Discrete Structures

Lecture # 3

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Recap

What is Logic and its used cases (Mathematics and CS)?

What is Propositional Logic (Proposition and Predicates)?

Examples of Proposition and Predicates

What are Logical Operators with examples?

What are Compound Statements and Basic Building blocks for Negation/Conjunction/Disjunction.

Today’s Topic

Exclusive Or with Examples

Logical Equivalence with Examples

De-Morgan’s Law with Examples

Tautology with Examples

Contradiction with Examples

Laws of Logic (Homework)!

The Exclusive Or Operator

The binary **exclusive-or** operator “$\oplus$” (XOR) combines two propositions to form their logical “exclusive or” (exjunction?).

e.g. p = “I will earn an A in this course”

q = “I will drop this course”

$p \oplus q$ = “I will either earn an A in this course, **or** I will drop it (but not both!)”

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!

- This operation is called **exclusive or**, because it **excludes** the possibility that both p and q are true.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note the difference from OR

Natural Language is Ambiguous

Note that <u>English</u> "or" can be <u>ambiguous</u> regarding the "both" case!

"Justin Bieber is a singer or
Justin Bieber is a writer" $\vee$

"John Cena is a man or
John Cena is a woman" $\oplus$

Need context to disambiguate the meaning!

For this class, assume "OR" means <u>inclusive</u>.

Logical Equivalence

- Two statement forms are called logically equivalent if and only if, they have identical truth values for all possible truth values for their statement variables.

- The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Truth Table – Example – Cont..

1. $\sim(\sim P) \equiv P$

P	~P	~(~P)
T	F	T
F	T	F

Same Truth Values in 1st and 3rd row

De Morgan’s Laws

- The negation of an **and / or** statement is logically equivalent to the **or / and** statement in which each component is negated.

- Symbolically:

$$\sim(P \wedge Q) \equiv \sim P \vee \sim Q \text{ and } \sim(P \vee Q) \equiv \sim P \wedge \sim Q$$

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	**Q**	**~P**	**~Q**	**P ∨ Q**	**~(P ∨ Q)**	**~P ∧ ~Q**
T	T					
T	F					
F	T					
F	F					

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∧ ~Q
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	**Q**	**~P**	**~Q**	**P ∨ Q**	**~(P ∨ Q)**	**~P ∧ ~Q**
T	T	F	F	T	F	
T	F	F	T	T	F	
F	T	T	F	T	F	
F	F	T	T	F	T	

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \land \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \lor \sim\boldsymbol{Q}$

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∧ ~Q
T	T	F	F			F
T	F	F	T			F
F	T	T	F			F
F	F	T	T			T

De Morgan's Laws – Example – 1

- Prove $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$

P	**Q**	**~P**	**~Q**	**P ∨ Q**	**~(P ∨ Q)**	**~P ∧ ~Q**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

De Morgan's Laws – Example – 2

- Prove $\sim(P \wedge Q)$ ***and*** $\sim P \wedge \sim Q$ are not equivalent

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∧ ~Q
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

De Morgan's Laws – Example – 2

- Prove $\sim(P \wedge Q)$ ***and*** $\sim P \wedge \sim Q$ are not equivalent

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∧ ~Q
T	T			T		
T	F			F		
F	T			F		
F	F			F		

De Morgan's Laws – Example – 2

- Prove $\sim(P \wedge Q)$ *and* $\sim P \wedge \sim Q$ are not equivalent

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∧ ~Q
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Exercise – 1

- Are the statements $(\boldsymbol{P} \wedge \boldsymbol{Q}) \wedge \boldsymbol{R}$ and $\boldsymbol{P} \wedge (\boldsymbol{Q} \wedge \boldsymbol{R})$ logically equivalent?

- Are the statements $(\boldsymbol{P} \wedge \boldsymbol{Q}) \vee \boldsymbol{R}$ and $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R})$ logically equivalent?

Tautology

- A tautology is a statement from that is always true regardless of the truth values of the statement variables.

- A tautology is represented by the symbol “t”.

Tautology – Example

- The statement $P \vee \sim P$ is Tautology.

$P \vee \sim P \equiv t$		
P	~P	P ∨ ~P
T	F	T
F	T	T

Contradiction

- A contradiction is a statement from that is always false regardless of the truth values of the statement variables.

- A contradiction is represented by the symbol “c”.

Contradiction – Example

- The statement $P \wedge \sim P$ is Contradiction.

$P \wedge \sim P \equiv c$		
P	~P	P ∧~P
T	F	F
F	T	F

Example – 1

$$(P \wedge Q) \vee (\sim P \vee (P \wedge \sim Q)) \equiv t$$

P	Q	~P	~Q	P ∧ Q	P ∧ ~Q	~P ∨ (P ∧ ~Q)	(P ∧ Q) ∨ (~P ∨ (P ∧ ~Q))
T	T	F	F	T	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	T	F	F	F	T	**T**
F	F	T	T	F	F	T	**T**

Example – 2

$$(P \wedge {\sim}Q) \wedge ({\sim}P \vee Q) \equiv c$$

P	Q	~P	~Q	P ∧ ~Q	~P ∨ Q	(P ∧ ~Q) ∧ (~P ∨ Q)
T	T	F	F	F	F	**F**
T	F	F	T	T	F	**F**
F	T	T	F	F	T	**F**
F	F	T	T	F	T	**F**

Laws of Logic

Commutative Law: $P \wedge Q \equiv Q \wedge P$ and $P \vee Q \equiv Q \vee P$

Associative Law: $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ and $(P \vee Q)$

Distributive Law: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (Q \wedge R)$ and $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (Q \vee R)$

Identity Law: $P \wedge t \equiv P$ and $P \vee c \equiv P$

Negation Law: $P \vee \sim P \equiv t$ and $P \wedge \sim P \equiv c$

Double Negation Law: $\sim(\sim P) \equiv P$

Idempotent Law: $P \wedge P \equiv P$ and $P \vee P \equiv P$

DeMorgan's Law: $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ and $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$

Universal Bound Law: $P \vee t \equiv t$ and $P \wedge c \equiv c$

Absorption Law: $P \vee (P \wedge Q) \equiv P$ and $P \wedge (P \vee Q) \equiv P$

Negations of "t" and "c": $\sim t \equiv c$ and $\sim c \equiv t$

Proving Equivalence via Truth Tables

Example: Prove that $p \vee q$ and $\neg(\neg p \wedge \neg q)$ are logically equivalent.

p	q	p∨q	~p	~q	~p∧~q	~(~p∧~q)
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Proving Equivalence using Logic Laws

Example: Show that $\neg (P \lor (\neg P \land Q))$ and $(\neg P \land \neg Q)$ are logically equivalent.

$$\neg (P \lor (\neg P \land Q))$$

$$\equiv \neg P \land \neg (\neg P \land Q) \quad \text{De Morgan}$$

$$\equiv \neg P \land (\neg(\neg P) \lor \neg Q) \quad \text{De Morgan}$$

$$\equiv \neg P \land (P \lor \neg Q) \quad \text{Double negation}$$

$$\equiv (\neg P \land P) \lor (\neg P \land \neg Q) \quad \text{Distributive}$$

$$\equiv \mathbf{F} \lor (\neg P \land \neg Q) \quad \text{Negation}$$

$$\equiv (\neg P \land \neg Q) \quad \text{Identity}$$

Proving Equivalence using Logic Laws

Example: Show that $\neg(\neg(P \to Q) \to \neg Q)$ is a contradiction.

$$\neg(\neg(P \to Q) \to \neg Q)$$

$$\equiv \neg(\neg(\neg P \vee Q) \to \neg Q) \text{ Equivalence}$$

$$\equiv \neg((P \wedge \neg Q) \to \neg Q) \text{ De Morgan}$$

$$\equiv \neg(\neg(P \wedge \neg Q) \vee \neg Q) \text{ Equivalence}$$

$$\equiv \neg(\neg P \vee Q \vee \neg Q) \text{ De Morgan}$$

$$\equiv \neg(\neg P \vee \mathbf{T}) \text{ Trivial Tautology}$$

$$\equiv \neg(\mathbf{T}) \text{ Domination}$$

$$\equiv \mathbf{F} \quad \text{Contradiction}$$

Application

Simplify: $p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	
$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan'sLaw
$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law
$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
$\equiv p \vee (\sim q)$	Indempotent Law

Which is the simplified statement form.

Application

Verify: $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

$\sim(\sim p \wedge q) \wedge (p \vee q)$	
$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
$\equiv p \vee (\sim q \wedge q)$	Distributive Law in Reverse
$\equiv p \vee c$	Negation Law
$\equiv p$	Identity Law

Simplifying a Statement

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains.:

<u>Solution: Let</u>

p = "You are hardworking'
q = "The sun shines"
r = "It rains"

The condition is then $(p \wedge q) \vee (p \wedge r)$

Exercise

Use Logical Equivalence to rewrite each of the following sentences more simply.

- It is not true that I am tired and you are smart.
- I forgot my pen or my bag and I forgot my pen or my glasses.
- It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.