Hypothesis testing for the difference of proportions

Now that we know how to find a confidence interval around the difference of proportions, let's look at how to conduct a hypothesis test with the difference of proportions, when we want to use the difference of sample proportions to make an inference about the difference of population proportions.

Building hypothesis statements

The null and alternative hypotheses will always be formulated in terms of the difference between the two population proportions, $p_1 - p_2$, and we can have three different scenarios.

In a two-tailed test, the null hypothesis will state that the proportions don't differ, whereas the alternative hypothesis states that there *is* a difference between proportions. So we write the hypothesis statements for a two-tailed test as

$$H_0 : p_1 - p_2 = 0$$

$$H_a : p_1 - p_2 \neq 0$$

or

$$H_0 : p_1 = p_2$$

$$H_a : p_1 \neq p_2$$

In an upper-tailed test, the alternative hypothesis states that the difference in proportions is positive, so we write

$$H_0: p_1 - p_2 \le 0$$

$$H_a: p_1 - p_2 > 0$$

or

$$H_0: p_1 \le p_2$$

$$H_a: p_1 > p_2$$

In a lower-tailed test, the alternative hypothesis states that the difference in proportions is negative, so we write

$$H_0: p_1 - p_2 \ge 0$$

$$H_a: p_1 - p_2 < 0$$

or

$$H_0: p_1 \ge p_2$$

$$H_a: p_1 < p_2$$

Calculating the test statistic

As long as we take independent random samples from each population, and $n_1\hat{p}_1 \ge 5$, $n_1(1-\hat{p}_1) \ge 5$, $n_2\hat{p}_2 \ge 5$, and $n_2(1-\hat{p}_2) \ge 5$, then the test statistic formula we'll use is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $\hat{p}_1$ and $\hat{p}_2$ are the sample proportions, p_1 and p_2 are the population proportions, n_1 and n_2 are the sample sizes, and $\hat{p}$ is the proportion of the combined sample, given by

$$\hat{p} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$$

which we can also write as

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

where x_1 and x_2 are the number of "successes" in each sample. We say that the null hypothesis always states a zero difference between population proportions, such that $p_1 - p_2 = 0$, so the test statistic formula actually simplifies to

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Let's rework the same example from the previous section.

Example

A team of scientists claims that a new cholesterol lowering drug is more affective than an older version. The team takes two random samples of 250

people, and for 3 months administer the new drug to the first group and the old drug to the second group. 120 people in the first group and and 107 people in the second group show decreased cholesterol levels. Can the team conclude at a 99% confidence level that the new drug is more affective than the old drug at lowering cholesterol?

If p_1 is the proportion of population 1 (the population that takes the new drug) whose cholesterol decreases, and p_2 is the proportion of population 2 (the population that takes the old drug) whose cholesterol decreases, then the null and alternative hypotheses are

$$H_0: p_1 - p_2 \leq 0$$

$$H_a: p_1 - p_2 > 0$$

The pooled proportion is

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$\hat{p} = \frac{120 + 107}{250 + 250}$$

$$\hat{p} = \frac{227}{500}$$

$$\hat{p} = 0.454$$

and the sample proportions are

$$\hat{p}_1 = \frac{120}{250} = 0.480$$

$$\hat{p}_2 = \frac{107}{250} = 0.428$$

So the test statistic is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z = \frac{0.480 - 0.428}{\sqrt{0.454(1-0.454)\left(\frac{1}{250} + \frac{1}{250}\right)}}$$

$$z = \frac{0.052}{\sqrt{0.454(0.546)\left(\frac{1}{125}\right)}}$$

$$z = \frac{0.052}{\sqrt{\frac{0.247884}{125}}}$$

$$z = 0.052\sqrt{\frac{125}{0.247884}}$$

$$z \approx 1.17$$

Now we need to determine the critical z-value. Our level of significance is $\alpha = 0.01$, and for a right-tailed test the corresponding z-value is $z = 2.33$.