lecture 10 – Sep 25

Heap operations

- insert : percolate up $\Theta(\log(N))$
- deleteMin. percolate down $\Theta(\log(N))$
- findMin : at root $\Theta(1)$
- findMax : check all leaves, i.e., half the nodes $\Theta(N)$
- decreaseKey (position, delta) : decreases heap entry at slot "position" by delta $\rightsquigarrow$ percolate up, $\Theta(\log(N))$
- increaseKey (position, delta) : increases heap entry at slot "position" by delta $\rightsquigarrow$ percolate down, $\Theta(\log(N))$
- remove (position) : decreaseKey (position, MAX); deleteMin()
  $\rightarrow \Theta(\log(N))$
- buildHeap (array) : changes array "array" to a heap with the same content

first (but sub-optimal) idea : sequence of N insertions

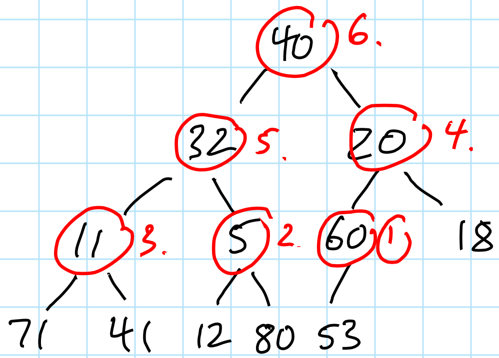
$T_{avg}(N) = O(N)$ but

$$
\begin{aligned}
\underline{T_{worst}}(N) &= \Theta(\log(1) + \log(2) + \ldots + \log(N)) \\
&= \Theta(\log(1 \cdot 2 \cdot 3 \cdot \ldots \cdot N)) = \Theta(\log(N!)) \\
&= \Theta(N \log N) \text{ by Stirling's approximation}
\end{aligned}
$$

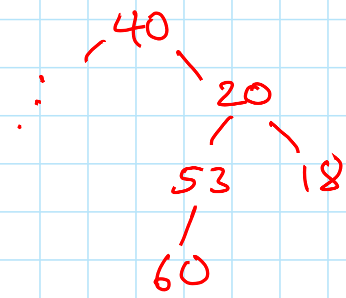
better idea : linear-time alg. ($\Theta(N)$)

buildHeap : for (i = N/2; i > 0; i--)
percolateDown (i);

Example 20.



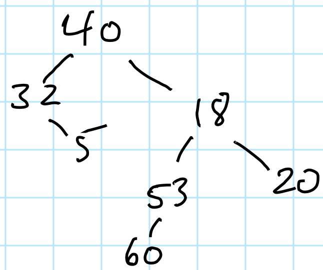
step 1. →



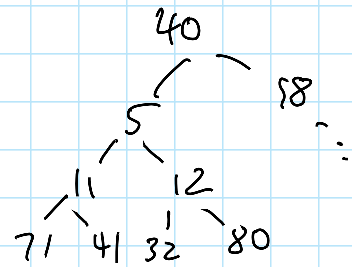
step 2. nothing happens

step 3. nothing happens

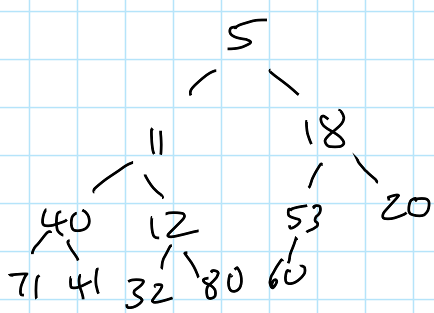
step 4. →



step 5. →



step 6. →



running time complexity ?

worst case. sum of the heights of the inner nodes

→ not depths

→ how far from leaf successor

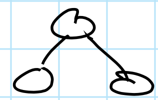
Theorem 6. The sum of the heights of the nodes of a complete BT with N nodes is $\Theta(N)$.

⇝ CONSEQUENCE : running time for buildHeap in worst case $\Theta(N)$.

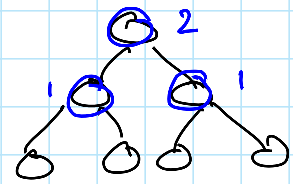
(1) First, consider the case of complete BT containing $N = 2^k - 1$ nodes for some $k \geq 1$.

$k=1$: $2^k - 1 = 1$

$k=2$: $2^k - 1 = 3$



$k=3$: $2^k - 1 = 7$



$2+1+1 = 4$

Claim: In such trees, the sum of the heights of the nodes is $\underbrace{2^k - 1}_{N} - k = N - k$.

$= \Theta(N - \log N) = \Theta(N)$

We prove that by induction on k

$\underline{k=1}$ sum of heights : 0

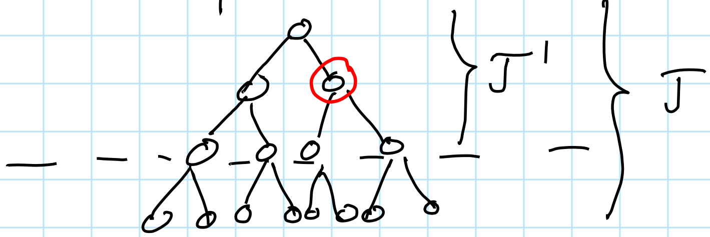
$2^k - 1 - k$: $2^1 - 1 - 1 = 0$ $\Big\} = \checkmark$

induction hypothesis For a fixed value of k:

The sum of the heights in a complete BT with $2^k - 1$ nodes is $2^k - 1 - k$.

induction step : $k \leadsto k+1$

Complete BT J with $2^{k+1} - 1$ nodes corresponds to



a complete BT J' with $2^k - 1$ nodes, extended by a level of 2^k further nodes (the leaves of J).

- The height of any of the 2^k-1 inner nodes in T is than its height in T'
- The height of any leaf in T is 0.

$\rightsquigarrow$ sum of heights of all nodes in T

$$= \text{sum of heights of all nodes in } T' + 2^k - 1$$

$$\overset{\text{i.h.}}{=} 2^k - 1 - k + 2^k - 1$$

$$= 2^{k+1} - 1 - (k+1) \quad \checkmark$$

This proves the claim.

(2) Now, consider general N (not just $N = 2^k - 1$ for some k)

Let k be smallest with $N \leq 2^k - 1$

e.g. if $N = 13$, then $k = 4$

$$\Rightarrow 2^k - 1 \leq 2N = \Theta(N)$$

Sum of heights $\leq$ sum of the heights in complete BT with 2^k-1 nodes

$$= \Theta(N).$$

$\square$.