ME7223: END SEMESTER EXAM

SAARTHAK SANDIP MARATHE | ME17B162

For my roll number, the values of constants are:

- A = 162/17
- B = 6
- C = 9
- D = 5
- E = 1

Question 1:

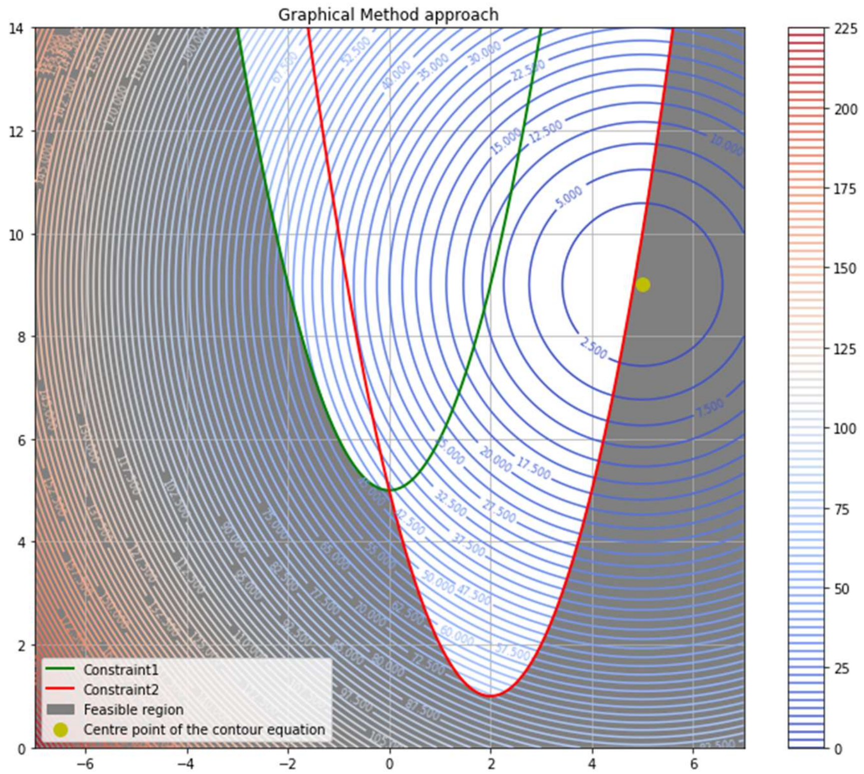
$$f(x_1, x_2) = (x_1 - 5)^2 + (x_2 - 9)^2$$

Subject to:

$$-x_1^2 + x_2 - 5 \leq 0$$

$$-(x_1 - 2)^2 + x_2 - 1 \leq 0$$

Graphical Method:



Contours are drawn as lines with their objective function values mentioned for easier interpretation of the graph and conclusion of the optimum point.

From the graph shown above we can see that:

- The feasible shaded region is in the convex part of the parabola (outside the cup area).
- Centre of the contour region has the lowest value of the objective function (which is zero)
- As the centre (x1=5, x2=9) is in the feasible region, our minima is the centre point.
- Thus, **optimum point: x1=5, x2=9 with f(5,9) = 0 as the optimum value**

KKT Method of question 1 done on the next page

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ME17B162

ME7223 : END SEMESTER EXAM

For my roll number = ME17B162 :

A = 162/17

B = 6

C = 9

D = 5

E = 1

Min : $f(x_1, x_2) = (x_1-5)^2 + (x_2-9)^2$

Subject to : $g_1(x_1, x_2) \Rightarrow -x_1^2 + x_2 - 5 \leq 0$

$g_2(x_1, x_2) \Rightarrow -(x_1-2)^2 + x_2 - 1 \leq 0$

We have :

$$\nabla f = \begin{bmatrix} 2(x_1-5) \\ 2(x_2-9) \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} -2x_1 \\ 1 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} -2(x_1-2) \\ 1 \end{bmatrix}$$

According to KKT conditions :

$$\nabla f + \sum_{j=1}^{2} \lambda_j \nabla g_j = 0$$

$$\left.\begin{array}{l} \lambda_j g_j = 0 \\ g_j \leq 0 \\ \lambda_j \geq 0 \end{array}\right\} \forall \; j = 1, 2$$

$\therefore$

$$2(x_1-5) - 2\lambda_1 x_1 - 2\lambda_2(x_1-2) = 0 \rightarrow (1)$$

$$2(x_2-9) + \lambda_1 + \lambda_2 = 0 \rightarrow (2)$$

$$\lambda_1(-x_1^2 + x_2 - 5) = 0 \rightarrow (3)$$

$$\lambda_2(-(x_1-2)^2 + x_2 - 1) = 0 \rightarrow (4)$$

$$-x_1^2 + x_2 - 5 \leq 0 \rightarrow (5)$$

$$-(x_1-2)^2 + x_2 - 1 \leq 0 \rightarrow (6)$$

$$\lambda_1, \lambda_2 \geq 0 \rightarrow (7)$$

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From (2):

$$x_2 = -\frac{(\lambda_1+\lambda_2)}{2} + 9$$

From (1):

$$x_1(2-2\lambda_1-2\lambda_2) - 10 + 4\lambda_2 = 0$$

$$\Rightarrow x_1 = \frac{5-2\lambda_2}{1-\lambda_1-\lambda_2}$$

Taking (3):

$$\lambda_1(-x_1^2+x_2-5) = 0$$

$$\therefore \lambda_1 = 0 \text{ or } -x_1^2+x_2-5 = 0$$

* If $\lambda_1 = 0 \Rightarrow x_2 = \frac{-\lambda_2}{2}+9$, $x_1 = \frac{5-2\lambda_2}{1-\lambda_2}$

Taking (4):

$$\lambda_2(-(x_1-2)^2+x_2-1) = 0$$

$$\therefore \lambda_2 = 0 \text{ or } -(x_1-2)^2+x_2-1 = 0$$

If $\lambda_2 = 0 \Rightarrow x_2 = 9,\ x_1 = 5$

$f(5,9) = 0$ conditions (5), (6), (7) are satisfied

If $-(x_1-2)^2+x_2-1 = 0 \Rightarrow$

$$\cancel{-\left(\frac{-\lambda_2}{2}+7\right)^2 + \frac{5-2\lambda_2}{1-\lambda_2} - 1 = 0}$$

$$-\left(\frac{5-2\lambda_2}{1-\lambda_2}-2\right)^2 - \frac{\lambda_2}{2} + 9 - 1 = 0$$

$$-2(5-\cancel{2\lambda_2}-2+\cancel{2\lambda_2})^2 - \lambda_2(1-\lambda_2)^2 + 16(1-\lambda_2)^2 = 0$$

$$-18 - \lambda_2^3 + 2\lambda_2^2 - \lambda_2 + 16 + 16\lambda_2^2 - 32\lambda_2 = 0$$

$$-\lambda_2^3 + 18\lambda_2^2 - 33\lambda_2 - 2 = 0$$

$\lambda_2 = 15.9191 3036$ or $2.1395 866$ or -0.05871823

Calculations of (5), (6), (7) conditions and checking feasibility in python code we get:

Only $\lambda_2 = 15.9191...$ is feasible with $f(x) = 73.6061$.

If $(-x_1^2+x_2-5)=0 \Rightarrow x_2 = 5+x_1^2$

Taking (4):

$\lambda_2\left(-(x_1-2)^2+x_2-1\right)=0$

If $\lambda_2 = 0 \Rightarrow x_2 = -\frac{\lambda_1}{2}+9, \quad x_1 = \frac{5}{1-\lambda_1}$

$\therefore -\frac{\lambda_1}{2}+9 = 5+\frac{25}{(1-\lambda_1)^2}$

$\Rightarrow -\lambda_1(1-\lambda_1)^2+18 = 10(1-\lambda_1)^2+50$

$\therefore -\lambda_1(1-2\lambda_1+\lambda_1^2)+18 = 10(1-2\lambda_1+\lambda_1^2)+50$

$\Rightarrow -\lambda_1^3+2\lambda_1^2-\lambda_1+18 = 10-20\lambda_1+10\lambda_1^2+50$

$-\lambda_1^3-8\lambda_1^2+19\lambda_1-42=0$

Solving this using python code

$\lambda_1 = -10.252,\ 1.126+1.68i,\ 1.126-1.68i$

All values are infeasible

If $-(x_1-2)^2+x_2-1=0 \Rightarrow$

$-(x_1-2)^2+5+x_1^2-1=0$

$-\cancel{x_1^2}+4x_1-4+5+\cancel{x_1^2}-1=0$

$4x_1 = 0 \Rightarrow x_1 = 0$

$\therefore x_2 = 5$

Satisfying all the conditions:

$f(0,5) = 41$

Comparing all the feasible solutions:

$f(5,9) = 0$

$f(\lambda_2 = 15.9191, \lambda_1 = 0) = 73.6061 = f(1.798, 1.04043)$

$f(0,5) = 41$

We find that $f(x_1 = 5, x_2 = 9) = 0$ is the minima and $x_1 = 5, x_2 = 9$ as optimum values.

PYTHON CALCULATIONS ATTACHED ON NEXT PAGE.

In [ ]:

```

```

KKT Method (Calculations)

In [64]:

```python
'''
Calculating roots of cubic equation for lambda2 values when lambda1=0 and -(x1-2)^2+x2-
Equation of lambda2 (l2) is: -(l2)^3 + 18(l2)^2 - 33*l2 - 2 = 0
'''
#calculating the solution of the above mentioned lambda2 equation
coeff = [-1, 18, -33, -2]
l2 = np.roots(coeff) #inbuild equation solver

#calculating g1, g2, f values for x1, x2 corresponding to each l2 values
def g1_cond(x):
    return -x[0]**2+x[1]-5

def g2_cond(x):
    return -(x[0]-2)**2+x[1]-1

x = [0]*2
for i in range(len(l2)):
    x[0] = (5-2*l2[i])/(1-l2[i])
    x[1] = -(l2[i]/2)+9
    if g1_cond(x)<=0 and g2_cond(x)<=0 and l2[i]>=0:
        print('Feasible value of lambda2=',l2[i],' with f(x):', f(x))
        print('x values for above are:', x)
    else:
        print('\nlambda2 value = ',l2[i],' infeasible')
```

```
Feasible value of lambda2= 15.91913036417428  with f(x): 73.60161734519403
x values for above are: [1.7989158934354523, 1.0404348179128604]

lambda2 value =  2.1395888625703603  infeasible

lambda2 value =  -0.05871922674463366  infeasible
```

In [63]:

```python
'''
Calculating roots of cubic equation for lambda1 values when lambda2=0 and -(x1)^2+x2-5=
Equation of lambda1 (l1) is: -(l1)^3 - 8(l1)^2 + 19*l1 - 42 = 0
'''
#calculating the solution of the above mentioned lambda1 equation
coeff = [-1, -8, 19, -42]
l1 = np.roots(coeff) #inbuild equation solver
l1
```

Out[63]:

```
array([-10.25271741+0.j        ,   1.1263587 +1.68160371j,
         1.1263587 -1.68160371j])
```

Question 2:

This question was done using **code** attached with the submission and the final values obtained are as below:

- No of experiments: 10
- The optimum region calculated: [-1.5542139077682984 , -1.5016037340195263]
- The optimum point is: -1.5279088208939124

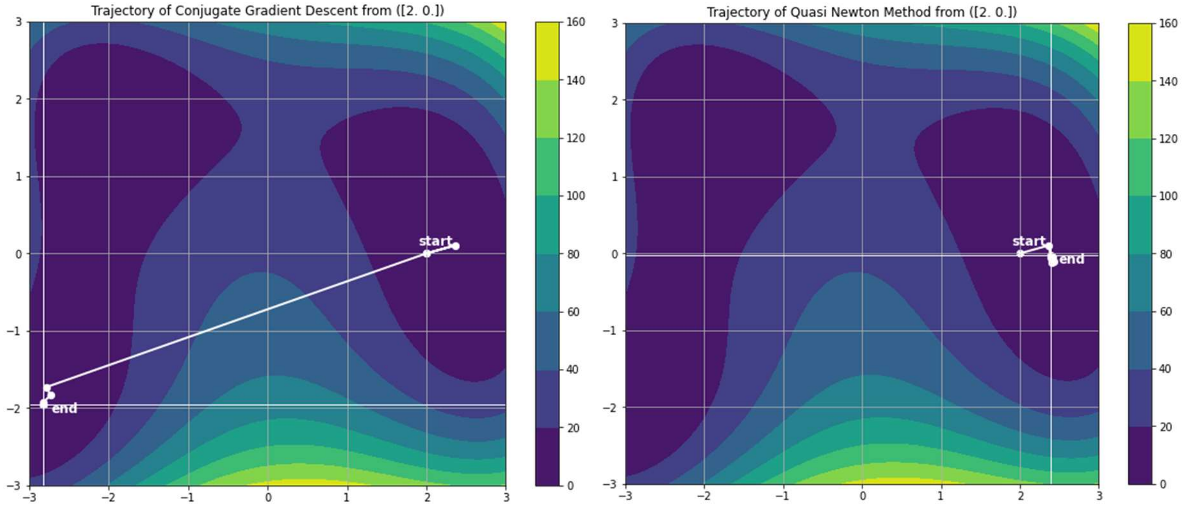
Question 3:

$$f(x_1, x_2) = (x_1^2 + x_2 - 6)^2 + (x_2^2 + x_1 - 1)^2$$

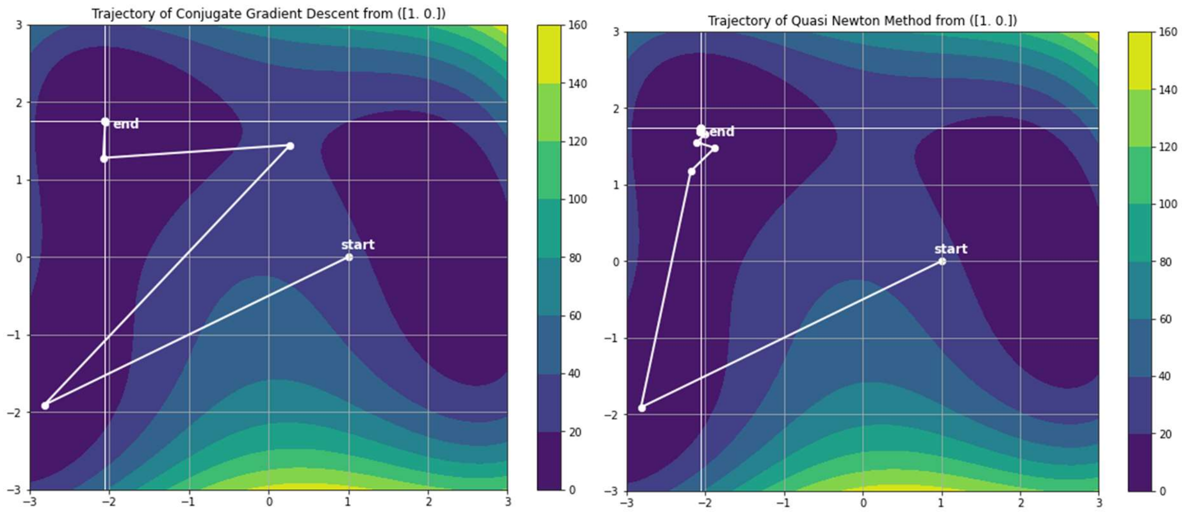
For the code part, the terminating conditions taken into consideration for both the methods are:

- Tolerance = 1e-03
- Max iterations = 10

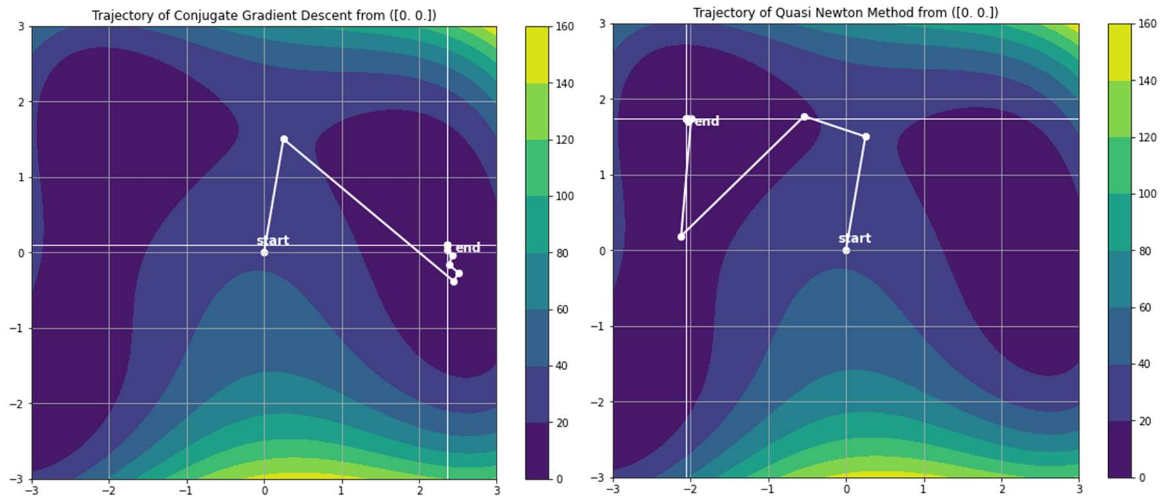
Comparison between Conjugate Gradient Descent Method and Quasi Newton Method:



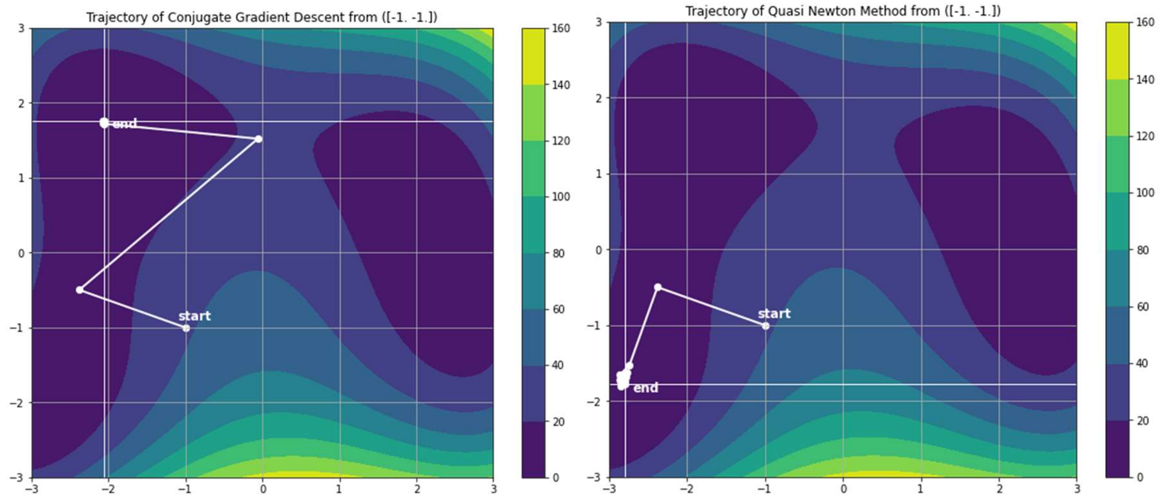
Both the methods for initial point = [2,0]



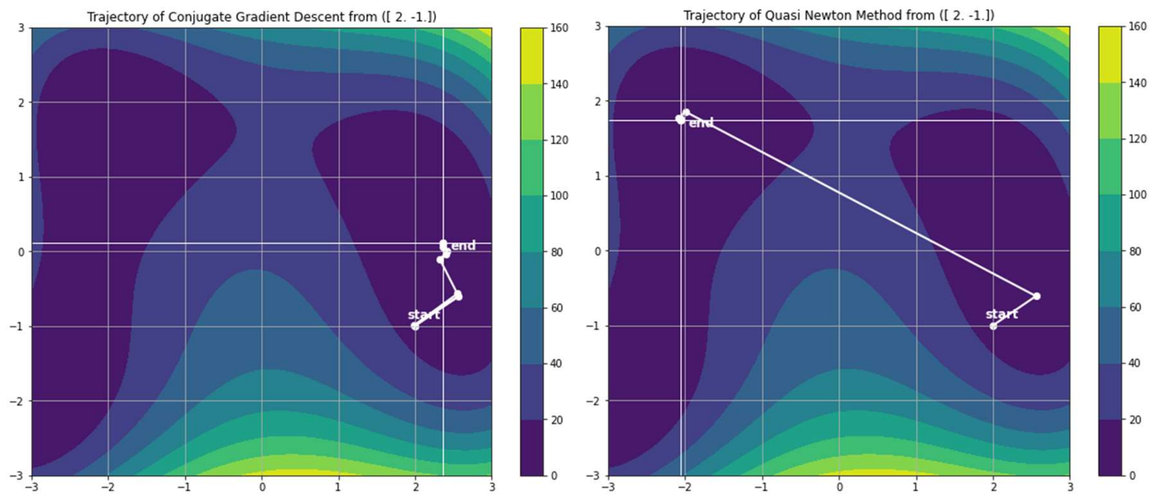
Both the methods for initial point = [1,0]



Both the methods for initial point = [0,0]



Both the methods for initial point = [-1,-1]



Both the methods for initial point = [2,-1]

The given objective function has three local minima regions out of which two (on left of the contour graph) give objective function values as 0 and act as global minimas while the third one is a local minima (on the right side of the contour graph)

Observations:

Start point	Conjugate Gradient Descent Method	Quasi Newton Method
[2,0]	• Converges near the **global** minima. [-2.8222216 -1.95702459] • Takes all 10 iterations	• Converges near the **local** minima. [2.39180183 -0.02756045] • Takes all 10 iterations
[1,0]	• Converges near the **global** minima. [-2.06252117 1.75920091] • Takes all 10 iterations	• Converges near the **global** minima. [-2.06326277 1.7429929] • Takes all 10 iterations
[0,0]	• Converges near the **local** minima. [2.36857251 0.10498207] • Takes all 10 iterations	• Converges near the **global** minima. [-2.05226283 1.74888669] • Takes all 10 iterations
[-1,-1]	• Converges near the **global** minima. [-2.05896247 1.76139139] • Takes all 10 iterations	• Converges near the **global** minima. [-2.7990231 -1.77445402] • Takes all 10 iterations
[2,-1]	• Converges near the **local** minima. [2.36178654 0.11800132] • Takes all 10 iterations	• Converges near the **global** minima. [-2.06161063 1.74975347] • Takes all 10 iterations

Inferences:

- Based on the starting point, the final point of convergence keeps on changing for both the methods
- Even, the methods don't converge to the same point for the same starting point except point [1,0]
- Among all the starting points, Quasi Newton Method converges to one of the global minimas than the Conjugate Gradient Descent Method
- Conjugate Descent method converges faster than Quasi Newton Method for the starting points which eventually reach the global minima
- These differences between the two methods are seen because both the methods proceed differently after i=1 iteration
- Conjugate Gradient Descent Method mainly depends on the **beta** value for updating S directions whereas the Quasi Newton Method depends on the **B** values calculated for updating S directions.

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4) $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$ ⇐ minimize

subject to : $g_1(x_1, x_2) \Rightarrow x_1 - x_2 - 5 = 0$

$g_2(x_1, x_2) \Rightarrow x_1 + x_2 - 0.5 \leq 0$

Using Lagrange multiplier method :

$$\mathcal{L}(x, \lambda, y) = f(x_1, x_2) + \lambda_1 (g_1(x_1, x_2)) + \lambda_2 (g_2(x_1, x_2) + y_2^2)$$

$$\therefore \mathcal{L}(x, \lambda, y) = (x_1 - 1)^2 + (x_2 - 1)^2 + \lambda_1 (x_1 - x_2 - 5) + \lambda_2 (x_1 + x_2 - 0.5 + y_2^2)$$

$$\frac{d\mathcal{L}}{d\lambda_1} = x_1 - x_2 - 5 = 0 \quad \rightarrow (1)$$

$$\frac{d\mathcal{L}}{d\lambda_2} = x_1 + x_2 - 0.5 + y_2^2 = 0 \quad \rightarrow (2)$$

$$\frac{d\mathcal{L}}{dx_1} = 2(x_1 - 1) + \lambda_1 + \lambda_2 = 0 \quad \rightarrow (3)$$

$$\frac{d\mathcal{L}}{dx_2} = 2(x_2 - 1) - \lambda_1 + \lambda_2 = 0 \quad \rightarrow (4)$$

$$\frac{d\mathcal{L}}{dy_2} = 2\lambda_2 y_2 = 0 \quad \rightarrow (5)$$

From (5) :

$\lambda_2 = 0$ or $y_2 = 0$

If $y_2 = 0 \Rightarrow$ $x_1 + x_2 = 0.5$

(1) : $x_1 - x_2 = 5$

$\therefore x_1 = 2.75, \; x_2 = -2.25$

(3) : $\lambda_1 + \lambda_2 = -3.5$

$-\lambda_1 + \lambda_2 = 6.5$

$\Rightarrow \lambda_2 = 1.5, \; \lambda_1 = -5$

If $\lambda_2 = 0 \Rightarrow \lambda_1 = -2(x_1 - 1)$

(4) : $2(x_2 - 1) + 2(x_1 - 1) = 0$

$\therefore \quad x_2 + x_1 = 2$

① : $x_1 - x_2 = 5$

$\Rightarrow \quad x_1 = 3.5, \; x_2 = -1.5$

⑥ : $3.5 - 1.5 - 0.5 + y_2^2 = 0$

$\Rightarrow y_2^2 = -1.5$ ✗

Which is not possible as y_2 is a real no.

$\therefore \lambda_2 \neq 0$ and $y_2 = 0$

Optimum values are : $\underline{x_1 = 2.75, \; x_2 = -2.25}$

$f(x_1 = 2.75, x_2 = -2.25) = \underline{\underline{13.625}}$