Optimization Methods for Mechanical Design - ME7223

Assignment 2

Max marks 20 **Due Date:** 10 Sep 2020

Instructions

- Answer all questions.
- Assume any missing data appropriately.
- Append the graphs to the scanned version of the answer sheets.
- Contact the TA (Sridatta Rapaka) if you have any questions.

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1. Consider a rectangular matrix M with independent columns. **(3)**

   (a) Show that $M^T M$ is positive definite.

   (b) Conversely, any positive definite matrix P can be expressed as $P = M^T M$, where M is defined as above. Show that $\mathbf{x}^T P\mathbf{x} > 0$ if $\mathbf{x} \neq \mathbf{0}$.

2. Express the function **(2)**

$$f\left(x_1, x_2, x_3\right) = -x_1^2 - x_2^2 + 2x_1x_2 - x_3^2 + 6x_1x_3 + 4x_1 - 5x_3 + 2$$

   in matrix form as

$$f(\mathbf{X}) = \frac{1}{2}\mathbf{X}^{\mathrm{T}}[A]\mathbf{X} + \mathbf{B}^{\mathrm{T}}\mathbf{X} + C,$$

   and determine whether the matrix A is positive definite, negative definite, or indefinite.

3. The potential energy of a particle moving along the x direction is given by, **(1)**

$$U(x) = 3x^2 - x^3.$$

   Plot the potential energy as a function of x. Identify all the possible equilibrium points, and label the stable equilibrium position.

4. Find the second-order Taylor's series approximation of the function **(2)**

$$f\left(x_1, x_2, x_3\right) = x_2^2x_3 + x_1e^{x3}$$

   at the point $(1, 0, -2)$.

5. It is possible to establish the nature of stationary points of an objective function based on its quadratic approximation. For this, consider the quadratic approximation of a two-variable function as **(3)**

$$f(\mathbf{X}) \approx a + \mathbf{b}^{\mathrm{T}}\mathbf{X} + \frac{1}{2}\mathbf{X}^{\mathrm{T}}[c]\mathbf{X},$$

where

$$\mathbf{X} = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}, \quad \mathbf{b} = \left\{ \begin{array}{c} b_1 \\ b_2 \end{array} \right\}, \quad \textit{and} \quad [c] = \left[\begin{array}{cc} c_{11} & c_{12} \\ c_{12} & c_{22} \end{array} \right].$$

If the eigenvalues of the Hessian matrix $[c]$, are denoted as β_1, β_2, identify the nature of the contours of the objective function and the type of stationary point in each of the following situations.

(a) $\beta_1, \beta_2 > 0$

(b) $\beta_1, \beta_2 < 0$

(c) $\beta_1 > 0, \beta_2 < 0$

(d) $\beta_1 > 0, \beta_2 = 0$

For each of the above cases, plot the contours for the function of your choice, and mark the stationary point. Interpret the contour plots and justify your previous answers. You may choose $a = b_1 = b_2 = 0$ for simplicity. With these assumptions, under what conditions will the contour lines of the function be circular ?

6. For a triangle ABC, find the maximum value of $\sin(A) + \sin(B) + \sin(C)$. Formulate it as a constrained optimization problem. **(1)**

7. Minimise **(3)**

$$f(\mathbf{X}) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$$

subject to

$$g_1(\mathbf{X}) = x_1 - x_2 = 0,$$
$$g_2(\mathbf{X}) = x_1 + x_2 + x_3 - 1 = 0$$

by

(a) direct substitution

(b) constrained variation

(c) Lagrange multiplier method.

8. (a) Find the dimensions of a rectangular box of volume V = 1000 m^3 for which the total length of the 12 edges is a minimum using the Lagrange multiplier method. **(2)**

(b) Find the change in the dimensions of the box when the volume is changed to 1200 m^3 by using the value of λ^* found in part (a).

(c) Compare the solution found in part (b) with the exact solution.

9. (a) Minimise the function **(3)**

$$f(x, y) = x^2 + y^2$$

subject to

$$g(x, y) = xy = 1,$$

using the Lagrange multiplier method. Find the solution point(s) and the corresponding Lagrange multiplier(s).

(b) Find ∇f and ∇g at the solution point. How are they related? What implication does this have on the contour lines of f and g ?

(c) How does the relation between the gradients ∇f and ∇g computed at the solution point compare with the Lagrange multiplier ?