SAARTHAK MARATHE

ME17B162

Date 10/9/19
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ME 7223 - ASSIGNMENT 2

1) M - matrix with independent columns.

a)

$\therefore M^T$ has linearly independent rows.

For any 'x' let $y = Mx$, x = eigenvector of M.

$\therefore y^T = x^T M^T$

$y^T y = x^T M^T M x \rightarrow$ (1)

(1) is zero iff y is zero. Then, x must also be zero.

$\therefore x^T M^T M x = 0$ iff x is zero.

But $x \neq 0$, the matrix is positive-definite.

b) P - positive definite matrix

$P = M^T M$, M = matrix with independent columns.

If x is eigenvector of P then $x \neq 0$.

$Px = \lambda x$

$x^T P x = x^T \lambda x = \lambda x^T x$

As $\lambda > 0$, then as $x^T x > 0 \Rightarrow$ we must have $x^T P x > 0$.

2) $f(x_1, x_2, x_3) = -x_1^2 - x_2^2 + 2x_1x_2 - x_3^2 + 6x_1x_3 + 4x_1 - 5x_3 + 2$

$f(X) = \frac{1}{2} X^T [A] X + B^T X + C$

[A] = hessian matrix.

$$[A] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 2 & 6 \\ 2 & -2 & 0 \\ 6 & 0 & -2 \end{bmatrix}$$

$\frac{\partial f}{\partial x_1} = -2x_1 + 2x_2 + 6x_3 + 4$, $\frac{\partial f}{\partial x_2} = -2x_2 + 2x_1$, $\frac{\partial f}{\partial x_3} = -2x_3 + 6x_1 - 5$

$\frac{\partial^2 f}{\partial x_1^2} = -2$, $\frac{\partial^2 f}{\partial x_2^2} = -2$, $\frac{\partial^2 f}{\partial x_3^2} = -2$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2$, $\frac{\partial^2 f}{\partial x_1 \partial x_3} = 6$, $\frac{\partial^2 f}{\partial x_2 \partial x_3} = 0$

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$J_1 = |-2| = -2 < 0$

$J_2 = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 4 - 4 = 0$

$J_3 = \begin{vmatrix} -2 & 2 & 6 \\ 2 & -2 & 0 \\ 6 & 0 & -2 \end{vmatrix} = -2(4) - 2(-4) + 6(12) = -8 + 8 + 72 = 72 > 0$

As $J_1 < 0$, $J_2 = 0$, $J_3 > 0$:

The given matrix A is indefinite.

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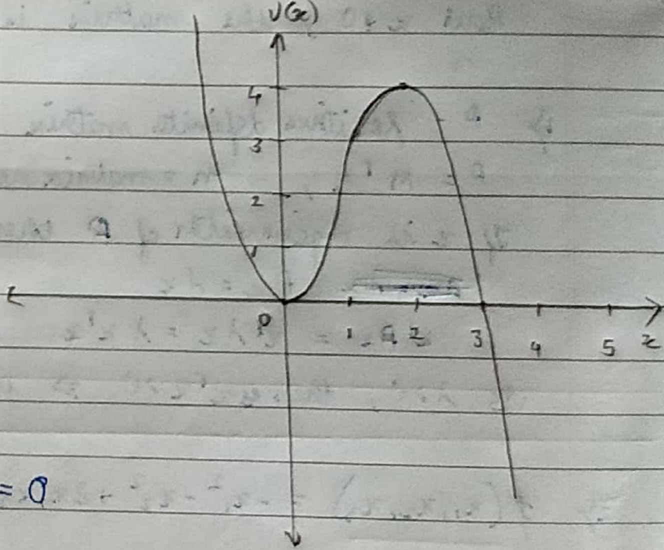
3) $V(x) = 3x^2 - x^3$

$\frac{dV}{dx} = 6x - 3x^2$

$\frac{d^2V}{dx^2} = 6 - 6x$



To find equilibrium points :

$\frac{dV}{dx} = 0$

$\therefore 6x - 3x^2 = 0 \Rightarrow 3x(2-x) = 0$

$\Rightarrow x = 0$ or $x = 2$.

$\left.\frac{d^2V}{dx^2}\right|_{x=0} = 6 > 0 \Rightarrow$ $x=0$ is point of minima

$\left.\frac{d^2V}{dx^2}\right|_{x=2} = 6 - 12 = -6 < 0 \Rightarrow x = 2$ is point of maxima.

From these points :

$x = 0$ is the point of stable equilibrium.

$x = 2$ is point of unstable equilibrium.

4) $f(x_1, x_2, x_3) = x_2^2 x_3 + x_1 e^{x_3}$

$x^* = (1, 0, -2)^T$

$$f(x) = f(x^*) + df(x^*) + \frac{1}{2!} d^2f(x^*)$$

$$f\begin{pmatrix}1\\0\\-2\end{pmatrix} = e^{-2}$$

$$df\begin{pmatrix}1\\0\\-2\end{pmatrix} = h_1 \frac{df}{dx_1}\begin{pmatrix}1\\0\\-2\end{pmatrix} + h_2 \frac{df}{dx_2}\begin{pmatrix}1\\0\\-2\end{pmatrix} + h_3 \frac{df}{dx_3}\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$= h_1 \cdot e^{-2} + h_2 \times 0 + h_3 \cdot e^{-2}$$

$$\therefore df\begin{pmatrix}1\\0\\-2\end{pmatrix} = h_1 e^{-2} + h_3 e^{-2}$$

$$d^2f\begin{pmatrix}1\\0\\-2\end{pmatrix} = \left[h_1^2 \frac{d^2f}{dx_1^2} + h_2^2 \frac{d^2f}{dx_2^2} + h_3^2 \frac{d^2f}{dx_3^2} + 2h_1h_2 \frac{d^2f}{dx_1 dx_2} + 2h_1h_3 \frac{d^2f}{dx_1 dx_3} + 2h_2h_3 \frac{d^2f}{dx_2 dx_3}\right]\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$= \left[h_1^2(0) + h_2^2(2x_3) + h_3^2(x_1 e^{x_3}) + 2h_1h_2(0) + 2h_2h_3(2x_2) + 2h_1h_3(e^{x_3})\right]\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$\therefore d^2f\begin{pmatrix}1\\0\\-2\end{pmatrix} = -4h_2^2 + e^{-2}h_3^2 + 2h_1h_3 e^{-2}$$

$$\boxed{f(x) \simeq e^{-2} + e^{-2}(h_1 + h_3) + \frac{1}{2!}\left(-4h_2^2 + e^{-2}h_3^2 + 2h_1h_3 e^{-2}\right)}$$

$h_1 = x_1 - 1$, $h_2 = x_2$, $h_3 = x_3 + 2$

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5) $f(X) = a + b^T X + \frac{1}{2} X^T [C] X$

$X = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$, $b = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$, $[C] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

β_1, β_2 are eigenvalues of $[C]$.

a) $\beta_1, \beta_2 > 0$ means eigenvalues are positive and hessian is positive definite. Stationary point is a minimum point.

b) $\beta_1, \beta_2 < 0$ means hessian is negative definite. Stationary point is a maximum point.

c) $\beta_1 > 0, \beta_2 < 0$ means hessian is indefinite. Stationary point is a saddle point.

d) $\beta_1 > 0$, $\beta_2 = 0$ means hessian is positive semi-definite. Stationary point is a local minimum but neither min / max over the overall domain.

Plots: (drawn on MATLAB and shown after this)

$f(X) = \frac{1}{2} X^T [C] X$

a) $\beta_1, \beta_2 > 0$

$C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow$ eigenvalues $= \{3, 3\}$

$\Rightarrow f(X) = \frac{1}{2} [x_1 \ x_2] \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{3}{2}(x_1^2 + x_2^2)$

b) $\beta_1, \beta_2 < 0$

$C = \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix} \Rightarrow$ eigenvalues $= \{-1, -5\}$

$f(X) = \frac{1}{2} [x_1 \ x_2] \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} [x_1 \ x_2] \begin{bmatrix} -3x_1 - 2x_2 \\ -2x_1 - 3x_2 \end{bmatrix}$

$\therefore f(X) = \frac{1}{2}\left(-3x_1^2 - 2x_1x_2 - 2x_1x_2 - 3x_2^2\right) = \frac{1}{2}\left(-3x_1^2 - 4x_1x_2 - 3x_2^2\right)$

c) $\beta_1 > 0$, $\beta_2 < 0$

$$[C] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow \text{eigenvalues} = \{3, -3\}$$

$$f(x) = \frac{3}{2}\left(x_1^2 - x_2^2\right)$$

d) $\beta_1 > 0$, $\beta_2 = 0$

$$[C] = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{eigenvalues} = \{0, 3\}$$

$$f(x) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3x_1 \\ 0 \end{bmatrix} = \frac{3}{2}x_1^2$$

★ For a circle contour, the $\beta_1, \beta_2 > 0$ or $\beta_1, \beta_2 < 0$ with non-diagonal values '0' for the hessian matrix.

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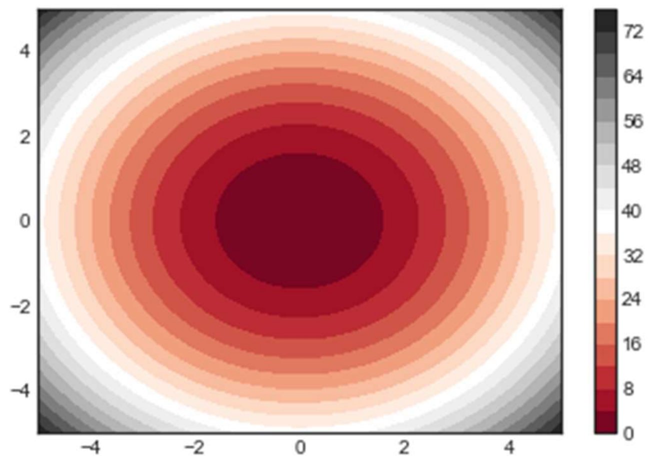
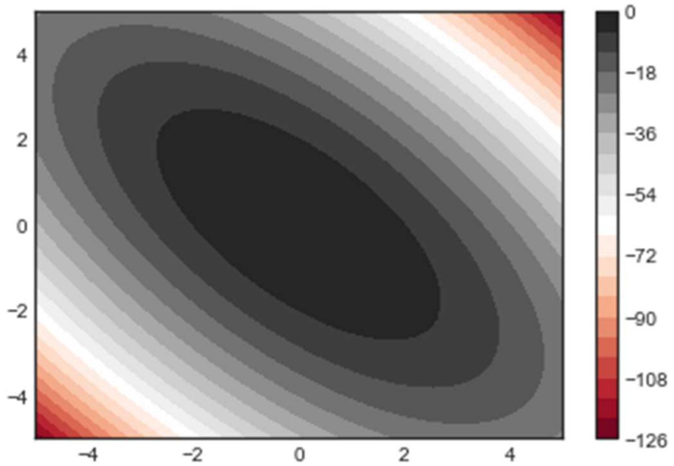
a. Positive Definite Hessian: (both positive eigenvalues)



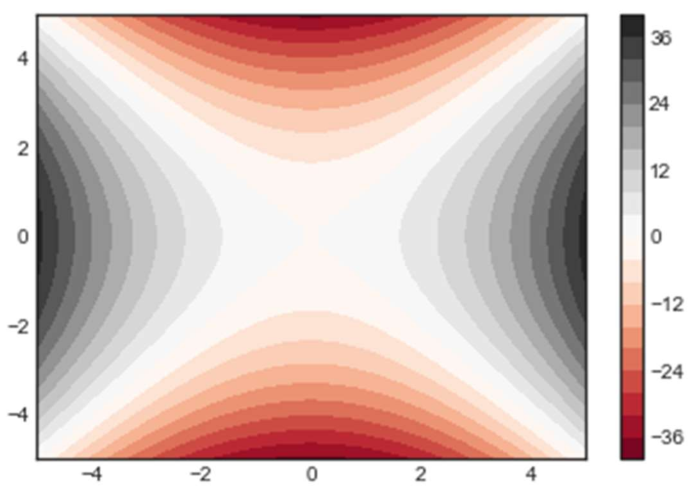
For a more generalised solution, both positive eigenvalues give elliptic contours (even though above shows a circle) with (0,0) as centres

b. Negative Definite Hessian: (both negative eigenvalues)



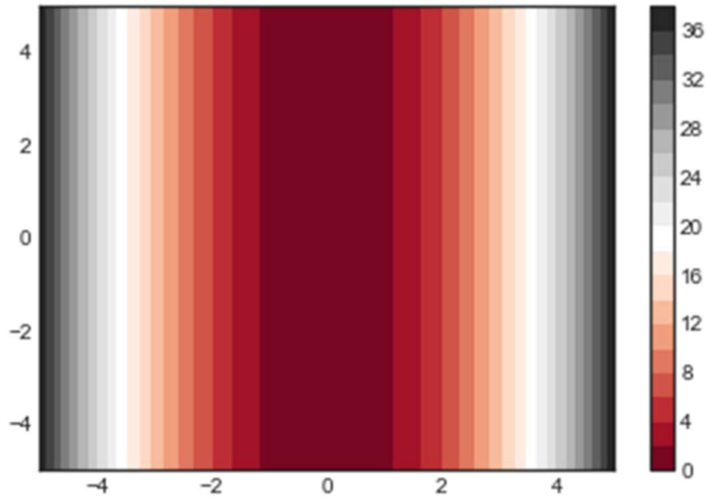
For a more generalised solution, both negative eigenvalues give elliptic contours with (0,0) as centres

c. Indefinite Hessian: (opposite signed eigenvalues)



As opposite signed eigenvalues give hyperbolic contours with saddle point centred at (0,0)

d. Positive semi-definite Hessian: (one positive and one zero eigenvalue)



Given condition gives us straight-line contours

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Maximise :

6) $f(x) = \sin(A) + \sin(B) + \sin(C)$, $x = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

Constraints :

$g : A + B + C - \pi = 0$

$\pi > A > 0$, $\pi > B > 0$, $\pi > C > 0$

Lagrange multipliers method :

$L = f + \lambda g$

$\frac{\partial L}{\partial A} = \frac{\partial f}{\partial A} + \lambda . \frac{\partial f}{\partial A} = \cos A + \lambda = 0$

$\frac{\partial L}{\partial B} = \cos B + \lambda = 0$

$\frac{\partial L}{\partial C} = \cos C + \lambda = 0$

$\therefore \; -\lambda = \cos A = \cos B = \cos C$

From this we can see, $A = B = C$.

$\therefore \; x^* = \begin{pmatrix} \pi/3 \\ \pi/3 \\ \pi/3 \end{pmatrix}$

$\Rightarrow f(x^*) = 3 \sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$.

7) $f(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

$g_1(x) = x_1 - x_2 = 0$

$g_2(x) = x_1 + x_2 + x_3 - 1 = 0$

a) By direct substitution :

$x_1 = x_2$, $x_3 = 1 - 2x_1$

$\Rightarrow f(x_1) = \frac{1}{2}\left(x_1^2 + x_1^2 + (1-2x_1)^2\right)$

$= \frac{1}{2}\left(x_1^2 + x_1^2 + 1 + 4x_1^2 - 4x_1\right) = \frac{1}{2}\left(6x_1^2 - 4x_1 + 1\right)$

$\frac{df}{dx_1} = 12x_1 - 4 = 0 \Rightarrow x_1 = \frac{1}{3} \Rightarrow x_2 = \frac{1}{3}, \; x_3 = \frac{1}{3}$

$f(x^*) = \frac{1}{2} \times \frac{\cancel{3} \times 1}{\cancel{9}\,3} = \frac{1}{6}$

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b) By constrained variation:

Let x_1 be the independent variable.

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

As the det $\neq 0$, we can go ahead with x_1 as the only independent var.

Necessary condition:

$$\begin{vmatrix} \partial f/\partial x_1 & \partial f/\partial x_2 & \partial f/\partial x_3 \\ \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \partial g_1/\partial x_3 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \partial g_2/\partial x_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x_1(-1) - x_2(1) + x_3(2) = 0 \Rightarrow x_1 + x_2 - 2x_3 = 0 \rightarrow (1)$$

With g_1, g_2 and (1):

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 = 0, \quad 2x_2 + x_3 = 1, \quad -3x_3 = -1$$

$$\Rightarrow x_3 = \frac{1}{3} \Rightarrow x_2 = \frac{1}{3} \Rightarrow x_1 = \frac{1}{3}$$

$$x^* = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \quad f(x^*) = \frac{1}{6}$$

c) By Lagrange multiplier method :

$$L = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) + \lambda_1 (x_1 - x_2) + \lambda_2 (x_1 + x_2 + x_3 - 1)$$

$\frac{\partial L}{\partial x_1} = x_1 + \lambda_1 + \lambda_2 = 0 \rightarrow (1)$

$\frac{\partial L}{\partial x_2} = x_2 - \lambda_1 + \lambda_2 = 0 \rightarrow (2)$

$\frac{\partial L}{\partial x_3} = x_3 + \lambda_2 = 0 \rightarrow (3)$

$\frac{\partial L}{\partial \lambda_1} = x_1 - x_2 = 0 \rightarrow (4)$

$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + x_3 - 1 = 0 \rightarrow (5)$

From (4) : $x_1 = x_2$

$\Rightarrow$ (5) : $x_3 = 1 - 2x_1$

(3) : $x_3 = -\lambda_2$

(2) and (1) : $-\lambda_1 - \cancel{\lambda_2} = \lambda_1 - \cancel{\lambda_2} \Rightarrow \lambda_1 = 0$

$\Rightarrow x_1 = -\lambda_2, \; x_3 = -\lambda_2, \; x_2 = -\lambda_2$

$\therefore -\lambda_2 = 1 + 2\lambda_2 \Rightarrow \lambda_2 = -\frac{1}{3}$

$\therefore x_1 = x_2 = x_3 = \frac{1}{3} \Rightarrow x^* = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \Rightarrow f(x^*) = \frac{1}{2} \times \cancel{3} \times \frac{1}{\cancel{9}_3} = \frac{1}{6}$

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a) $f(x) = 4(x_1 + x_2 + x_3)$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$g(x) = x_1 x_2 x_3 - 1000 = 0$

Lagrange Multiplier :

$L = 4(x_1+x_2+x_3) + \lambda(x_1 x_2 x_3 - 1000)$

$\frac{\partial L}{\partial x_1} = 4 + \lambda x_2 x_3 = 0 \rightarrow (1)$

$\frac{\partial L}{\partial x_2} = 4 + \lambda x_1 x_3 = 0 \rightarrow (2)$

$\frac{\partial L}{\partial x_3} = 4 + \lambda x_1 x_2 = 0 \rightarrow (3)$

$\frac{\partial L}{\partial \lambda} = x_1 x_2 x_3 - 1000 = 0 \rightarrow (4)$

$\therefore \; -\frac{4}{\lambda} = x_2 x_3 = x_1 x_3 = x_1 x_2$

$\Rightarrow$ As $x_1 x_2 x_3 = 1000$:

$-\frac{4}{\lambda} = \frac{1000}{x_1} = \frac{1000}{x_2} = \frac{1000}{x_3} \Rightarrow x_1 = x_2 = x_3 = -250\lambda$

$\Rightarrow (-250\lambda)^3 = 1000 \Rightarrow \lambda^3 = \left(-\frac{10}{250}\right)^3 \Rightarrow \lambda^* = -0.04$

$\Rightarrow x_1^* = x_2^* = x_3^* = 10\,m, \quad X^* = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$

b) $g(x) = x_1 x_2 x_3 - b = 0$ Here, b changed from -1000 to -1200.
Taking into consideration the above optimal values, Changes in 'f' occurs as follows :

$df = \lambda^* db$

$\lambda^* = -0.04, \quad db = -1200 - (-1000) = -200$

$\therefore df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = 4(dx_1 + dx_2 + dx_3) = -0.04 \times -200$

$\Rightarrow df = 8 = 4(dx_1 + dx_2 + dx_3)$

As $dx_1 = dx_2 = dx_3 \Rightarrow dx_1 = dx_2 = dx_3 = \frac{2}{3} \Rightarrow x_{1\,new} = x_{2\,new} = x_{3\,new} = \frac{32}{3}$

$\therefore (x_1 = x_2 = x_3)_{New} = \underline{10.667\ m}$

c) As following the previous optimal solution of $x_1 = x_2 = x_3$, we would have each side : $x_1 = x_2 = x_3 = (1200)^{1/3} = 10.6265\,m$

$\therefore$ Error $= \dfrac{10.667 - 10.6265}{10.6265} \times 100 = \underline{0.38\,\%}$

d) $f(x,y) = x^2+y^2$

$g(x,y) = xy - 1 = 0$

<u>Lagrange multiplier method:</u>

$L = x^2+y^2+\lambda(xy-1)$

$\dfrac{\partial L}{\partial x} = 2x + \lambda y = 0 \rightarrow (1)$

$\dfrac{\partial L}{\partial y} = 2y + \lambda x = 0 \rightarrow (2)$

$\dfrac{\partial L}{\partial \lambda} = xy - 1 = 0 \rightarrow (3)$

(3) : $x = \dfrac{1}{y}$

(1) : $\dfrac{2}{y} + \lambda y = 0 \Rightarrow$ $y = \sqrt{\dfrac{-2}{\lambda}}$

$\Rightarrow x = \sqrt{\dfrac{-\lambda}{2}}$

(2) : $2\sqrt{\dfrac{-2}{\lambda}} + \lambda \cdot \sqrt{\dfrac{-\lambda}{2}} = 0 \Rightarrow 4\left(\dfrac{+2}{\lambda}\right) = +\lambda^2\left(\dfrac{\lambda}{2}\right) \Rightarrow \lambda^4 = 16$

$\therefore \lambda = 2$ or -2

But $\lambda = 2$ is invalid as $\sqrt{\dfrac{-2}{2}} \Rightarrow \sqrt{-1}$ is imaginary.

$\therefore \lambda = -2 \Rightarrow x = \pm 1, y = \pm 1$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\Rightarrow \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} +1 \\ -1 \end{bmatrix} \text{ is invalid as } xy = 1\right)$

b) $\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$\nabla g = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

∇f is linearly magnified as compared to ∇g.

$\nabla f = 2 \nabla g$

This tells that f, g have their optimal at tangent to each other.

c) $\nabla f = 2 \nabla g$

as $\lambda = -2$

$\nabla f = -\lambda \nabla g$

$\therefore$ gradient of objective function is a linear combination of gradient of constraints.

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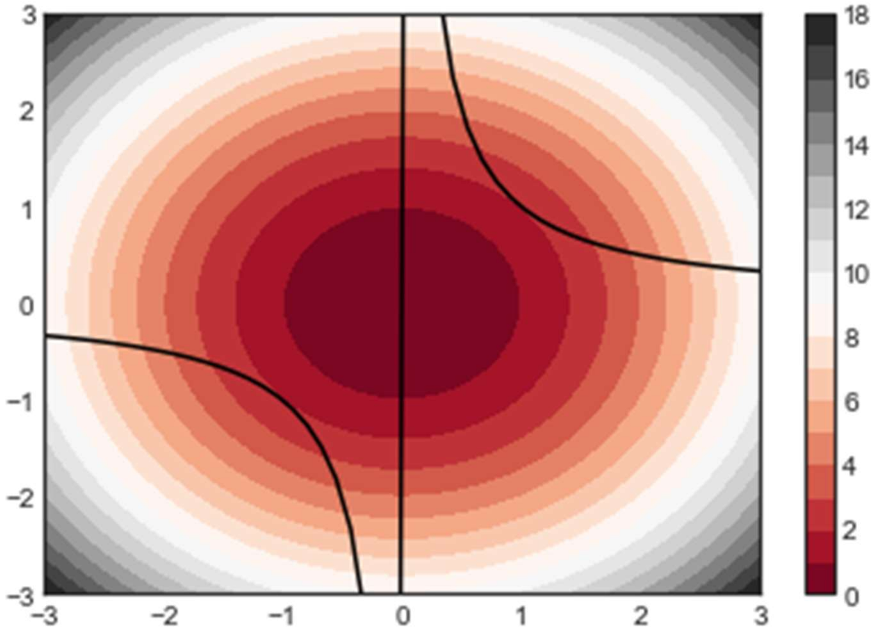
9.



Here the given ‘g’ constraint equation intersects at the tangent to circle with sqrt(2) as the radius with (1,1) and (-1,-1) being the optimal solution.