Random Numbers

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CONTENTS

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
   Solution: Download the following files and execute the C program.

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/1.1.c
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/coeffs.
    h
gcc 1.1.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \le x\right) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/1.2.py
python3 1.2.py
```

1.3 Find a theoretical expression for $F_U(x)$.
   Solution:
   The PDF of U is

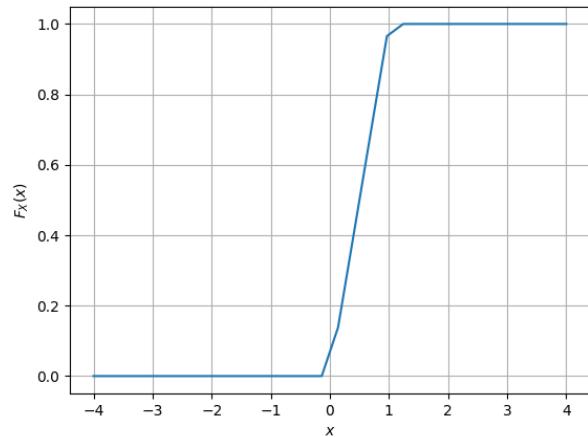$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

Fig. 1.2: The CDF of U

The CDF of U is given by

$$F_U(x) = \Pr\left(U \le x\right) = \int_{-\infty}^{x} p_U(x)\, dx \quad (1.3)$$

case.1: if $x < 0$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\, dx \quad (1.4)$$

$$= \int_{-\infty}^{x} 0\, dx \quad (1.5)$$

$$= 0 \quad (1.6)$$

case.2: if $x \in [0, 1]$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\, dx \quad (1.7)$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} 1\, dx \quad (1.8)$$

$$= x \quad (1.9)$$

case.3: if $x > 1$

$$F_U(x) = \int_{-\infty}^{x} p_U(x) \, dx \qquad (1.10)$$

$$= \int_{-\infty}^{0} 0 \, dx + \int_{0}^{1} 1 \, dx + \int_{1}^{x} 0 \, dx \qquad (1.11)$$

$$= 1 \qquad (1.12)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0, 1] \\ 1 & x > 1 \end{cases} \qquad (1.13)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.14)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \qquad (1.15)$$

Write a C program to find the mean and variance of U.

Solution:
Download and execute the following C program

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/1.4.c
gcc 1.4.c
./a.out
```

Output:

$$mean = 0.500031 \qquad (1.16)$$

$$variance = 0.083247 \qquad (1.17)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k \, dF_U(x) \qquad (1.18)$$

Solution:

$$E[U] = \int_{-\infty}^{\infty} x \, dF_U(x) \qquad (1.19)$$

$$\qquad (1.20)$$

From (1.13) $\qquad (1.21)$

$$dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & x \in [0, 1] \\ 0 & x > 1 \end{cases} \qquad (1.22)$$

$$E[U] = 0 + \int_{0}^{1} x \, dF_U(x) + 0 \qquad (1.23)$$

$$= 0 + \int_{0}^{1} x \, dx + 0 \qquad (1.24)$$

$$= \frac{1}{2} = 0.5 \qquad (1.25)$$

$$E[U^2] = 0 + \int_{0}^{1} x^2 \, dF_U(x) + 0 \qquad (1.26)$$

$$= 0 + \int_{0}^{1} x^2 \, dx + 0 \qquad (1.27)$$

$$= \frac{1}{3} \qquad (1.28)$$

$$var = E[U^2] - (E[U])^2 \qquad (1.29)$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \qquad (1.30)$$

$$var = \frac{1}{12} = 0.083333 \qquad (1.31)$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.1.c
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/coeffs.
    h
gcc 2.1.c
./a.out
```
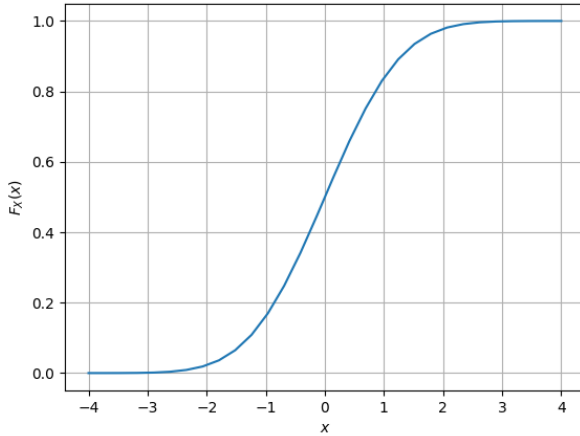
Fig. 2.2: The CDF of X
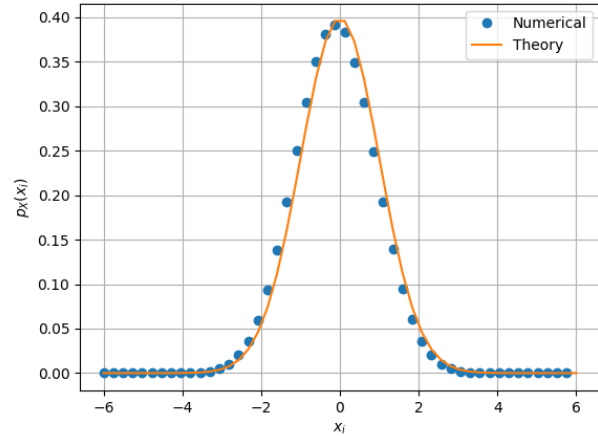
Fig. 2.3: The PDF of X

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The following code plots Fig. 2.2

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.2.
    py
python3 2.2.py
```

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.3.py
python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:
Download and execute the following C program

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.4.c
gcc 2.4.c
./a.out
```

Ouput of the program is:

$$mean = 0.000685 \tag{2.3}$$
$$variance = 1.000025 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \, dx \tag{2.6}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) \, dx \tag{2.7}$$
$$= 0 \tag{2.8}$$

$\because x \exp\left(-\frac{x^2}{2}\right)$ is an odd function.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) \, dx \tag{2.9}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) \, dx \tag{2.10}$$

Integrating by parts

$$E[X^2]$$
$$= \frac{1}{\sqrt{2\pi}} x \int x \exp\left(\frac{-x^2}{2}\right) - \int \frac{d(x)}{dx} \int x \exp\left(\frac{-x^2}{2}\right) \tag{2.11}$$
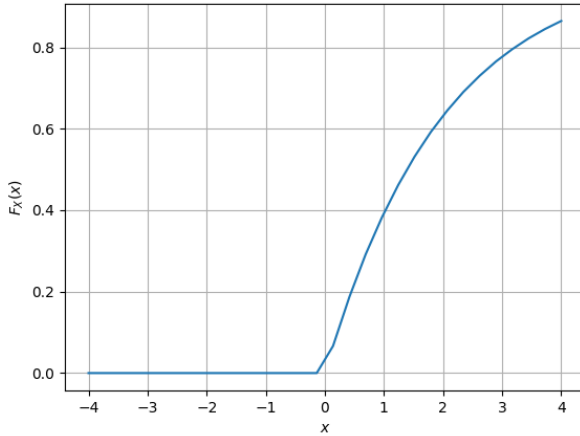
Fig. 3.1: The CDF of V

$$E\left[X^2\right] = \frac{\left(\left(-x\exp\left(-\frac{x^2}{2}\right)\right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2}\right)\right)}{\sqrt{2\pi}}$$

(2.12)

$$= \frac{0 + \sqrt{2\pi}}{\sqrt{2\pi}}$$

(2.13)

$$= 1$$

(2.14)

$$var(X) = E\left[X^2\right] - (E[X])^2$$

(2.15)

$$= 1 - 0 = 1$$

(2.16)

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U)$$

(3.1)

and plot its CDF.
Solution:
To generate samples of V,download and run the following code.

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/v.py
python3 v.py
```

The following code plots the CDF of V Fig.3.1 using the samples generated above.

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/3.1
    _cdf.py
python3 3.1_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:
The CDF of V is given by

$$F_V(x) = \Pr(V \le x)$$

(3.2)

$$= \Pr(-2\ln(1 - U) \le x)$$

(3.3)

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right)$$

(3.4)

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right)$$

(3.5)

$$= F_V\left(1 - \exp\left(-\frac{x}{2}\right)\right)$$

(3.6)

For

$$x \le 0 \qquad 1 - \exp\left(-\frac{x}{2}\right) \le 0$$

(3.7)

$$x > 0 \qquad 0 < 1 - \exp\left(-\frac{x}{2}\right) < 1$$

(3.8)

$$\implies F_V(x) = \begin{cases} 0 & x \le 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x > 0 \end{cases}$$

(3.9)