Topology Packet
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Contents

Chapter 1

1.1 Exercise 1.2

Among the following, some are topologies on the set $\mathbb{Z}$ and some are not. Which ones are? If an example is not a topology, but you can modify it slightly to make it into a topology, do so. If an example is a topology, and you can generalize it into more examples, do so.

Question 1: Exercise 1.2.a

$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\}$ In other words, a set is open iff it contains 0.}

Solution: This is not a topology on $\mathbb{Z}$ because τ must contain φ, and φ does not contain zero.

Question 2: Exercise 1.2.b

$\tau = \{V \subseteq \mathbb{Z} \mid 0 \notin V\}$ In other words, a set is open iff it does not contain 0.}

Solution: This is not a valid topology on X, because if 0 is not included, the total set will not be an open set, therefore τ is not valid.

Question 3: Exercise 1.2.c

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ or } 1 \in V\}$$

. ***Solution:*** This is a topology on $\mathbb{Z}$ because it contains the empty set and the total set, and a union or intersection of 2 open sets will be also be considered an open set.

Question 4: Exercise 1.2.d

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ and } 1 \in V\}$$

. ***Solution:*** This is a topology on $\mathbb{Z}$ because the empty set will be included because the elements 0 and 1 do not have to be in every open set.

Question 5: Exercise 1.2.e

$$\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is finite}$$

Solution: This is not a topology on $\mathbb{Z}$ as it requires that all sets inside must be finite and the total set is infinite.

Question 6: Exercise 1.2.f

$$\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is infinite}$$

Solution: The total set is an infinite set, therefore this topology is invalid.

1.2 Exercise 1.3

Among the following, which ones are topologies on the set $\mathbb{R}$ and which ones are not?

Question 7: 1.3.a

$$\tau = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

Solution: We can prove this topology is not valid by a proof of contradiction. Consider the claim to be true. Then, the union of 2 open sets must also be an open set. Consider the 2 sets where the first set has a starting value of a and the second set has a starting value of b. Consider $a < b$. In this situation, the union of these 2 sets will contain all values from a to ∞, not inclusive. This set will not include b though, therefore there is a missing value between a and ∞ and the union of these two sets cannot be an open sets. This satisfies that (a) cannot be a valid topology.

Question 8: 1.3.b

$$\tau = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

.

Solution: We can prove this topology is valid by satisfying all 3 axioms that define a topological space. The first axiom is automatically satisfied as the set is in union with the total set and the empty set. Continuing on, the union of 2 sets must also be an empty set. This can be proved by considering 2 sets, one with an a value of a and another with an a value of b. If $a = b$, then their union is itself and the resulting set is open. Continuing on, if $a < b$, then the set will include all values from a to ∞, including b, therefore this set would also be an open set. The same logic can be applied to the situation in which $b < a$, and for the 3rd axiom, regarding intersections of 2 sets.

1.3 Exercise 1.4

Let X be any set.

Question 9: 1.4.a

What is the topology on X that has the most open sets? This is called the *discrete* topology on X.

Solution: The topology with the most amount of open sets on X will include the total set, the empty set, and every subset, and the power set $\mathbb{X}$ that includes all the subsets of X.

Question 10: 1.4.b

What is the topology on X that has the least open sets? This is called the *indiscrete* topology on X.

Solution: The topology with the least amount of open sets would only contain the total set, the empty set, and the union of those 2 sets.

1.4 Exercise 1.5

Let X be an arbitrary set. Which ones of the following are topologies?

Question 11: 1.5.a

The *cofinite* topology: A set

$$V \subseteq X \text{ is open iff } [X \backslash V \text{ is finite or } V = \phi]$$

.

Solution: We must prove this topology τ is in order with the 3 defining points of a topology on X. Now, if we have two sets in τ being V_i and V_j, then $X - (V_i \cup V_j)$ must also be infinite. This is from the second axiom of a topology. From De Morgan's Laws, we know that

$$X - (V_i \cup V_j) = (X - V_i) \cap (X - V_j).$$

This indicates the intersection of 2 infinite sets, which can be either infinite, finite or empty. Therefore, there exists a counterexample and the *coinfinite* topology does not exist for all sets X.

Question 12: 1.5.c

The *cocountable* topology: A set $V \subseteq X$ is open iff

$$[X \backslash V \text{ is countable or } V = \phi]$$

.

Solution: To prove this, we must prove that the topology τ exists only for countable sets. That is, we have already proved when X is finite in the first case. Let K_i be equal to $X \backslash V_i$. Therefore, τ is equal to the collection of sets V_i, V_j, $\ldots$. Therefore, we must prove that the union of V_i and V_j is also an open set. To do this, we use De Morgan's laws.

$$X - (V_i \cup V_j) = (X - V_i) \cap (X - V_j) \implies$$
$$X - (V_i \cup V_j) = (K_i) \cap (K_j)$$

. By definition, both K_i and K_j are countable, therefore their union is countable. The same logic and be applied to the 3rd axiom of a topology, proving that τ is a valid topology. A very nice question!

1.5 Exercise 1.6

Let (X, τ) be a topological space. Prove each of the following statements true or false.

Question 13: 1.6.a

The intersection of any 3 sets is open.

Solution: Let A, B, and C be open sets. By the definition of a topology, $A \cup B$ is open. Name this set D. By the definition of topology, $D \cup C$ is also an open set, therefore the intersection of 3 open sets is also an open set.

Question 14: 1.6.b

The intersection of finitely many sets is open.

Solution: Consider x sets, X_a, X_b, $\ldots$, X_z such that each of these sets are in τ. Using the answer to the question above, the intersection of these sets will be an open set, until you reach the x^{th} set.

Question 15: 1.6.c

The intersections of open sets is open.

Solution: Consider infinite sets. The intersection of infinite open sets does not neccesarily have to be open. If X is $\mathbb{R}$, then an open set in τ is an interval, (a, b). If the intersection of these infinite amount of open sets is a single point, c, then this would not be considered an open set.

1.6 Exercise 1.8

Question 16: 1.8

Describe geometrically what a ball is in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

Solution: In $\mathbb{R}$, a ball is an open interval. In $\mathbb{R}^2$, a ball is a circle of radius y that excludes all the points lying on the outside of the circle. In $\mathbb{R}^3$, a ball is a sphere that does not include the outermost layer of points on the sphere.

1.7 Exercise 1.9

Definition 1.7.1

Let $x \in \mathbb{R}^N$ and let $\epsilon > 0$. The *ball* centered at x with radius ϵ is

$$B_\epsilon(x) := y \in \mathbb{R}^N \mid d(x, y) < \epsilon,$$

where $d(x, y)$ is the Euclidean distance between the points x and y.

1.8 Exercise 1.10

Question 17: 1.10

Prove that the topology in Definition 1.9 is actually a topology.

Solution: To prove this, we must show that the standard topology satisfied the 3 axioms of the definition of a topology. First, the empty set exists in this topology. If there are no x in φ, then there is no condition that can checked and φ is in τ. An open set V is defined to be a proper subset or subset of $\mathbb{R}^N$, therefore the total set can be included in τ. Now, we must prove that the intersection and union of two open sets is also an open set. Consider the following two lines.

INSERT DRAWING HERE

For every element x in an open set V, there exists a ball centered at x with radius $\epsilon > 0$. Call the two open sets U, and V. Define their union as A. There exists a set A in τ such that the ball centered at some x with radius $\epsilon > 0$ contains the endpoints that set U and V touch. Therefore, $U \cup V$ will an open set. This same logic applies to the intersection of two sets, claiming that there exists a set A such that A touches the first point of intersection to the second point of intersection of sets U and V.

1.9 Example 11

Show which one of the following examples are open according to Definition 1.9.

Question 18: 1.11.a

The set 1 in $\mathbb{R}$.

Solution: There exists no ball centered at 1 with raidus $\epsilon < 0$ such that this ball is a subset of 1, therefore this set is not an open set in the standard topology on $\mathbb{R}$.

Question 19: 1.11.b

The interval $(2, 5)$ in $\mathbb{R}$.

Solution: There exists a ball centered at 3 with radius $\epsilon = 1.5$, therefore this set is an open set in the standard topology on $\mathbb{R}$.

Question 20: 1.11.c

The ball $B_\epsilon(x)$ in $\mathbb{R}^N$ for any $x \in \mathbb{R}^N$ and any $\epsilon > 0$.

Solution: This set is an open set, because for all x in this open set V, there exists a ball centered at this x that can have a $\epsilon > 0$. This is because this open set will be an interval from $x - \epsilon$ to $x + \epsilon$, therefore there will always a ball for points between these endpoints centered at x with radius $| \ x - \epsilon \ |$.

Question 21: 1.11.d

The interval $[0, 1)$ in $\mathbb{R}$.

Solution: There exists no ball that is centered at $x = 1$ for this open set, because if $\epsilon > 0$ then the range of this ball will stretch past 1, which is outside the interval of the set. There exists no ball $B_{\epsilon>)}(1)$, therefore this set cannot be open.

Question 22: 1.11.e

The set $\{(x, y) \in \mathbb{R}^2 \mid x > y\}$ in $\mathbb{R}^2$.

Solution: This is not an open set in $\mathbb{R}^2$ because visually, a ball in $\mathbb{R}^2$ visually is a set containing all points in a circle, except those lying on the circumference of the circle. If the x value is greater than the y value for all (x, y) in the circle, then it is not possible for there to be a ball centered at the point with the utmost x value, becuase this point will house a ball (circle) that extends outside of the open set's range.

1.10 Exercise 1.12

Question 23: 1.12

Find all the topologies on the set $X = \{0, 1, 2\}$.

Solution:

- $(X, \tau) = \{\phi, X\}$
- $(X, \tau) = \{\phi, X, \{0\}\}$
- $(X, \tau) = \{\phi, X, \{0, 1\}\}$
- $(X, \tau) = \{\phi, X, \{0, 2\}\}$

- $(X, \tau) = \{\phi, X, \{1,2\}\}$
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1.11 Fuzzy 1.13

Fuzzy 1.13 Look back at your answer to Exercise 1.12. Some of those topologies are very similar. One could even say that they are practically "the same topology" with different names. Come up with a definition of what *practically the same topology* could mean. Also, come up with a better name. With this definition, how many essentially different topologies are there on $\{0, 1, 2\}$?

Solution: Each topology on X contains X by definition. Excluding this property, each topology contains from 0 to 7 different subsets of X, and we call classify the topologies of X based on the amount of subsets existing in τ excluding X and ϕ. We can call this topology an *n-topology*, where n represents the amount of subsets in each topology of X.

This would mean there are

- 3 1-topologies
- 11 2-topologies
- 1 3-topology

Chapter 2

2 Sequences and Limits

This chapter deals with limits and accumulation points of sequences in any topological space.

Definition 2.0.1: 2.1

Let X be a set. A *sequence* in X is a *map* $x : \mathbb{N} \to X$. Consider $0 \in \mathbb{N}$. As notation, x_n refers to an element in X. We may also write (x_n) to refer to the whole sequence.

Definition 2.0.2: 2.2

Let $P(n)$ be a statement that depends on a natural number $n \in \mathbb{N}$. We say that "$P(n)$ is eventually true for all n" if there exists $n_o \in \mathbb{N}$ such that $P(n)$ is true for all $n \geqslant n_o$. If there is no abiguity, we will say simply that "$P(n)$ is eventually true".

Definition 2.0.3: 2.3

Let (X, τ) be a topological space. Let (x_n) be a sequence in X. Let $a \in X$. We say that a is a *limit* of the sequence when the following statement is true: "If $V \subseteq X$ is an open set such that $a \in V$, then $x_n \in V$ eventually for all n." In this case we say that the sequence *converges* to a. In words, this means that every open set containing a has to contain all the sequence, except for the first few terms. We say a sequence is *convergent* if it has at least one limit.

Question 24: Exercise 2.4

Consider the sequence $0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6 \dots$ on the set $X = \mathbb{R}$ For each of the following topologies, find all of its limits.

1. the discrete topology ***Solution:***
2. the indiscrete topology ***Solution:***