Introduction to Machine Learning

Section 1

Linear Algebra

1.a

$\Rightarrow$ symmetric matrix A is PSD such that $v^t Av = (v^t u)diag(\lambda)(u^t v) = \sum_i \lambda_i (Vu^t)^2 \geq 0$ where λ is the Eigenvalue of A.
and matrix A can be decomposed as:

$$A = QDQ^t = Q * diag(\lambda_1, \lambda_2 \ldots \lambda_n) * Q^t =$$

$$Q * diag(\sqrt{\lambda_1}, \sqrt{\lambda_2} \ldots \sqrt{\lambda_n}) * diag(\sqrt{\lambda_1}, \sqrt{\lambda_2} \ldots \sqrt{\lambda_n}) * Q^t = XX^t$$

$\Leftarrow$ A can be written as $v^t XX^t v$ we get:

$$v^t Av = v^t XX^t v = (X^t v)^t (X^t v) = \parallel X_v^t \parallel \geq 0$$

1.b

for a given PSD matrix A and $\alpha \in R$
(*) $v^t(\alpha A) \geq 0 \Rightarrow v^t(uA) \geq 0$ when $u, A \geq 0$
then for PSD matrix's A,B when $A, B \geq 0$, $A + B \geq 0$
now let's apply (*) on (A+B) we will get (**)

$$v^t(A + B)v = v^t Av + v^t Bv \geq 0$$

then from both (*) and (**) immediately get $\alpha A + \beta B \geq 0$
the set of all n $\times$ n PSD matrices over R is not a vector space over R because its not apply the closures to multiplication in scalar property , for $\lambda < 0$ and

$$A \geq 0 \rightarrow \lambda A < 0 \rightarrow \lambda A \notin \{PSD\}$$

Calculus and Probability

1.a

for $x_1, x_2 \ldots x_n$ i.i.d $U([0,1])$ continuous random variables, lets write the Order Statistics such as $\overline{x_1}, \overline{x_2} \ldots \overline{x_n}$ when $\forall$ i , $\overline{x_i} \leq \overline{x_{i+1}}$ first lets find the CDP of $Y = MAX\{x_1, x_2 \ldots x_n\} = \overline{x_n}$:

$$F_y(x) = F_{\overline{x_n}} = \Pr(\overline{x_n} \leq k) = \Pr(\overline{x_1} \leq k, \overline{x_2} \leq k \ldots \overline{x_n} \leq k)$$

because they i.i.d

$$\Pr(\overline{x_1} \leq k) \Pr(\overline{x_2} \leq k) \ldots \Pr(\overline{x_n} \leq k) = [\Pr(\overline{x_i} \leq k)]^n = [F_x(k)]^n$$

$$(*)F(x_i) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x/1 & \text{for} \quad x \in \{0,1\} \\ 1 & \text{for} \quad x > 1 \end{cases} \quad f(x) = \begin{cases} 1 & \text{for} \quad x \in \{0,1\} \\ 0 & \text{for} \quad x \notin \{0,1\} \end{cases}$$

we get:

$$f_y(k) = f_{\overline{x_n}}(k) = \frac{d}{dk}(F_{\overline{x_n}}(k))^n = n(F(k))^{n-1} f(k)$$

now lets set values in (*) and get :

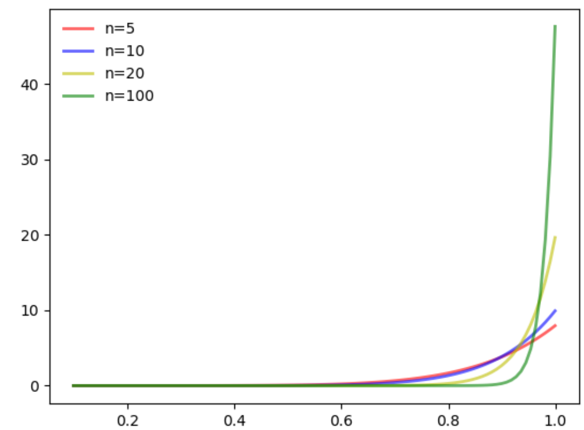
$$f_y(k) = nk^{n-1} f(k) = nk^{n-1} f(k) = nk^{n-1} I_{(1,0)} \sim Beta(n,1)$$

therefore :

$$\lim(E[y])_{n \to \inf} = \lim(\frac{n}{n+1})_{n \to \infty} \longrightarrow 1$$

and :

$$(var[y])_{n \to \inf} = \lim(\frac{n}{(n+1)^2(n+2)})_{n \to \inf} \to 0$$



2

$$E[|x-\alpha|] = \int_{-\infty}^{+\infty} |x-\alpha| f(x)\, dx = \int_{-\infty}^{\alpha} |x-\alpha| f(x)\, dx + \int_{\alpha}^{+\infty} |x-\alpha| f(x)\, dx$$

when $\alpha \in argmin$:

$$\underbrace{(\alpha - x) f(x)|}_{\to 0} + \int_{-\infty}^{\alpha} f(x)\, dx + \underbrace{(x-\alpha) f(x)|}_{\to 0} - \int_{\alpha}^{+\infty} f(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\alpha} f(x)\, dx = \int_{\alpha}^{+\infty} f(x)\, dx \Rightarrow \Pr(x \le \alpha) = \Pr(x \ge \alpha) \Leftrightarrow$$

.

$$\Pr(x \le \alpha) = 1/2$$

Optimal Classifiers and Decision Rules

1.a

Let X and Y be random variables where Y can take values in $Y = \{1, \ldots, L\}$, and Let ℓ be the 0-1 loss function defined in class , hence:

$$E[\triangle(y, f(x))] = \sum_{k=1}^{L} Pr(X = \hat{x}, Y = k)\triangle(k, f(x))$$

using bayes :

$$\sum_{k=1}^{L} \Pr(X = \hat{x}) \Pr(Y = k|X = \hat{x})\triangle(k, f(k)) = \Pr(X = \hat{x}) \sum_{k=1}^{L} \Pr(y = k|X = \hat{x})\triangle(k, f(k))$$

$$L(h) = Arg \min_{f:X \to Y} \{\Pr(x = \hat{x}) \sum_{y \neq k, y \in \{1 \ldots L\}} \Pr(y = k|X = \hat{x})\} = f(\hat{x}) = h(x) = k$$

$$\Rightarrow h(\hat{x}) = Arg \max\{\Pr(x = \hat{x})(1 - \Pr(y = k|X = \hat{x})\} : h(\hat{x}) = k$$

$$h(\hat{x}) = Arg \max_{y \in Y} \Pr(y = i|x = \hat{x})$$

Optimal Classifiers and Decision Rules

1.b

To find decision rule for:

$$\Pr[y = 1|X] > \Pr[y = 0|X]$$

lets apply bayes rule on both sides. we get:

$$\frac{f_{X|Y=1}(x)Pr[Y = y]fX(X)}{f_x} > \frac{f_{X|Y=0}(x)Pr[Y = y]fX(X)}{f_x}$$

$$pf_1(x, \mu_1, \sum) > (1 - p)f_0(x, \mu_o, \sum)$$

$$\frac{exp(-(1/2)(x - \mu_1)^T \sum^{-1}(x - \mu_1)}{exp(-(1/2)(x - \mu_0)^T \sum^{-1}(x - \mu_0)} > \frac{1 - p}{p}$$

$$(x-\mu_0)^T \sum^{-1}(x-\mu_0) - (x-\mu_1)^T \sum^{-1}(x-\mu_1) > 2ln(\frac{1-p}{p})$$

where $(x-\mu)^T \sum^{-1}(x-\mu)$ is the square Mahalanobis Distance between x and μ
so our simpler Decision rule will be

$$h(x) = \begin{cases} 1 & \text{for} \quad d^2{}_{\mathbf{m}}(x,\mu_0) > d^2{}_{\mathbf{m}}(x,\mu_1) + 2ln(\frac{1-p}{p}) \\ 0 & \quad \text{otherwise} \end{cases}$$

1.c

when d=1 the general Matrix $\sum$ size will be size d X d, so the shape of the decision shape boundery will be just dot,in the same way when d=2 we will have a line , and for general d its might be d-dimenonal shape...

1.d

For $d = 1, \mu_0 = \mu_1 = \mu$ and $\sigma_1 \neq \sigma_0$we looking for equation in the decision rule formula we go had above:

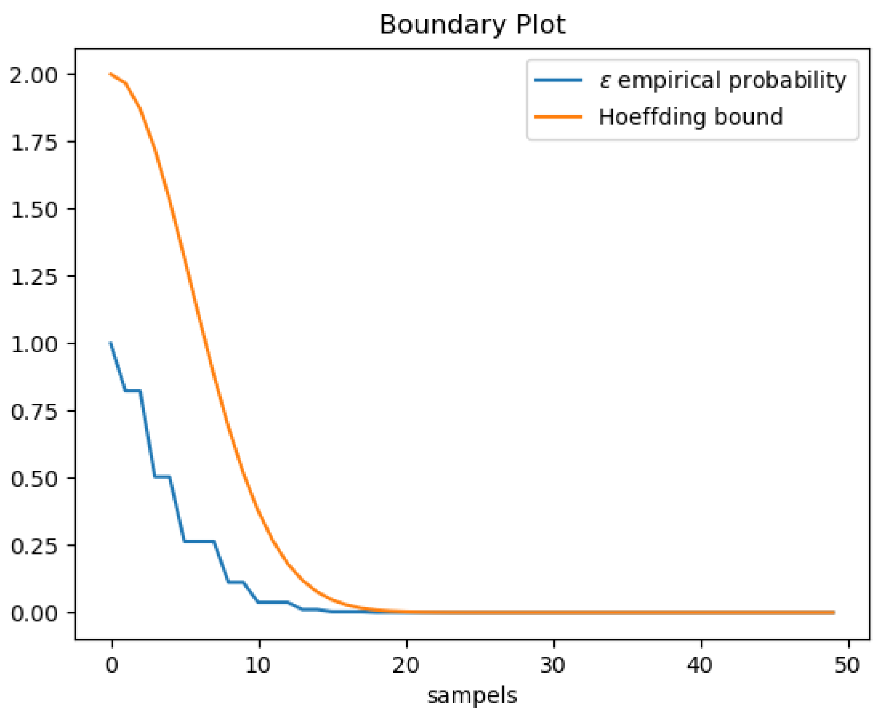
$$d^2{}_{\mathbf{m}}(x,\mu_0) - d^2{}_{\mathbf{m}}(x,\mu_1) = 2ln(\frac{1-p}{p})$$

$$(x-\mu)^2(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}) = 2ln(\frac{1-p}{p}) \Rightarrow (x-\mu)^2 = (\sigma_0^2 - \sigma_1^2)2ln(\frac{1-p}{p})$$

$$(x-\mu) = +-\sqrt{(\sigma_0^2 - \sigma_1^2)2ln(\frac{1-p}{p})} \Rightarrow x = \mu + -\sqrt{(\sigma_0^2 - \sigma_1^2)2ln(\frac{1-p}{p})}$$

Programming Assignment

Visualizing the Hoeffding bound:



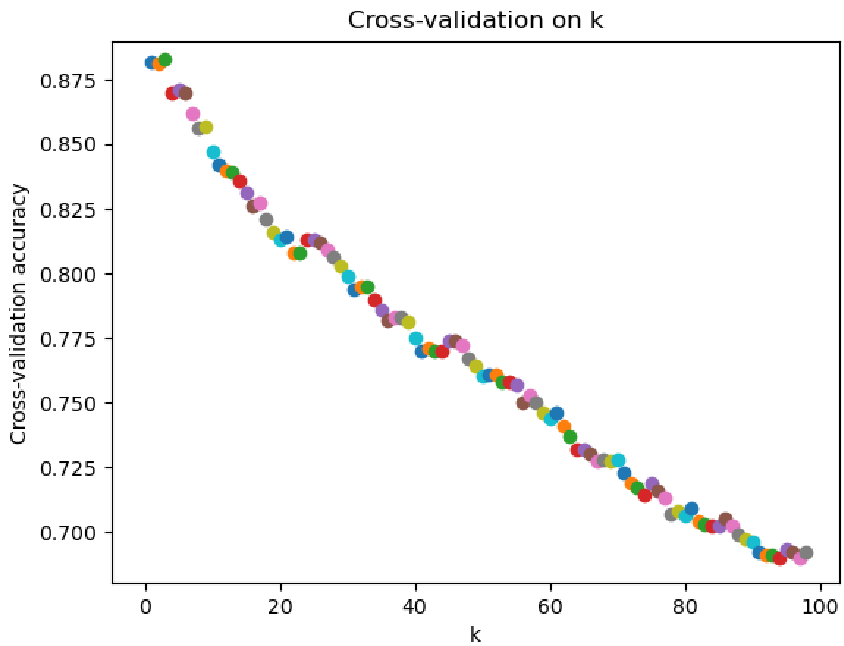
Nearest Neighbor:

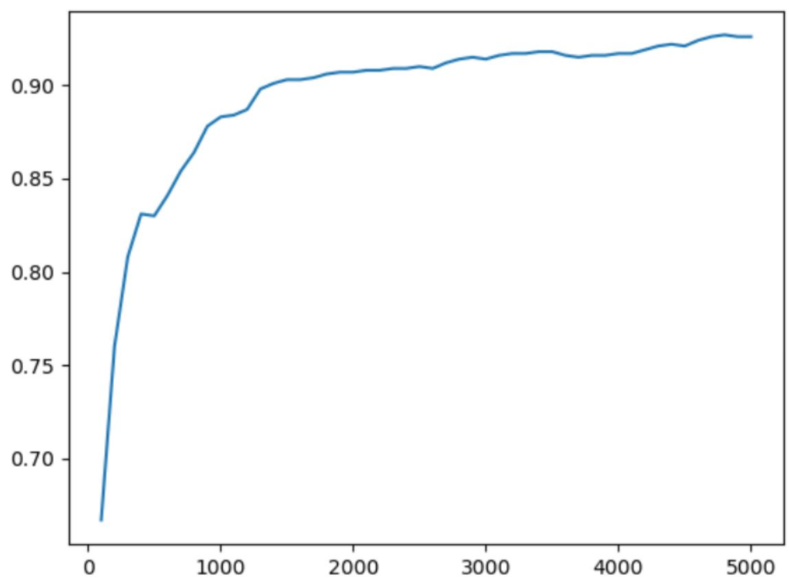
1

.The KNN accuracy for $k = 10$. its got 882/1000 correct labeling. and the accuracy rate is 0.882000

2

The best K found is $k = 4$ with 883/1000 correct labeling.and the accuracy rate is 0.883000





It was my first time ”LaTeXing” hope it was find (: