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CS383, Algorithms
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HW10 Solutions

1. (a) Prove that $(x_1, x_2) = (51, 49)$ is the solution to the optimization problem in HW9 task 2, resulting in 5510 units being produced (that is, 51 employees should be hired from company A and 49 from company B in order to maximize the number of units). Do this by directly manipulating the inequality constraints from that task, in the same way that we did in class for a 2-D version of the cargo plane task. *You should explicitly find a suitable weighted combination of some or all of the inequality constraints that produce a tight upper bound on the objective function, of the form* $60x_1 + 50x_2 \leq 5510$. Explain.

Solution

The proposed solution $(x_1, x_2) = (51, 49)$ occurs at the intersection point of the boundaries of the following two constraints:

$$5x_1 + 3x_2 \leq 402, \quad x_1 + x_2 \leq 100$$

We try to find a linear combination of these two inequalities that provides the desired upper bound. Letting s and t be multipliers to be determined, we obtain the following bound from the above constraints:

$$s(5x_1 + 3x_2) + t(x_1 + x_2) \leq 402s + 100t$$

Grouping x_1 and x_2 terms separately, this inequality becomes:

$$(5s + t)x_1 + (3s + t)x_2 \leq 402s + 100t$$

In order for this inequality to provide a bound on the objective function $60x_1 + 50x_2$, we need:

$$5s + t \geq 60, \quad 3s + t \geq 50$$

Consider the case of equality in these inequalities, and subtract the second from the first. The result is:

$$2s = 10$$

Substitute the resulting value $s = 5$ into the second equality, yielding:

$$t = 35$$

The resulting upper bound on the objective function is:

$$60x_1 + 50x_2 \leq 5510,$$

which is exactly what is needed. This bound proves optimality of the proposed solution $(x_1, x_2) = (51, 49)$, because the value of the objective function at the proposed solution point is exactly 5510, a value that the upper bound ensures cannot be exceeded anywhere.

(b) Construct the dual of the linear programming problem from HW9 task 2. State the dual in matrix form. Explain the relationship between the matrices that define the dual and those that define the original problem from HW9 task 2.

Solution

In matrix form, the primal problem is of the form

$$\max_x c'x \quad \text{subject to } Ax \leq b,\ x \geq 0$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 5 & 3 \\ 1 & 1 \\ 1 & -6 \\ -6 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 66 \\ 72 \\ 402 \\ 100 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 60 \\ 50 \end{bmatrix}$$

In the dual problem, b and c change roles, so that now b defines the objective function $b'x$, while c plays the role of right-hand side for the inequality constraints. Also, the objective function is now to be minimized rather than maximized. The coefficient matrix for the inequality constraints is now the transposed coefficient matrix A', and the inequalities change from $\leq$ to $\geq$. The dual problem is thus stated as follows:

$$\min_y b'y \quad \text{subject to } A'y \geq c,\ y \geq 0$$

The variable y in the dual problem may be interpreted as a vector of multipliers for the constraints of the primal problem.

(c) How are the weights that you found in 1a related to the dual problem that you constructed in 1b? Explain in detail.

Solution

The weights $s = 5,\ t = 35$ found in 1a are two components of a full vector y of multipliers for all of the primal inequality constraints:

$$y = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 35 \\ 0 \\ 0 \end{bmatrix}$$

All other components of the y vector are 0 because the associated constraints were not used in deriving the upper bound in 1a. This vector y is a solution to the dual problem. Indeed, y satisfies the dual constraints:

$$A'y = \begin{bmatrix} 1 & 0 & 5 & 1 & 1 & -6 \\ 0 & 1 & 3 & 1 & -6 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 5 \\ 35 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5*5+1*35 \\ 3*5+1*35 \end{bmatrix} = \begin{bmatrix} 60 \\ 50 \end{bmatrix} \geq c$$

Furthermore, the value of the dual objective function at y is:

$$b'y = \begin{bmatrix} 66 & 72 & 402 & 100 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 5 \\ 35 \\ 0 \\ 0 \end{bmatrix} = 5510,$$

which is the minimum possible value for any vector y that satisfies the constraints since any feasible value will be an upper bound for the primal objective function and 5510 is known to be an attainable value for the primal problem.

2. The computational task of *1-D linear approximation* involves finding a linear function (straight line) $y = ax + b$ that produces values that are as close as possible to the output terms of a given sequence of (input, output) pairs $(x_1, y_1) \cdots (x_n, y_n)$. The degree of closeness can be measured using one of several alternative metrics. One of the simplest metrics is the *maximum absolute error*, or MAE:

$$\max_{1 \leq i \leq n} |y_i - (ax_i + b)|$$

In words, the goal of minimum MAE 1-D linear approximation is to find a straight line for which the maximum vertical deviation between the line and the given target points (x_i, y_i) is as small as possible. You will solve this task by casting it as a linear programming task.

(a) Cast the minimum MAE linear approximation task for the specific target points given below as a linear programming task. Explain how many variables are needed, and what their meanings are. Specify all of the entries of the A, b, c matrices in the standard formulation discussed in class. Explain.

$$(1,4),\ (2,8),\ (3,7),\ (4,10),\ (5,14),\ (7,16),\ (8,21),\ (9,20)$$

Hint. The key is to cast the MAE objective in linear terms. Think along the lines of our analysis of two-person zero-sum games.

Solution

The central issue is to cast the nonlinear optimization problem associated with minimizing the maximum absolute error:

$$\max_{1 \leq i \leq 8} |y_i - (ax_i + b)|$$

as a *linear* programming problem. As discussed in class, the trick is to introduce a new variable that represents the nonlinear optimization target and to introduce enough constraints so that the new variable conforms to this target.

Let z be the new variable described above. In order to force z to "track" the maximum of the desired quantities, we introduce the following eight constraints on z:

$$z \geq |y_i - (ax_i + b)|, \quad i = 1...8$$

These constraints are not linear. However, by examining the graph of the absolute value function $|w|$, it becomes clear that $|w|$ is the maximum of the two linear functions w and $-w$, so the original constraints may be split, yielding the following set of 16 linear constraints:

$$z \geq y_i - ax_i - b, \quad z \geq -y_i + ax_i + b, \quad i = 1...8$$

Note that the variables in this problem are the line parameters a, b and the new variable z. The values x_i, y_i are *constants* that represent the coordinates of given data points. See the next subtask below for the actual numerical values.

In summary, the resulting linear programming problem is:

$$\min_{a,b,z} z, \quad \text{subject to the above constraints}$$

(b) Use Matlab to solve the LP task from 2a. Include a printout of the Matlab commands used. State the resulting solution clearly.

Solution

Taking into account the above discussion, the target linear programming problem involves the following vector variable u:

$$u = [a \ b \ z]'$$

The optimization problem is:

$$\min_u c'u \quad \text{subject to } Au \leq b \text{ (different } b \text{ here)},$$

for the following matrices:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & -1 \\ 2 & 1 & -1 \\ -2 & -1 & -1 \\ 3 & 1 & -1 \\ -3 & -1 & -1 \\ 4 & 1 & -1 \\ -4 & 1 & -1 \\ 5 & 1 & -1 \\ -5 & -1 & -1 \\ 7 & 1 & -1 \\ -7 & -1 & -1 \\ 8 & 1 & -1 \\ -8 & -1 & -1 \\ 9 & 1 & -1 \\ -9 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -4 \\ 8 \\ -8 \\ 7 \\ -7 \\ 10 \\ -10 \\ 14 \\ -14 \\ 16 \\ -16 \\ 21 \\ -21 \\ 20 \\ -20 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matlab's linprog function finds the following solution:

$$a = 2.1667, \quad b = 2.0833, \quad z = 1.5833$$

A plot of the resulting linear function

$$y = 2.1667 + 2.0833x$$

confirms that there is a good fit to the given data points.

3. In class we discussed two-person zero-sum games. Consider such a game with the following payoff matrix:

$$G = \begin{bmatrix} 7 & -6 \\ -4 & 3 \end{bmatrix}$$

Each player has two possible moves. As in class, the convention is that rows correspond to the first player's moves and columns to the second player's moves; the value in position (i, j) is player 1's gain (and player 2's loss) if player 1 chooses move i and player 2 chooses move j.

(a) Suppose that, in a long string of many repeated instances of this game, player 1 decides to *randomly* select move 1 with probability 1/3 and move 2 with probability 2/3. Player 1's move selection for any particular instance of the game is independent of the selection in all other instances (no "memory" or "sequencing" of moves). Suppose also that player 2 is aware of the precise numerical probabilities with which player 1 selects between the two moves. Explain carefully how to determine player 2's optimal strategy in this situation. Derive player 2's optimal strategy and expected payoff.

Solution

Player 2 must choose between two possible moves. If she chooses move 1, the payoff in any particular instance of the game will be either 7 if player 1 chooses move 1, or -4 if player 1 chooses move 2. The *expected* payoff is the weighted average of these two payoffs, with weights equal to the probabilities with which player 1 selects the respective moves, that is, $1/3(7) + 2/3(-4) = -1/3$. Similarly, if player 2 chooses move 2, the expected payoff will be a weighted average of the values in column 2 of the payoff matrix: $1/3(-6) + 2/3(3) = 0$. Since the payoff (to player 1) is smaller in the first case, player 2's optimal strategy is to always choose the first move. There would be no advantage to randomizing player 2's moves, because the resulting expected payoff would be an interpolation between the expected payoffs for always choosing move 1 and always choosing move 2; such an intermediate value would necessarily be higher than the minimum value of $-1/3$ which is obtained by always choosing move 1.

(b) Now assume that it is known in advance that the *other* player (player 2) will randomly select move 1 one quarter of the time and move 2 three quarters of the time. Derive player 1's optimal strategy and expected payoff in this case. Explain.

Solution

Similar to the preceding subtask. The expected payoff if player 1 chooses move 1 is $1/4(7) + 3/4(-6) = -11/4$, while if player 1 chooses move 2 the expected payoff is $1/4(-4) + 3/4(3) = 5/4$. Since player 1 wishes to maximize the payoff, he should always choose move 2 under these circumstances, with an expected payoff of 5/4.

(c) Assume that each player will select a move randomly based on the result of tossing a suitably biased coin (Heads = move 1, Tails = move 2). Each player seeks the best possible payoff from his/her point of view. For each of the two players, state the optimization problem that should be solved in order to determine that player's optimal move 1 probability. Use the following notation: player 1's Heads probability is p, player 2's Heads probability is q.

Solution

Player 1 seeks to maximize the expected payoff. He knows that player 2 will seek to minimize the payoff for whatever probabilities p and $1-p$ player 1 uses for his two moves. Hence, the expected payoff resulting from player 2's counter-strategy will be:

$$\min\{7p + (-4)(1-p), \quad -6p + 3(1-p)\}$$

Player 1's strategy should maximize this expected payoff by adjusting p optimally. Player 1's optimization problem is therefore the following:

$$\max_{0 \le p \le 1} \min\{7p + (-4)(1-p), \quad -6p + 3(1-p)\}$$

Similarly, player 2 knows that player 1 will seek to maximize his payoff for player 2's move probabilities q and $1-q$, resulting in the following expected payoff from player 1's counter-strategy:

Player 2's strategy should minimize this expected payoff by optimally choosing her move probabilities y_1 and y_2. Here is player 2's optimization problem:

$$\min_{0 \leq q \leq 1} \max\{7q + (-6)(1-q), \quad -4q + 3(1-q)\}$$

(d) Solve the optimization problems in the preceding subtask. Explain. Determine the optimal values of the Heads probabilities p and q. Calculate the shared value of the game (expected payoff when both players use their optimal random strategy). Which of the two players has the advantage on average? Explain.

Solution

Simplifying, player 1's optimization goal is the following:

$$\max_p \min\{11p - 4, \quad -9p + 3\}, \quad \text{subject to } 0 \leq p \leq 1$$

As discussed in class, the solution occurs at the intersection point of the two linear functions in the min:

$$11p - 4 = -9p + 3 \Rightarrow p = 7/20$$

The simplified optimization goal for player 2 is:

$$\min_q \max\{13q - 6, \quad -7q + 3\}, \quad \text{subject to } 0 \leq q \leq 1$$

The solution occurs at the intersection point of the two linear functions in the max:

$$13q - 6 = -7q + 3 \Rightarrow q = 9/20$$

The expected payoff when these two optimal strategies are pitted against one another is the common value that occurs at the optimal point for either of the two players (the result is the same):

$$E[\text{payoff}] = 11(7/20) - 4 = 13(9/20) - 6 = -3/20$$

Since this expected payoff of $-3/20$ is negative, player 2 has the advantage. Note that the expected payoff when both players play optimally is higher than in 3a, where player 1's poorly chosen strategy increases his loss, but lower than in 3b, where player 2's poor strategy actually gives her opponent the advantage.