Lecture 4 – September 11 check: Handout 1; Assignment 1

Definition 2 Let $T: \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f: \mathbb{N} \to \mathbb{R}^{\geq 0}$ be any two functions.

(a) $T(N) = O(f(N))$ if and only if there are positive constants c and n_0 such that

$$T(N) \leq c \cdot f(N) \quad \text{for all } N \geq n_0.$$

"$T(N)$ is asymptotically upper-bounded by $f(N)$."

(b) $T(N) = \Omega(f(N))$ if and only if there are positive constants c and n_0 such that

$$T(N) \geq c \cdot f(N) \quad \text{for all } N \geq n_0$$

"$T(N)$ is asymptotically lower-bounded by $f(N)$."

(c) $T(N) = \Theta(f(N))$ if and only if
$T(N) = O(f(N))$ AND $T(N) = \Omega(f(N))$
"$T(N)$ has growth rate class $f(N)$."

(d) $T(N) = o(f(N))$ if and only if
$T(N) = O(f(N))$ AND $T(N) \neq \Theta(f(N))$
"$T(N)$ is negligible compared to $f(N)$"

Often "$T(N) \in O(f(N))$" used instead "$T(N) = O(f(N))$"

Example 5.

$$T(N) = \underline{0.05\,N^2} + 1{,}000\,N$$

Note: $1{,}000\,N \leq N^2$ for $N \geq 1{,}000$

- $T(N) = O(N^2)$

because with $c = 2$ and $n_0 = 1{,}000$ we get:

$$\begin{aligned} T(N) &= 0.05\,N^2 + \underline{1{,}000\,N} \\ &\leq 0.05\,N^2 + \underline{N^2} \\ &= 1.05 \cdot N^2 \\ &\leq 2N^2 = c \cdot N^2 \end{aligned}$$

for $N \geq n_0$

- $T(N) = \Omega(N^2)$

because with $c = 0.05$ and $n_0 = 1$, we get:

$$T(N) = 0.05\,N^2 + 1000\,N \geq c\,N^2$$

for all $N \geq n_0$.

- $T(N) = \Theta(N^2)$ since $T(N) = O(N^2)$ and $T(N) = \Omega(N^2)$
- $T(N) = O(N^3)$ because $T(N) = O(N^2)$ and $N^2 = O(N^3)$ (rule?)
- $T(N) \neq \Omega(N^3)$
- $T(N) \neq \Theta(N^3)$, because $T(N) \neq \Omega(N^3)$
- $T(N) = o(N^3)$, because $T(N) = O(N^3)$ and $T(N) \neq \Theta(N^3)$.

Example 6.

(a) $T(N) = 12 \cdot N + 4N^3 \cdot \log(N) + 2 + \underbrace{2\sqrt{N} \cdot N^2 \cdot N \cdot \sqrt{N}}_{2 \cdot N^4}$
$+ 1{,}500 N^2$
$= \Theta(N^4)$

$T(N) = O(N^5)$, $T(N) \neq O(N^3)$, $T(N) = \Omega(N^3)$

(b) $T(N) = 4\sqrt{N}(3 + 2N) = 12\sqrt{N} + 8N \cdot \sqrt{N}$
$= \Theta(N \cdot \sqrt{N})$

(c) $T(N) = 4^N = \Theta(4^N)$
$T(N) \neq O(2^N)$, $T(N) = \Omega(2^N)$

Rules.

Theorem 1. Let $f: \mathbb{N} \to \mathbb{R}^{\geq 0}$, $g: \mathbb{N} \to \mathbb{R}^{\geq 0}$.

(a) $f(N) = \Theta(g(N)) \iff g(N) = \Theta(f(N))$

(b) $f(N) = O(g(N)) \iff g(N) = \Omega(f(N))$

Proof of (b)

$f(N) = O(g(N)) \iff$ there are $c, n_0 \in \mathbb{R}^{\geq 0}$ s.t.
$f(N) \leq c \cdot g(N)$ for all $N \geq n_0$

$\iff$ there are $c, n_0 \in \mathbb{R}^{\geq 0}$ s.t. $\frac{1}{c} \cdot f(N) \leq g(N)$ for $N \geq n_0$

$c' = \frac{1}{c}$

$\iff$ there are $c', n_0 \in \mathbb{R}^{\geq 0}$ s.t. $g(N) \geq c' \cdot f(N)$ for $N \geq n_0$

$\iff g(N) = \Omega(f(N))$

Proof of (a)

$$f(N) = \Theta(g(N)) \iff f(N) = O(g(N)) \text{ and } f(N) = \Omega(g(N))$$
$$\overset{(b)}{\iff} g(N) = \Omega(f(N)) \text{ and } g(N) = O(f(N))$$
$$\iff g(N) = \Theta(f(N))$$

Theorem 2. Suppose $T_1(N) = O(f_1(N))$
$T_2(N) = O(f_2(N))$

Then:

(a) $T_1(N) + T_2(N) = O(\max(f_1(N), f_2(N)))$

(b) $T_1(N) \cdot T_2(N) = O(f_1(N) \cdot f_2(N))$

REMEMBER

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- If $T(N)$ is a polynomial of N of degree k, then $T(N) = \Theta(N^k)$
- $\log^k(N) = O(N)$ for any constant k
  $\log^k(N) = o(N)$