CS 340 – lecture 30 – Nov 23

5.3. (IN)TRACTABALITY

Can we use the running time of reduction functions to make conclusions about the efficiency with which a problem can be solved?

focus only on yes/no problems "decision problems"

Example 50. Assume decision problem $\mathcal{P}$ is reducible to decision problem $\mathcal{P}'$ via "red" with $T^{red}_{worst}(N) = \Theta(N^2)$.
Assume A' is an algorithm solving $\mathcal{P}'$ with $T^{A'}_{worst}(N) = \Theta(N^2)$.
(N is the size of the problem instance)

$$\begin{tikzcd}[column sep=huge, row sep=huge] \text{instance } x \text{ of } \mathcal{P} \arrow[r, "\text{red}"] & \text{instance } red(x) \text{ of } \mathcal{P}' \arrow[d, "A'"] \\ \text{solution to } x & \text{solution to } red(x) \arrow[l] \end{tikzcd}$$

$size(x) = N$

$size(red(x)) = O(N^2)$

A' produces solution to $red(x)$ in time
$O(size(red(x))^2) = O(N^4)$

algorithm A solving $\mathcal{P}$ in time $O(N^4)$.

"Definition" 19.

(a) A decision problem $\mathcal{P}$ is polynomially reducible to a decision problem $\mathcal{P}'$ if $\mathcal{P}$ is reducible to $\mathcal{P}'$ via a mapping "red" in $O(p(N))$ for some polynomial p.

(b) A decision problem $\mathcal{P}$ is in (the complexity class) $\mathbb{P}$ if there is a (deterministic) algorithm in $O(p(N))$ solving $\mathcal{P}$, for some polynomial p. ($\rightsquigarrow$ tractable problems)

NOTE: If $\mathcal{P}'$ is in $\mathbb{P}$ and
$\mathcal{P}$ is polynomially reducible to $\mathcal{P}'$
then $\mathcal{P}$ is in $\mathbb{P}$.

$\Longrightarrow$ (i) polyn reduction can prove that certain problems are "tractable" / "easy".

(ii) polyn. reduction can also prove a problem to be at least as "hard" as a known "hard" problem.

Example 51. Hamiltonian Cycle Problem (HCP)

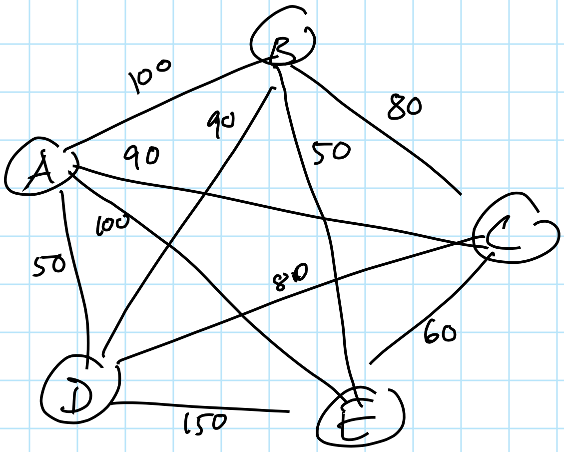
given: an undirected graph $G=(V,E)$

question: does G contain a Hamiltonian Cycle, i.e., a simple cycle containing all $v \in V$.

Example 52. Traveling Salesman Problem (TSP)

given.

question: is there a simple cycle in G, containing all $v \in V$, such that the sum of its edge weights is at most k? ("a TS tour of length $\le k$")



$k = 350$

is there a round-trip, visiting every city exactly once, at a total distance of at most 350?

C, A, D, B, E, C $\rightarrow$ 340

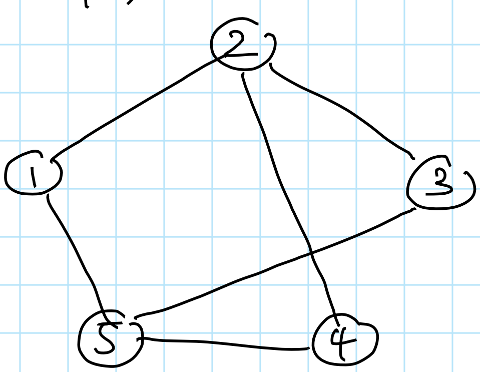
90 50 90 50 60

yes!

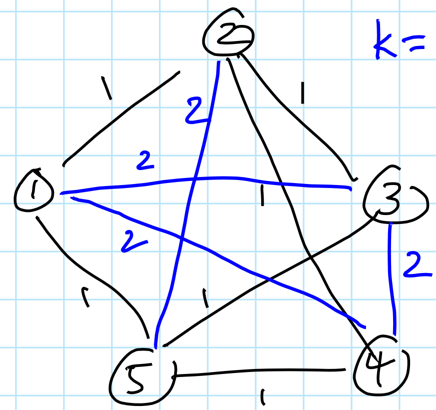
HCP is polyn. reducible to TSP

instance of HCP

graph $G = (V, E)$



$\xrightarrow{\text{red}}$



$k = |V|$