

Random Numbers

Saanvi Amrutha
AI21BTECH11026

CONTENTS

1	Uniform Random Numbers	2
2	Central Limit Theorem	3
3	From Uniform to Other	5

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/saanviamrutha/
  Assignment_rand/blob/main/codes/1.1.c
wget https://github.com/saanviamrutha/
  Assignment_rand/blob/main/codes/coeffs.
  h
gcc 1.1.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1) \quad \text{case.1: if } x < 0$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/saanviamrutha/
  Assignment_rand/blob/main/codes/1.2.py
python3 1.2.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution:

The PDF of U is

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

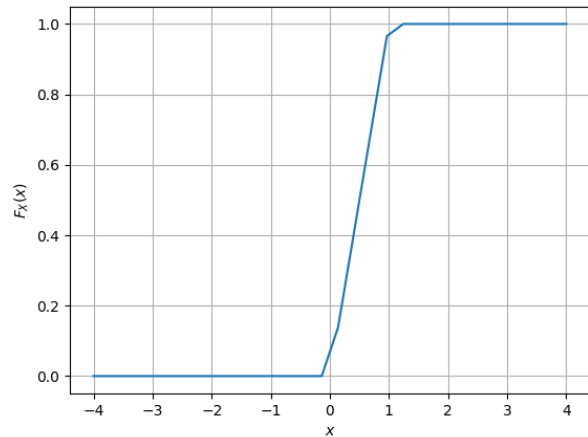


Fig. 1.2: The CDF of U

The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.4)$$

$$= \int_{-\infty}^x 0 dx \quad (1.5)$$

$$= 0 \quad (1.6)$$

case.2: if $x \in [0, 1]$

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.7)$$

$$= \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (1.8)$$

$$= x \quad (1.9)$$

case.3: if $x > 1$

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.10)$$

$$= \int_{-\infty}^0 0 dx + \int_0^1 1 dx + \int_1^x 0 dx \quad (1.11)$$

$$= 1 \quad (1.12)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0, 1] \\ 1 & x > 1 \end{cases} \quad (1.13)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.14)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.15)$$

Write a C program to find the mean and variance of U .

Solution:

Download and execute the following C program

```
wget https://github.com/saanviamrutha/
Assignment_rand/blob/main/codes/1.4.c
gcc 1.4.c
./a.out
```

Output:

$$\text{mean} = 0.500031 \quad (1.16)$$

$$\text{variance} = 0.083247 \quad (1.17)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.18)$$

Solution:

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.19)$$

$$\quad (1.20)$$

$$\text{From (1.13)} \quad (1.21)$$

$$dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & x \in [0, 1] \\ 0 & x > 1 \end{cases} \quad (1.22)$$

$$E[U] = 0 + \int_0^1 x dF_U(x) + 0 \quad (1.23)$$

$$= 0 + \int_0^1 x dx + 0 \quad (1.24)$$

$$= \frac{1}{2} = 0.5 \quad (1.25)$$

$$E[U^2] = 0 + \int_0^1 x^2 dF_U(x) + 0 \quad (1.26)$$

$$= 0 + \int_0^1 x^2 dx + 0 \quad (1.27)$$

$$= \frac{1}{3} \quad (1.28)$$

$$\text{var} = E[U^2] - (E[U])^2 \quad (1.29)$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.30)$$

$$\text{var} = \frac{1}{12} = 0.083333 \quad (1.31)$$

2 CENTRAL LIMIT THEOREM

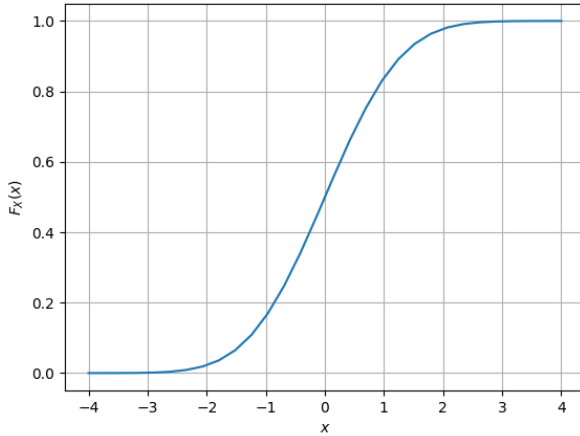
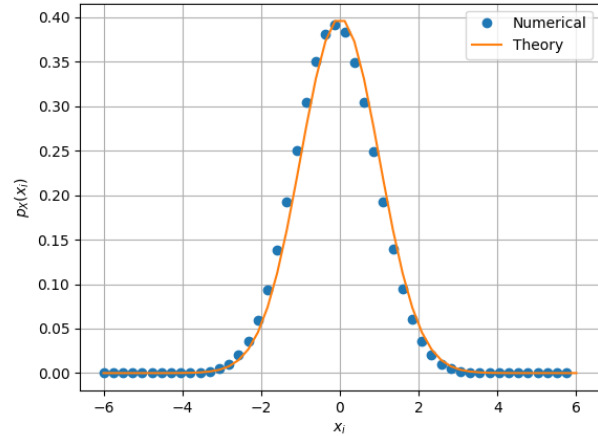
2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program

```
wget https://github.com/saanviamrutha/
Assignment_rand/blob/main/codes/2.1.c
wget https://github.com/saanviamrutha/
Assignment_rand/blob/main/codes/coeffs.
h
gcc 2.1.c
./a.out
```

Fig. 2.2: The CDF of X Fig. 2.3: The PDF of X

2.2 Load `gau.dat` in python and plot the empirical CDF of X using the samples in `gau.dat`. What properties does a CDF have?

Solution: The following code plots Fig. 2.2

```
wget https://github.com/saanviamrutha/
  Assignment_rand/blob/main/codes/2.2.
  py
python3 2.2.py
```

2.3 Load `gau.dat` in python and plot the empirical PDF of X using the samples in `gau.dat`. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/saanviamrutha/
  Assignment_rand/blob/main/codes/2.3.py
python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.

Solution:

Download and execute the following C program

```
wget https://github.com/saanviamrutha/
  Assignment_rand/blob/main/codes/2.4.c
gcc 2.4.c
./a.out
```

Output of the program is:

$$\text{mean} = 0.000685 \quad (2.3)$$

$$\text{variance} = 1.000025 \quad (2.4)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= 0 \quad (2.8)$$

$\therefore x \exp\left(-\frac{x^2}{2}\right)$ is an odd function.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.9)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

Integrating by parts

$$\begin{aligned} E[X^2] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \left[x \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \int_{-\infty}^{\infty} \frac{d(x)}{dx} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \right] \end{aligned} \quad (2.11)$$

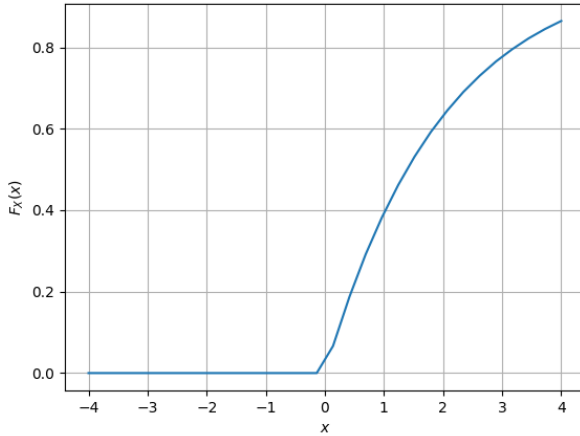


Fig. 3.1: The CDF of V

$$E[X^2] = \frac{\left((-x \exp(-\frac{x^2}{2}))\right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2})}{\sqrt{2\pi}} \quad (2.12)$$

$$= \frac{0 + \sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.13)$$

$$= 1 \quad (2.14)$$

$$\text{var}(X) = E[X^2] - (E[X])^2 \quad (2.15)$$

$$= 1 - 0 = 1 \quad (2.16)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution:

To generate samples of V, download and run the following code.

```
wget https://github.com/saanviamrutha/
Assignment_rand/blob/main/codes/v.py
python3 v.py
```

The following code plots the CDF of V Fig.3.1 using the samples generated above.

```
wget https://github.com/saanviamrutha/
Assignment_rand/blob/main/codes/3.1
_cdf.py
python3 3.1_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

The CDF of V is given by

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= F_V\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

For

$$x \leq 0 \quad 1 - \exp\left(-\frac{x}{2}\right) \leq 0 \quad (3.7)$$

$$x > 0 \quad 0 < 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad (3.8)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x > 0 \end{cases} \quad (3.9)$$