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CS383, Algorithms
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HW10

1. This task reexamines the HW9 factory hiring task from the point of view of linear duality.

   (a) Prove that $(x_1, x_2) = (51, 49)$ is the solution to the optimization problem in HW9 task 2, resulting in 5510 units being produced (that is, 51 employees should be hired from company A and 49 from company B in order to maximize the number of units). Do this by directly manipulating the inequality constraints from that task, in the same way that we did in class for a 2-D version of the cargo plane task. *You should explicitly find a suitable weighted combination of some or all of the inequality constraints that produce a tight upper bound on the objective function, of the form* $60x_1 + 50x_2 \leq 5510$. Explain.

   (b) Construct the dual of the linear programming problem from HW9 task 2. State the dual in matrix form. Explain the relationship between the matrices that define the dual and those that define the original problem from HW9 task 2.

   (c) How are the weights that you found in 1a related to the dual problem that you constructed in 1b? Explain in detail.

2. The computational task of *1-D linear approximation* involves finding a linear function (straight line) $y = ax + b$ that produces values that are as close as possible to the output terms of a given sequence of (input, output) pairs $(x_1, y_1) \cdots (x_n, y_n)$. The degree of closeness can be measured using one of several alternative metrics. One of the simplest metrics is the *maximum absolute error*, or MAE:

$$\max_{1 \leq i \leq n} |y_i - (ax_i + b)|$$

   In words, the goal of minimum MAE 1-D linear approximation is to find a straight line for which the maximum vertical deviation between the line and the given target points (x_i, y_i) is as small as possible. You will solve this task by casting it as a linear programming task.

   (a) Cast the minimum MAE linear approximation task for the specific target points given below as a linear programming task. Explain how many variables are needed, and what their meanings are. Specify all of the entries of the A, b, c matrices in the standard formulation discussed in class. Explain.

$$(1, 4),\ (2, 8),\ (3, 7),\ (4, 10),\ (5, 14),\ (7, 16),\ (8, 21),\ (9, 20)$$

   Hint. The key is to cast the MAE objective in linear terms. Think along the lines of our analysis of two-person zero-sum games.

(b) Use Matlab to solve the LP task from 2a. Include a printout of the Matlab commands used. State the resulting solution clearly.

3. In class we discussed two-person zero-sum games. Consider such a game with the following payoff matrix:

$$G = \begin{bmatrix} 7 & -6 \\ -4 & 3 \end{bmatrix}$$

Each player has two possible moves. As in class, the convention is that rows correspond to the first player's moves and columns to the second player's moves; the value in position (i, j) is player 1's gain (and player 2's loss) if player 1 chooses move i and player 2 chooses move j.

(a) Suppose that, in a long string of many repeated instances of this game, player 1 decides to *randomly* select move 1 with probability 1/3 and move 2 with probability 2/3. Player 1's move selection for any particular instance of the game is independent of the selection in all other instances (no "memory" or "sequencing" of moves). Suppose also that player 2 is aware of the precise numerical probabilities with which player 1 selects between the two moves. Explain carefully how to determine player 2's optimal strategy in this situation. Derive player 2's optimal strategy and expected payoff.

(b) Now assume that it is known in advance that the *other* player (player 2) will randomly select move 1 one quarter of the time and move 2 three quarters of the time. Derive player 1's optimal strategy and expected payoff in this case. Explain.

(c) Assume that each player will select a move randomly based on the result of tossing a suitably biased coin (Heads = move 1, Tails = move 2). Each player seeks the best possible payoff from his/her point of view. For each of the two players, state the optimization problem that should be solved in order to determine that player's optimal move 1 probability. Use the following notation: player 1's Heads probability is p, player 2's Heads probability is q.

(d) Solve the optimization problems in the preceding subtask. Explain. Determine the optimal values of the Heads probabilities p and q. Calculate the shared value of the game (expected payoff when both players use their optimal random strategy). Which of the two players has the advantage on average? Explain.