

Discrete Structures

Lecture # 06

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EXERCISE

A number of computer users are surveyed to find out if they have a **printer**, **modem** or **scanner**. Draw separate **Venn diagrams** and shade the areas, which represent the following configurations. **Keep in mind that all three are not mutually exclusive/disjoint.**

- (i) **modem and printer but no scanner**
- (ii) **scanner but no printer and no modem**
- (iii) **scanner or printer but no modem.**
- (iv) **no modem and no printer.**

SOLUTION

We have the sets

Computer users having a **printer**.

Computer users having a **modem**.

Computer users having a **scanner**.

SOLUTION

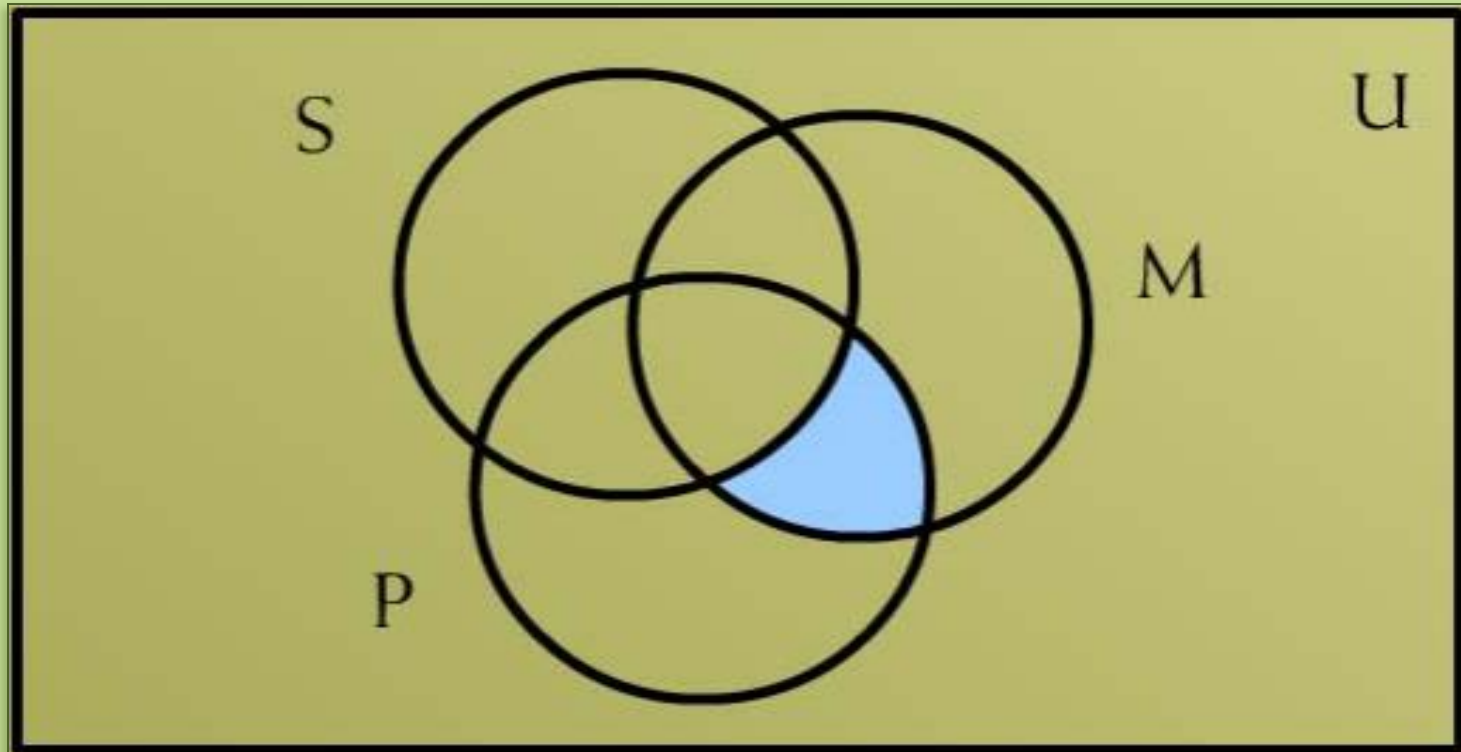
P represent the set of computer users having printer.

M represent the set of computer users having modem.

S represent the set of computer users having scanner.

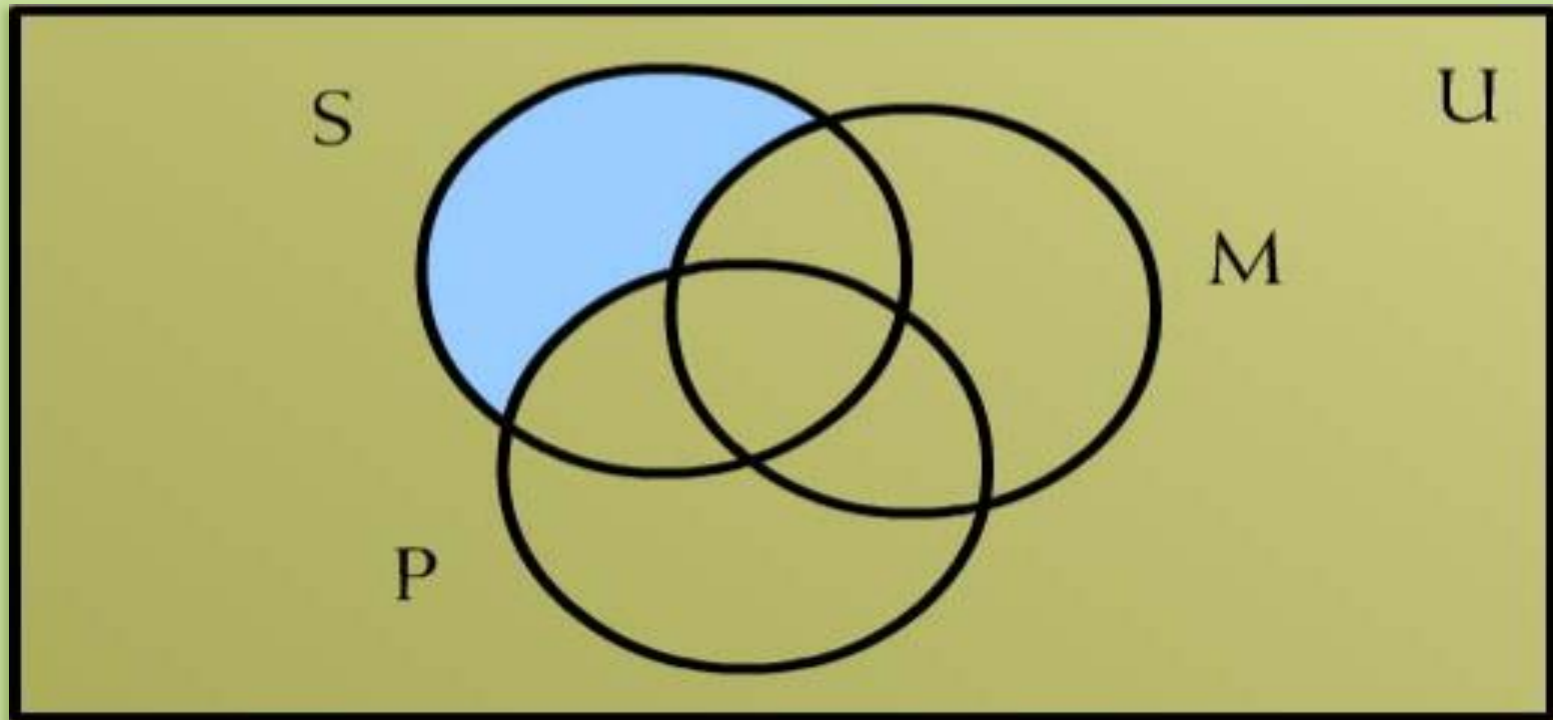
SOLUTION

(i) **modem** and **printer** but no **scanner**.



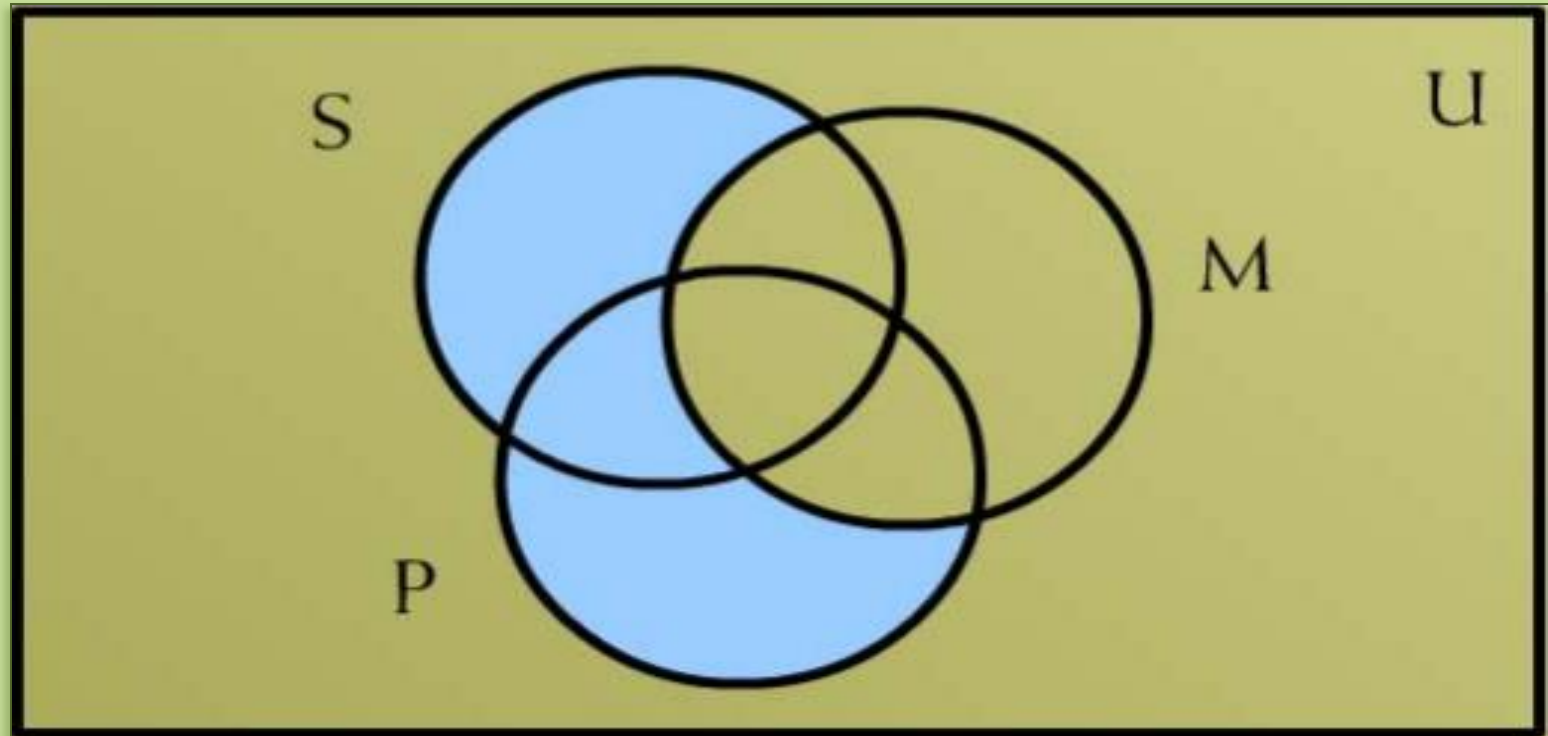
SOLUTION

(ii) **scanner** but no **printer** and no **modem**.



SOLUTION

(iii) **scanner** or **printer** but no **modem**.

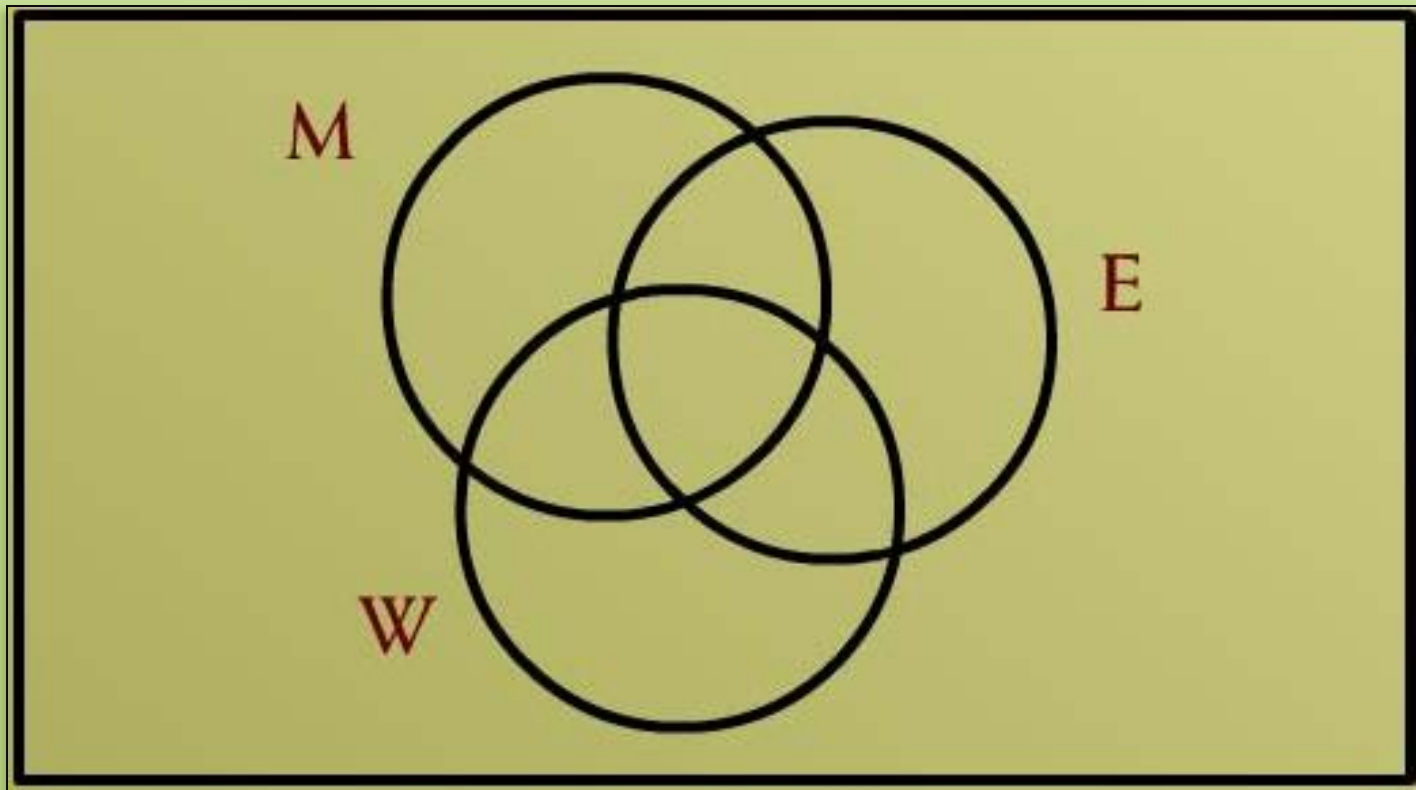


EXERCISE

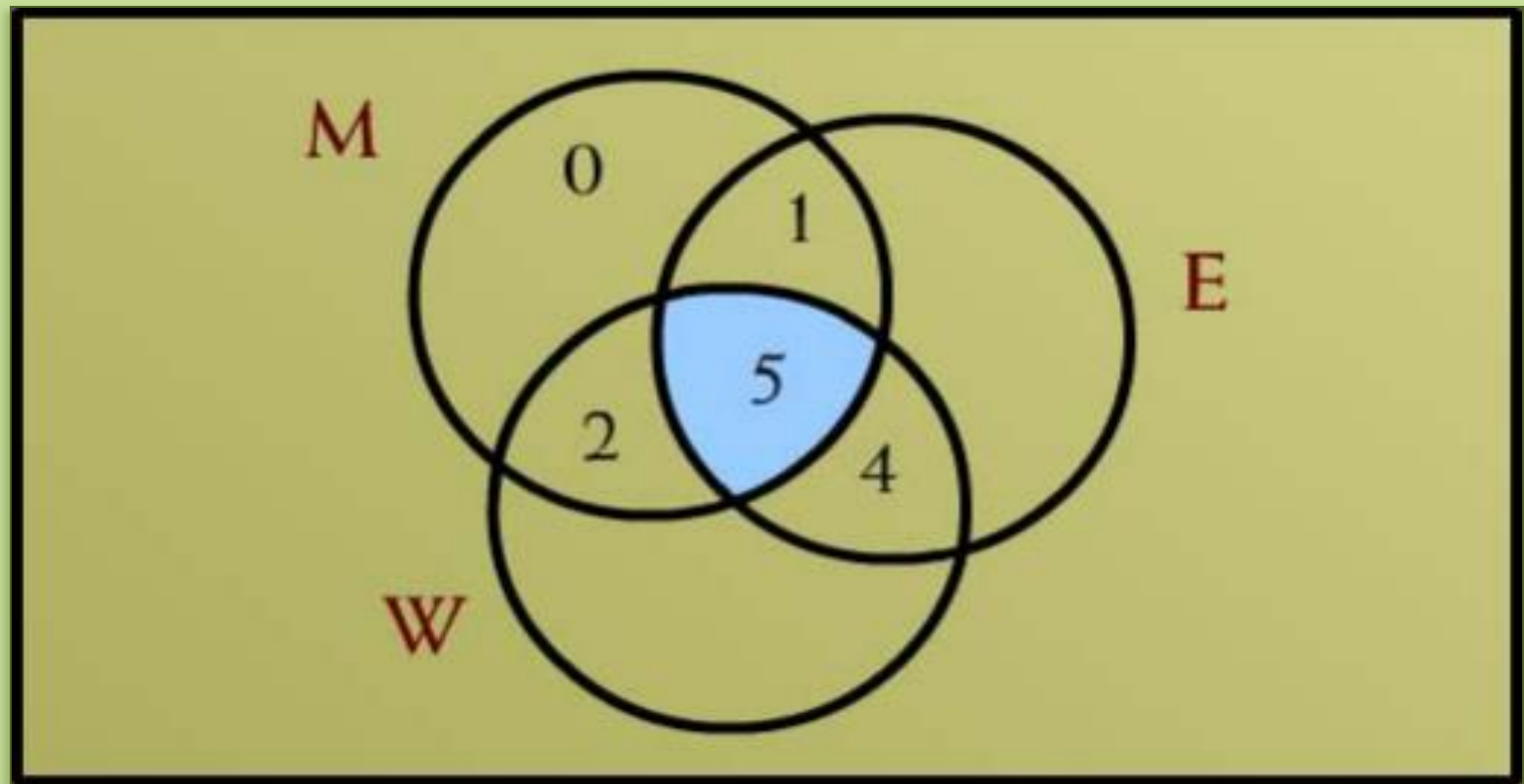
21 typists are in an office, 5 use all **manual typewriters (M)**, **electronic typewriters (E)** and **word processors (W)**; 9 use **E** and **W**; 7 use **M** and **W**; 6 use **M** and **E**; but no one uses **M** only.

- (i) Represent this information in a **Venn Diagram**.
- (ii) If the same number of typists use **electronic** as use **word processors**, then
 - (a) how many use **word processors** only,
 - (b) how many use **electronic typewriters**?

SOLUTION

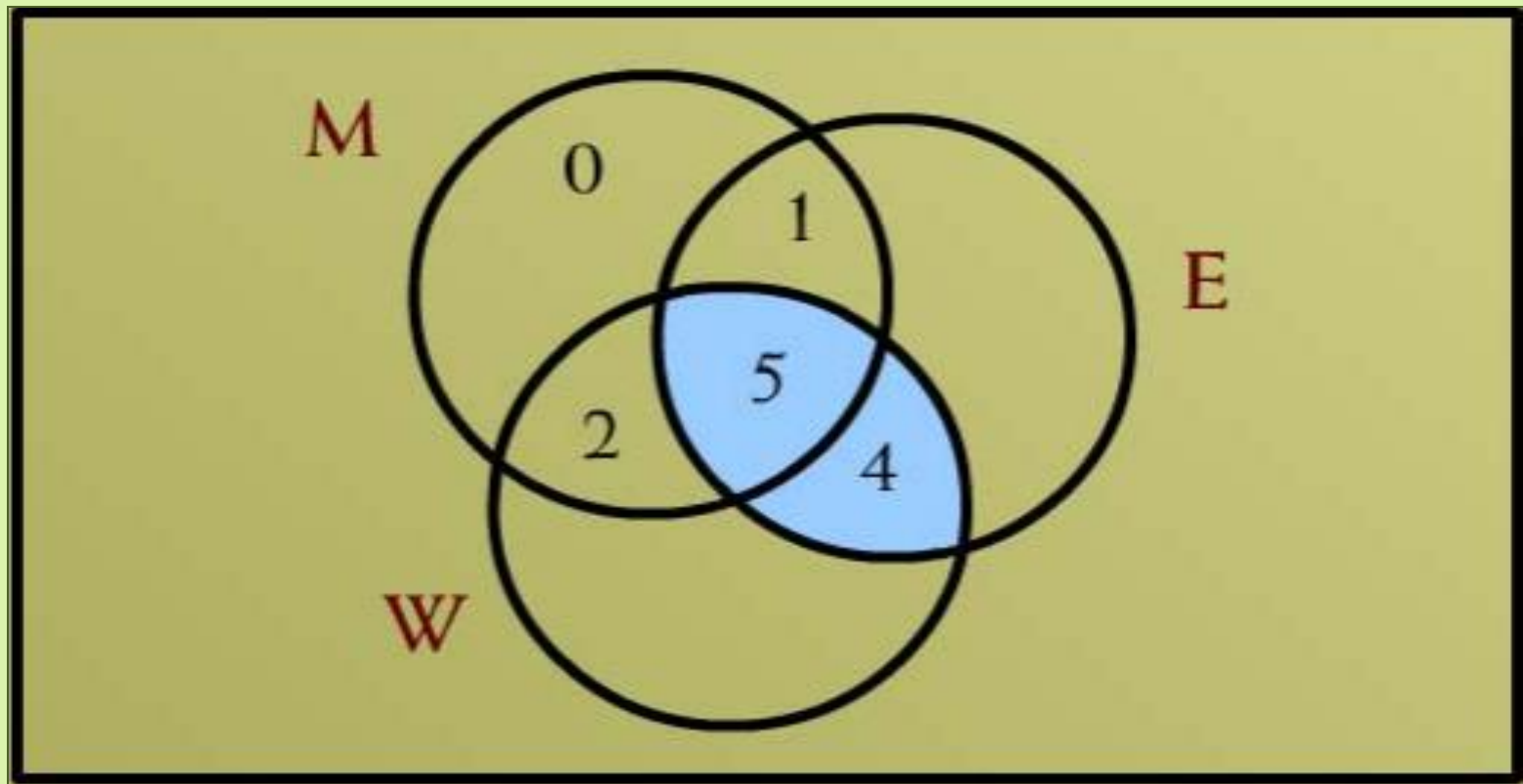


SOLUTION



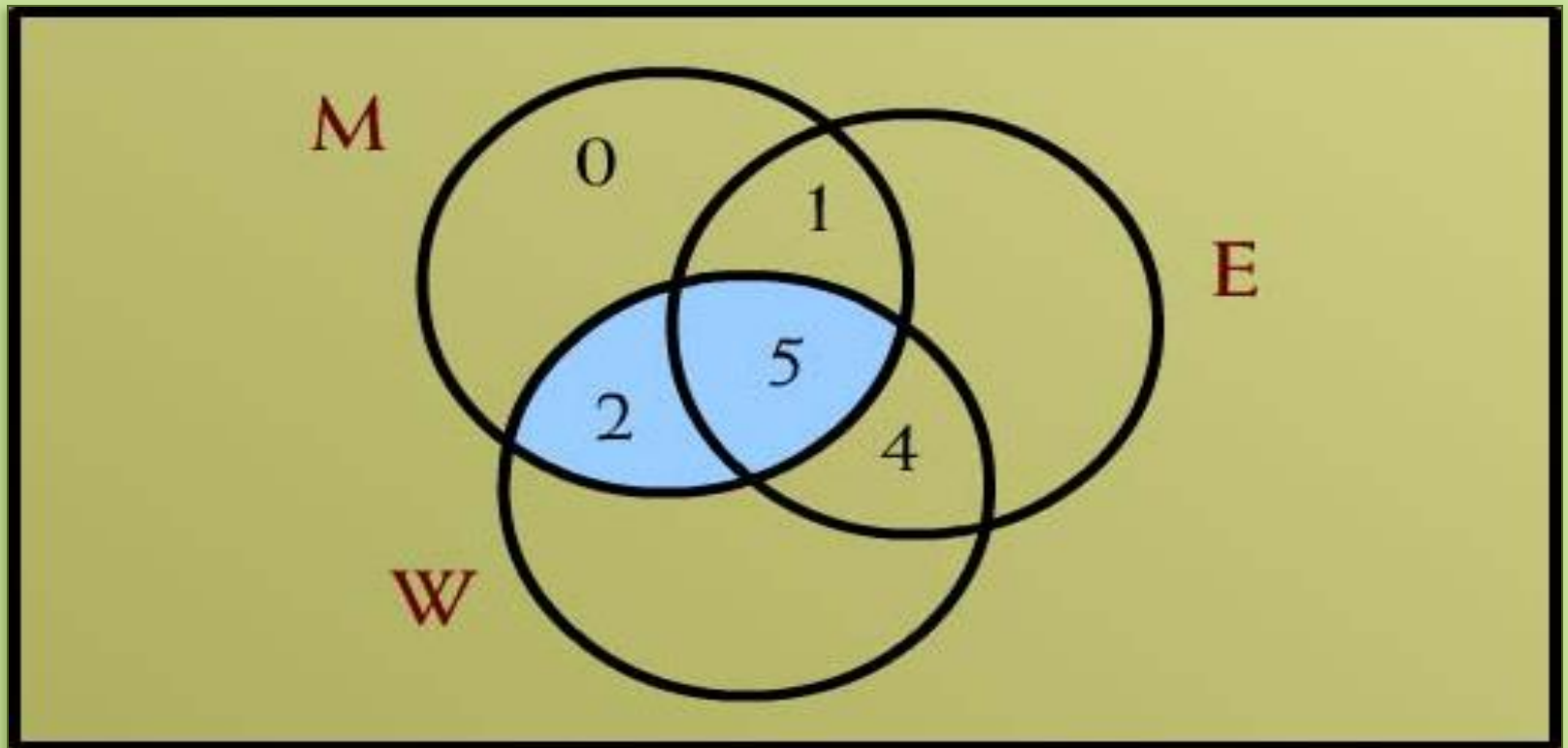
5 use all M, E and W

SOLUTION



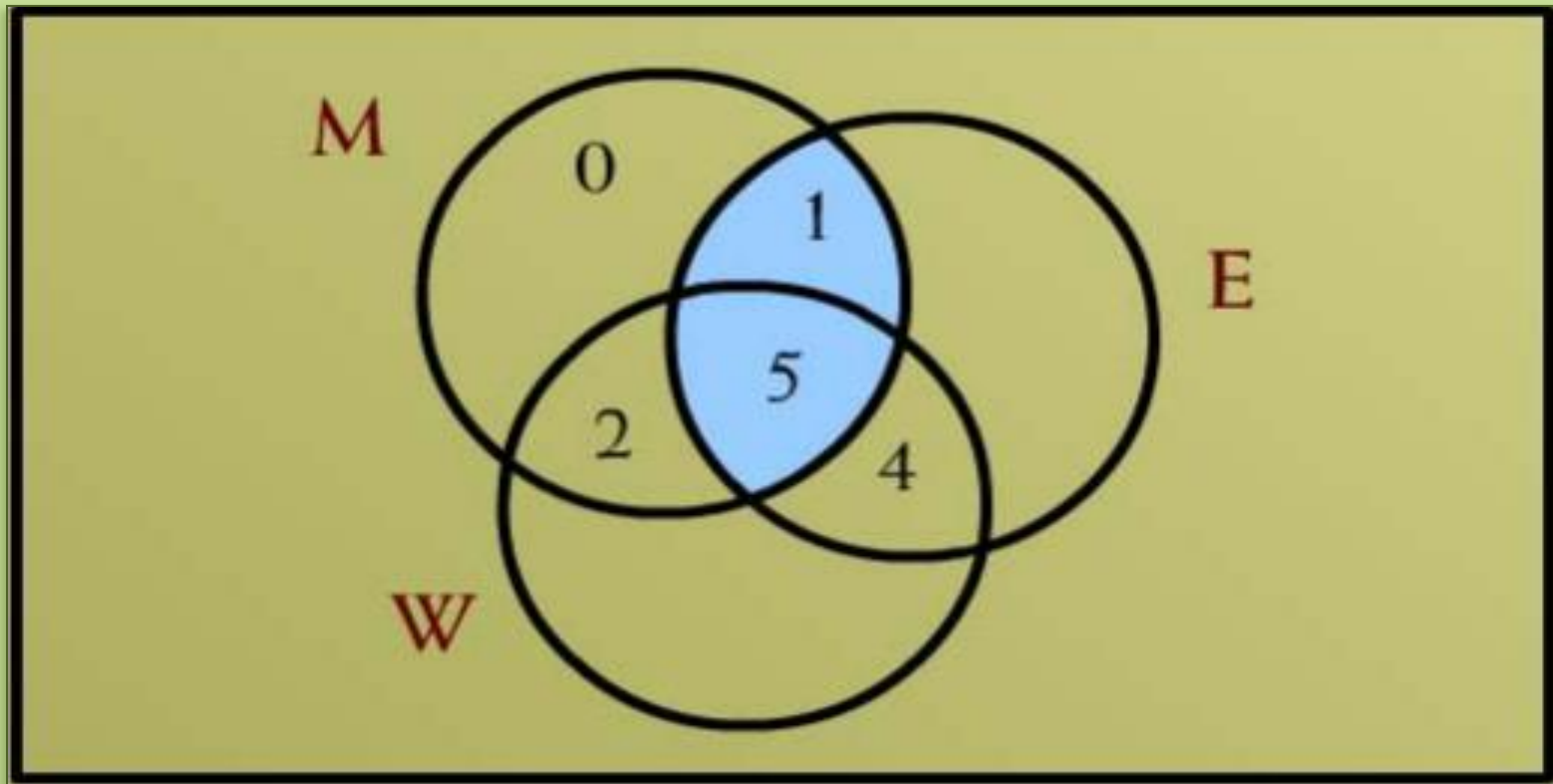
9 use E and W

SOLUTION



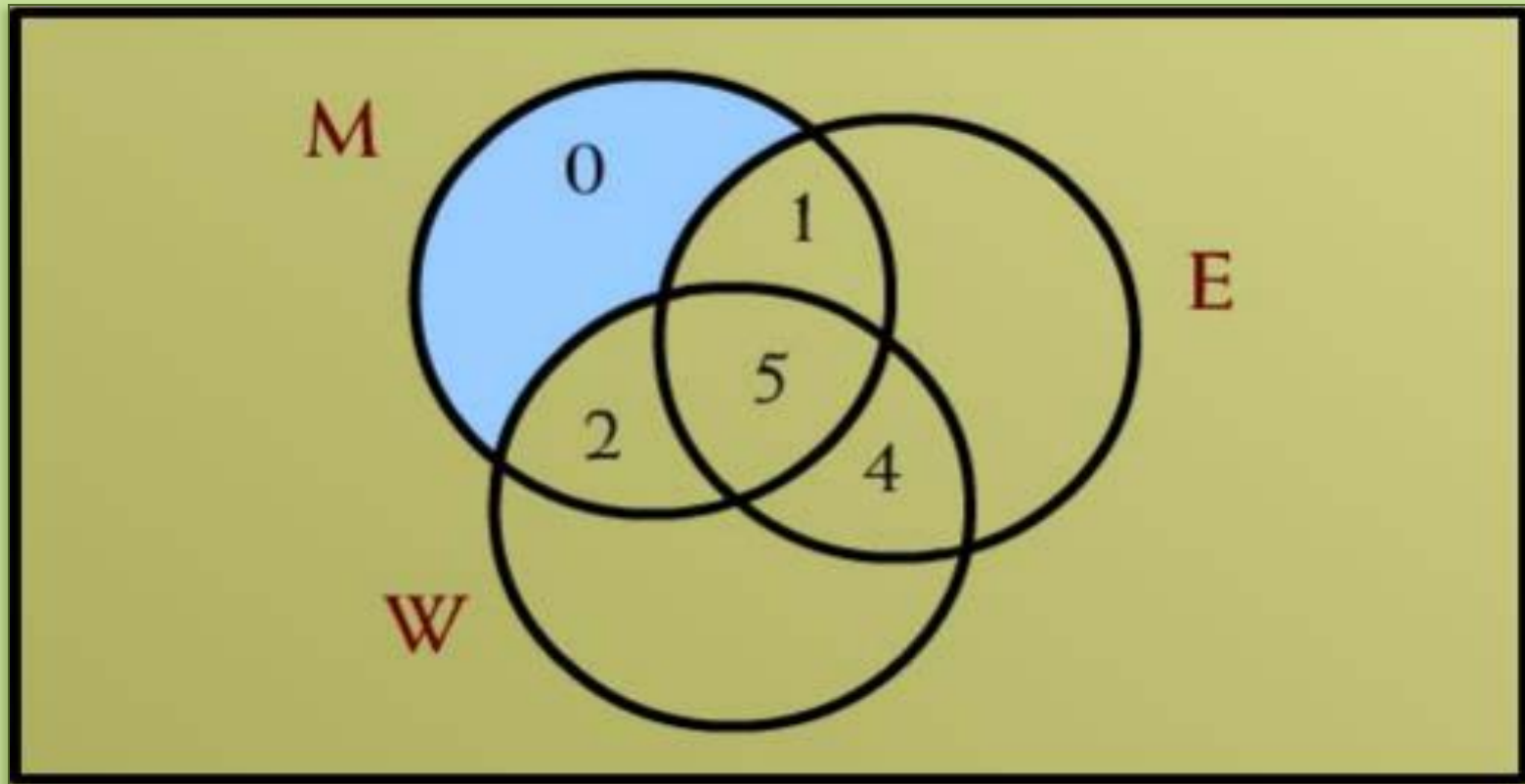
7 use M and W

SOLUTION



6 use **M** and **E**

SOLUTION



no one uses **M** only

SOLUTION

(ii) - (a) If same number of typists use **electronic typewriters** as **word processors**, how many use **word processors** only?

SOLUTION:

Let

x = number of typists using **electronic typewriters** (E)

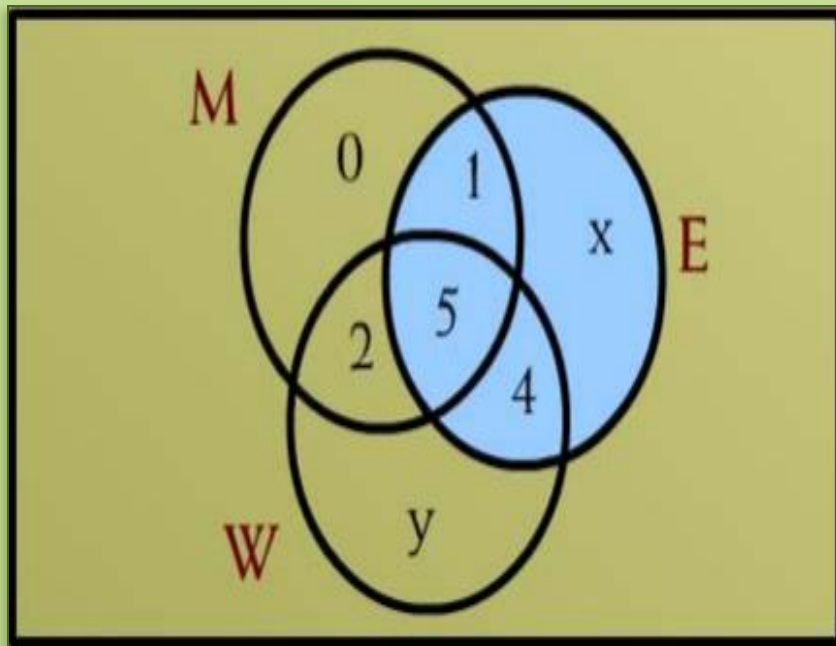
y = number of typists using **word processors** (W)

Solution contd...

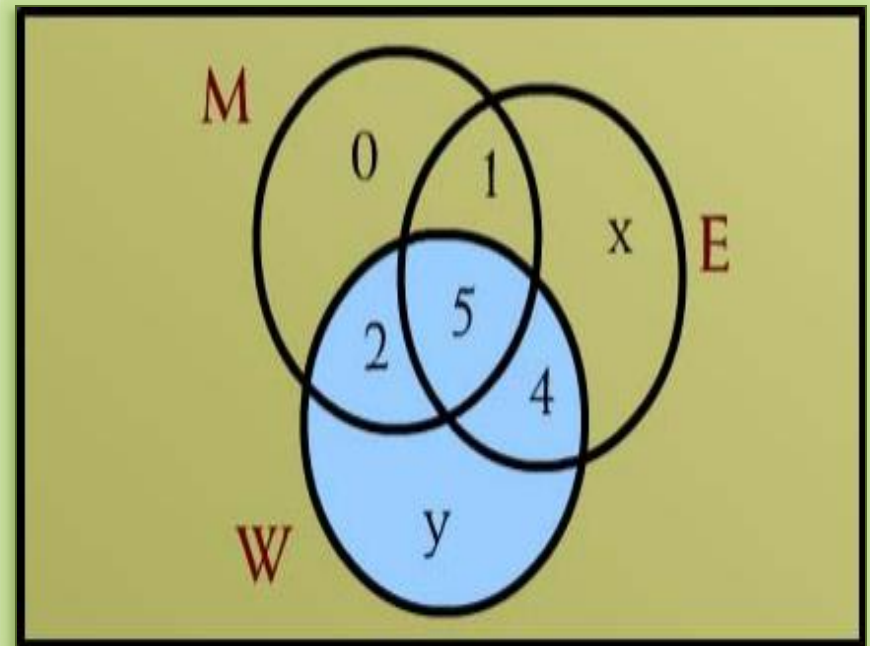
Total number of typists using **E** =

Total number of typists using **W**

$$1 + 5 + 4 + x = 2 + 5 + 4 + y$$



Electronic users only (In diagram)



Word processor users only (In diagram)

Solution contd...

$$\text{or} \quad x - y = 1 \dots \dots \dots (1)$$

$$\text{total number of typists} = 21$$

$$\Rightarrow 0 + x + y + 1 + 2 + 4 + 5 = 21$$

$$\text{or} \quad x + y = 9 \dots \dots \dots (2)$$

Solving (1) & (2), we get

$$x = 5, y = 4$$

\therefore Number of typists using **word processor**
only is **$y = 4$**

SOLUTION

(ii)-(b) How many typists use **electronic typewriters**?

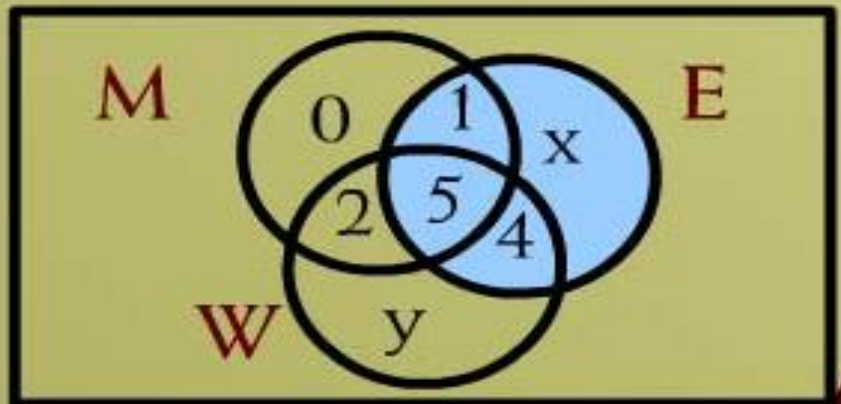
SOLUTION:

Typists using **electronic typewriters**
= No. of elements in **E**

$$= 1 + 5 + 4 + x$$

$$= 1 + 5 + 4 + 5$$

$$= 15$$



EXERCISE

In a school, 100 students have access to three software packages, A, B and C

28 did not use any software

8 used only packages A

26 used only packages B

7 used only packages C

10 used all three packages

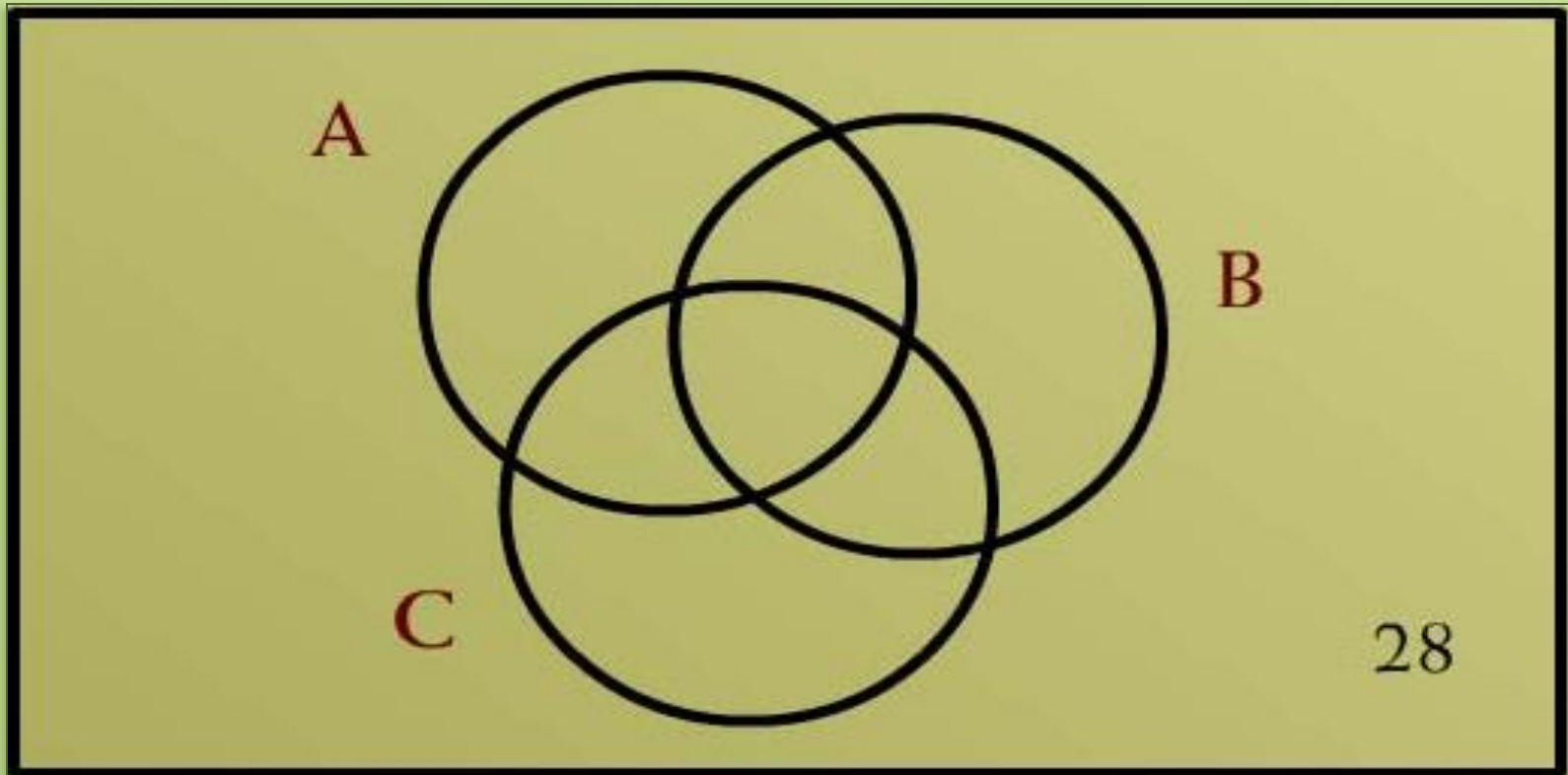
13 used both A and B

EXERCISE

- (i) Draw a **Venn diagram** with all sets enumerated as far as possible. Label the two subsets which cannot be enumerated as x and y , in any order.
- (ii) If twice as many students used **package B** as **package A**, write down a pair of simultaneous equations in x and y .
- (iii) Solve these equations to find x and y .
- (iv) How many students used **package C**?

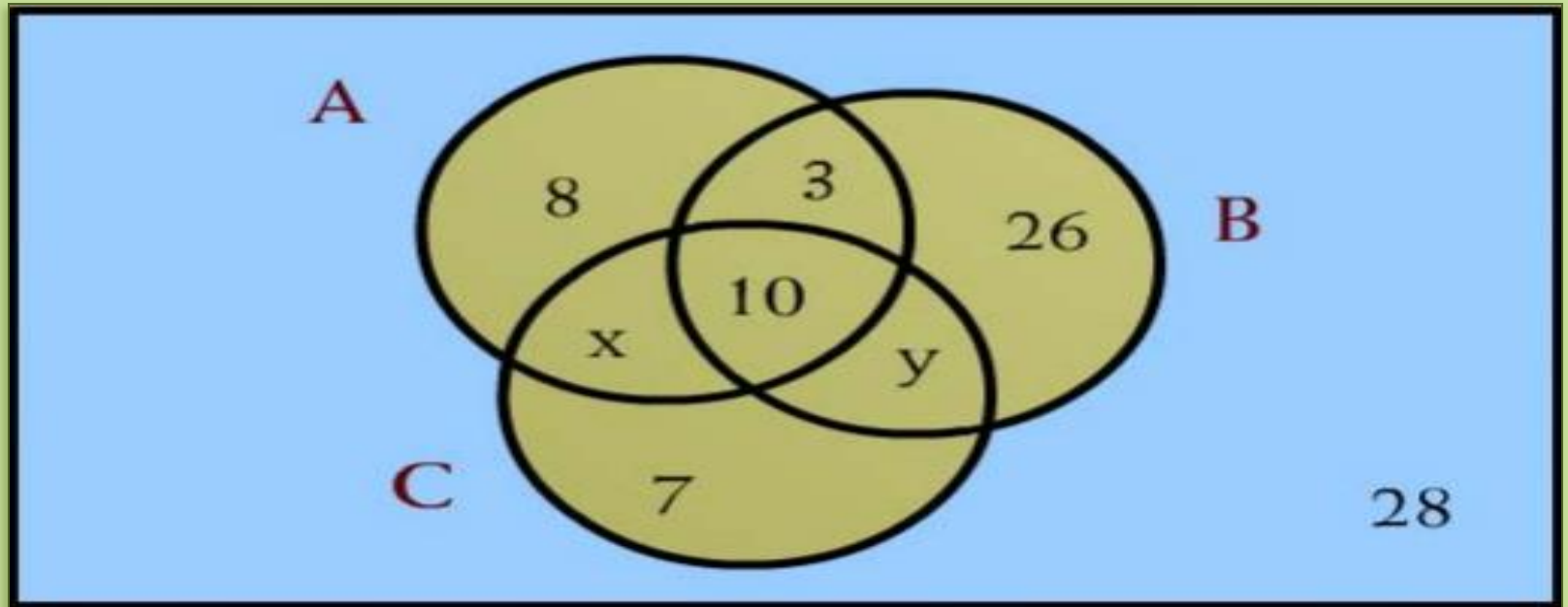
SOLUTION

(i) **Venn Diagram** with all sets enumerated.



SOLUTION

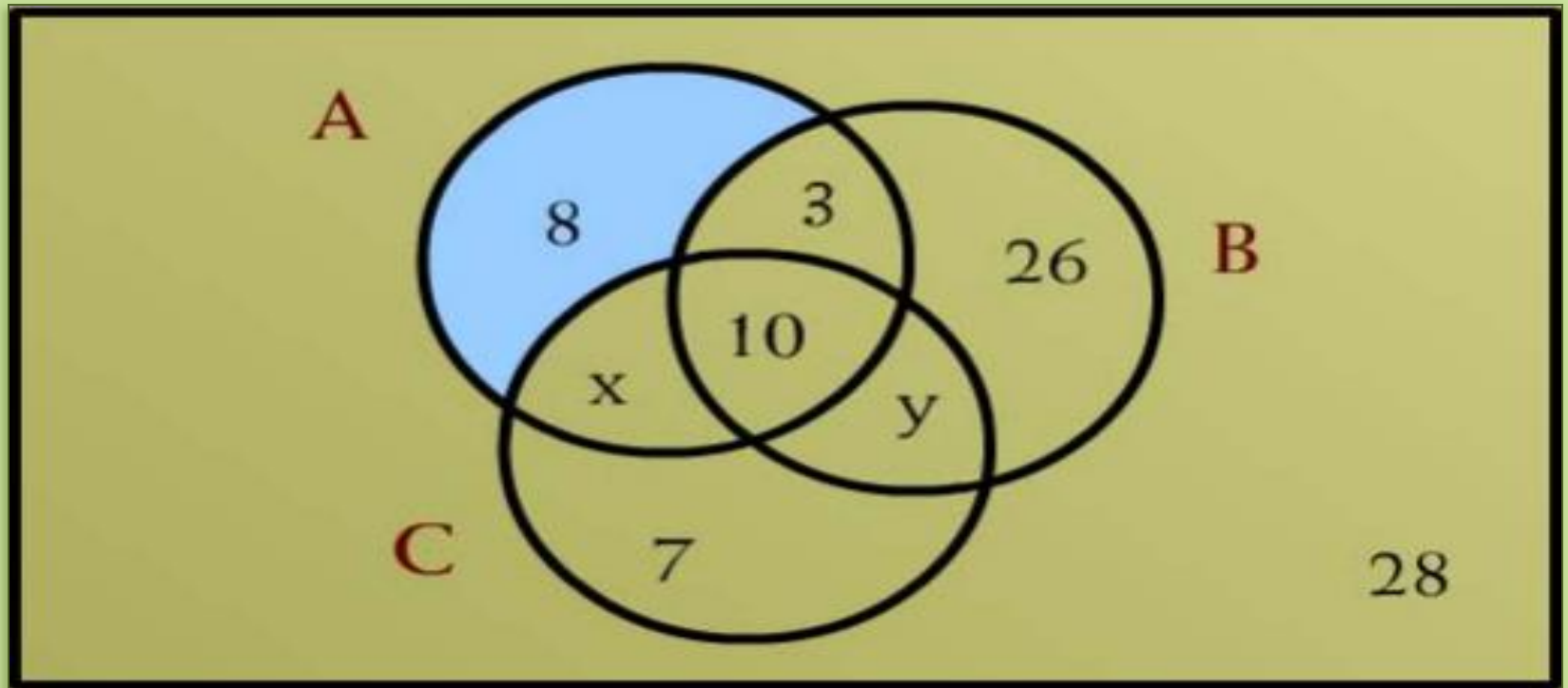
(i) **Venn Diagram** with all sets enumerated.



28 did not use any **software**

SOLUTION

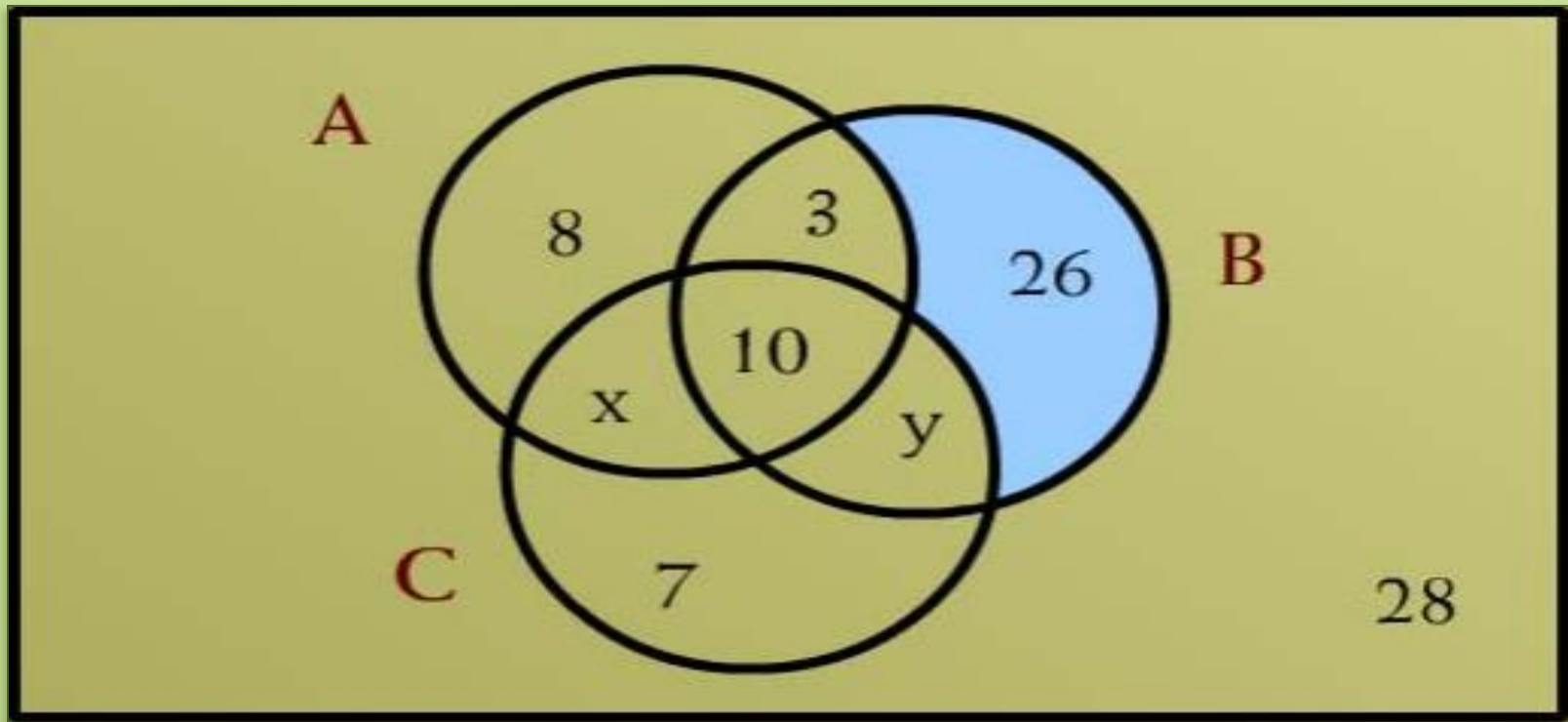
(i) **Venn Diagram** with all sets enumerated.



8 used only **package A**

SOLUTION

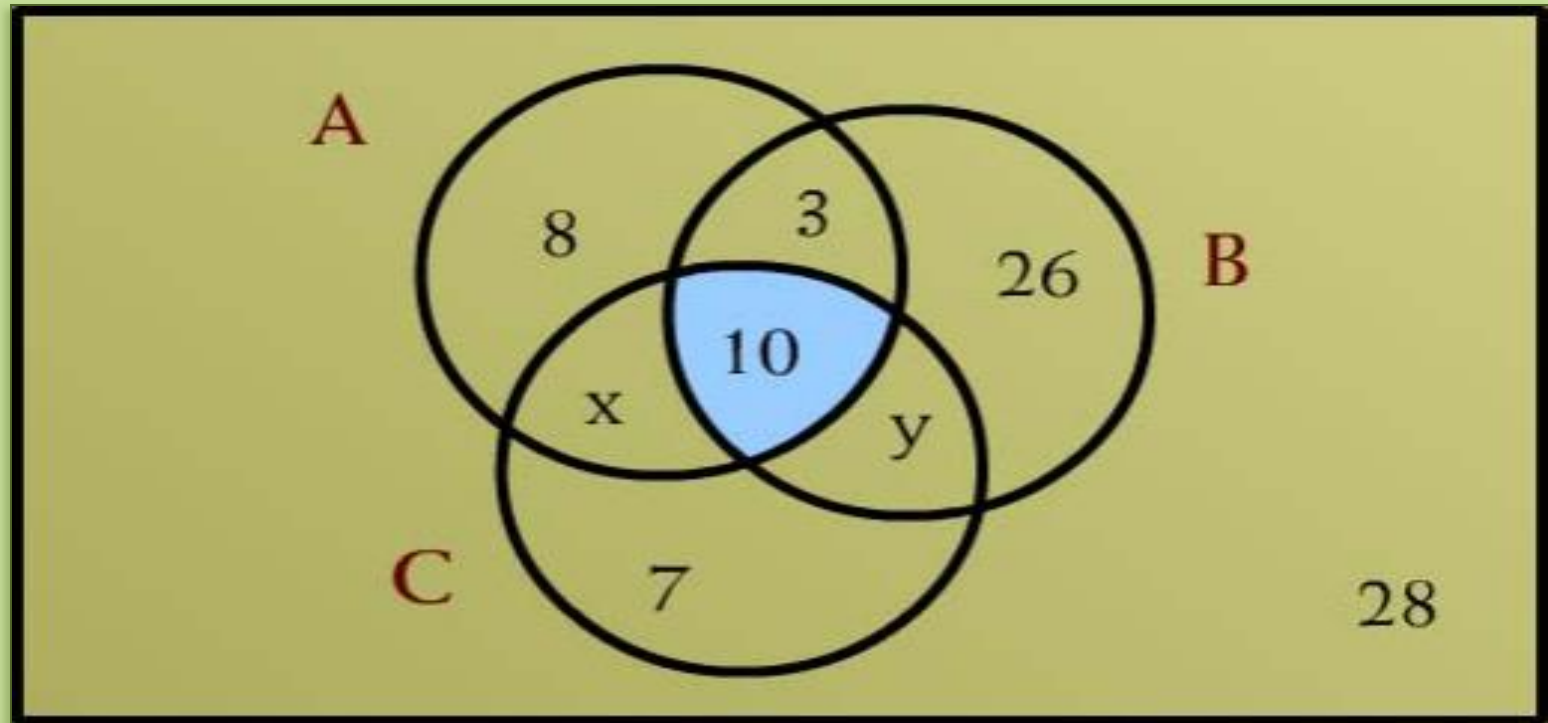
(i) **Venn Diagram** with all sets enumerated.



26 used only **package B**

SOLUTION

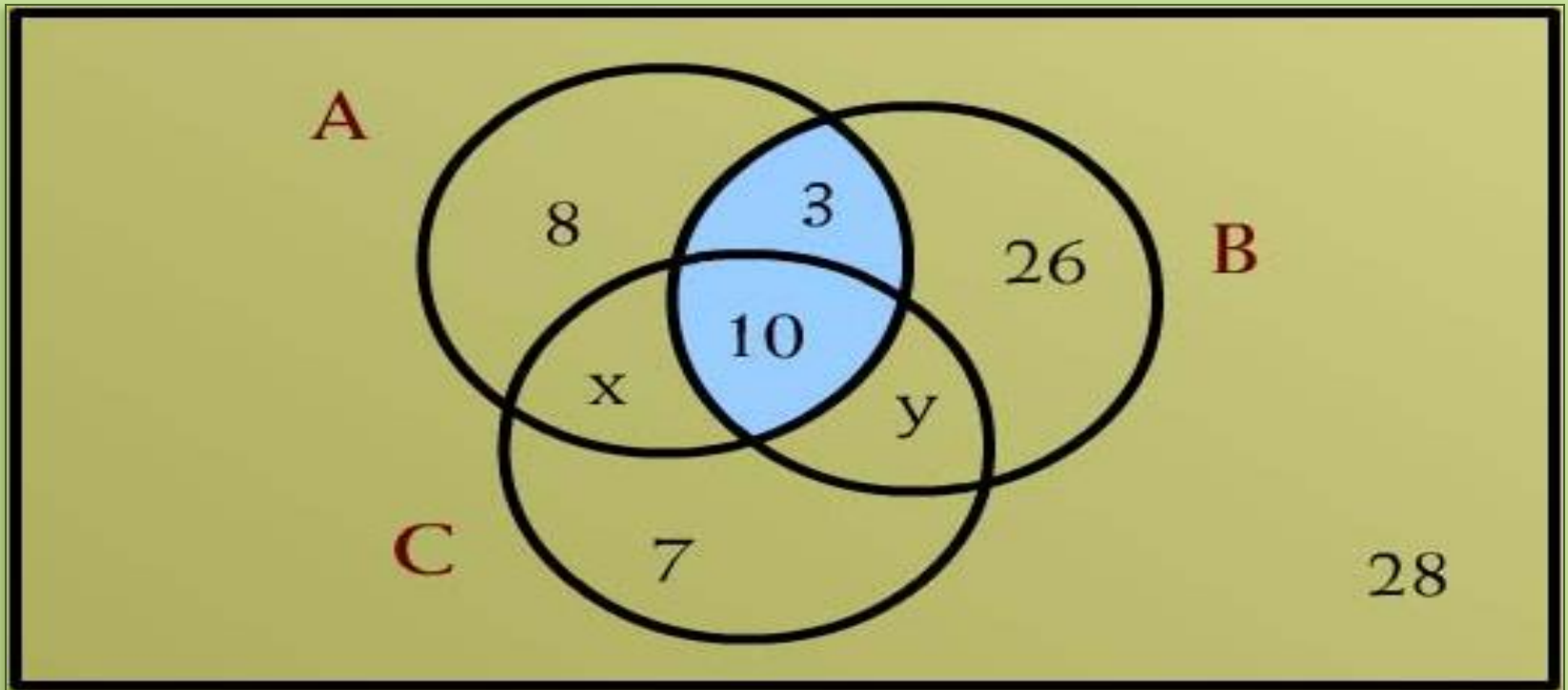
(i) **Venn Diagram** with all sets enumerated.



10 used all three **packages**

SOLUTION

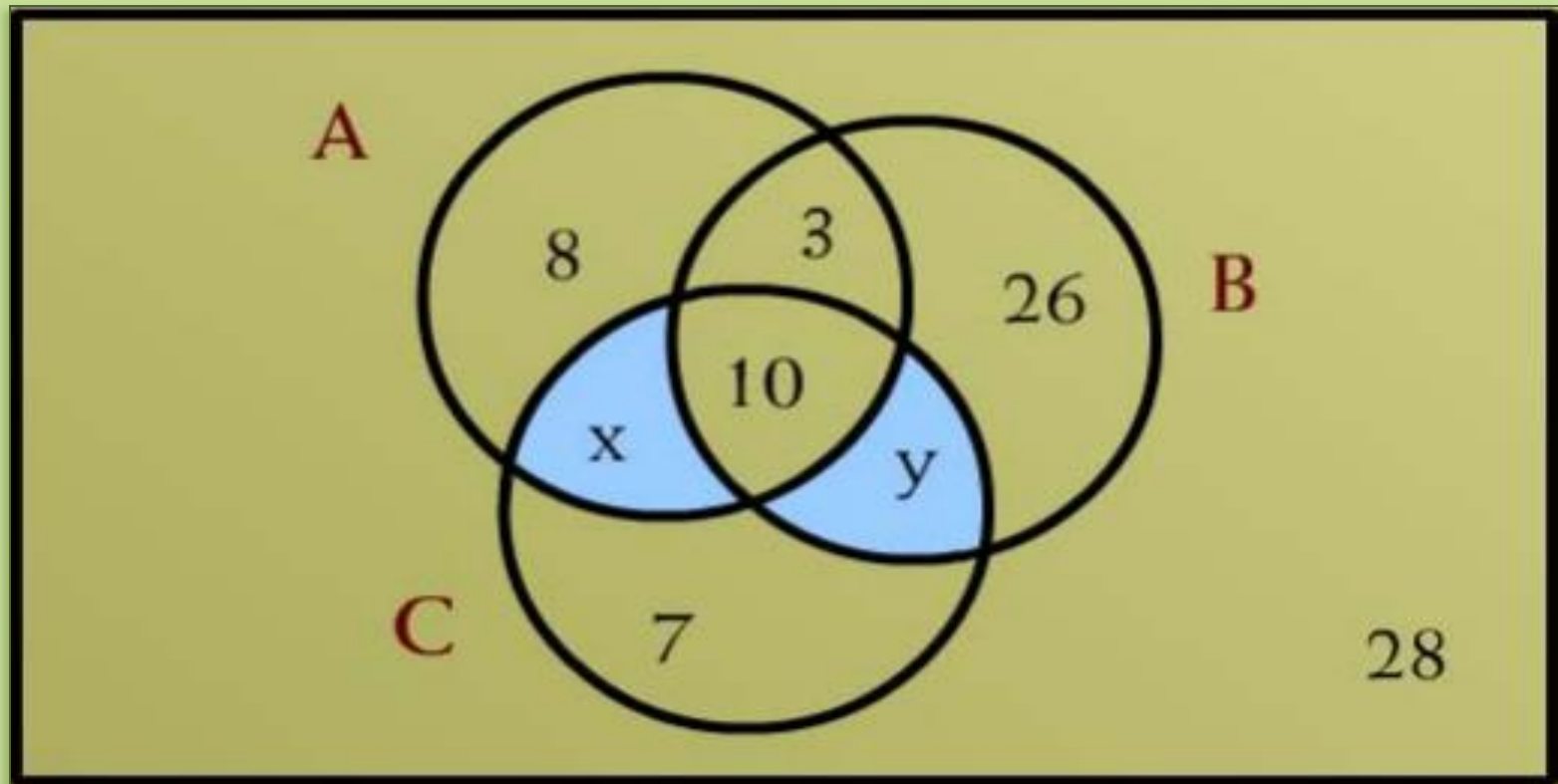
(i) **Venn Diagram** with all sets enumerated.



13 used both **A** and **B**

SOLUTION

(i) **Venn Diagram** with all sets enumerated.



Solution contd...

(ii) If twice as many students used **package B** as **package A**, write down a pair of simultaneous equations in x and y .

SOLUTION:

students using **package B**
 $= 2(\text{\# students using package A})$

$$\Rightarrow 3 + 10 + 26 + y = 2(8 + 3 + 10 + x)$$

Solution contd...

$$\Rightarrow 39 + y = 42 + 2x$$

$$\text{or } y = 2x + 3 \dots \dots \dots (1)$$

Also, total number of students = 100.

Hence,

$$8 + 3 + 26 + 10 + 7 + 28 + x + y = 100$$

$$\text{or } 82 + x + y = 100$$

$$\text{or } x + y = 18 \dots \dots \dots (2)$$

Solution contd...

(iii) Solving simultaneous equations for x and y.

SOLUTION:

$$y = 2x + 3 \dots\dots\dots (1)$$

$$x + y = 18 \dots\dots\dots (2)$$

Using (1) in (2), we get,

$$x + (2x + 3) = 18$$

$$\text{or} \qquad 3x + 3 = 18$$

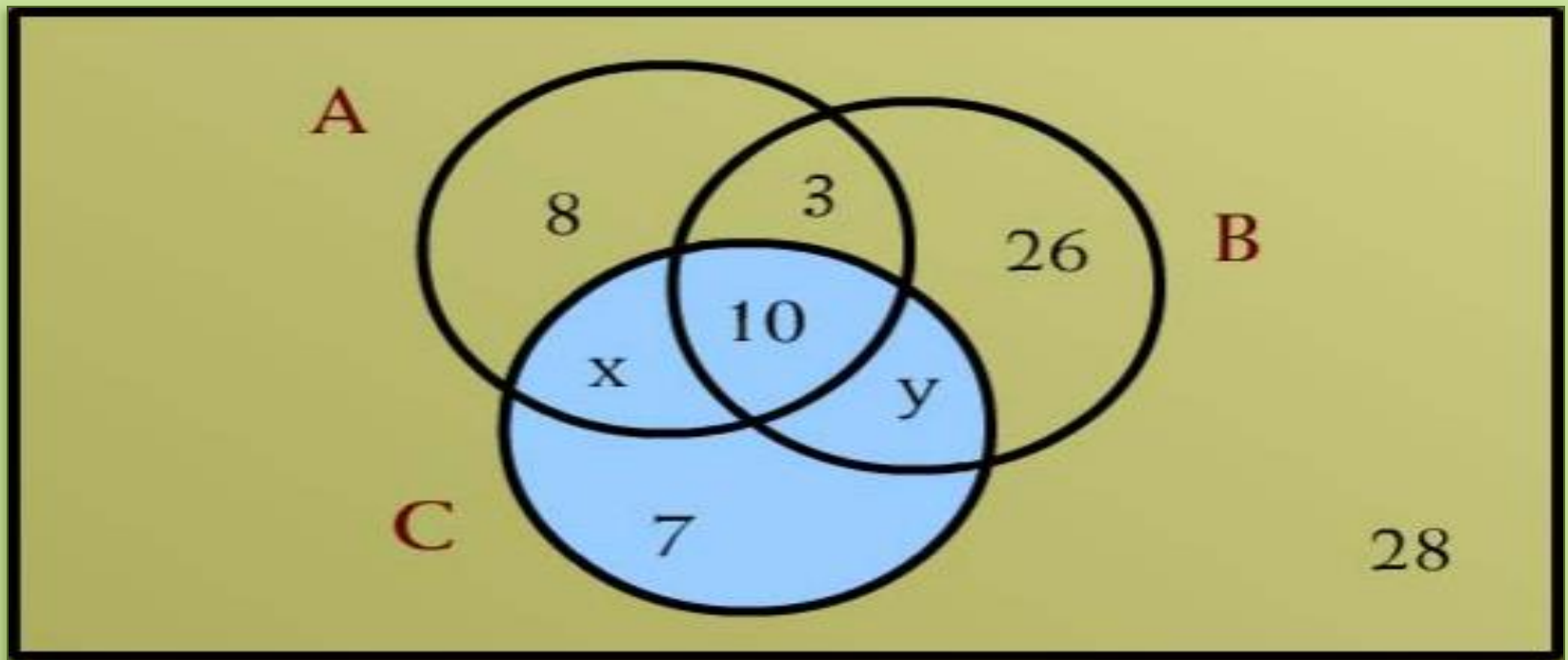
$$\text{or} \qquad 3x = 15$$

$$\Rightarrow \qquad x = 5$$

$$\text{Consequently} \qquad y = 13$$

SOLUTION

(iv) How many students used **package C**?



$$x + 10 + y + 7 = 35$$

PARTITION OF A SET

A set may be divided up into its **disjoint subsets**. Such division is called a **partition**.

More precisely,

A partition of a set A is a collection of non-empty subsets $\{A_1, A_2, \dots, A_n\}$ of A , such that

PARTITION OF A SET

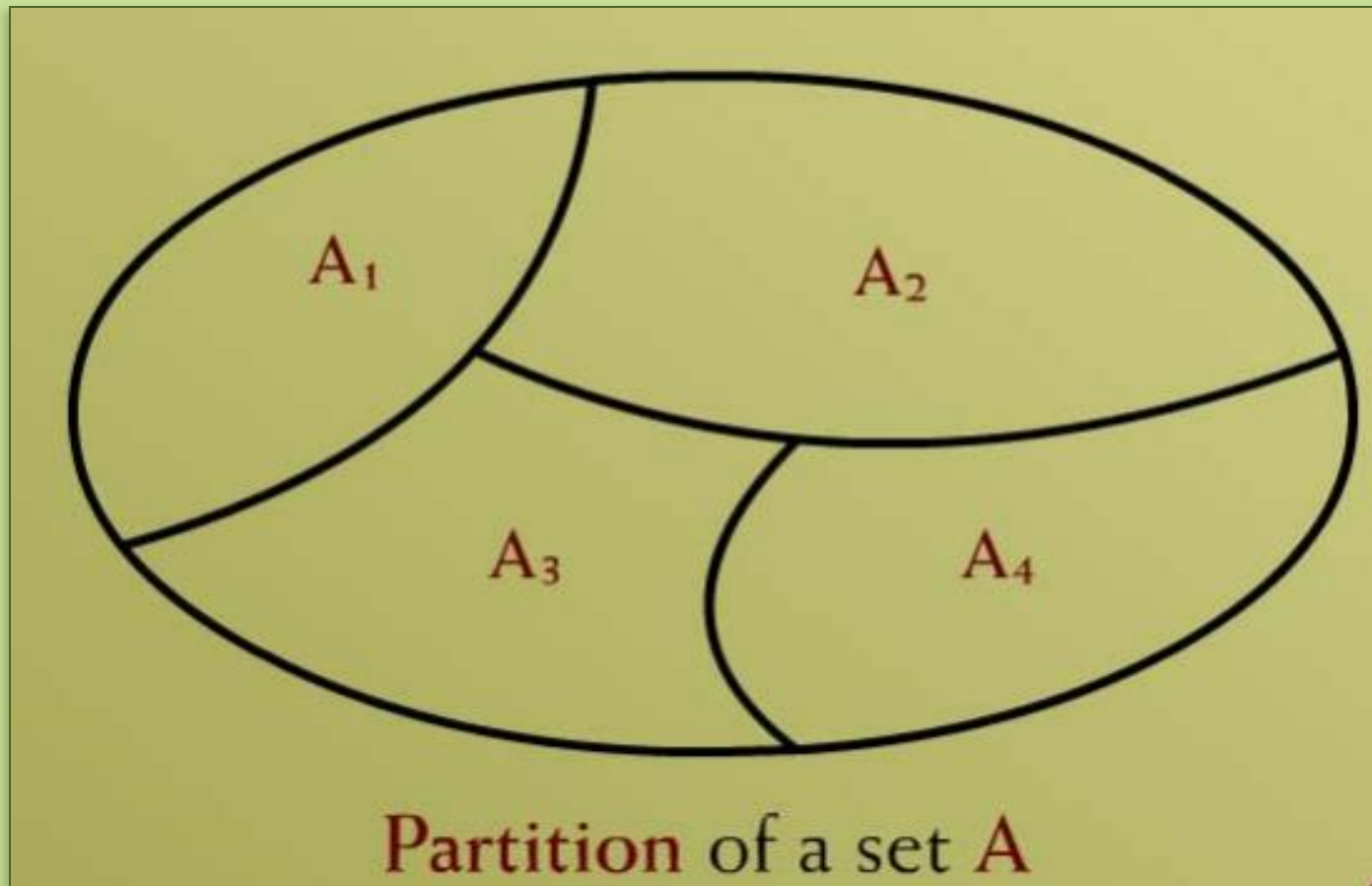
1. $A = A_1 \cup A_2 \cup \dots \cup A_n$

2. A_1, A_2, \dots, A_n are **mutually disjoint**
(or pair wise disjoint),

i.e., $\forall i, j = 1, 2, \dots, n$

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j$$

PARTITION OF A SET



EXAMPLE

Let $A = \{1, 2, 3, 4, 5, 6\}$

$$A_1 = \{1, 2\}$$

$$A_2 = \{3, 4, 5\}$$

$$A_3 = \{6\}$$

Then

$$\begin{aligned} A_1 \cup A_2 \cup A_3 &= \{1, 2\} \cup \{3, 4, 5\} \cup \{6\} \\ &= A \end{aligned}$$

EXAMPLE

$$A_1 \cap A_2 = \emptyset$$

$$A_1 \cap A_3 = \emptyset$$

and

$$A_2 \cap A_3 = \emptyset$$

i.e. A_1, A_2, A_3 are **mutually disjoint**.

$\{A_1, A_2, A_3\}$ is a **partition** of A .

EXAMPLE

Let **E** be the set of all **even integers** and **O** be the set of all **odd integers**. Is $\{\mathbf{E}, \mathbf{O}\}$ a partition of **Z**, the set of all integers? Justify your answer.

SOLUTION

$$Z = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$$

$$E = \{0, \pm 2, \pm 4, \pm 8, \dots\}$$

$$O = \{\pm 1, \pm 3, \pm 5, \pm 7, \dots\}$$

$$E \cup O = Z$$

$$E \cap O = \emptyset$$

Then $\{E, O\}$ is a **partition** of Z .

POWER SET

The **power set** of a set A is the set of all **subsets** of A , denoted by $P(A)$.

EXAMPLE:

Let $A = \{1, 2\}$, then

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

REMARK:

If A has **n elements** then $P(A)$ has **2^n elements**.

ORDERED PAIR

An **ordered pair** (a, b) consists of two elements “**a**” and “**b**” in which “**a**” is the **first element** and “**b**” is the **second element**.

ORDERED n -TUPLE

The **ordered n -tuple**, (a_1, a_2, \dots, a_n) consists of elements a_1, a_2, \dots, a_n together with the ordering: first a_1 , second a_2 , and so forth up to a_n .

In particular, an ordered **2-tuple** is called an **ordered pair**, and an ordered **3-tuple** is called an **ordered triple**.

EQUALITY OF ORDERED n-TUPLE

Two ordered n-tuples, (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are equal if the corresponding elements are equal, that is

$$a_1 = b_1$$

$$a_2 = b_2$$

.....

.....

$$a_n = b_n$$

CARTESIAN PRODUCT OF TWO SETS

Let **A** and **B** be sets. The **Cartesian product** of **A** and **B**, denoted **A x B** (read “**A cross B**”) is the set of all **ordered pairs** (**a**, **b**), where **a** is in **A** and **b** is in **B**.

Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

EXAMPLE

$$A = \{1, 2\} \quad B = \{a, b, c\}$$

$$\begin{aligned} A \times B &= \{ (x, y) \in A \times B \mid x \in A \text{ and } y \in B \} \\ &= \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\} \end{aligned}$$

$$\begin{aligned} B \times A &= \{ (x, y) \in A \times B \mid x \in B \text{ and } y \in A \} \\ &= \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\} \end{aligned}$$

EXAMPLE

$$A = \{1, 2\} \quad B = \{a, b, c\}$$

$$A \times A = \{ (x, y) \in A \times A \mid x \in A \text{ and } y \in A \}$$

$$= \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$B \times B = \{ (x, y) \in B \times B \mid x \in B \text{ and } y \in B \}$$

$$= \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), \\ (c, a), (c, b), (c, c)\}$$

REMARKS

1. $A \times B \neq B \times A$

for non-empty and unequal sets A and B .

2. $A \times \emptyset = \emptyset \times A = \emptyset$

3. $|A \times B| = |A| \times |B|$

CARTESIAN PRODUCT OF MORE THAN TWO SETS

The **Cartesian product** of sets A_1, A_2, \dots, A_n , denoted $A_1 \times A_2 \times \dots \times A_n$, is the set of all ordered **n-tuples** (a_1, a_2, \dots, a_n) where $a_1 \in A_1$, $a_2 \in A_2, \dots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i, \text{ for } i = 1, 2, \dots, n\}$$

RELATION

Let A and B be sets. A (binary) **relation** R from A to B is a subset of $A \times B$.

When $(a, b) \in R$, we say a is related to b by R , written $a R b$.

Otherwise if $(a, b) \notin R$, we write $a \nR b$. $a \nR b$ means that a is not related to b by R .

The total no. of relations that can be defined from a set A to a set B is the no. of possible subsets of $A \times B$. If $n(A)=p$ and $n(B)=q$ then $n(A \times B)=pq$ and the total no. of relations is $2^{(pq)}$.

EXAMPLE

$$A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

$$A \times B =$$

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$

$$R_1 = \{(1, 1), (1, 3), (2, 2)\}$$

$$R_2 = \{(1, 2), (2, 1), (2, 2), (2, 3)\}$$

$$R_3 = \{(1, 1)\}$$

$$R_4 = A \times B$$

$$R_5 = \emptyset$$

DOMAIN OF A RELATION

The **domain** of a relation **R** from **A** to **B** is the set of all first elements of the **ordered pairs** which belong to **R** denoted **Dom(R)**

Symbolically:

$$\text{Dom}(R) = \{a \in A \mid (a, b) \in R\}$$

RANGE OF RELATION

The **range** of **A** relation **R** from **A** to **B** is the set of all **second elements** of the **ordered pairs** which belong to **R** denoted **Ran(R)**.

Symbolically:

$$\text{Ran}(R) = \{b \in B \mid (a, b) \in R\}$$

EXERCISE

$$A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

$$R = \{(a, b) \in A \times B \mid a < b\}$$

Then

- Find the **ordered pairs** in **R**.
- Find the **Domain** and **Range** of **R**.
- Is **1R3**, **2R2**?

SOLUTION

$$A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

$$A \times B =$$

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

$$\text{a. } R = \{(a,b) \in A \times B \mid a < b\}$$

$$R = \{(1,2), (1,3), (2,3)\}$$

Solution contd...

b. $\text{Dom}(R) = \{1, 2\}$

$$\text{Dom}(R) = A$$

and

$$\text{Ran}(R) = \{2, 3\}$$

$$\text{Ran}(R) \subseteq B$$

c. Since $(1, 3) \in R$ so $1R3$

But $(2, 2) \notin R$ so $2 \not R 3$

EXAMPLE

Let $A = \{\text{eggs, milk, corn}\}$

and

$B = \{\text{cows, goats, hens}\}$

Define a relation R from A to B by $(a, b) \in R$ iff a is produced by b .

EXAMPLE

Let $A = \{\text{eggs, milk, corn}\}$

and

$B = \{\text{cows, goats, hens}\}$

Define a relation R from A to B by $(a, b) \in R$ iff a is produced by b .

Then **Define $A \times B$ first**

$R = \{(\text{eggs, hens}), (\text{milk, cows}), (\text{milk, goats})\}$

eggs R hens,

milk R cows, milk R goats etc.

EXERCISE

$$A = \{0, 1\}$$

$$B = \{1\}$$

Find all **binary relations** from **A** to **B**

SOLUTION:

$$A \times B = \{(0, 1), (1, 1)\}$$

All **binary relations** from **A** to **B** are in fact all subsets of $A \times B$, which are:

$$R_1 = \emptyset$$

$$R_2 = \{(0, 1)\}$$

$$R_3 = \{(1, 1)\}$$

$$R_4 = \{(0, 1), (1, 1)\} = A \times B$$

RELATION ON A SET

A **relation** on the set **A** is a **relation** from **A** to **A**
In other words, a **relation** on a set **A** is a **subset**
of $A \times A$.

EXAMPLE:

Let $A = \{1, 2, 3, 4\}$

$(a,b) \in R$ iff **a divides b** {symbolically written
as $a \mid b$ }

Define $A \times A$ first

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2),$
 $(2,4), (3,3), (4,4)\}$

REMARK

For any set A

1. $A \times A$ is known as the **universal relation**.
2. \emptyset is known as the **empty relation**.