A Better Approximation Ratio for the Vertex Cover Problem

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Abstract. We reduce the approximation factor for the vertex cover to $2 - \Theta(\frac{1}{\sqrt{\log n}})$ (instead of the previous $2 - \Theta\frac{\ln\ln n}{2\ln n}$ obtained by Bar-Yehuda and Even [1985] and Monien and Speckenmeyer [1985]). The improvement of the vanishing factor comes as an application of the recent results of Arora et al. [2004] that improved the approximation factor of the sparsest cut and balanced cut problems. In particular, we use the existence of two big and well-separated sets of nodes in the solution of the semidefinite relaxation for balanced cut, proven by Arora et al. [2004]. We observe that a solution of the semidefinite relaxation for vertex cover, when strengthened with the triangle inequalities, can be transformed into a solution of a balanced cut problem, and therefore the existence of big well-separated sets in the sense of Arora et al. [2004] translates into the existence of a big independent set.

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1. *Introduction*

One of the most well-studied problems in combinatorial optimization is the vertex cover (VC) problem: Given a graph $G = (V, E)$, we look for a minimum size subset of vertices such that for every $(u, v) \in E$, at least one of u, v belongs to this subset. In the *weighted* version of VC, each vertex has an integral weight, and we are looking for the minimum total weight subset of vertices with the property given before.

Since the complexity of VC has been heavily studied since Karp's original proof of its NP-completeness [Karp 1972], the related bibliography is vast and cannot be

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covered, of course, in this introductory note. We mention here that VC is known to be APX-complete [Papadimitriou and Yannakakis 1991], and moreover it cannot be approximated within a factor of 1.36 [Dinur and Safra 2005], unless P=NP. Moreover, if the *unique 2-prover-1-round game conjecture* [Khot 2002] holds, Khot and Regev [2008] show that VC is hard to approximate within any constant factor better than 2. A 2-approximation, on the other hand, can be trivially obtained by taking all the vertices of a maximal matching in the graph.

Improving this simple 2-approximation algorithm has been quite a nontrivial task. The best approximation algorithms known before this work were published 20 years ago by Bar-Yehuda and Even [1985], and Monien and Speckenmeyer [1985]. They achieved an approximation factor of $2 - \frac{\ln \ln n}{2 \ln n}$, where n is the number of vertices. If Δ is the maximum degree of the graph, Halperin [2002] showed that a factor of $2 - (1 - o(1))\frac{2 \ln \ln \Delta}{\ln \Delta}$ can be achieved by using the semidefinite programming (SPD) relaxation of VC.

In this work we use a stronger SDP relaxation to improve the approximation factor achieved in polynomial time to $2 - \Theta(\frac{1}{\sqrt{\log n}})$. We observe that the introduction of all the so-called *triangle inequalities* to the standard SDP relaxation of VC is, in fact, very similar to the balanced cut SDP relaxation used by Arora et al. [2004]. Then we use one of the main results of Arora et al. [2004], which asserts that in the solution of this SDP, there are two big and well-separated vertex subsets. At the same time, we show that edges that were not covered by a trivial initial rounding are too big to have both of their endpoints in either of these two sets. Hence, one of these two big subsets has to be a big independent set which can be excluded. We show this process first for the unweighted VC, and then we show how it can be extended to the weighted case in a straightforward manner.

Hence the main idea in this improvement of the approximation factor is the transformation of the "classic" SDP formulation of VC to a formulation that corresponds to a balanced cut problem with the addition of the "antipodal" points of the original points. This appears to be a very general technique that can find application to other problems, and brings the Arora et al. [2004] improved approximation factor for balanced cuts to other contexts as well. Indeed, subsequently to our work, Agarwal et al. [2005] used this stronger SDP formulation and the negative symmetric metric that it implies, together with a clever iterative application of the separation algorithm of Arora et al. [2004], and several other ideas like volume arguments to improve the approximation factor of several problems (Min UnCut, Min 2CNF Deletion, Directed Balanced Separator, Directed Sparsest Cut) to $O(\sqrt{\log n})$.

2. *The Unweighted Case*

The following is a semidefinite programming relaxation of unweighted Vertex Cover (VC) for a graph $G = (V, E)$ with n nodes.

$$\min \sum_{i=1}^{n} \frac{1 + v_0 v_i}{2} \text{ such that} \tag{SDP}$$

$$(v_0 - v_i)(v_0 - v_j) = 0, \qquad \forall (i, j) \in E \tag{1}$$

$$(v_i - v_j)(v_i - v_k) \geq 0, \qquad \forall i, j, k \in V \cup \{0\} \tag{2}$$

$$v_i^2 = 1, \qquad \forall i \in V \cup \{0\}, \tag{3}$$

where $v_i \in \mathbb{R}^{n+1}$. Constraints (2) are *triangular inequalities* which must be satisfied by the vertex cover. In an "integral" solution of (SDP) (which would correspond to a vertex cover of G), vectors for vertices that are picked coincide with v_0, while vectors for vertices that are not picked coincide with $-v_0$. In general though, an optimal solution of (SDP) will not be "integral".

In fact, one can strengthen this SDP relaxation for VC by adding all the so-called triangle inequalities.

$$
\begin{aligned}
&\min \sum_{i=1}^{n} \frac{1+v_0 v_i}{2} \text{ such that} && \\
&(v_0 - v_i)(v_0 - v_j) = 0, && \forall (i, j) \in E \\
&(v_i - v_j)(v_i - v_k) \geq 0, && \forall i, j, k \in V \cup \{0\} \\
&(v_i + v_j)(v_i - v_k) \geq 0, && \forall i, j, k \in V \cup \{0\} \\
&(v_i + v_j)(v_i + v_k) \geq 0, && \forall i, j, k \in V \cup \{0\} \\
&v_i^2 = 1, && \forall i \in V \cup \{0\}.
\end{aligned}
$$

This relaxation is in fact equivalent to the following relaxation: We add n more "shadow" points to (SDP) so that for every unit vector $v_i,\ i = 1, \ldots, n$ we add unit vector v_i' which is the antipodal of v_i, namely, $v_i v_i' = -1,\ \forall i$. Note that this also implies that $v_i = -v_i',\ \forall i$. Let V' be the set of shadow points. Note that in an integral solution of (SDP), exactly half (n) of the points in $V \cup V'$ coincide with v_0 and the other half coincide with $-v_0$. Therefore the following must hold:

$$\sum_{i,j \in V \cup V'} |v_i - v_j|^2 = 4n^2,$$

where every pair (i, j) appears only once in the sum. (Hence the set $V \cup V'$ is 1/2-spread in the terminology of Arora et al. [2004].) In addition, the triangular inequalities (2) must also hold when we extend V with V'. Hence we have the following strengthened SDP.

$$
\begin{aligned}
&\min \sum_{i=1}^{n} \frac{1+v_0 v_i}{2} \text{ such that} && && \text{(SDP}'\text{)} \\
&(v_0 - v_i)(v_0 - v_j) = 0, && \forall (i, j) \in E && (4) \\
&(v_i - v_j)(v_i - v_k) \geq 0, && \forall i, j, k \in V \cup V' \cup \{0\} && (5) \\
&v_i^2 = 1, && \forall i \in V \cup V' \cup \{0\} && (6) \\
&v_i v_i' = -1, && \forall i \in V && (7) \\
&\sum_{i,j \in V \cup V'} |v_i - v_j|^2 = 4n^2, && && (8)
\end{aligned}
$$

where $v_i, v_i' \in \mathbb{R}^d$ for some $d \gg \log n$. Constraint (8) is in fact unnecessary since it is always satisfied by a set of points and their antipodals, but we include it in order to point out that this relaxation defines a *spread metric* as defined in Arora et al. [2004]. Now we can use results of Arora et al. [2004] to find an approximate VC.

For any $\varepsilon > 0$, we define the following two sets of graph vertices.

$$S_1 := \{v \in V : v_0 v > \varepsilon\}$$
$$S_2 := \{v \in V \cup V' : -\varepsilon \le v_0 v \le \varepsilon\}.$$

For now, we concentrate our attention on S_2. Note that in S_2 we have included also shadow points. In fact, note that if $v_i \in V$ belongs to S_2 then its shadow $v_i' \in V'$ belongs to S_2 as well, and vice versa. In other words, S_2 contains both original points and their shadows.

LEMMA 1.

$$\sum_{i,j \in S_2} |v_i - v_j|^2 = |S_2|^2.$$

PROOF. Note that for a particular pair $i, j \in S_2 \cap V$, $i \neq j$ we have $v_i v_j' = v_i' v_j = -v_i v_j$. So if we group summation terms according to pairs of vertices $i, j \in S_2 \cap V$, each pair contributes 8 to the sum due to cancellation of terms. Since $|S_2 \cap V| = |S_2|/2$, there are $\binom{|S_2 \cap V|}{2} = \frac{|S_2|^2 - 2|S_2|}{8}$ such pairs, for a total of $|S_2|^2 - 2|S_2|$. The remaining $|S_2|/2$ terms of the summation are of the form $|v_i - v_i'|^2 = 4$, for all $i \in S_2 \cap V$, for a total of $2|S_2|$. The lemma follows. $\square$

Let $\Delta, \sigma > 0$ be two parameters, where Δ is determined later in the article, and σ will determine the "spread" between a random unit vector u and the members of the two sets defined as follows.

$$S_u := \left\{v \in S_2 : uv \ge \frac{\sigma}{\sqrt{d}}\right\},$$
$$T_u := \left\{v \in S_2 : uv \le -\frac{\sigma}{\sqrt{d}}.\right\}.$$

We will use the fact that Arora et al. [2004] prove the existence of "large" such sets for σ properly chosen (refer Lemma 4 that follows). Since $v_i = -v_i'$, it is easy to prove the following.

LEMMA 2. *If $v_i \in S_u$ for some $v_i \in V$, then $v_i' \in T_u$, and vice versa, if $v_i' \in T_u$, then $v_i \in S_u$. The same holds with the roles of S_u, T_u interchanged.*

As a result of Lemma 2, $S_u \cup T_u$ contains only pairs of points in V with their shadow points, and each such pair is separated between S_u, T_u, and $|S_u| = |T_u|$. Moreover, the following easy fact also holds.

LEMMA 3. *Let $f(v)$ be the antipodal vector of v.[1] Then*

$$v \in S_u \wedge w \in T_u \wedge |v - w|^2 \le \Delta \Rightarrow f(w) \in T_u \wedge f(v) \in S_u \wedge |f(v) - f(w)|^2 \le \Delta.$$

Let $c' > 0$ be another parameter which will be defined later. We modify the procedure SET-FIND of Arora et al. [2004] as follows.

[1]Hence $f(v_i) = v_i'$ and $f(v_i') = v_i$.

—If $|S_u| < 2c'|S_2|$ or $|T_u| < 2c'|S_2|$ then we HALT (just like in Arora et al. [2004]).

—Otherwise, pick any $x \in S_u$, $y \in T_u$ such that $|x - y|^2 \leq \Delta$. Then, because of Lemma 3, the corresponding pair of antipodal points $y' \in S_u$, $x' \in T_u$ also satisfy $|x' - y'|^2 \leq \Delta$. Delete x, x', y, y'. Repeat until no such x, y can be found.

Note that initially T_u contains the antipodal points of S_u (Lemma 2), and every deletion eliminates two points from each of S_u, T_u, and these four actually form two (a point in V, its shadow point in V') pairs. Therefore, in the end, the remaining points in S_u are *exactly* the antipodal points of T_u (or both S_u, T_u are empty). As in Arora et al. [2004], $|x - y|^2 > \Delta$, $\forall x \in S_u, y \in T_u$. One can define the parameters c', σ so that, initially, S_u, T_u are big with high probability.

LEMMA 4 (LEMMA 4 IN ARORA ET AL. [2004]). *For every positive $c < 1/3$, there are $c', \sigma > 0$ such that the probability (over the choice of u) is at least $c/8$ that the initial sets S_u, T_u defined before have size at least $2c'|S_2|$.*

PROOF. From Lemma 1 and application of Lemma 4 of Arora et al. [2004]. □

From now on, c' is the constant defined in Lemma 4. One of the main results of Arora et al. [2004] is to show that, with high probability over u, not many points are deleted before SET-FIND terminates. We match the points removed to form a matching (at every step, x is matched to y, and x' is matched to y'). Theorem 5 in Arora et al. [2004] shows that, with $\Delta = O(\log^{-2/3} n)$, the probability that SET-FIND removes a matching of size $c'|S_2|$ is $o(1)$. Hence the final S_u, T_u of SET-FIND have size $\geq c'|S_2|$ with probability $\Omega(1)$, and $|S_u| = |T_u|$. In what follows, we assume that S_u, T_u are the big final sets we get with high probability from SET-FIND.

LEMMA 5. *If $\varepsilon \leq \Delta/4$, then there is no edge $(i, j) \in E$ such that $v_i, v_j \in V$ belong both to S_u or both to T_u.*

PROOF. Without loss of generality, suppose that there is $(i, j) \in E$ such that $v_i, v_j \in S_u$. Then their shadow (antipodal) points belong to T_u, namely, $v_i', v_j' \in T_u$. Since $v_i, v_j \in S_2$ and constraint (4) holds, we have that

$$v_i v_j = v_0 v_i + v_0 v_j - 1 \leq -(1 - 2\varepsilon). \tag{9}$$

Since $v_i' \in T_u$ and $v_j \in S_u$ are not deleted in SET-FIND, $|v_i' - v_j|^2 > \Delta$, or, equivalently, $|v_i + v_j|^2 > \Delta$. This implies that

$$v_i v_j > -1 + \frac{\Delta}{2}. \tag{10}$$

But (9) and (10) together imply that $\varepsilon > \Delta/4$ which contradicts the hypothesis. □

From now on we set $\varepsilon := \Delta/4 > 0$. Since $|S_u| = |T_u| \geq c'|S_2|$, and the two sets contain antipodal points, one of them (without loss of generality let's assume that this is S_u), contains at least $\frac{c'|S_2|}{2}$ points from V. Let I be this set of points from V. Lemma 5 implies that I is an *independent set* of G of size at least $c_0|S_2|$, where $c_0 := c'/2 > 0$. We return the set $S := S_1 \cup (S_2 \backslash (I \cup V'))$ as our vertex cover.

LEMMA 6. *S is a vertex cover of G.*

PROOF. If there is $(i, j) \in E$ with $v_i, v_j \in V \setminus (S_1 \cup S_2)$, we have (by the definition of S_1, S_2) that $v_0 v_i < -\varepsilon$ and $v_0 v_j < -\varepsilon$, which implies that $v_0 v_i +$

$v_0 v_j - 1 < -1-2\varepsilon$. Then constraint (4) implies that $v_i v_j < -1-2\varepsilon$, a contradiction. Also, since I is an independent set, not both of v_i, v_j can belong to it. If $v_i \in I$ and $v_j \in V \backslash (S_1 \cup S_2)$, then $v_0 v_i \leq \varepsilon$ and $v_0 v_j < -\varepsilon$, therefore constraint (4) implies that $v_i v_j < -1$, a contradiction. We conclude that every edge must have at least one of its endpoints in S. $\square$

A weaker version of our main result is the following.

THEOREM 1. $|S| \leq (2 - \Theta(\frac{1}{\log^{2/3} n}))VC(G)$.

PROOF. We follow the analysis of Halperin [2002]. From (SDP$'$) and the definition of S_1, S_2 we have that

$$VC(G) \geq |S_1|\frac{1+\varepsilon}{2} + |S_2 \backslash V'|\frac{1-\varepsilon}{2},$$

or, equivalently,

$$|S_1| \leq \frac{2 \cdot VC(G)}{1+\varepsilon} - |S_2 \backslash V'|\frac{1-\varepsilon}{1+\varepsilon}. \tag{11}$$

Hence

$$|S| = |S_1| + |S_2 \backslash V'| - |I| \overset{(11)}{\leq} \frac{2}{1+\varepsilon}VC(G) + |S_2 \backslash V'|\left(\frac{2\varepsilon}{1+\varepsilon} - c_0\right).$$

Note that for $\Delta = \Theta(\log^{-2/3} n)$, $\frac{2\varepsilon}{1+\varepsilon} = \Theta(\log^{-2/3} n) < c_0$, for big enough n. Therefore,

$$|S| \leq \frac{2}{1+\varepsilon}VC(G) = (2 - \Theta(\log^{-2/3} n)) \cdot VC(G). \quad \square$$

Recently, Lee [2005] proved that the SET-FIND algorithm of Arora et al. [2004] can also be used to obtain their stronger result, that is, Δ can be as big as $\Theta(1/\sqrt{\log n})$. Therefore we can get the following strengthening of Theorem 1.

THEOREM 2. $|S| \leq (2 - \Theta(\frac{1}{\sqrt{\log n}}))VC(G)$.

3. *The Weighted Case*

The following is a semidefinite programming relaxation of weighted Vertex Cover (VC) for a graph $G = (V, E)$ with n nodes.

$$\min \sum_{i=1}^{n} w_i \cdot \frac{1 + v_0 v_i}{2} \text{ such that} \tag{WSDP}$$
$$\begin{aligned} (v_0 - v_i)(v_0 - v_j) &= 0, & \forall (i, j) \in E \\ (v_i - v_j)(v_i - v_k) &\geq 0, & \forall i, j, k \in V \cup \{0\} \\ v_i^2 &= 1, & \forall i \in V \cup \{0\}, \end{aligned}$$

where w_i is the *integral* weight of node i. Let $W := \sum_{i=1}^{n} w_i$.

In order to apply the methods of Section 2, we solve (SDP$'$) with the weights incorporated in the objective function, and replace every v_i by w_i copies of v_i (v_i' is also replaced by w_i copies of v_i'). In fact, we don't need to do this replacement in practice, but this mental experiment is helpful in order to see how the unweighted case applies here, too. Note that this new set of vectors still satisfies the triangular inequalities, and Lemmata 4 through 6 in Section 2 apply here as well with $n := W$. Note that SET-FIND can be made to run in polynomial time in this case (recall that we don't really do the replacement of v_i with w_i, all we need to do is to keep track of how much weight remains for each node after each matching). Now Theorem 1 (and hence Theorem 2) can be proven in the same way as before, if we replace the cardinality of sets $|\cdot|$ with their weights $w(\cdot)$.

4. *Open Problems*

Obviously one of the biggest open problems in theoretical computer science is the exact determination of the approximability of VC. There is a big gap between the hardness and the approximability results. Unfortunately, the SET-FIND procedure of Arora et al. [2004] is limited to a gap of at most $\Theta(1/\sqrt{\log n})$ by the embedding of the $\log n$-dimensional hypercube: In this case any two subsets of linear size are closer than $\Theta(1/\sqrt{\log n})$ (this simple fact was pointed out to us by J. R. Lee).

We couldn't extend our techniques to other problems related to VC, for example, the maximum independent set problem (IS), and we don't know whether this is possible (Halperin [2002] techniques, on the contrary, can be applied to IS). Another extension of VC is the vertex cover problem in hypergraphs. We don't know how to extend our techniques to this problem as well. Therefore we leave the application of the preceding results to these and other problems as an open question.

A weaker formulation than the strengthened SDP relaxation (SDP′) used earlier, that doesn't contain all the triangle inequalities but is equivalent to Schrijver's θ' function [Goemans and Kleinberg 1998], was proven to have an integrality gap of $2-\varepsilon$ for any constant $\varepsilon > 0$ by Charikar [2002]. His example was lately extended to an even stronger relaxation than (SDP′) by Hatami et al. [2007] that contains pentagonal inequalities as well as the triangle ones. It would be interesting to show the same result for even stronger SDP formulations. On the same vein, recently there has been great progress in proving that a lift-and-project procedure by Lovasz and Schrijver [1991] does not produce VC relaxations with integrality gap better than 2. Arora et al. [2006] initially showed that such a good ratio cannot be achieved in $O(\log n)$ rounds of the LS procedure of Lovász and Schrijver [1991], and Tourlakis [2006] showed a lower bound of $1.5-\epsilon$ for any constant $\epsilon > 0$ in $O(\log^2 n)$ rounds. Currently, the best result has been achieved by Georgiou et al. [2007] that shows a lower bound of $2-o(1)$ for $\Omega(\sqrt{\log n/\log\log n})$ rounds of the more powerful LS_+ procedure of Lovász and Schrijver [1991]. Also, Georgiou et al. [2008] shows that this lower bound for the approximation ratio remains even if one strengthens the SDP relaxation we use by adding all hypermetric inequalities supported on at most k vertices.[2] Hopefully such results will shed more light in the open problem of the approximability of VC.

It is worthy also to consider the approximation of VC in special graphs. For some cases the approximability remains the same as the general case (e.g., graphs with perfect matching [Chen and Kanj 2005]). But there are cases for which any improvement of the VC approximation factor translates directly to better approximation

[2] The SDP we use includes the hypermetric inequalities on 3 vertices, namely, the triangle inequalities.

factor for other important problems. Such an example is the recent reduction of the single machine precedence constrained scheduling problem to VC by Ambuhl and Mastrolilli [2006].

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