# Topology Packet
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# Contents

# Chapter 1

## 1.1 Exercise 1.2

Among the following, some are topologies on the set $\mathbb{Z}$ and some are not. Which ones are? If an example is not a topology, but you can modify it slightly to make it into a topology, do so. If an example is a topology, and you can generalize it into more examples, do so.

**Question 1: Exercise 1.2.a**

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ In other words, a set is open iff it contains 0.}\}$$

***Solution:*** This is not a topology on $\mathbb{Z}$ because all sets must contain the element 0, therefore the empty set will not be included and $\tau$ is not a valid topology.

**Question 2: Exercise 1.2.b**

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \notin V\} \text{ In other words, a set is open iff it does not contain 0.}\}$$

***Solution:*** This is not a valid topology on $X$, because if 0 is not included, the total set will not be an open set, therefore $\tau$ is not valid.

**Question 3: Exercise 1.2.c**

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ or } 1 \in V\}$$

. ***Solution:*** This is a topology on $\mathbb{Z}$ because it contains the empty set and the total set, and a union or intersection of 2 open sets will be also be considered an open set.

**Question 4: Exercise 1.2.d**

$$\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\} \text{ and } 1 \in V\}$$

. ***Solution:*** This is a topology on $\mathbb{Z}$ because the empty set will be included because the elements 0 and 1 do not have to be in every open set.

**Question 5: Exercise 1.2.e**

$$\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is finite}$$

***Solution:*** This is not a topology on $\mathbb{Z}$ as it requires that all sets inside must be finite and the total set is infinite.

**Question 6: Exercise 1.2.f**

$$\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is infinite}$$

***Solution:*** The total set is an infinite set, therefore this topology is invalid.

## 1.2 Exercise 1.3

Among the following, which ones are topologies on the set $\mathbb{R}$ and which ones are not?

**Question 7: 1.3.a**

$$\tau = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$$

***Solution:*** We can prove this topology is not valid by a proof of contradiction. Consider the claim to be true. Then, the union of 2 open sets must also be an open set. Consider the 2 sets where the first set has a starting value of $a$ and the second set has a starting value of $b$. Consider $a < b$. In this situation, the union of these 2 sets will contain all values from $a$ to $\infty$, not inclusive. This set will not include $b$ though, therefore there is a missing value between $a$ and $\infty$ and the union of these two sets cannot be an open sets. This satisfies that (a) cannot be a valid topology.

**Question 8: 1.3.b**

$$\tau = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$$

.

***Solution:*** We can prove this topology is valid by satisfying all 3 axioms that define a topological space. The first axiom is automatically satisfied as the set is in union with the total set and the empty set. Continuing on, the union of 2 sets must also be an empty set. This can be proved by considering 2 sets, one with an $a$ value of $a$ and another with an $a$ value of $b$. If $a = b$, then their union is itself and the resulting set is open. Continuing on, if $a < b$, then the set will include all values from $a$ to $\infty$, including $b$, therefore this set would also be an open set. The same logic can be applied to the situation in which $b < a$, and for the 3rd axiom, regarding intersections of 2 sets.

## 1.3 Exercise 1.4

Let $X$ be any set.

**Question 9: 1.4.a**

What is the topology on $X$ that has the most open sets? This is called the *discrete* topology on $X$.

***Solution:*** The topology with the most amount of open sets on $X$ will include the total set, the empty set, and every subset, and the power set $\mathbb{X}$ that includes all the subsets of $X$.

**Question 10: 1.4.b**

What is the topology on $X$ that has the least open sets? This is called the *indiscrete* topology on $X$.

***Solution:*** The topology with the least amount of open sets would only contain the total set, the empty set, and the union of those 2 sets.

## 1.4 Exercise 1.5

Let $X$ be an arbitrary set. Which ones of the following are topologies?

**Question 11: 1.5.a**

The *cofinite* topology: A set $V \subseteq X$ is open iff $[X \backslash V$ is finite or $V = \phi]$.

***Solution:*** We must prove this topology $\tau$ is in order with the 3 defining points of a topology on $X$. Now, if we have two sets in $\tau$ being $V_i$ and $V_j$, then $X-(V_i \cup V_j)$ must also be infinite. This is from the second axiom of a topology. From De Morgan's Laws, we know that

$$X-(V_i \cup V_j) = (X-V_i) \cap (X-V_j).$$

This indicates the intersection of 2 infinite sets, which can be either infinite, finite or empty. Therefore, there exists a counterexample and the *coinfinite* topology does not exist for all sets $X$.

**Question 12: 1.5.c**

The *cocountable* topology: A set $V \subseteq X$ is open iff

$$[X \backslash V \text{ is countable or } V = \phi.$$

***Solution:*** To prove this, we must prove that the topology $\tau$ exists only for countable sets. That is, we have already proved when $X$ is finite in the first case. Let $K_i$ be equal to $X \backslash V_i$. Therefore, $\tau$ is equal to the collection of sets $V_i$, $V_j$, $\ldots$. Therefore, we must prove that the union of $V_i$ and $V_j$ is also an open set. To do this, we use De Morgan's laws.

$$X-(V_i \cup V_j) = (X-V_i) \cap (X-V_j) \implies$$
$$X-(V_i \cup V_j) = (K_i) \cap (K_j)$$

. By definition, both $K_i$ and $K_j$ are countable, therefore their union is countable. The same logic and be applied to the 3rd axiom of a topology, proving that $\tau$ is a valid topology. A very nice question!