Discrete Structures

Lecture # 13

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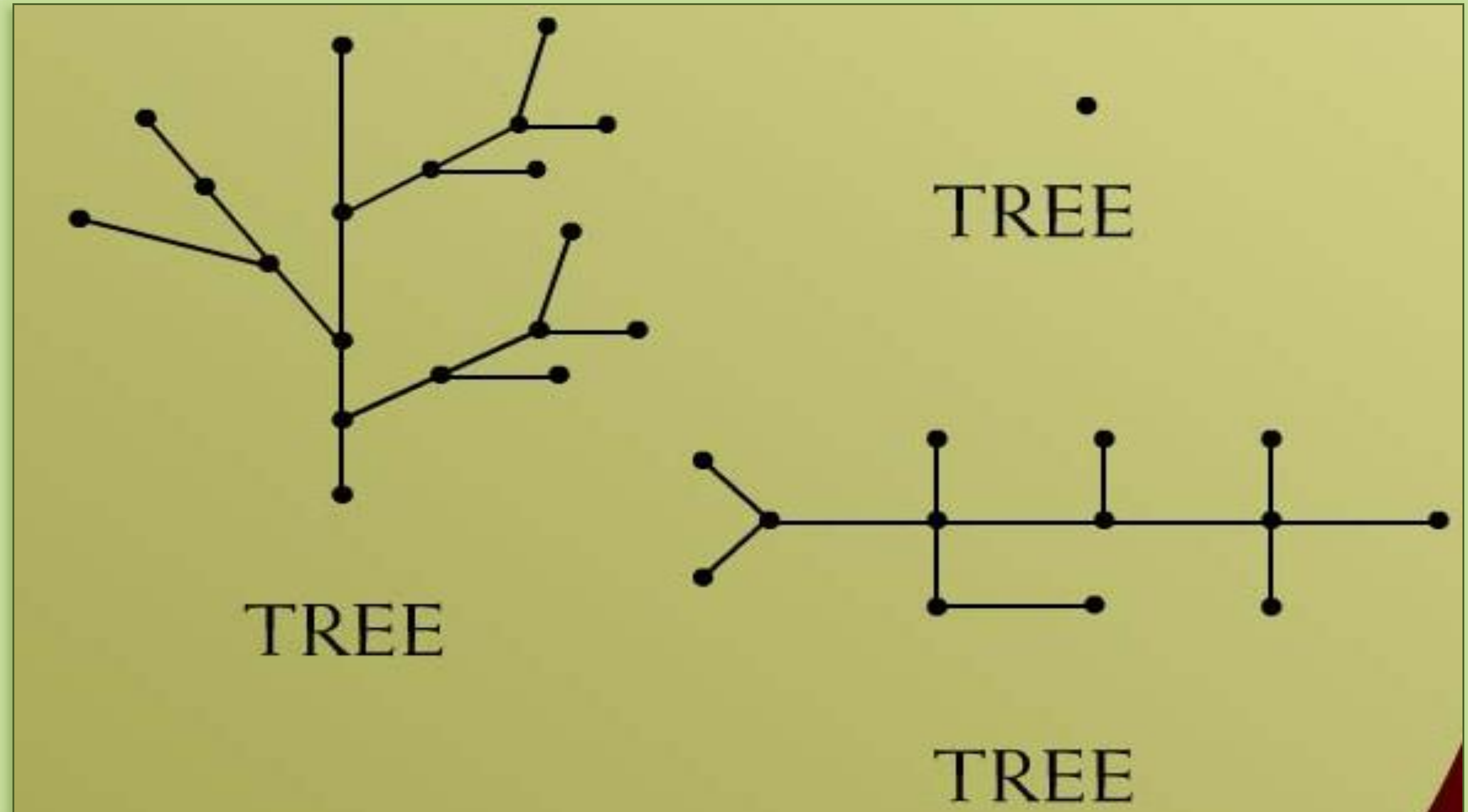
TREE

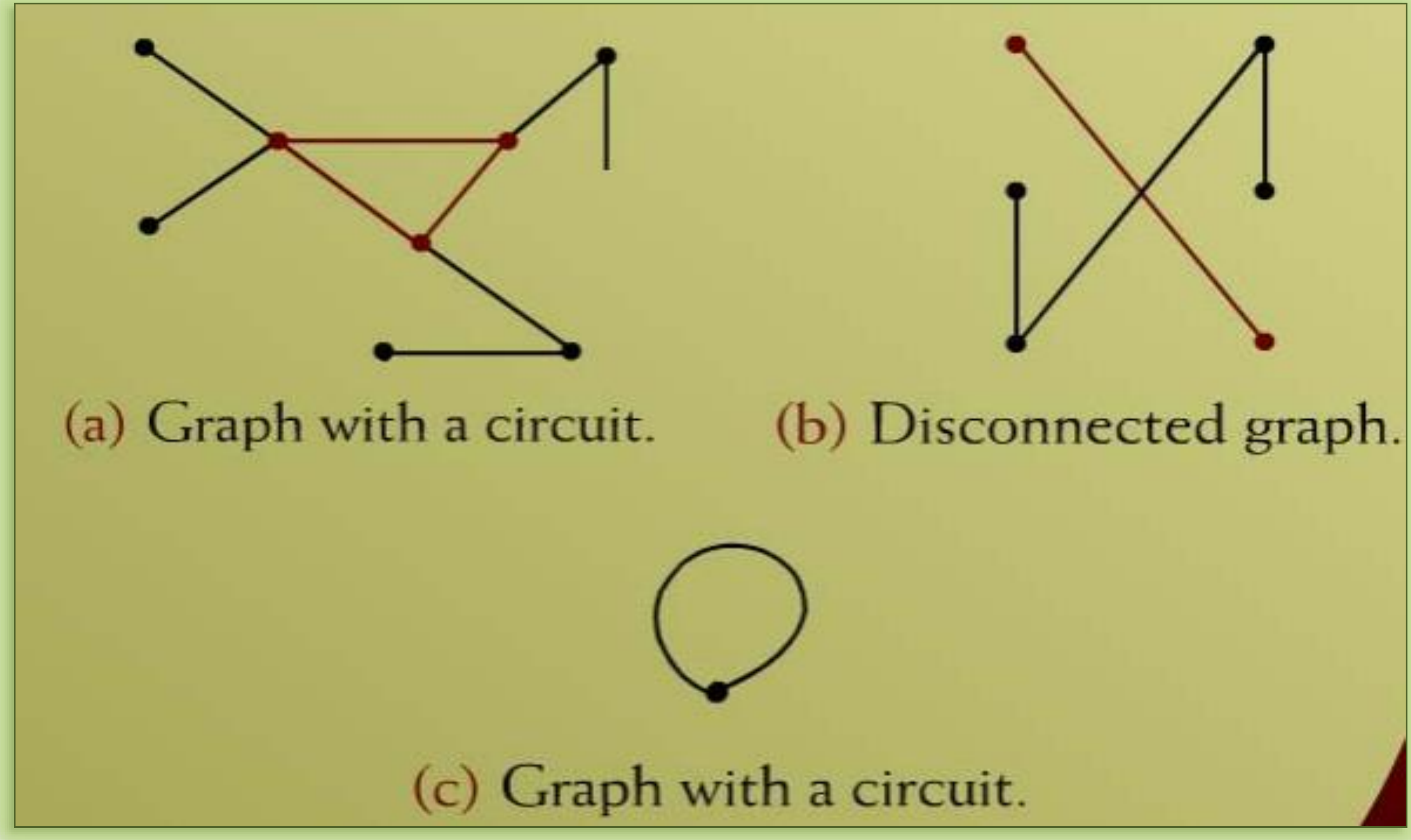
A tree is a connected graph that does not contain any nontrivial circuit. (it is circuit-free).

A trivial circuit is one that consists of a single vertex.

EXAMPLE



EXAMPLES OF NON TREES



SOME SPECIAL TREES

1. TRIVIAL TREE

   A graph that consists of a single vertex is called a trivial tree or degenerate tree.

2. EMPTY TREE

   A tree that does not have any vertices or edges is called an empty tree.

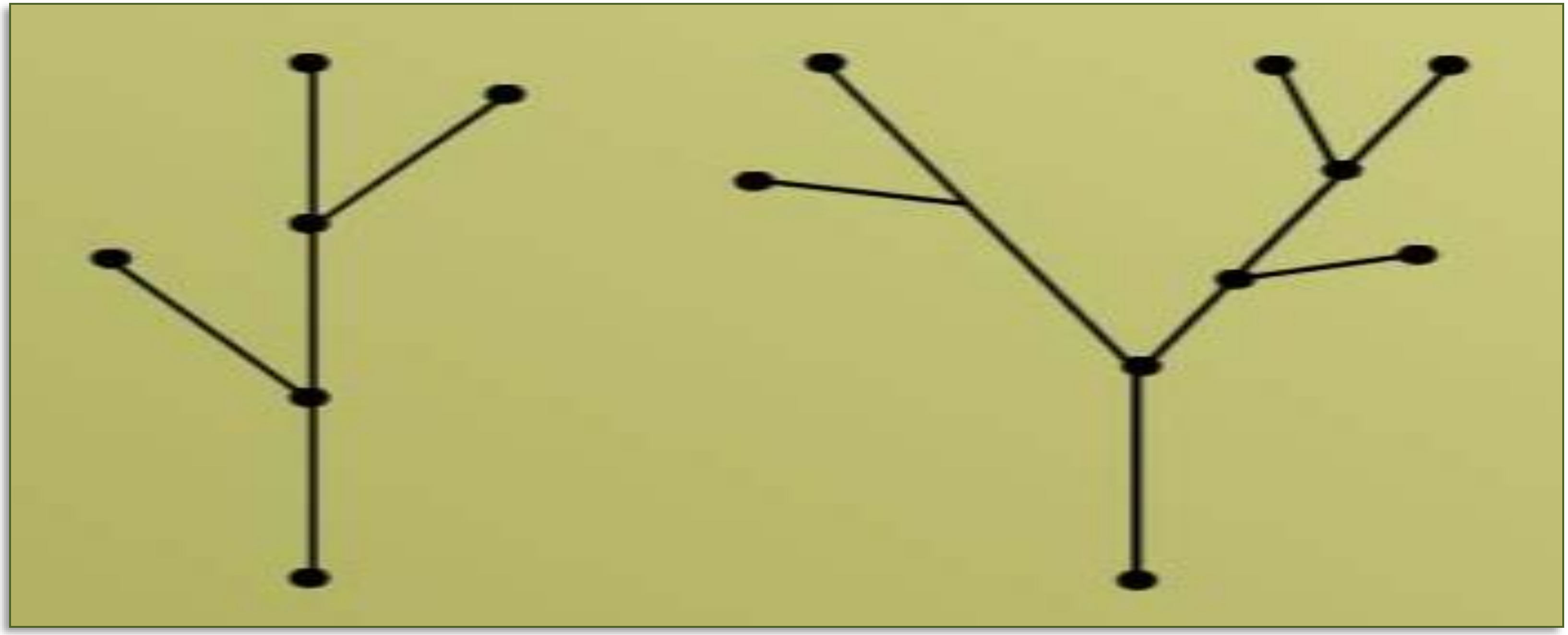
SOME SPECIAL TREES

FOREST:

A **graph** is called a **forest** if, and only if, it is **circuit-free**.

Hence, the **connected components** of a **forest** are **trees**.

FOREST



Forest has only two trees in it.

PROPERTIES OF TREES

1. A tree with n vertices has n − 1 edges.

2. Any connected graph with n vertices and n − 1 edges is a tree.

3. A tree has no nontrivial circuit; but if one new edge is added to it, then the resulting graph has exactly one nontrivial circuit.

PROPERTIES OF TREES

4. A **tree** is connected, but if any **edge** is deleted from it, then the **resulting graph** is **not connected**.

5. Any **tree** that has more than one **vertex** has at least **two vertices** of **degree 1**.

6. A **graph** is a **tree** iff there is a **unique path** between any two of its **vertices**.

EXERCISE

Explain why graphs (Tree) with the given specification do not exist.

1. Tree with twelve vertices and fifteen edges.
2. Trees with five vertices and total degree 10.

SOLUTION:

1. Any tree with 12 vertices will have $12 - 1 = 11$ edges.
   Hence solution of first is not possible.

SOLUTION

Tree with five vertices and total degree 10.

Any tree with five vertices will have 5 - 1 = 4 edges.

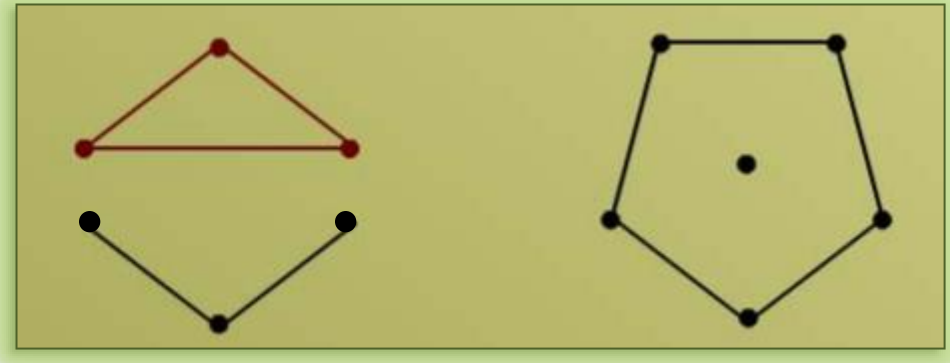
We are given total degree of graph is 10. So it must have edges 10/2 = 5.

The two conditions contradict each other.

SOLUTION

Draw a graph with six vertices, five edges that is not a tree.

SOLUTION:

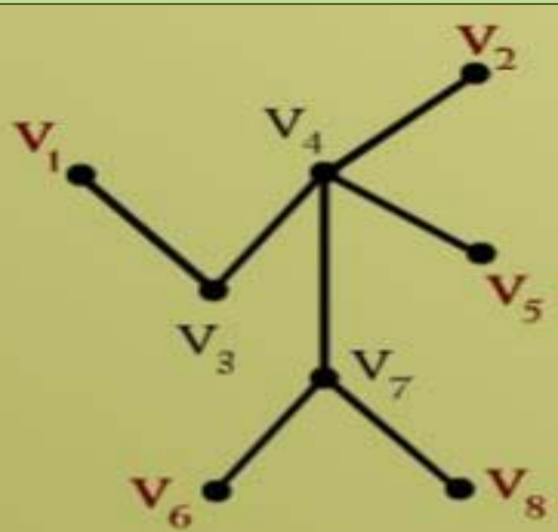


First is not tree because it is not connected and also has a circuit similarly for second.

TERMINAL AND INTERNAL VERTEX

A vertex of degree '1' in a tree is called terminal vertex or a leaf (leaf has no children) and a vertex of degree greater than '1' in a tree is called an internal vertex or a branch vertex.

EXAMPLE



v_1, v_2, v_5, v_6 and v_8 are terminal vertices.

v_3, v_4, v_7 are internal vertices.

ROOTED TREE

A **rooted tree** is a **tree** in which **one vertex** is **distinguished** from the others and is called the **root**.

The **level of a vertex** is the number of **edges** along the **unique path** between it and the **root**.

The **height of a rooted tree** is the maximum level to **any vertex** of the **tree**.

ROOTED TREE

The children of any internal vertex v are all those vertices that are adjacent to v and are one level farther away from the root than v. If w is a child of v, then v is called the parent of w.

Two vertices that are both children of the same parent are called siblings.

Given vertices v and w, if v lies on the unique path between w and the root, then v is an ancestor of w and w is a descendant of v.

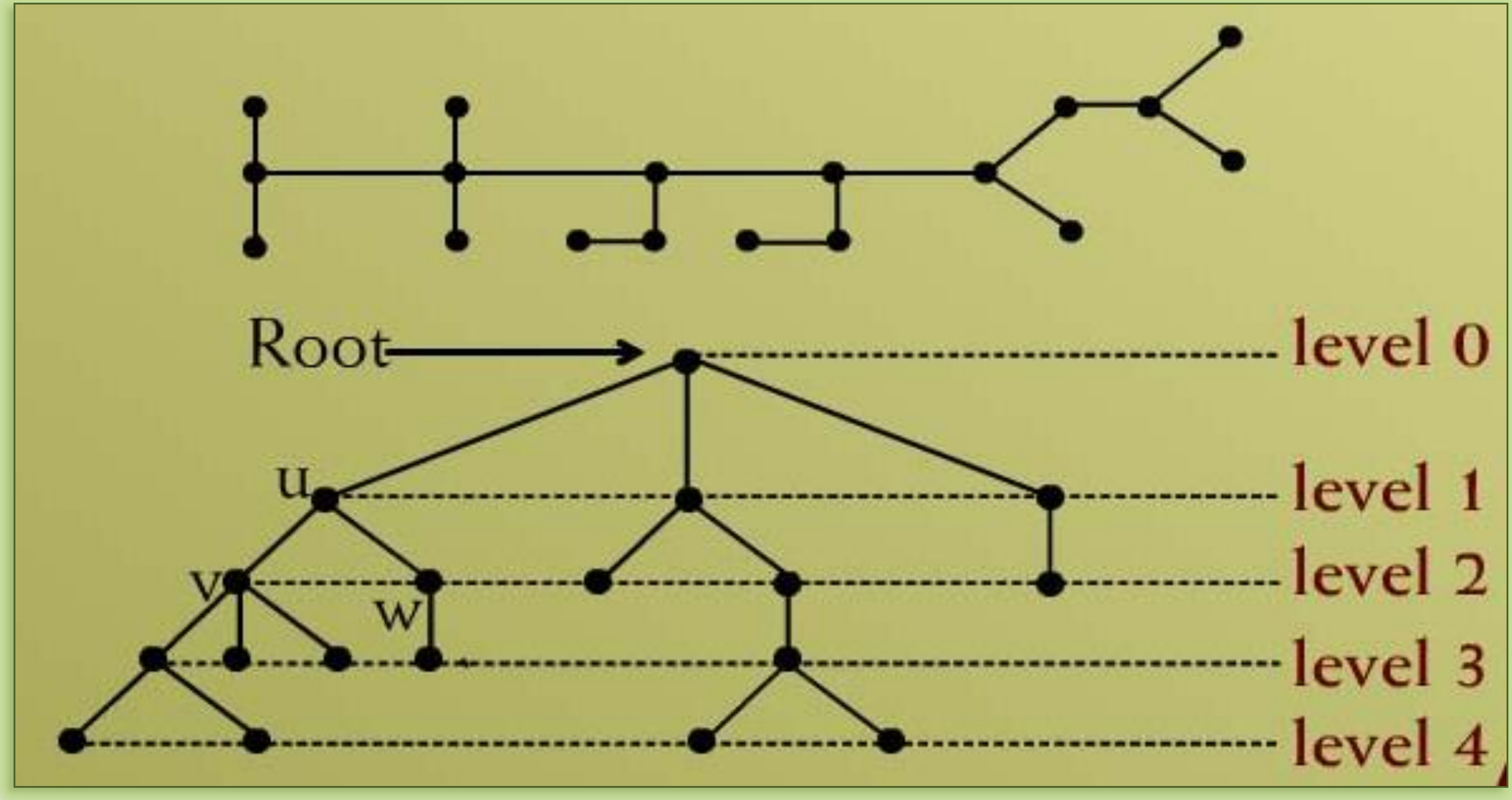
ROOTED TREE

The root is an internal vertex unless it is the only vertex in the graph, and in that case it will be a leaf.

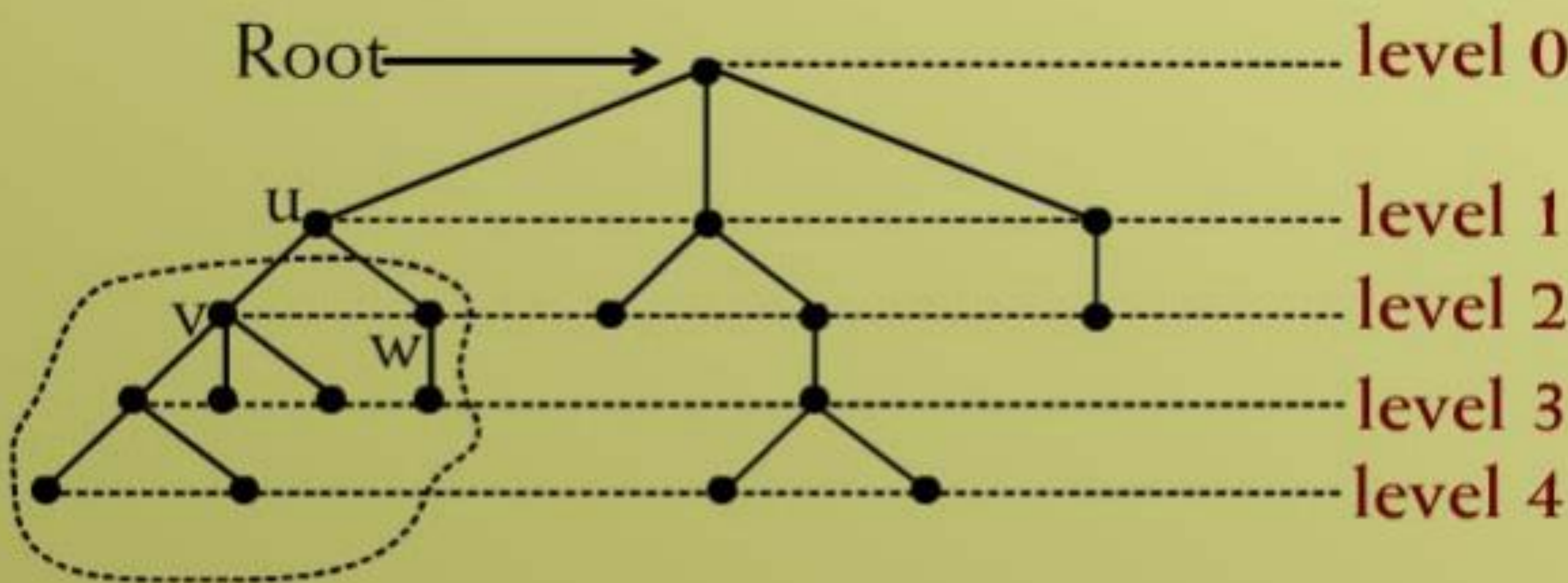
Subtree

If *a* is a vertex in a tree, the subtree with *a* as its root is the subgraph of the tree consisting of *a* and its descendants and all edges incident to these descendants.

EXAMPLE

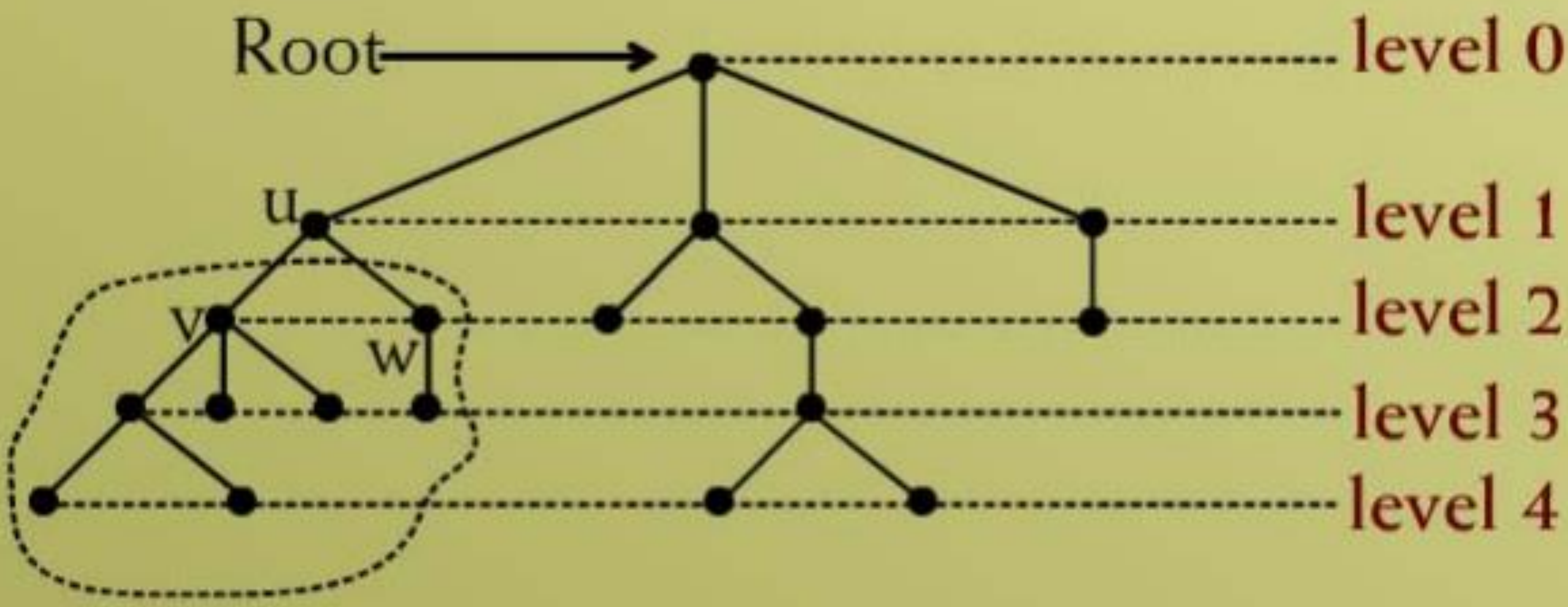


EXAMPLE



Vertices in enclosed region are descendants of u, which is an ancestor of each.

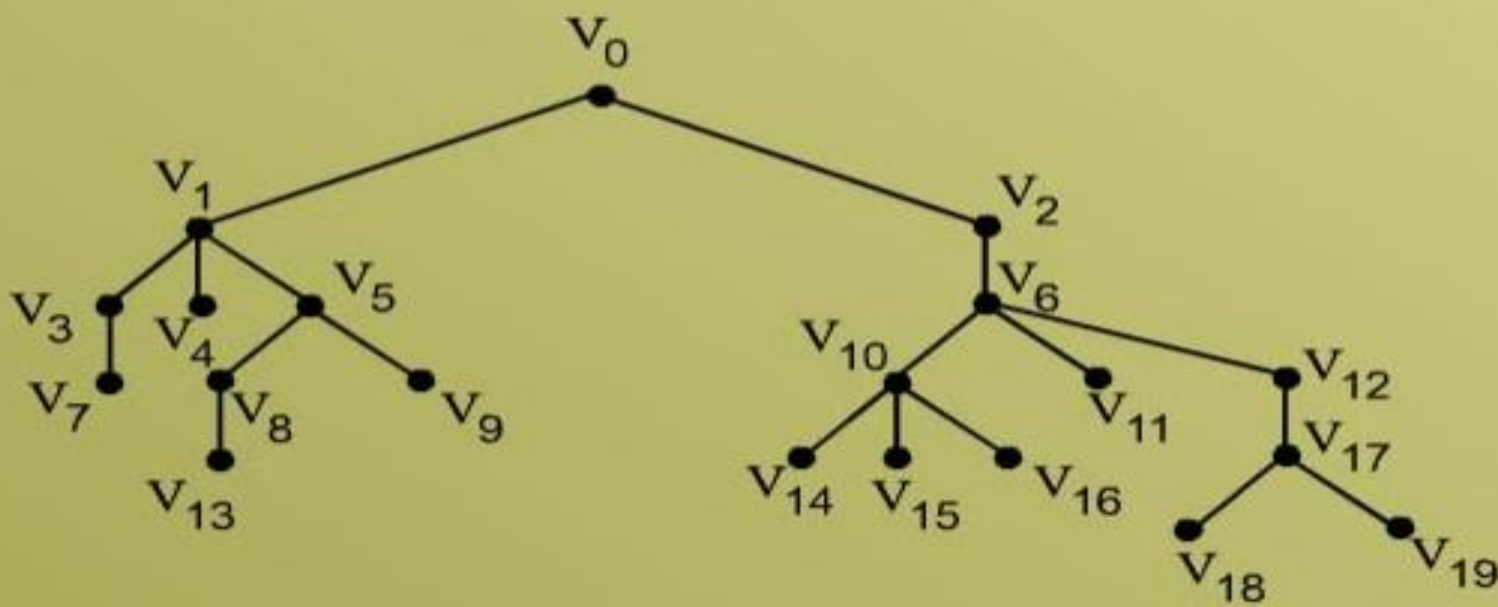
EXAMPLE



v is a child of u,u is the parent of v,v and w are sublings,height = 4.

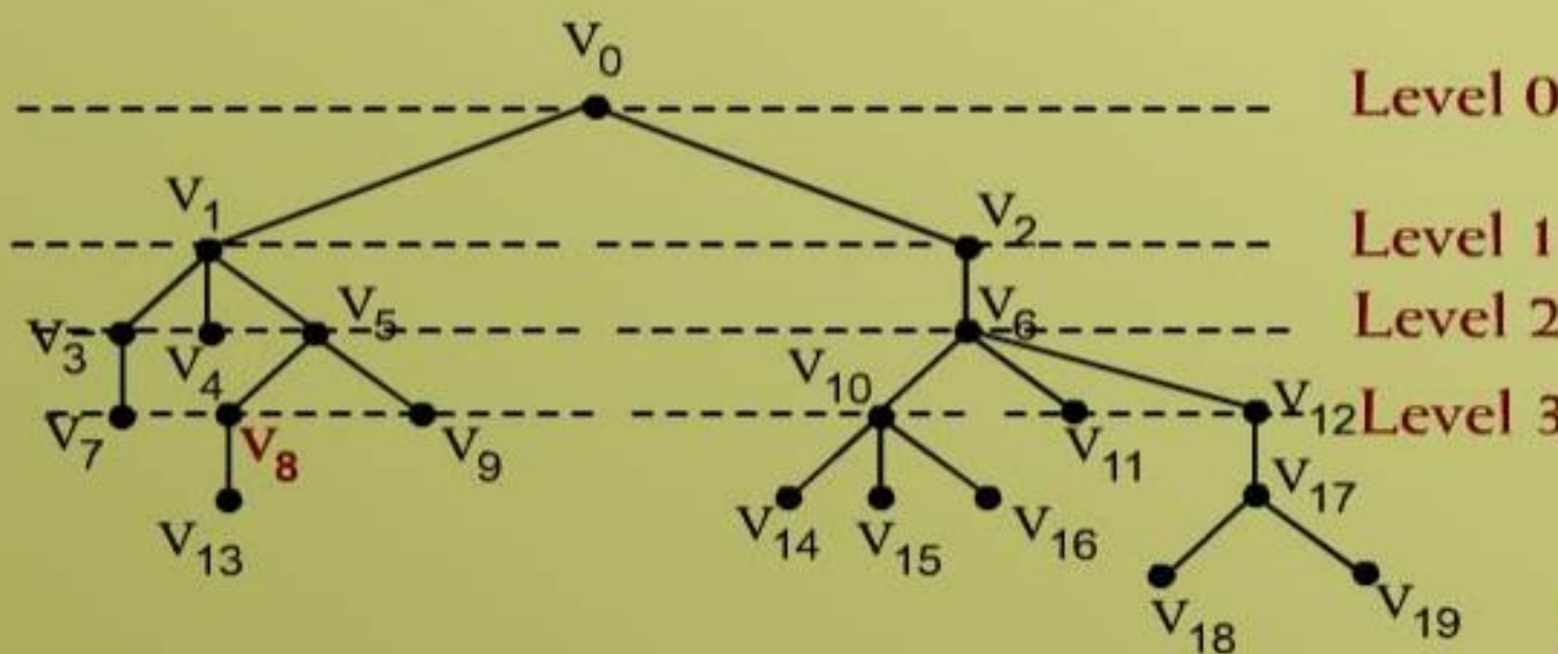
EXERCISE

Consider the **rooted tree** shown below with root v_0.



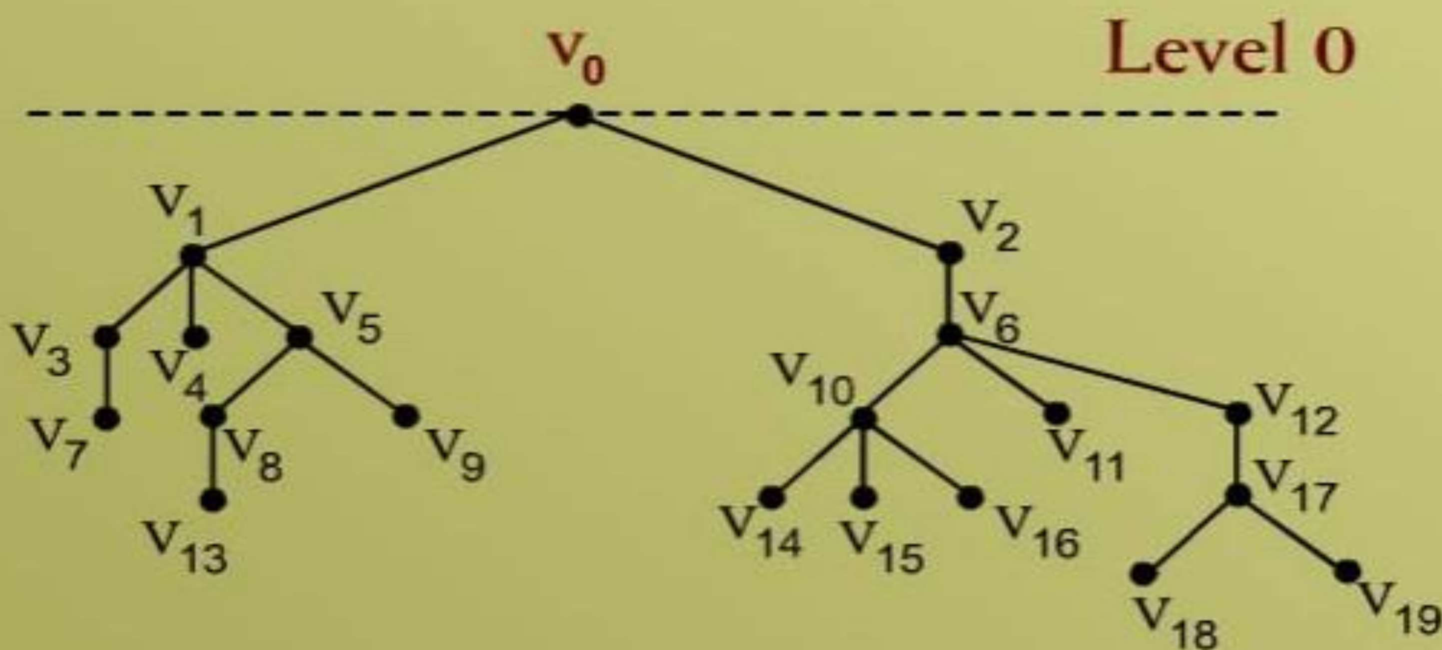
EXERCISE

a. What is the level of v_8?



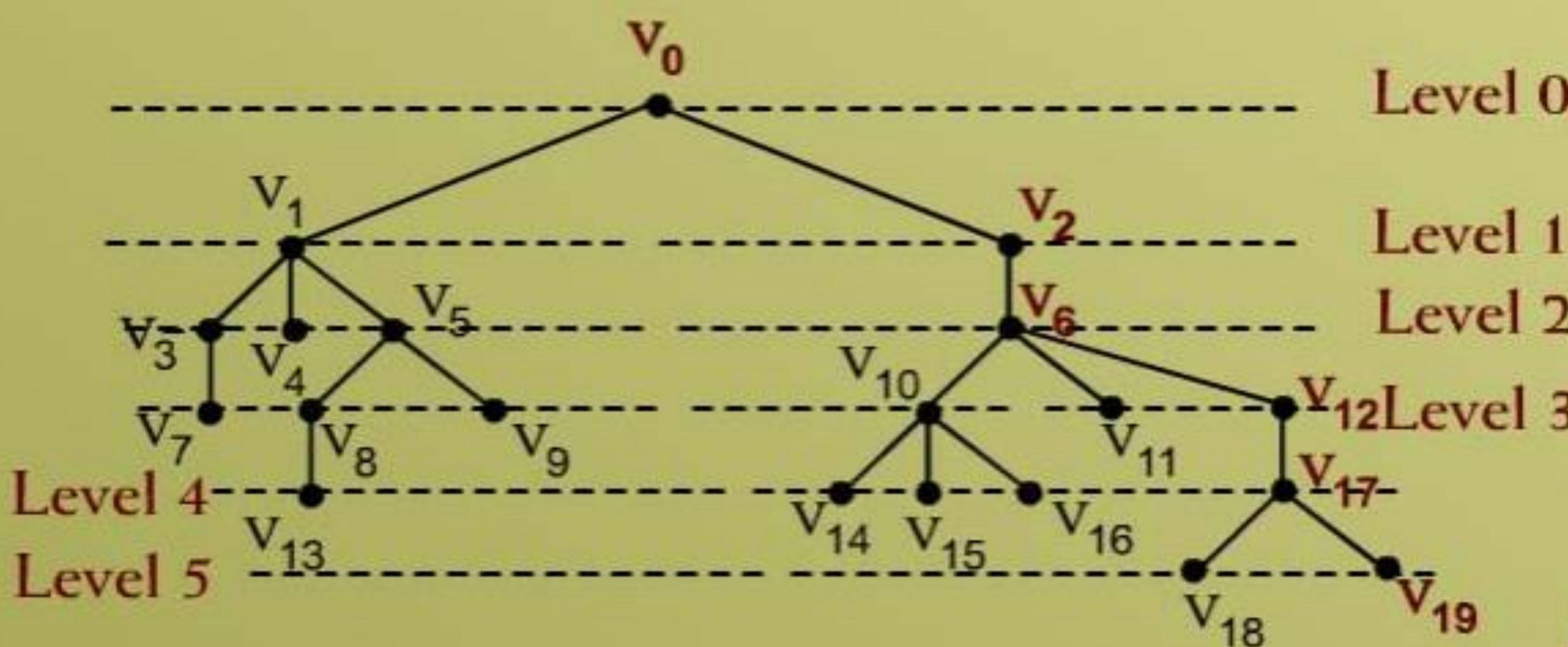
EXERCISE

b. What is the level of v_0?



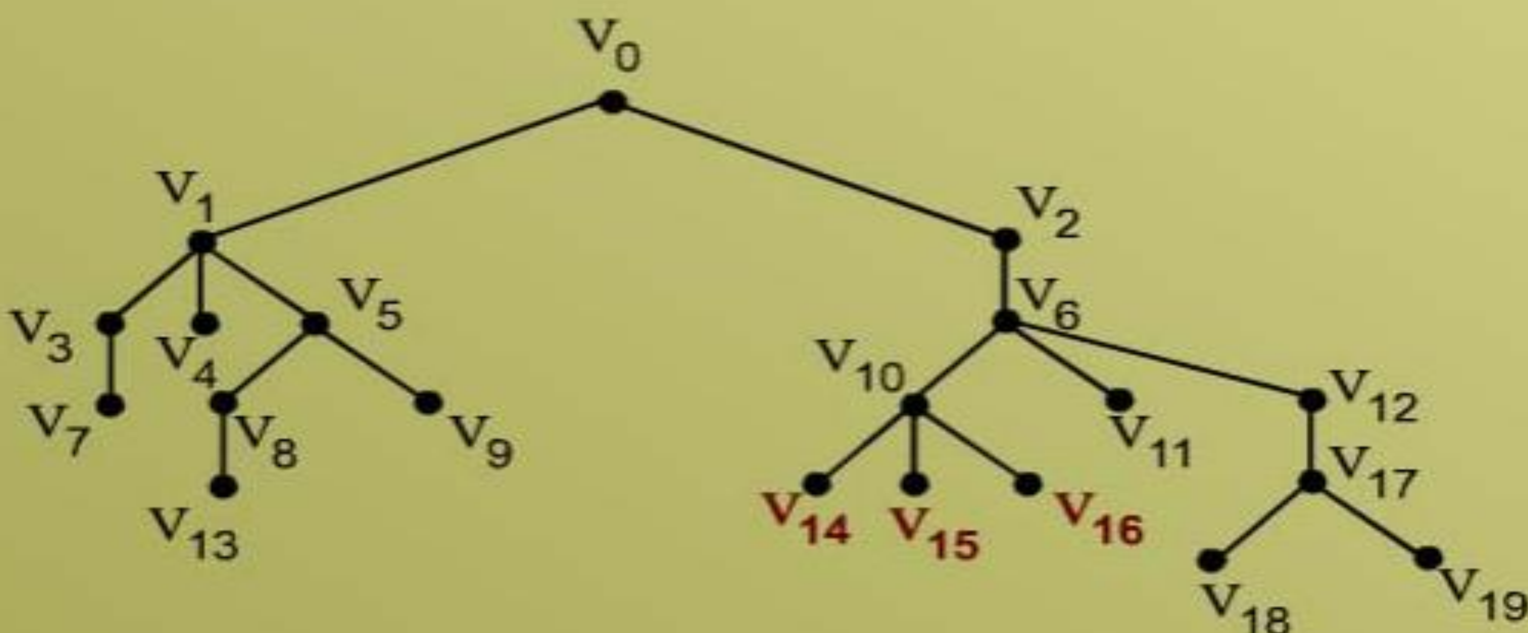
EXERCISE

c. What is the **height of this tree**?



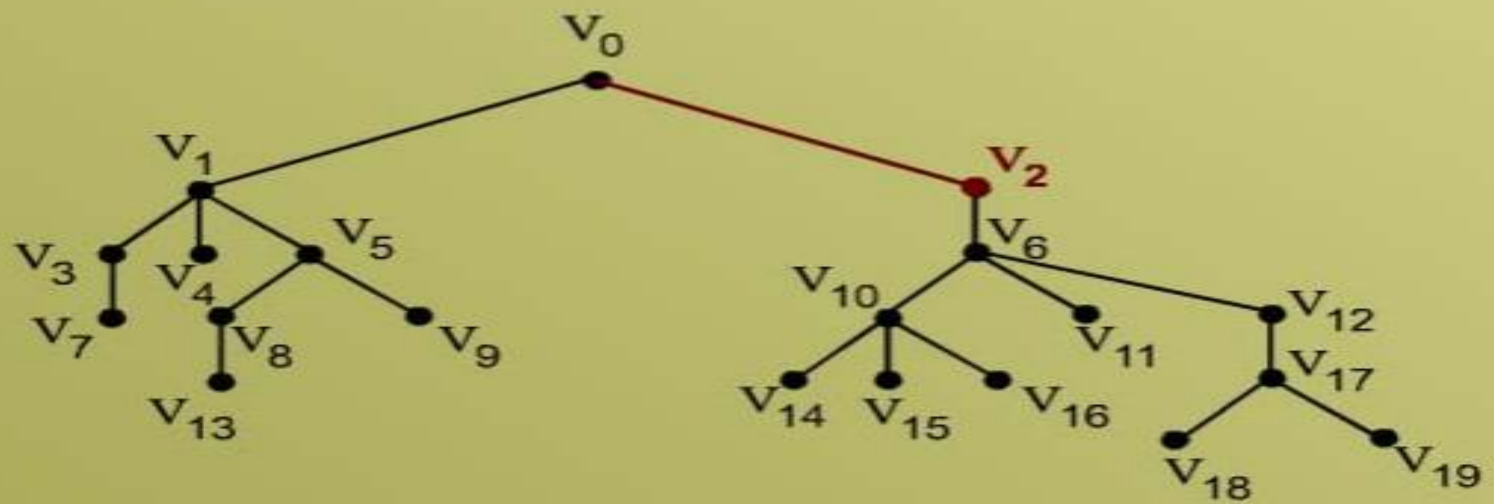
EXERCISE

d. What are the children of v_{10}?



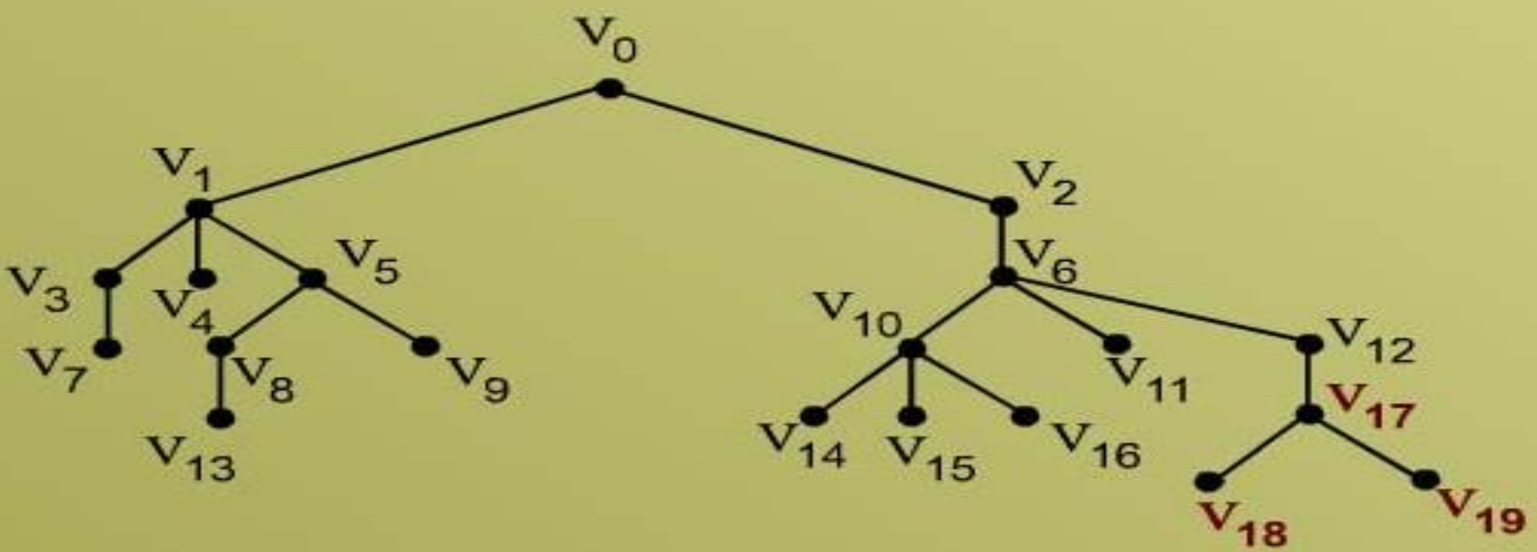
EXERCISE

e. What are the siblings of v_1?



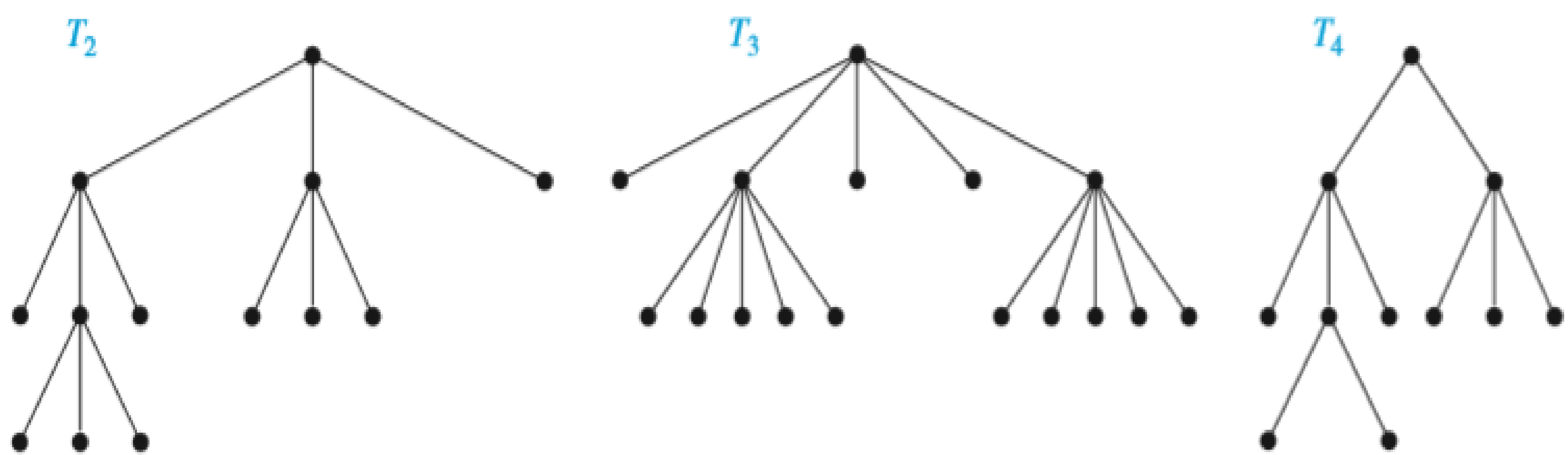
EXERCISE

f. What are the **descendants** of v_{12}?



m-ary TREE

- A rooted tree is called *m-ary* tree if every internal vertex has no more than *m* children.
- The tree is called *full m-ary tree* if every internal vertex has exactly *m* children.
- An *m-ary* tree with *m=2* is called a binary tree.



BINARY TREE

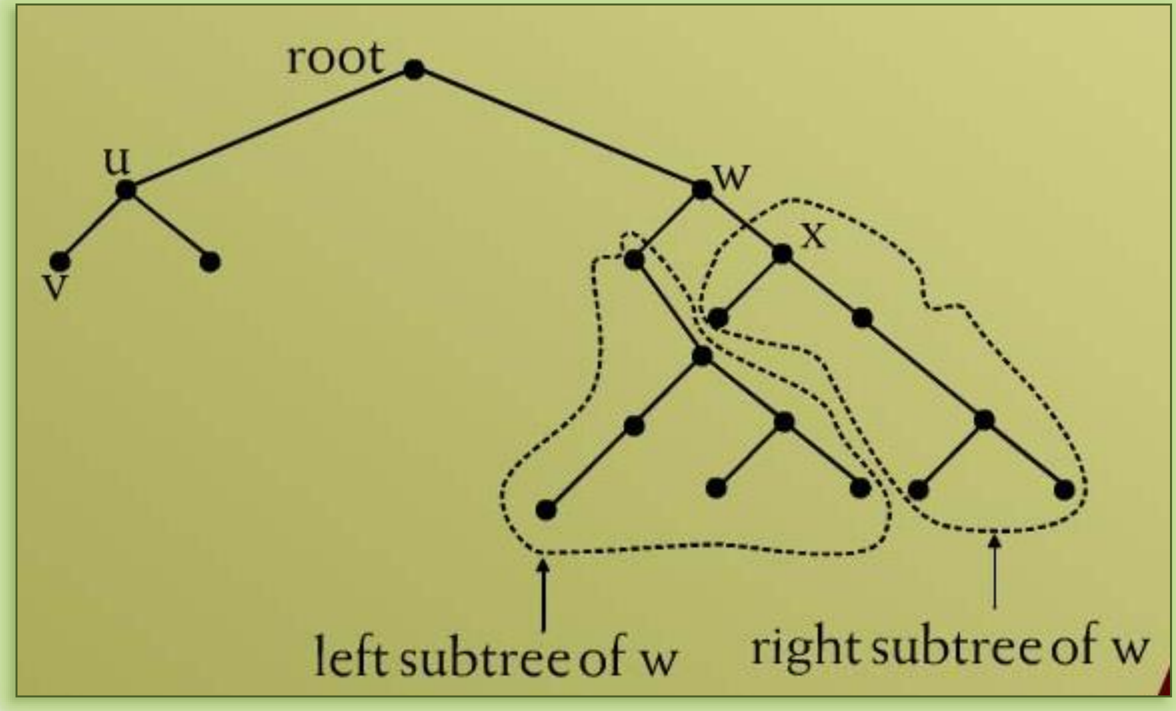
A binary tree is a rooted tree in which every internal vertex has at most two children.

Every child in a binary tree is designated either a left child or a right child.

FULL BINARY TREE

A **full binary tree** is a **binary tree** in which each **internal vertex** has **exactly two children**.

EXAMPLE



THEOREM

A *full m-ary* tree with k internal vertices contains $n=mk+1$ vertices.

1. If k is a positive integer and T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.

THEOREM

A *full m-ary* tree with

1. n vertices has $k= (n-1)/m$ internal vertices and $l=[(m-1)n + 1]/m$ leaves.
2. k internal vertices has $n=mk+1$ vertices and $l=(m-1)k + 1$ leaves.
3. l leaves has $n=(ml-1)/(m-1)$ vertices and $k=(l-1)/(m-1)$ internal vertices.

THEOREM

There are at most m^h leaves (terminal vertices) in an *m-ary* tree of height h.

THEOREM

2. If T is a binary tree that has t terminal vertices and height h, then

$$t \leq 2^h$$

Equivalently,

$$\log_2 t \leq h$$

EXERCISE

Explain why graphs with the given specification do not exist.

1. Full binary tree with nine vertices and five internal vertices.

2. Binary tree with height 4 and eighteen terminal vertices.

SOLUTION

1. Any full binary tree with five internal vertices has six terminal vertices, for a total of eleven, not nine vertices in all.

   Thus there is no full binary tree with the given properties.

2. Any binary tree of height 4 has at most 2^4 = 16 terminal vertices.

   Hence, there is no binary tree that has height 4 and eighteen terminal vertices.

EXERCISE

Draw a full binary tree with seven vertices.

SOLUTION:

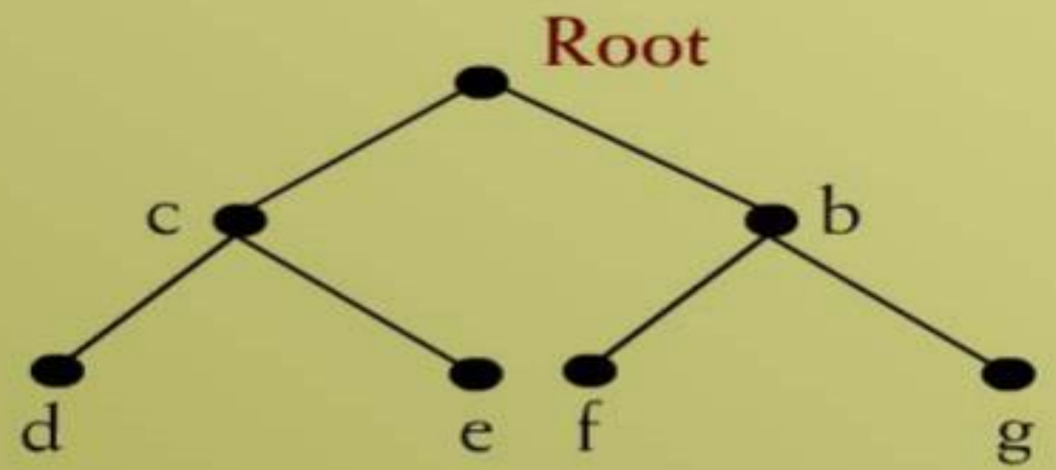
Total vertices $= 2k + 1 = 7$

$\Rightarrow \quad k = 3$

Hence, number of internal vertices $= k = 3$

Number of terminal vertices $= k + 1 = 4$

EXERCISE

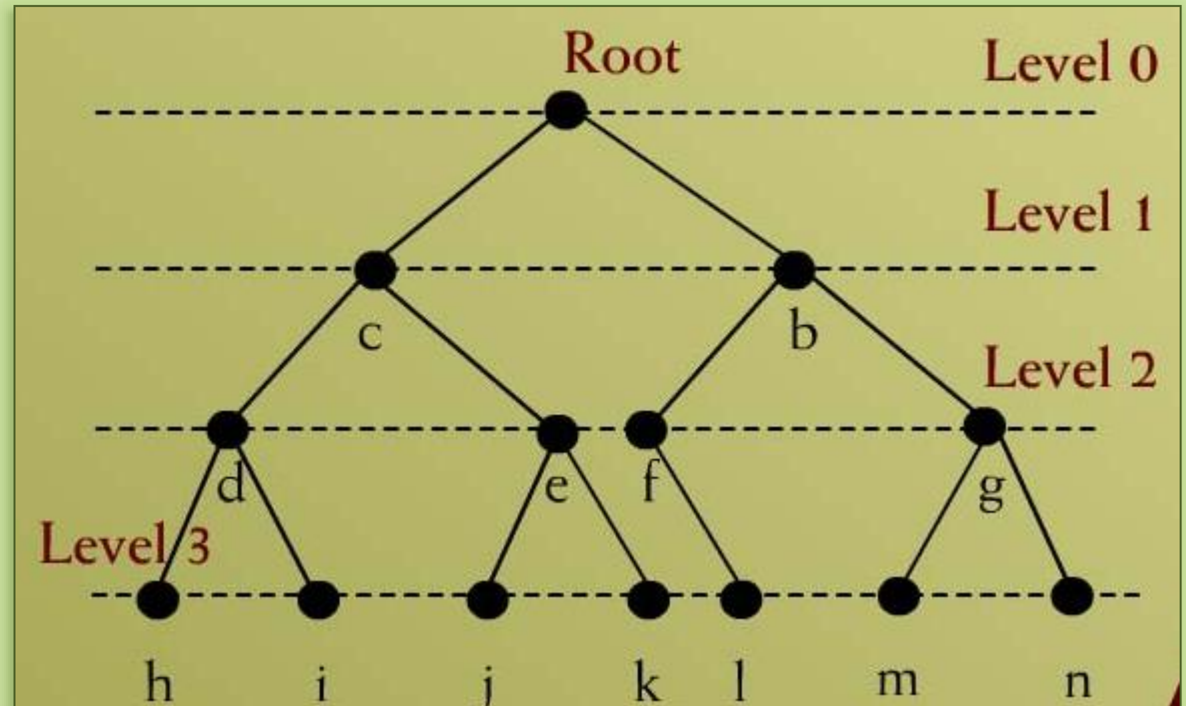


Which is required full binary tree with seven vertices.

EXAMPLE

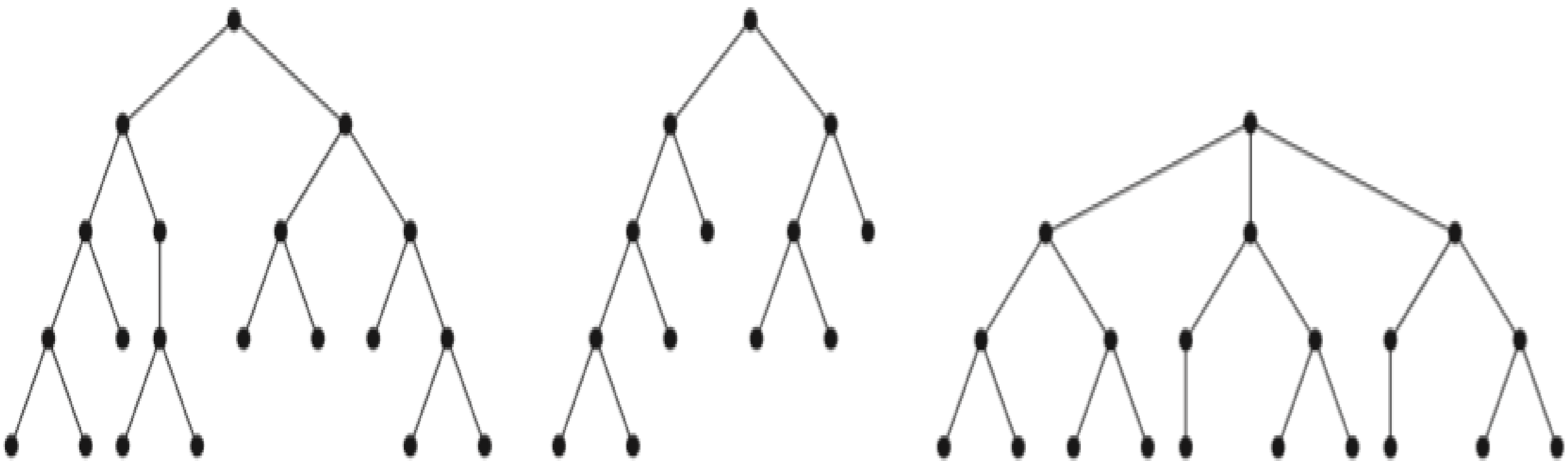
Draw a binary tree with height 4 (level 3) and having seven terminal vertices.

SOLUTION



Balanced Rooted Tree

A rooted *m-ary* tree of height *h* is balanced if all leaves are at levels *h* or *h-1*.



T_1 T_2 T_3