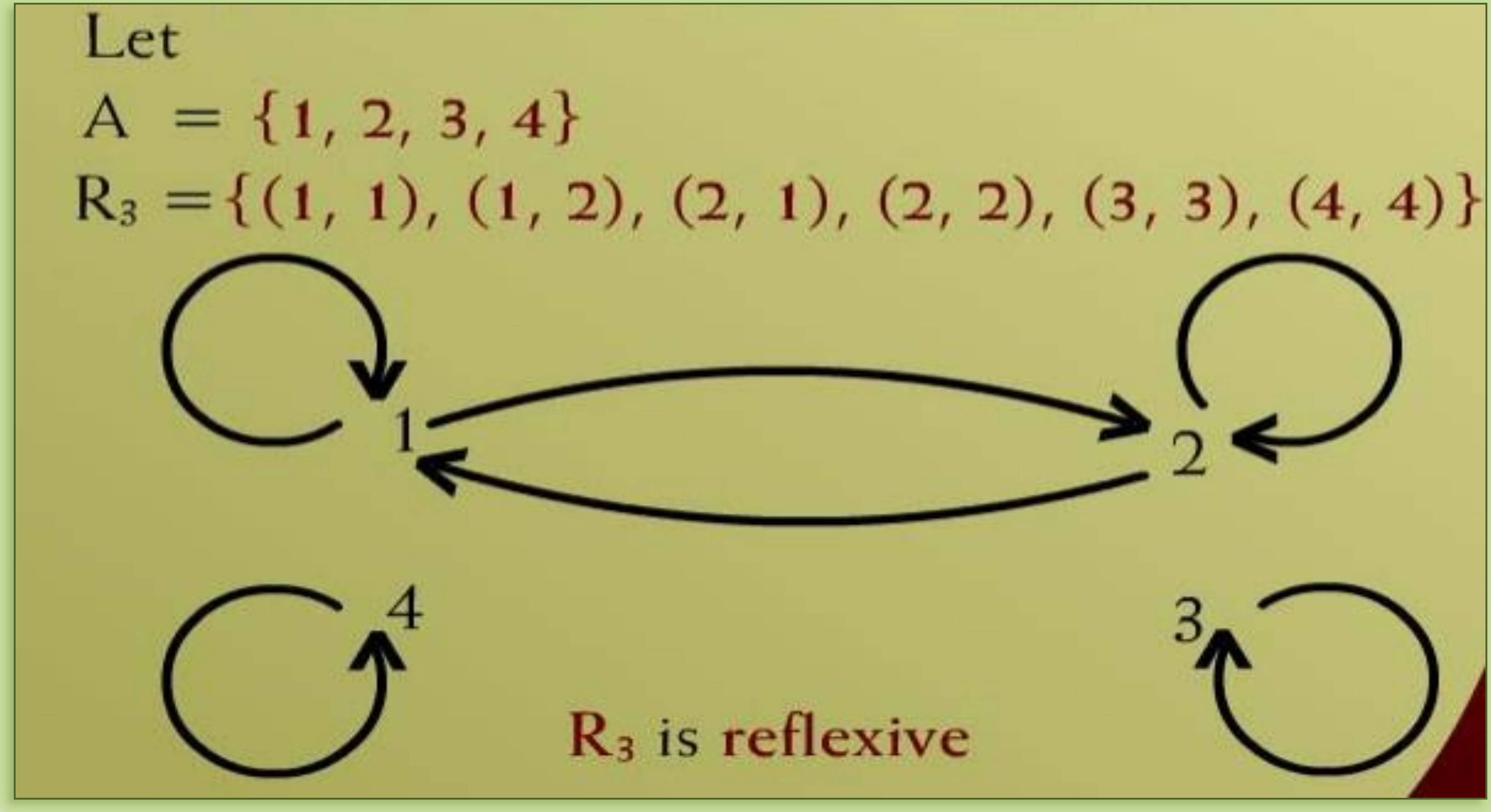


Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

R_2 is not **reflexive**, as there is no loop at 4.

DIRECTED GRAPH OF A REFLEXIVE RELATION



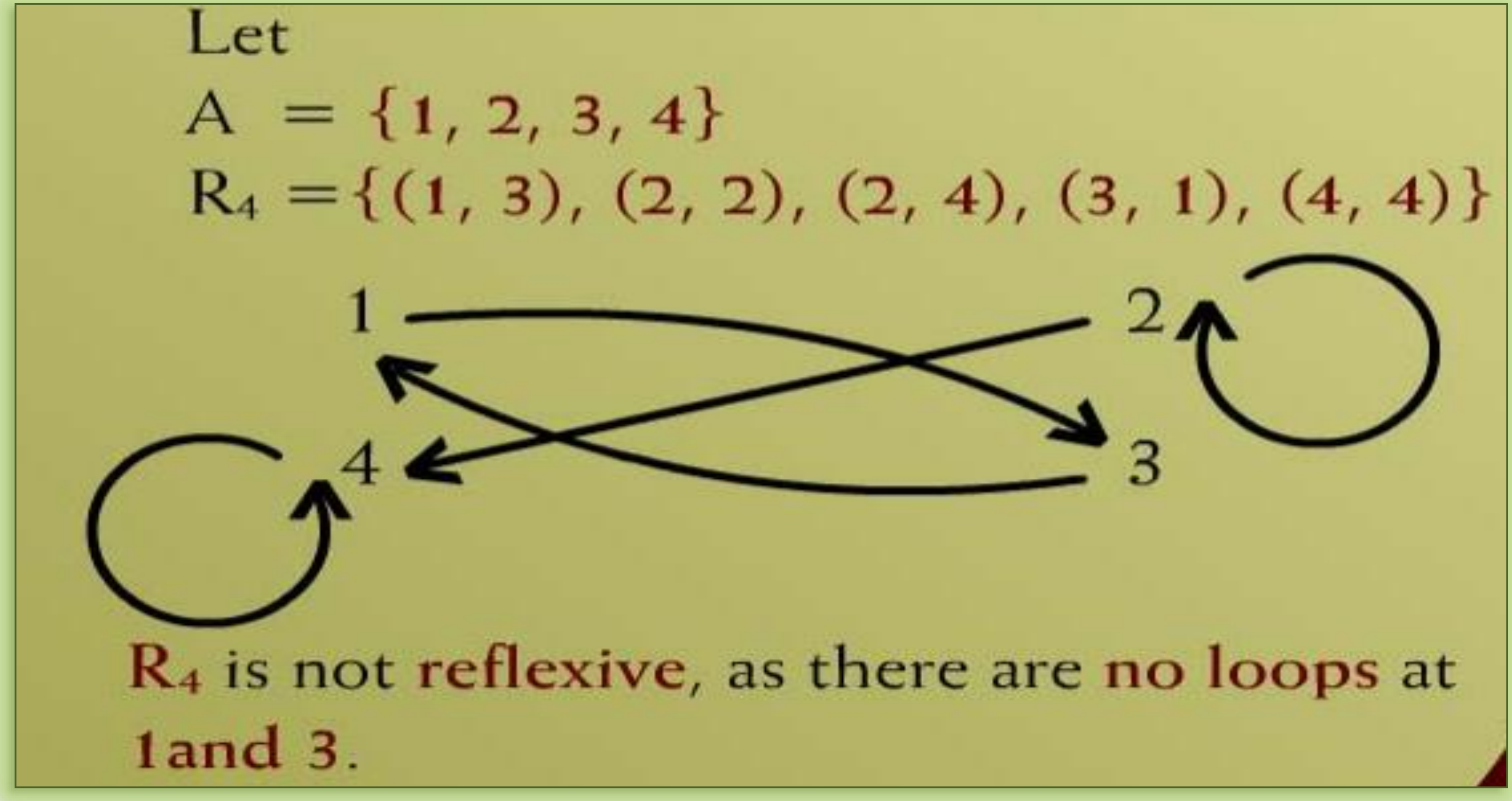
Let

$A = \{1, 2, 3, 4\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

R_3 is reflexive

DIRECTED GRAPH OF A REFLEXIVE RELATION



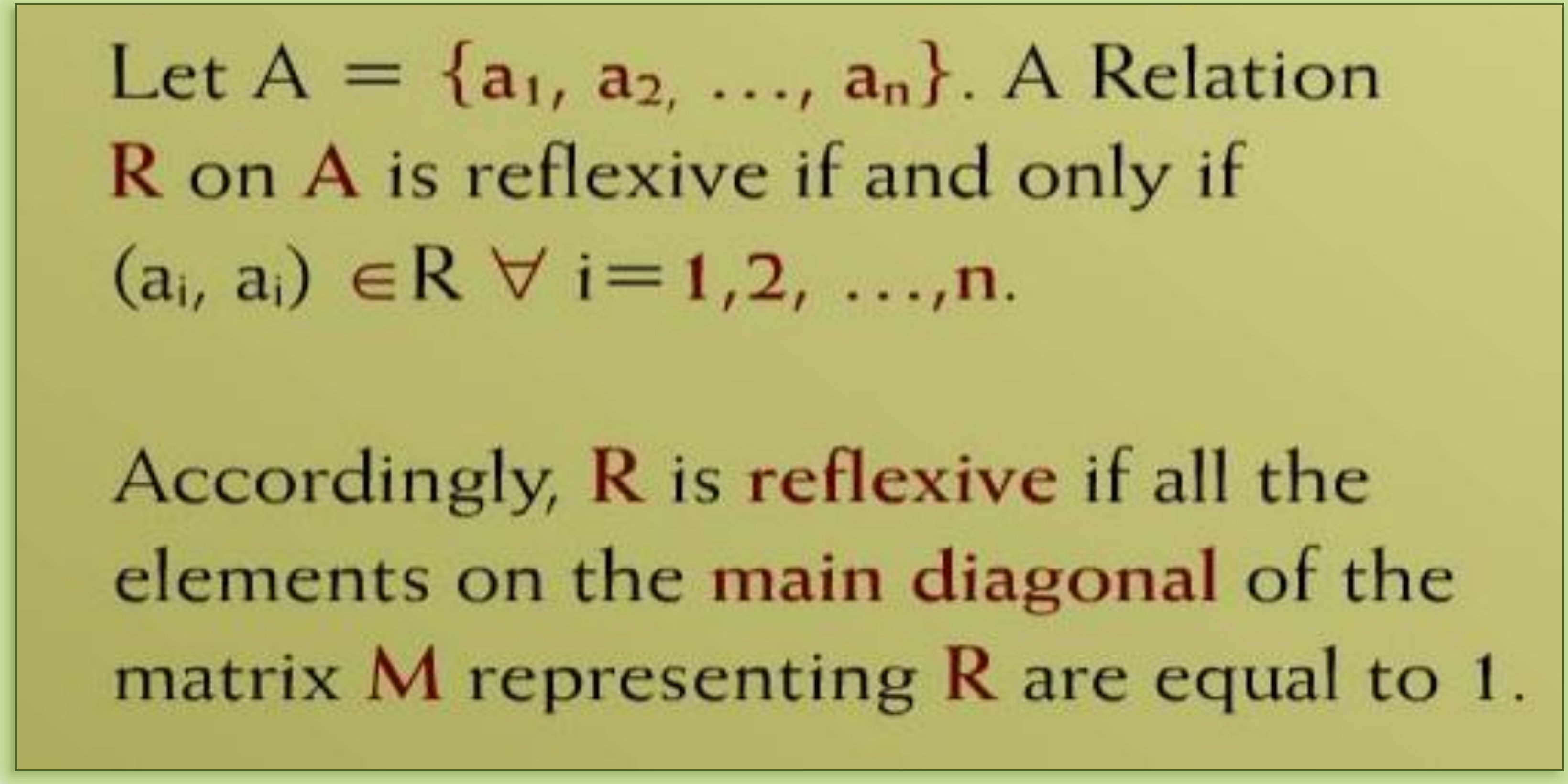
Let

$A = \{1, 2, 3, 4\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

R_4 is not **reflexive**, as there are **no loops** at 1and 3.

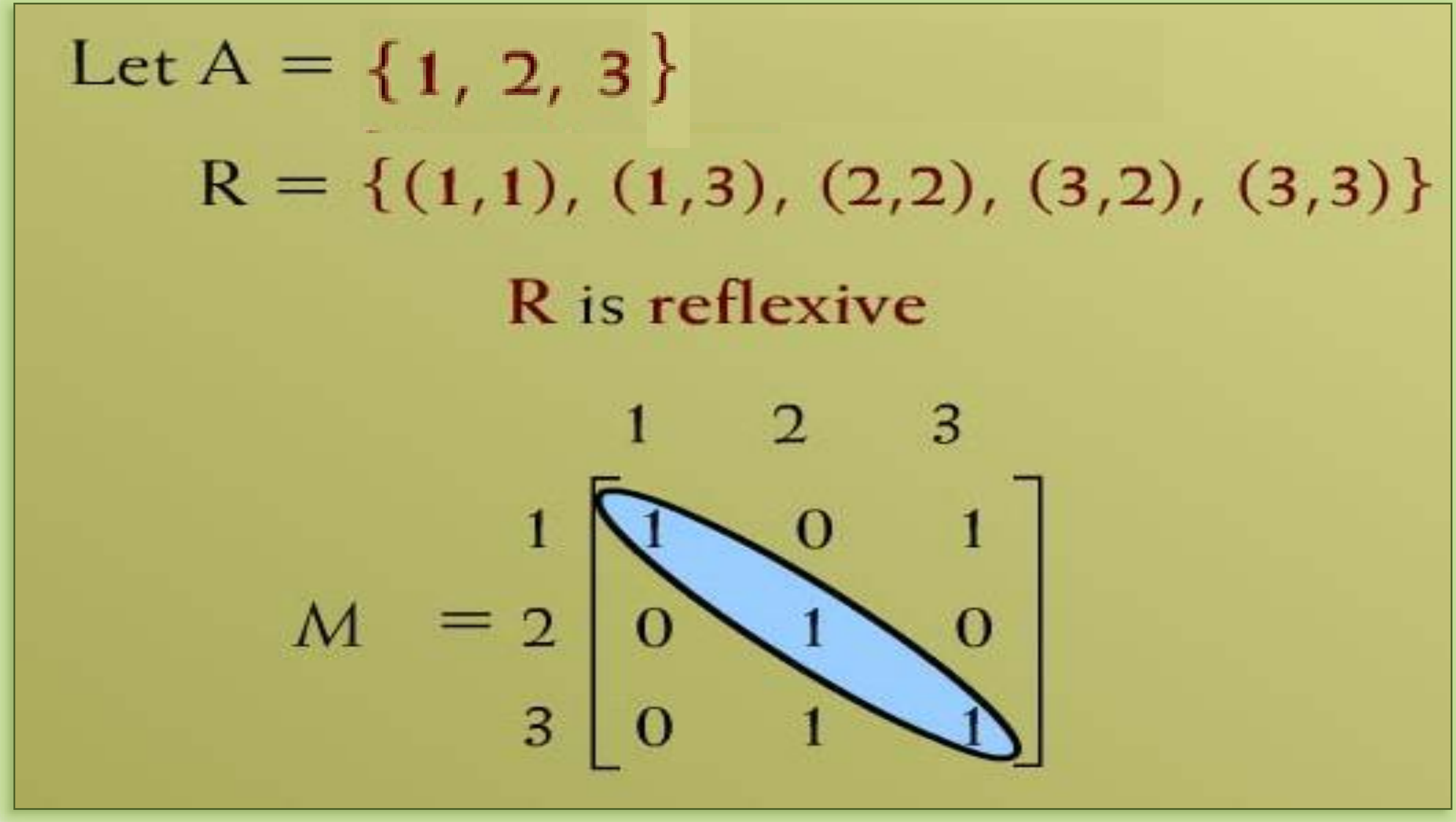
MATRIX REPRESENTATION OF A REFLEXIVE RELATION



Let $A = \{a_1, a_2, \ldots, a_n\}$. A Relation **R** on **A** is reflexive if and only if $(a_i, a_i) \in R \ \forall \ i = 1, 2, \ldots, n$.

Accordingly, **R** is **reflexive** if all the elements on the **main diagonal** of the matrix **M** representing **R** are equal to 1.

EXAMPLE



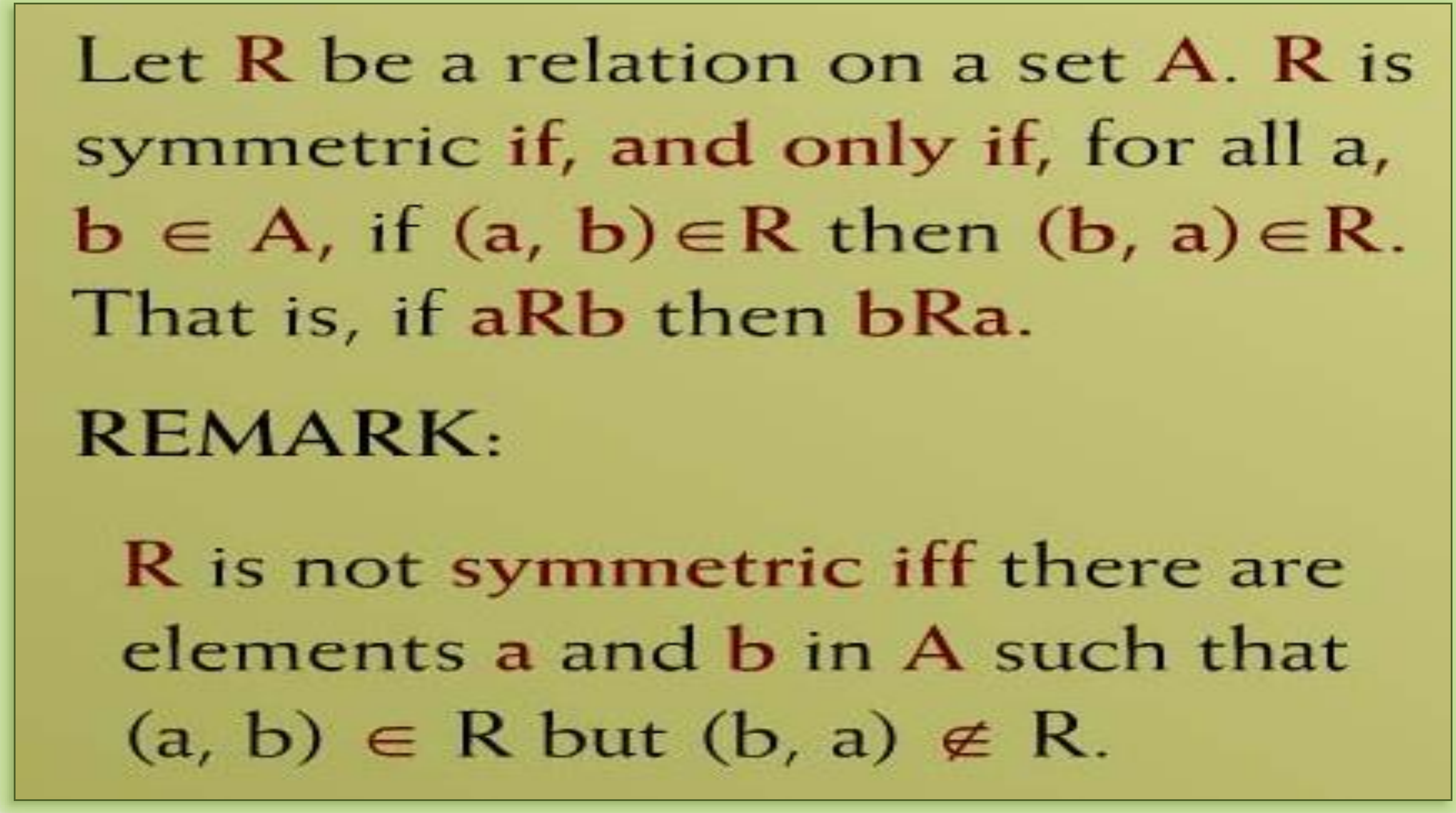
Let A = {1, 2, 3}

R = {(1,1), (1,3), (2,2), (3,2), (3,3)}

R is reflexive

$$M = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ \hline \end{array}$$

SYMMETRIC RELATION

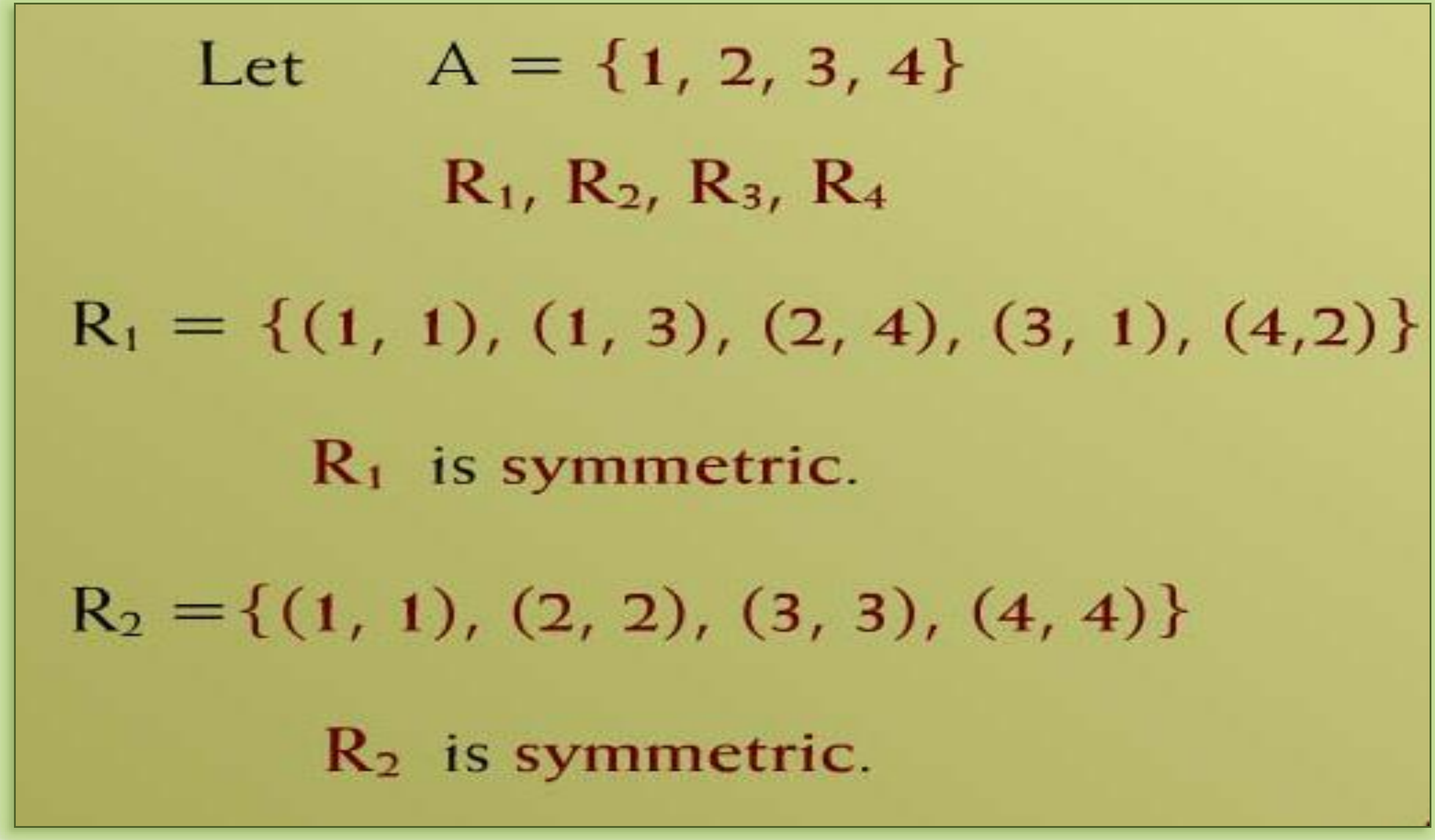


Let R be a relation on a set A. R is symmetric **if, and only if,** for all a, b $\in$ A, if $(a, b) \in R$ then $(b, a) \in R$. That is, if aRb then bRa.

REMARK:

R is not symmetric iff there are elements a and b in A such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE



Let $A = \{1, 2, 3, 4\}$

R_1, R_2, R_3, R_4

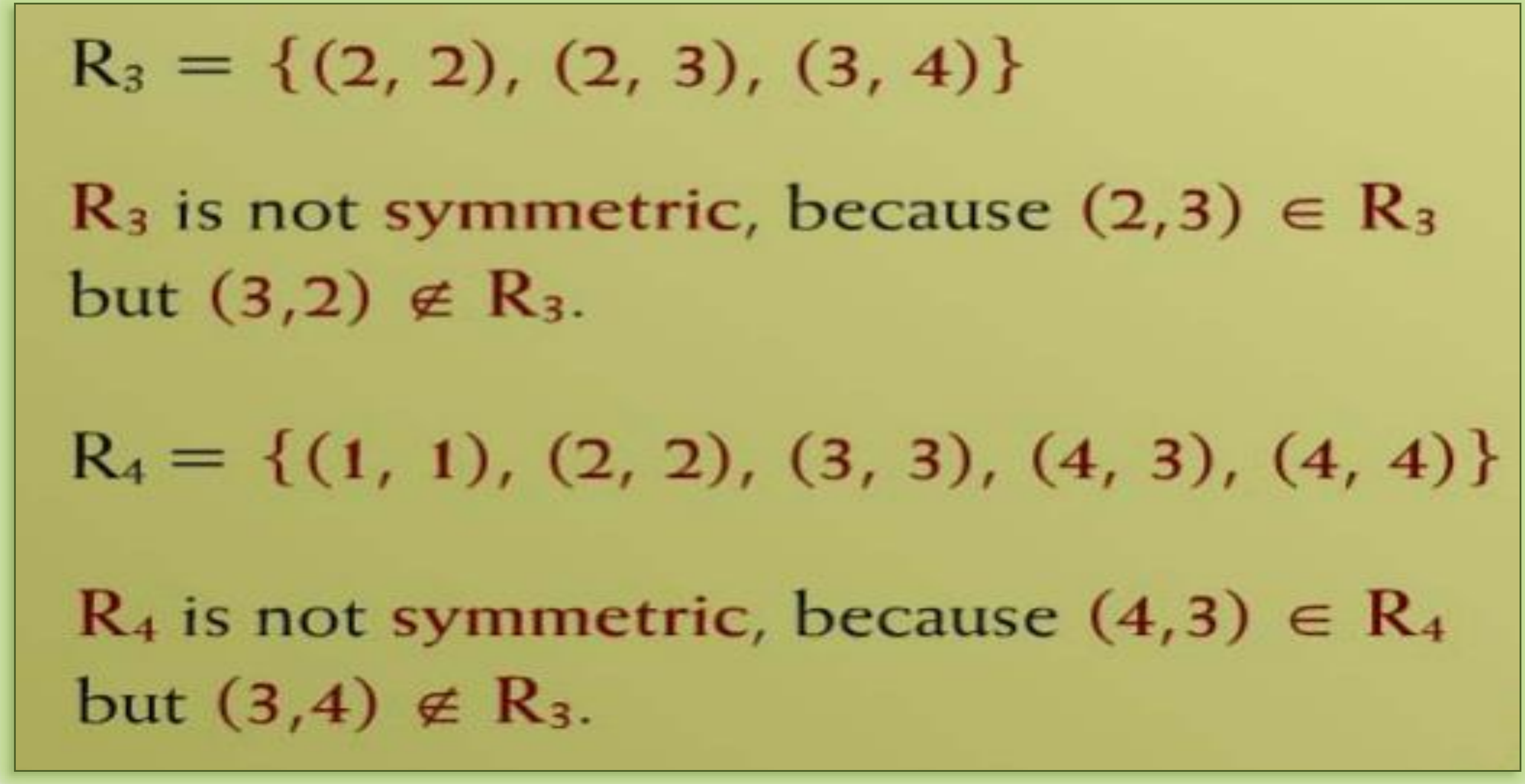
$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$

R_1 is symmetric.

$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

R_2 is symmetric.

EXAMPLE



$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

R_3 is not symmetric, because $(2,3) \in R_3$ but $(3,2) \notin R_3$.

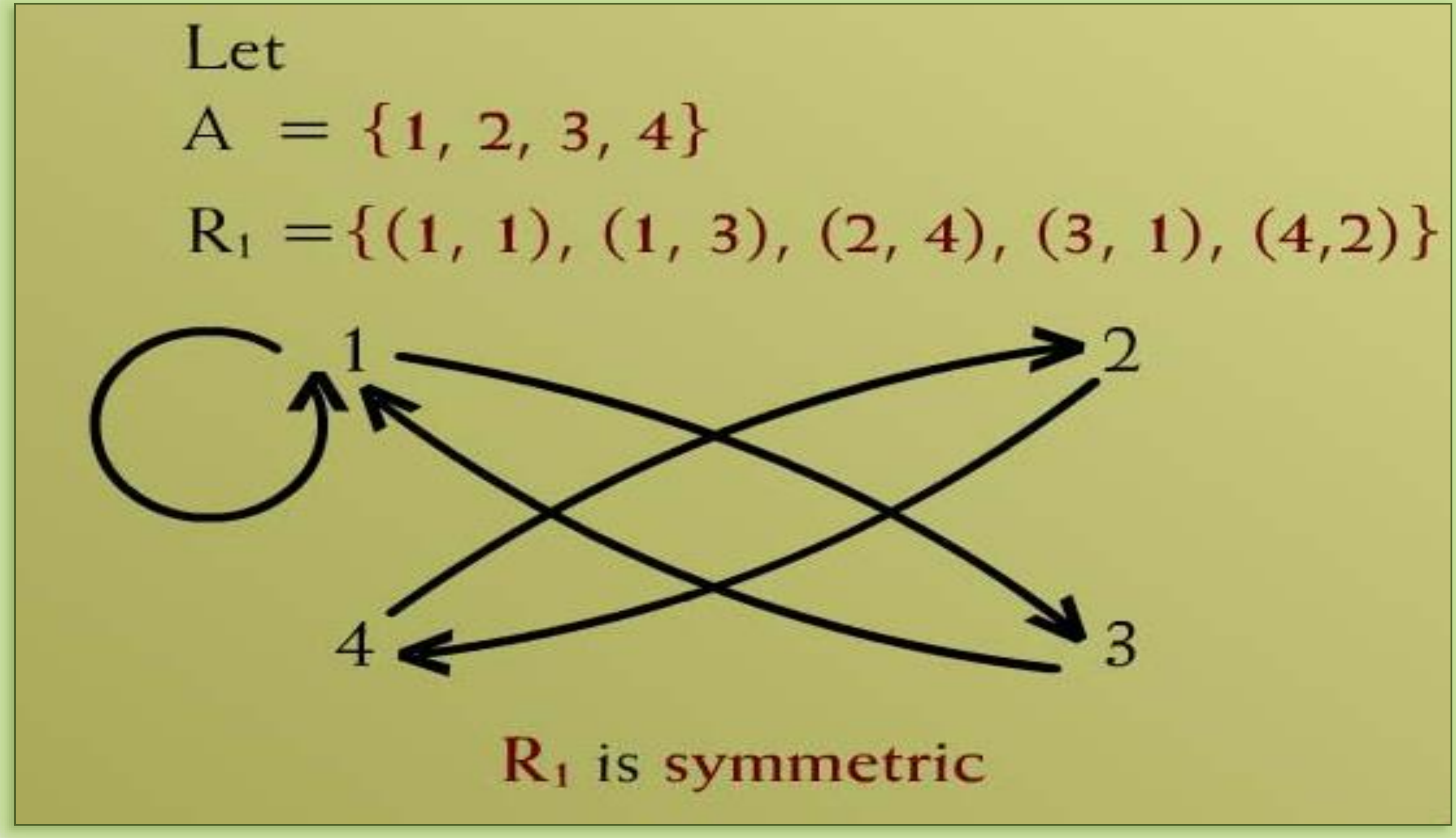
$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

R_4 is not symmetric, because $(4,3) \in R_4$ but $(3,4) \notin R_3$.

DIRECTED GRAPH OF A SYMMETRIC RELATION

For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

EXAMPLE



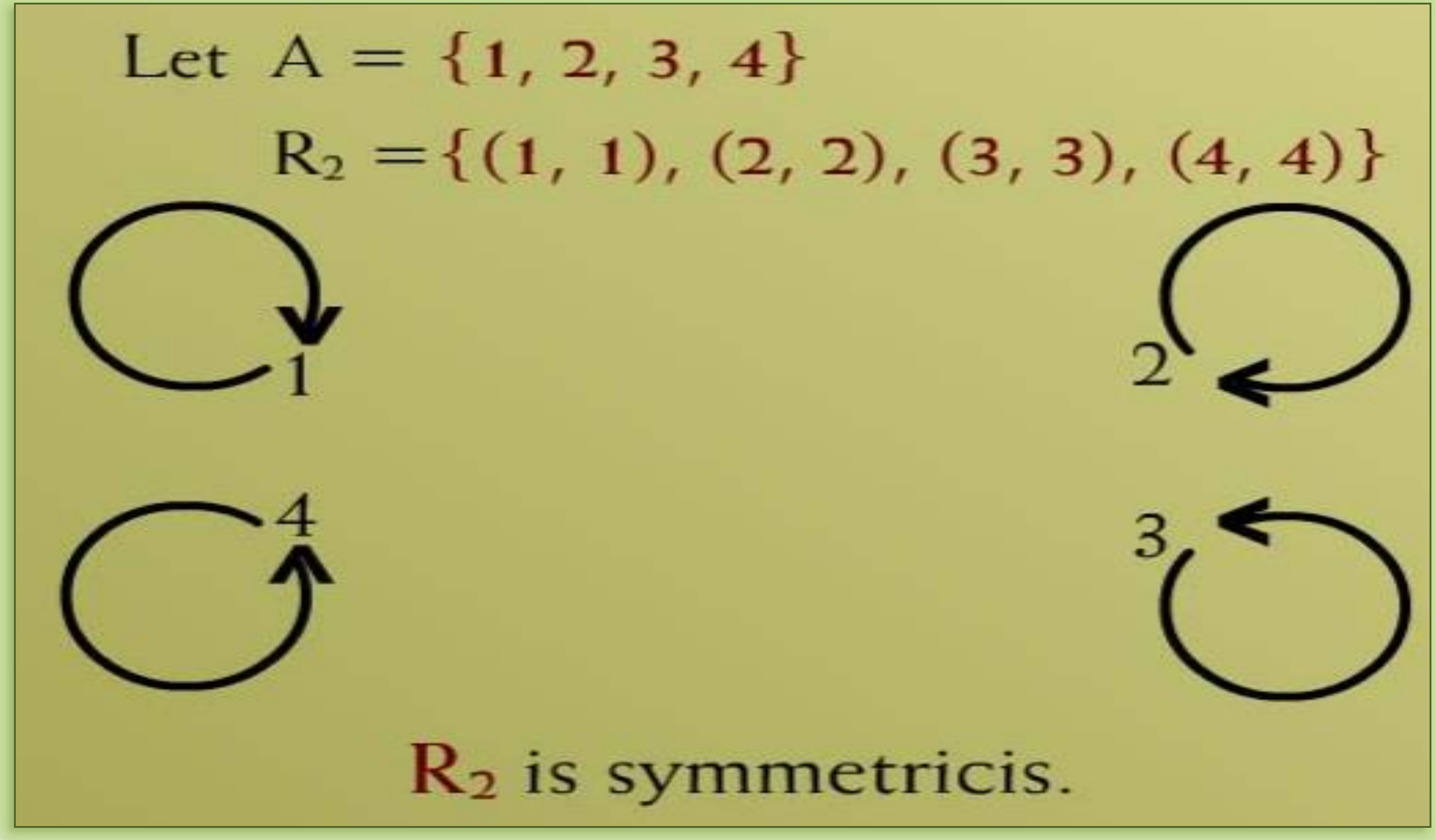
Let

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$

R_1 is symmetric

EXAMPLE

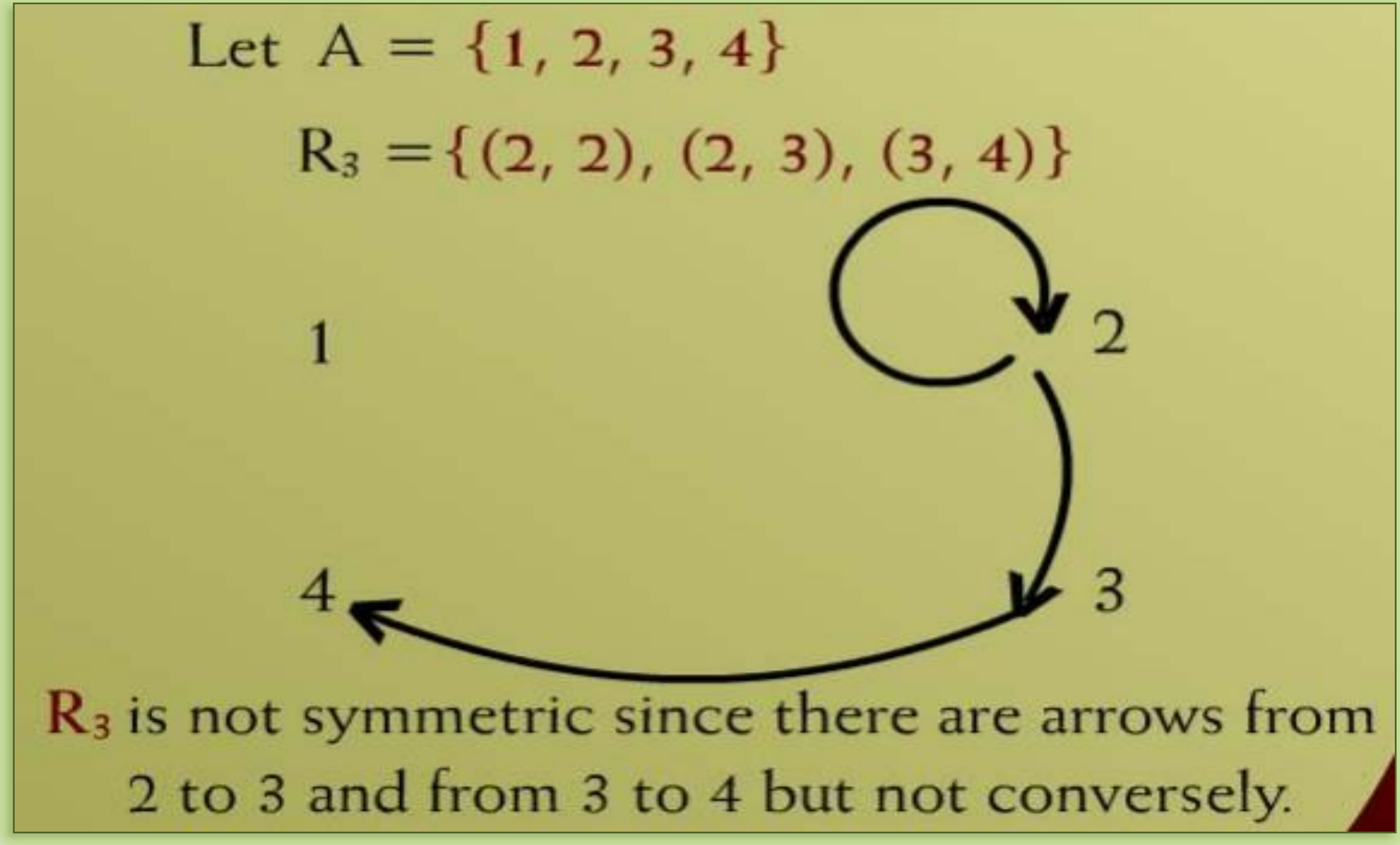


Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

R_2 is symmetricis.

EXAMPLE

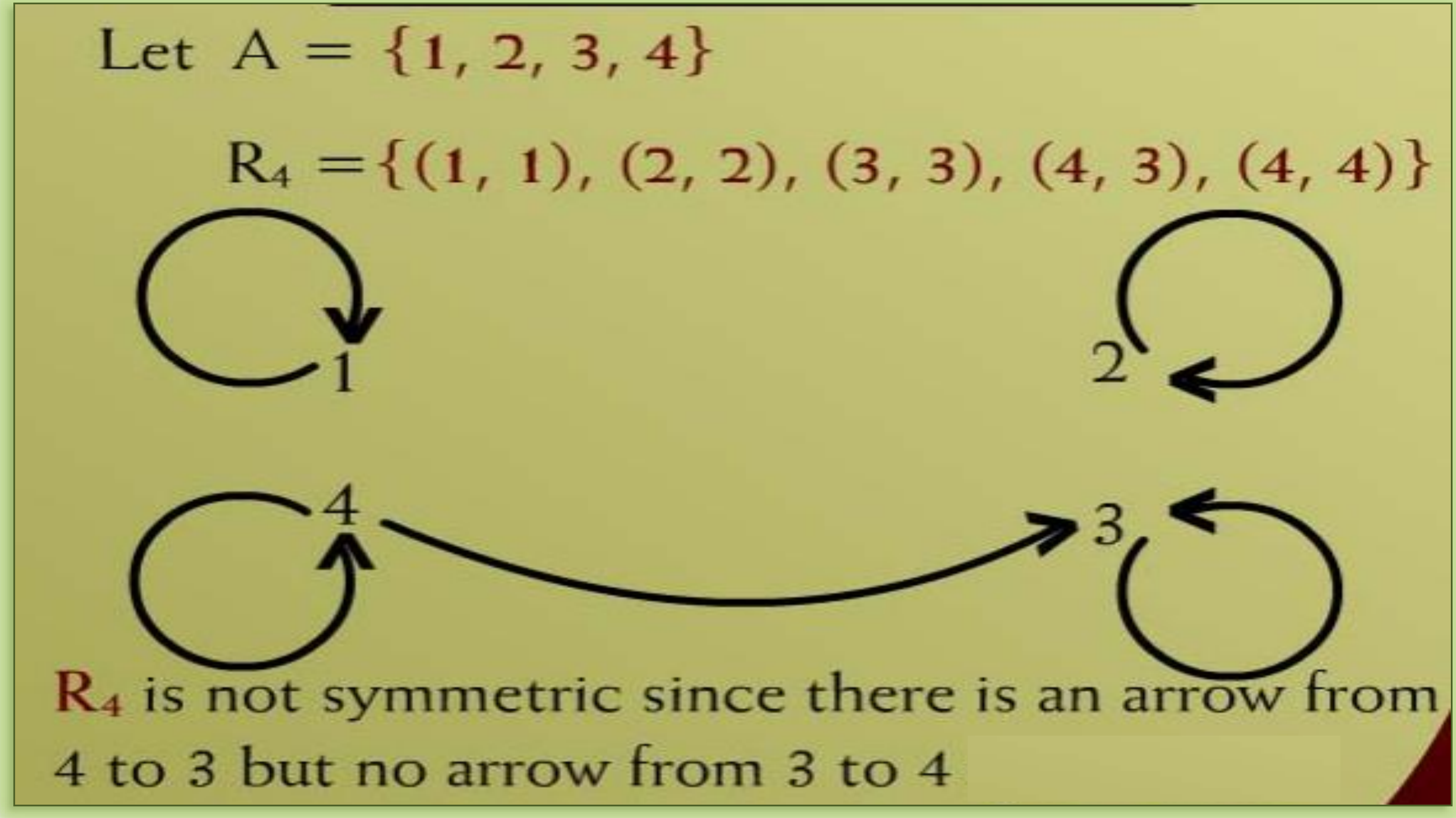


Let $A = \{1, 2, 3, 4\}$

$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

R_3 is not symmetric since there are arrows from 2 to 3 and from 3 to 4 but not conversely.

EXAMPLE

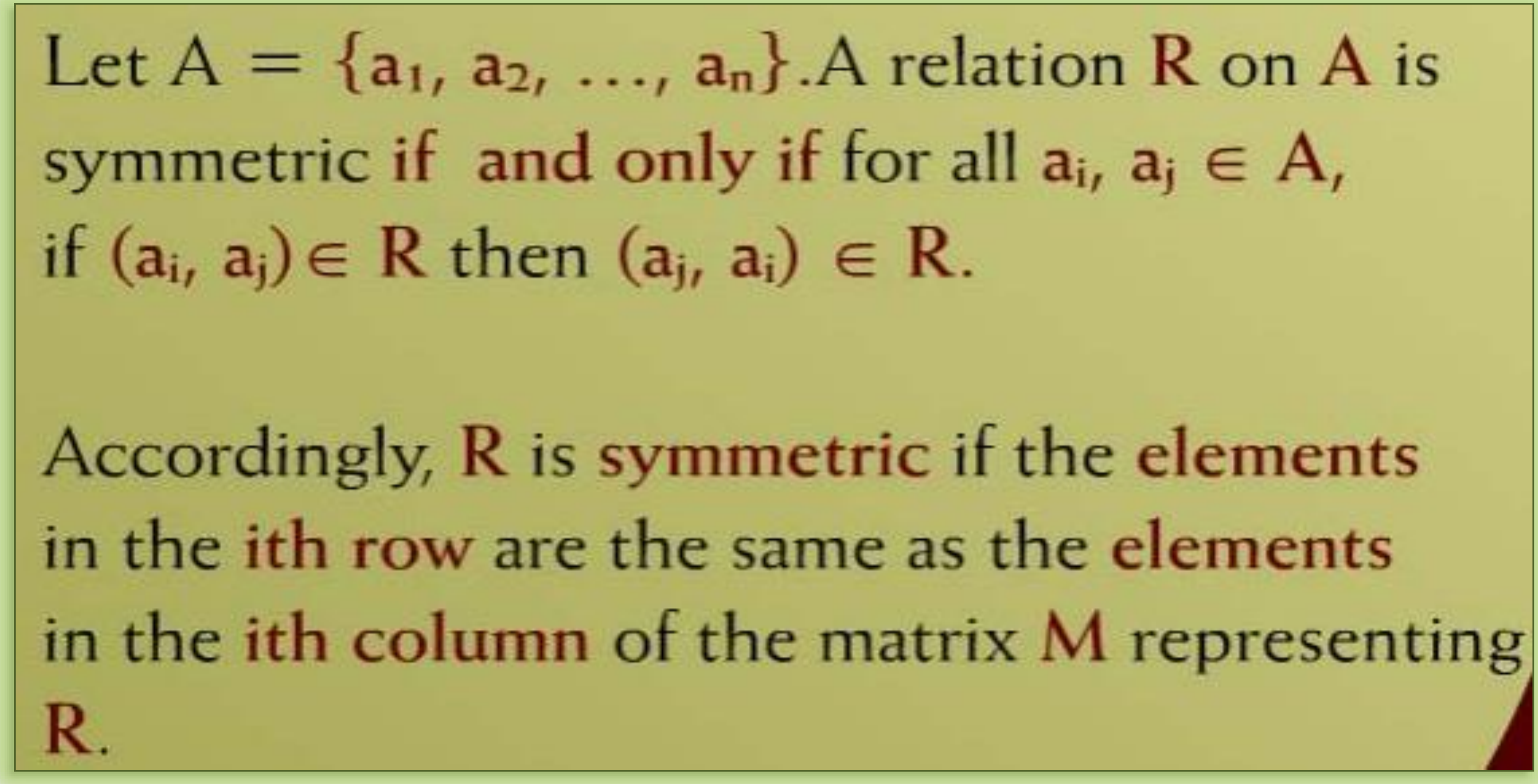


Let A = {1, 2, 3, 4}

R_4 = {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}

R_4 is not symmetric since there is an arrow from 4 to 3 but no arrow from 3 to 4

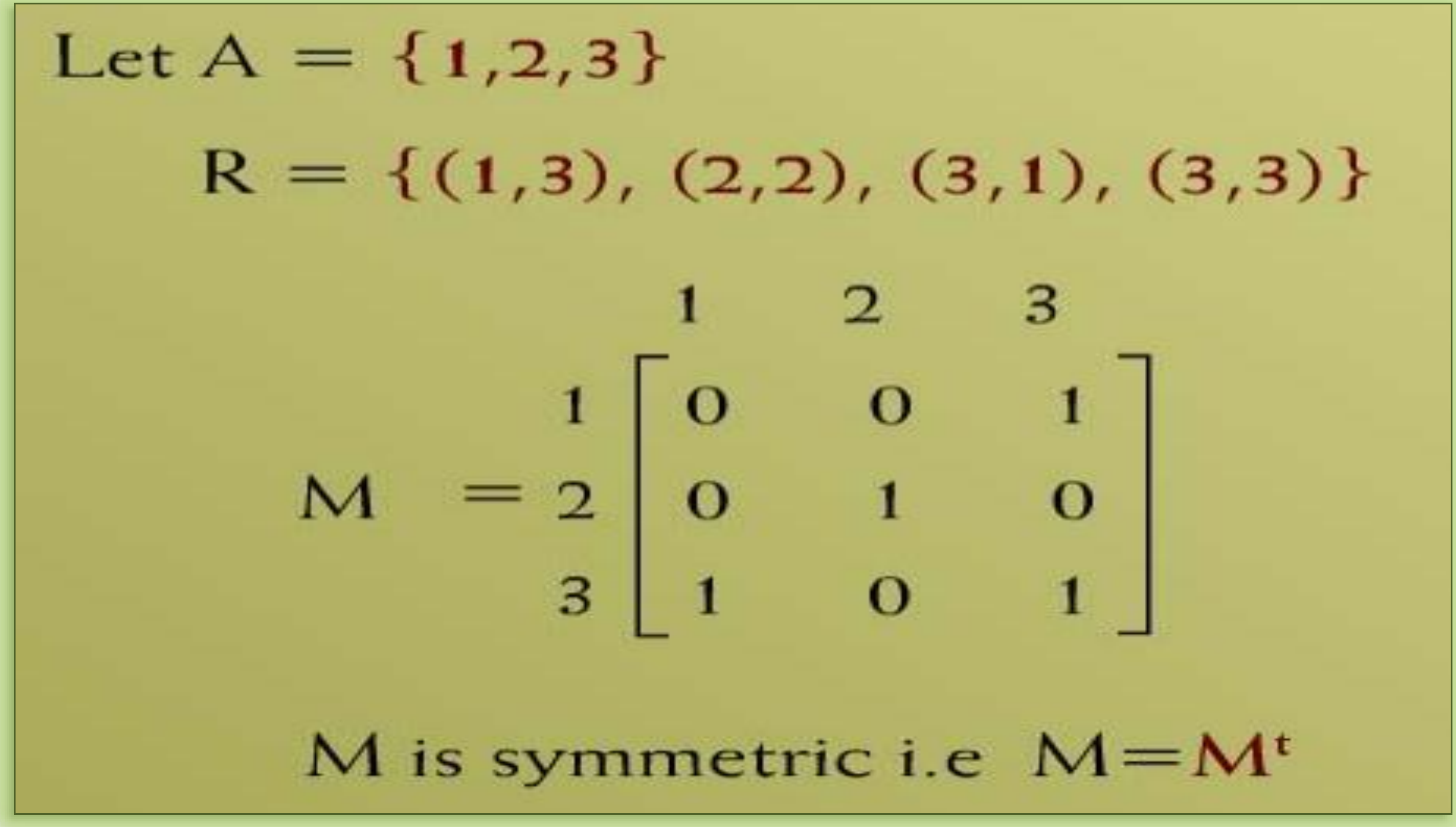
MATRIX REPRESENTATION OF A SYMMETRIC RELATION



Let $A = \{a_1, a_2, \ldots, a_n\}$. A relation R on A is symmetric **if and only if** for all $a_i, a_j \in A$, if $(a_i, a_j) \in R$ then $(a_j, a_i) \in R$.

Accordingly, R is **symmetric** if the **elements** in the **ith row** are the same as the **elements** in the **ith column** of the matrix M representing R.

EXAMPLE



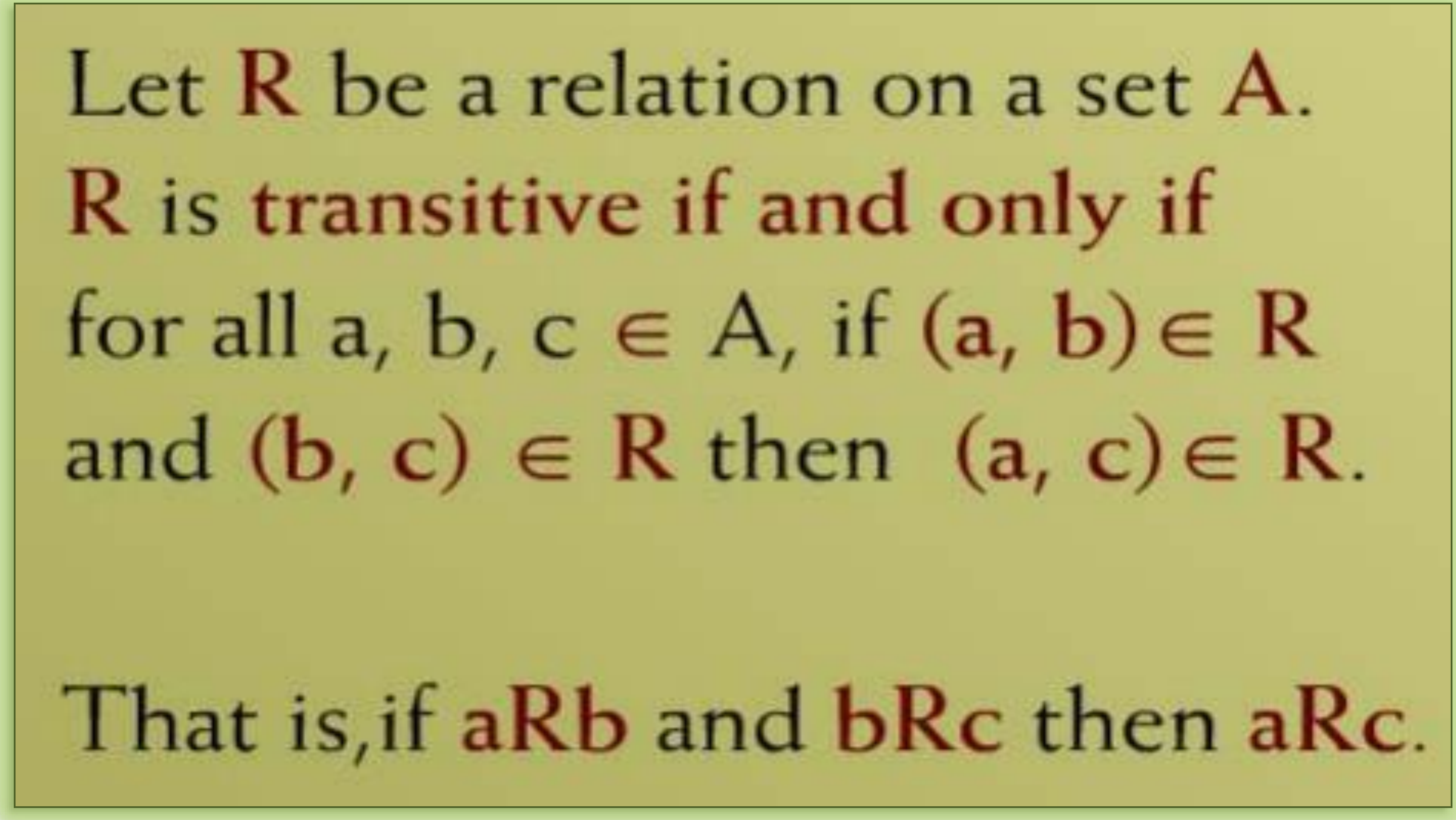
Let $A = \{1,2,3\}$

$R = \{(1,3), (2,2), (3,1), (3,3)\}$

$$M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}$$

M is symmetric i.e $M = M^t$

Transitive Relation



Let R be a relation on a set A.

R is **transitive if and only if**

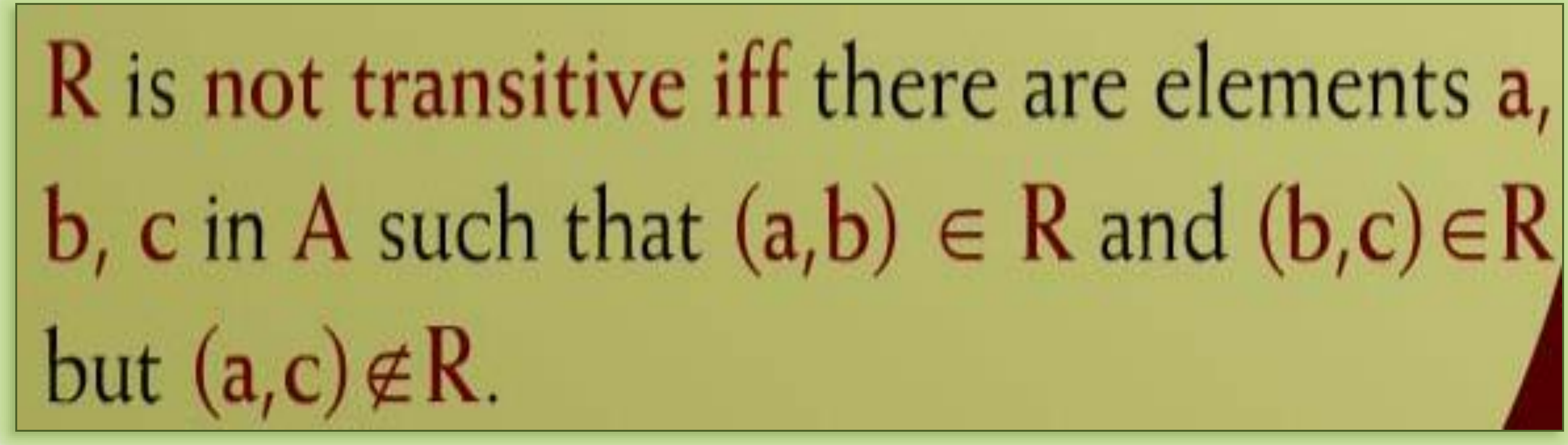
for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

That is, if aRb and bRc then aRc.

EXAMPLE

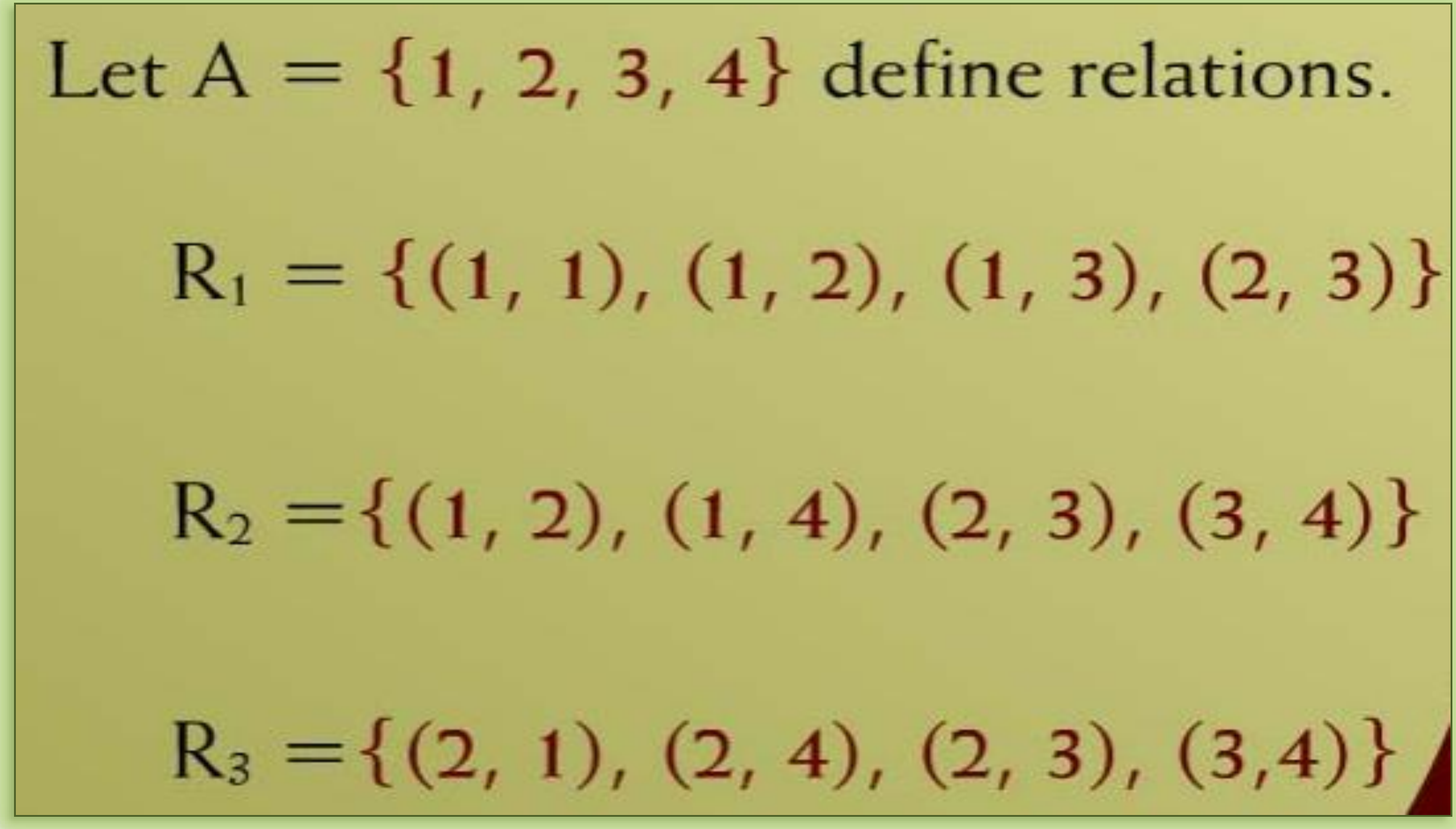
In words, if any one element is related to a second and that second element is related to a third, then the first is related to the third.

REMARKS:



R is not transitive iff there are elements a, b, c in A such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

EXAMPLE



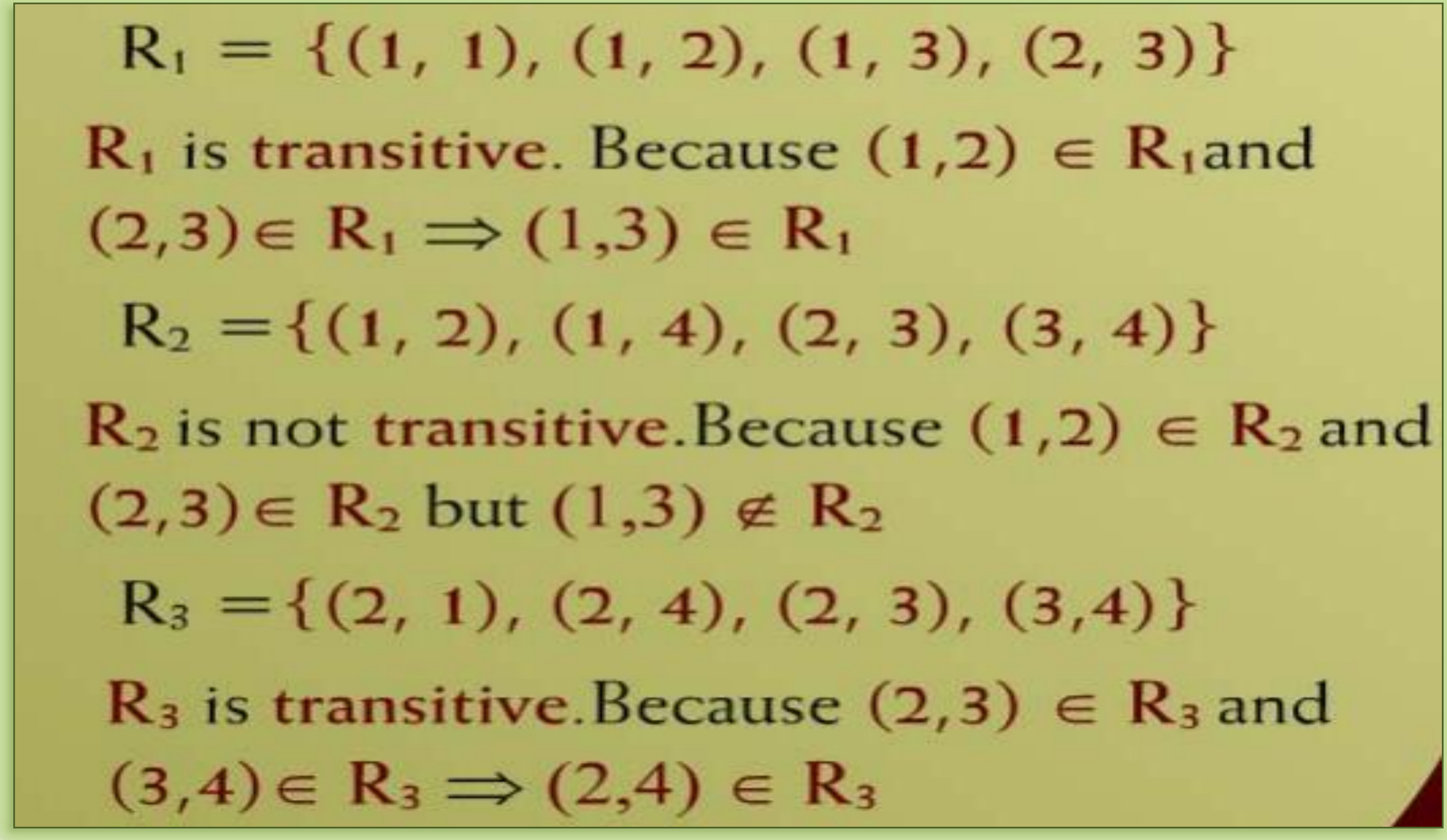
Let $A = \{1, 2, 3, 4\}$ define relations.

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

EXAMPLE



$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

R_1 is transitive. Because $(1,2) \in R_1$ and $(2,3) \in R_1 \Rightarrow (1,3) \in R_1$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

R_2 is not transitive. Because $(1,2) \in R_2$ and $(2,3) \in R_2$ but $(1,3) \notin R_2$

$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

R_3 is transitive. Because $(2,3) \in R_3$ and $(3,4) \in R_3 \Rightarrow (2,4) \in R_3$

DIRECTED GRAPH OF A TRANSITIVE RELATION

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

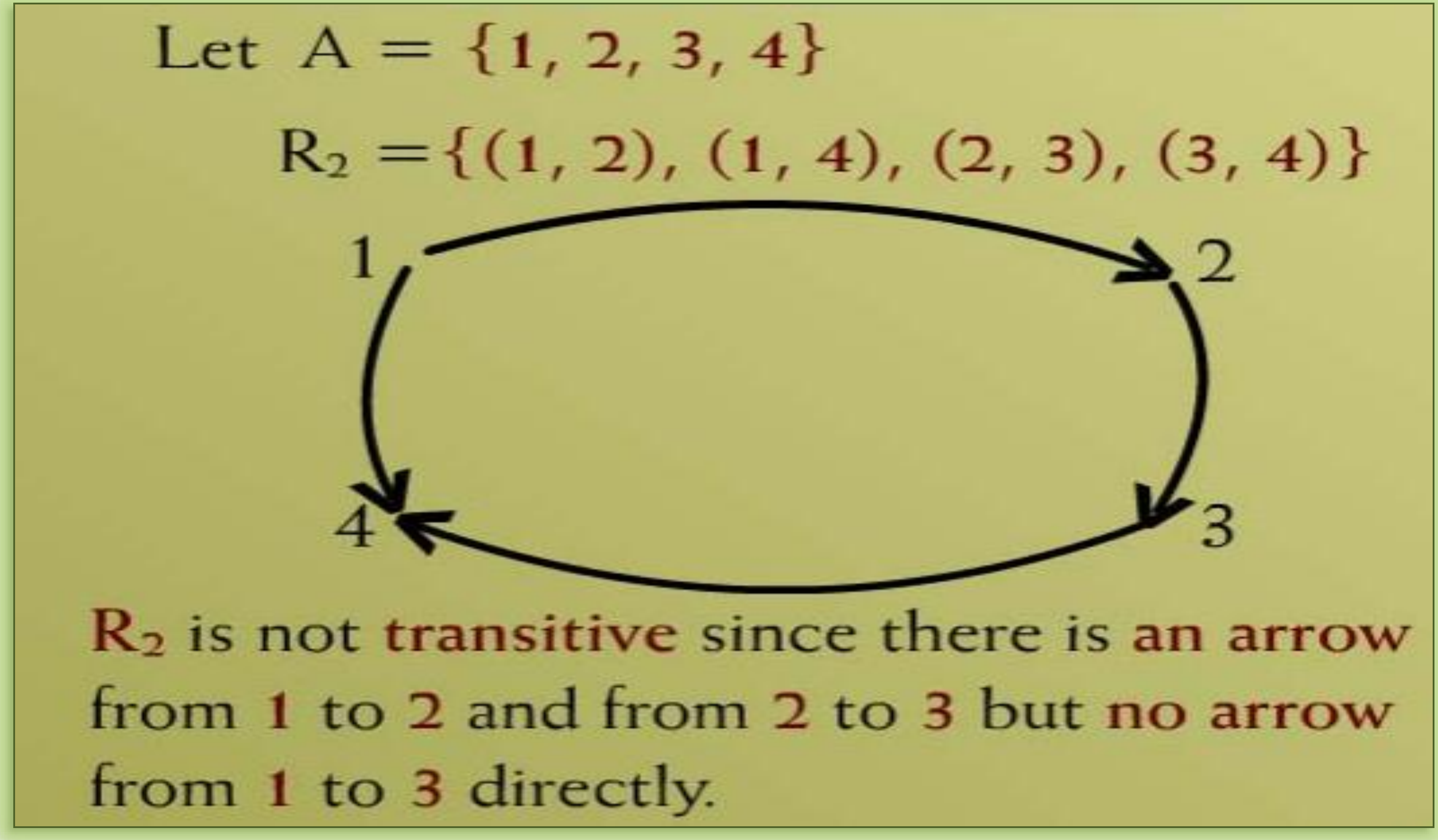
EXAMPLE

Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

R_1 is **transitive.**

EXAMPLE

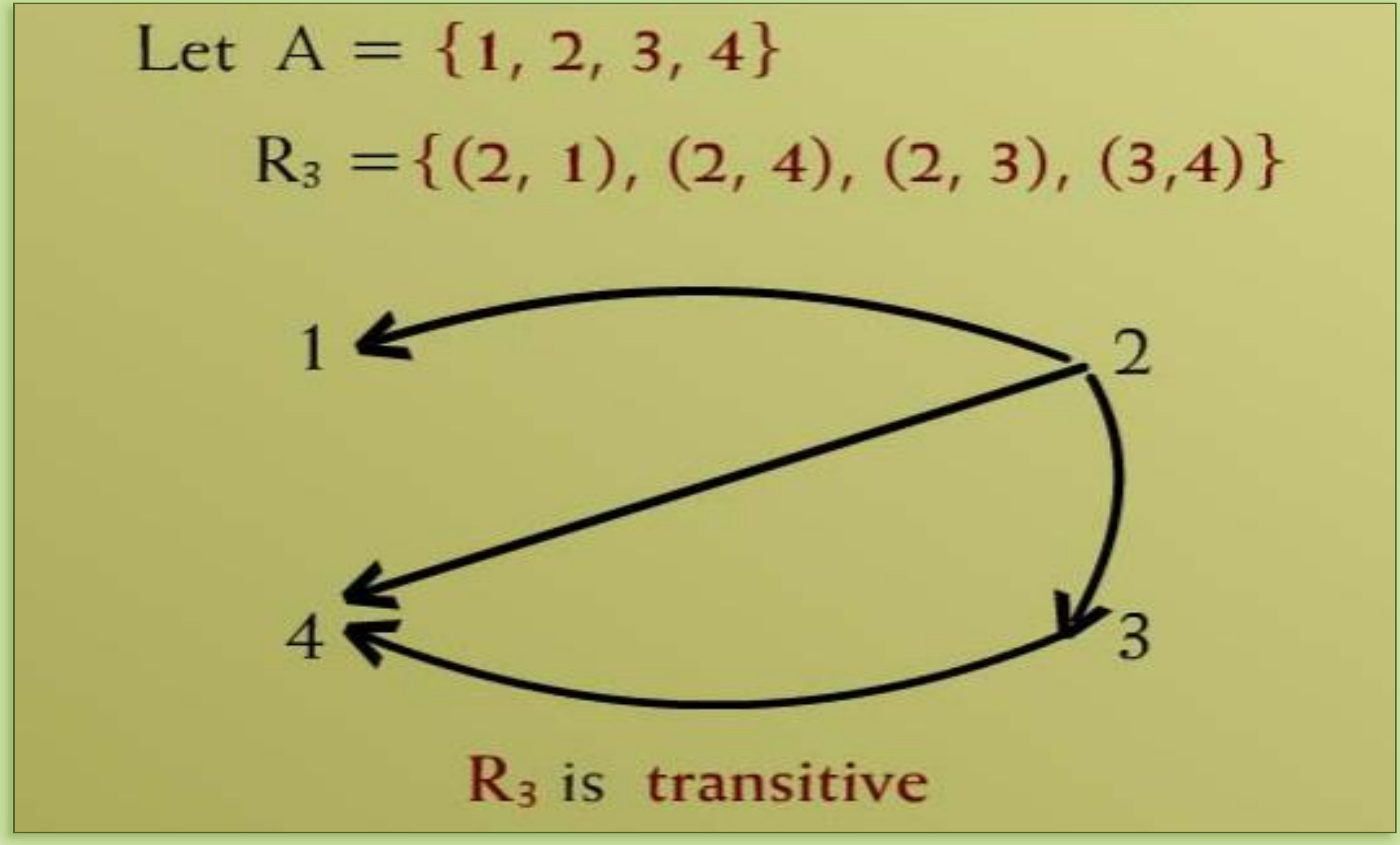


Let $A = \{1, 2, 3, 4\}$

$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

R_2 is not **transitive** since there is **an arrow** from **1** to **2** and from **2** to **3** but **no arrow** from **1** to **3** directly.

EXAMPLE



Let $A = \{1, 2, 3, 4\}$

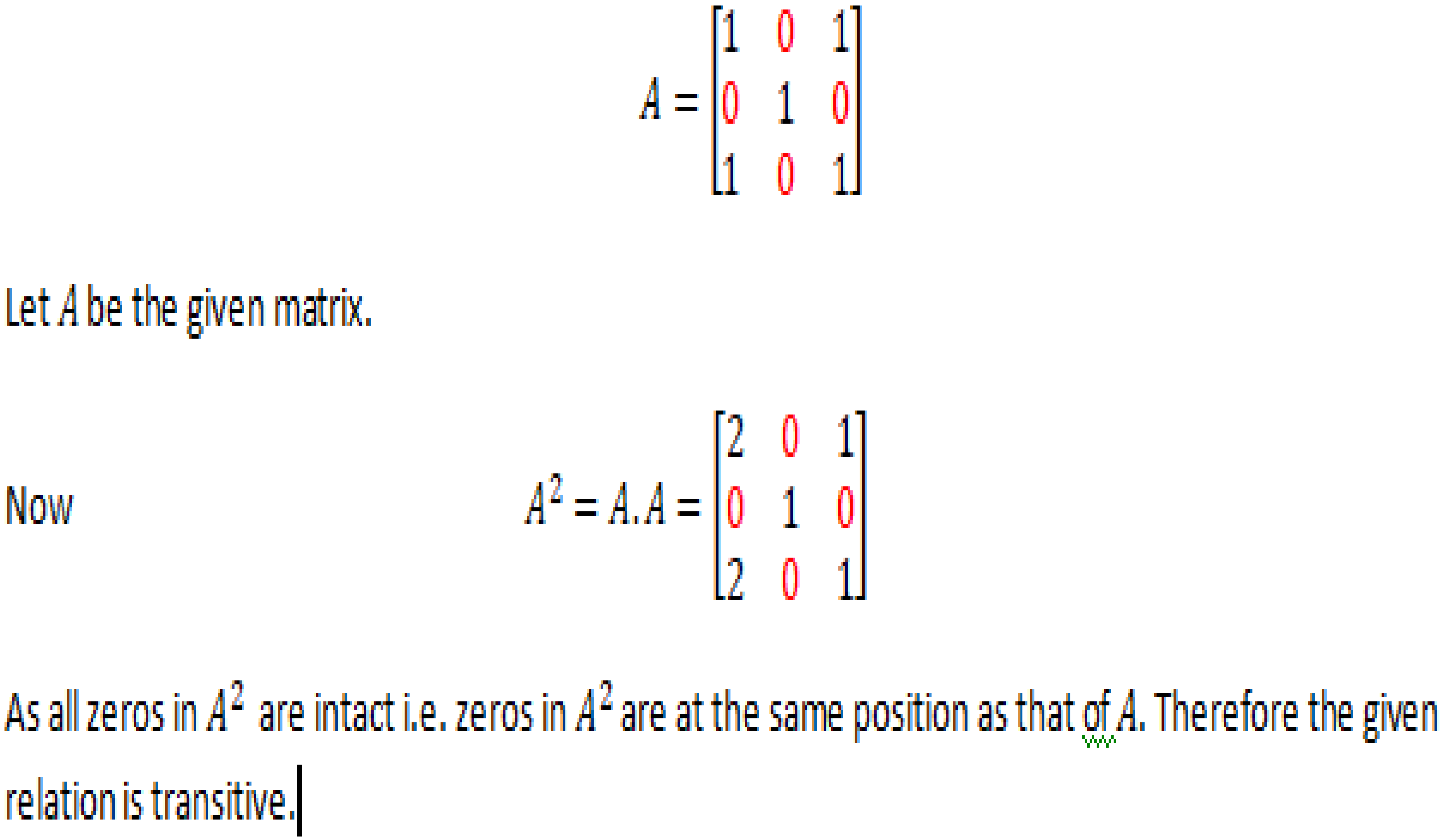
$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

R_3 is **transitive**

Matrix Representation of Transitive Relation

- Let A be the given matrix.
- Identify all zeros of the A .
- Take the square of the given matrix A .
- The relation is transitive if and only if the squared matrix has no nonzero entry where the original had a zero.

Example



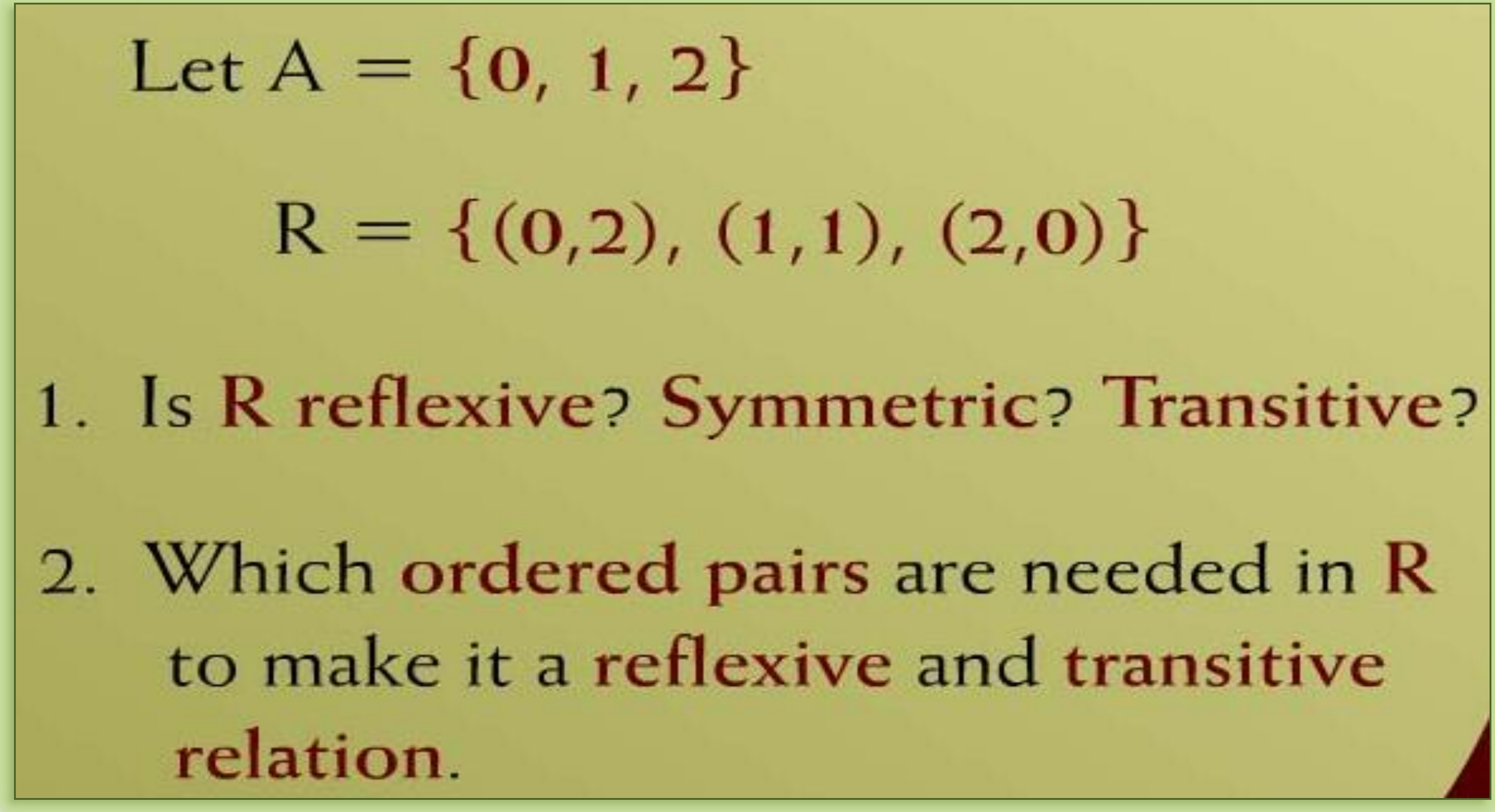
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Let A be the given matrix.

Now $$A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

As all zeros in A^2 are intact i.e. zeros in A^2 are at the same position as that of A. Therefore the given relation is transitive.

EXERCISE

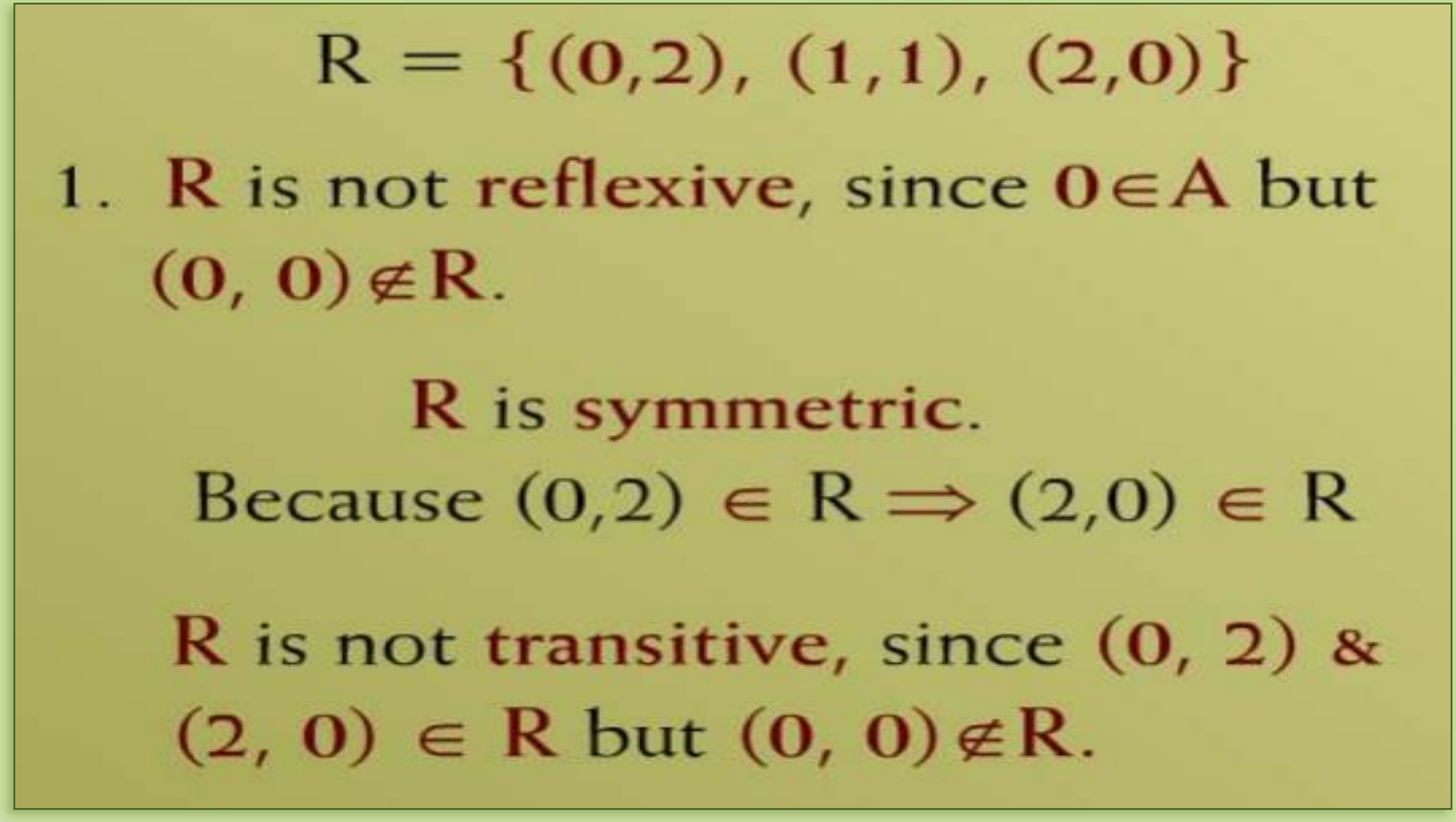


Let $A = \{0, 1, 2\}$

$$R = \{(0,2), (1,1), (2,0)\}$$

1. Is R reflexive? Symmetric? Transitive?
2. Which ordered pairs are needed in R to make it a reflexive and transitive relation.

SOLUTION



$R = \{(0,2), (1,1), (2,0)\}$

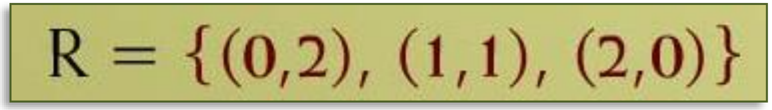
1. R is not reflexive, since $0 \in A$ but $(0, 0) \notin R$.

R is symmetric.

Because $(0,2) \in R \Rightarrow (2,0) \in R$

R is not transitive, since $(0, 2)$ & $(2, 0) \in R$ but $(0, 0) \notin R$.

SOLUTION

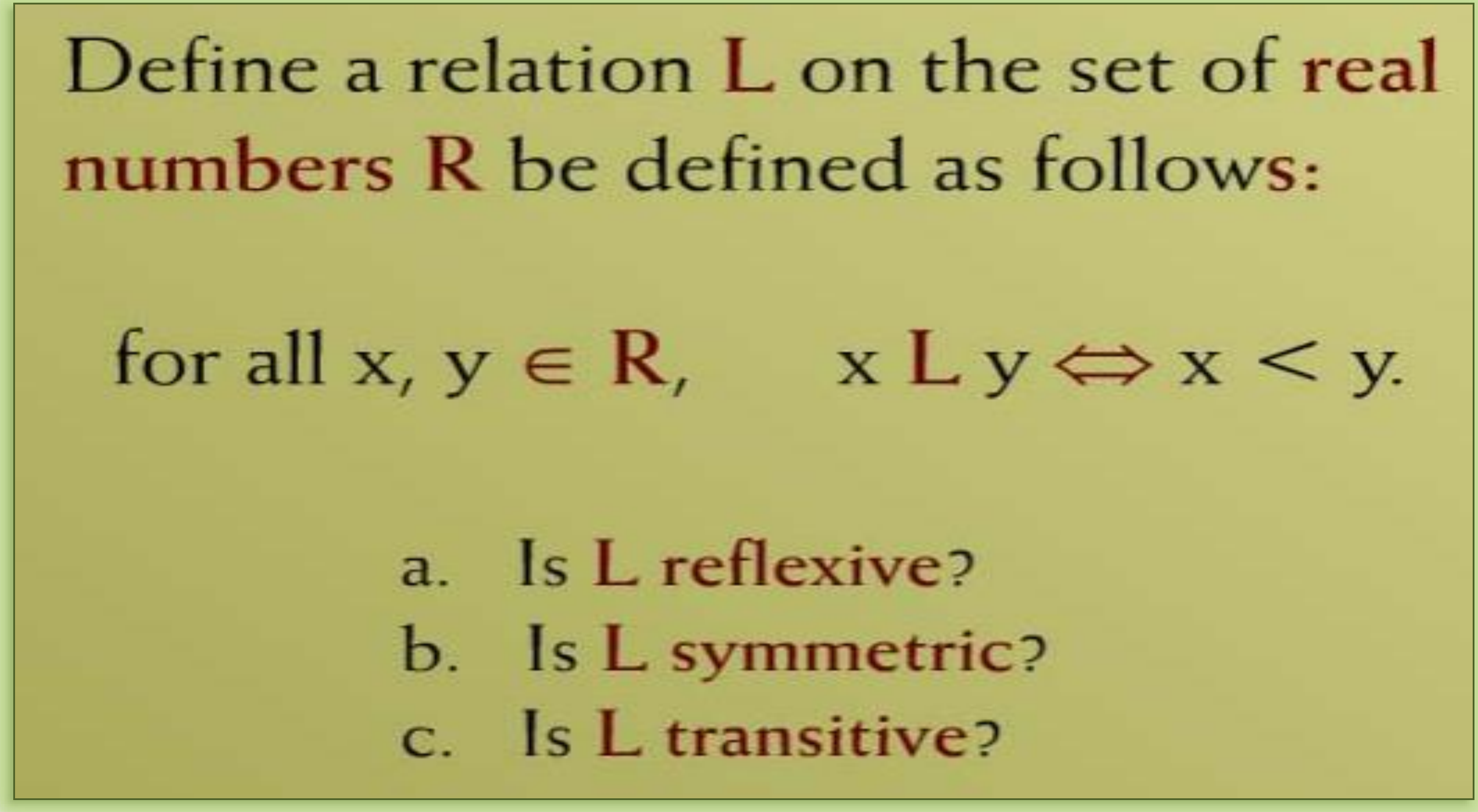


$R = \{(0,2), (1,1), (2,0)\}$

1. R is not reflexive.
   R to be reflexive it must contains (0, 0) and (2, 2).

2. R is not transitive.
   For R to be transitive it must contain (0, 0) and (2, 2).

EXERCISE



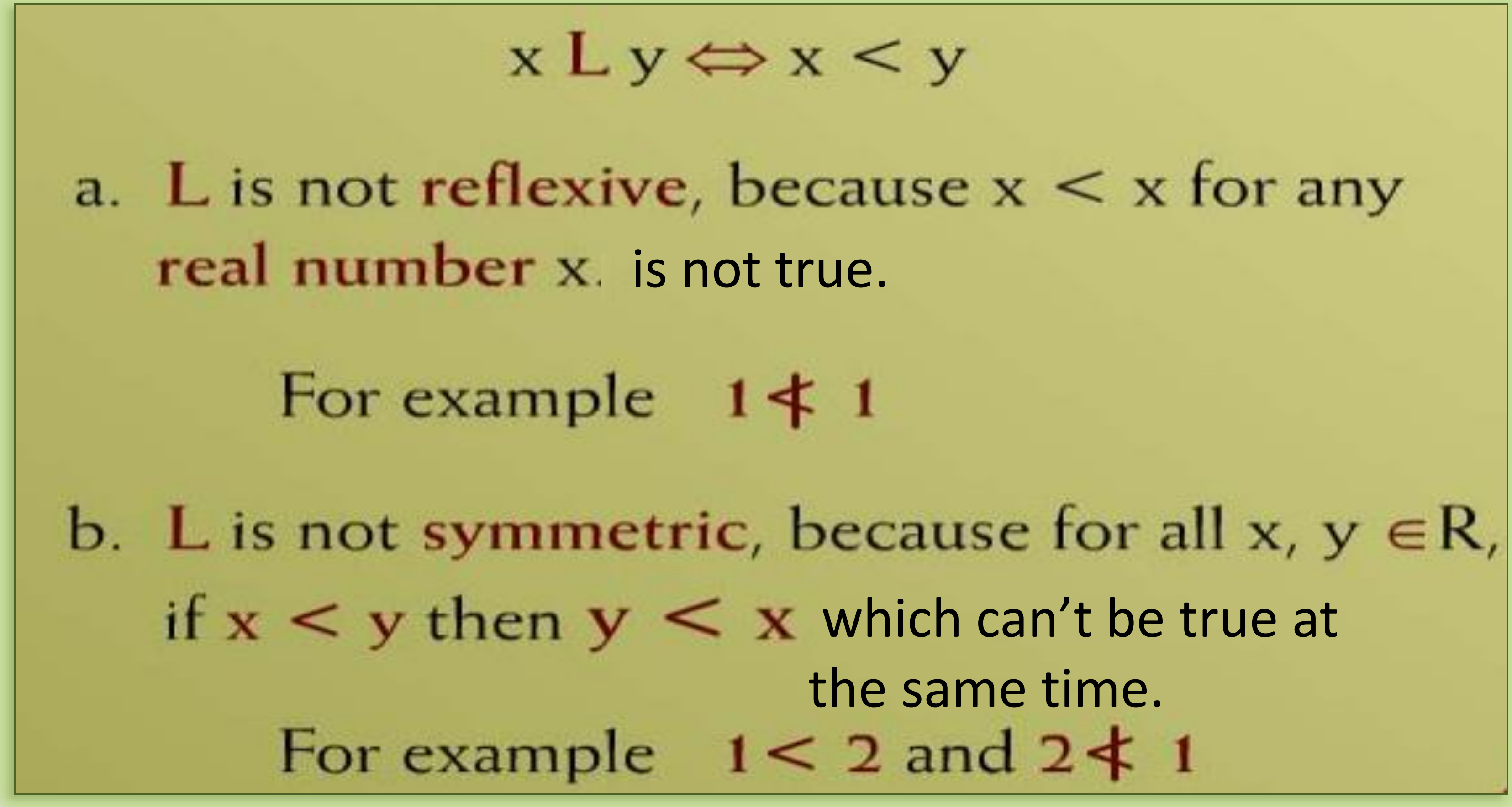
Define a relation L on the set of real numbers R be defined as follows:

for all $x, y \in R$, $x \, L \, y \Leftrightarrow x < y$.

a. Is L reflexive?
b. Is L symmetric?
c. Is L transitive?

R here is representing Real Numbers and L is representing relation.

SOLUTION



$$x \, L \, y \Leftrightarrow x < y$$

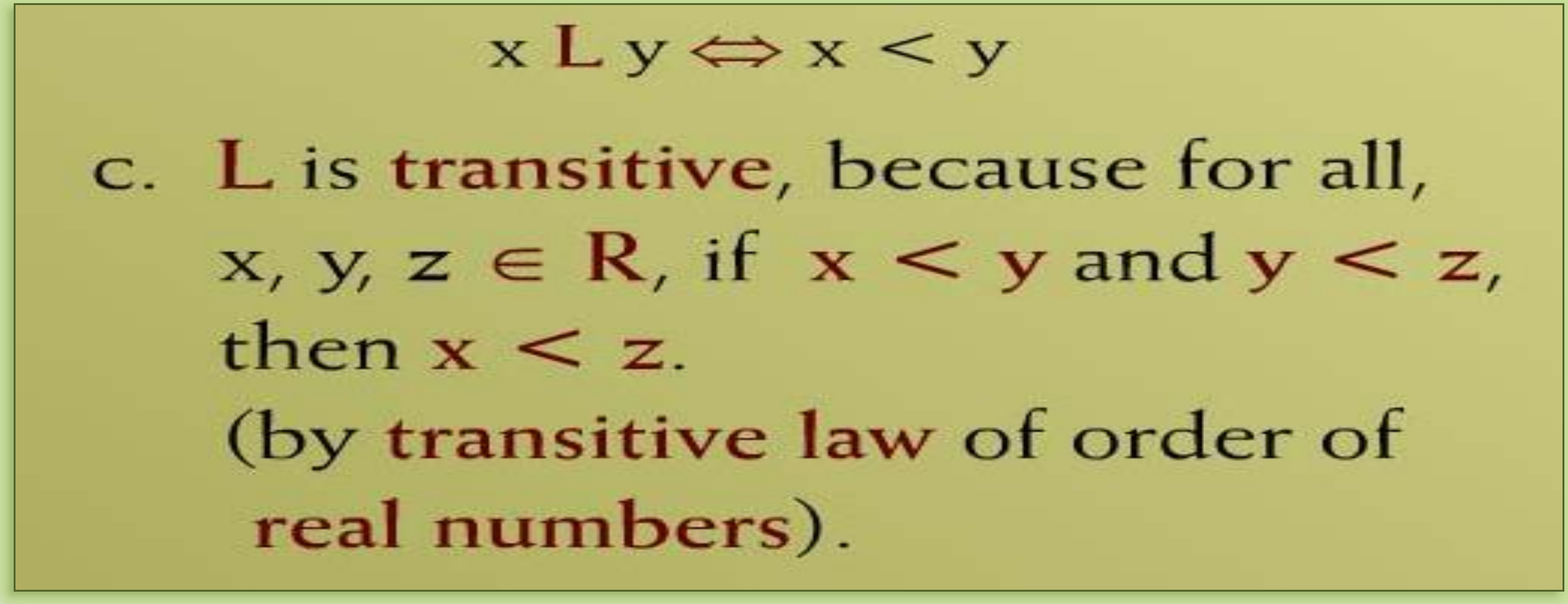
a. **L** is not **reflexive**, because $x < x$ for any **real number** x. is not true.

For example $1 \nless 1$

b. **L** is not **symmetric**, because for all $x, y \in R$, if $x < y$ then $y < x$ which can’t be true at the same time.

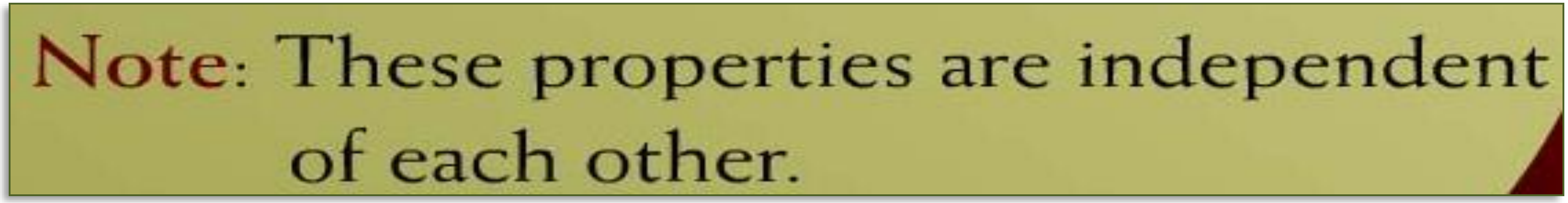
For example $1 < 2$ and $2 \nless 1$

SOLUTION



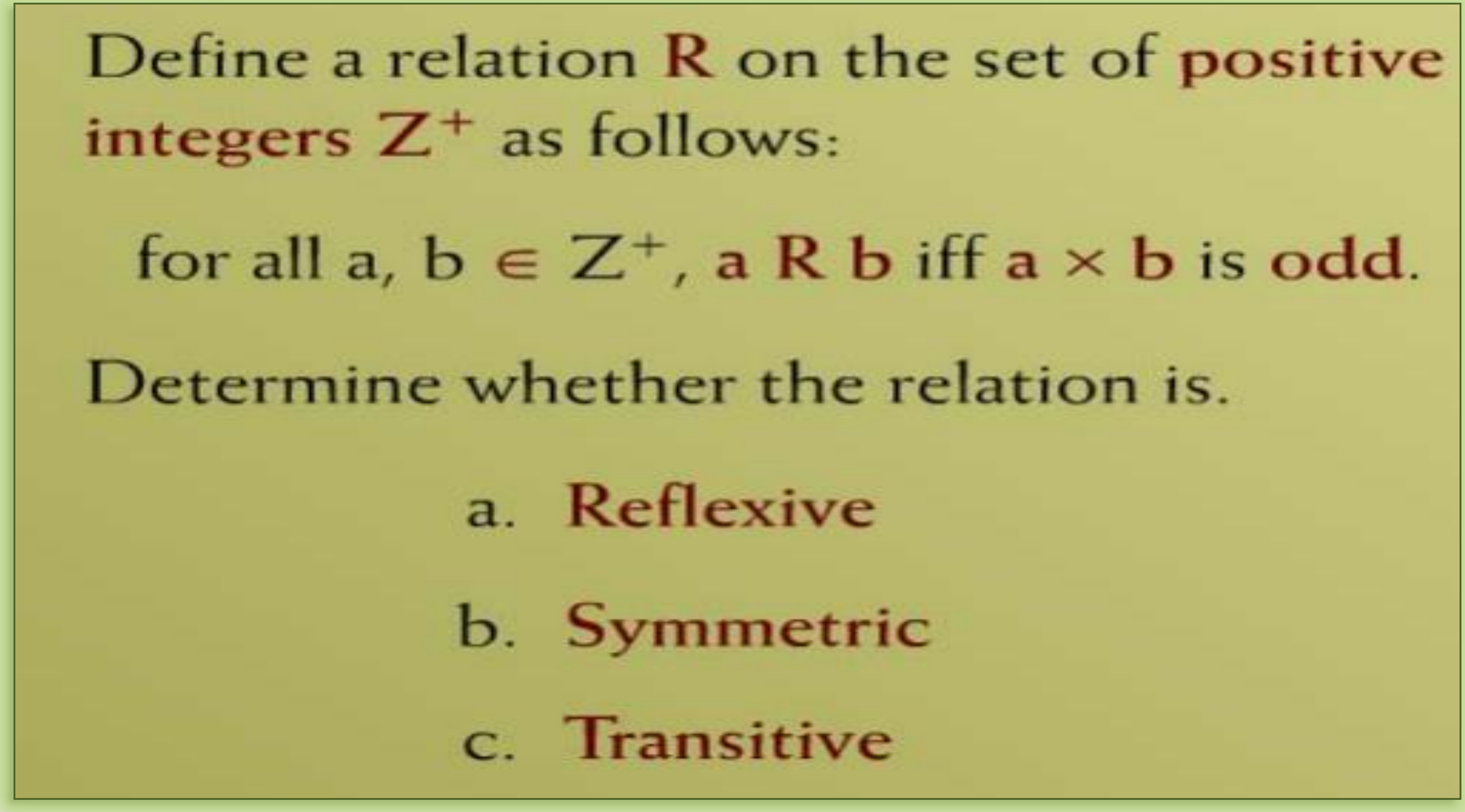
$$x \, L \, y \Leftrightarrow x < y$$

c. **L** is **transitive**, because for all, $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$.
(by **transitive law** of order of **real numbers**).



Note: These properties are independent of each other.

EXAMPLE



Define a relation R on the set of **positive integers** Z^+ as follows:

for all a, b $\in Z^+$, **a R b** iff **a × b** is **odd**.

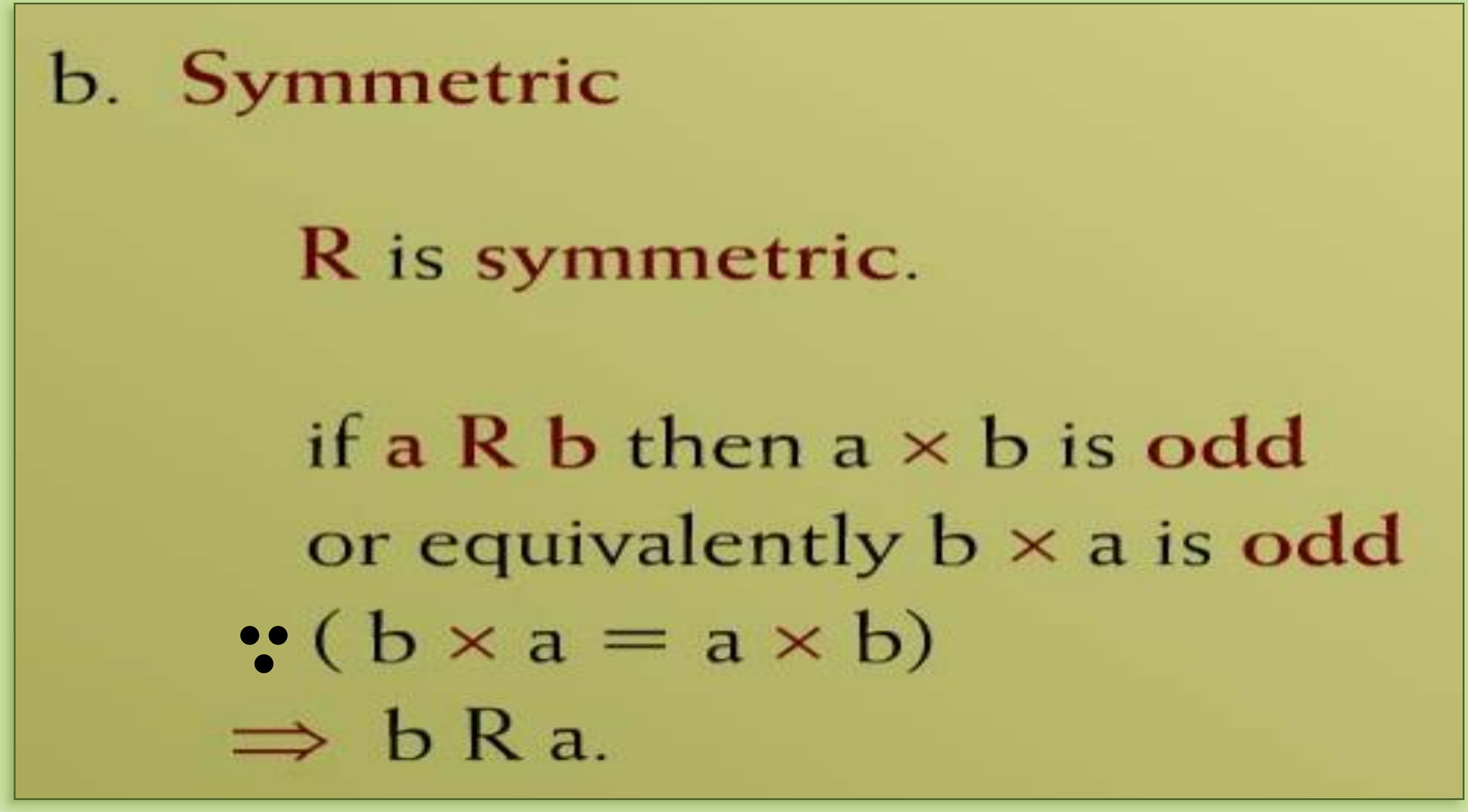
Determine whether the relation is.

a. **Reflexive**
b. **Symmetric**
c. **Transitive**

SOLUTION

For reflexivity, we have to show that,
aRa iff a x a = odd number.
And it contradicts the definition of reflexivity. Because 2R2 ↔ 2x2=4 which is not odd.
Hence, R is not reflexive.

SOLUTION



b. **Symmetric**

R is symmetric.

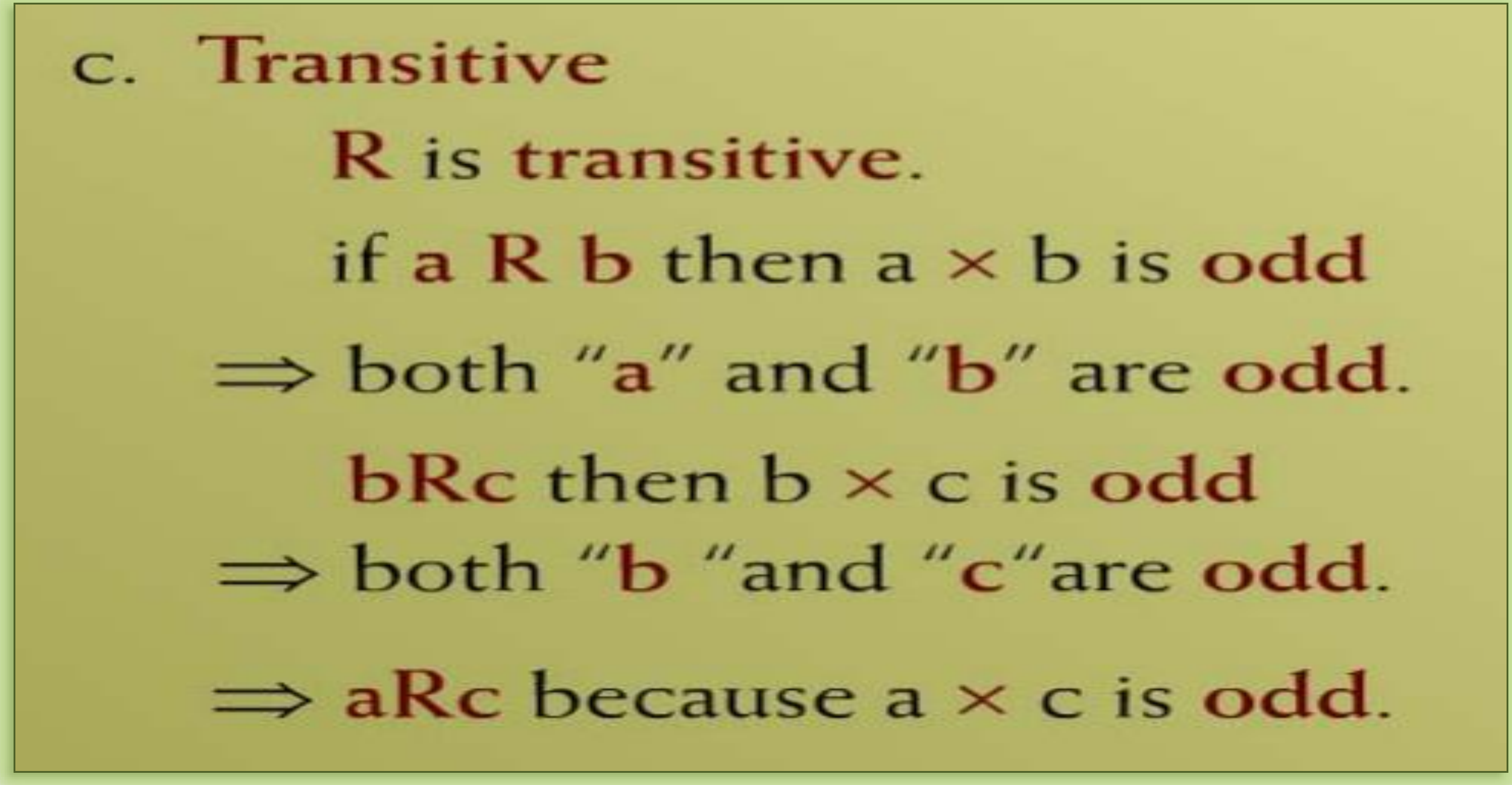
if a R b then a × b is odd

or equivalently b × a is odd

$\because (b \times a = a \times b)$

$\Rightarrow$ b R a.

SOLUTION



c. **Transitive**

R is **transitive**.

if a R b then a × b is **odd**

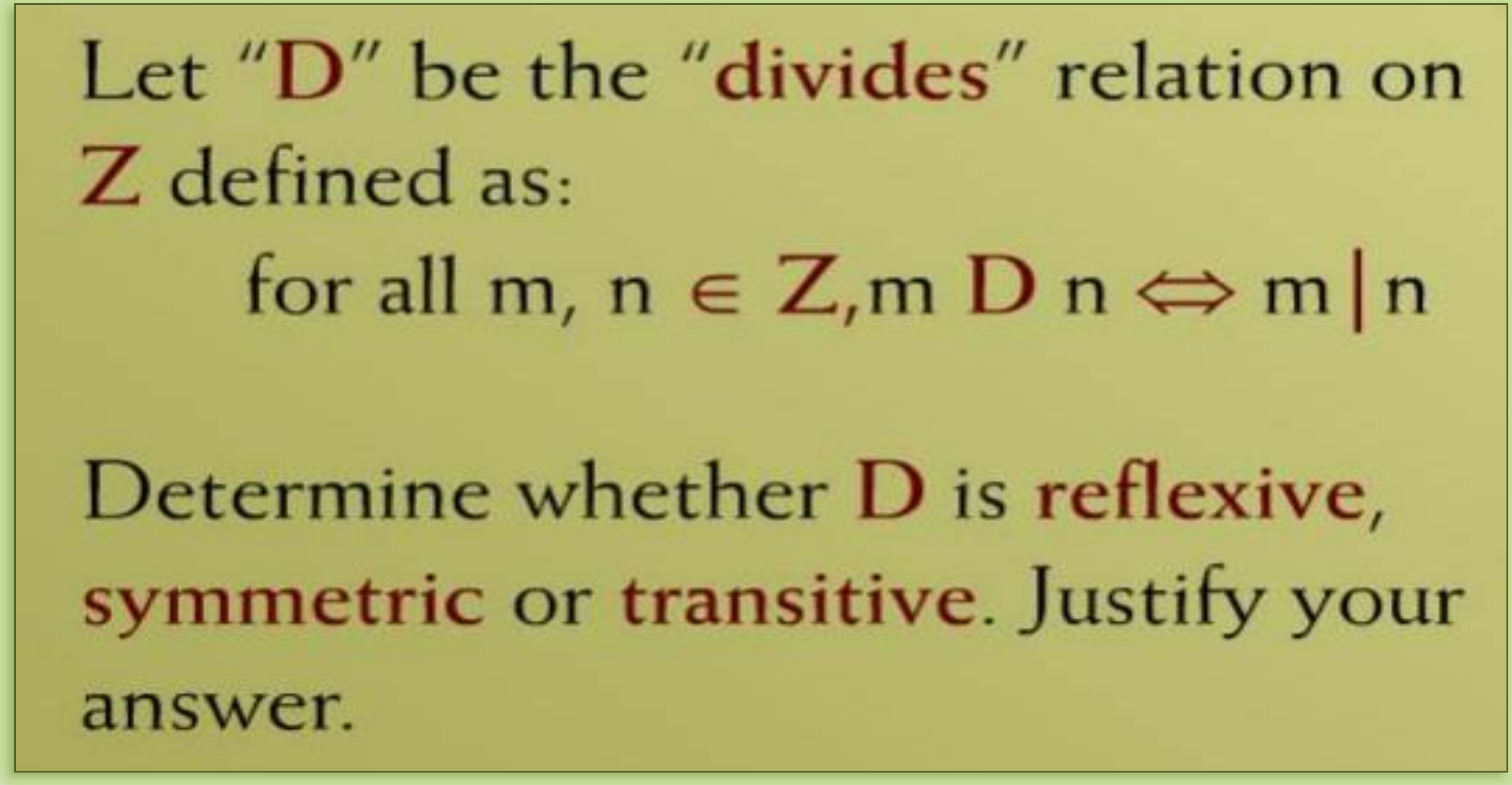
⇒ both “a” and “b” are **odd**.

bRc then b × c is **odd**

⇒ both “b “and “c“are **odd**.

⇒ **aRc** because a × c is **odd**.

EXAMPLE



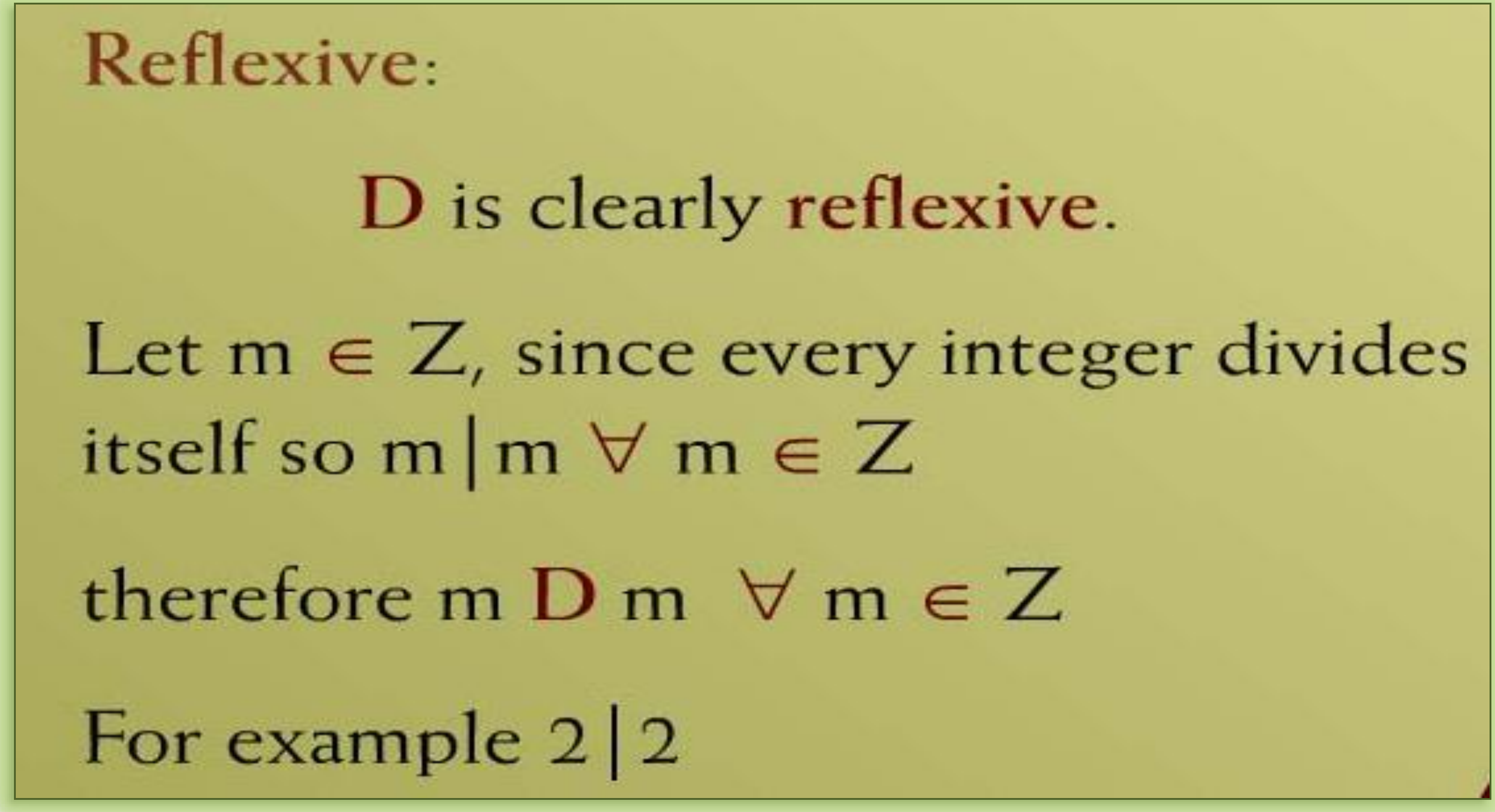
Let "**D**" be the "**divides**" relation on **Z** defined as:

for all $m, n \in Z, m \; D \; n \Leftrightarrow m \mid n$

Determine whether **D** is **reflexive**, **symmetric** or **transitive**. Justify your answer.

Note: Here the set of integers do not include 0 i.e. Z-{0}

SOLUTION



Reflexive:

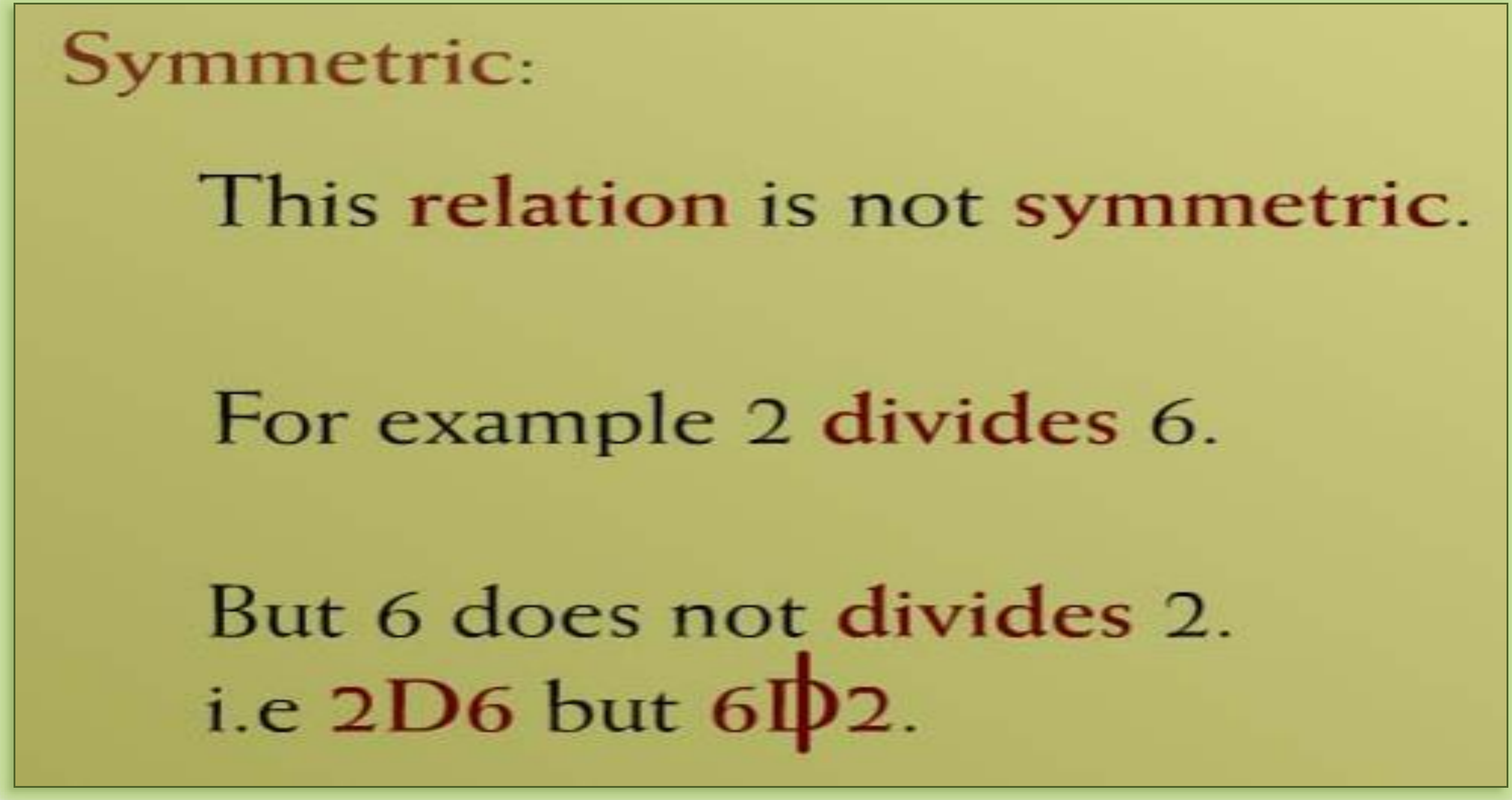
D is clearly **reflexive**.

Let $m \in Z$, since every integer divides itself so $m|m \ \forall m \in Z$

therefore $m \, D \, m \ \ \forall m \in Z$

For example $2|2$

SOLUTION



Symmetric:

This relation is not symmetric.

For example 2 divides 6.

But 6 does not divides 2.
i.e 2D6 but $6\not{D}2$.

SOLUTION

Transitive:

If m **divides** n and n **divides** p.

Then m **divides** p.

EQUIVALENCE RELATION

Let A be a non-empty set and R a binary relation on A. R is an equivalence relation if and only if, R is reflexive, symmetric, and transitive.

EXAMPLE

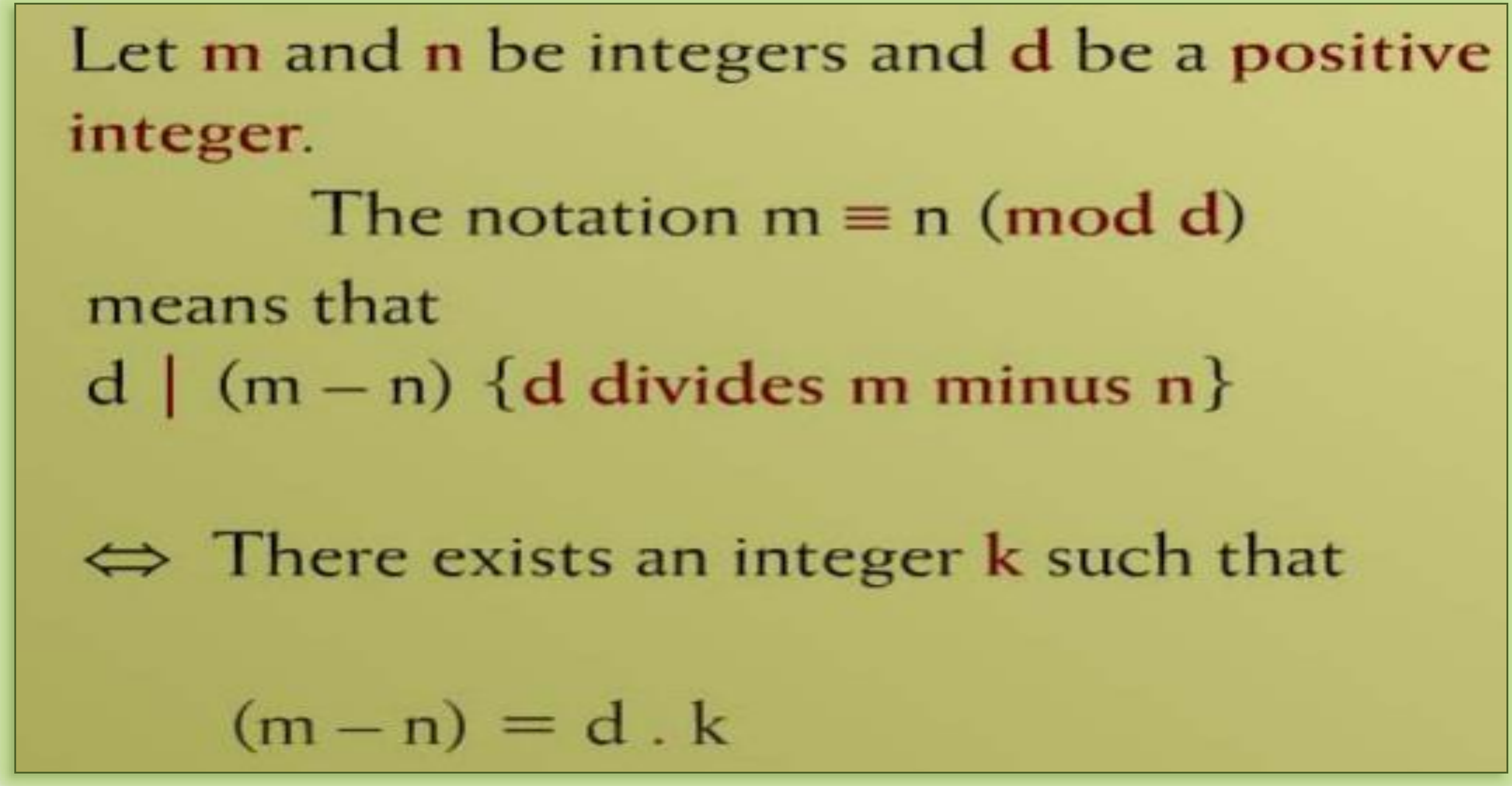
Let

A = {1, 2, 3, 4}

R = { (1, 1) , (2, 2) , (2, 4) , (3, 3) , (4, 2) , (4, 4) }

R is reflexive, symmetric and transitive.

CONGRUENCES RELATION



Let m and n be integers and d be a positive integer.

The notation $m \equiv n \pmod{d}$

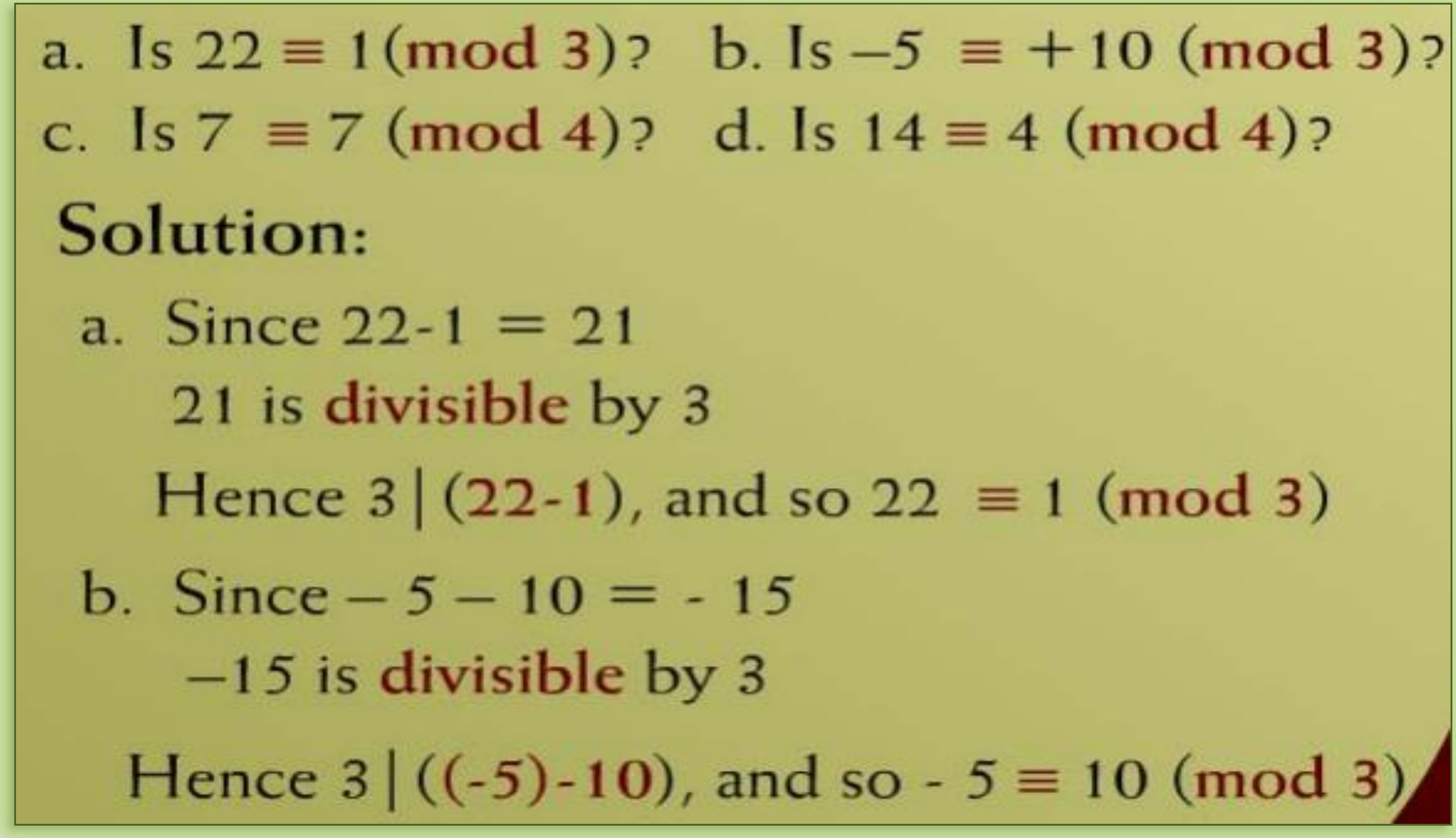
means that

$d \mid (m - n)$ {d divides m minus n}

⇔ There exists an integer k such that

$$(m - n) = d \, . \, k$$

EXAMPLE



a. Is $22 \equiv 1 (\mathrm{mod}\ 3)$? b. Is $-5 \equiv +10\ (\mathrm{mod}\ 3)$?

c. Is $7 \equiv 7\ (\mathrm{mod}\ 4)$? d. Is $14 \equiv 4\ (\mathrm{mod}\ 4)$?

Solution:

a. Since $22-1 = 21$

21 is **divisible** by 3

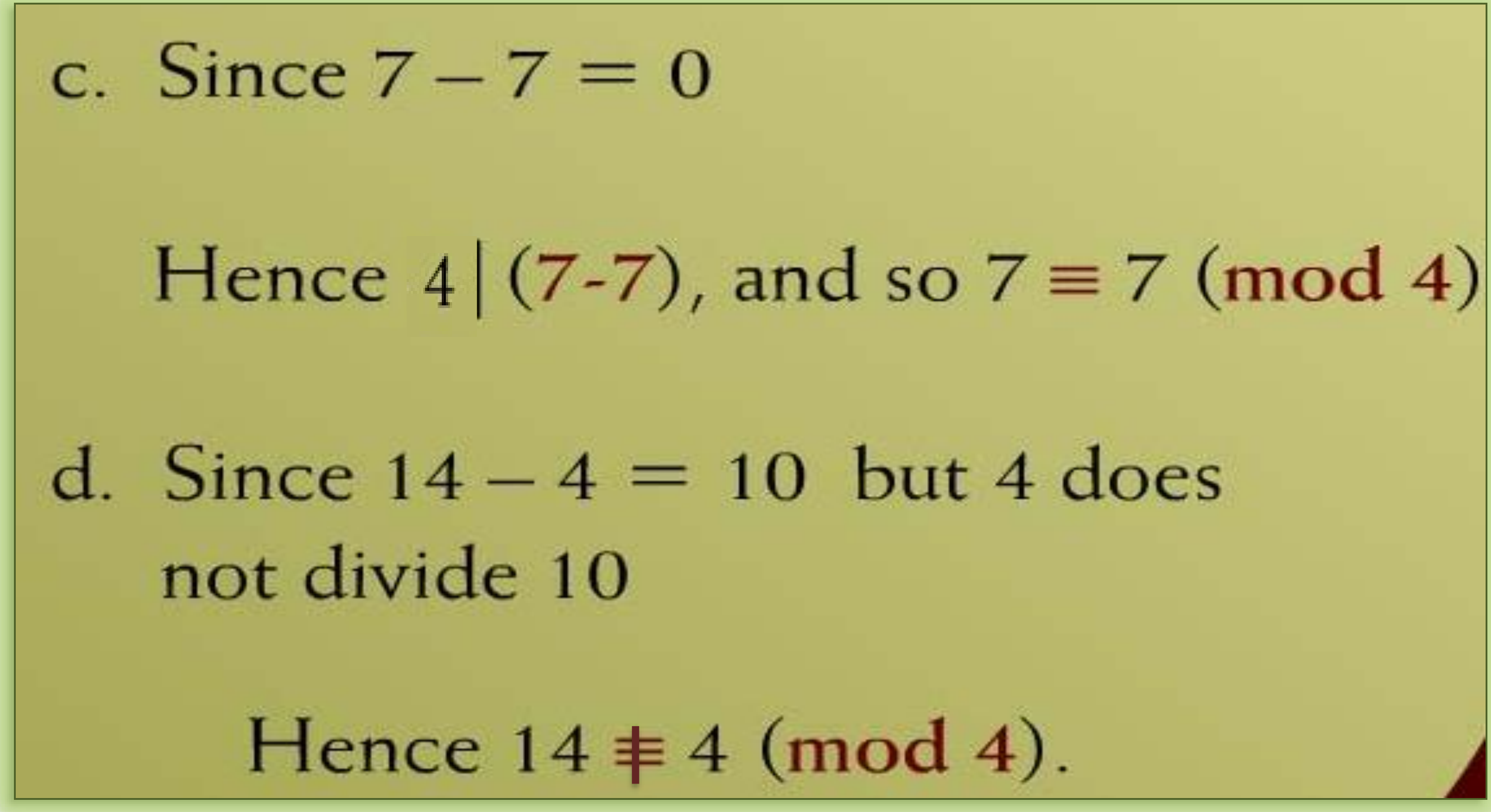
Hence $3 \mid (22-1)$, and so $22 \equiv 1\ (\mathrm{mod}\ 3)$

b. Since $-5-10 = -15$

-15 is **divisible** by 3

Hence $3 \mid ((-5)-10)$, and so $-5 \equiv 10\ (\mathrm{mod}\ 3)$

Solution contd…



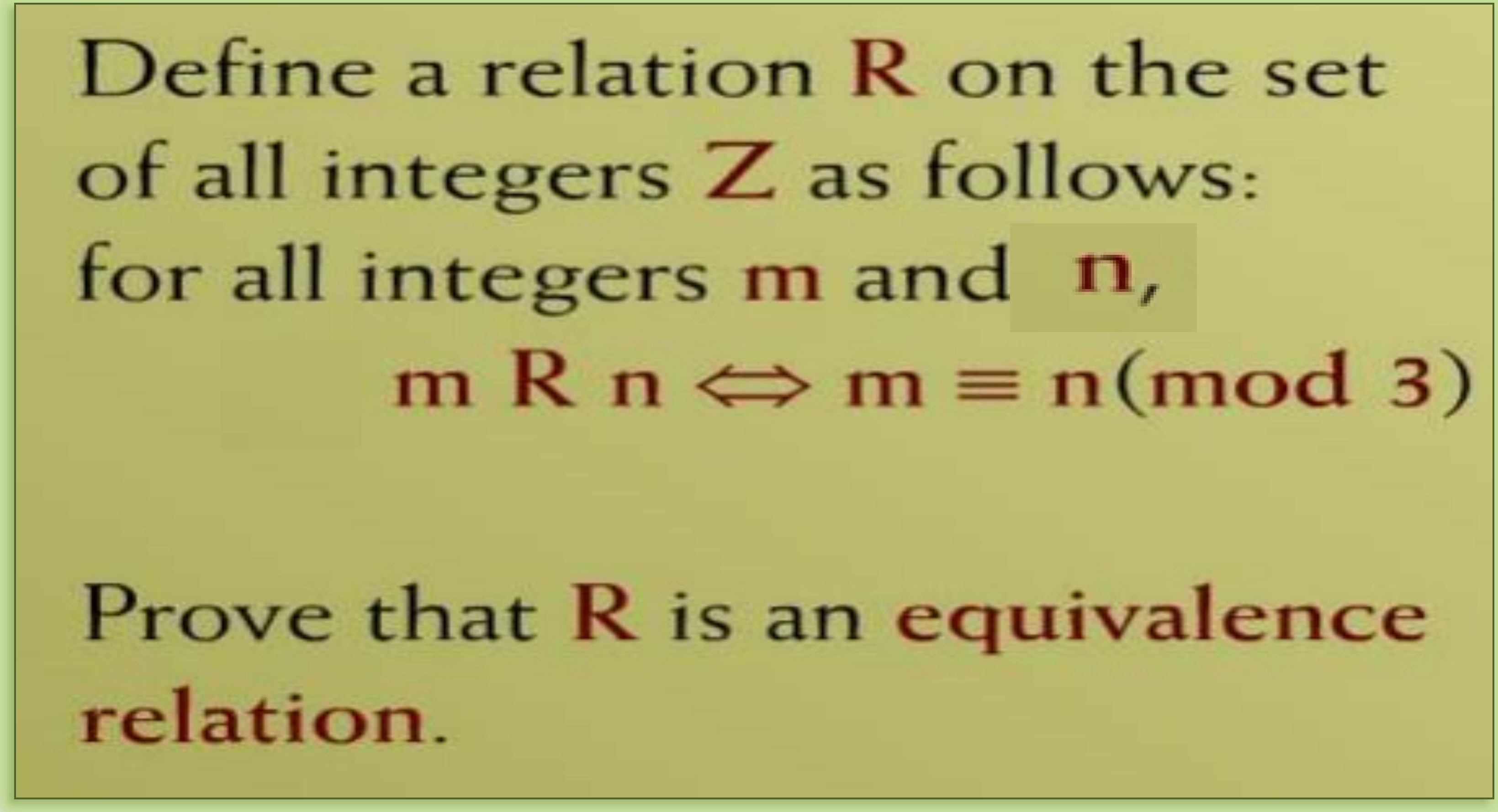
c. Since $7 - 7 = 0$

Hence $4 \mid (7-7)$, and so $7 \equiv 7 \pmod{4}$

d. Since $14 - 4 = 10$ but 4 does not divide 10

Hence $14 \not\equiv 4 \pmod{4}$.

EXERCISE

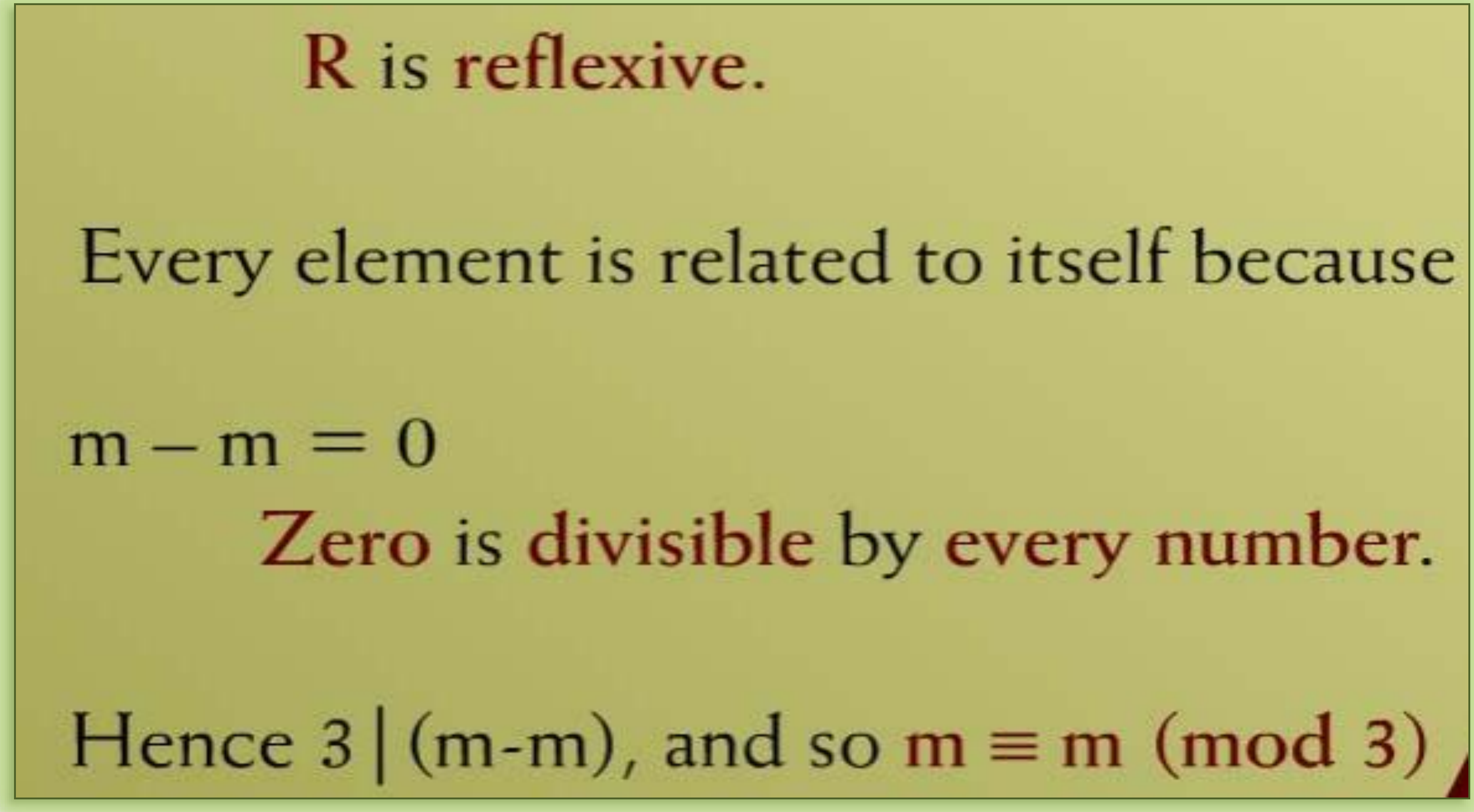


Define a relation R on the set of all integers Z as follows: for all integers m and n,

$$m \; R \; n \Leftrightarrow m \equiv n \pmod{3}$$

Prove that R is an equivalence relation.

SOLUTION



R is reflexive.

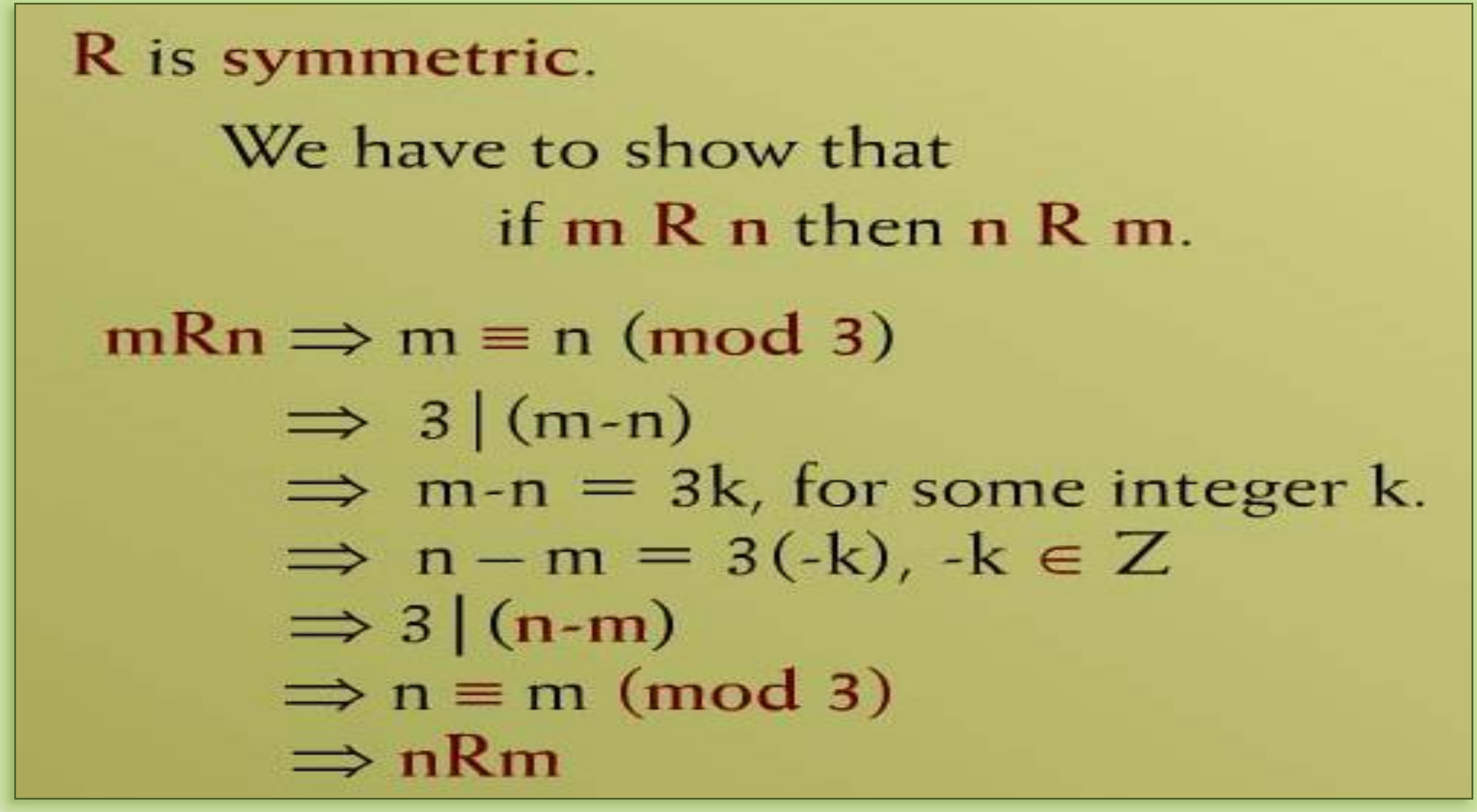
Every element is related to itself because

$m - m = 0$

Zero is divisible by every number.

Hence $3 \mid (m-m)$, and so $m \equiv m \pmod{3}$

SOLUTION



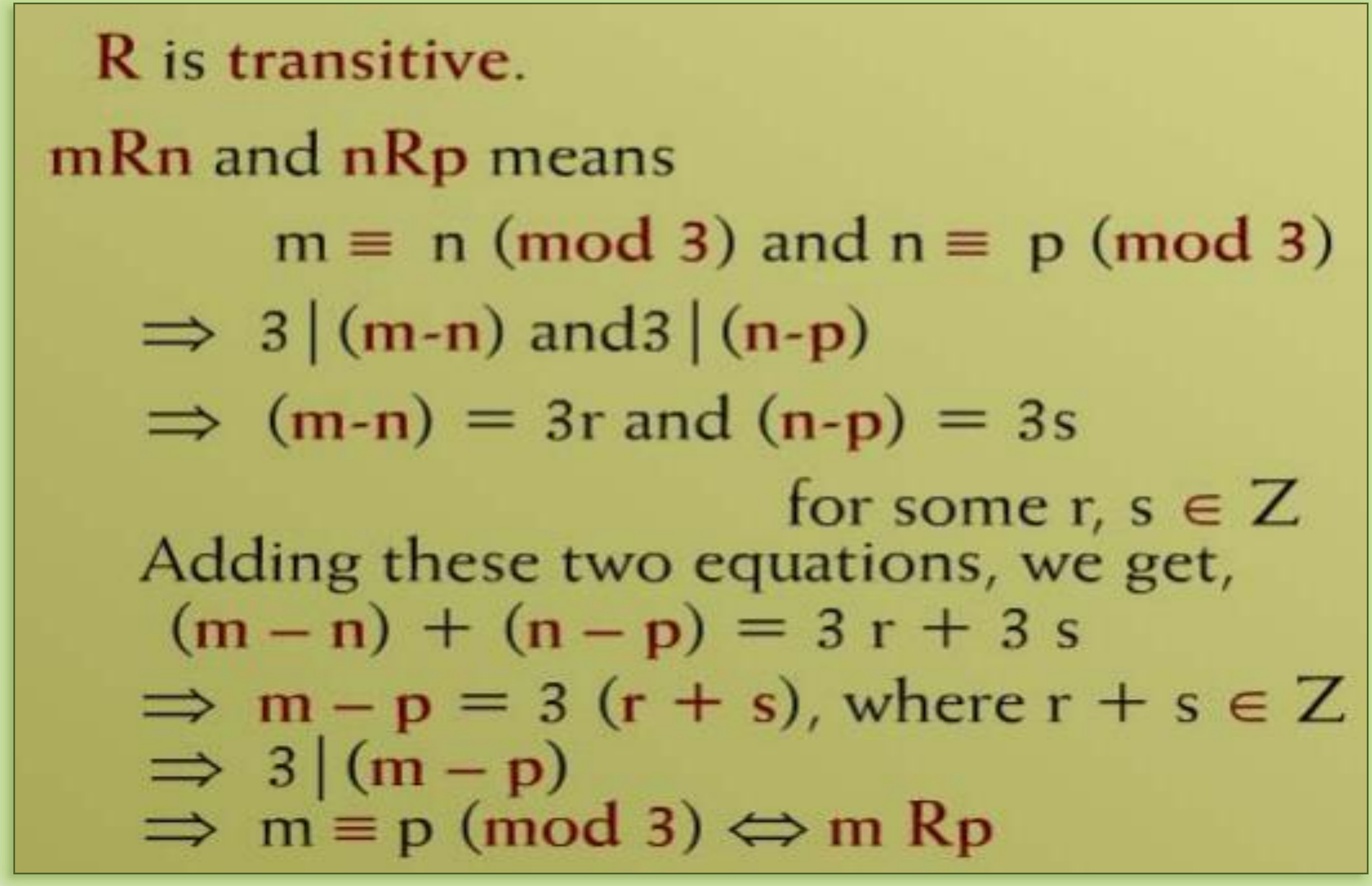
R is **symmetric.**

We have to show that

if $m\ R\ n$ then $n\ R\ m$.

$$\begin{aligned} mRn &\Rightarrow m \equiv n \pmod{3} \\ &\Rightarrow 3 \mid (m-n) \\ &\Rightarrow m-n = 3k, \text{ for some integer } k. \\ &\Rightarrow n - m = 3(-k),\ -k \in Z \\ &\Rightarrow 3 \mid (n-m) \\ &\Rightarrow n \equiv m \pmod{3} \\ &\Rightarrow nRm \end{aligned}$$

SOLUTION



R is transitive.

mRn and nRp means

$m \equiv n \pmod 3$ and $n \equiv p \pmod 3$

$\Rightarrow 3 | (m\text{-}n)$ and$3 | (n\text{-}p)$

$\Rightarrow (m\text{-}n) = 3r$ and $(n\text{-}p) = 3s$

for some $r, s \in Z$

Adding these two equations, we get,

$(m - n) + (n - p) = 3\,r + 3\,s$

$\Rightarrow m - p = 3\,(r + s)$, where $r + s \in Z$

$\Rightarrow 3 | (m - p)$

$\Rightarrow m \equiv p \pmod 3 \Leftrightarrow m\,Rp$