Eigenvectors

Definition

A vector whose direction remains unchanged when a linear transformation is applied to it.
Let A be a square matrix, and $\triangledown$ a non-zero vector. In general, the eigenvector $\triangledown$ of A is the vector for which the following holds true:

$$\mathbf{A}\triangledown = \boldsymbol{\lambda}\triangledown$$

where, λ is a scalar value called the eigenvalue.

Motivation

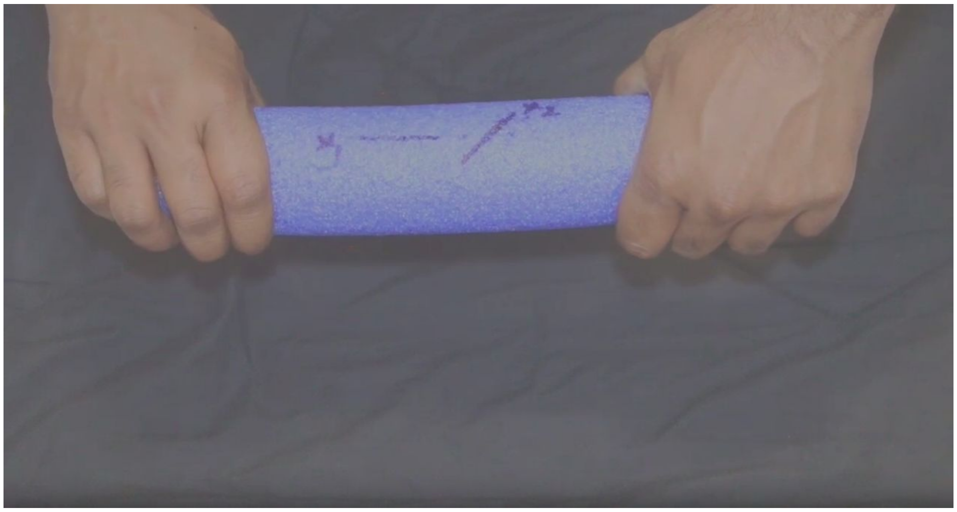
The mathematical understanding of eigenvectors helps in modelling various problems spread across various engineering fields, be it the collapse of the Original Tacoma Narrows bridge, the movement of shafts in a vehicle or in machine learning applications. Hence, it is important to analyse the theoretical as well as applied concept of eigenvectors.

Bird's Eye View

It is an extremely important yet undervalued (not undervector-ed) topic, especially in applications of computer vision and machine learning in general.

Perhaps a visual application will help understand it better.

Consider the vectors x_1 and x_2 given below:



<source: https://www.youtube.com/watch?v=R13Cwgmpuxc>

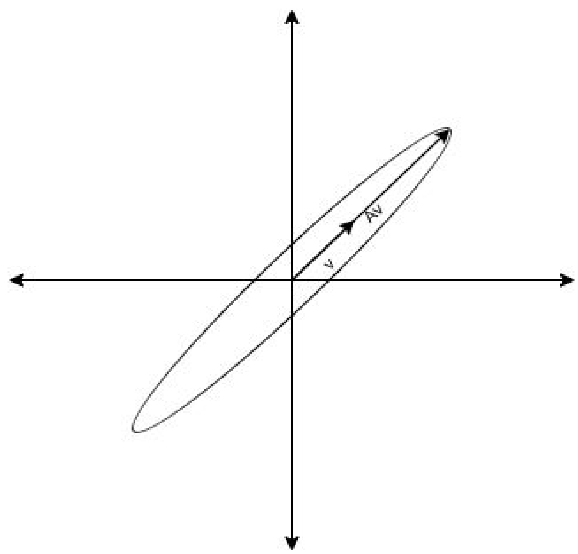
When rotated along an axis, notice the change in the direction of each of the vectors.

<source: https://www.youtube.com/watch?v=R13Cwgmpuxc>

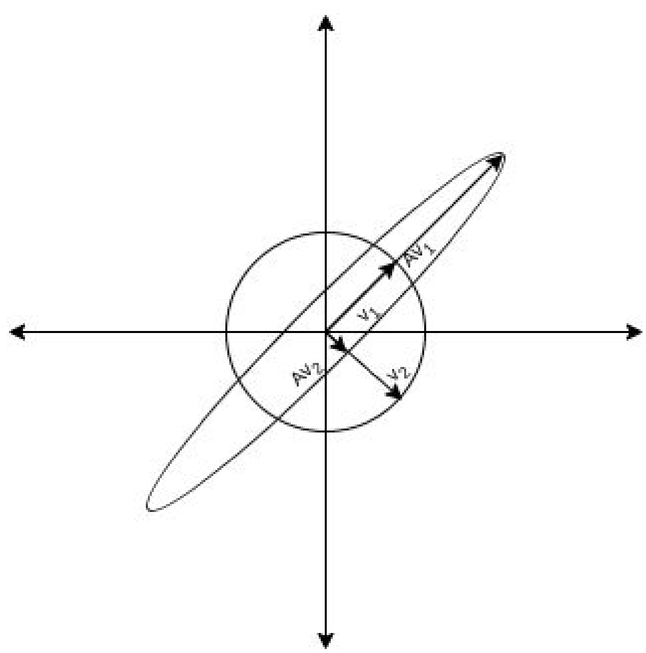
The direction of vector x_2 remains the same, while that of x_1 does not. This ‘rotation’ is called a linear transformation, and x_2 is called an eigenvector, as its direction remains unchanged after it undergoes said linear transformation.

Context of the Definition

By definition, the scalar λ and vector v are the eigenvalue and eigenvector of A respectively if A▿ = λ ▿. Visually, A▿ lies along the same line as ▿.

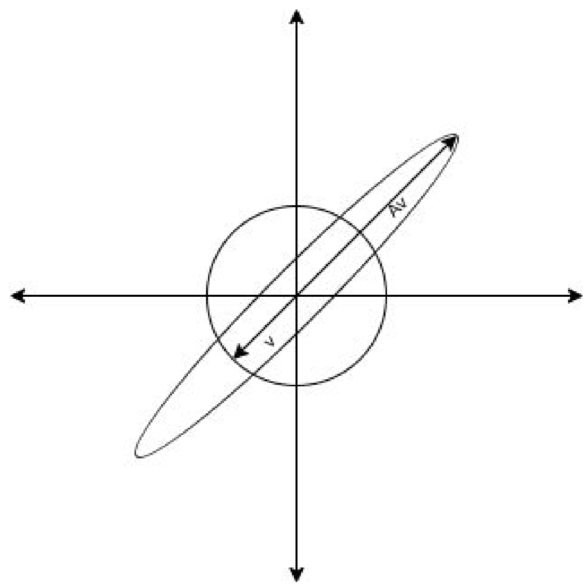


If the eigenvalue is greater than one, corresponding A▿ expands. If it is smaller than one, it shrinks.



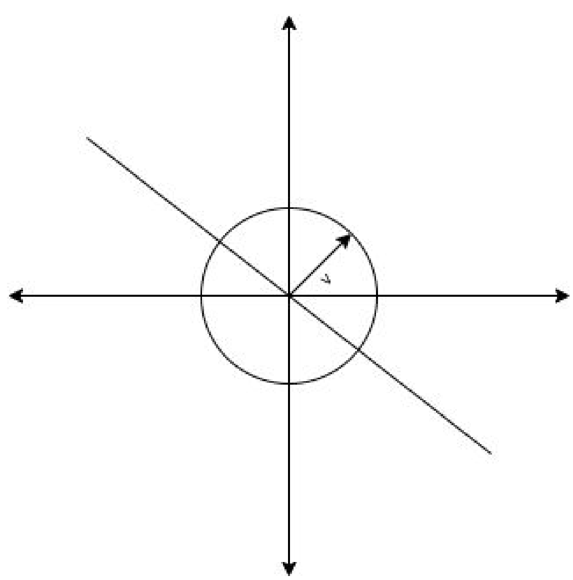
Here, the eigenvalue for vector v_2 is less than one and that for vector v_1 is greater than one. Hence, after applying the linear transformation A, v_2 shrinks while v_1 expands.
Here are some interesting visualizations:

1. Vector ▿ and Image A▿ are in the opposite direction.



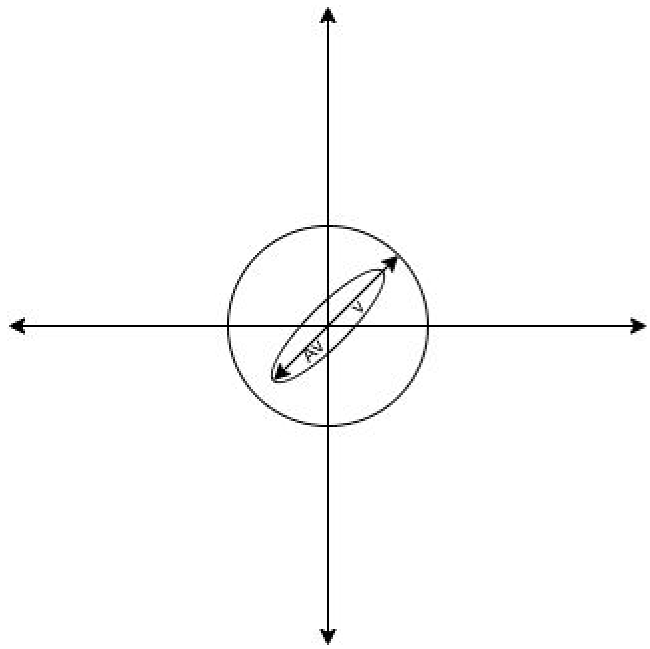
Implication: Eigenvalue is negative

2. Image of the circle (of radius |▿|) is a line.



Implication: One eigenvalue is zero

3. Image of the circle is inside the circle.



Implication: Eigenvalues are negative.

We know,

$$\mathbf{A}\triangledown = \boldsymbol{\lambda}\triangledown$$
$$\Rightarrow \mathbf{A}\triangledown - \boldsymbol{\lambda}\triangledown = \mathbf{0}$$

Hence, $(A - \lambda I).\triangledown = 0$, where I is an identity matrix of the same dimensions as A.

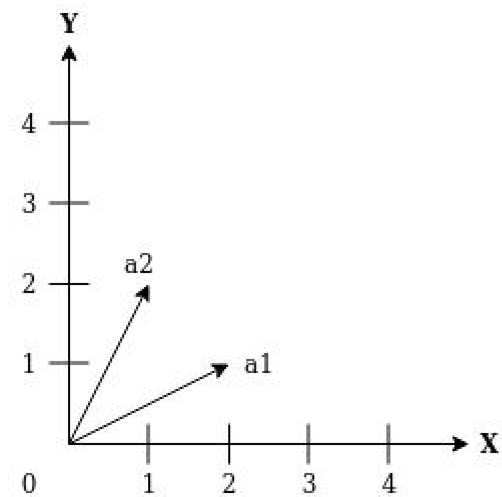
Assuming that $\triangledown$ is not a null vector, the above equation can only be defined if $(A - \lambda I)$ is not invertible. Hence, its determinant must be equal to zero. Therefore, to find the eigenvectors of A, we simply solve the equation

$$\mathbf{det(A - \lambda I) = 0}$$

Consider the following example.

Let A be a matrix with columns a1 and a2 (shown as arrows).

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



We know det(A - λ I) = 0, i.e.,

$$det(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}) = 0$$

(where L ≡ λ)

$$=> det(\begin{bmatrix} 2-L & 1 \\ 1 & 2-L \end{bmatrix}) = 0$$

⇒ $(2 - \lambda)^2 - (1)^2 = 0$

⇒ $(2 - \lambda) = \pm 1$

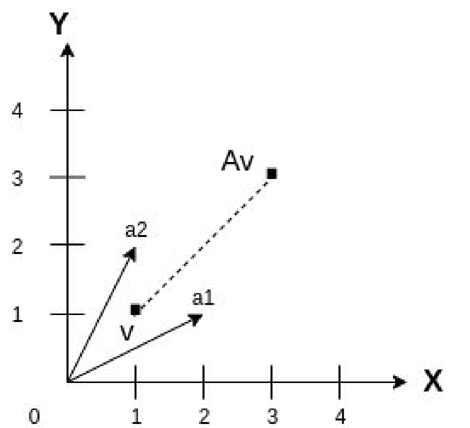
⇒ $\lambda = 1, 3$

Hence, we get the eigenvalues λ = 1, 3.

To find the eigenvectors:

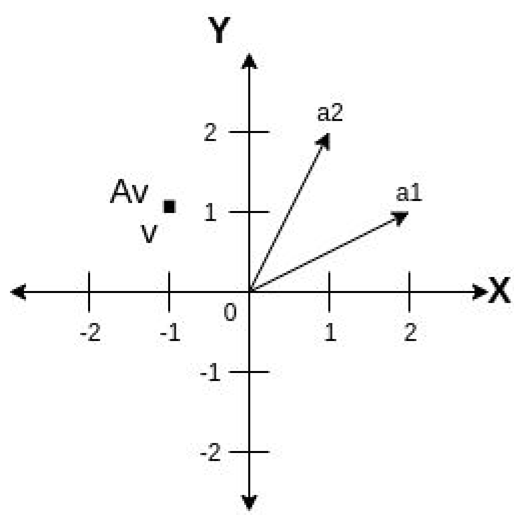
a) Calculate with λ = 3

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = 3 * \begin{bmatrix} x1 \\ x2 \end{bmatrix} \quad => v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} => Av = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$



b) Calculate with λ = 1

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = 1 * \begin{bmatrix} x1 \\ x2 \end{bmatrix} \quad => v = \begin{bmatrix} -1 \\ 1 \end{bmatrix} => Av = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Applications

- Principal Component Analysis (PCA) involves linear dimensionality reduction using Singular Value Decomposition of the data to project it to a lower dimensional space - used in Machine Learning applications.
- Image Segmentation - partitioning a digital image into multiple segments
- Google's PageRank - an algorithm used to rank web pages in search results.
- Spectral Clustering - used to identify communities of nodes in a graph based on the edges connecting them.

History

- 18^{th} century:
  - Euler - studied rotational motion of a rigid body, importance of principal axes
  - Lagrange - principal axes are eigenvectors of inertia matrix
- 19^{th} century:
  - Cauchy - classified quadric surfaces, generalized it to arbitrary dimensions; coined the term *racine caractéristique*, now called eigenvalue.
  - Fourier - solved heat equation by separation of variables
  - Sturm - developed Fourier's ideas, combined with Cauchy's - conclusion: real symmetric matrices have real eigenvalues.
  - Hermite - extension: Hermitian matrices
  - Brioschi - eigenvalues of orthogonal matrices lie on the unit circle
  - Sturm–Liouville theory
- 20^{th} century:
  - Hilbert - viewed integral operators as infinite matrices; first to use the German word '*eigen*'.
  - Von Mises - first numerical algorithm for computing eigenvalues and eigenvectors
  - John G.F. Francis, Vera Kublanovskaya - QR Algorithm

Pause and Ponder

- How essentially was the concept of eigenvectors identified and mathematically transcribed?
- Think about situations where you see eigenvectors play an essential role in day to day life.

References and Further Reading

References

[1] https://www.khanacademy.org/math/linear-algebra/alternate-bases/Eigen-everything
[2] http://www.math.iitb.ac.in/~ars/MA106/week7.pdf
[3] https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Further Reading

[1] https://math.mit.edu/~gs/linearalgebra/ila0601.pdf
[2] http://setosa.io/ev/eigenvectors-and-eigenvalues/
[3] https://graphics.stanford.edu/courses/cs205a-13-fall/assets/notes/chapter5.pdf