

# Discrete Structures

## Lecture # 1

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FAST -- National University of Computer and Emerging  
Sciences. CFD Campus

Do not worry about your difficulties in Mathematics. I can assure you, mine are still  
greater!

-Albert Einstein-

# About me

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## **Research interests:**

- Machine Learning, Computer Vision, Remote Sensing, Hyperspectral Imaging, Wearable Computing.

# Outline

- ❑ What is DS and when can it be used?
- ❑ Why study DS?
- ❑ Topics
- ❑ Grading
- ❑ Today's Lecture

# Discrete Structures

## □ Study of Discrete Objects

Consisting of Distinct Objects

## □ Problems Solved Using Discrete Structures

- How many ways are there to choose a valid password?
- What's the probability of winning a lottery?
- How to encrypt a message?
- What is the shortest path b/w two cities?
- How to sort a list of integers?
- How to prove that an algorithm works correctly?
- .
- .
- .

# Why Study Discrete Structures???

❑ Ability to understand and create mathematical arguments

❑ Gateway to more advanced courses

- Algorithms
- Database theory
- Automata theory
- Compiler theory
- Computer security
- Operating system

# Topics we'll study

- ☐ Logic and Proofs
- ☐ Mathematical Induction
- ☐ Sequences and Recursion
- ☐ Set Theory
- ☐ Functions
- ☐ Relations
- ☐ Counting and Probability
- ☐ Graphs and Trees
- ☐ Machine Learning (Tentative)

# Course organization

- **Class Schedule**

- **Lecture # 1:** Check the Timetable ☺
- **Lecture # 2:** Check the Timetable ☺

- Class will be conducted using Slides

- **Text Book**

- Susana Epp, *Discrete Mathematics with Applications*, (4<sup>th</sup> Edition)
- Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 4th Edition

# Grading

- ☐ **Two One Hour Mid Tests (30%)**
- ☐ **Final Exam (40%)**
- ☐ **Assignments (04)/Quizzes (04) (16%)**
- ☐ **Project (10%)**
- ☐ **Class Participation (4%)**



# Other Info

☐ **Cooperation Policy**

☐ **Feedback**

☐ **Bonus Points**

☐ **Cheating Policy**

# Policies

- No re-take of quizzes in any case except that the leave is officially approved by department
- Mid-term retakes are to be governed as per department policy
- Office hours will be announced in a couple of weeks. It is preferred to contact within office hours. Most of the problems can be dealt via email or in class.
- Cheating in one assignment= ZERO in two assignments
- Cheating in Quiz = ZERO in two quizzes
- Cheating in Project= ZERO in project
- Cheating in mid-terms and final exams are to be dealt as per department rules.

# Today's Lecture

## ☐ **Integers:**

- Arithmetic Properties
- Powers
- Divisibility
- Primes of Composite Numbers

## ☐ **Rational Numbers**

- Equivalent fractions
- Operating with fractions
- Decimals

## ☐ **Irrational Numbers**

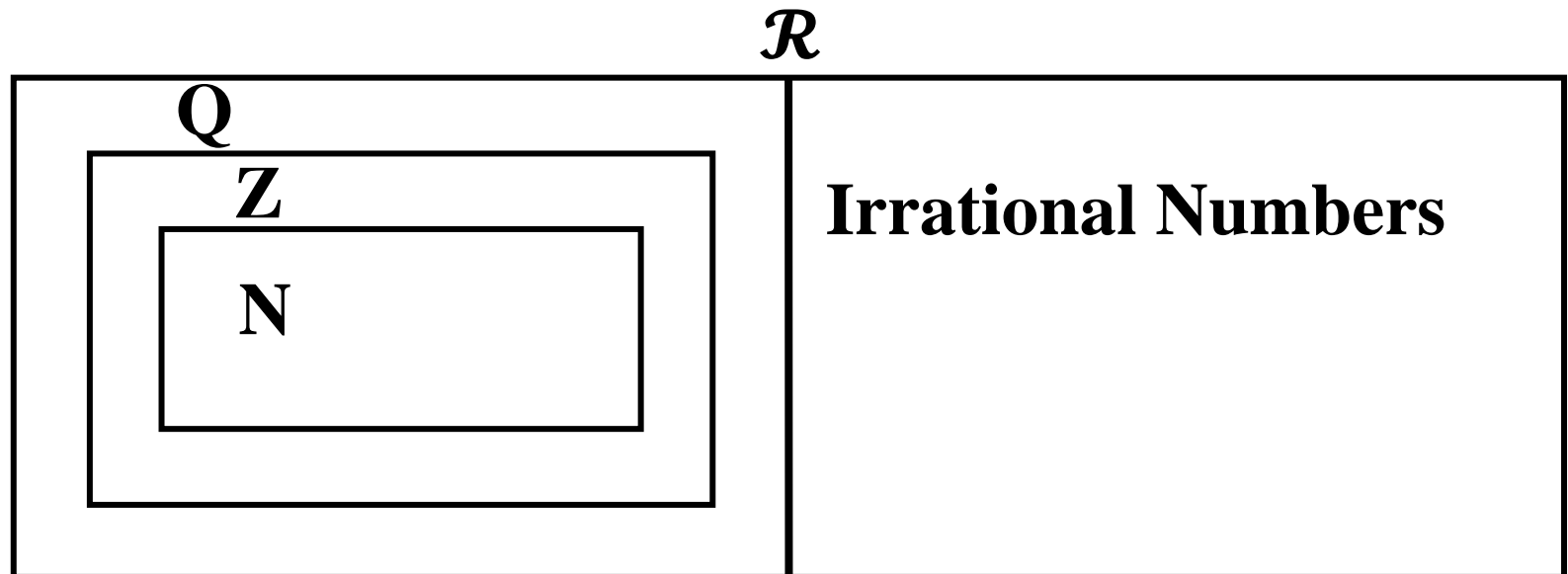
## ☐ **Real Numbers**

- Square roots
- N-th roots
- Logarithms
- Inequalities

## ☐ **Oder of Operations**

# Numbers

- ❑  $\mathbf{N} = \{1, 2, 3, \dots\}$  The set of Natural Numbers
- ❑  $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  The set of Integers
- ❑  $\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, \text{ and } q \neq 0\}$  The set of rational numbers
- ❑  $\mathcal{R}$ , the set of real number. e.g. Real Space



# Integers

## □ Simple rule of Addition

- For an integer  $a$ ,
- $0+a = a+0=a$
- $a+(-a)=0$ , and  $(-a)+a=0$
- $-a$  is the additive inverse of  $a$ .

We use “Minus  $a$ ” rather than “Negative  $a$ ”

# Integers

## □ Rules of Addition

### □ Commutativity

- If  $a$  and  $b$  are integers, then
  - $a + b = b + a$

### □ Associativity

- If  $a$ ,  $b$  and  $c$  are integers, then
  - $(a + b) + c = a + (b + c)$

# Integers

## □ Rules of Addition

- If  $a + b = 0$ , then  $b = -a$  and  $a = -b$

- Proof

$$a + b = 0$$

Add  $-a$  to both sides

$$-a + a + b = 0 - a$$

$$0 + b = 0 - a$$

$$\mathbf{b = - a}$$

As desired.

Similarly we can find ----  $a = - b$

# Integers

## □ Rules of Addition

- If  $a$ ,  $b$  are positive integers, then  $a + b$  is also positive integer.
- If  $a$ ,  $b$  are negative integers, then  $a + b$  is also negative integer.
- If we have the relationship b/w three integers.
  - $a + b = c$

Then we can drive other relationships b/w them.

$$\mathbf{a = c - b} \qquad \mathbf{b = c - a}$$

## □ Example: Solve for $x$ .

$$\mathbf{x + 3 = 5}$$

$$\mathbf{x = 5 - 3}$$

$$\mathbf{x = 2}$$



# Integers

## □ Rules of Addition

- Cancellation rule for addition
  - If  $a + b = a + c$ , then  $b = c$

Exercise:

Prove that if  $a + b = a$ , then  $b = 0$ ?

# Integers

## □ Rules of Multiplication

### □ Commutativity

- If  $a$  and  $b$  are integers, then
  - $a * b = b * a$

### □ Associativity

- If  $a$ ,  $b$  and  $c$  are integers, then
  - $(a * b) * c = a * (b * c)$
- For any integer  $a$ 
  - $1 * a = a$  and  $0 * a = 0$

# Integers

## □ Rules of Multiplication

### □ Distributivity

- $\mathbf{a * (b + c) = a * b + a * c}$
- $\mathbf{(b + c) * a = b * a + c * a}$

Using all these properties

- $\mathbf{-1 * a = -a}$
- $\mathbf{-(a * b) = (-a) * (b) \text{ or } -(a * b) = a * (-b)}$
- $\mathbf{(-a) * (-b) = a * b}$

# Integers

## □ Powers

- An exponent is used to indicate repeated multiplication.
- Tells how many times the base is used as a factor.

- $a * a = a^2$
- $a * a * a = a^3$

In general if  $n$  is a positive integer,

- $a^n = a * a * a \dots a$  (product is taken  $n$  times)

We say  $a^n$  is the  $n$ -th power of  $a$ .

If  $m, n$  are positive integers, then

- $a^{m+n} = a^m * a^n$

# Integers

## □ Powers

- $(a^m)^n = a^{m * n}$

Some important formulas

- $(a + b)^2 = a^2 + b^2 + 2ab$

- $(a - b)^2 = a^2 + b^2 - 2ab$

- $(a + b)(a - b) = a^2 - b^2$

# Integers

## □ Even and Odd integers

□ An even integer is an integer which can be written in the form  **$2n$**  for some integer  **$n$**

- $2 = 2 * 1$
- $4 = 2 * 2$
- $6 = 2 * 3$

□ An odd integer is an integer that differs from an even integer by 1.

□ It can be written in the form  **$2m \pm 1$**  for some integer  **$m$** .

- $1 = (2 * 1) - 1$
- $3 = (2 * 2) - 1$
- $7 = (2 * 3) + 1$

# Integers

## □ Theorem

- Let  $a, b$  be integers,
  - If  $a$  is even and  $b$  is also even, then  $a + b$  is also even
  - If  $a$  is even and  $b$  is odd, then  $a + b$  is odd
  - If  $a$  is odd and  $b$  is even, then  $a + b$  is odd
  - If  $a$  is odd and  $b$  is also odd, then  $a + b$  is also even

## □ Exercise.

- Let's prove the Second statement ??

# Integers

## □ Divisibility

□ Given two integers  $a$  and  $b$ , with  $a \neq 0$ , we say that  **$a$  divides  $b$** , or that  **$b$  is divisible by  $a$**  if there is an integer  $c$ , such that  **$b = a * c$** .

□ Remember that every integer is divisible by 1 because we can always write

$$\bullet \quad n = 1 * n$$

□ Also, every positive integer is divisible by itself.



# Rational Numbers

□ By a rational numbers, we mean a fraction as  $\frac{m}{n}$ ,

where  $m$  and  $n$  are integers,  $n \neq 0$ .

- $m$  is called numerator
- $n$  is called denominator

□ Improper fraction

- $m$  larger than or equal to  $n$

□ Proper fraction

- $m$  smaller than  $n$

# Rational Numbers

## □ Equivalent Fractions

□ Two fractions that represent the same value.

- $\frac{1}{2} = \frac{2}{4}$

How can we know whether two fractions are equivalent?

## □ Rule for cross-Multiplication

- Let  $m, n, r, s$  be integers and assume that  $n \neq 0$  and  $s \neq 0$ . Then

- $\left(\frac{m}{n}\right) = \left(\frac{r}{s}\right)$ , iff  $m * s = r * n$

# Rational Numbers

## ❑ Simplifying Fractions

❑ We can simplify four special fractions forms

❑ Fractions that have the same numerator and denominator.

$$\bullet \quad 1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \dots$$

❑ Fractions that have a denominator of 1.

$$\bullet \quad \frac{5}{1} = 5, \quad \frac{24}{2} = 24, \quad \frac{-6}{1} = -6$$

❑ Fractions that have a numerator of 0.

$$\bullet \quad \frac{0}{8} = 0, \quad \frac{0}{71} = 0, \quad \frac{0}{-10} = 0$$

❑ Fractions that have a denominator of 0

$$\bullet \quad \frac{7}{0} = \infty, \quad \frac{-17}{0} = \infty, \quad (\infty = \text{Infinity} = \text{Undefined Value})$$

# Rational Numbers

## □ Simplifying Fractions

### □ Cancellation Rule for Fractions

□ Let  $a$  be a non-zero integer. Let  $m$ ,  $n$  be integers, and  $n \neq 0$ , then

$$\bullet \quad \frac{am}{an} = \frac{m}{n}$$

□ Proof: By applying the rule for cross-multiplication and using the associativity and commutativity laws.

# Rational Numbers

## □ Simplifying Fractions

□ A fraction is in simplest form when the numerator and denominator have no common factors (or divisors) other than 1.

□ Theorem:

□ “Any positive rational number has an expression as a fraction in the lowest form.”

# Rational Numbers

## □ Operating with Fractions

□ Addition (or Subtraction) with same denominator.

$$\bullet \quad \frac{a}{d} + \frac{b}{d} = \frac{a+b}{d} \quad \underline{\text{or}} \quad \frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$$

□ With different denominator:

$$\bullet \quad \frac{m}{n} + \frac{r}{s} = \frac{ms+rn}{ns} \quad \underline{\text{or}} \quad \frac{m}{n} - \frac{r}{s} = \frac{ms-rn}{ns}$$

□ Follows the same basic rules as addition of integers (commutativity and association)

# Rational Numbers

## □ Multiplication:

□ Let  $a = \frac{m}{n}$

- Then for any positive integer  $k$ , such that

- $a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$

- Follows the same basic rules as multiplication of integers.

# Rational Numbers

## □ Division:

□ If  $a$  is a rational number and  $a \neq 0$ , then there exists ( $\exists$ ) a rational number, denoted by

- $a^{-1}$  such that
- $a^{-1} * a = a * a^{-1} = 1$

□ Note that if  $a = \frac{m}{n}$  then  $a^{-1} = \frac{n}{m}$

□  $a^{-1}$  is called the multiplicative inverse of  $a$ .



# Rational Numbers

## □ Decimals:

□ Finite decimals (and periodic) give us examples of rational numbers.

- $1.4 = \frac{14}{10}$
- $1.41 = \frac{141}{100}$
- $0.2 = \frac{1}{5}$
- $0.75 = \frac{3}{4}$
- $0.3333 = 0.\bar{3} = \frac{1}{3}$
- ...

# Irrational Numbers

- ❑ A number that cannot be expressed as fraction of  $\frac{p}{q}$  for any integers  $p$  and  $q$ .
- ❑ Have decimal expressions that neither terminate nor become periodic
  - $\sqrt[2]{2} = 1.41421356237 \dots$
  - $\sqrt[2]{3} = 1.73205080757 \dots$
  - $\pi = 3.14159265359 \dots$
  - $\dots$

# Irrational Numbers

□ Is  $\sqrt[2]{25}$  an irrational number?

- No!
- Because  $\sqrt[2]{25} = \pm 5$

□ Is  $\sqrt[2]{-1}$  an irrational number?

- No or Yes??? In both cases HOW?

# Real Numbers

- ❑ Integers, Rational and Irrational Numbers are part of a larger system.
- ❑ Real Numbers can be described as all the numbers that consist of a decimal expansion, possibly infinite.

# Real Numbers

## □ Properties of Real Numbers:

### □ Addition:

- $a + b = b + a$
- $a + (b + c) = (a + b) + c$
- For all ( $\forall$ ) real numbers  $a$ ,  $b$ , and  $c$ .

### □ Multiplication

- $a * b = b * a$
- $a * (b * c) = (a * b) * c$
- $\forall$  real numbers  $a$ ,  $b$ ,  $c$ .

### □ Also

- $a * (b + c) = a * b + a * c$
- $(b + c) * a = b * a + c * a$

# Real Numbers

## □ Absolute Value

□ The non-negative values of a real number without regard to its sign.

- $|a| = a$  for a positive  $a$ .
- $|a| = -a$  for a negative  $a$   
(in which case  $-a$  is positive).
- $|0| = 0$

# Real Numbers

## □ Square Roots

□ If  $a > 0$ , then there exists ( $\exists$ ) a number  $b$  such that (s.t).

- $b^2 = a$

## □ N-th Roots

□ There exists a unique real number  $r$  such that

- $r^n = a$

It is called the  $n$ -th root of  $a$ , and is denoted by

- $a^{1/n}$  or  $\sqrt[n]{a}$

# Logarithms

- ❑ Can be seen as the reverse operation of the exponentiation.
- ❑ The logarithm of a number is the exponent to which another fixed value, the **Base** must be raised to produce that number.
  - $\log_{10}(10000) = 4$ , because  $10^4=10000$
  - $\log_2(16) = 4$ , because  $2^4=16$
  - $\log_3\left(\frac{1}{3}\right) = -1$ , because  $3^{-1}=\frac{1}{3}$



# Logarithms

## ❑ Properties of Logarithms:

### ❑ Product:

- $\log_b(x * y) = \log_b(x) + \log_b(y)$

### ❑ Quotient:

- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

### ❑ Power:

- $\log_b(x^p) = p * \log_b(x)$

### ❑ Change of Base:

- $\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$

# Inequalities

Symbol	Meaning	Example
$>$	Greater Than	$(X + 3) > 2$ , for any X
$<$	Less Than	$(7X) < 28$ , $X = \{ \dots, -2, -1, 0, 1, 2, 3 \}$
$\geq$	Greater Than or Equal	$5 \geq (X - 1)$ , $X = \{ \dots, -2, -1, 0, 1, \dots, 5, 6 \}$
$\leq$	Less Than or Equal	$(2Y + 1) \leq 7$ , $Y = \{ \dots, -2, -1, 0, 1, 2, 3 \}$

□ Let a, b, c be real numbers,

- If  $a > b$  and  $b > c$  then  $a > c$ . (Transitivity)
- If  $a > b$  and  $c > 0$  then  $a * c > b * c$ .
- If  $a > b$  and  $c < 0$  then  $a * c < b * c$ .