Discrete Structures

Lecture # 08

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EXERCISE

Suppose R and S are binary relations on a set A.

a) If R and S are reflexive, is $R \cap S$ reflexive.

b) If R and S are symmetric, is $R \cap S$ symmetric.

c) If R and S are transitive, is $R \cap S$ transitive.

SOLUTION

a) $R \cap S$ is reflexive:

Since R and S are reflexive.

Then by definition of reflexive relation

$$\forall a \in A \quad (a,a) \in R \text{ and } (a,a) \in S$$

$$\Rightarrow \forall a \in A \quad (a,a) \in R \cap S$$

(by definition of intersection)

Accordingly, $R \cap S$ is reflexive.

SOLUTION

b) $R \cap S$ is symmetric.

Suppose R and S are symmetric.
To prove $R \cap S$ is symmetric we need to show that

$$\forall\ a, b \in A, \text{ if } (a,b) \in R \cap S$$

then

$$(b,a) \in R \cap S$$

SOLUTION

Suppose $(a,b) \in R \cap S$.

$\Rightarrow (a,b) \in R$ and $(a,b) \in S$

Since R is symmetric, so if $(a,b) \in R$ then $(b,a) \in R$

Also S is symmetric, so if $(a,b) \in S$ then $(b,a) \in S$.

SOLUTION

Thus $(b,a) \in R$ and $(b,a) \in S$

$(b,a) \in R \cap S$

(by definition of intersection)

Accordingly, $R \cap S$ is symmetric.

SOLUTION

Suppose $(a,b) \in R \cap S$ and $(b,c) \in R \cap S$

$\Rightarrow (a,b) \in R$ and $(a,b) \in S$ and $(b,c) \in R$ and $(b,c) \in S$

Since R is **transitive**, therefore

if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

Also S is **transitive**, so $(a,c) \in S$

Hence $(a,c) \in R$ and $(a,c) \in S \Rightarrow (a,c) \in R \cap S$

$R \cap S$ is transitive.

IRREFLEXIVE

Let R be a binary relation on a set A. R is **irreflexive** iff for all $a \in A$, $(a,a) \notin R$.

That is, R is **irreflexive** if no element in A is related to itself by R.

R is **reflexive** if every element related to itself.

R is not **irreflexive** iff there is an element $a \in A$ such that $(a,a) \in R$.

EXAMPLE

Let $A = \{1,2,3,4\}$ and define the following relations on A:

$R_1 = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,2), (2,3), (3,3), (3,4)\}$

EXAMPLE

$R_1 = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$

$(1,1) \notin R_1$, $(2,2) \notin R_1$, $(3,3) \notin R_1$, $(4,4) \notin R_1$

-R_1 is irreflexive

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$(1,1) \in R_2$

R_2 is not irreflexive. It is however, reflexive.

EXAMPLE

$R_3 = \{(1,2), (2,3), (3,3), (3,4)\}$

$(3,3) \in R_1$

R_3 is not irreflexive and R_3 is not reflexive.

A relation may be neither reflexive nor irreflexive.

DIRECTED GRAPH OF A IRREFLEXIVE RELATION

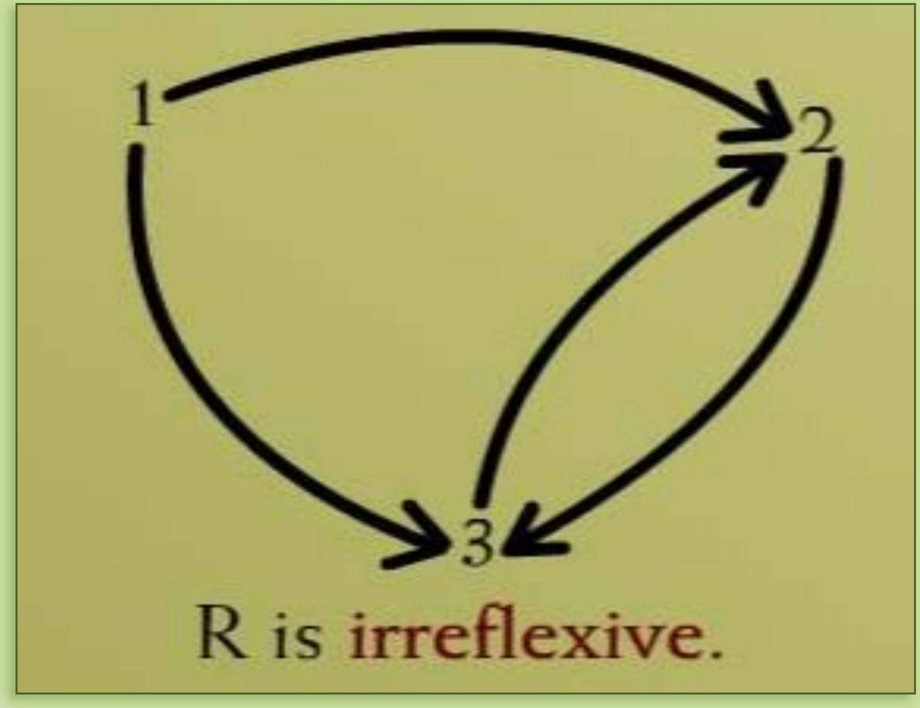
Let R be an irreflexive relation on a set A. Then by definition, no element of A is related to itself by R.

Accordingly, there is no loop at each point of A in the directed graph of R.

EXAMPLE

Let A = {1, 2, 3}

R = { (1, 3) , (1,2) , (2, 3) , (3, 2) }



COMPARISON

Graphical difference between reflexive and irreflexive relation is

The graph of reflexive relation has loop on every element of set A.

The graph of irreflexive relation has no loop on any element of set A.

MATRIX REPRESENTATION OF AN IRREFLEXIVE RELATION

Let R be an irreflexive relation on a set A. Then by definition, no element of A is related to itself by R. Since the self related elements are represented by 1's on the main diagonal of the matrix representation of the relation, so for irreflexive relation R, the matrix will contain all 0's in its main diagonal.

EXAMPLE

$A = \{1,2,3\}$

$R = \{(1,3),\ (2,1),\ (2,3),\ (3,2)\}$

Matrix Representation

$$M = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

R is irreflexive

EXERCISE

Let R be the relation on the set of integers Z defined as:

for all $a,b \in Z$, $(a,b) \in R \Leftrightarrow a > b$.

Is R irreflexive ?

SOLUTION

R is irreflexive

if for all $a \in Z$, $(a,a) \notin R$.

Now by the definition of given relation R,

for all $a \in Z$, $(a,a) \notin R$ since $a \ngtr a$.

Hence R is irreflexive.

ANTISYMMETRIC RELATION

Let R be a binary relation on a set A.

R is antisymmetric iff

$\forall$ a, b $\in$ A

if (a,b) $\in$ R and (b,a) $\in$ R then a = b.

Alternatively, $\forall\, a, b \in A$,

$if\ a \neq b, then\ either\ (a, b) \notin R\ or\ (b, a) \notin R.$

REMARK

1) R is not antisymmetric iff there are elements a and b in A such that $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$.

2) The properties of being symmetric and being anti-symmetric are not negative of each other.

EXAMPLE

Let $A = \{1,2,3,4\}$ and define the following relations on A.

$R_1 = \{(1,1),(2,2),(3,3)\}$

$R_2 = \{(1,2),(2,2), (2,3), (3,4), (4,1)\}$

$R_3 = \{(1,3),(2,2), (2,4), (3,1), (4,2)\}$

$R_4 = \{(1,3),(2,4), (3,1), (4,3)\}$

SOLUTION

$R_1 = \{(1,1),(2,2),(3,3)\}$

R_1 is anti-symmetric and symmetric

$R_2 = \{(1,2),(2,2), (2,3), (3,4), (4,1)\}$

R_2 is anti-symmetric but not symmetric

SOLUTION

$R_3 = \{(1,3), (2,2), (2,4), (3,1), (4,2)\}$

R_3 is symmetric but not anti-symmetric.

since $(1,3)$ & $(3,1) \in R_3$ but $1 \neq 3$.

$R_4 = \{(1,3), (2,4), (3,1), (4,3)\}$

Neither anti-symmetric nor symmetric

MATRIX REPRESENTATION OF AN ANTISYMMETRIC RELATION

Let R be an anti-symmetric relation on a set $A = \{a_1, a_2, \ldots, a_n\}$. Then if $(a_i, a_j) \in R$ for $i \neq j$ then $(a_j, a_i) \notin R$.

Thus in the matrix representation of R there is a 1 in the ith row and jth column iff the jth row and ith column contains 0.

EXAMPLE

Let A = {1, 2, 3}

R = { (1, 1) , (1, 2) , (2, 3) , (3, 1) }

$$M = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array}$$

$$M = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

DIRECTED GRAPH OF AN ANTISYMMETRIC RELATION

Let R be an anti- symmetric relation on a set A. Then by definition, no two distinct elements of A are related to each other.

Accordingly, there is no pair of arrows between two distinct elements of A in the directed graph of R.

EXAMPLE

Let $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,3), (3,1)\}$

R is anti-symmetric

PARTIAL ORDER RELATION

Let R be a binary relation defined on a set A. R is a partial order relation, if and only if, R is

a. reflexive,
b. anti-symmetric and,
c. transitive.

EQUIVALENCE RELATION

Let R be a binary relation on A. R is an equivalence relation if and only if, R is

a. reflexive,
b. symmetric and,
c. transitive.

EXAMPLE

Let $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2),(3,3),(4,4)\}$

$R_2 = \{(1,1),(1,2), (2,1), (2,2), (3,3),(4,4)\}$

$R_3 = \{(1,1),(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$

EXAMPLE

$R_1 = \{(1,1),(2,2),(3,3),(4,4)\}$

R_1 is reflexive.

R_1 is antisymmetric.

R_1 is Transitive.

R_1 is partial order relation.

EXAMPLE

$R_2 = \{(1,1),(1,2), (2,1), (2,2), (3,3),(4,4)\}$

R_2 is reflexive.

R_2 is not antisymmetric.
As $(1,2),(2,1) \in R_2$ but $1 \neq 2$.

R_2 is Transitive.

EXAMPLE

$R_3 = \{(1,1),(1,2), (1,3), (1,4), (2,2), (2,3),(2,4), (3,3), (3,4)(4,4)\}$

R_3 is reflexive.

R_3 is antisymmetric.

R_3 is Transitive.

R_3 is partial order relation.

EXERCISE

Let R be the set of real numbers and define the "less than or equal to" relation, $\leq$, on R as follows:

for all real numbers x and y in R.

$$x \leq y \Leftrightarrow x < y \text{ or } x = y$$

Show that $\leq$ is a partial order relation.

SOLUTION

$\leq$ is reflexive

Because for all $x \in R$

$$x = x \quad \Rightarrow \quad x \, R \, x$$

$\leq$ is anti-symmetric

if

$$x \leq y \quad \text{and} \quad y \leq x \quad \text{then}$$

$$x = y$$

SOLUTION

$\leq$ is transitive

$\forall$ x,y,z $\leq$ R

if x $\leq$ y and y $\leq$ z then x $\leq$ z

$\leq$ is a partial order

EXAMPLE

Let "$|$" be the "divides" relation on a set A of positive integers.

That is, for all $a, b \in A$,

$$a \mid b \Leftrightarrow b = k \cdot a \text{ for some integer } k.$$

Prove that $|$ is a partial order relation on A.

SOLUTION

"|" is reflexive.

Since every integer divides itself i.e

$$a \mid a$$

In this case we have $K = 1$

$$a = 1 \, . \, a$$

EXAMPLE

"$|$" is anti-symmetric

We must show that

for all $a, b \in A$,

if $a|b$ and $b|a$ then $a=b$

SOLUTION

Suppose $a|b$ and $b|a$

By definition of divides there are integers k_1, and k_2 such that

$b = k_1 . a$ and $a = k_2 . b$

Now $b = k_1 . a$

$= k_1 . (k_2 . b)$ (by substitution)

$= (k_1 . k_2) . b$

Dividing both sides by b gives

$1 = k_1 . k_2$

Since a, b $\in$ A, where A is the set of positive integers, so the equations

$$b = k_1.a \quad \text{and} \quad a = k_2 .b$$

implies that k_1 and k_2 are both positive integers. Now the equation

$$k_1.k_2 = 1$$

can hold only when

$$k_1 = k_2 = 1$$

Thus $a = k_2.b = 1 .b = b$ i.e., $a = b$

SOLUTION

We have to show that $\forall$ a,b,c $\in$ A

if $a|b$ and $b|c$ than $a|c$

Suppose $a|b$ and $b|c$

By definition of divides, there are integers k_1 and k_2 such that

$$b = k_1 . a$$

and

$$c = k_2 . b$$

SOLUTION

$= k_2 . (k_1 . a)$ (by substitution)

$= (k_2 . k_1) . a$ (by associative law under multiplication)

$= k_3 . a$ where $k_3 = k_2 . k_1$ is an integer

$\Rightarrow a | c$ by definition of divides

Thus "|" is a partial order relation on A.

INVERSE OF A RELATION

Let R be a relation from A to B. The **inverse relation** R^{-1} from B to A is defined as:

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$$

More simply, the **inverse relation** R^{-1} of R is obtained by **interchanging** the **elements** of all the **ordered pairs** in R.

EXAMPLE

Let $A = \{2, 3, 4\}$, $B = \{2,6,8\}$ and let **R** be the "**divides**" relation from A to B

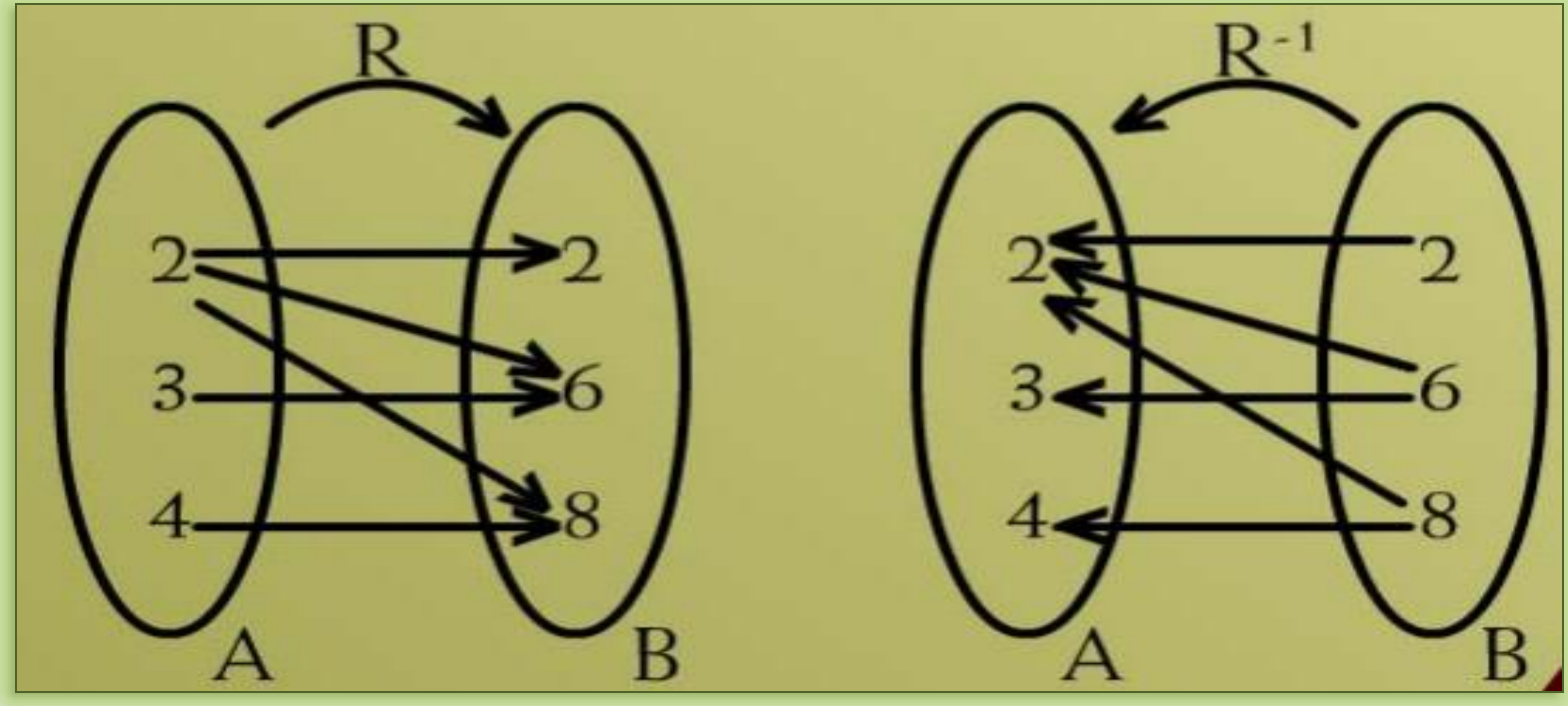
i.e. for all $(a,b) \in A \times B$, $a\,R\,b \Leftrightarrow a \mid b$
(a **divides** b)

$$R = \{(2,2), (2,6), (2,8), (3,6), (4,8)\}$$

$$R^{-1} = \{(2,2), (6,2), (8,2), (6,3), (8,4)\}$$

ARROW DIAGRAM OF AN INVERSE RELATION

$R = \{ (2, 2) , (2, 6) , (2, 8) , (3, 6) , (4, 8) \}$



MATRIX REPRESENTATION OF INVERSE RELATION

R = { (2, 2) , (2, 6) , (2, 8) , (3, 6) , (4, 8) } From A = {2, 3, 4} to B = {2, 6, 8}

$$M = \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 2 & 6 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad M^t = \begin{array}{c} \\ 2 \\ 6 \\ 8 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

COMPLEMENTRY RELATION

Let R be a relation from a set A to a set B. The complementary relation R of R is the set of all those ordered pairs in A x B that do not belong to R.

Symbolically:

$$\begin{aligned}\overline{R} &= A \times B - R \\ &= \{(a,b) \in A \times B \mid (a,b) \notin R\}\end{aligned}$$

EXAMPLE

Let

$A = \{1, 2, 3\}$

$A \times A = \{ (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) \}$

$R = \{ (1, 1), (1, 3), (2, 2), (2, 3), (3, 1) \}$

Then

$$\overline{R} = \{(1,2), (2,1), (3,2), (3,3)\}$$

COMPOSITE RELATION

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S denoted $S \circ R$ is the relation from A to C, consisting of ordered pairs (a,c) where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$.

Symbolically:

$$S \circ R = \{(a,c) \mid a \in A,\ c \in C, \exists\ b \in B,\ (a,b) \in R \text{ and } (b,c) \in S\}$$

EXAMPLE

Let $A = \{a,b,c\}$

$B = \{1,2,3,4\}$

$C = \{x,y,z\}$

$$R = \{(a,1), (a,4), (b,3), (c,1), (c,4)\}$$

$$S = \{(1,x), (2,x), (3,y), (3,z)\}$$

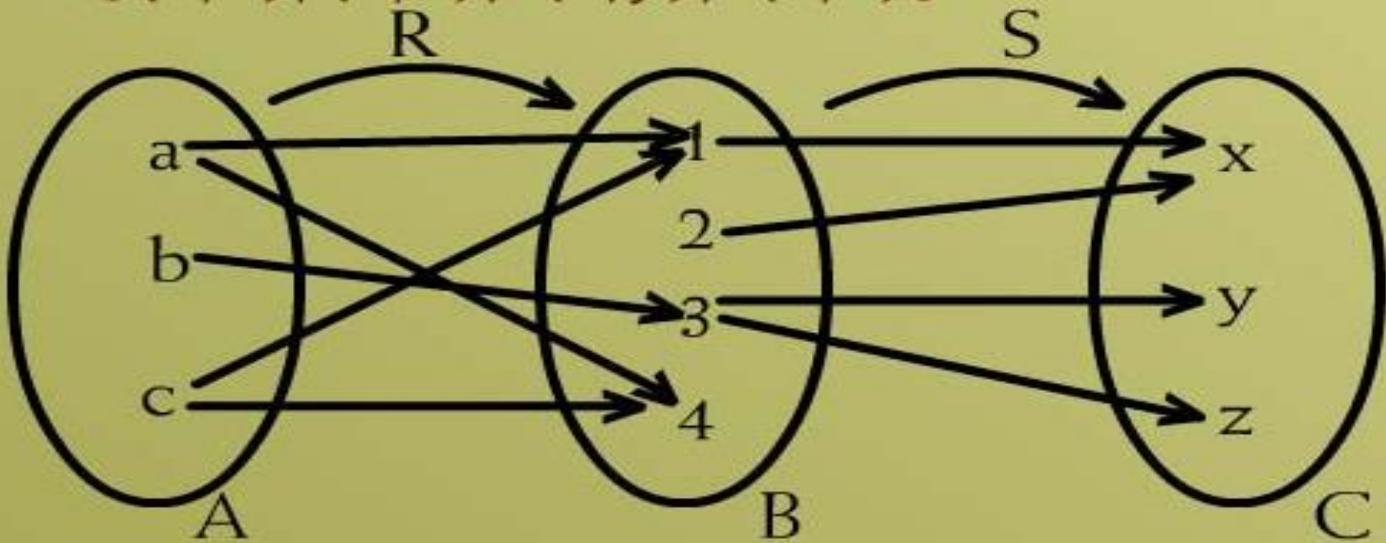
$$S \circ R = \{(a,x), (b,y), (b,z), (c,x)\}$$

COMPOSITE RELATION FROM ARROW DIAGRAM

Let $A = \{a,b,c\}$ $B = \{1,2,3,4\}$ $C = \{x,y,z\}$

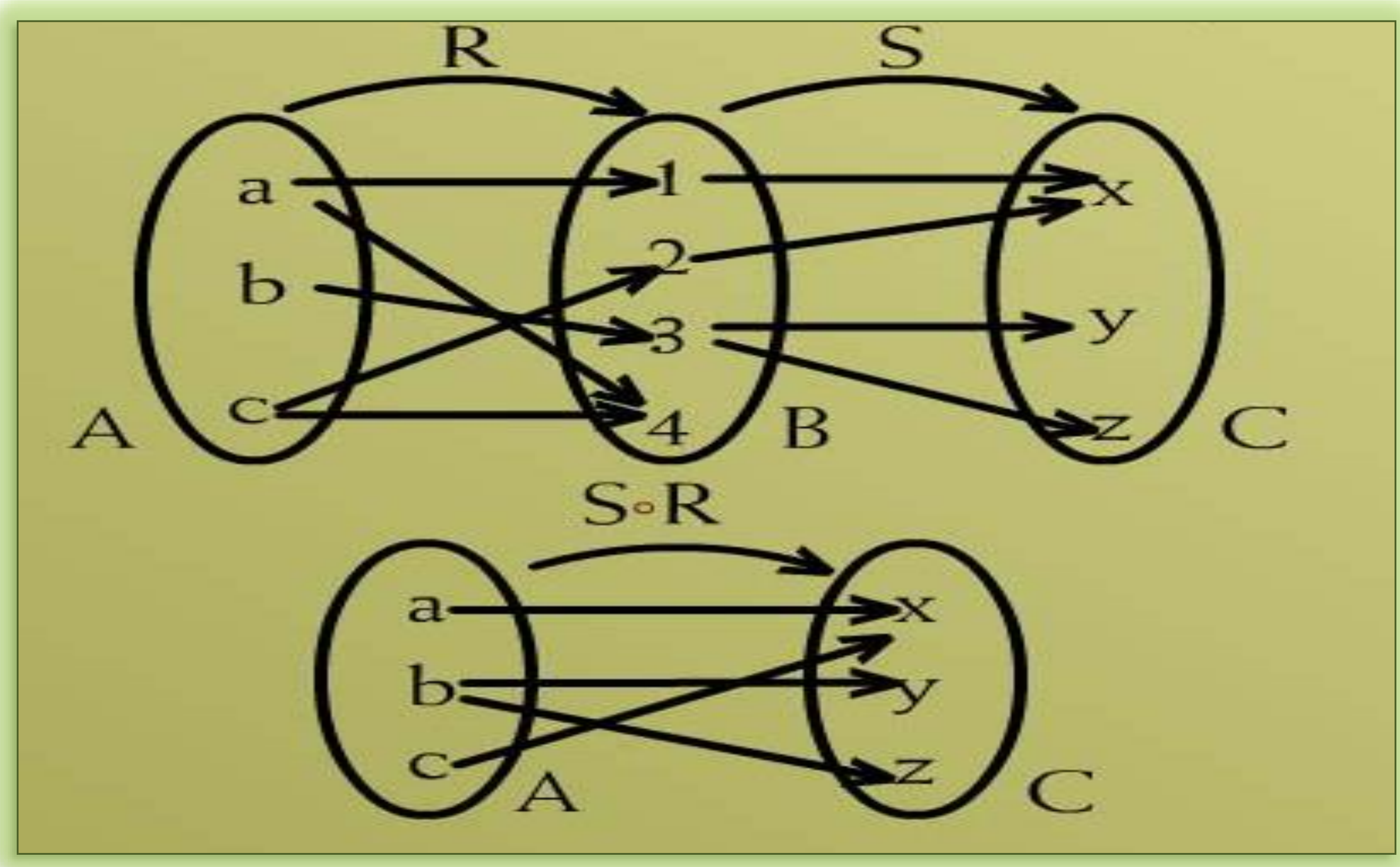
$R = \{(a,1), (a,4), (b,3), (c,1), (c,4)\}$

$S = \{(1,x), (2,x), (3,y), (3,z)\}$



A = {a, b, c}

C = {x, y, z}



MATRIX REPRESENTATION OF COMPOSITE RELATION

The matrix representation of the composite relation can be found using the Boolean product of the matrices for the relations. Thus if M_R and M_S are the matrices for relations R (from A to B) and S (from B to C), then

$$M_{SoR} = M_R \odot M_S$$

is the matrix for the composite relation SoR from A to C.

BOOLEAN ALGEBRA

BOOLEAN ADDITION	BOOLEAN MULTIPLICATION
(a) $1 + 1 = 1$	(a) $1 . 1 = 1$
(b) $1 + 0 = 1$	(b) $1 . 0 = 0$
(c) $0 + 0 = 0$	(c) $0 . 0 = 0$

EXERCISE

We are given relations R and S in matrix form as:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

SOLUTION

$$M_{SoR} = M_R \odot M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$