Chap 4.5

There is a close relationship between the z-transform and the DTFT

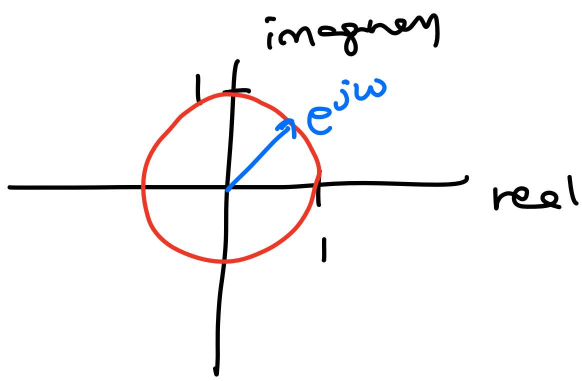
z-transform

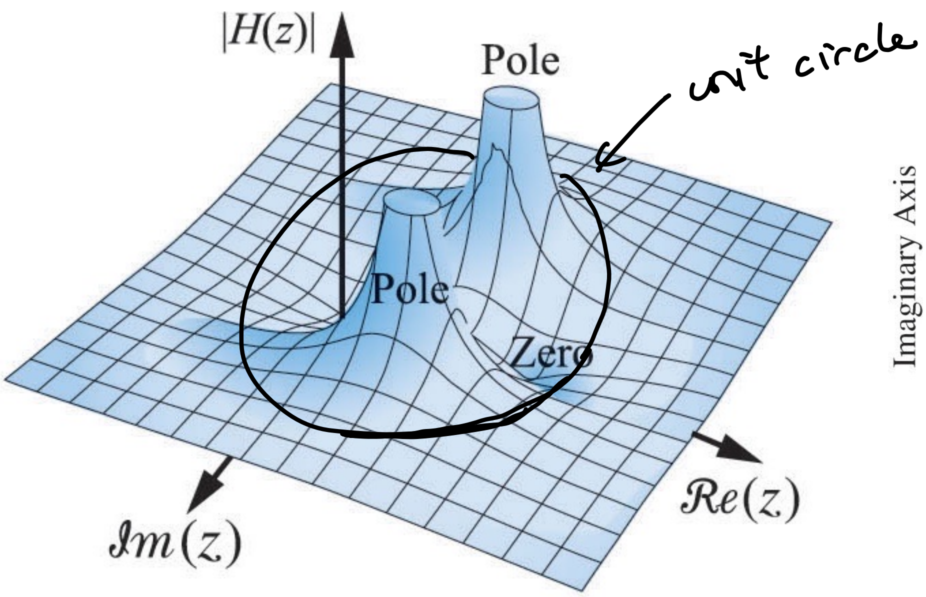
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

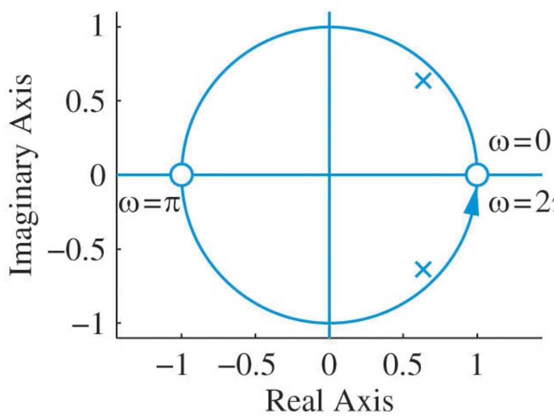
DTFT is the z-transform when $z = e^{jw}$

DTFT

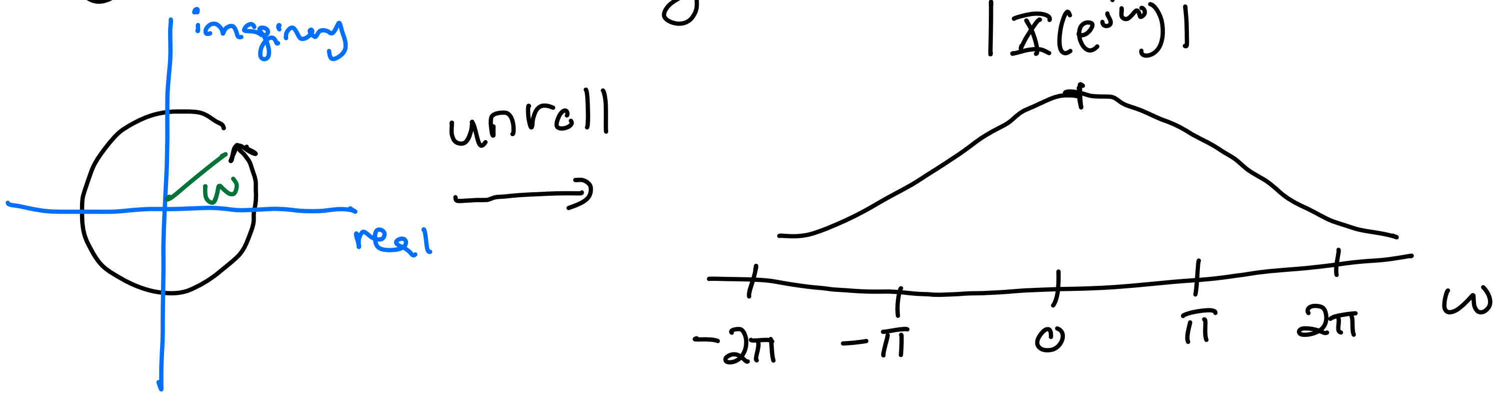
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$







Since the DTFT is only on the unit circle



The behavior of the DTFT is the behavior of the z-transform on the unit circle.

If ROC of the z-transform does not contain the unit circle, then the DTFT does not exist

Strangely, it is possible that the DTFT exist but not the z-transform

Since DTFT is a special case of the z-transform, all the properties of the z-transform translate to similar properties for the Fourier transform

Linearity

$$a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{\text{DTFT}} a_1 X_1(e^{jw}) + a_2 X_2(e^{jw})$$

Time shift

If $x[n] \xleftrightarrow{\text{DTFT}} X(e^{jw})$, then

$$x[n-k] \xleftrightarrow{\text{DTFT}} e^{-jwk} X(e^{jw})$$

Conjugation of a complex sequence

$$x^*[n] \xleftrightarrow{\text{DTFT}} X^*(e^{-jw})$$

Convolution of sequences

$$y[n] = x[n] * h[n] \xleftrightarrow{\text{DTFT}} Y(e^{jw}) = X(e^{jw}) H(e^{jw})$$

multiplication / windowing

$$y[n] = x[n]\,w[n] \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) * W(e^{j\omega})$$

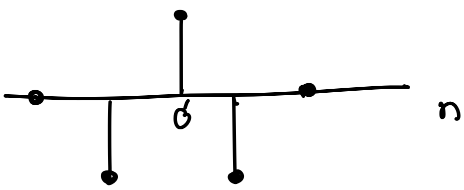
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,W(e^{j(\omega-\theta)})\,d\theta$$

	Property	Sequence	Transform
		$x[n]$	$\mathcal{F}\{x[n]\}$
1.	Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(e^{j\omega})+a_2X_2(e^{j\omega})$
2.	Time shifting	$x[n-k]$	$e^{-jk\omega}X(e^{j\omega})$
3.	Frequency shifting	$e^{j\omega_0 n}x[n]$	$X[e^{j(\omega-\omega_0)}]$
4.	Modulation	$x[n]\cos\omega_0 n$	$\frac{1}{2}X[e^{j(\omega+\omega_0)}]+\frac{1}{2}X[e^{j(\omega-\omega_0)}]$
5.	Folding	$x[-n]$	$X(e^{-j\omega})$
6.	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
7.	Differentiation	$nx[n]$	$-j\dfrac{dX(e^{j\omega})}{d\omega}$
8.	Convolution	$x[n]*h[n]$	$X(e^{j\omega})H(e^{j\omega})$
9.	Windowing	$x[n]w[n]$	$\dfrac{1}{2\pi}\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta$
10.	Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \dfrac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$	
11.	Parseval's relation	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \dfrac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

Some back of the envelop tricks on the DTFT on Symmetry properties

Even function

$x[n]$ is an even sequence if $x[n] = x[-n]$



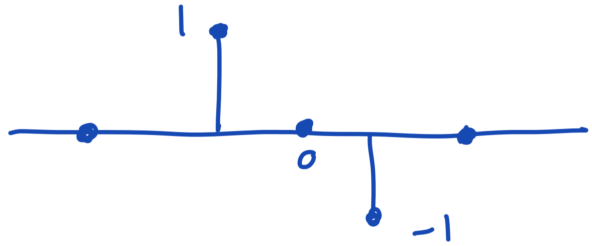
$F(e^{jw})$ is an even function if

$$F(e^{jw}) = F(e^{-jw})$$

The magnitude spectrum $|X(e^{jw})|$ is always an even function of w

Odd function

$x[n]$ is an odd sequence if $x[n] = -x[n]$



$F(e^{jw})$ is an odd function if

$$F(e^{jw}) = -(e^{-jw})$$

The phase spectrum $\angle X(e^{jw})$ is always an odd function of w

In addition, consider

$$x[n] = x_R[n] + j\, x_I[n]$$

if $x[n]$ is a real signal, that means $x_I[n] = 0$

if $x[n]$ is an imaginary signal, that means $x_R[n] = 0$

Similarly

$$X(e^{jw}) = X_R(e^{jw}) + j X_I(e^{jw})$$

if $X(e^{jw})$ is a real function, that means $X_I(e^{jw}) = 0$

if $X(e^{jw})$ is an imaginary function, that means

$X_R(e^{jw}) = 0$

It turns out

$X_R(e^{jw})$ is always an even function,

so $X_R(e^{-jw}) = X_R(e^{jw})$

$X_I(e^{jw})$ is always an odd function,

so $X_I(e^{-jw}) = -X_I(e^{jw})$

$|X(e^{jw})|$ is always an even function,

so $|X(e^{-jw})| = |X(e^{jw})|$

$\angle X(e^{jw})$ is always an odd function,

so $\angle X(e^{-jw}) = -\angle X(e^{jw})$