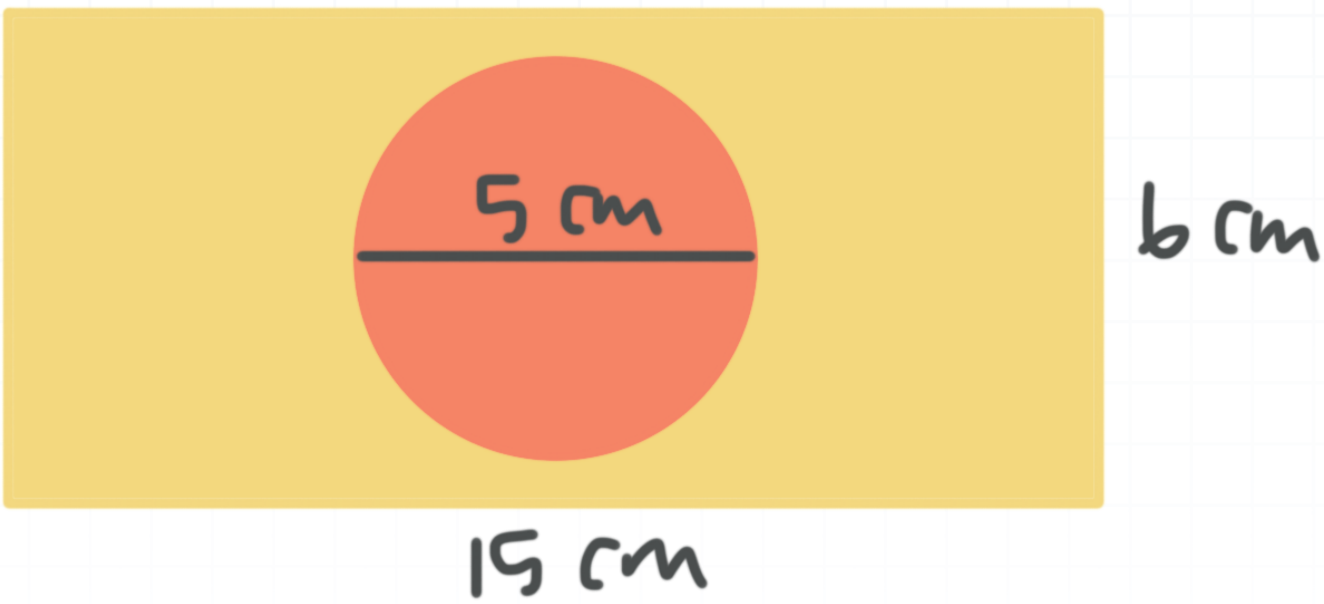
Probability & Statistics Workbook Solutions

Probability

krista king MATH

SIMPLE PROBABILITY

1. A child drops a marble onto a board. Suppose that it is equally likely for it to fall anywhere on the board. What is the probability, to the nearest percent, that it lands on the red circle?



Solution:

We want to know the probability that the marble falls on the red area of the board. So we need to know

$$P(\text{red circle}) = \frac{\text{area of red circle}}{\text{area of full rectangle}}$$

This means we need to find the area of the circle,

$$A_{\text{circle}} = \pi r^2$$

$$A_{\text{circle}} = \pi (2.5)^2$$

$$A_{\text{circle}} \approx 19.63 \text{ cm}^2$$

and the rectangle.

$$A_{\text{rectangle}} = lw$$

$$A_{\text{rectangle}} = (15)(6)$$

$$A_{\text{rectangle}} = 90 \text{ cm}^2$$

So the probability that the marble lands on the red circle is

$$P(\text{red circle}) = \frac{19.63 \text{ cm}^2}{90 \text{ cm}^2} \approx 0.22$$

There's a 22% chance the marble lands on the blue circle.

■ 2. A 12-sided number cube is rolled 60 times. Use the table to calculate $P(\text{rolling an } 11)$. Is this theoretical or experimental probability? Why?

Number rolled	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	5	8	2	0	10	1	6	5	2	8	12	1

Solution:

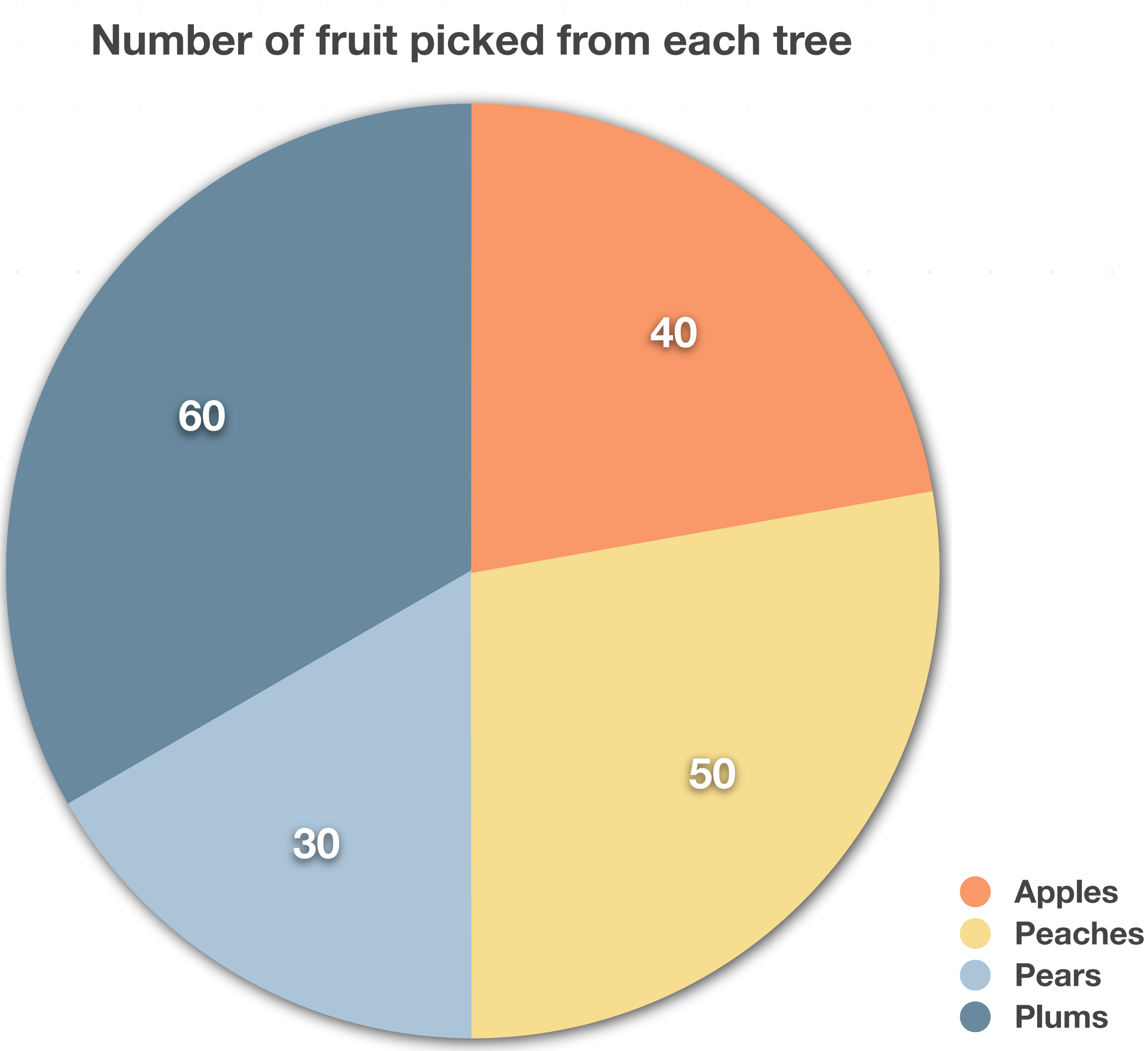
This is an experimental probability because it's based on the results of actual trials. From the table, we can see that we rolled an 11 on the dice 12 times out of the 60 total rolls.

Number rolled	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	5	8	2	0	10	1	6	5	2	8	12	1

So P(rolling an 11) is

$$P(\text{rolling an } 11) = \frac{12}{60} = \frac{1}{5} = 0.2 = 20\,\%$$

■ 3. Monica's class went on a trip to an orchard. At the end of the trip they put all of the fruit they picked into one big basket. The chance of picking any fruit from the basket is equally likely. Monica's teacher picks out a fruit for her to eat at random. What is the probability that it's a plum (Monica's favorite)? Is this an experimental or theoretical probability? Why?



Solution:

This is a theoretical probability because it was calculated based on the knowledge of the sample space. Monica didn't perform repeated trials, so there was no experiment.

In this case, the outcomes that meet our criteria are the 60 plums. All possible outcomes can be found by found by adding all of the types of fruit together.

$$60 + 40 + 30 + 50 = 180$$

Therefore, the probability of getting a plum is

$$P(\text{event}) = \frac{\text{outcomes that meet our criteria}}{\text{all possible outcomes}}$$

$$P(\text{plum}) = \frac{60}{180} = \frac{1}{3}$$

Monica has a $1/3 \approx 33\%$ chance of getting a plum.

4. Jamal surveyed the people at his local park about their favorite hobby and recorded his results in a table. Based on the survey, what's the probability that someone who visits the park will choose Art as their favorite hobby? Is this a theoretical or experimental probability? Why?

Hobby	Count
Reading	14
Sports	28
Art	15
Total	**57**

Solution:

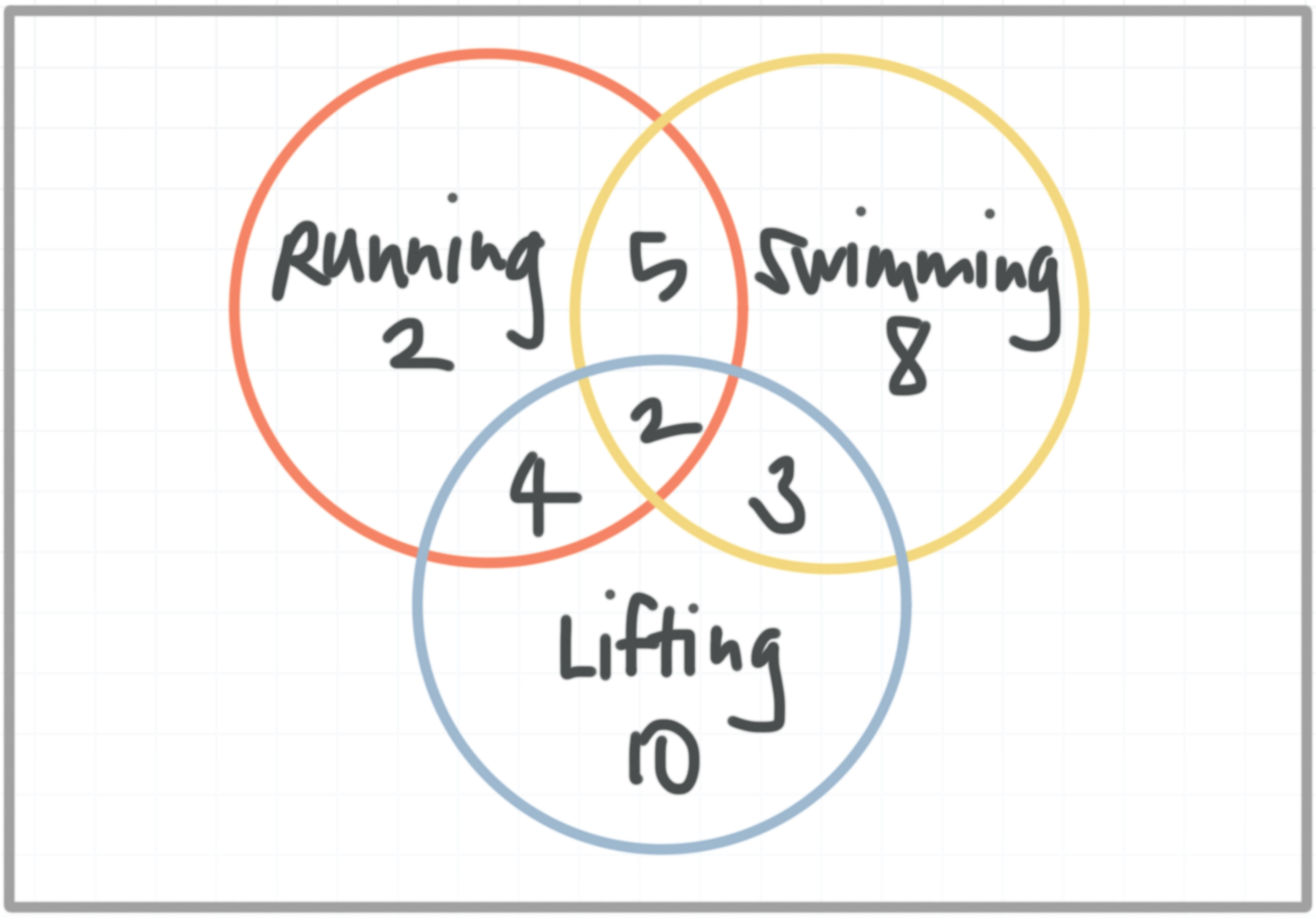
Jamal is not likely to have surveyed everyone who visits the park or everyone who will visit the park in the future. A survey is most often a sample of a larger population, so the results are an experimental probability.

In this case, the outcomes that meet our criteria are the 15 people who selected Art as their favorite hobby. The total possible outcomes are the number of people surveyed, 57. Therefore, the probability that someone in Jamal's survey chooses Art is

$$P(\text{event})=\frac{\text{outcomes that meet our criteria}}{\text{all possible outcomes}}$$

$$P(\text{Art})=\frac{15}{57}=\frac{5}{19}$$

■ 5. What is the probability that someone's favorite exercise was weight lifting only?



Solution:

In this case, the outcomes that meet our criteria are the 10 people whose favorite exercise was weight lifting. The total of all possible outcomes are the total number of people included in the Venn diagram:

$$2 + 5 + 8 + 4 + 2 + 3 + 10 = 34$$

So the probability that someone in the survey chose weight lifting as their favorite exercise is

$$P(\text{event}) = \frac{\text{outcomes that meet our criteria}}{\text{all possible outcomes}}$$

$$P(\text{weight lifting}) = \frac{10}{34} = \frac{5}{17}$$

6. What is the sample space for rolling two six-sided dice (the list of all possible outcomes)? What's the probability that the sum of the two dice is an odd number? Is this a theoretical or experimental probability? Why?

Solution:

We're asked to list the sample space for rolling two six-sided dice. This means we want to make a list of all the possible ways we could roll the dice (the total outcomes).

A nice way to make sure we include every combination is to make a table. We can represent one die by the top row and one die by the far-left column and then write down all of the combinations to find the sample space.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The rolls that give an odd sum are

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

There are 36 total rolls in the sample space, and 18 that give an odd sum, so the probability of rolling an odd sum is

$$P(\text{event}) = \frac{\text{outcomes that meet our criteria}}{\text{all possible outcomes}}$$

$$P(\text{odd sum}) = \frac{18}{36} = \frac{1}{2} = 0.5 = 50\,\%$$

This is an example of theoretical probability because we used the probability formula and did not perform an experiment.

THE ADDITION RULE, AND UNION VS. INTERSECTION

1. Given the probabilities $P(A)=0.3$, $P(B)=0.6$ and $P(A\cap B)=0.05$, what is $P(A\cup B)$? Are A and B mutually exclusive events? Why or why not?

Solution:

Events A and B are not mutually exclusive events because sometimes they can happen at the same time. The problem even tells us that $P(A\cap B)=0.05$, which means there's a 5% chance that both events happen at the same time. To find $P(A\cup B)$, we'll use

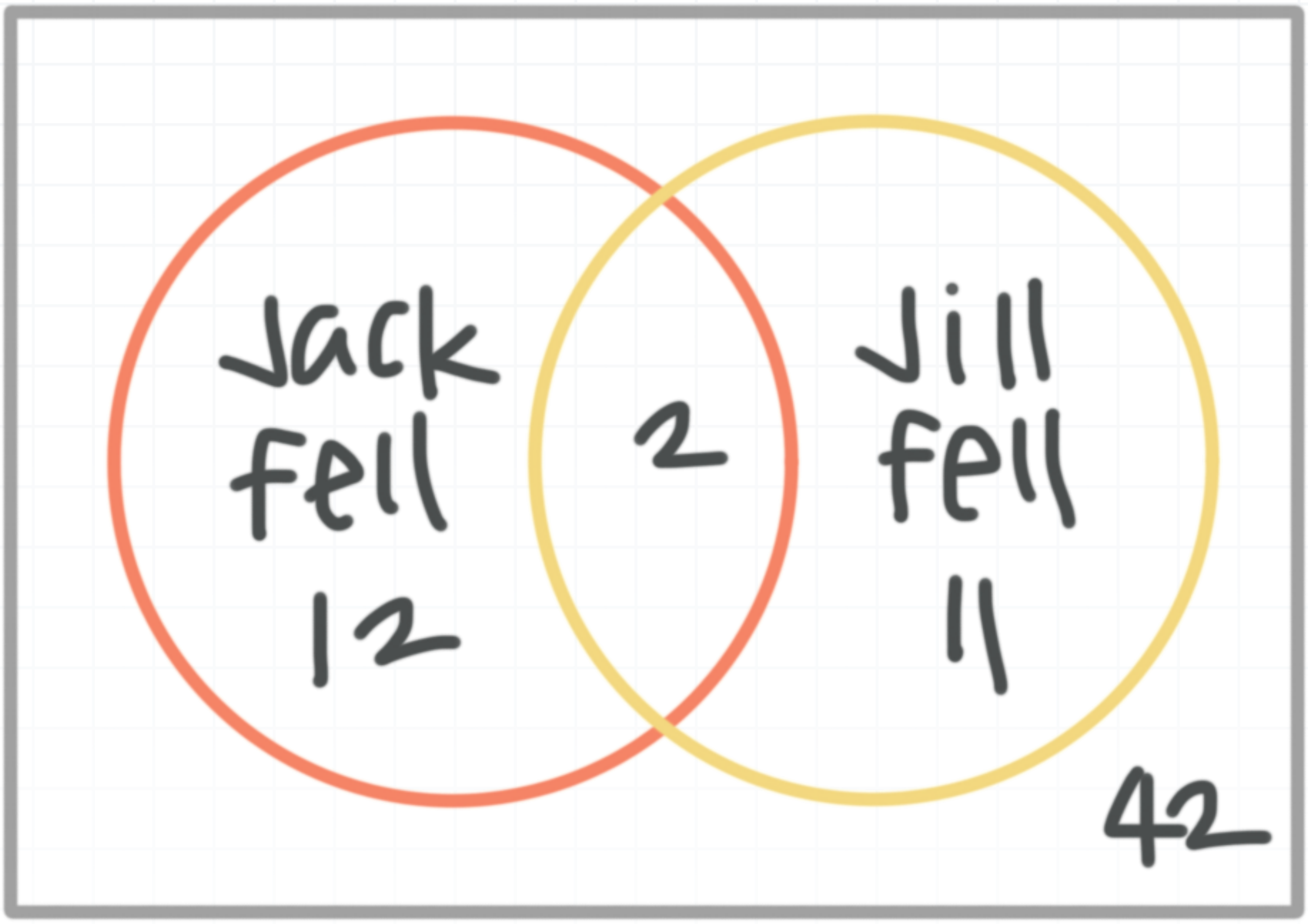
$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

and plug in $P(A)=0.3$, $P(B)=0.6$, and $P(A\cap B)=0.05$.

$$P(A\cup B)=0.3+0.6-0.05$$

$$P(A\cup B)=0.85$$

2. Jack and Jill are taking multiple trips up a hill together. The Venn diagram shows the number of times Jack and Jill fell down on their various trips up the hill. What is the probability that Jack and Jill both fell down on any particular trip, and what is the probability that only Jack fell down or only Jill fell down on any particular trip?



Solution:

From the Venn diagram, we can add the numbers from each of the four sections to see that Jack and Jill made

$$12 + 2 + 11 + 42 = 67$$

trips up the hill together. From the 2 in the center of the Venn diagram where the circles overlap, we can tell that Jack and Jill both fell down on 2 of the trips up the hill. So the probability that Jack fell down and Jill fell down is

$$P(\text{Jack fell down} \cap \text{Jill fell down}) = \frac{2}{67}$$

From the Venn diagram, we know that they took 12 trips where only Jack fell down, and 11 trips where only Jill fell down. So the probability that either only Jack fell down or only Jill fell down is

$$P(\text{only Jack fell down} \cup \text{only Jill fell down}) = \frac{12}{67} + \frac{11}{67}$$

$$P(\text{only Jack fell down} \cup \text{only Jill fell down}) = \frac{23}{67}$$

■ 3. When people buy a fish at a pet store the cashier can check off the color of the fish as mostly red, mostly orange or mostly yellow. Currently the probability of buying a red fish is 0.31, the probability of buying an orange fish is 0.23, and the probability of buying a mostly yellow fish is 0.13 (there are colors of fish other than red, orange, and yellow).

Are the events buying a mostly red fish and buying a mostly orange fish mutually exclusive? Find the probability that the purchase of a randomly selected fish is either mostly red or mostly orange.

Solution:

The events of buying a mostly red fish and buying a mostly orange fish are mutually exclusive because a single fish must be either mostly red or mostly orange. It can't be both, so there's no overlap in the two events.

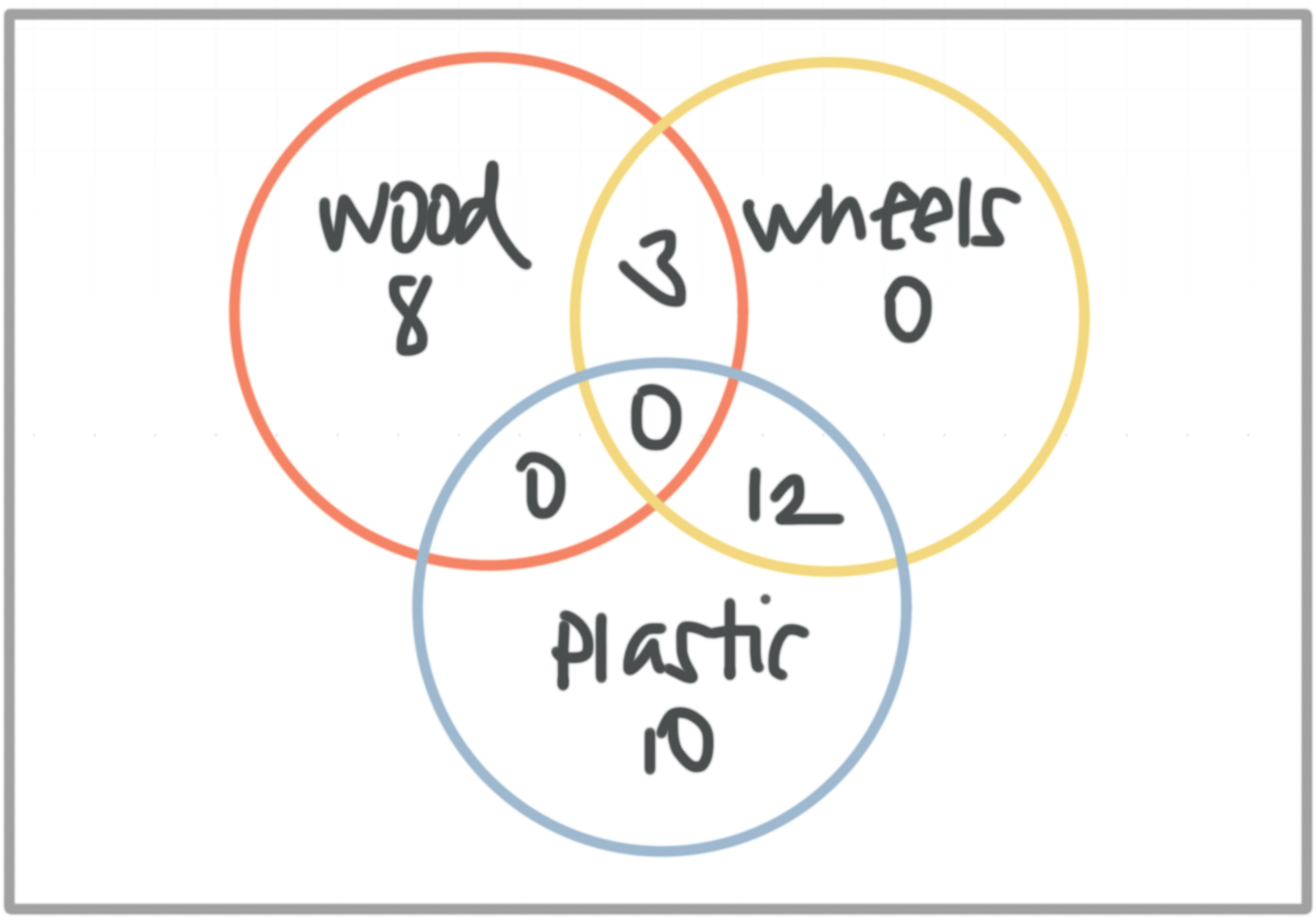
The probability that the purchase of a randomly selected fish is either mostly red or mostly orange is

$$P(\text{mostly red} \cup \text{mostly orange}) = P(\text{mostly red}) + P(\text{mostly orange})$$

$$P(\text{mostly red} \cup \text{mostly orange}) = 0.31 + 0.23$$

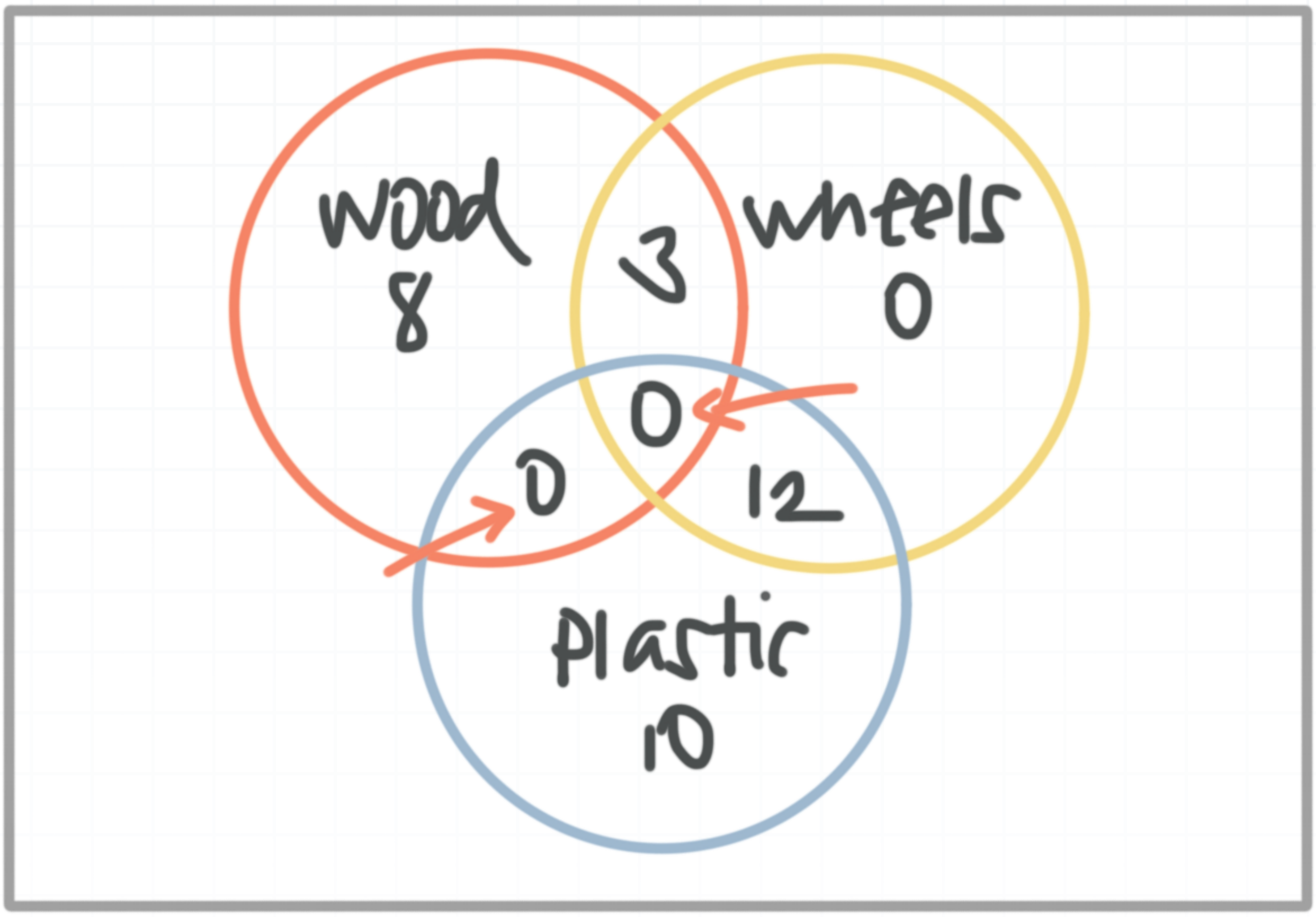
$$P(\text{mostly red} \cup \text{mostly orange}) = 0.54$$

■ 4. The Venn diagram shows Mason's toy car collection. Are the events "plastic" and "wood" mutually exclusive? What is the probability that a vehicle is made from plastic or wood? Are the events "wood" and "wheels" mutually exclusive? What is the probability that a vehicle is made from wood and has wheels?



Solution:

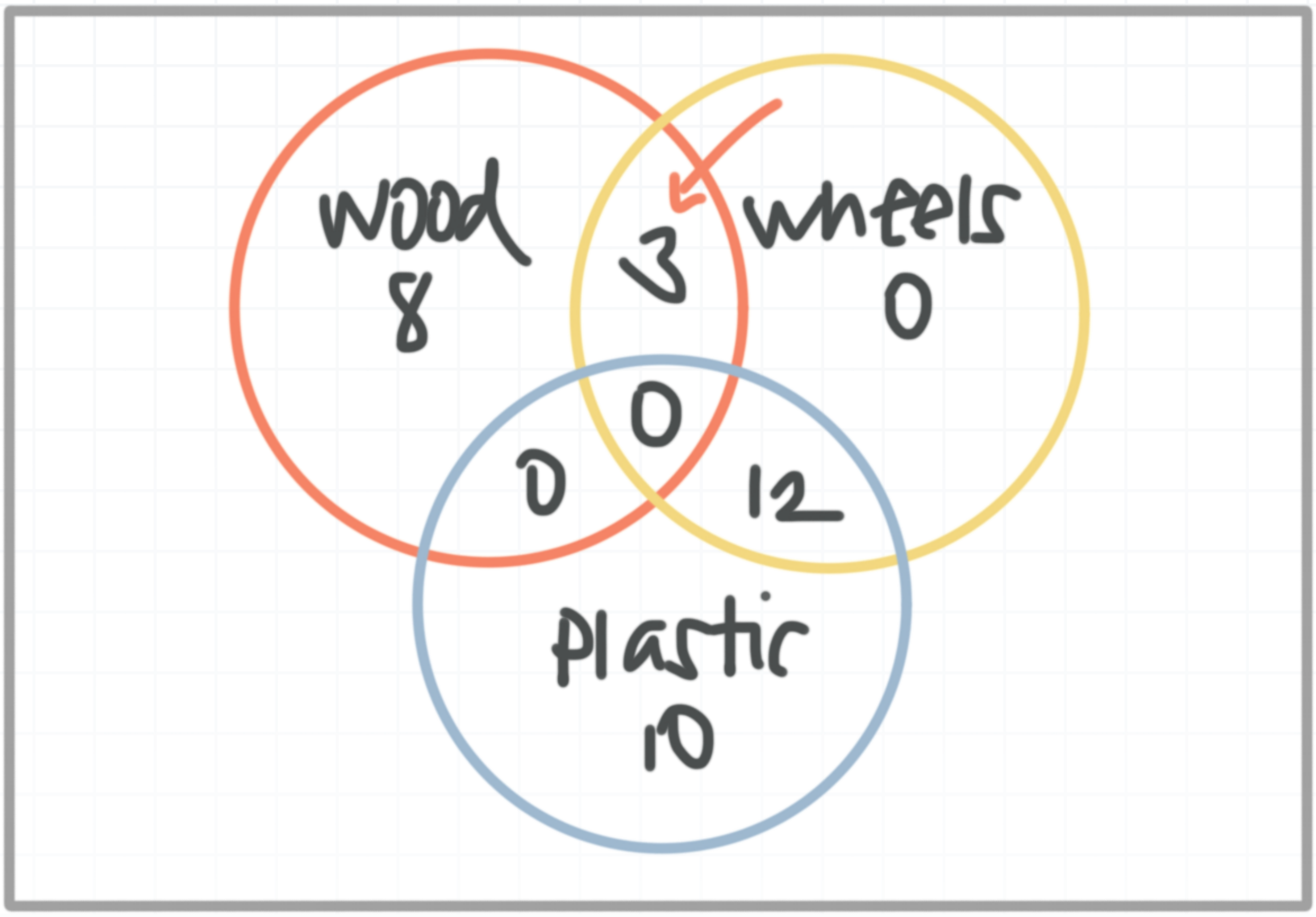
The events "plastic" and "wood" are mutually exclusive, because the intersection between them is 0.



The probability that a vehicle is made from plastic or wood is represented by $P(\text{plastic } \cup \text{ wood})$. There are $12+10+8+3=33$ total cars in the Venn diagram, and $8+3=11$ of them are made with wood, while $10+12=22$ of them with plastic. Which means that the probability that a car is made with wood or plastic is

$$P(\text{plastic} \cup \text{wood}) = \frac{11+22}{33} = \frac{33}{33} = 1 = 100\,\%$$

The events "wood" and "wheels" are not mutually exclusive because they have a non-zero number in their intersection.



The probability that a vehicle is made from wood and has wheels is represented by $P(\text{wood} \cap \text{wheels})$. Of all the vehicles in the Venn diagram, 3 are made from wood and have wheels, so

$$P(\text{wood} \cap \text{wheels}) = \frac{3}{33} = \frac{1}{11} \approx 9\,\%$$

5. Every student at a certain high school needs to choose exactly one fine arts elective. The frequency table shows the enrollment of electives for all students. Are the events "junior" and "architecture" mutually exclusive? What is the probability that a student is taking architecture and a junior? What is the probability that a student is a junior or is taking architecture?

		Extracurricular activities			
		Art	Architecture	Music	Total
Grade	Freshmen	40	25	55	**120**
	Sophomore	52	12	71	**135**
	Junior	56	45	54	**155**
	Senior	30	60	20	**110**
	Total	**178**	**142**	**200**	**520**

Solution:

The events "junior" and "architecture" are not mutually exclusive events because it's possible for a student to be both a junior and enrolled in architecture.

The probability that a student is a junior and is taking architecture is given by

$$P(\text{junior} \cap \text{architecture}) = \frac{45}{520} = \frac{9}{104}$$

		Extracurricular activities			
		Art	Architecture	Music	Total
Grade	Freshmen	40	25	55	**120**
	Sophomore	52	12	71	**135**
	Junior	56	45	54	**155**
	Senior	30	60	20	**110**
	Total	**178**	**142**	**200**	**520**

The probability that a student is a junior or is taking architecture is given by

$$P(\text{junior} \cup \text{architecture}) = P(\text{junior}) + P(\text{architecture})$$

$$-P(\text{junior} \cap \text{architecture})$$

$$P(\text{junior} \cup \text{architecture}) = \frac{155}{520} + \frac{142}{520} - \frac{45}{520}$$

$$P(\text{junior} \cup \text{architecture}) = \frac{252}{520} = \frac{63}{130}$$

		Extracurricular activities			
		Art	Architecture	Music	Total
Grade	Freshmen	40	25	55	**120**
	Sophomore	52	12	71	**135**
	Junior	56	45	54	**155**
	Senior	30	60	20	**110**
	Total	**178**	**142**	**200**	**520**

These are not mutually exclusive events, which is why we need to subtract the overlap.

6. James tosses a coin and rolls a six-sided die. What is the sample space for this situation? What is the probability the coin lands on heads and the die lands on a 2 or a 3?

Solution:

We're asked to list the sample space for flipping a coin and rolling a six-sided die. This means we want to make a list of all the possible ways we could flip the coin and roll the die (the total outcomes). A nice way to make sure we include every combination is to make a table. We can represent one die in the top row and the coin in the far-left column. Then we can write down all of the combinations to find the sample space, in a similar way that we would make a multiplication table.

	1	**2**	**3**	**4**	**5**	**6**
Heads	Heads, 1	Heads, 2	Heads, 3	Heads, 4	Heads, 5	Heads, 6
Tails	Tails, 1	Tails, 2	Tails, 3	Tails, 4	Tails, 5	Tails, 6

Next, we're interested in the probability that the coin lands on heads and the die lands on a 2 or a 3. This means we need to find $P(\text{heads} \cap 2 \text{ or } 3)$. There are only two values from the sample space that give heads and a 2 or a 3.

	1	2	3	4	5	6
Heads	Heads, 1	Heads, 2	Heads, 3	Heads, 4	Heads, 5	Heads, 6
Tails	Tails, 1	Tails, 2	Tails, 3	Tails, 4	Tails, 5	Tails, 6

And there are 12 possible outcomes. So the probability is

$$P(\text{heads} \cap \ 2 \text{ or } 3) = \frac{2}{12} = \frac{1}{6}$$

INDEPENDENT AND DEPENDENT EVENTS AND CONDITIONAL PROBABILITY

■ 1. What is the probability of getting four heads in a row when we flip a fair coin four times?

Solution:

Each coin flip is an independent event. The probability of getting a head on each flip is $1/2$ (there's one way to get a head out of two possible ways, heads or tails). Therefore,

$$P(HHHH)=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

■ 2. An old dog finds and eats $60\,\%$ of food that's dropped on the floor. A toddler wanders through the house and drops 10 pieces of cereal. What's the probability the dog finds and eats all 10 pieces?

Solution:

The dog's success rate of finding dropped food is 60%. We can calculate the probability the dog finds all the pieces by saying that the dog finding the next piece of food is independent from finding the piece before. Then,

$$P(FFFFFFFFFF) = (0.6)(0.6)(0.6)(0.6)(0.6)(0.6)(0.6)(0.6)(0.6)(0.6)$$

$$P(FFFFFFFFFF) = (0.6)^{10}$$

$$P(FFFFFFFFFF) \approx 0.006$$

There's a 0.6% chance the dog will find and eat all of the dropped cereal.

3. Amelia is choosing some pretty stones from the gift shop at the museum. The gift shop has a grab bag that contains 5 amethyst stones, 6 fluorite stones, 2 pink opals, and 7 yellow calcite stones. Amelia looks into the bag and takes out two stones, one at a time, at random. What is the probability that she gets an amethyst first and then a pink opal?

Solution:

There are a total of $5 + 6 + 2 + 7 = 20$ stones. If Amelia pulls one stone from the grab bag, the probability of taking out an amethyst is

$$P(\text{amethyst}) = \frac{5}{20} = \frac{1}{4}$$

Once an amethyst is pulled out, there are only 19 stones left in the bag, 2 of which are pink opals, so the chance of pulling a pink opal is

$$P(\text{pink opal} \mid \text{amethyst}) = \frac{2}{19}$$

We can therefore say that the probability of pulling both stones in that specific order (these are dependent events) is

$$P(\text{amethyst then pink opal}) = \frac{1}{4} \cdot \frac{2}{19} = \frac{2}{76} = \frac{1}{38}$$

4. Emily counted the shape and type of blocks that her little sister owns and organized the information into a frequency table.

		Block Shape		
		Cube	Rectangular Prism	Total
Block Color	Red	5	9	**14**
	Blue	4	10	**14**
	Total	**9**	**19**	**28**

Are events A and B dependent or independent events? Use the formula to explain the answer.

Event A is that the block is a cube.

Event B is that block is red.

Let $P(A)$ be the probability that a block drawn at random is a cube.

Let $P(B)$ be the probability that a block drawn at random is red.

Solution:

The events are independent if we can show that $P(A \text{ and } B) = P(A)P(B)$. $P(A)$ is the probability that a block drawn at random is a cube. $P(A) = 9/28$.

		Block Shape		
		Cube	Rectangular Prism	Total
Block Color	Red	5	9	**14**
	Blue	4	10	**14**
	Total	**9**	**19**	**28**

$P(B)$ is the probability that a block drawn at random is red.

$P(B) = 14/28 = 1/2$.

		Block Shape		
		Cube	Rectangular Prism	Total
Block Color	Red	5	9	**14**
	Blue	4	10	**14**
	Total	**9**	**19**	**28**

$P(A \text{ and } B)$ is the probability that the chosen block is both red and a cube.

$P(A \text{ and } B) = 5/28$.

		Block Shape		
		Cube	Rectangular Prism	Total
Block Color	Red	5	9	**14**
	Blue	4	10	**14**
	Total	**9**	**19**	**28**

Now we can check for independence by showing $P(A \text{ and } B) = P(A)P(B)$.

$$P(A \text{ and } B) = P(A)P(B)$$

$$\frac{5}{28} = \frac{9}{28} \cdot \frac{1}{2}$$

$$\frac{5}{28} = \frac{9}{56}$$

$$\frac{10}{56} = \frac{9}{56}$$

Because the values are unequal, $P(A)$ and $P(B)$ are dependent events.

5. A bag has 4 cinnamon candies, 6 peppermint candies, and 12 cherry candies. Sasha draws 3 candies at random from the bag one at a time without replacement. Does the situation describe dependent or independent events? What is the probability of drawing a cinnamon first, then a cherry, and then a peppermint?

Solution:

These events are dependent events, because removing a candy from the bag changes what's inside and effects the probability of subsequent pulls.

We want to find the probability of drawing a cinnamon first, then a cherry, and then a peppermint last. There are $4+6+12=22$ total candies in the bag. Let's look at the probability of getting a cinnamon first. Since there are 4 cinnamon candies, the probability of getting a cinnamon is

$$P(\text{cinnamon})=\frac{4}{22}=\frac{2}{11}$$

Now there are 21 total candies remaining, 12 of which are cherry, so the probability of getting cherry next is

$$P(\text{cherry})=\frac{12}{21}=\frac{4}{7}$$

Now there are 20 total candies remaining, 6 of which are peppermint, so the probability of getting peppermint next is

$$P(\text{peppermint})=\frac{6}{20}=\frac{3}{10}$$

Therefore, the probability of drawing these three flavors in this particular order is

$$P(\text{Ci},\text{Ch},\text{Pe})=\frac{2}{11}\cdot\frac{4}{7}\cdot\frac{3}{10}$$

$$P(\text{Ci},\text{Ch},\text{Pe})=\frac{24}{770}$$

$$P(\text{Ci, Ch, Pe}) = \frac{12}{385}$$

■ 6. Nyla has 12 stuffed animals, 7 of which are elephants (4 of the elephants play music and light up) and 5 of which are bears (2 of the bears play music and light up). Her mother randomly selects an animal to bring with them on vacation. Let A be the event that she selects an elephant and B be the event that she selects an animal that plays music and lights up.

Find $P(A)$, $P(B)$, $P(A\,|\,B)$, and $P(B\,|\,A)$. State if events A and B are dependent or independent events, then find $P(A$ and $B)$.

Solution:

There are $7+5=12$ total stuffed animals. $P(A)$ is the probability of selecting an elephant, and there are 7 elephants.

$$P(A) = \frac{7}{12}$$

$P(B)$ is the probability of selecting an animal that plays music and lights up. There are $4+2=6$ animals that play music and light up.

$$P(B) = \frac{6}{12} = \frac{1}{2}$$

$P(A\,|\,B)$ is the probability of selecting an elephant, given that the animal plays music and lights up. There are 4 elephants that play music and light up out of $4+2=6$ total animals that play music and light up.

$$P(A \mid B) = \frac{4}{6} = \frac{2}{3}$$

$P(B \mid A)$ is the probability of picking a toy that plays music and lights up given that the toy is an elephant. There are 4 elephants that play music and light up out of 7 total elephants.

$$P(B \mid A) = \frac{4}{7}$$

Because $P(A) \neq P(A \mid B)$ and $P(B) \neq P(B \mid A)$, A and B are dependent events. $P(A \text{ and } B)$ is the probability of choosing an elephant that plays music and lights up. We know the events are dependent events, so

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

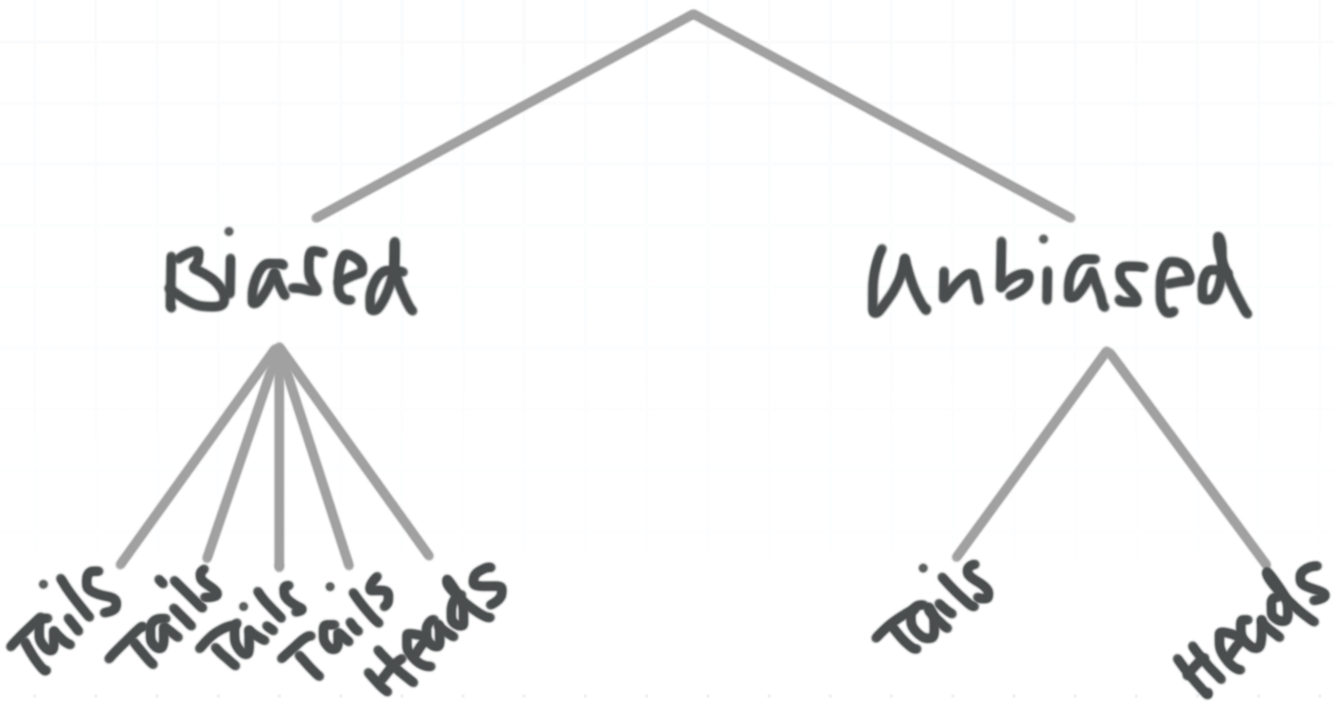
$$P(A \text{ and } B) = \frac{7}{12} \cdot \frac{4}{7}$$

$$P(A \text{ and } B) = \frac{28}{84} = \frac{4}{12}$$

$$P(A \text{ and } B) = \frac{1}{3}$$

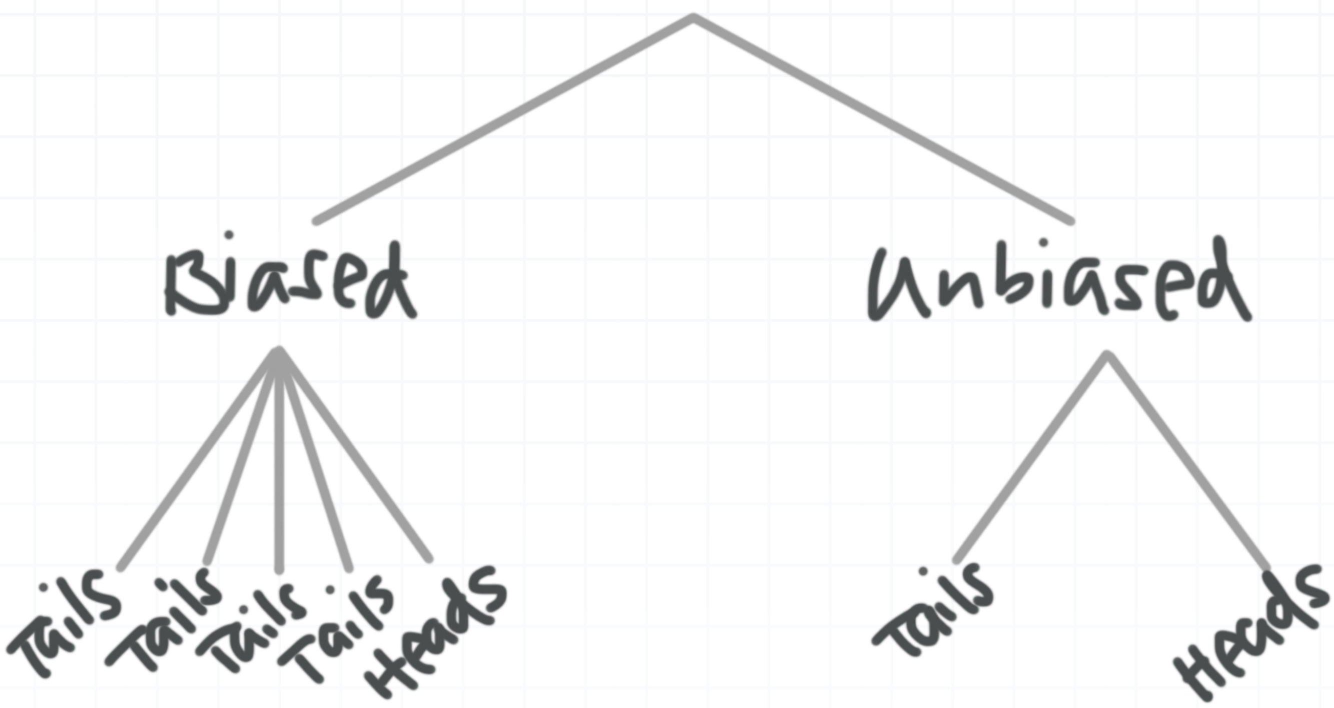
BAYES' THEOREM

1. We have two coins. One is fair and the other one is weighted to land on tails $4/5$ of the time. Without knowing which coin we're choosing, we pick one at random, toss the coin and get tails. What is the probability we flipped the biased coin? Complete the tree diagram to answer the question.

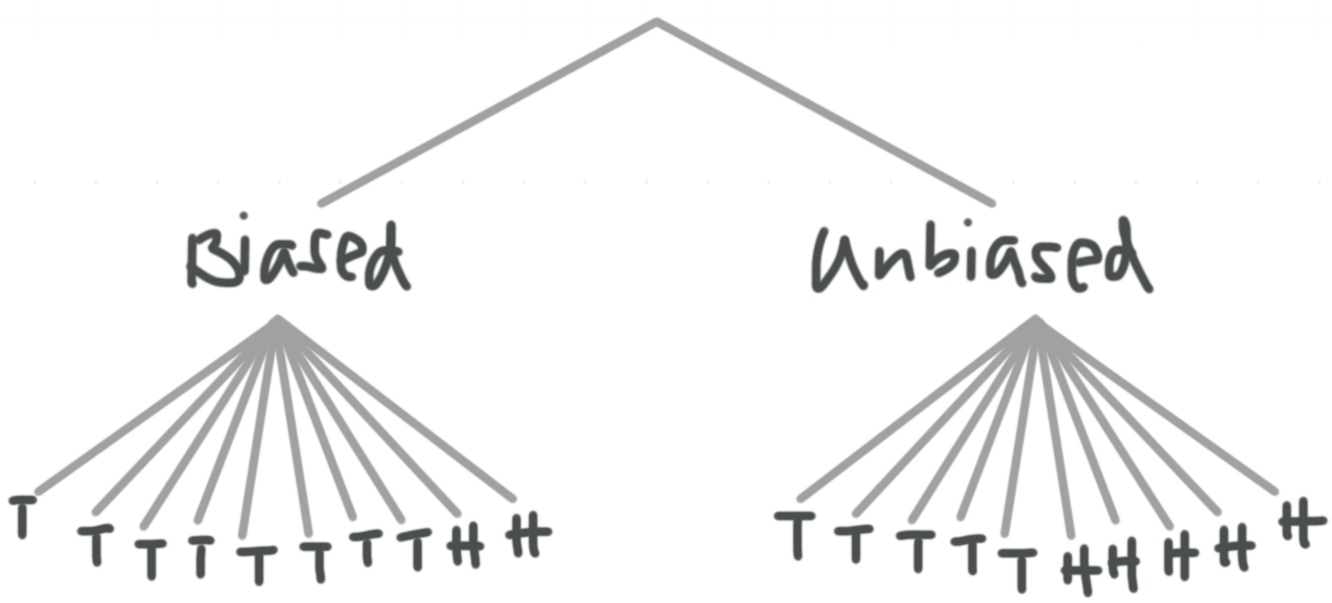


Solution:

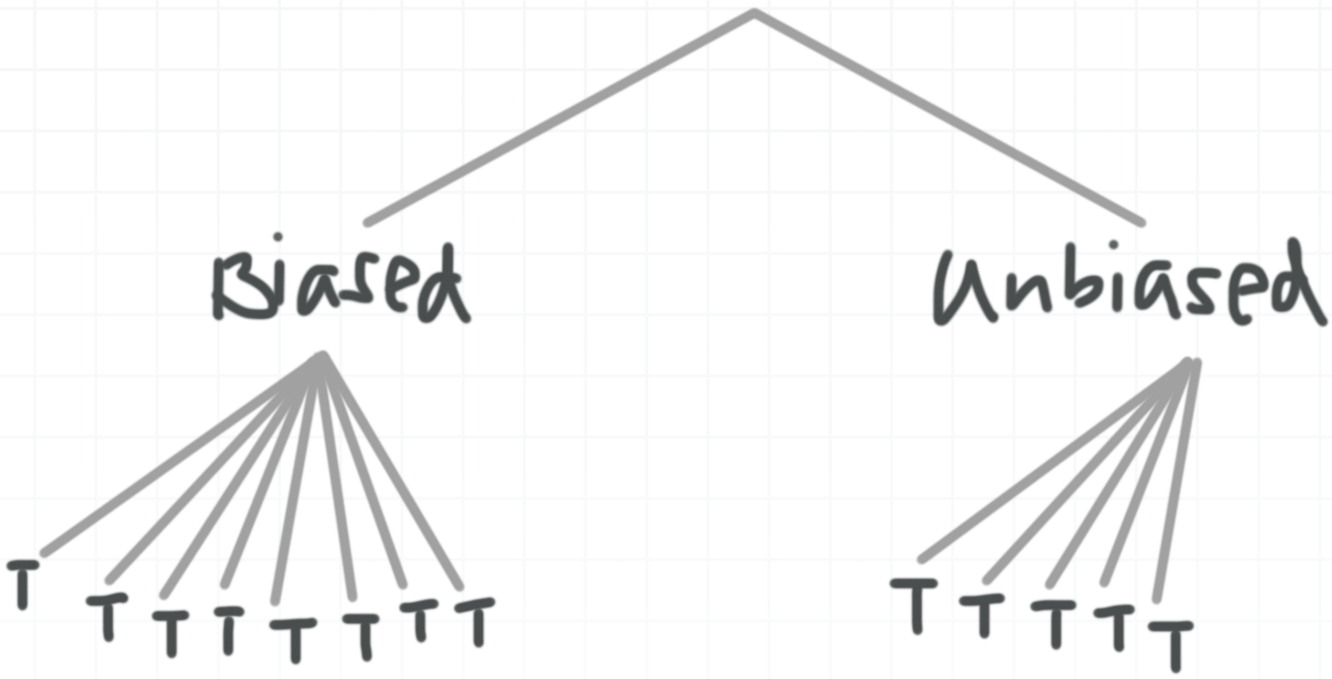
We're looking for the probability that the coin is biased given that we already flipped a tails, so we're looking for $P(\text{biased}\,|\,\text{tails})$.



The next step for the tree diagram is to make sure the branches are balanced. We use equivalent fractions to do this. For the biased side we know that we get tails 4 out of 5 times. This is the same as 8 out of 10 times. For the unbiased coin, we get tails 1 out of 2 times, which is the same as 5 out of 10 times.



We're only interested in tails, so now we need to trim the tree.



Now we're looking for the probability that we tossed the biased coin. 8 of the tails came from the biased coin and 5 did not.

$$P(\text{biased}) = \frac{8}{8+5} = \frac{8}{13}$$

The probability we tossed the biased coin, knowing that it landed on tails, is $8/13$.

■ 2. We have two dice. One is fair and the other is biased. The biased die is weighted to land on 6 every 1 out of 36 rolls. There's an equal probability for all of the other five faces on the biased die. Without knowing which one we're choosing, we pick one of the dice, roll it, and get a 6.

Calculate the following and use them to answer the question: What is the probability that we rolled the fair die?

$$P(6\,|\,\text{fair})$$

$$P(\text{fair})$$

$$P(6)$$

Solution:

$P(6\,|\,\text{fair})$ is the probability of rolling a 6, given that the die was fair. Since all outcomes are equally likely on the fair die, we have a 1 in 6 chance of rolling a 6.

$$P(6\,|\,\text{fair}) = \frac{1}{6}$$

$P(\text{fair})$ is the probability of choosing the fair die. Each of the 2 dice has an equally likely chance of being chosen, so the probability of choosing the fair die is 1 in 2.

$$P(\text{fair}) = \frac{1}{2}$$

$P(6)$ is the probability of rolling a 6. This is the probability of choosing the biased die and rolling a 6 or the probability of choosing the fair die and rolling a 6. Let's find the probability that the die is fair and we roll a 6.

$$P(\text{fair and } 6) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Now let's find the probability the die is biased and we roll a 6.

$$P(\text{biased and } 6) = \frac{1}{2} \cdot \frac{1}{36} = \frac{1}{72}$$

Therefore, the probability of rolling a 6 is

$$P(6) = \frac{1}{12} + \frac{1}{72}$$

$$P(6) = \frac{6}{72} + \frac{1}{72}$$

$$P(6) = \frac{7}{72}$$

Now we want to answer the question: "What is the probability that we rolled the fair die?" We're looking for $P(\text{fair} \mid 6)$, and we have everything we need to use Bayes' Theorem.

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$$P(\text{fair} \mid 6) = \frac{P(6 \mid \text{fair}) \cdot P(\text{fair})}{P(6)}$$

$$P(\text{fair} \mid 6) = \frac{\frac{1}{6} \cdot \frac{1}{2}}{\frac{7}{72}}$$

$$P(\text{fair} \mid 6) = \frac{\frac{1}{12}}{\frac{7}{72}} = \frac{1}{12} \cdot \frac{72}{7} = \frac{72}{84} = \frac{6}{7}$$

The probability we rolled the fair die given that we rolled a 6 is $6/7$.

3. Charlie knows that, at his school,

$$P(\text{senior}) = 0.40$$

$$P(\text{playing soccer}) = 0.15$$

$$P(\text{soccer and senior}) = 0.05$$

Solve for the probability $P(\text{senior} \mid \text{soccer})$, then state whether or not Bayes' Theorem can be used to solve the problem.

Solution:

Let's look to see if we can use Bayes' Theorem to find the probability. First let's take Bayes' Theorem and write it in terms of our problem. We want to solve for the probability $P(\text{senior} \mid \text{soccer})$, so

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$$P(\text{senior} \mid \text{soccer}) = \frac{P(\text{soccer} \mid \text{senior}) \cdot P(\text{senior})}{P(\text{soccer})}$$

Remember that the multiplication rule says that $P(B \text{ and } A) = P(B \mid A) \cdot P(A)$. So we can also say that $P(\text{soccer and senior}) = P(\text{soccer} \mid \text{senior}) \cdot P(\text{senior})$. Then we can use Bayes' Theorem.

$$P(\text{senior} \mid \text{soccer}) = \frac{P(\text{soccer and senior})}{P(\text{soccer})}$$

Now we can use the information we've been given to solve the problem.

$$P(\text{soccer and senior}) = 0.05$$

$$P(\text{playing soccer}) = 0.15$$

$$P(\text{senior}\,|\,\text{soccer}) = \frac{0.05}{0.15} = \frac{1}{3} \approx 33\,\%$$

We could have also used a Venn diagram, instead of Bayes' Theorem, to solve this problem.

4. We have two coins. One is fair and the other is weighted to land on tails $3/4$ of the time. Without knowing which coin we're choosing, we pick one at random, toss the coin, and get tails. What's the probability we flipped the biased coin?

Solution:

We're looking for the probability that the coin is biased, given that we already flipped a tails, so we're looking for $P(\text{biased}\,|\,\text{tails})$. We can solve this problem using Bayes' Theorem, or by creating a tree diagram. Let's use Bayes' Theorem.

$$P(A\,|\,B) = \frac{P(B\,|\,A) \cdot P(A)}{P(B)}$$

That means that to use Bayes' Theorem, we have $P(A) = P(\text{biased})$ and $P(B) = P(\text{tails})$. Then we need to find these values to plug into the formula:

$$P(\text{tails}\,|\,\text{biased})$$

$$P(\text{biased})$$

$$P(\text{tails})$$

We know from the problem that $P(\text{tails}\,|\,\text{biased})=3/4$. There are two coins, and it's equally likely that we choose either one, so $P(\text{biased})=1/2$. The probability of flipping a tails is the probability of flipping the biased coin and landing on tails or the probability of flipping the unbiased coin and landing on tails. Let's find the probability the coin is biased and it lands on tails.

$$P(\text{biased and tails})=\frac{1}{2}\cdot\frac{3}{4}=\frac{3}{8}$$

Now let's find the probability the coin is fair and lands on tails.

$$P(\text{fair and tails})=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

So the probability of flipping a tails is

$$P(\text{tails})=\frac{3}{8}+\frac{1}{4}$$

$$P(\text{tails})=\frac{3}{8}+\frac{2}{8}$$

$$P(\text{tails})=\frac{5}{8}$$

Putting these values into Bayes' Theorem, we get

$$P(A\,|\,B)=\frac{P(B\,|\,A)\cdot P(A)}{P(B)}$$

$$P(A\,|\,B)=\frac{\frac{3}{4}\cdot\frac{1}{2}}{\frac{5}{8}}=\frac{\frac{3}{8}}{\frac{5}{8}}=\frac{3}{8}\cdot\frac{8}{5}=\frac{3}{5}$$

The probability that we flipped the biased coin is $3/5$.

5. A company is giving a drug test to all of its employees. The test is $90\,\%$ accurate, given that a person is using drugs, and $85\,\%$ accurate, given that the person is not using drugs. It's also known that $10\,\%$ of the general population of employees uses drugs. What is the probability that an employee was actually using drugs, given that they tested positive?

Let P represent a positive test for an individual.

Let N represent a negative test for an individual.

Let D represent the event that an employee is a drug user.

Solution:

We're asked to determine the probability that an employee was using drugs, given that they tested positive, or $P(D\,|\,P)$. Let's use Bayes' Theorem.

$$P(A\,|\,B) = \frac{P(B\,|\,A) \cdot P(A)}{P(B)}$$

$$P(D\,|\,P) = \frac{P(P\,|\,D) \cdot P(D)}{P(P)}$$

$P(P\,|\,D)$ is the probability that an employee tests positive, given that they are a drug user. From the problem, we know that $P(P\,|\,D) = 90\,\%$. $P(D)$ is the probability that an employee is a drug user, and from the problem, we know that $P(D) = 10\,\%$. $P(P)$ is the probability of testing positive, regardless of whether the result was accurate or inaccurate.

Let's find the probability the an employee tests positive, and the result is accurate, because they're a drug user. We know $10\,\%$ of employees are drug users, and we know that $90\,\%$ of drug users will test positive.

$$(0.10)(0.90) = 0.09$$

Now let's calculate the probability that an employee tested positive, but wasn't a drug user. The problem tells us that the test is $85\,\%$ accurate for non drug users, which means that $15\,\%$ of those who aren't using drugs will still test positive. Since $10\,\%$ of the employees are drug users, $90\,\%$ are not. So the probability of a false positive from a non drug user is

$$(0.90)(0.15) = 0.135$$

Now we can calculate $P(D\,|\,P)$.

$$P(D\,|\,P) = \frac{(0.90)(0.10)}{0.09 + 0.135}$$

$$P(D\,|\,P) = \frac{0.09}{0.225}$$

$$P(D\,|\,P) = 40\,\%$$

This means that, for an employee who tests positive, there's a $40\,\%$ chance that employee is actually using drugs.

6. Two factories A and B produce heaters for car seats. A customer received a defective car seat heater and the manager at factory B would like to know if it came from her factory. Use the table below to determine the probability that the heater came from factory B.

Factory	% of production	Probability of defective heaters
A	0.55	0.020 P(DIA)
B	0.45	0.014 P(DIB)

Solution:

The manager wants to know the probability the heater came from her factory, given it was defective. So she's looking for $P(B|D)$. We can use Bayes' Theorem to find the probability. Substituting in with the given events, we get

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$P(B|D) = \frac{P(D|B) \cdot P(B)}{P(D)}$$

Let's find $P(D|B)$, $P(B)$, and $P(D)$. $P(D|B)$ is the probability the heater is defective, given it came from factory B. We have this probability in the table as $P(D|B) = 0.014$. $P(B)$ is the probability the heater came from factory B. We also have this in the table as $P(B) = 0.45$. Next, we need $P(D)$, which is the probability the heater is defective. This is made of the probability the heater comes from factory A and is defective and the probability it came from factory B and is defective. So we need to find $P(A \cap D) + P(B \cap D)$.

First let's find the probability that the heater comes from factory A and is defective.

Factory	% of production	Probability of defective heaters
A	0.55	0.020 P(D|A)

$$P(A \cap D) = P(D \mid A) \cdot P(A)$$

$$P(A \cap D) = (0.55)(0.020)$$

$$P(A \cap D) = 0.011$$

Next let's find the probability the heater comes from factory B and is defective.

Factory	% of production	Probability of defective heaters
B	0.45	0.014 P(D|B)

$$P(B \cap D) = P(D \mid B) \cdot P(B)$$

$$P(B \cap D) = (0.45)(0.014)$$

$$P(B \cap D) = 0.0063$$

Now we can find $P(D)$.

$$P(D) = P(A \cap D) + P(B \cap D)$$

$$P(D) = 0.011 + 0.0063$$

$$P(D) = 0.0173$$

Putting these values into Bayes' Theorem, we get

$$P(B \mid D) = \frac{P(D \mid B) \cdot P(B)}{P(D)}$$

$$P(B \mid D) = \frac{(0.014) \cdot (0.45)}{0.0173}$$

$$P(B \mid D) = \frac{0.0063}{0.0173}$$

$$P(B \mid D) \approx 36\,\%$$

There is about a $36\,\%$ chance the defective heater came from factory B.