Notes for Topology: Munkres

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January 3, 2024

Abstract

Welcome! This document will be used to create notes on Topology for the first 3 chapters of Munkres. Problems and exercises will be completed in a different document. All notes will be written by Saahil Sharma. Text Editor being used is Nvim with WSL 2022. Notes are first dated at 12/12/2023. Thank you.

0.1 Chapter 1

- Notation

  If an object a belongs to a set A, we denote this by writing

  $$a \in A$$

  . If the converse is true - The element a does not belong to a set A, we denote this by writing

  $$a \notin A$$

  . We say that A is a subset of B if all of the elements of A are also in B, denoted by writing

  $$A \subset B$$

  . If A is not the same as B, we call A a proper subset of B, denoted with

  $$A \subseteq B$$

  . These relations are called inclusion and proper inclusion, respectively. It can be read as "B contains A".

  How can one specify a set?

  If a set is short, one can simply list the elements of the set:

  $$A = \{a, b, c\}.$$

- Set Builder Notation:

  If a set is too long, or possibly infintely long, we can use set builder notation to list the items within the set. To define the set B using set builder notation, we denote B as

  $$B = \{x \mid \text{x is an even integer }\}.$$

  In this example, B would be the set of all even integers.

  The union of two sets is defined as

  $$A \cup B = \{x \mid x \in A \text{or} x \in B\}.$$

  The intersection of two sets in defined as

  $$A \cap B = \{x \mid x \in A \text{and} x \in B\}.$$

  What if A and B have no elements in common? To denote this set, we use the empty set, which is written as

  $$\varnothing = \{\}.$$

  This fact can also be expressed by saying A and B are ***disjoint***.

- The Difference of 2 Sets

  Think about this as litterally subtracting out the terms that exist in A and B. To define this with rigor, we can write

  $$A - B = \{\, x \mid x \in A \text{ and } x \notin B \,\}.$$

  This can sometimes be called the ***complement*** of B relative to A.

- Rules of Set Theory
  - 1
    $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
    This sort of resembles the distributive property.
  - 2
    $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
    This also shares a resemblence to the distrubitive property.
  - 3
    $$A - (B \cup C) = (A - B) \cap (A - C)$$
  - 4
    $$A - (B \cap C) = (A - B) \cup (A - C)$$
    An efficient way to memorize this:
    * 1 *The complement of the union equals the intersection of the complements.*
    * 2 *The complement of the intersection equals the union of the complements.*
- Sets

  A set can consist of anything, from other numbers to sets themselves. You can have a set of of all the subsets in a set:
  $$A = \{\, A \mid A \text{ is a subset of } B \,\}.$$
  This is sometimes called the power set of B, denoted as $\mathcal{B}$. A set whose elements are set is commonly referred to as a collection of sets.

  We now make a distinction in notation. To illustrate, if A is the set $\{1, 2, 3\}$, then the following are all equivalent.
  $$a \in A, \{a\} \subset A, \text{and}\{a\} \in \mathcal{P}(A).$$
  Given a collection of sets $\mathcal{A}$, the **union** of $\mathcal{A}$ is defined by the equation
  $$\bigcup_{x \in \mathcal{A}} A = \{\, x \mid x \in A \text{ for at least one } A \in \mathcal{A} \,\}.$$
  The intersection of the elements a is defined by the equation
  $$\bigcap_{x \in \mathcal{A}} A = \{\, x \mid x \in A \text{ for every } A \in \mathcal{A} \,\}.$$
  This intersection is not defined when $\mathcal{A}$ is empty.
- Cartesian Products Given sets A and B, we can define their cartesian product to be
  $$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$
  The cartesian product is the set of all ordered pairs, (a, b), such that a is an element of A and b is an element of B.

- Rule of Assignment

  A Rule of Assignment r is of $C \times D$ is a rule of assignment if

  $$[(c, d) \in r \text{ and } (c, d') \in r] \Longrightarrow [d = d'].$$

  We think of r as a way of assigning, to the element c of C, the element d of D, for which (c, d) $\in r$.

  Given a rule of assignment r, the ***domain*** of r is defined to be the subset of C consisting of all the first coordinates of elements of r, and the **image set** of r is defined as the subset of D consisting of all second coordinates of elements of r. Formally,

  $$\text{domain } r = \{c \mid \text{ there exists } d \in D \text{ such that } (c, d) \in r\}, \text{image } r = \{d \mid \text{ there exists } c \in C \text{ such that } ($$

- Function A function $f$is a rule of assignment r, together with a set B, that contains the image set of r. The domain A of the rule r is also called the domain of the function f; the image set of r is also called the **image set** of f; and the set B is called the **range** of f.

  If f is a function having domain A and range B, we express this fact by writing

  $$f : A \Longrightarrow B,$$

  which is read "f is a function from A to B, or "f is a mapping from A into B," or simply "f maps A into B." One sometimes visualizes f as a geometric transformation phsyically carrying the points of A to points of B.

  If $f : A \Longrightarrow B$ and if a is an element of A, we denote by $f(a)$ the unique element of B that the rule determining f assigns to a; it is called the $textbf value$ of f at a, or sometimes the **image** of a under f. Formally, if r is the rule of the function f, then $f(a)$ denotes the unique element of B such that $(a, f(a)) \in r$. Using this notation, one can go back to defining functions almost as one did before, with no lack of rigor. For instance, one can write (letting $\mathbb{R}$ denote the real numbers)

  $$\text{"Let } f \text{be the function whose rule is } \{(x, x^3+1) \mid x \in \mathbb{R} \text{ and whose range is } \mathbb{R}\text{"},$$

  or one can equally write

  $$\text{"Let } f : \mathbb{R} \Rightarrow \mathbb{R}\text{be the function such that } f(x) = x^3 + 1.\text{"}$$

  The sentence let f be the function $f(x) = x^3 + 1$ is no longer adequate for specifying a function becuase it neither specifies the domain or range of f.

- Restriction IF $f : A \Rightarrow B$ and if A_0 is a subset of A, we define the restriction of f to A_0 to be the function mapping A_0 into B whose rule is

  $$\{(a, f(a)) \mid a \in A_0\}.$$

  It is denoted by $f/mid A_0$, which is read "f restricted to A_0.

- Composite Functions Given functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, we define the **composite** $g \circ f$ of f and g as the function $g \circ f : A \longrightarrow C$ defined by the equation $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f : A \longrightarrow C$ is the function whose rule is

$$\{(a, c) \mid \text{For some } b \in B, f(a) = band g(b) = c\}.$$

In your mind, you can picture this as the point a moving to the point $f(a)$, and then to the point $g(f(a))$. Note that $g \circ f$ is only defined when the range of f *equals* the domain of g.

- Bijective, Injective, and Surjective A function $f : A \longrightarrow B$ is said to be **injective** (or one to one) if for each pair of distinct points of A, their images under f are distinct. It is said to be **surjective** (or f is said to map $A\textbf{onto}B$) if every element of B is the image of some element of A under the function f. IF f is both injective and surjective, it is said to be **bijective** (or is called a **one to one correspondence**). More formally, f is injective if

$$[f(a) = f(a')] \Longrightarrow [a = a'],$$

and f is surjective if

$$[b \in B] \Longrightarrow [b = f(a) \text{ for at least one } a \in A].$$

Injectivity of f depends only on the rule of f; surjectivity depends on the range of f as well. You can check that the composite of two injective functions is injective, and the composite of two sujective functions is surjective; it follows that the composite of two bijective functions is bijective.

If f is bijective, there exists a function from B to A called the **inverse** of f. IT is denoted by f^-1, and is the standard inverse of f as you have learned. IF f is surjective, this implies there exists such an element in the domain that yeilds an inverse relationship, and if f is injective then there is *only one* such element that exists.

Lemma 2.1. Let $f : A \longrightarrow B$. If there are functions $g : B \longrightarrow A$ and $h : B \longrightarrow A$ such that $g(f(a)) = a$ for every a in A and $f(h(b)) = b$ for every b in B, then f is bijective and $g = h = f^-1$.

- Image Let $f : A \longrightarrow B$. If A_0 is a subset of A, we denote by $f(A_0)$ the set of all images of points of A_0 under the function f; this set is called the **image** of A_0 under f. Formally,

$$f(A_0) = \{b \mid b = f(a) \text{ for at least one } a \in A_0\}.$$

On the other hand, if B_0 is a subset of B, we denote by $f^{-1}(B_0)$ the set of all elements of A whose images under f lie in B_0; it is called the **preimage** of B_0 under f, also called the "counterimage" or the "inverse image". Formally,

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\}.$$