Discrete Structures

Lecture # 4

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Recap

- Conditional Statements
- Bi-conditional Statements
- Conversion of NLP to Argument and vice versa
- Inverse / Converse / Contrapositive of Bi-conditional Statements
- Examples / Exercise

Argument

An argument is a list of statement called premises (or assumptions or hypotheses) followed by a statement called the conclusion.

Valid and Invalid Arguments

- Propositional logic can be used as a math model to investigate the validity of arguments.
- As argument is a sequence of statements.
- All but the final statements are called premises.
- Final statement is called conclusion.
- Valid Argument: If the premises are all true then the conclusion is also true.
  - $P_1 \wedge \ldots \wedge P_n \vdash Q$

  i.e. Premises logically implies the conclusion.

Argument Validity

- ❑Two Ways:
  - ❑Using truth Tables
  - ❑Reason at a higher level using generally valid rules (inference values).

Argument

P1	Premise
P2	Premise
P3	Premise
P4	Premise
……..	……………..
∴ **C**	Conclusion

Valid Argument

An argument is valid if the conclusion is true when all the premises are true.

Invalid Argument

An argument is invalid if the conclusion is false when all the premises are true.

Example of Valid Argument

Show that the following argument form is valid:

$p \rightarrow q$	premise
p	premise
$\therefore q$	Conclusion

		Premise		Conclusion
p	**q**	**$p \rightarrow q$**	**p**	**q**
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

The given argument is valid.

Example of Invalid Argument

Show that the following argument form is valid:

$p \to q$	premise
q	premise
$\therefore p$	Conclusion

p	q	$p \to q$ (Premise)	q (Premise)	p (Conclusion)
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

The given argument is Invalid.

Example

Show that the following argument form is valid:

$p \vee q$ premise

Premise

$p \rightarrow \sim q$ premise

Conclusion

$p \rightarrow r$ premise

$\therefore r$ Conclusion

p	**q**	**r**	**$p \vee q$**	**$p \rightarrow \sim q$**	**$p \rightarrow r$**	**r**

Example

Premise | Conclusion

p	q	r	p ∨ q	p → ~q	p → r	r
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

The given argument is invalid.

Exercise – 1

If Tariq is not on team A, then Hameed is on team B

If Hameed is not on team B, then Tariq is on team A.

∴ If Hameed is not on team B, then Tariq is not on team A.

Solution:

Let t = Tariq is on team A

h = Hameed is on team B

Exercise – 1 – Cont..

Solution: Let

t = Tariq is on team A

h = Hameed is on team B

1. If Tariq is not on team A, then Hameed is on team B -- $(\sim t \rightarrow h)$

2. If Hameed is not on team B, then Tariq is on team A -- $(\sim h \rightarrow t)$

3. $\therefore$ If Hameed is not on team A, then Tariq is not on team B

   B -- $\sim h \rightarrow \sim t$

Exercise – 1 – Cont..

1. $(\sim t \rightarrow h)$
2. $(\sim h \rightarrow t)$
3. $\therefore \sim h \rightarrow \sim t$

		Premise		Conclusion
t	**h**	$\sim t \rightarrow h$	$\sim h \rightarrow t$	$\sim h \rightarrow \sim t$
T	T	T	T	T
T	F	T	T	F
F	T	T	T	T
F	F	F	F	T

The given argument is invalid.

Exercise – 2

If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.

Neither of these two numbers is divisible by 6.

$\therefore$ The product of these two numbers is not divisible by 6.

Solution: Let

d = at least one of these two numbers is divisible by 6

p = product of these two numbers is divisible by 6

Exercise – 2 – Cont..

Solution: Let,

d = at least one of these two numbers is divisible by 6

p = product of these two numbers is divisible by 6.

1. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6 -- $(d \rightarrow p)$

2. Neither of these two numbers is divisible by 6 -- $\sim d$

3. $\therefore$ The product of these two numbers is not divisible by 6 -- $\sim p$

Exercise – 2 – Cont..

Solution:

1. $(d \rightarrow p)$
2. $\sim d$
3. $\therefore \sim p$

		Premise		Conclusion
d	**p**	**$d \rightarrow p$**	**$\sim d$**	**$\sim p$**
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

The given argument is invalid.

Exercise – 3

If I got an Eid Bonus, I'll buy a stereo.

If I sell my motorcycle, I'll buy a stereo.

∴ If I get Eid bonus or I sell my motorcycle, then I'll buy a stereo.

Solution: Let

e = I got an Eid Bonus

s = I'll buy a stereo

m = I sell my motorcycle

Exercise – 3 – Cont..

Solution: Let, e = I got an Eid Bonus; s = I'll buy a stereo; m = I sell my motorcycle

1. If I got an Eid Bonus, I'll buy a stereo -- $(e \rightarrow s)$

2. If I sell my motorcycle, I'll but a stereo -- $(m \rightarrow s)$

3. ∴ If I get Eid bonus or I sell my motorcycle, then I'll buy a stereo – $e \vee m \rightarrow s$

e	s	m	$e \rightarrow s$	$m \rightarrow s$	$e \vee m$	$e \vee m \rightarrow s$

Exercise – 3 – Cont..

Solution: $(e \rightarrow s)$; $(m \rightarrow s)$; $\therefore e \vee m \rightarrow s$

e	s	m	$e \rightarrow s$	$m \rightarrow s$	$e \vee m$	$e \vee m \rightarrow s$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	F	T	F
F	F	F	T	T	F	T

The given argument is valid.

Inference Rule

- To helps showing that a conclusion follows logically from a set of premises we may apply inference rules on the form,

$$p_1 \dots p_n \mid \therefore q$$

- The validity of the rule is ensured

  If $(p_1 \wedge \dots \wedge p_n) \rightarrow q$ is a Tautology

  - A tautology is a statement which is always true. E.g. $p \vee \sim p$.

Inference Rule

- **Modus Ponens**

$$\begin{array}{c} p \\ \underline{p \to q} \\ \therefore q \end{array} \quad \text{(Based on } [p \wedge (p \to q) \to q]\text{)}$$

- **Modus Tollens**

$$\begin{array}{c} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array} \quad \text{(Based on } [(p \to q) \wedge \sim q \to \sim p]\text{)}$$

- **Generalization**

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array} \;,\; \begin{array}{c} q \\ \hline \therefore p \vee q \end{array}$$

Inference Rule

- **Specialization** $\dfrac{p \wedge q}{\therefore p}$, $\dfrac{p \wedge q}{\therefore q}$

- **Elimination** $\dfrac{\begin{array}{c} p \vee q \\ \sim q \end{array}}{\therefore p}$, $\dfrac{\begin{array}{c} p \vee q \\ \sim p \end{array}}{\therefore q}$

- **Conjunction** $\dfrac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$

- **Transitivity** $\dfrac{\begin{array}{c} p \to q \\ q \to r \end{array}}{\therefore p \to r}$

Inference Rule -- Application – An Example

- Example: You are about to leave for University in the morning and discover that you don't have your glasses. You know the following statements are true.

  A. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

  B. If my glasses are on the kitchen table, then I saw them at breakfast.

  C. I did not see my glasses at breakfast.

  D. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

  E. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses??

Inference Rule -- Application – An Example

Assume, RK= Reading the newspaper in the kitchen.

GK= Glasses are on the kitchen table.

SB= I saw my glasses at breakfast.

RL= Reading the newspaper in the living room.

GC= Glasses are on the coffee table.

So by rules of inference,

1.
$$\frac{\begin{array}{ll} RK \rightarrow GK & \text{(by A)} \\ GK \rightarrow SB & \text{(by D)} \end{array}}{\therefore RK \rightarrow SB \text{ (Transiticity)}}$$

2.
$$\frac{\begin{array}{ll} RK \rightarrow SB & \text{(by 1)} \\ \sim SB & \text{(by C)} \end{array}}{\therefore \sim RK \quad \text{(by modus tollens)}}$$

3.
$$\frac{\begin{array}{ll} RL \vee RK & \text{(by D)} \\ \sim RK & \text{(by 2)} \end{array}}{\therefore RL \quad \text{(by elimination)}}$$

4.
$$\frac{\begin{array}{ll} RL \rightarrow GC & \text{(by C)} \\ RL & \text{(by 3)} \end{array}}{\therefore GC \quad \text{(by modus ponens)}}$$

So the Glasses are on the Coffee table.

Contradiction and Valid Arguments.

- Contradiction Rule

  Suppose p is some statement whose truth you wish to deduce.

  If you can show that the supposition that p is false leads logically to a contradiction, then you can conclude that p is true.

Contradiction and Valid Arguments.

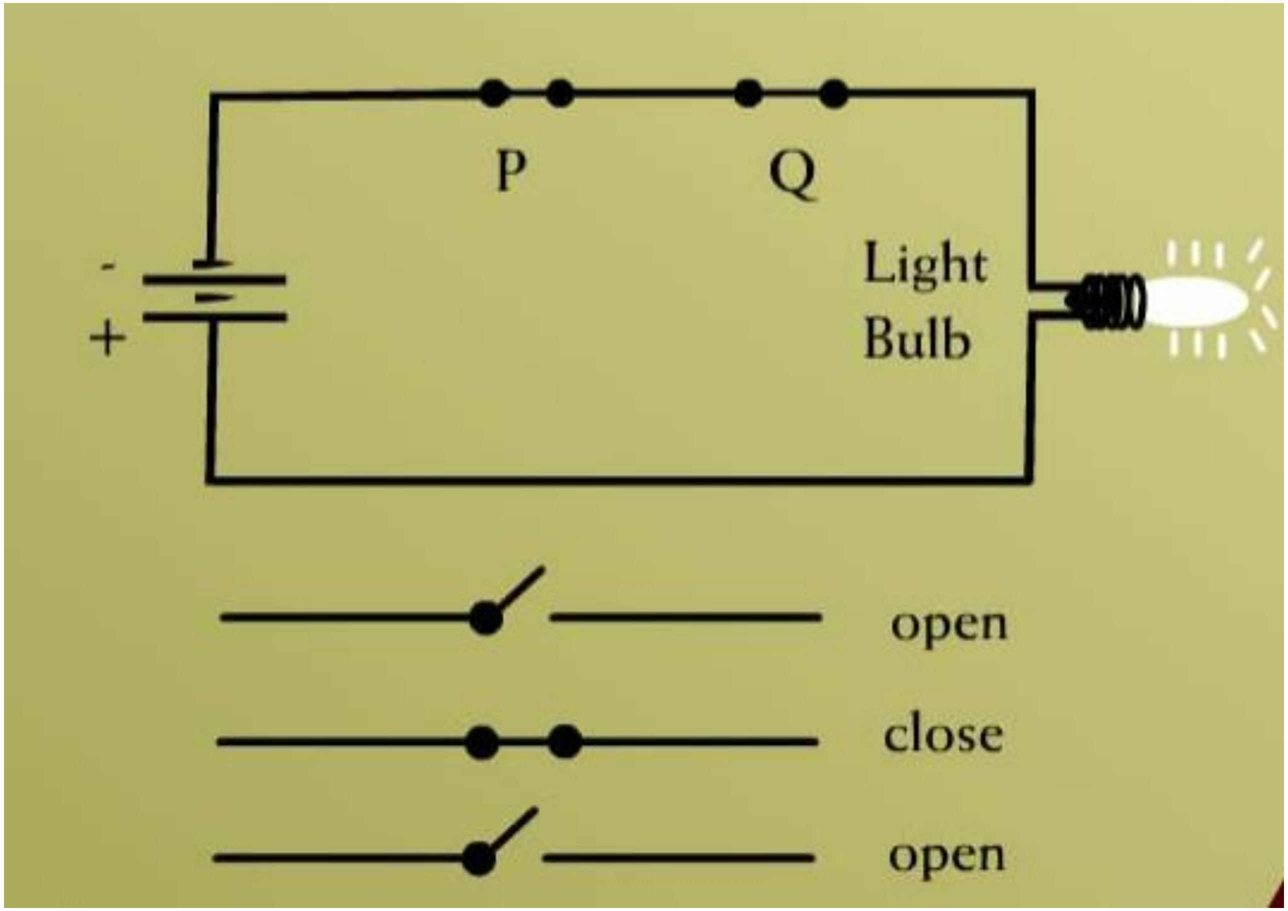
- Contradiction Rule

$$\begin{array}{l} \sim p \to c \\ \hline \therefore p \end{array}, \text{ where c is a contradiction}$$

p	$\sim p$	c	$\sim p \to c$	p
T	F	F	T	T
F	T	F	F	F

Logical heart of the method of proof by contradiction.

Switches in Series



Switches in Series

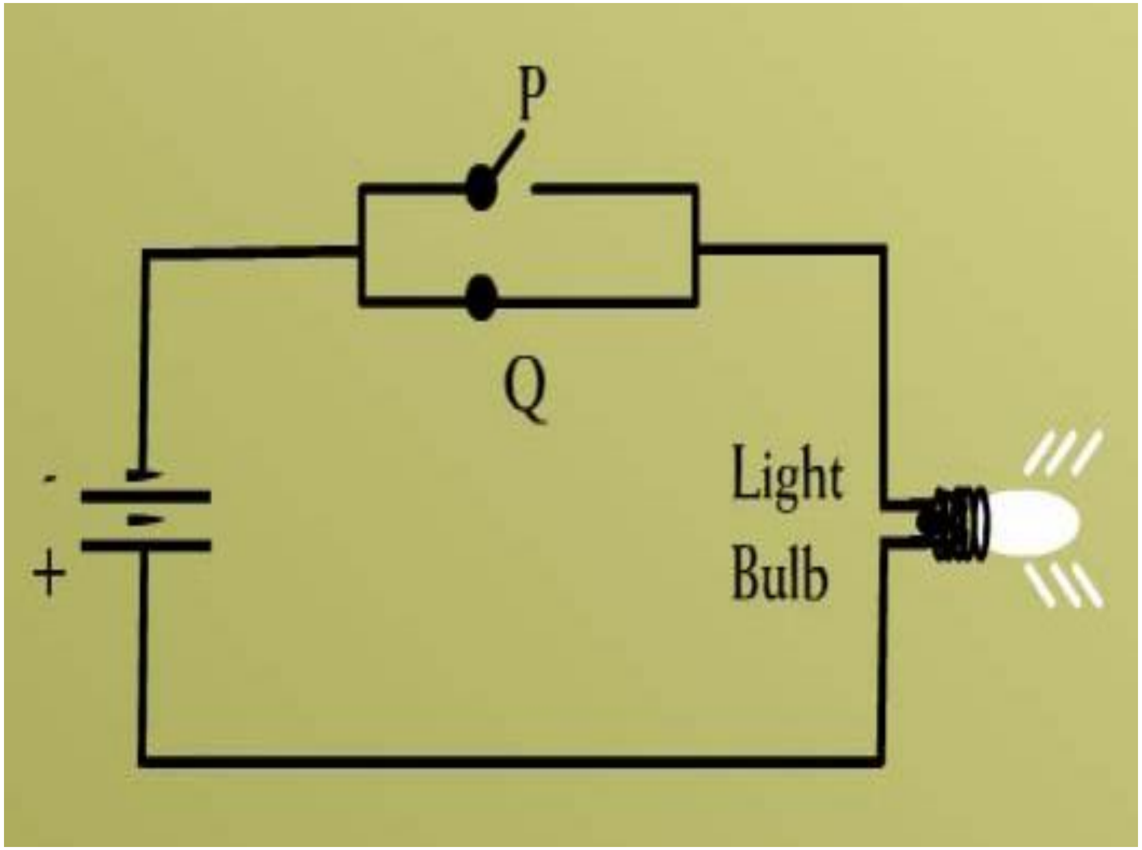
Switches		Light Bulb
P	Q	State
Open	Open	Off
Open	Closed	Off
Closed	Open	Off
Closed	Closed	On

Switches in Series

Switches		Light Bulb
P	Q	State
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Switches in Parallel



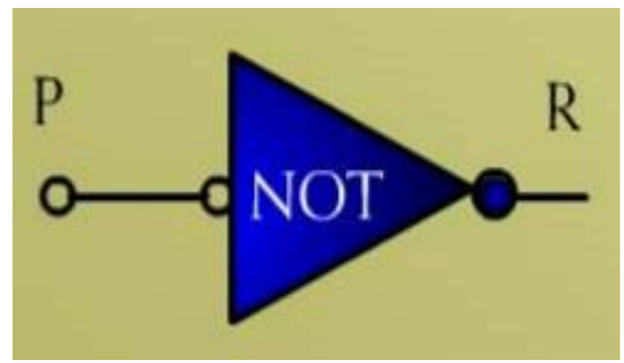
Switches in Parallel

Switches		Light Bulb
P	Q	State
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

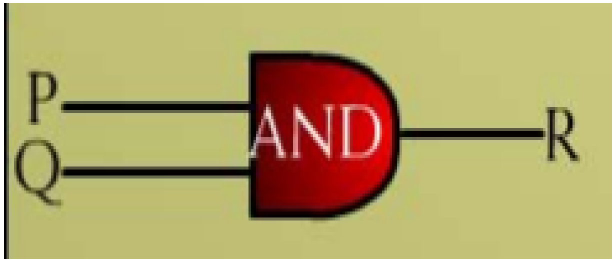
Not Gate or Inverter

Input	Output
P	Q
1	0
0	1



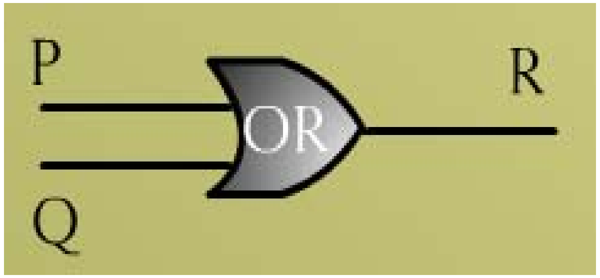
AND Gate

Input		Output
P	Q	R
1	1	1
1	0	0
0	1	0
0	0	0

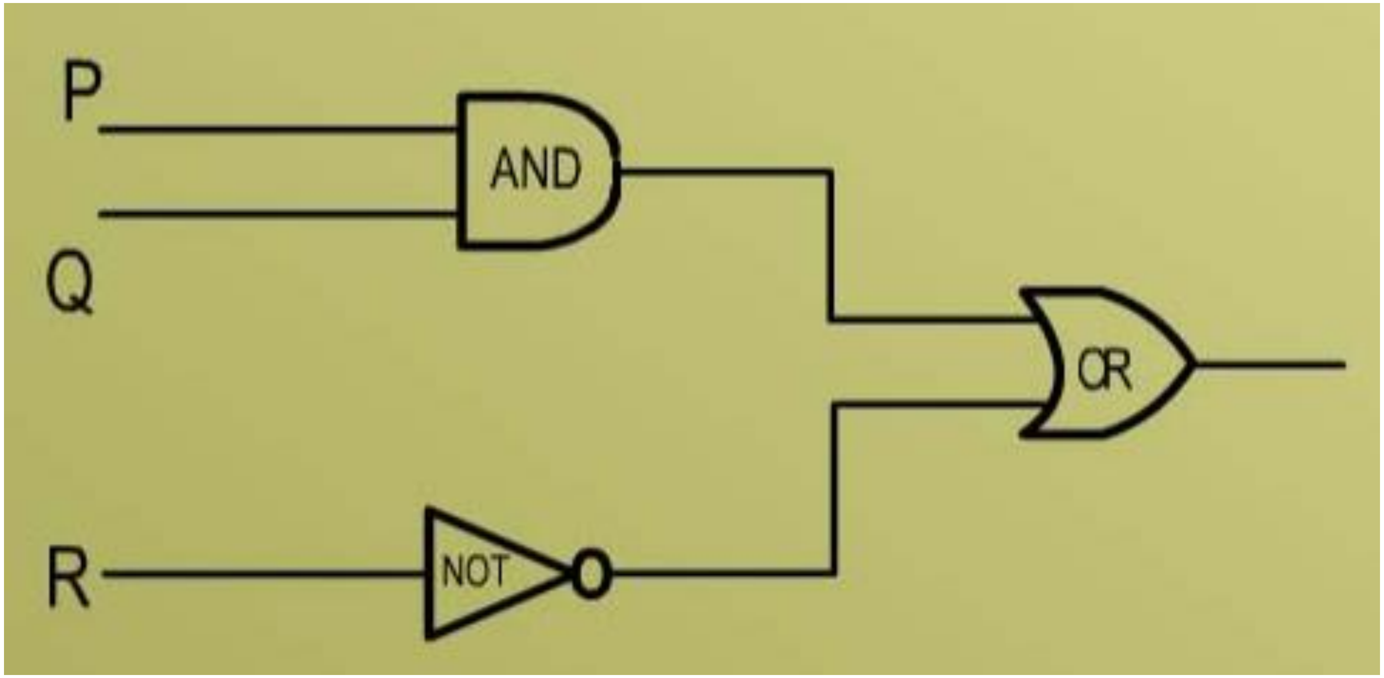


OR Gate

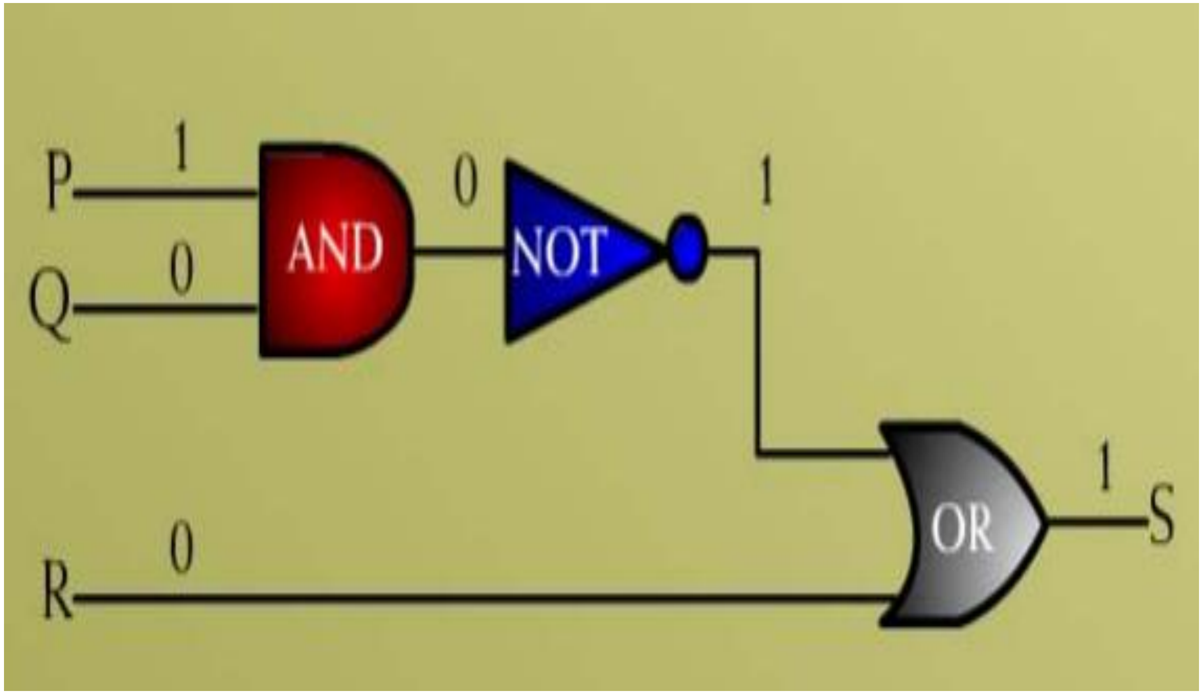
Input		Output
P	Q	R
1	1	1
1	0	1
0	1	1
0	0	0



Combinational Circuit



Output for a given Input



Input / Output table for a circuit

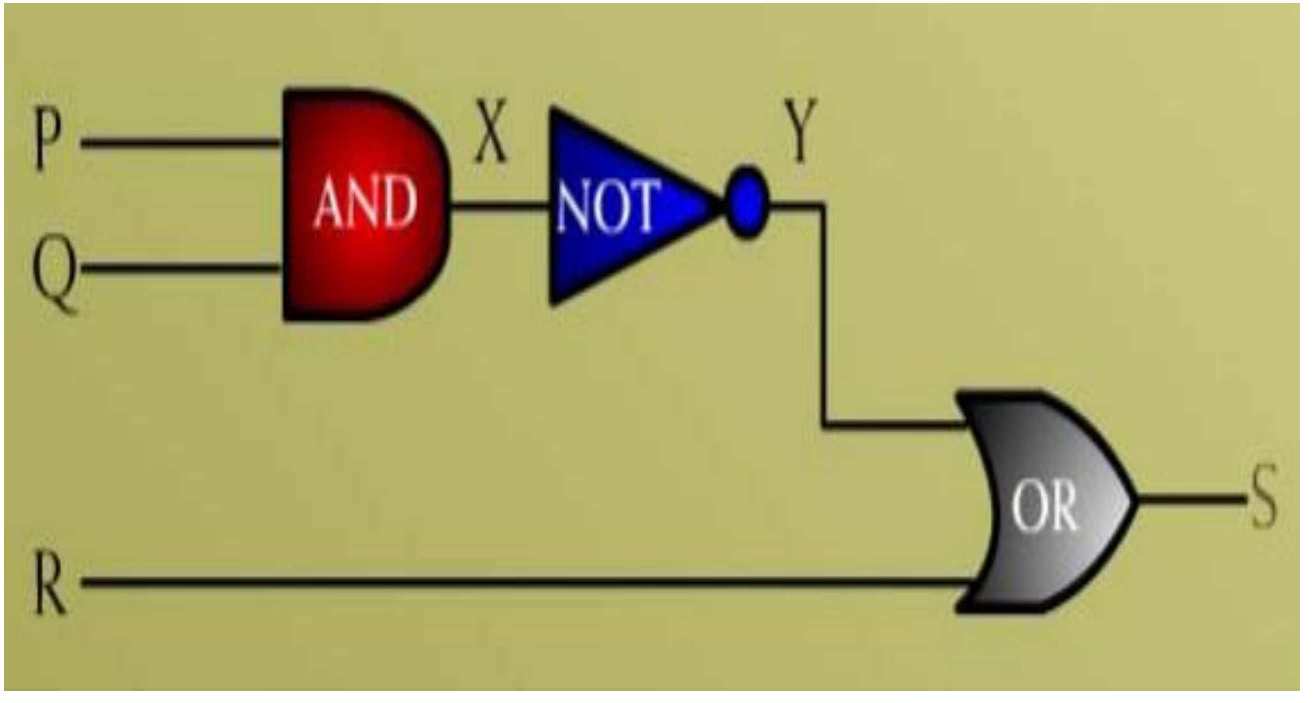


Table for a circuit

P	Q	R	X	Y	S
1	1	1			
1	1	0			
1	0	1			
1	0	0			
0	1	1			
0	1	0			
0	0	1			
0	0	0			

Table for a circuit – Cont.

P	Q	R	X	Y	S
1	1		1		
1	1		1		
1	0		0		
1	0		0		
0	1		0		
0	1		0		
0	0		0		
0	0		0		

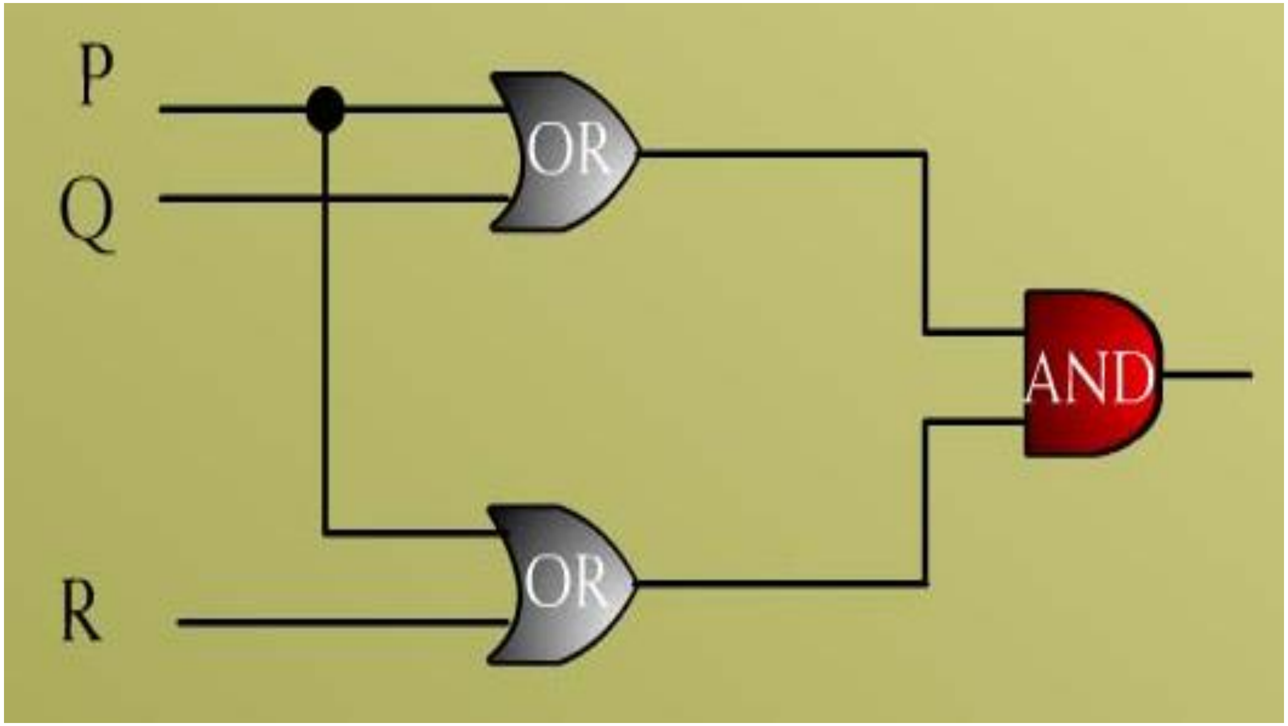
Table for a circuit – Cont.

P	Q	R	X	Y	S
			1	0	
			1	0	
			0	1	
			0	1	
			0	1	
			0	1	
			0	1	
			0	1	

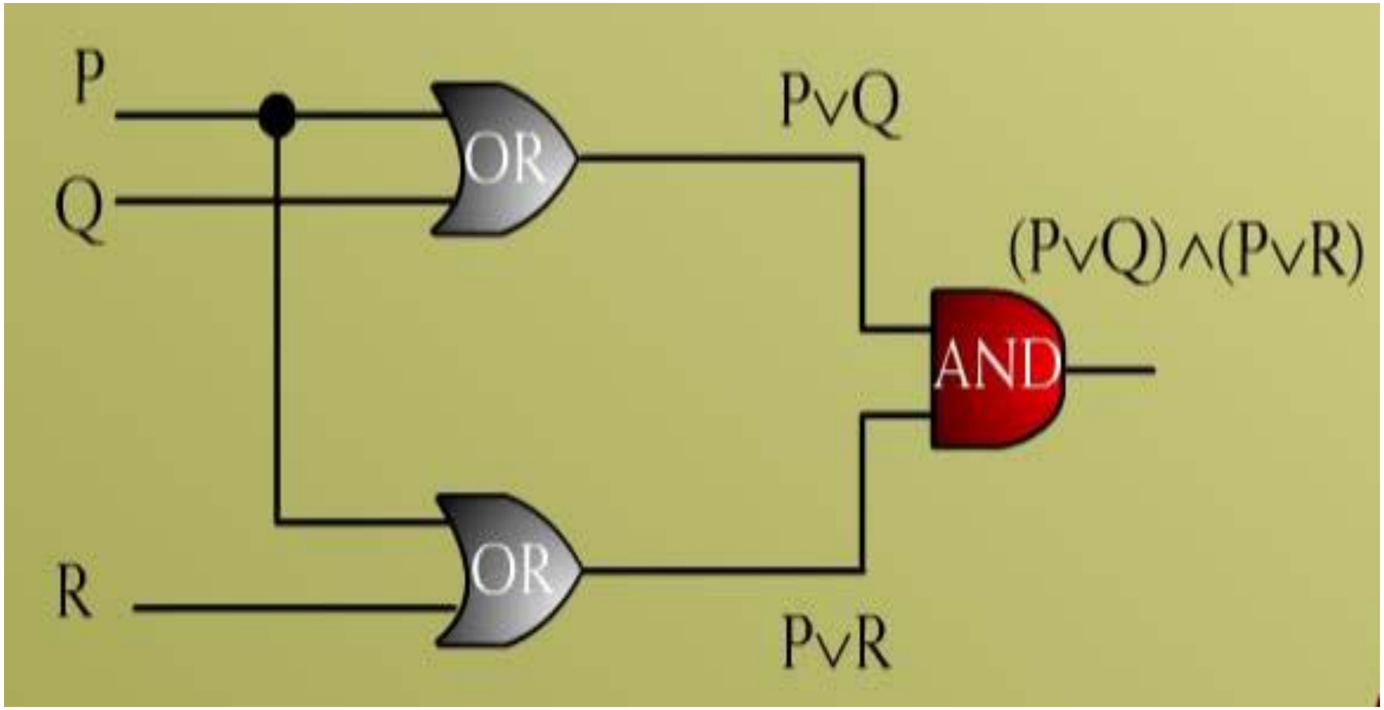
Table for a circuit – Cont.

P	Q	R	X	Y	S
		1		0	1
		0		0	0
		1		1	1
		0		1	1
		1		1	1
		0		1	1
		1		1	1
		0		1	1

Boolean Expression for a Circuit

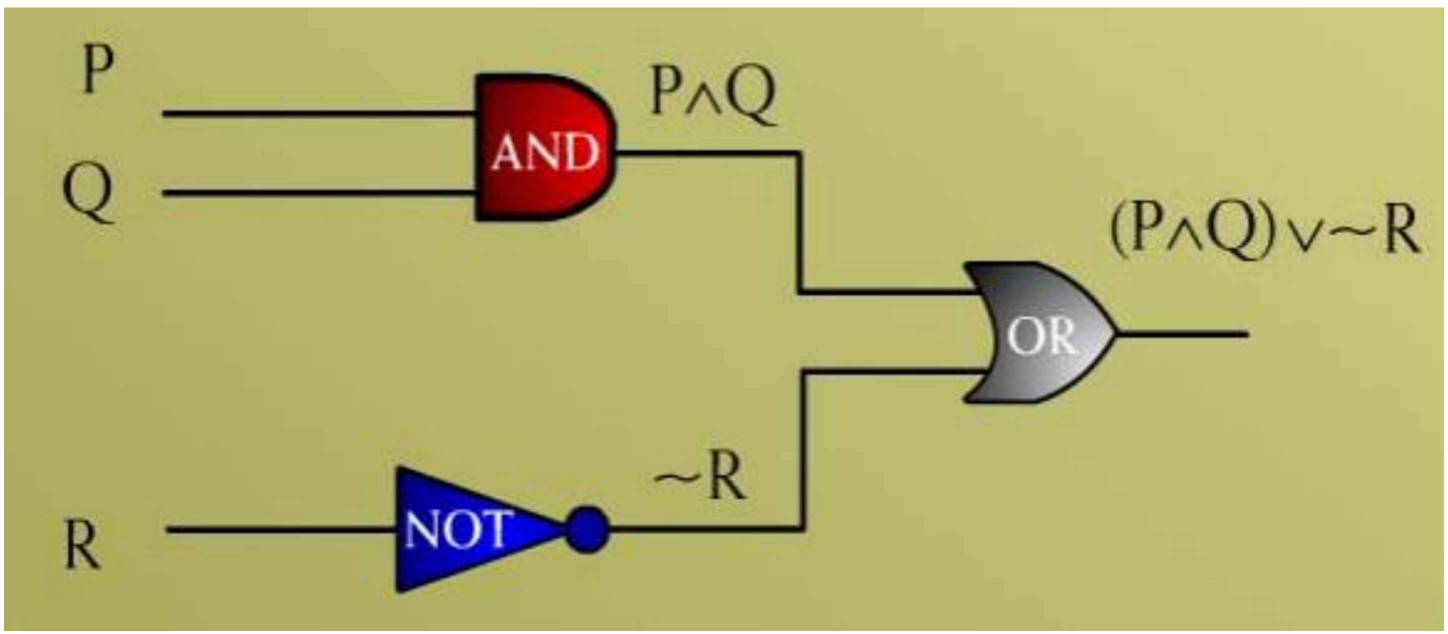


Boolean Expression for a Circuit



Circuit for a Boolean Expression

$$(P \wedge Q) \vee \sim R$$



Circuit for Input / Output Table

INPUTS			OUTPUTS
P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

Circuit for Input / Output Table – Sol.

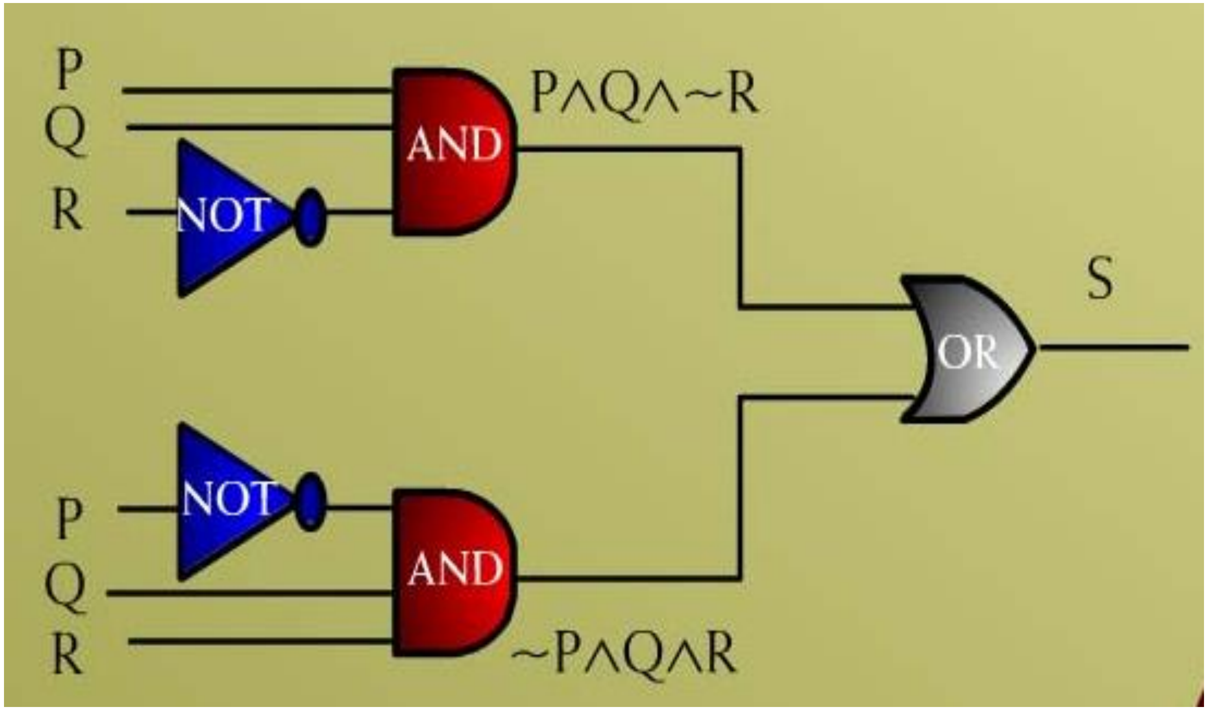
INPUTS			OUTPUTS
P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

$P \wedge Q \wedge \sim R$

$\sim P \wedge Q \wedge R$

Circuit Diagram

$$(P \land Q \land \sim R) \lor (\sim P \land Q \land R) = S$$



Exercise – 1

Design a circuit to take input signals P,Q, and R and output a 1 if, and only if, P and Q have the same value and Q and R have opposite values.

Exercise – 1: Sol.

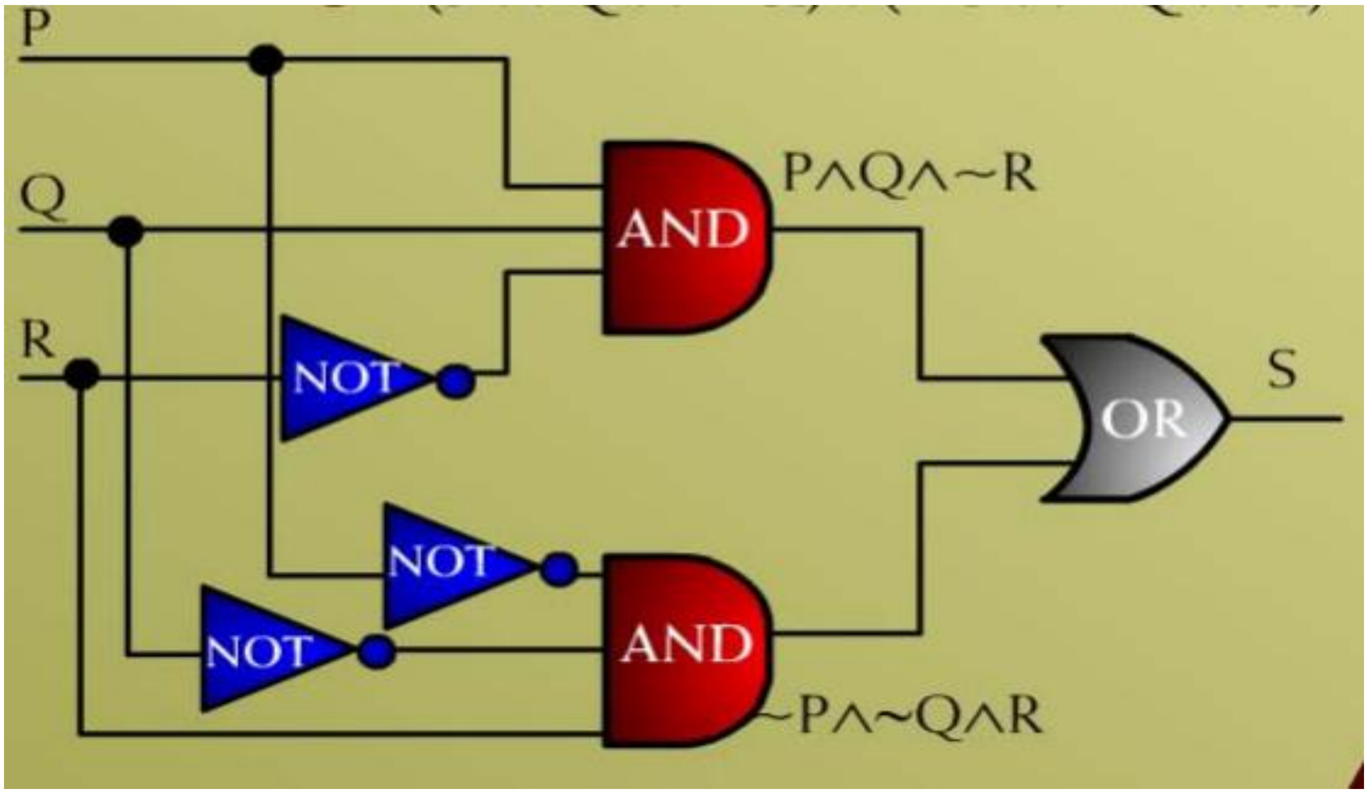
INPUTS			OUTPUTS
P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

P ^ Q ^ ~R

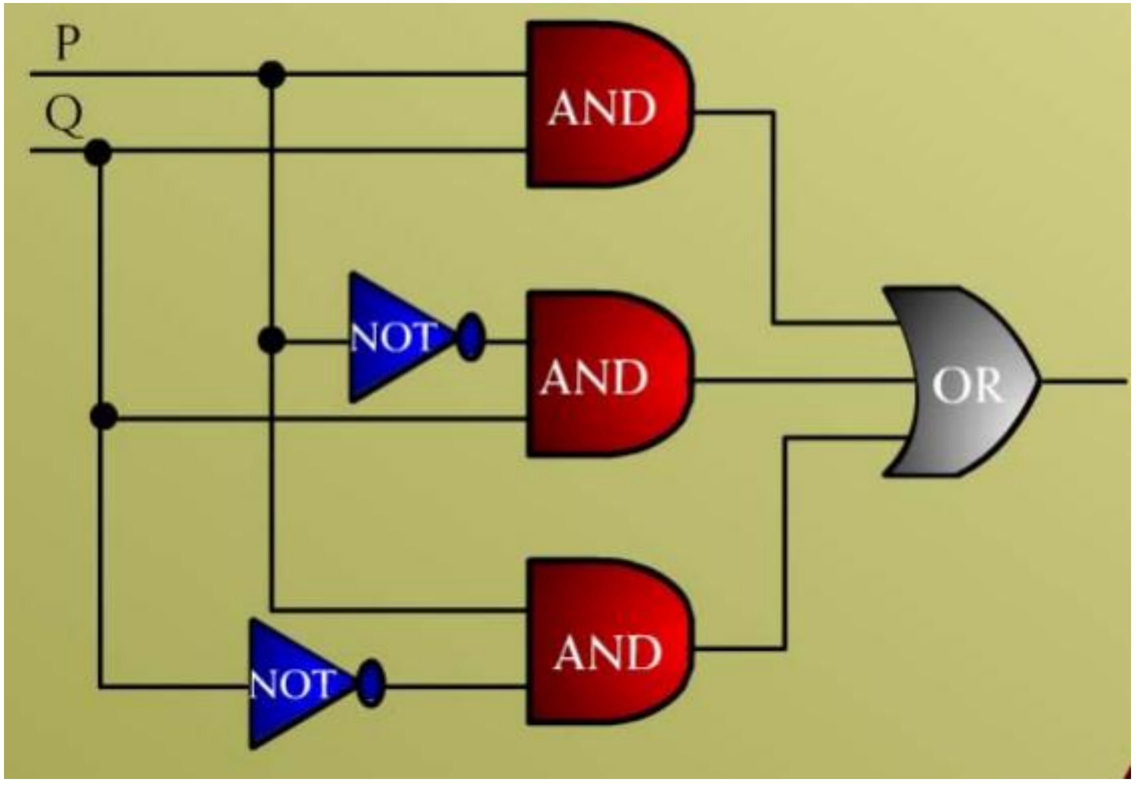
~P ^ ~Q ^ R

Exercise – 1: Sol.

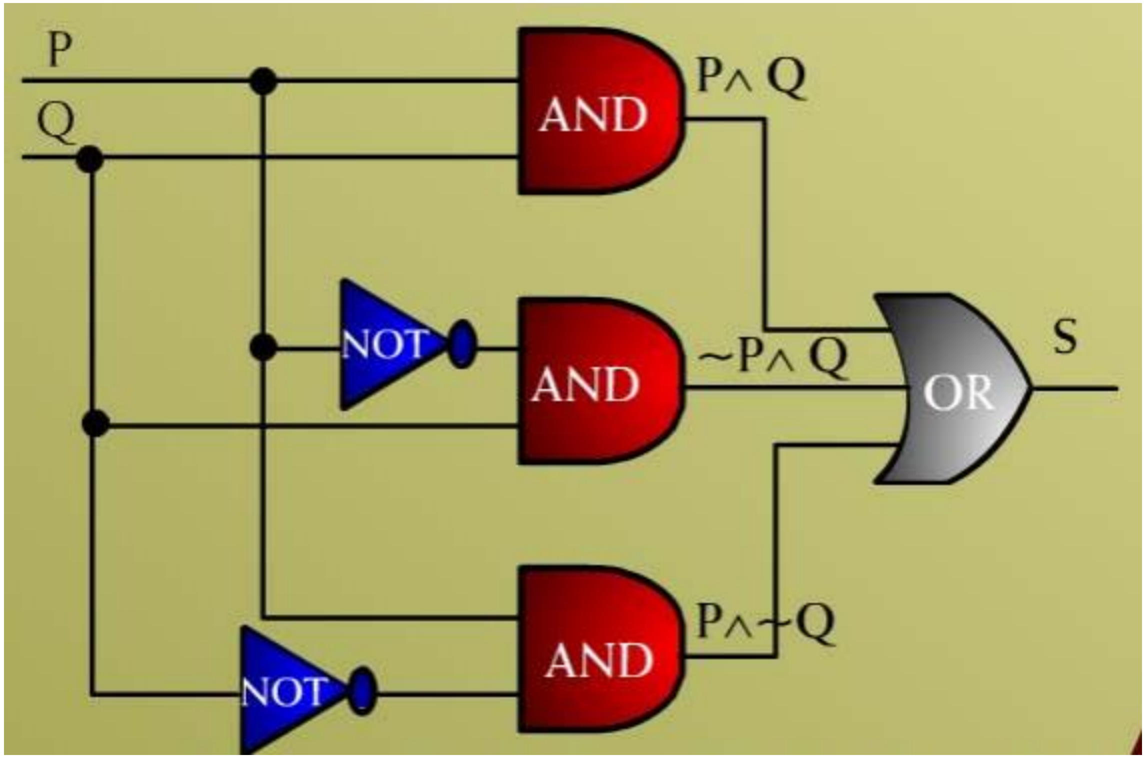
$$S = (P \land Q \land \sim R) \lor (\sim P \land \sim Q \land R)$$



Exercise – 2



Exercise – 2 : Sol.



OUTPUT: $S = (P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$

Exercise – 2 : Sol.

Statement	Reason
$(P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$	
$\equiv (P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$	
$\equiv (P \vee \sim P) \wedge Q \vee (P \wedge \sim Q)$	Distributive law
$\equiv t \wedge Q \vee (P \wedge \sim Q$	Negation law
$\equiv Q \vee (P \wedge \sim Q)$	Identity law
$\equiv (Q \vee P) \wedge (Q \vee \sim Q)$	Distributive law

Exercise – 2 : Sol.

Statement	Reason
$\equiv (Q \vee P) \wedge t$	Negation law
$\equiv Q \vee P$	Identity law
$\equiv Q \vee P$	Commutative law

Thus $(P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q) \equiv P \vee Q$