Discrete Structures

Lecture # 9

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“The Consequences of an Act Affect the Probability of its Occurring Again!”
- B. F. Skinner -

Material Partially Collected by Prof. Dr. A.M. Khan, Slides Prepared by Dr. M. Ahmad

Probability---Introduction

- One of the most important disciplines in Computer Science (CS).
- Algorithm Design and Game Theory
- Information Theory
- Signal Processing
- Cryptography
- Etc.…

Probability---Introduction---Cont.

But it is also probably the least well understood

- Human intuition and Random events

Goal: To try our best to teach you how to easily and confidently solve problem involving probability

- "What is the probability that … ?"

Probability

- **Contents**

  - Basic definitions and an elementary 4-step process
  - Conditional probability and the concept of independence
  - Random variables and distributions
  - Expected value and deviation from it

Probability

❑ Let's Make a Deal

- **The famous fame show** (you might have seen this problem)

- Participant is given a choice of three doors. Behind one door is a car, behind the others, useless stuff. The participant picks a door (**say door 1**). The host, who knows what is behind the doors, opens another door (**say door 3**) which has the useless stuff. He then asks the participant if he would like to switch (**pick door 2**)?

Is it to participant's advantage to switch or not?

Probability

❑ Precise Description

- The car is equally likely to be hidden behind one of the three doors.
- The player is equally likely to pick each of the doors.
- After the player picks a door, the host must open a different door (with the useless thing behind it) and offer the player to switch.
- When a host has a choice of which door to pick, he is equally likely to pick each of them.

Now here comes the question: "What is the probability that a player who switches wins the car?"

Probability

- **Solving Problems Involving Probability**

  - Model the situation mathematically

  - Solve the resulting mathematical problem

  - Solving Problems Involving Probability

Probability

❑ Solving Problems Involving Probability

Step 1: Finding the sample space

- Set of all possible outcomes of a random process

To find this, we must understand the quantities involve in the random process

Quantities in the above problem:

- The door concealing the car
- The door initially chosen by the player
- The door that host opens to reveal the useless thing

Probability

- **Finding the Sample Space**

Every possible outcome of these quantities is called an outcome.

And (as said earlier) the set of all possible outcomes is called the sample space

A tree structure (Possibility tree) is a useful tool for keeping track of all outcomes

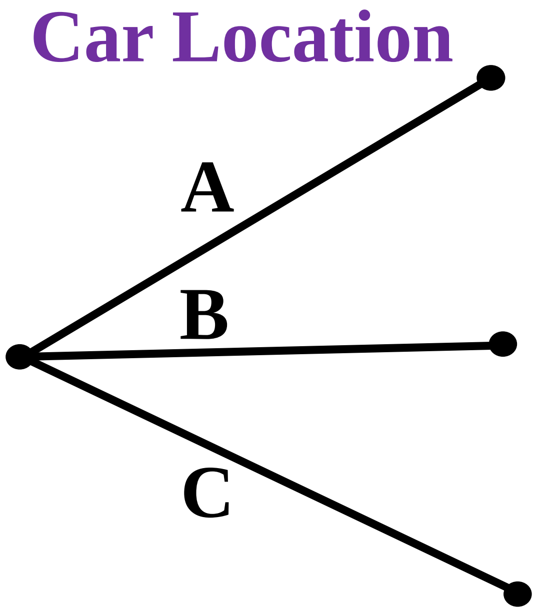
- When the number of possible outcomes is not too large

Probability

❑ Possibility Tree

The first quantity in our example is the door concealing the car

Represent this as a root of tree with three branches (three doors)



Probability

❑ Possibility Tree --- Cont.

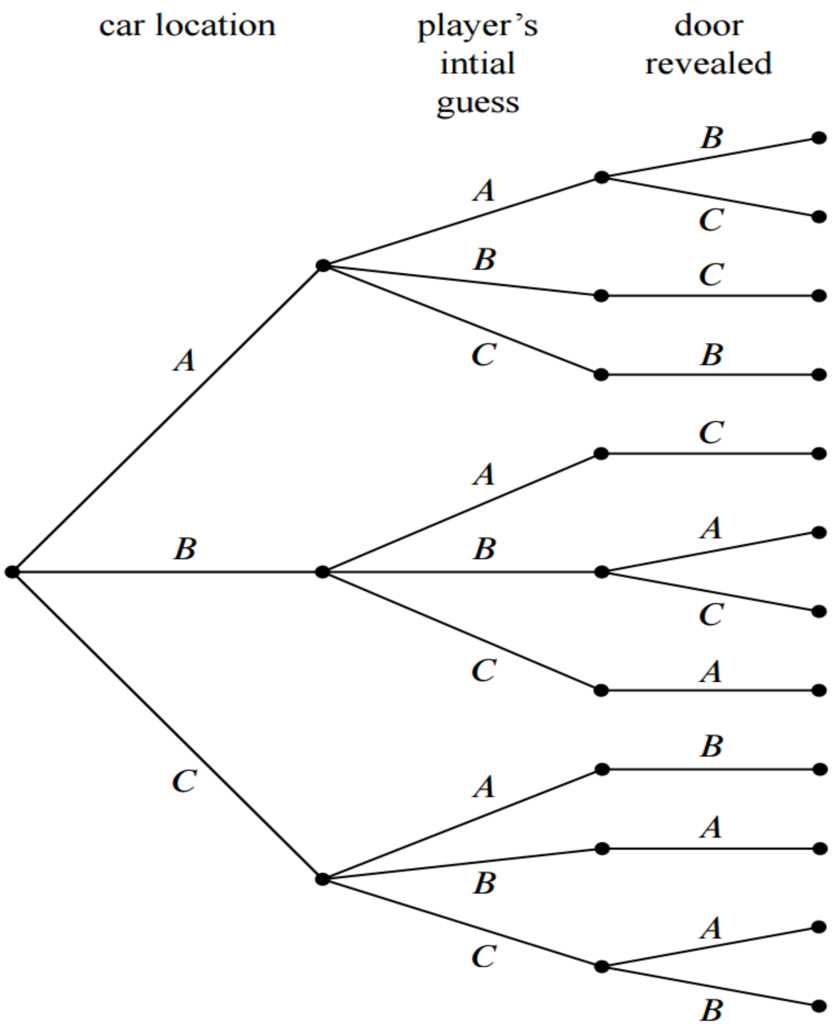
- For each possible location of the car, the player can choose any of the three doors

- Then the final possibility is regarding the host opening a door to reveal the useless thing

- Overall tree turns out to be

Probability

❑ Possibility Tree --- Cont.



Probability

❑ Finding The Sample Space

The leaves of the possibility tree represent the outcomes of a random process

The set of all leaves represent the sample space

In our example, if we represent the leaves as a sequence of "labels" of intermediate nodes including the leaf node then,

$$\boldsymbol{S} = \begin{Bmatrix} (A, A, B), (A, A, C), (A, B, C), (A, C, B), (B, A, C), (B, B, A), \\ (B, B, C), (B, C, A), (C, A, B), (C, B, A), (C, C, A), (C, C, B) \end{Bmatrix}$$

Probability

❑ Solving Problems Involving Probability

Step 2: Defining the Events of Interest:

Remember, we want to answer the questions of type:

- "What is the probability that … ?"

Replacing the "…" with some specific event. For example,

- "What is the probability that the car is behind door C?"

Doing this reduces S to some specific outcomes, called event of interest.

Probability

❑ Event of Interest

For the event,

- “What is the probability that the car is behind door C?”

The set of possible outcomes reduces to

$$\{(C;\ A;\ B);\ (C;\ B;\ A);\ (C;\ C;\ A);\ (C;\ C;\ B)\}$$

- Simply speaking, **an event is a subset of S**

Probability

❑ Solving Problems Involving Probability

Coming back to our example

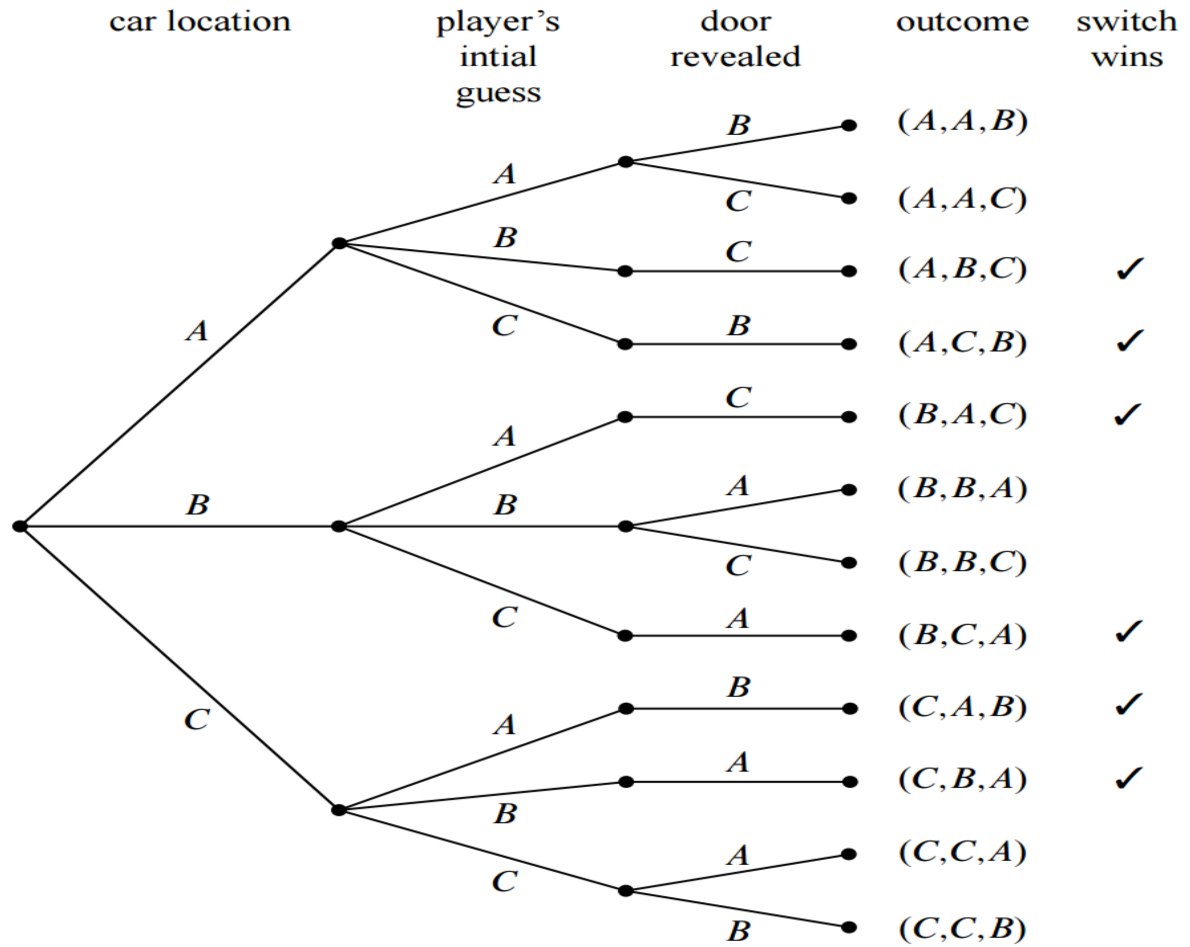
We want to know:

“What is the probability that the player will win by switching?”

This event can be represented as the following set

Probability

❑ Solving Problems Involving Probability---Cont.



Note: Half of the outcomes are checked. Does this mean that the player wins by switching in half of all outcomes?

Probability

❑ **Solving Problems Involving Probability---Cont.**

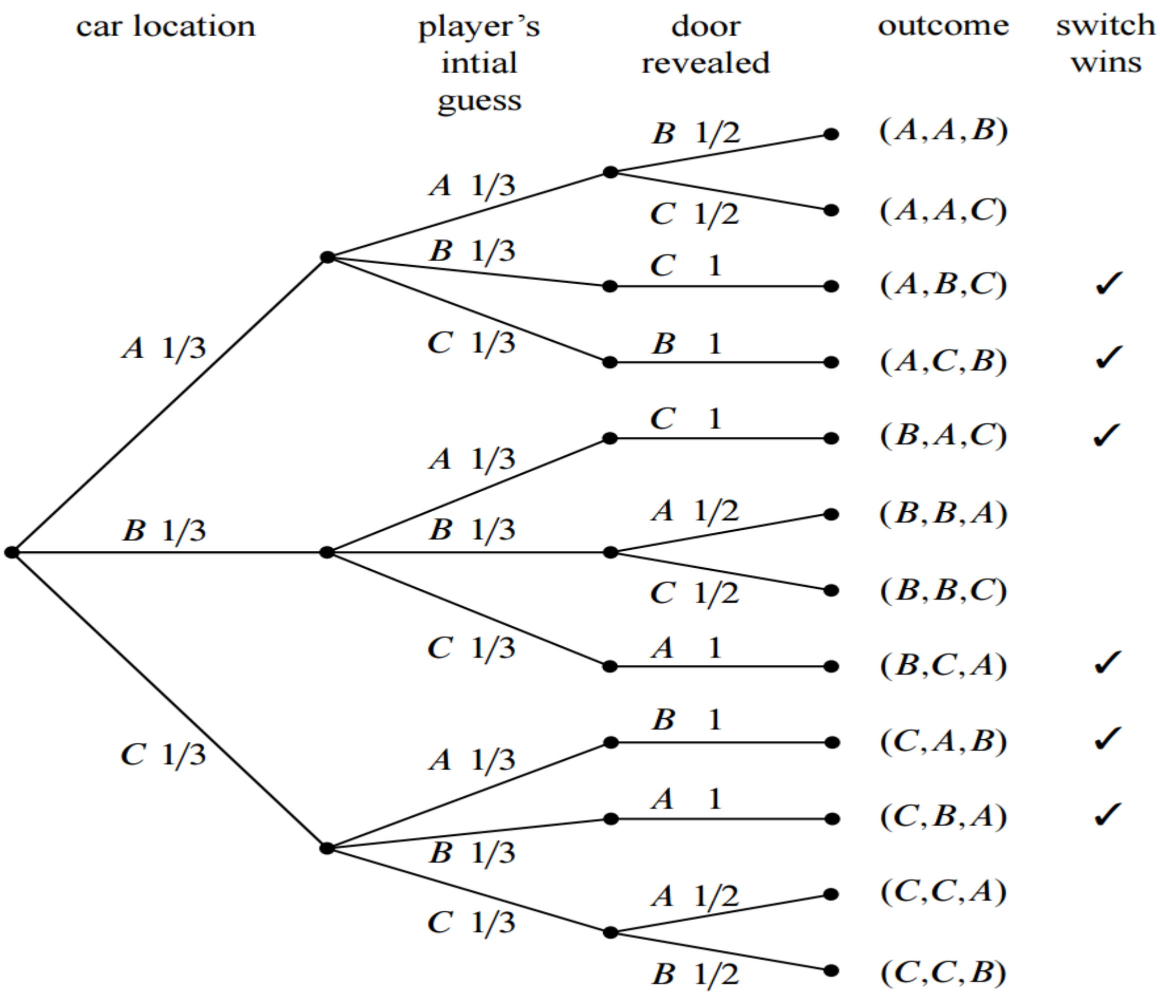
Step 3: Determining Outcome Probability

- ❖ **Assign Edge Probabilities**

- ❖ **Compute Outcome Probabilities**

Probability

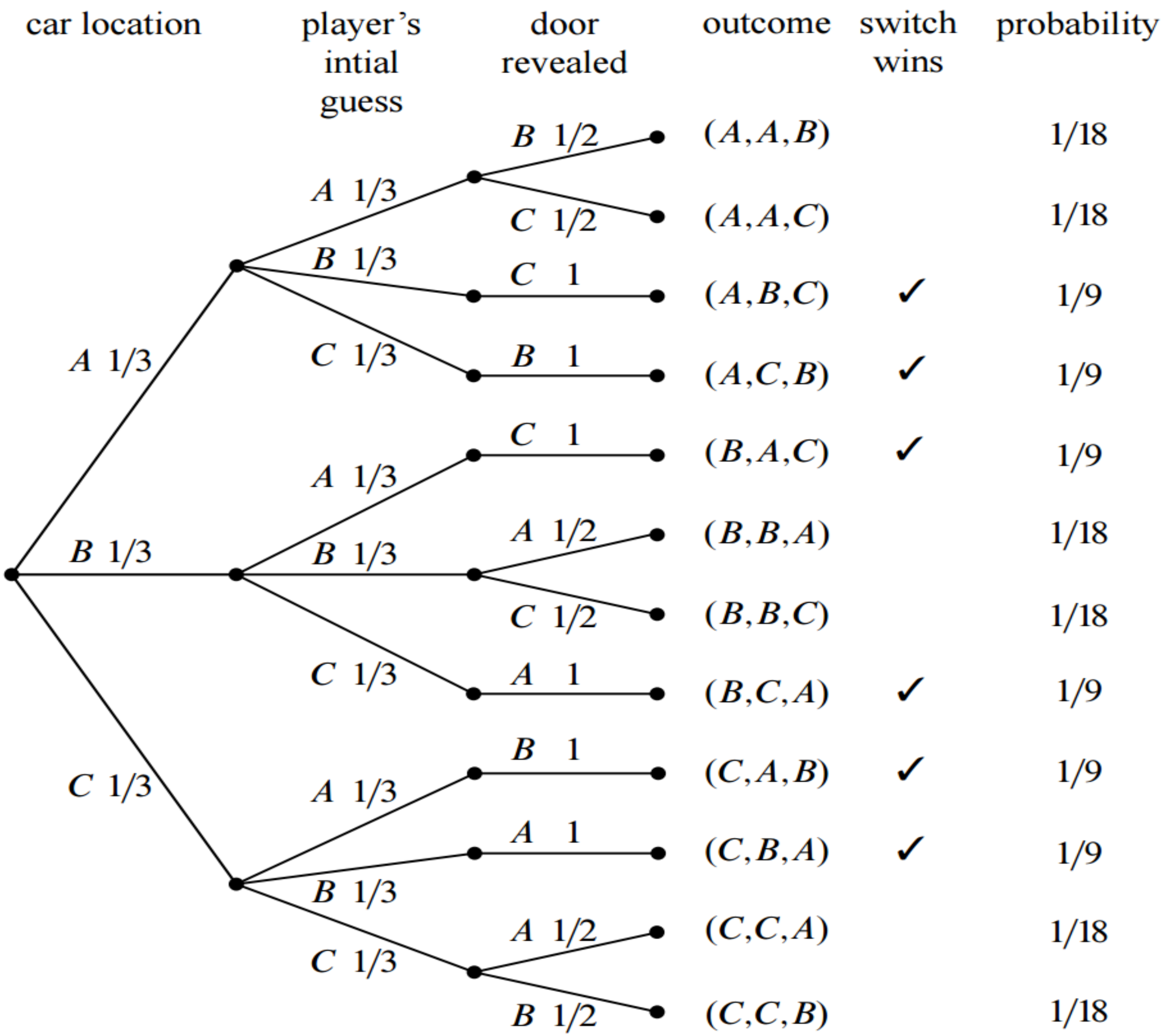
❑ Edge Probabilities



To understand, let's analyze the path leading to the leaf node (A, A, B)!

Probability

❑ Outcome Probabilities



Probability

❑ Solving Problems Involving Probability---Cont.

Step 4: Compute Event Probability

$$Pr[\text{Switching Wins}] = Pr[(A, B, C)] + Pr[(A, C, B)] +$$

$$Pr[B, A, C)] + Pr[(C, A, B)] + Pr[(C, B, A)]$$

$$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{2}{3}$$

Probability

- **Summary**

To solve problems involving probability, that is, "**what is the probability that … ?**"

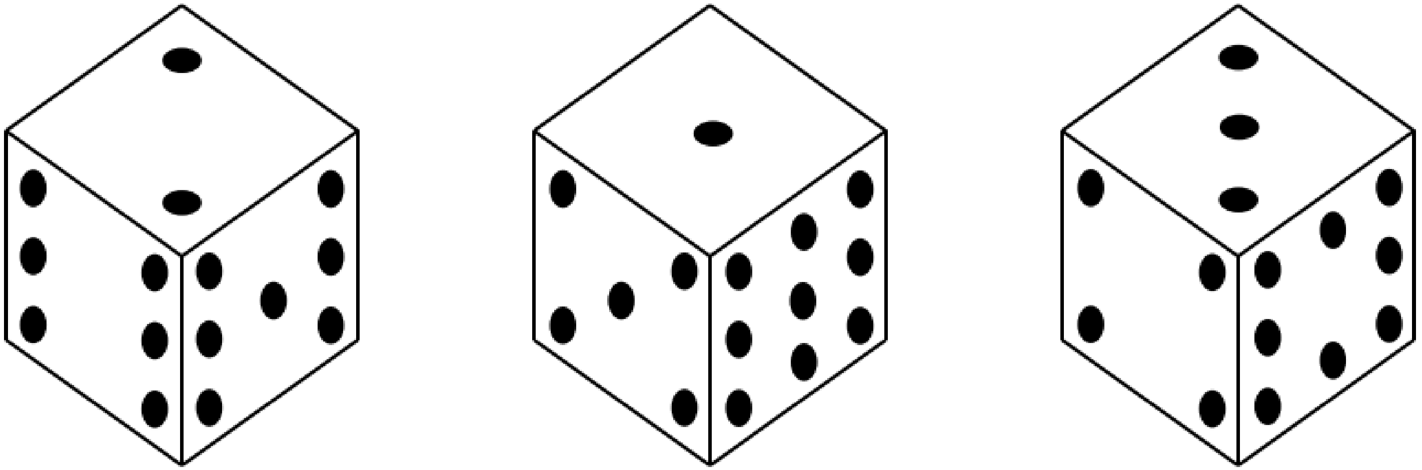
Perform the following **four steps**:

- Find the sample space
- Define event of interest
- Compute outcome probabilities
- Compute event probability

Probability

❑ Uniform Sample Space

Strange Dice

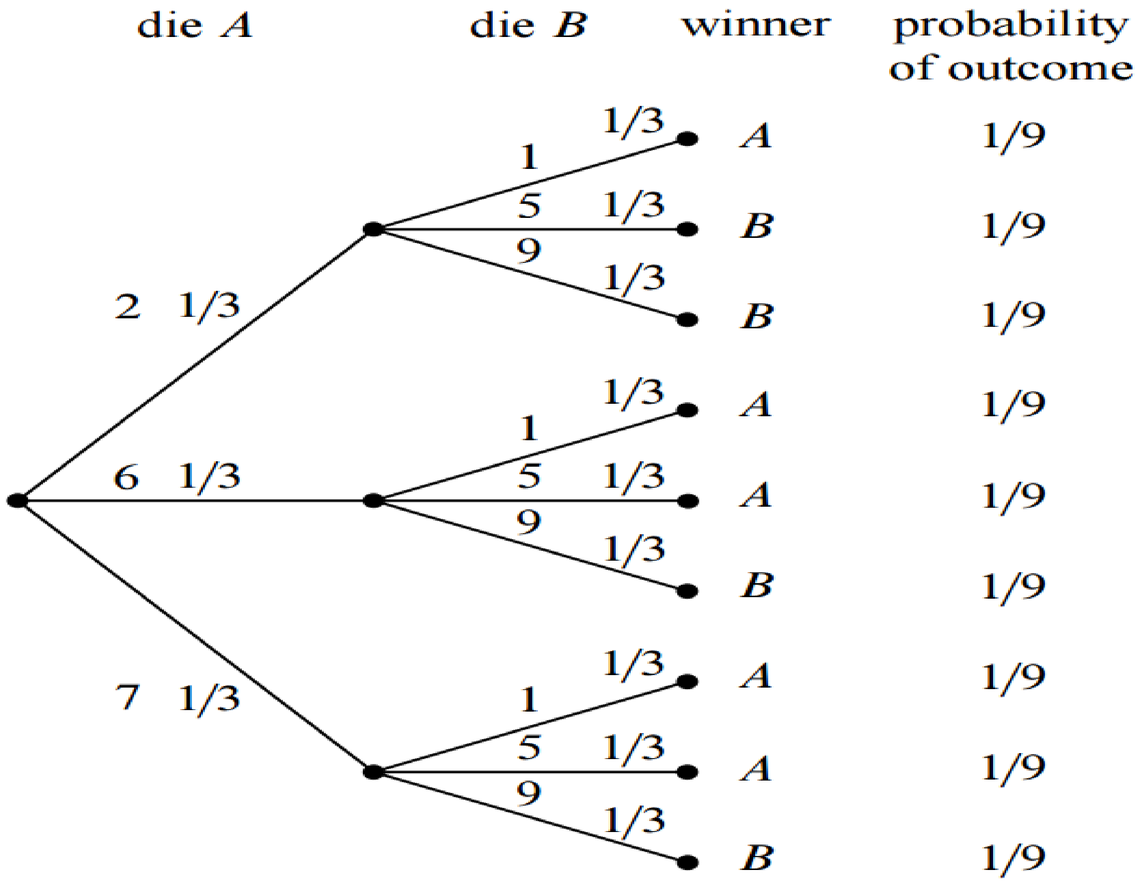


a *b* *c*

If we picked dices (a) and (b), rolled them once, what is the probability that (a) beats (b) (has higher value)?

Probability

❑ Applying Four-Step Method



When the probability of every outcome is the same, we say such a sample space is uniform

Probability

❑ Applying Four-Step Method

- **Example--- Cont.**

So what is the **probability that (a) beats (b)??**

$$\boldsymbol{Pr}[\boldsymbol{E}] = \frac{|\boldsymbol{E}|}{|\boldsymbol{S}|}$$

It is also called **“Equally likely probability formula”.**

Which in this case = $\frac{5}{9}$

(a) Beats (b) more than half of the time.

Probability

- **Applying Four-Step Method**

  - **Example--- Cont.**

What about the following:

- (a) vs. (c)
- (b) vs. (c)

Homework!

Probability

- Counting

Rules of counting the elements in a set

Probability

❑ The Addition Rule

- The basic rule underlying the calculation of the number of elements in a union or difference or intersection is the addition rule.

- This rule states that the number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Theorem 9.3.1:

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, A_2, …., A_k. Then

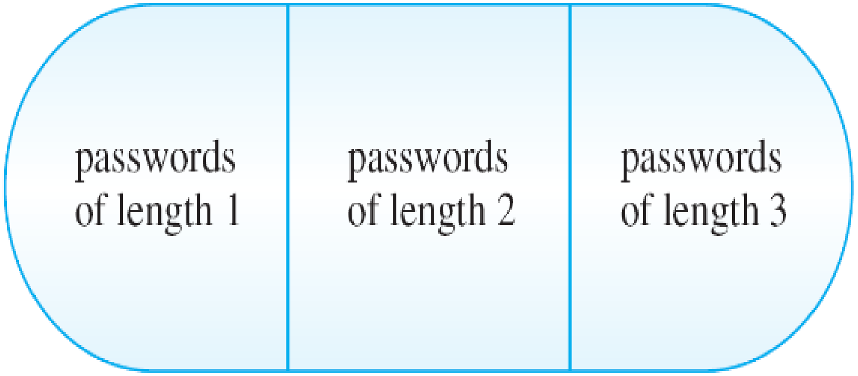
$$N(A) = N(A_1)+N(A_2)+\ldots+ N(A_k)$$

Probability

❑ The Addition Rule---Cont.

Example: A computer access password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different passwords are possible?

Solution: The set of all passwords can be partitioned into subsets consisting of those of length 1, those of length 2, and those of length 3 as shown in the figure below.



Probability

❑ The Addition Rule---Cont.

By the addition rule, the total number of passwords equals the number of passwords of length 1, plus the number of passwords of length 2, plus the number of passwords of length 3.

Now the,

Number of passwords of length 1= 26

Because there are 26 letters in the alphabet

Number of passwords of length 2 $=26^2$

Because forming such a word can be thought of as a two step process in which there are 26 ways to perform each step

Probability

❑ The Addition Rule---Cont.

Number of passwords of length 3 $=26^3$

Because forming such a word can be thought of as a three step process in which there are 26 ways to perform each step

Hence the total number of passwords$= 26^1+26^2+26^3=18,278$

Probability

❑ The Difference Rule

An important consequence of the addition rule is the fact that if the number of elements in a set *A* and the number in a subset *B* of *A* are both known, then the number of elements that are in *A* and not in *B* can be computed.

- **Theorem 9.3.2:** The Difference Rule:

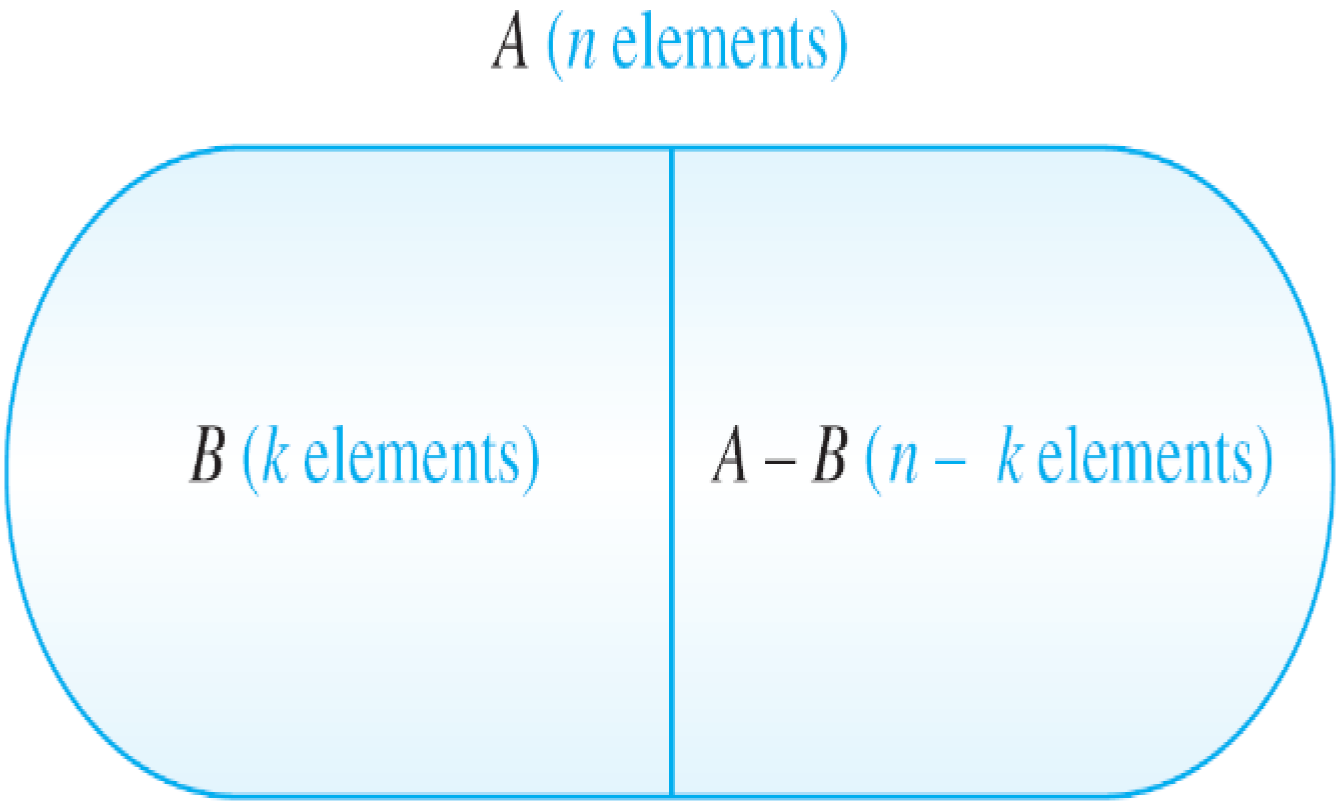
If A is finite set and B is a subset of A, then

$$N(A-B) = N(A) - N(B)$$

Probability

❑ The Difference Rule---Cont.

The difference rule is illustrated below.



Probability

❑ The Difference Rule---Cont.

- The difference rule holds for the following reason: If B is a subset of A, then the two sets B and $A - B$ have no elements in common and $B \cup (A - B) = A$. Hence, by the addition rule,

$$N(B) + N(A - B) = N(A).$$

Subtracting $N(B)$ from both sides gives the equation

$$N(A - B) = N(A) - N(B).$$

Probability

- **The Difference Rule---Cont.**

  - **Example:**

    A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed.

    a. How many PINs contain repeated symbols?

    b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

Probability

- **The Difference Rule---Cont.**

  - **Example --- Cont.:**

    There are $36^4 = 1{,}679{,}616$ PINs when repetition is allowed, and there are

    $$36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$$

    PINs when repetition is not allowed.

    Thus, by the difference rule, there are

    $$1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$$

    PINs that contain at least one repeated symbol.

Probability

- **The Difference Rule---Cont.**

  - **Example --- Cont.:**

There are 1,679,616 PINs in all, and by part (a) 265,896 of these contain at least one repeated symbol.

Thus, by the equally likely probability formula, the probability that a randomly chosen PIN contains a repeated symbol is

$$\frac{\mathbf{265{,}896}}{\mathbf{1{,}679{,}616}} \cong \mathbf{0.158 = 15.8\%}$$

Probability

❑ The Difference Rule---Cont.

An alternative solution to Example 3(**b**) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S - A$ is the set of all PINs with at least one repeated symbol.

It follows that

$$
\begin{aligned}
P(S-A) &= \frac{N(S-A)}{N(S)} && \boxed{\textit{By definition of Probability in the equally likely case}} \\
&= \frac{N(S)-N(A)}{N(S)} && \boxed{\textit{By the difference rule}} \\
&= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} && \boxed{\textit{By the law of fractions}} \\
&= 1 - P(A) && \boxed{\textit{By the definition of probability in the equally likely case}}
\end{aligned}
$$

Probability

❑ The Difference Rule---Cont.

We know that the probability that a PIN chosen at random contains no repeated symbol is

$$P(A) = \frac{1{,}413{,}720}{1{,}679{,}616} \cong 0.8417$$

And hence

$$P(S - A) \cong 1 - 0.8417$$

$$\cong 0.158$$

$$= 15.8\%$$

Probability

- **The Difference Rule---Cont.**

This solution illustrates a more general property of probabilities: that the probability of the complement of an event is obtained by subtracting the probability of the event from the number 1.

Formula for the **Probability of the Complement** of an event!

If S is a finite sample space and A is an event in S, then

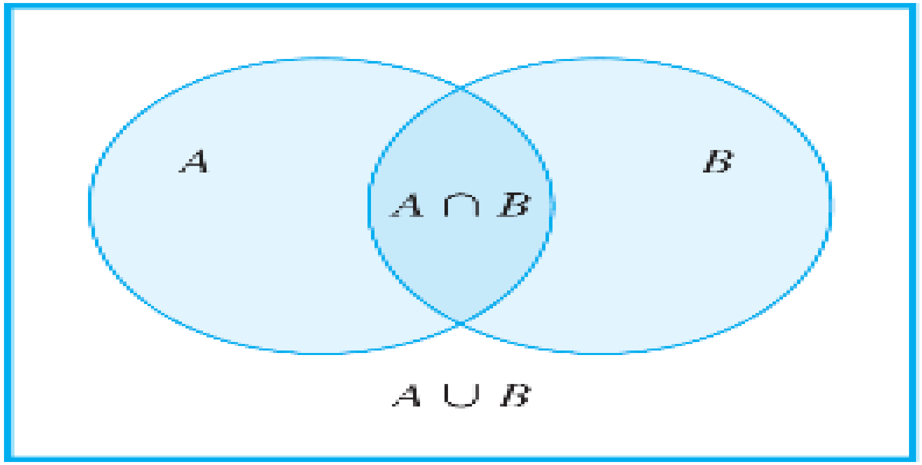
$$P(A^c) = 1 - P(A).$$

Probability

❑ The Inclusion/Exclusion Rule

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a union of sets when some of the sets overlap.

For simplicity, begin by looking at a union of two sets A and B, as shown below.



Probability

- **The Inclusion/Exclusion Rule--- Cont.**

To get an accurate count of the elements in $A \cup B$, it is necessary to subtract the number of elements that are in both A and B. Because these are the elements in $A \cap B$.

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

Probability

❑ Counting Rules in terms of Probabilities

If $\{E_0, E_1, \ldots.\}$ is collection of disjoint events, then

$$\Pr\left[\bigcup_{n\in N} E_n\right] = \sum_{n\in N} \Pr[E_n]$$

Probability

- **Counting Rules in terms of Probabilities---Cont.**

Complement Rule:

$$\Pr[\bar{A}] = 1 - \Pr[A]$$

Probability

- **Counting Rules in terms of Probabilities---Cont.**

$$\Pr[\boldsymbol{B} - \boldsymbol{A}] = \boldsymbol{Pr}[\boldsymbol{B}] - \Pr[\boldsymbol{A} \cap \boldsymbol{B}]$$

$\boxed{\boldsymbol{Difference\ Rule}}$

$$\Pr[\boldsymbol{A} \cup \boldsymbol{B}] = \boldsymbol{Pr}[\boldsymbol{A}] + \boldsymbol{Pr}[\boldsymbol{B}] - \Pr[\boldsymbol{A} \cap \boldsymbol{B}]$$

$\boxed{\boldsymbol{Inclusion - Exclusion}}$

Probability

❑ Formal Definitions

Definition: A countable samples space S is a nonempty countable set. An element $w \in S$ is called an outcome. A subset of S is called an event.

Definition: A probability function on a sample space S is total function $Pr: S \to \mathcal{R}$ such that

- $\Pr[w] \geq 0, for\ all\ w \in S$ and
- $\sum_{w \in S} \Pr[w] = 1$

A sample space together with a probability function is called **probability space**. For any event $E \subseteq S$, the probability of E is defined to be the sum of the probabilities of the outcomes in E.

$$\Pr[E] ::= \sum_{w \in S} \Pr[w]$$

Probability

❑ Formal Definitions---Cont.

- An immediate consequence of the definition of event probability is that for disjoint events E and F,

$$\boldsymbol{Pr[E \cup F] = Pr[E] + Pr[F]}$$

The Sum Rule: If $\{E_0, E_1, \ldots.\}$ is collection of disjoint events, then

$$\Pr\left[\bigcup_{n \in N} E_n\right] = \sum_{n \in N} \Pr[E_n]$$

Probability

❑ Counting Subsets of a Set: Combinations:

Look at these examples:

- In how many ways, can I select 5 books from my collection of 100 to take on vacation?
- How many different ways 13-card Bridge hands can be dealt from a 52-card deck?
- In how many ways, can I select 5 toppings for my pizza if there are 14 available?

What is common in all these questions?

Each is trying to find “how many k-element subsets of an n-element set are there?”

Probability

❑ Counting Subsets of a Set: Combinations---Cont.

$\binom{n}{k}$
$::=$The number of k–element sunsets of an n elements.

Is read as "n choose k"

Probability

❑ Counting Subsets of a Set: Combinations---Cont.

- How to calculate “n choose k”??
    - Permutations
    - Division rule

Probability

❑ Permutations:

- A permutation of a set S is a sequence that contains every element of S exactly once.

- **For example**, Let S = {a, b, c}, then here are all its permutations

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \quad (\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}) \quad (\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c})$$
$$(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}) \quad (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}) \quad (\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{a})$$

Probability

❑ Permutations---Cont.

- How many permutations of an n-element set are there?

Simple:

- There are n choices for the first element, n-1 for the second, n-2 for the third, and so forth.

- Thus there are a total of

- $n.\ (n-1).\ n(n-2).\ .\ .\ .\ 3.\ 2.\ 1 = n!$

Probability

❑ The Division Rule

- How many students are in this room?
- One stupid but workable strategy: count how many ears are in this room and then divide by 2
- A k-to-1 function maps exactly k elements of the domain to every element of the domain

$$\boldsymbol{if\ f: A \rightarrow B} \text{ is k–to–1, then } |\mathrm{A}| = \mathrm{k} \cdot |\mathrm{B}|$$

Equivalently:

$$|\mathrm{B}| = \frac{|A|}{2}$$

That is what we did above: the number of people is half the number of ears.

Probability

- **Why Division Rule??**
  - Interestingly, many counting problems are solved:
    - By counting every item multiple times
    - Then, correcting the answer using the division rule

Probability

- **Back to Combinations: How to calculate $\binom{n}{k}$**

Lets start with permutations of an "n-element" set then map these permutations to a k-elements subset by taking its first "k" elements.

$$a_1, a_2, a_3, \ldots, a_n \rightarrow \{a_1, a_2, a_3, \ldots, a_k\}$$

Note: Any other permutation having **<u>the same</u>** "***k***" elements in any order and remaining "***n-k***" elements will also map to the same set.

Total permutations for a

"k"-elements" set = $k!$

"n-k" elements set = $(n-k)!$

Thus we can conclude that $\boxed{k!\,(n-k)!}$ permutations of n-elements set map to a particular subset.

That is, the mapping from permutations to k-elements subset is $\boxed{k!\,(n-k)! \rightarrow 1}$

Probability

- **Back to Combinations: How to calculate $\binom{n}{k}$**

For a second, lets go back to our "Ears of People" example:

A= {Set of Ears in a room}

B= {Set of persons in the same room}

Mapping from ***A* to *B*** = ***2 →1***

Thus, $|A|= 2.B$

Lets come back to the problem in hand

A= {Set of permutations of an n-element set}

B= {Set of k-subsets formed by taking the first "k" elements of each permutation}

Then mapping from ***A*** to ***B*** = $k!\,(n-k)! \rightarrow 1$

Thus $|A|=k!(n\text{-}k)!\,.\,|B|$

$$n! = k!\,(n-k)!\,.\binom{n}{k}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}}$$

Probability

- **Practice:**

Please do this at home!

Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Probability

- **Why Did we learn about Counting??**

  Let's get back to probability

  Remember this: $\mathbf{Pr}[\boldsymbol{E}] = \frac{|\boldsymbol{E}|}{|\boldsymbol{S}|}$

  Here, we are interested in knowing the cardinalities of ***E*** and ***S***.

  The concepts that we just learned, help us do this exact thing.

Probability

- **Example:**

Suppose we select 5 cards at random from a deck of 52 cards.

What is the probability that we will end up having a full house?

Doing this using the possibility tree will take some effort. But we can do this as follows:

$$|S| = \binom{n}{k}$$

$$|E| = 13.\binom{4}{3}.12.\binom{4}{2}$$

So, $$\Pr[E] = \frac{13.12.\binom{4}{3}.\binom{4}{2}}{\binom{13}{5}} = \frac{13.12.4.6.5.4.3.2}{52.51.50.49.48} = \frac{18}{12495} \approx \frac{1}{694}$$