Proofs

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Abstract

This document will be used to practice writing proofs and using latex.

- 1. Prove that for all x, there exists a y such that $y^2 > x$.

  Proof: Begin by defining y to be $(x + 1)$. Now, y^2 can be defined as

  $$(x + 1)^2 = x^2 + 2x + 1.$$

  This follows that the above equation will always be greater than x because it contains the term$x^2 + 2x$, which is greater than x for all x.

- 2. Consider the peicewise function

  $$f(x) = \left\{ \begin{array}{ll} e^x, & \text{if } x \leq 0 \\ 1 + x, & \text{if } x > 0 \end{array} \right\} \tag{1}$$

  - a) Prove that -1 is not in the range of f. That is, show that there does not exist an x such that $f(x) = -1$. Proof: Consider the two functions seperately. First, the e^x function can never contain a negative value in its range, as raising a constant to an exponent will never produce a negative number. Continuing on, the function $x+1$ will only contain a negative value when the function has x values inputted that are less than -1. This peicewise function specifies that only values of x that are greater than 0 will go through the $1 + x$ function, therefore it is not possible for this peicewise function to ontain -1 within the range.
  - b) Prove that f is continuous. Proof: For all values x less than or equal to 0, this peicewise function will send that x value to the function e^x. This function will continue up to the value $x = 0$, where the function stops and is at the y-value of 1. Now, the function $1+x$ that is used for all $x > 0$, this function will start at 0 but not include 0. The function $x+1$ at 0 is equal to 1, the same value of the previous function at 0. Therefore, these functions will meet at $x = 0$ and the function will continue in both directions, as x approaches positive and negative infinity.
  - c) Prove that f is differentiable. Proof: To prove that f is differentiable, we must proof that it is differentiable at all points. First, both functions that make up the peicewise function are differentiable at all points, so now, one must verify that the point at which the peicewise function meets is differentiable for both functions and that the derivative of the function at that point is the same. The derivative of e^x at $x = 0$ is 1, and the derivative of $x + 1$ at $x = 0$ is also 1, therefore this function is differentiable at all points.

- 4. Prove that the composition of any two decreasing functions is increasing. Proof: If a function is decreasing, then the function must approach $-\infty$. Therefore, when 2 functions that both are decreasing are composed together, the function will approach the negative of $-\infty$, which is $+\infty$. Therefore, the composition of the two functions must be increasing.

- 5. Prove that the sum of any two decreasing functions is a decreasing function. Proof: If $f(x)$ and $g(x)$ are considered decreasing functions, than both functions approach $-\infty$. Therefore the sum of $f(x) + g(x)$ is going to also approach $-\infty$, therefore the sum of the two decreasing functions must also be decreasing.