Optimization Methods for Mechanical Design (ME7223)

Assignment 3

September 13, 2020

Max marks: 20 Due date: 17th Sept 2020

Instructions

- Answer all questions.
- Assume any missing data appropriately.
- Append the graphs to the scanned version of the answer sheets.
- Contact the TA (Mayank Raj) if you have any questions.

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1. A symmetric matrix $\mathbf{M} \in \mathcal{R}^{n\times n}$ is positive definite iff $x^T\mathbf{M}x > 0,\ \forall x \in \mathcal{R}^{n\times 1}$. Find a vector $\mathbf{x}$ to show that the following matrix in not positive definite: (1.5)
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 5 \\ 3 & 5 & 1 \end{bmatrix}$$
Conclude that a diagonal element of a positive definite matrix can not be negative.

2. The Hessian matrix $\mathbf{H}(\mathbf{x}) \in \mathcal{R}^{n\times n}$ corresponding to a function $f(\mathbf{x})\,;\ \mathbf{x} \in \mathcal{R}^{n\times 1}$, is defined such that $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Given that both the function $f(\mathbf{x})$ and it's derivative $\nabla f(\mathbf{x})$ are continuous and diffrentiable, answer the following questions with respect to the function $f(\mathbf{x})$ and its Hessian $\mathbf{H}$: (2.5)

   (a) Is $\mathbf{H}$ always a symmetric matrix?

   (b) For what kind of functions, $\mathbf{H}$ would be independent of spatial coordinates $\mathbf{x}$, give an example of such function.

   (c) For what kind of functions, $\mathbf{H}$ would be a function of spatial coordinates $\mathbf{x}$.

   (d) What does the quantity $\mathbf{Hy}$ mean physically, where $\mathbf{y}$ is a small change in vector $\mathbf{x}$.

   (e) What does the quantity $\mathbf{y^T H y}$ mean physically, where $\mathbf{y}$ is a small change in vector $\mathbf{x}$.

3. Consider the following problem: (3)
$$\min_{x_1,x_2} f = (x_1-2)^2 + (x_2-1)^2$$
subject to
$$2 \geq x_1 + x_2$$

$$x_2 \geq x_1^2$$

Using Kuhn-Tucker (KKT) conditions, find which of the following vectors are local minima:

$$\mathbf{x_1} = \begin{Bmatrix} 1.5 \\ 0.5 \end{Bmatrix}, \quad \mathbf{x_2} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \mathbf{x_3} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

4. Using Kuhn-Tucker conditions, find the value(s) of β for which the point $x_1^* = 1, x_2^* = 2$ will be optimal to the problem: (2)

$$\max_{x_1, x_2} f(x_1, x_2) = 2x_1 + \beta x_2$$

subject to

$$g_1(x_1, x_2) = x_1^2 + x_2^2 - 5 \leq 0$$

$$g_2(x_1, x_2) = x_1 - x_2 - 2 \leq 0$$

Verify your result using a graphical procedure.

5. Find a usable and feasible direction $\mathbf{S}$ at (a) $\mathbf{x_1} = \begin{Bmatrix} -1, & 5 \end{Bmatrix}^T$ (b) $\mathbf{x_2} = \begin{Bmatrix} 2, & 3 \end{Bmatrix}^T$ for the following problem: (3)

$$\min f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 5)^2$$

subject to

$$g_1(\mathbf{x}) = -x_1^2 + x_2 - 4 \leq 0$$

$$g_2(\mathbf{x}) = -(x_1 - 2)^2 + x_2 - 3 \leq 0$$

6. Determine and show if the following functions are convex: (2)

   (a) $f(x_1, x_2) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

   (b) $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$

7. Determine and show if the following optimization problem is convex: (3)

$$\min_{x_1, x_2} f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_1 - 4$$

subject to

$$x_1^2 + x_2^2 + 2x_1 \geq 16$$

8. Consider the function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{x} + c$, where: (3)

$$\mathbf{A} = \begin{bmatrix} 18\alpha & 2-2\alpha \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \alpha^2 & 1 \end{bmatrix}$$

$$c = \alpha$$

(a) Is the Hessian of $f(\mathbf{x})$ symmetric? Justify.

(b) Is a quadratic function always bound to be either convex or concave?

(c) Find the range of α for which $f(\mathbf{x})$ is strictly convex along the direction $x_2 = (1-\alpha)x_1$ **and** strictly concave along the direction $x_1 = x_2$.

(d) Will the minima obtained from the follwing optimization problem, for $\alpha = -8$, be the global minima (in context of the constraints)?

$$\min_{\mathbf{x}} f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{x} + c$$

subject to

$$x_2 = (1-\alpha)\,x_1$$