Pingala Series

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Contents

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution: Download the following python code which verifies the above equation.

```
https://github.com/saanviamrutha/EE3900/tree/
    main/Pingala/codes/1.1.py
```

Execute the following command.

```
python3 1.1.py
```

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution: Download the following python code which verifies the above equation.

```
https://github.com/saanviamrutha/EE3900/tree/
    main/Pingala/codes/1.2.py
```

Execute the following command.

```
python3 1.2.py
```

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution: Download the following python code which verifies the above equation.

```
https://github.com/saanviamrutha/EE3900/tree/
    main/Pingala/codes/1.3.py
```

Execute the following command.

```
python3 1.3.py
```

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download the following python code which verifies the above equation.

```
https://github.com/saanviamrutha/EE3900/tree/
    main/Pingala/codes/1.4.py
```

Execute the following command.

```
python3 1.4.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: Download the following code that plots Fig.2.2.

```
https://github.com/saanviamrutha/EE3900/tree/
    main/Pingala/codes/2.2.py
```

Execute the following command.

```
python3 2.2.py
```

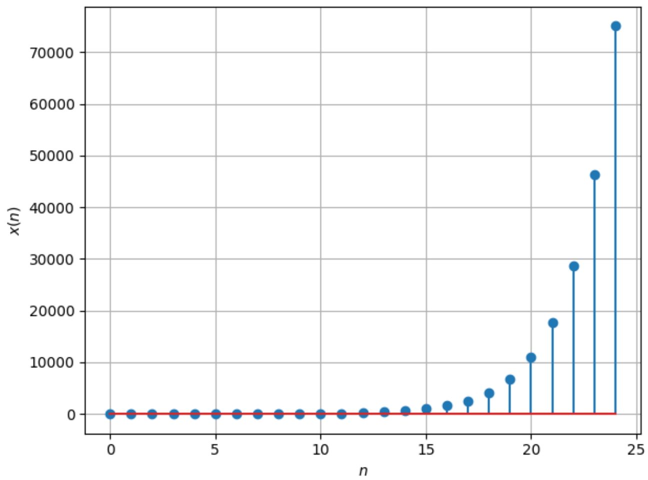

Fig. 2.2: Plot of $x(n)$

2.3 Find $X^{+}(z)$.
Solution: Consider the one sided Z-transform of (2.2)

$$\mathcal{Z}^{+}\left[x(n+2)\right] = \mathcal{Z}^{+}\left[x(n+1)\right] + \mathcal{Z}^{+}\left[x(n)\right] \tag{2.3}$$

$$z^2X^{+}(z) - z^2x(0) - zx(1) = zX^{+}(z) - zx(0) + zX^{+}(z) \tag{2.4}$$

$$X^{+}(z) = \frac{z^2}{z^2 - z - 1} \tag{2.5}$$

$$X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

$$X^{+}(z) = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)} \tag{2.7}$$

2.4 Find $x(n)$.
Solution: Consider the following eq of $X^{+}(z)$

$$X^{+}(z) = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)} \tag{2.8}$$

$$= \frac{1}{\alpha - \beta}\left[\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}}\right] \tag{2.9}$$

$$= \frac{1}{\alpha - \beta}\sum_{n=0}^{\infty}\left(\alpha^{n+1} - \beta^{n+1}\right)z^{-n} \tag{2.10}$$

From (2.1)we get,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) = a_{n+1}u(n) \tag{2.11}$$

2.5 Sketch

$$y(n) = x\left(n-1\right) + x\left(n+1\right), \quad n \geq 0 \tag{2.12}$$

Solution: Download the following code that plots Fig.2.5.

```
https://github.com/saanviamrutha/EE3900/tree/
    main/Pingala/codes/2.5.py
```

Execute the following command.

```
python3 2.5.py
```

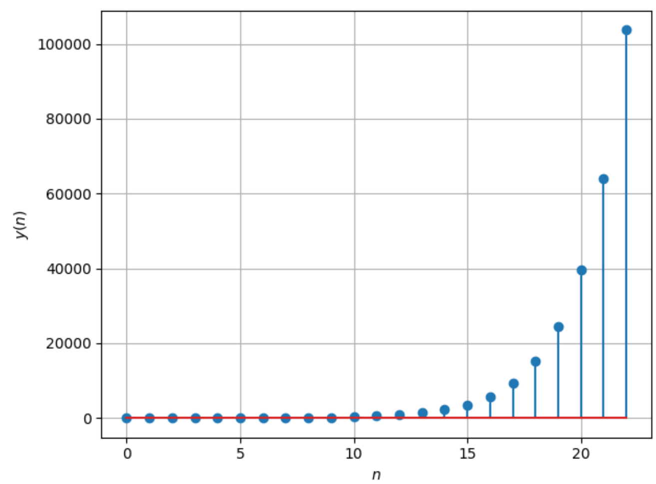

Fig. 2.5: Plot of $y(n)$

2.6 Find $Y^{+}(z)$.
Solution: Consider one sided Z-transform of (2.12).

$$\mathcal{Z}^{+}\left[y(n)\right] = \mathcal{Z}^{+}\left[x(n-1)\right] + \mathcal{Z}^{+}\left[x(n+1)\right] \tag{2.13}$$

$$Y^{+}(z) = z^{-1}X^{+}(z) + zx(-1) + zX^{+}(z) - zx(0) \tag{2.14}$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.15}$$

$$Y^{+}(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.16}$$

since x(n) = 0, $\forall n < 0$

2.7 Find $y(n)$.

Solution: Using the eq. 2.7

$$Y^+(z) = \left(1 + 2z^{-1}\right) \sum_{n=0}^{\infty} x(n) z^{-n} \tag{2.17}$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n} + \sum_{n=1}^{\infty} 2x(n-1) z^{-n} \tag{2.18}$$

$$= x(0) + \sum_{n=1}^{\infty} \left(x(n) + 2x(n-1)\right) z^{-n} \tag{2.19}$$

Using the fact that α and β are the roots of the eq $z^2 - z - 1 = 0$

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + 2\left(\alpha^n + \beta^n\right)}{\alpha - \beta} \tag{2.20}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + \left(\alpha^n + \beta^n\right)}{\alpha - \beta} \tag{2.21}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta\left(\alpha^n + \beta^n\right)}{\alpha - \beta} \tag{2.22}$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{(\alpha - \beta)} \tag{2.23}$$

$$y(n) = \alpha^{n+1} + \beta^{n+1} \tag{2.24}$$

on comparing with the equation of b_n we get

$$y(n) = b_{n+1}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution: Using eq. 2.11

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k) u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$\left[x(n+1) - 1\right] u(n) \tag{3.7}$$

Solution: From eq. 2.11

$$a_{n+2} - 1 = \left[x(n+1) - 1\right], \quad n \geq 0 \tag{3.8}$$

using the definition of $u(n)$,

$$a_{n+2} - 1 = \left[x(n+1) - 1\right] u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.12}$$

$$= \frac{1}{10} X^+(10) \tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.16}$$

and find $W(z)$.
Solution: Substitute n = k+1 in the given exp

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{3.17}$$

Hence, it can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.18}$$

As per the definition of $u(n)$, it can be concluded that one sided Z-transform of both $w(n)$ and $y(n)$ are equal.

$$\therefore W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.19}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.20}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.21}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.22}$$

$$= \frac{1}{10} Y^{+}(10) \tag{3.23}$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.24}$$

3.6 Solve the JEE 2019 problem.
Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.25}$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z) z^{-1} U(z) \tag{3.26}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{3.27}$$

$$= z \left(\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right) \tag{3.28}$$

$$= z \sum_{n=0}^{\infty} \left(x(n) - 1\right) z^{-n} \tag{3.29}$$

$$= \sum_{n=0}^{\infty} \left(x(n) - 1\right) z^{-n+1} \tag{3.30}$$

$$= \sum_{n=0}^{\infty} \left(x(n+1) - 1\right) z^{-n} \tag{3.31}$$

$$\tag{3.32}$$

From (2.11) and using inverse Z-transform, we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.33}$$