ϕ and X are in $\tau(0)$.1.5.14tcb@cnt@claim.1.5.1 @finishall

Topology Packet
Alfonso Gracia-Saz

Saahil Sharma

Contents

Chapter 1

The Definition of Topology

1.1 Definition 1.1

Definition 1.1.1: Topology

Let X be a set. A *topology* on X is a family of τ subsets of X which satisfies three properties. We will say that a subset of X is an open set iff it is an element of τ. The three properties are:

- The total set and the empty set are open sets.
- The intersection of any 2 open sets is an open set.
- The union of any 2 open sets is an open set.

A topological space is a pair (X, τ) where X is a set and τ is a topology on X.

1.2 Exercise 1.2

Question 1: Exercise 1.2

Among the following, some are topologies on the set $\mathbb{Z}$ and some are not. Which ones are? If an example is not a topology, but you can modify it slightly to make it into a topology, do so. If an example is a topology, and you can generalize it into more examples, do so.

Question 2: Exercise 1.2.a

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ In other words, a set is open iff it contains 0.}\}$$

Solution: This is not a topology on $\mathbb{Z}$ because all sets must contain the element 0, therefore the empty set will not be included and τ is not a valid topology.

Question 3: Exercise 1.2.b

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ In other words, a set is open iff it does not contain 0.}$$

Solution: This is a topology on $\mathbb{Z}$ because it contains the empty set and the total set.

Question 4: Exercise 1.2.c

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V \text{ and } 1 \in V\}$$

. ***Solution:*** This is not a topology because if 0 and 1 are in every open set, the empty set will not be an open set.

Question 5: Exercise 1.2.d

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V \text{or} 1 \in V\}$$

.

Solution: This is a topology on $\mathbb{Z}$ because the empty set will be included because the elements 0 and 1 do not have to be in every open set.

Question 6: Exercise 1.2.e

$$\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is finite}\}$$

. ***Solution:*** This is not a topology on $\mathbb{Z}$ as it requires that all sets inside must be finite and the total set is infinite.

Question 7: Exercise 1.2.f

$$\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is infinite }\}$$

. ***Solution:*** This is not a topology on $\mathbb{Z}$ because the empty set will not be included.

Note:-

We can further generalize this by saying that all topologies on $\mathbb{Z}$ that require a certain element to exist cannot be topologies because they lack the existence of the empty set. Continuing on, one can also say that all topologies τ that require a certain element does not exist within all sets of τ is considered a valid topology. Furthermore, one can also say that if all sets within the topology τ must be infinite or finite, these will not be considered valid topologies.

1.3 Exercise 1.3

Question 8: 1.3

Among the following, which ones are topologies on the set $\mathbb{R}$ and which ones are not?

Question 9: 1.3.a

$$\tau = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$$

Solution: We can prove this topology is not valid by a proof of contradiction. Consider the claim to be true. Then, the union of 2 open sets must also be an open set. Consider the 2 sets where the first set has a starting value of a and the second set has a starting value of b. Consider $a < b$. In this situation, the union of these 2 sets will contain all values from a to ∞, not inclusive. This set will not include b though, therefore there is a missing value between a and ∞ and the union of these two sets cannot be an open sets. This satisfies that (a) cannot be a valid topology.

Question 10: 1.3.b

$$\tau = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$$

.

Solution: We can prove this topology is valid by satisfying all 3 axioms that define a topological space. The first axiom is automatically satisfied as the set is in union with the total set and the empty set. Continuing on, the union of 2 sets must also be an empty set. This can be proved by considering 2 sets, one with an a value of a and another with an a value of b. If $a = b$, then their union is itself and the resulting set is open. Continuing on, if $a < b$, then the set will include all values from a to ∞, including b, therefore this set would also be an open set.

The same logic can be applied to the situation in which $b < a$, and for the 3rd axiom, regarding intersections of 2 sets.

1.4 Exercise 1.4

Let X be any set.

Question 11: 1.4.a

What is the topology on X that has the most open sets? This is called the *discrete* topology on X.

Solution: The topology with the most amount of open sets on X will include the total set, the empty set, and every subset, and the power set $\mathbb{X}$ that includes all the subsets of X.

Question 12: 1.4.b

What is the topology on X that has the least open sets? This is called the *indiscrete* topology on X.

Solution: The topology with the least amount of open sets would only contain the total set, the empty set, and the union of those 2 sets.

1.5 Exercise 1.5

Let X be an arbitrary set. Which ones of the following are topologies?

Question 13: 1.5.a

The *cofinite* topology: A set $V \subseteq X$ is open iff $[X \backslash V$ is finite or $V = \phi]$.

Solution: We must prove this topology τ is in order with the 3 defining points of a topology on X.

Claim 1.5.1

I

t is known that ϕ is in τ by the definition of τ. The total set is also in this topology because if V is X, then $X \backslash X$ is finite, therefore X is also in this topology. The union of 2 open sets in τ is also an open set.

Claim 1.5.2

I

f a set V is in τ, then by definition V is finite. The union between any 2 finite sets is also finite, therefore the union of all sets in τ is also an open set.

Claim 1.5.3

I

f a set V is in τ, then by definition V is finite. The intersection between any 2 finite sets must either be finite or ϕ, therefore the intersection between any 2 sets in τ is also an open set.

Question 14: 1.5.b

The *coinfinite* topology: A set $V \subseteq X$ is open iff

$$[X \backslash V \text{ is infinite or } V = \phi \text{ or } V = X$$

.

Solution: The first defining term of the definition of a topology is satisfied by the definition of τ, saying that both X andϕ exist within τ. Now, if we have two sets in τ being V_i and V_j, then $X - (V_i \cup V_j)$ must also be infinite. This is from the second axiom of a topology. From De Morgan's Laws, we know that

$$X - (V_i \cup V_j) = (X - V_i) \cap (X - V_j).$$

This indicates the intersection of 2 infinite sets, which can be either infinite, finite or empty. Therefore, there exists a counterexample and the *coinfinite* topology does not exist for all sets X.

Question 15: 1.5.c

The *cocountable* topology: A set $V \subseteq X$ is open iff

$$[X \backslash V \text{ is countable or } V = \phi.$$

Solution: To prove this, we must prove that the topology τ exists only for countable sets. That is, we have already proved when X is finite in the first case. Let K_i be equal to $X \backslash V_i$. Therefore, τ is equal to the collection of sets V_i, V_j, $\ldots$. Therefore, we must prove that the union of V_i and V_j is also an open set. To do this, we use De Morgan's laws.

$$X - (V_i \cup V_j) = (X - V_i) \cap (X - V_j) \Longrightarrow$$

$$X - (V_i \cup V_j) = (K_i) \cap (K_j)$$

. By definition, both K_i and K_j are countable, therefore their union is countable. The same logic and be applied to the 3rd axiom of a topology, proving that τ is a valid topology. A very nice question!