Random Numbers

Saanvi Amrutha
AI21BTECH11026

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/1.1.c
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/coeffs.
    h
gcc 1.1.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/1.2.py
python3 1.2.py
```

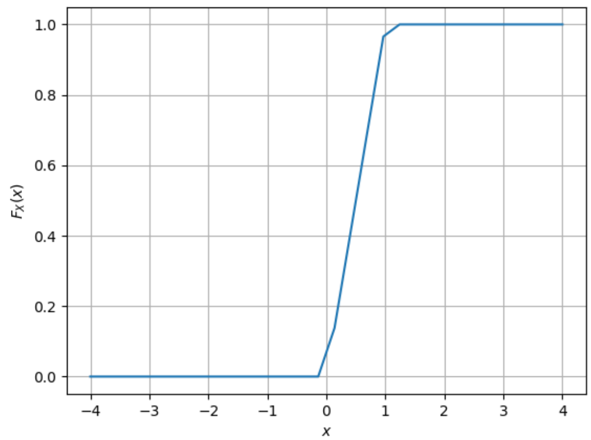

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:
The PDF of U is

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.3}$$

case.1: if $x < 0$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.4}$$
$$= \int_{-\infty}^{x} 0\, dx \tag{1.5}$$
$$= 0 \tag{1.6}$$

case.2: if $x \in [0,1]$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.7}$$
$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} 1\, dx \tag{1.8}$$
$$= x \tag{1.9}$$

case.3: if $x > 1$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\ \mathrm{d}x \tag{1.10}$$

$$= \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{1} 1\ \mathrm{d}x + \int_{1}^{x} 0\ \mathrm{d}x \tag{1.11}$$

$$= 1 \tag{1.12}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases} \tag{1.13}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.14}$$

and its variance as

$$\mathrm{var}[U] = E[U - E[U]]^2 \tag{1.15}$$

Write a C program to find the mean and variance of U.
Solution:
Download and execute the following C program

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/1.4.c
gcc 1.4.c
./a.out
```

Output:

$$mean = 0.500031 \tag{1.16}$$

$$variance = 0.083247 \tag{1.17}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k\ \mathrm{d}F_U(x) \tag{1.18}$$

Solution:

$$E[U] = \int_{-\infty}^{\infty} x\ \mathrm{d}F_U(x) \tag{1.19}$$

$$\tag{1.20}$$

From (1.13) (1.21)

$$\mathrm{d}F_U(x) = \begin{cases} 0 & x < 0 \\ \mathrm{d}x & x \in [0,1] \\ 0 & x > 1 \end{cases} \tag{1.22}$$

$$E[U] = 0 + \int_{0}^{1} x\ \mathrm{d}F_U(x) + 0 \tag{1.23}$$

$$= 0 + \int_{0}^{1} x\ \mathrm{d}x + 0 \tag{1.24}$$

$$= \frac{1}{2} = 0.5 \tag{1.25}$$

$$E\left[U^2\right] = 0 + \int_{0}^{1} x^2\ \mathrm{d}F_U(x) + 0 \tag{1.26}$$

$$= 0 + \int_{0}^{1} x^2\ \mathrm{d}x + 0 \tag{1.27}$$

$$= \frac{1}{3} \tag{1.28}$$

$$var = E\left[U^2\right] - (E[U])^2 \tag{1.29}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.30}$$

$$var = \frac{1}{12} = 0.083333 \tag{1.31}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.1.c
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/coeffs.
    h
gcc 2.1.c
./a.out
```

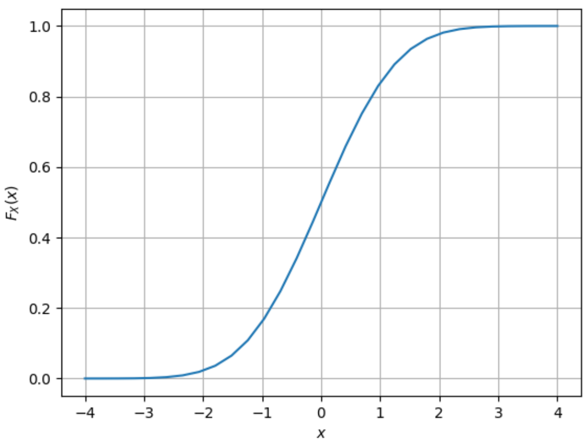

Fig. 2.2: The CDF of X

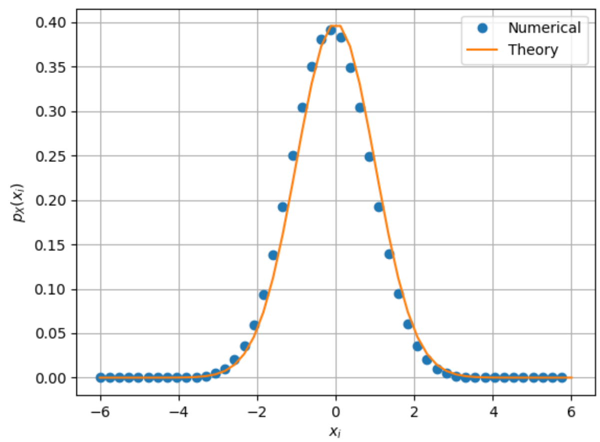

Fig. 2.3: The PDF of X

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The following code plots Fig. 2.2

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.2.
    py
python3 2.2.py
```

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.3.py
python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:
Download and execute the following C program

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/2.4.c
gcc 2.4.c
./a.out
```

Ouput of the program is:

$$mean = 0.000685 \tag{2.3}$$
$$variance = 1.000025 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution:

$$E\left[X\right] = \int_{-\infty}^{\infty} x p_X\left(x\right)\, \mathrm{d}x \tag{2.6}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right)\, \mathrm{d}x \tag{2.7}$$
$$= 0 \tag{2.8}$$

$\because x \exp\left(-\frac{x^2}{2}\right)$is an odd function.

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X\left(x\right)\, \mathrm{d}x \tag{2.9}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right)\, \mathrm{d}x \tag{2.10}$$

Integrating by parts

$$E\left[X^2\right]$$
$$= \frac{1}{\sqrt{2\pi}} x \int x \exp\left(\frac{-x^2}{2}\right) - \int \frac{d\left(x\right)}{\mathrm{d}x} \int x \exp\left(-\frac{-x^2}{2}\right) \tag{2.11}$$

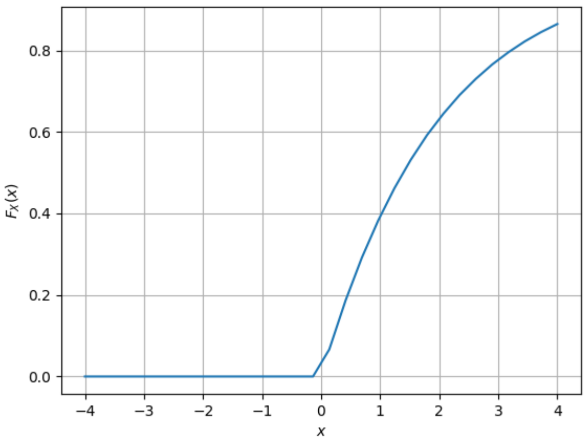

Fig. 3.1: The CDF of V

$$E\left[X^2\right] = \frac{\left(\left(-x\exp\left(-\frac{x^2}{2}\right)\right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2}\right)\right)}{\sqrt{2\pi}} \tag{2.12}$$

$$= \frac{0+\sqrt{2\pi}}{\sqrt{2\pi}} \tag{2.13}$$

$$= 1 \tag{2.14}$$

$$var(X) = E\left[X^2\right] - (E[X])^2 \tag{2.15}$$

$$= 1 - 0 = 1 \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:
To generate samples of V,download and run the following code.

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/v.py
python3 v.py
```

The following code plots the CDF of V Fig.3.1 using the samples generated above.

```
wget https://github.com/saanviamrutha/
    Assignment_rand/blob/main/codes/3.1
    _cdf.py
python3 3.1_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:
The CDF of V is given by

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$

$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= F_V\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

For

$$x \le 0 \qquad 1 - \exp\left(-\frac{x}{2}\right) \le 0 \tag{3.7}$$

$$x > 0 \qquad 0 < 1 - \exp\left(-\frac{x}{2}\right) < 1 \tag{3.8}$$

$$\implies F_V(x) = \begin{cases} 0 & x \le 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x > 0 \end{cases} \tag{3.9}$$