Markov Switching

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Abstract

Markov switching models are a popular family of models that introduces time-variation in the parameters in the form of their state- or regime-specific values. Importantly, this time-variation is governed by a discrete-valued latent stochastic process with limited memory. More specifically, the current value of the state indicator is determined only by the value of the state indicator from the previous period, thus the Markov property, and the transition matrix. The latter characterizes the properties of the Markov process by determining with what probability each of the states can be visited next period, given the state in the current period. This setup decides on the two main advantages of the Markov switching models. Namely, the estimation of the probability of state occurrences in each of the sample periods by using filtering and smoothing methods and the estimation of the state-specific parameters. These two features open the possibility for improved interpretations of the parameters associated with specific regimes combined with the corresponding regime probabilities, as well as for improved forecasting performance based on persistent regimes and parameters characterizing them.

The most commonly applied models from this family are the Markov switching ones that presume a finite number of regimes and exogeneity of the Markov process which is defined as its independence of the model unpredictable innovations. In many such applications, desired properties of the Markov switching model have been obtained either by imposing appropriate restrictions on transition probabilities or by introducing the time-dependence of these probabilities determined by explanatory variables or functions of the state indicator. One of the recent extensions of this basic specification includes Infinite Hidden Markov models that grant great flexibility and improved forecasting performance by allowing the number of states to go to infinity. Another one, namely, endogenous Markov switching model, explicitly relates the state indicator to model's innovations making it more interpretable and offering promising avenues for future developments.

Keywords: Transition Probabilities, Exogenous Markov Switching, Infinite Hidden Markov Model, Endogenous Markov Switching, Markov Process, Finite Mixture Model, Change-point Model, Non-homogeneous Markov Switching, Time Series Analysis, Business Cycle Analysis

1. Introduction

The Markov switching (MS) methodology was introduced by the seminal work of Hamilton (1989). It is directly applicable to time series analysis for its dynamic nature. This section shows the benchmark model and corresponding notation for the data and model parameters. For a more comprehensive textbook exposition, see Chapter 22 by Hamilton (1994), Krolzig (1997), Kim et al. (1999) and Frühwirth-Schnatter (2006).

The dependent variable at time t is denoted by y_t for $t = 1, ..., T$ and T is the number of periods in the sample. y_t can be either a scalar, vector, or matrix. The independent variable is denoted by x_t, where it can be a scalar, vector, or matrix. x_t does not need to have the same dimension as y_t, and it might include lagged values of y.

1.1. Markov process

A latent state at time t is unobservable to an econometrician and denoted by s_t. It takes the value $k \in \{1, 2, ..., K\}$, where K is a positive integer representing the total number of states. Such a latent variable s_t indicates in which state the current system is at time t. Hence, it can be called *state indicator* or *regime indicator*. The name *state* and *regime* are often used interchangeably in the literature.

From the definition, the state indicator is a scalar. However, it generalizes any vector or matrix representation of states as long as the number of states is finite. For example, assume a vector of states $z_t = (z_{1t}, z_{2t})$, in which z_{1t} takes values from set $\{1, ..., K_1\}$ and z_{2t} from $\{1, ..., K_2\}$. Such a tuple z_t can be simply collapsed to a scalar indicator s_t, which takes value of $1, ..., K_1 K_2$. Each element in s_t corresponds to a vector z_t. Sometimes, a vector form of the state variable is preferred in specific applications. However, knowing a scalar representation is generally enough to learn the basic framework.

The dynamics of the state indicator is governed by a Markov process. The probability distribution of s_t given the whole path $\{s_{t-1}, s_{t-2}, ..., s_1\}$ only depends on the most recent state s_{t-1}. Define such a *transition probability* as

$$\Pr(s_t = j \mid s_{t-1} = i, s_{t-2}, ..., s_1) = \Pr(s_t = j \mid s_{t-1} = i) = p_{ij}, \tag{1}$$

where $i, j = 1, ..., K$. Given that in period $t - 1$ the process was in state i the probability that the state will switch to state j in period t is equal to p_{ij}. A *transtion matrix* organizes these transition probabilities in a $K \times K$ matrix and is definded as

$$P = [p_{ij}]_{K \times K}, \tag{2}$$

where p_{ij} is the element on the ith row and jth column such that the elements in each of the rows of matrix P sum to one.

A vector of unconditional state probabilities, denoted by $\pi \equiv E[s_t]$, is defined from

$$P'\pi = \pi. \tag{3}$$

The equation above indicates time-invariance of distribution π, that is, an iteration over one period performed by the premultiplication by the transition probability P' does not change vector π. The solution to equation (3) for π given $\iota'_K \pi = 1$, where ι_K is a K-vector of ones, is given in

Hamilton (1994, Chapter 22) and expresses π as a function of P. Define a $(K + 1) \times K$ matrix $\mathbf{P} = \begin{bmatrix} I_K - P' \\ \iota'_K \end{bmatrix}$. Hamilton's solution for π is given by the $(K + 1)$th column of $(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'$.

The distribution of the initial state at $t = 0$, denoted by s_0, is represented by the following K-vector

$$\pi_0 \equiv [\Pr(s_0 = k)]_{K \times 1}. \tag{4}$$

For an ergodic Markov process, the initial distribution can be simply set to the stationary distribution π. If the Markov process is non-stationary, there typically exists theory-guided information for π_0. For example, a change-point model requires that the initial state is in the first state, namely, $s_0 = 1$.

1.2. Measurement Equation

A measurement equation lays out the probability law of the observations and is given by

$$y_t \sim F(x_t, s_t), \tag{5}$$

where F represents the distribution of y_t conditional on the observation x_t and the latent state s_t. There is no restriction on F. It can be a discrete, continuous, or mixture distribution depending on the structure of the data y_t. For example, consider a time series of asset returns and assume that each state admits an autoregressive process of order one with Gaussian innovations. Then F can be expressed as

$$F(x_t, s_t) \equiv N(\mu_{s_t} + \beta_{s_t} y_{t-1}, \sigma^2_{s_t}),$$

where $\mu_{s_t} + \beta_{s_t} y_{t-1}$ is the mean and $\sigma^2_{s_t}$ the variance of this conditional distribution given s_t and y_{t-1}. The same assumption also implies a regression form of equation (5) given by

$$y_t = \mu_{s_t} + \beta_{s_t} y_{t-1} + \sigma_{s_t} \epsilon_t, \quad \epsilon_t \sim N(0, 1). \tag{6}$$

Equation (1) and (5) comprise the foundation of the Markov switching framework. Together with the initial condition (4), the likelihood function $p(Y|\Theta)$ is available through the filter technique from Hamilton (1989) called the *Hamilton filter*. The notation Y denotes a collection of all y_t for $t = 1, ..., T$; and Θ is the collection of all time-invariant parameters. For instance, $\Theta \equiv \{\mu_k, \beta_k, \sigma^2_k\}_{k=1}^K$ in the example above.

1.3. Filtering the States

The Hamilton filter gives the conditional distribution of the state s_t on the data up to time t, $Y_{1:t}$, denoted by $p(s_t \mid Y_{1:t})$. It is used to compute the one-period ahead probability density of y_t from

the following formula.

$$p(y_{t+1} \mid Y_{1:t}) = \sum_{s_{t+1}=1}^{K} p(y_{t+1}, s_{t+1} \mid Y_{1:t}) \tag{7a}$$

$$= \sum_{s_{t+1}=1}^{K} p(y_{t+1} \mid s_{t+1}, Y_{1:t}) p(s_{t+1} \mid Y_{1:t}) \tag{7b}$$

$$= \sum_{s_{t+1}=1}^{K} \left[p(y_{t+1} \mid s_{t+1}, Y_{1:t}) \sum_{s_t=1}^{K} p(s_{t+1}, s_t \mid Y_{1:t}) \right] \tag{7c}$$

$$= \sum_{s_{t+1}=1}^{K} \left[p(y_{t+1} \mid s_{t+1}, Y_{1:t}) \sum_{s_t=1}^{K} p(s_{t+1} \mid s_t, Y_{1:t}) p(s_t \mid Y_{1:t}) \right] \tag{7d}$$

$$= \sum_{s_{t+1}=1}^{K} \left[p(y_{t+1} \mid s_{t+1}, Y_{1:t}) \sum_{s_t=1}^{K} p(s_{t+1} \mid s_t) p(s_t \mid Y_{1:t}) \right] \tag{7e}$$

$$= \sum_{s_{t+1}=1}^{K} \sum_{s_t=1}^{K} p(y_{t+1} \mid s_{t+1}, Y_{1:t}) p(s_{t+1} \mid s_t) p(s_t \mid Y_{1:t}), \tag{7f}$$

where $p(y_{t+1}|s_{t+1}, Y_{1:t})$ is obtained from the measurement equation (5) and $p(s_{t+1}|s_t)$ is the transition probability in equation (1). The derivation above utilizes the forecasted state probability conveniently decomposed into the transition and filtered probabilities

$$p(s_{t+1} \mid Y_{1:t}) = \sum_{s_t=1}^{K} p(s_{t+1} \mid s_t) p(s_t \mid Y_{1:t}), \tag{8}$$

where the filtered probability for s_{t+1} is given by

$$p(s_{t+1} = k \mid Y_{1:t+1}) = \frac{p(s_{t+1}, y_{t+1} \mid Y_{1:t})}{p(y_{t+1} \mid Y_{1:t})}$$

$$= \frac{p(y_{t+1} \mid s_{t+1}, Y_{1:t}) p(s_{t+1} \mid Y_{1:t})}{p(y_{t+1} \mid Y_{1:t})}$$

$$= \frac{p(y_{t+1} \mid s_{t+1}, Y_{1:t}) \sum_{s_t=1}^{K} p(s_{t+1} \mid s_t, Y_{1:t}) p(s_t \mid Y_{1:t})}{p(y_{t+1} \mid Y_{1:t})}.$$

The filtered probability of s_{t+1} is easy to compute as long as the filtered probability of s_t is known. The conditional density $p(y_{t+1}|Y_{1:t})$ has been derived above. Given the initial distribution of the first-period state, π_0, the filtered probability of s_t and the associated one-period ahead probability density can be calculated iteratively. The likelihood function is constructed as the product of conditional probability densities:

$$p(Y \mid \Theta) = \prod_{t=1}^{T} p(y_t \mid Y_{1:t-1}, \Theta), \tag{9}$$

where $Y_{1:0}$ is treated as known and the state indicators, s_t for $t = 1, ..., T$, are integrated out. An analytical expression for the likelihood function is available and can be found in Frühwirth-Schnatter (2006). It is based on equations (9) and (7b).

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1.4. Parameter Estimation

The maximum likelihood estimation is based on an iterative expectation-maximization (EM) algorithm (see Hamilton, 1990). In each of its iterations, a filtering-smoothing algorithm is used to propose the current estimate of S, a collection of all s_t for $t = 1, \ldots, T$, and a maximization step is applied to compute an estimate of Θ and P. The maximum likelihood estimation is straightforward if regularity conditions are satisfied. However, for larger models, it might become cumbersome due to unbounded likelihood function.

Bayesian estimation relies on a data augmentation technique that requires the specification of a complete-data likelihood function $p(Y, S|\Theta) = p(Y|S, \Theta)p(S|\Theta)$, where the objects on the right-hand side of the expression are easily obtainable without integration. The complete-data likelihood function is subsequently used to specify the full-conditional posterior distributions $p(\Theta|Y, S)$ and $p(S|Y, \Theta)$. Sampling from the joint posterior distribution of the parameters and states is performed via Markov Chain Monte Carlo (MCMC) methods. Since the former conditional distribution takes S as given, it can be sampled from using standard techniques. The sampling from the latter relies on the *forward filtering and backward sampling* (FFBS) method from Chib (1996).

1.5. Inference

The first results for the asymptotic normality of the maximum likelihood estimator of the MS model parameters were proposed by Lindgren (1978) and Lehmann (1983), whereas the finite sample properties of this estimator are analyzed in Psaradakis & Sola (1998). The frequentist solution to a problem of selection of the number of states of the Markov process K relies on information criteria such as, for instance, Akaike Information Criterion (see a detailed study by Psaradakis & Spagnolo, 2003). Testing of a hypothesis that $K = 1$ against an alternative hypothesis that $K = 2$ is highly cumbersome since the MS model is not identified under the null hypothesis, and the solutions require sophisticated inferential methods some of which are provided by Carrasco et al. (2014) and more recently by Meitz & Saikkonen (2017).

Bayesian model selection based on marginal data densities provides the solution to the number of states determination problem, including a model with $K = 1$ (see Frühwirth-Schnatter, 2006). However, the main challenge for correct Bayesian inference in MS models is the problem of label switching which is defined as the invariance of the likelihood function to the labeling of the states. Consider an example in equation (6) with two states characterized by state-specific parameter vectors of state A and state B, denoted respectively by Θ_A and Θ_B. The label switching problem states that irrespective of whether the vector of parameters (Θ_1, Θ_2) is set to (Θ_A, Θ_B) or (Θ_B, Θ_A) the value of the likelihood function evaluated at these parameters stays invariant. Frühwirth-Schnatter (2001) proposes to analyze a multimodal global shape of the likelihood function and the posterior distribution (see Frühwirth-Schnatter, 2004, for the application of this approach to the computation of marginal data densities). Implementing ordering restrictions on the state-dependent parameters of the model that would provide a unique classification of the states is a solution that is only applicable if such restrictions do not bind the posterior distribution. Alternatively, Geweke (2007) proposes to base inference on label-switching invariant characteristics such as predictive densities.

1.6. Interpretations

The interpretation of the MS models is based on the combined analysis of the parameter estimates and one of the available estimates of the state probabilities. The latter should be chosen amongst

the forecasted, filtered, or smoothed probabilities, denoted by $\Pr[s_t|y_{t-1}, y_{t-2}, \dots]$, $\Pr[s_t|y_t, y_{t-1}, \dots]$ and $\Pr[s_t|Y]$, respectively, that are obtained through the FFBS algorithm, depending on a particular application and the objective of investigation.

The first application of the MS models in economics included the analysis of business cycles and could be presented as follows. Consider data on gross domestic product growth rates to which the autoregressive model from equation (6) is fitted. The business cycle interpretation of the model was proposed by Hamilton (1989). It was acclaimed because the state-dependent constant term estimates were positive and negative, respectively, in the two states, which smoothed probabilities had high values for the periods commonly related to as economic expansions and recessions, respectively. Similar reasoning was applied to financial markets characterized by bull and bear markets subsequently occurring one after another as in Hamilton & Lin (1996).

Another such example includes a multivariate model of the effects of monetary policy on the real economy in the U.S. with conditional heteroskedasticity modeled with the MS process proposed by Sims & Zha (2006). In this model, the subsequent volatility states occurred to have high values of smoothed probabilities in the periods corresponding to the terms of subsequent Chairs of the Federal Reserve. For instance, the state with the highest value of the volatility had high probabilities of occurrence in the period of Paul Volcker's chairmanship. In contrast, the lowest volatility state spanned the term of Alan Greenspan's.

Many applications in economics rely on some concept of causality. Two such popular concepts are Granger causality proposed by Granger (1969) and Sims (1980) that relates the causal link between variables to their predictive power, and another considers causal links between variables to be based on a structural model of an economy. An approach that investigates Granger causality for a specific state of an MS Vector Autoregressive model was proposed by Psaradakis et al. (2005) whereas the framework that is unconditional on the states was proposed by Droumaguet et al. (2017). An explicit form of dependence between two or more Markov processes describing country or regional business cycles was proposed by Owyang et al. (2005), Hamilton & Owyang (2012), and more recently by Leiva-Leon (2017).

Finally, a new class of structural MS rational expectations models utilizes the MS rule to determine the time-variation of the parameters of the Dynamic Stochastic General Equilibrium model. In this framework, the considered agents are rational and, thus, know the MS rule and, consequently, take it into account in their decision-making problem. Farmer et al. (2009) provides the theory behind such a formulation of the model while Liu et al. (2011) provides a model for macroeconomic fluctuations. Farmer et al. (2011) propose a method of solving these models.

2. Exogenous Markov Switching

The MS model was introduced in Section 1 through the definition of the likelihood function in which the predictive densities of the data $p(y_{t+1}|s_{t+1}, Y_{1:t})$ are weighted by the forecasted state probabilities $p(s_{t+1}|Y_{1:t})$ as in equation (7b). The decomposition of the latter into a product of the transition times the filtered probabilities as in equation (8) reveals that it does not depend on the contemporaneous observation y_{t+1} but only on the previous state s_t and past observations $Y_{1:t}$. The independence of forecasted state probability $p(s_{t+1}|Y_{1:t})$ from the error component of the measurement equation (5) defines a popular family of exogenous MS models with finite number of states, K, that is considered in this section. In Section 3, models with an infinite number of states, K, are considered while Section 4, discusses endogenous MS models in which this dependence is introduced.

2.1. A Family of Markov Switching Models

The properties of the latent Markov process are driven by the form of the transition matrix P. A general way of imposing restrictions on the transition matrix was proposed by Sims et al. (2008) and Woźniak & Droumaguet (2015). Let P_i denote the ith row of matrix P that is represented in terms of its unrestricted elements, collected in an $1 \times r_i$ vector p_i whose elements sum to one, as follows

$$P_i = p_i W_i, \tag{10}$$

where W_i are predetermined $r_i \times K$ matrices for $i = 1, \ldots, K$. The definition of the transition matrix as in the equation above is used below to demonstrate various types of this matrix and the implied properties of the Markov process. Note that if $r_i = 1$ for some i, then the only element of p_i is equal to one.

A general stationary and aperiodic MS process in which each of the states can be revisited at any time t presumes $r_i = K$ and $W_i = I_K$ for each i, where I_K is an identity matrix of order K. In such a model, all of the K^2 elements of the transition matrix are estimated, there is no absorbing state, all of the elements of the ergodic probabilities vector π are nonzero, and the probabilities of the initial state are most often set to π. This type of MS model is the most frequently applied in economics and finance.

A class of finite mixture models is nested within the MS models by setting $W_i = I_K$ and $p_i = \pi'$ for each i, where all of the elements of π are strictly positive. In this model, the forecasted state probabilities are time-invariant and equal to $p(s_{t+1}|Y_{1:t}) = \pi$. However, the clustering of observations is facilitated through the smoothed probabilities that are allowed to change over time. Finite mixture models provide a convenient way of modeling nonstandard distributions that are often required for the error terms in economic and finance applications (see Norets, 2010). It can be shown that any distribution of a random variable defined on a real scale can be approximated by a mixture of normal distributions, while distribution of a random variable defined on a positive real scale can be approximated by a mixture of gamma distributions.

Change-point models can be used to introduce monotonic regime changes as in a model proposed by Chib (1998). The process is initiated in the first state $s_0 = 1$, with probability p_{11} it remains unchanged, and with probability p_{21} it switches to the other regime. The first state is never to be revisited and, thus, $p_{12} = 0$. In general, given that at some period t the Markov process is in state $s_t = k < K$, it remains in this state with probability p_{kk} and is only allowed to switch to the next regime with probability $p_{k+1,k}$. Finally, when the process reaches the Kth state, it stays there forever. In the change-point models the transition matrix is obtained by setting $r_i = 2$ and W_i to a $2 \times K$ matrix whose columns contain the ith and $(i+1)$th rows of I_K for $i = 1, \ldots, K-1$, as well as $r_K = 1$ and a row vector W_K contains the last row of I_K. An example of such a transition matrix for $K = 3$ is given by

$$\begin{bmatrix} p_{11} & p_{12} & 0 \\ 0 & p_{22} & p_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, these models are capable of estimating the time at which the regime changes occur. Consequently, they attracted considerable attention in economic and finance applications despite some recent evidence that maintaining the transition matrix unrestricted is capable of capturing similar patterns in data with non-deteriorating in- and out-of-sample fit. Finally, the change-point models introduce non-stationarity in the Markov-process and, thus, their ergodic probabilities are all equal to zero except for the last element of π that is equal to one. Therefore, the last state is

the absorbing state that gains 100 percent of the probability mass asymptotically with $T \to \infty$. Finally, Frühwirth-Schnatter (2006) gives a detailed discussion of the nuances of the estimation of stationary and non-stationary Markov processes.

In a simple and popular deterministic change-point models, s_t is assumed to be known and provided by the econometrician. In many applications, this model is used in order to estimate state-dependent parameters in pre-determined subsamples. Moreover, it is straightforward to set s_t to obtain monotonic regime changes. Nevertheless, in deterministic change-point models, the properties of the Markov process are not driven by the transition matrix, which is redundant given fixed s_t. The regime-change dates are not estimated and, unless the econometrician knows the data generating process and sets s_t accordingly, the model fit deteriorates heavily compared to other models considered in this section.

A more elaborate form of matrices W_i may lead to the desired application-specific properties of the Markov process. Consider a model used by Sims (2001) who introduces symmetric jumping among adjacent regimes. In this example, the desired transition matrix for the case of $K = 4$ states is

$$\begin{bmatrix} p_{11} & (1-p_{11}) & 0 & 0 \\ (1-p_{22})/2 & p_{22} & (1-p_{22})/2 & 0 \\ 0 & (1-p_{33})/2 & p_{33} & (1-p_{33})/2 \\ 0 & 0 & (1-p_{44}) & p_{44} \end{bmatrix},$$

which can be easily obtained by setting $r_i = 2$ for each i and

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 \frac{1}{2} & 0 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In another example, Sims et al. (2008) consider a generalization of this model inspired by the developments in Cogley & Sargent (2005) that allows for potentially many states and the estimation of all of the parameters in matrix P, while keeping the number of its free parameters relatively small. In this model, the desired transition matrix has the following form

$$\begin{bmatrix} p_{11} & a_1\alpha_1(1-p_{11}) & a_1\alpha_1^2(1-p_{11}) & \cdots & a_1\alpha_1^{K-1}(1-p_{11}) \\ a_2\alpha_2(1-p_{22}) & p_{22} & a_2\alpha_2(1-p_{22}) & \cdots & a_2\alpha_2^{K-2}(1-p_{22}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_K\alpha_K^{K-1}(1-p_{KK}) & a_K\alpha_K^{K-2}(1-p_{KK}) & a_K\alpha_K^{K-3}(1-p_{KK}) & \cdots & p_{KK} \end{bmatrix},$$

that can be obtained by setting $r_i = 2$ for each i, matrices W_i accordingly to

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & a_1\alpha_1 & a_1\alpha_1^2 & \ldots & a_1\alpha_1^{K-1} \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ a_2\alpha_2 & 0 & a_2\alpha_2 & \ldots & a_2\alpha_2^{K-2} \end{bmatrix},$$

$$\ldots$$

$$W_K = \begin{bmatrix} 0 & 0 & 0 & \ldots & 1 \\ a_K\alpha_K^{K-1} & a_K\alpha_K^{K-2} & a_K\alpha_K^{K-3} & \ldots & 0 \end{bmatrix},$$

where $\alpha_1, \ldots, \alpha_K$ are positive free parameters to be estimated, and $a_1, \ldots, a_K$ are such that the columns of matrices W_i sum to one. This model uses scarce parameterization of the transition

matrix that is capable of capturing occasional discontinuous shifts in the values of the regime-dependent parameters when K is small, as well as frequent, incremental changes in these parameters for larger K. The choice of K is a matter of empirical investigation.

Finally, the Markov property of the latent process might be extended by introducing the dependence of the current state, s_t^*, not only on one but several recent realizations of the latent process. The original model by Hamilton (1989) assumes the dependence of the model parameters on the current and previous regime, s_t^* and s_{t-1}^* respectively. This dependence can be modeled by a new four-state Markov process, s_t, through the following state representation:

$$
\begin{aligned}
s_t &= 1 &&\text{if } s_t^* = 1 \text{ and } s_{t-1}^* = 1, \\
s_t &= 2 &&\text{if } s_t^* = 2 \text{ and } s_{t-1}^* = 1, \\
s_t &= 3 &&\text{if } s_t^* = 1 \text{ and } s_{t-1}^* = 2, \\
s_t &= 4 &&\text{if } s_t^* = 2 \text{ and } s_{t-1}^* = 2,
\end{aligned}
$$

and an appropriate form of the transition matrix:

$$
\begin{bmatrix}
p_{11} & p_{12} & 0 & 0 \\
0 & 0 & p_{21} & p_{22} \\
p_{11} & p_{12} & 0 & 0 \\
0 & 0 & p_{21} & p_{22}
\end{bmatrix},
$$

that can be specified by setting $r_1 = r_2 = 2$, $p_1 = p_3$ and $p_2 = p_4$ and

$$
W_1 = W_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad W_2 = W_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

2.2. Independent Markov Processes

In this class of MS models, various groups of parameters of the model depend on separate and independent Markov processes (see Phillips, 1991; Ravn & Sola, 1995, for some early applications). The examples of such models include models in which the parameters of the conditional mean process depend on a different Markov process than conditional variances (see, e.g. Sims et al., 2008), or structural models in which the money demand equation depends on different Markov process than other parameters of the models (see, e.g. Sims & Zha, 2006). Consider L such independent processes s_{lt} each parameterized by a $K^l \times K^l$ transition matrix P^l for $l = 1, \ldots, L$. This model can be represented by a composite Markov process $s_t = (s_{1t}, \ldots, s_{Lt})$ with $\prod_{l=1}^{L} K^l$ states and the corresponding transition matrix given by

$$
P = P^1 \otimes \cdots \otimes P^L,
$$

where $\otimes$ denotes the Kronecker product. The gain from the tensor product representation of the transition matrix above, introducing nonlinear restrictions, is an economic parameterization facilitating the estimation. Note that each of the transition matrices P^l might be subject to restrictions as that in equation (10).

2.3. Non-homogeneous Markov Switching

Finally, this survey of parameterizations of transition matrices is concluded by presentation of a non-homogenous MS model in which the transition probabilities change over time (see Filardo, 1994; Diebold et al., 1994). The introduction of this time variation is often combined with the dependence on some variables v_t that might contain x_t (as in e.g. Filardo, 1994) as well as the state indicators s_t (as in Otranto, 2005) or a measure of a duration of the state (as in Durland & McCurdy, 1994). The restriction imposed on the columns of the transition matrix leads to the parameterization of the transition probabilities through the logistic regression in the following form

$$p_{ij} \equiv P(s_t = j \mid s_{t-1} = i) = \frac{\exp(v_t \gamma_{ij})}{\sum_{j=1}^{K} \exp(v_t \gamma_{ij})},$$

where γ_{ij} are parameter vectors to be estimated the dimensions of which correspond to vector v_t. Note that for the identification of transition probabilities for each i there is a j so that γ_{ij} is a vector of zeros (see Koki et al., 2020, for a recent exposition on the setup of the model). The selection of the variables in v_t determines the time-dependence in transition probabilities and is subject to empirical verification. As long as the establishment of the factors determining transition probabilities might be of interest in itself the non-homogeneous MS models often lead to similar estimates of filtered and smoothed state probabilities as the simple MS model in many data sets.

3. Infinite Hidden Markov Model

The infinite hidden Markov model (IHMM) was developed by Beal et al. (2002) and Teh et al. (2006). It builds on the Dirichlet process mixture model (DPM) from Escobar & West (1995) and extends the finite number of states of the MS model to the case in which this number goes to infinity, $K \rightarrow \infty$. Such an extension introduces a fundamental advancement of econometric modeling by transforming the parametric MS framework into a nonparametric structure.

A direct consequence is that the transition matrix P implied by equation (2) has an infinite dimension and can be presented as

$$P \equiv [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $i, j = 1, 2, 3, \ldots$ and $p_{ij} \geq 0$. From the definition, each row of P must sum up to 1, $\sum_{j=1}^{\infty} p_{ij} = 1$. The time-invariant parameters that describe the kth state are defined as θ_k, and there is an infinite number of them. Similarly to the finite-state MS models, the parameter space comprises of the state indicator $S \equiv \{s_t\}_{t=1}^{T}$, the time-invariant parameters $\Theta \equiv \{\theta_k\}_{k=1}^{\infty}$ and the transition matrix $P = [p_{ij}]_{\infty \times \infty}$.

3.1. Estimation

Because of the parameter saturation problem, the IHMM cannot be estimated by classical methods without regularization. On the contrary, the Bayesian approach is coherent, more appropriate for inference, and thus, the existing research on the IHMM uses mostly this framework.

Three popular ways could be used to draw inference from the IHMM. The first is to integrate out the transition probability P based on the Chinese restaurant representation of the Dirichlet process as in Fox et al. (2011). This method works directly on states, but the derivation is complicated. The second is to apply the beam sampler to stochastically truncate the number of states to a finite one during the MCMC as in Van Gael et al. (2008). This method provides an exact inference similarly to the first method. Besides, it utilizes conditional independence by keeping the transition matrix P in the parameter space, which allows parallel computations and is usually much faster than the first approach.

The last method applies the degree-K weak limit approximation from Ishwaran & Zarepour (2002) as in Bauwens et al. (2017). It uses a truncated Dirichlet process so that the IHMM resembles an appropriate finite-state MS model. In practice, Song (2014) found that standard MS models with a large number of states performs similarly to the IHMM, where the number of inactive states should be nonzero, where an inactive state is a state without data being assigned to it. This approach renders the IHMM easier to execute. Although two caveats exist. First, the prior distribution on the transition matrix must be chosen according to Ishwaran & Zarepour (2002). Namely, the concentration parameter from the truncation approximation should be consistent with the concentration parameter from the IHMM. Otherwise, the approximation is not valid. Second, the number of states in the MCMC must be monitored to avoid poor approximation. A simple rule would be that the number of active states, that is those with nonzero number of observations classified into it, must always be less that the total number states K in the approximation. For execution of the algorithm, see Bauwens et al. (2017).

3.2. Flexibility of IHMM

The IHMM originated from the machine learning literature where it was used to reveal dynamic clustering in applications, including dialogue summary and motion capture. For the last decade, it has been applied to various fields in economics and finance. The seminal developments include Song (2014), Jochmann (2015) and Dufays (2015). The motivation for using the IHMM is that it demonstrates well the trade-off between heuristic economic interpretations and competitive forecast accuracy. This feature constitutes an advantage over many of the machine learning methods that hardly allow for structural interpretations.

Similarly to the conventional MS models, the IHMM maintains its first-order Markov chain property. However, due to the differences in the prior distribution setup, and the problem of doubling the states, a phenomenon consisting of the possibility of producing an additional state that mimics an already existing one, the DPM should not be considered a device for detecting the number of states of the mixture or MS models as it was suggested in some early approaches such as by Otranto & Gallo (2002). Miller & Harrison (2013) states the argument formally.

An attractive feature of the IHMM is that it captures regime-switching and structural break jointly. Therefore it grants more flexibility and accommodates data dynamics upon the arrival of new observations. The regime-switching module pools data with similar behavior to borrow statistical strength from each other. In addition, the structural break dynamics can generate a new state when the arriving new observations exhibit a new law of motion. A well-known example is the global financial crisis in 2007 and 2008. It began with the subprime mortgage crisis in the U.S. and became the most severe financial crisis since 1930. Such an asset market downturn does not have the same source as any bear market regime in history. Any standard MS model for bull and bear markets such as Maheu & McCurdy (2000), Lunde & Timmermann (2004) and Maheu et al. (2012) is incapable of capturing the new phenomenon, because they are limited by

the data history and do not allow structural changes. The IHMM is an appropriate vehicle to achieve such a goal because the capability of generating a new unprecedented state is a feature of the latent process.

Another advantage of the IHMM lies in its superior forecasting performance, which is an empirical observation with plausible intuition. Benchmark models such as autoregressions or linear regression have a rigid assumption of the functional forms. Instead, the IHMM is more flexible and hence robust to model misspecification. Its flexibility comes from its ability to sort data into clusters endogenously, so that behaviorally different data will not pollute each other's inference. One can find the kinship between this idea and the piece-wise linear model in a univariate framework. Meanwhile, the IHMM is easily extendible to the multivariate framework, and the nodes are inferred jointly with the other model coefficients. Two vital components to achieve better prediction performance are the hierarchical prior structure (Song, 2014) and certain regime persistence (Fox et al., 2011) in economic and finance applications. The hierarchical structure exploits more data information by letting the regimes to inform one another. At the same time, regime persistence reflects salient stylized facts that economic time series are prone to local dynamic stability.

The IHMM provides a convenient tool for a control approach in grand modeling frameworks. Consider a modeling framework that utilizes independent variables x_t and error term ϵ_t. Any incorrect distributional assumption about ϵ_t could potentially adversely affect the inference on the functional form of x_t. If such distribution is not the focus of the application, simple estimators as (nonlinear) least-squares are perfectly competent. If the distribution matters for inference such as in risk analysis, these methods are no longer useful, and a semi-parametric approach has its advantages by imposing a nonparametric distribution on ϵ_t. Such an assumption releases ϵ_t from any potential misspecification. Moreover, the parametric part related to x_t is free from contamination of any parametric assumption imposed in ϵ_t. Lastly, the curse of dimensionality is not an issue because ϵ_t is usually univariate, while x_t is not. For examples, see Jensen & Maheu (2010), who modeled ϵ_t as a DPM model. The most recent research treating ϵ_t as an IHMM includes Hou (2017) and Dufays et al. (2019).

3.3. Extensions of IHMM

The IHMM provides a basis for the burgeoning academic literature on its extensions. An IHMM with DPM emission can be found in Fox et al. (2011), but it does not allow sharing particles among states. A block IHMM by Stepleton et al. (2009) generalizes the stick structure of Fox et al. (2011) by introducing more in-state dynamics to capture finer structures. In this approach, each state is a distinct MS model, that makes it suitable in the application to the bull and bear markets modeling as in Maheu et al. (2012). To capture long memory, Gael et al. (2009) proposed the factorial IHMM. Examples of its applications include Nakano et al. (2011) and Heller et al. (2009). The factorial IHMM does not impose any restrictions for identification and, thus, structural interpretations are hardly possible, which limits the scope of applications in economics and finance. Many interesting modeling frameworks arise from incorporating various components into the IHMM structure, and some of such ideas can be found in Gray (1996), Ehrmann et al. (2003) and Haas et al. (2004). Alternatively, a modeling strategy consisting of applying the IHMM structure to existing interpretable parametric models can be found in Liu & Maheu (2018), Jin et al. (2019) and Jin et al. (2019). Finally, there is a substantial body of work that follows up and extends the IHMM in different directions and some examples include Shi & Song

(2014), Bauwens et al. (2017), Maheu & Yang (2016), Yang (2019), Hou (2017) and Luo et al. (2019).

4. Endogenous Markov Switching

Recent literature proposed models that question the assumption of exogeneity of the Markov process and make an explicit link between the measurement equation error term and this latent process. In consequence, some form of endogeneity of the Markov process is introduced. Kim et al. (2008) argue that in many applications, endogeneity of the Markov process corresponds to the data properties and theoretical considerations to a larger extent than the exogenous one. While the initial proposal by Kim et al. (2008) implements endogeneity in the MS model, this section focuses on a specification proposed by Chang et al. (2017). In this model, a discrete-valued latent process, s_t, is driven by a real-valued process, w_t, that is related to the measurement equation error term and then conveniently discretized.

4.1. Introducing Endogeneity

Define the dynamics of the real-valued latent factor by an autoregressive process

$$w_t = \alpha w_{t-1} + v_t, \tag{11}$$

where α is the persistence coefficient such that $|\alpha| \leq 1$ and v_t is a standard normal error term. The initial value of the process, w_0, is recommended to be normally distributed with the zero mean and variance equal to $1/(1 - \alpha^2)$ for $|\alpha| < 1$, or equal to zero, $w_0 = 0$, for $\alpha = 1$. The process in equation (11) is discretized by defining a threshold parameter, τ, and the discrete-valued state indicator for a two-state model, $K = 2$, as

$$s_t = \left\{ \begin{array}{ll} 1 & \text{for } w_t < \tau, \\ 2 & \text{for } w_t \geq \tau. \end{array} \right. \tag{12}$$

Therefore, as long as the process w_t is subject to interpretation, its primary role is to define the Markov process s_t.

Chang et al. (2017) define the measurement equation in a general form that makes an explicit link between the potential dependence of the conditional mean and standard deviation on the independent variables, x_t, and the latent processes

$$y_t = m(x_t, w_t) + \sigma(x_t, w_t)\epsilon_t = m(x_t, s_t) + \sigma(x_t, s_t)\epsilon_t, \tag{13}$$

where ϵ_t, conditionally on x_t and w_t (or s_t), is a serially uncorrelated standard normal error term. Endogeneity of the Markov process is formally introduced in this model by an appropriate specification of the joint distribution of the error terms from the state and measurement equations (11) and (13) respectively that is given by

$$\begin{bmatrix} \epsilon_t \\ v_{t+1} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right), \tag{14}$$

where ρ is the correlation coefficient. It can be shown that the exogenous MS model that is discussed in Section 2 can be obtained by setting $\rho = 0$. In other words, the restricted endogenous and the exogenous MS models are observationally equivalent, that is, they both lead to the same

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value of the likelihood function given the values of Θ and S. However, for the values of ρ that are different from zero the relationship between ϵ_t and w_{t+1} and, consequently, between ϵ_t and s_{t+1} is established. Expressions for the implied transition probabilities that in this model are changing over time are given in the original paper.

A different form of endogeneity was proposed by Kim et al. (2008) who specified the joint distribution of error terms, such as the one in equation (14), for vector (ϵ_t, v_t). This model presumes a contemporaneous effect of ϵ_t on w_t and s_t which Chang & Kwak (2017) point out to be misspecified. Kim et al. (2008) consider an application to modeling a volatility feedback effect in financial time series.

4.2. Interpretations Considering Correlation Coefficient

To illustrate the interpretation of non-zero ρ consider a model in equation (6) with s_t specified by the endogenous Markov process and with $|\beta_k| < 1$ for $k = 1, 2$. A possible application in finance includes the modeling of the leverage effect defined as the negative correlation between the current innovation and future conditional variance of the return on a financial asset. Let $\rho < 0$ and $\sigma_1 < \sigma_2$. In this case, a negative realization of ϵ_t increases the probability of the second state in period $t+1$ and leads to an increase in the conditional volatility.

A simple application in time series analysis includes the modeling of the mean reversion that works differently for ρ of different signs. Let $\mu_1 < \mu_2$ and $\rho < 0$. Then, a positive realization of ϵ_t decreases the probability of the second state in the period $t+1$. Therefore, the mean reversion is also obtained at the level of the future conditional expected value that now decreases. In the opposite case of $\rho > 0$, a positive realization of ϵ_t increases the probability of the second state in period $t+1$ and, consequently, increases the conditional expected value of y_t. Therefore, the forecasts of y revert to a mean that is higher, which has a destabilizing effect. Examples of more elaborate applications in economics include the endogenous switching of the parameters modeling the effects of monetary and fiscal policies proposed by Chang et al. (2018) and Chang & Kwak (2017), respectively.

5. Future Developments

Two suggested paths for future developments in the MS and IHMM frameworks include efficient algorithms with the potential for parallel computations and interpretability incurred through sparsity. Improvements on both of these fronts are required to make the analysis of big data sets possible by combining sufficient flexibility of the model with scarce parameterization.

Existing approaches to MS models rely on the FFBS technique, discussed in Section 1. This filtering and smoothing method consists of an iterative procedure that implies serial computations. With an increasing number of observations and states required to capture the features of data, the FFBS algorithm becomes computationally too requiring for practical implementations. Vectorization, tensor algebra, and efficient numerical algorithms provide some of the solutions. Still, they are far from being as computationally fast as available algorithms for real-values state-space models such as, for instance, the precision sampler by Chan & Jeliazkov (2009).

Moreover, an increasing interest in heterogenous MS models in which individual parameters follow independent Markov processes calls for new methods of allowing sparsity in the modeling. In many such studies, the question of whether time-variation is required for a particular parameter and, if yes, then how many MS regimes are required to model it stay unanswered due to the lack of appropriate algorithms. It is worth mentioning that recent studies provide such solutions for

real-values state-space models, and some examples include Frühwirth-Schnatter & Wagner (2010), Bitto & Frühwirth-Schnatter (2019), and Cadonna et al. (2019).

Similar considerations apply to the IHMM, although it should be emphasized that the IHMM provides certain solutions to the challenges singled out above for the MS models. Here, as the number of observations increases, the number of states, K, to be modeled in a particular iteration of the estimation algorithm following the beam sampler step shall increase as well. This fact increases the computation time required geometrically and calls for a more time-efficient estimation method. Variational inference offers faster algorithms at the cost of approximating the posterior distribution at an unknown precision (see Wainwright et al., 2008, for further reference). Variational Bayes captures correctly the central tendency of the approximated distribution. However, it underestimates the posterior variances (see Wang & Blei, 2019, and references therein). Applications of Variational Bayes estimation to Dirichlet process and IHMM can be found in Blei et al. (2006), Kurihara et al. (2007), Teh et al. (2008), and Wang et al. (2011). Alternative approaches may seek to improve the computation through parallelization such as Fearnhead (2004), Rodriguez (2011), Williamson et al. (2013) and Tripuraneni et al. (2015).

Notable recent developments granting sparsity in Dirichlet process models were proposed by Frühwirth-Schnatter & Malsiner-Walli (2019). They build foundations for new developments for the IHMM. In this approach, the sparsity is accompanied by choice of concentration parameters, and as an extra benefit, it offers significantly simplified computations as the sparse structure heavily penalizes the number of states.

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