Chap 2.5

Convolution allows us to compute the output of an LTI system with impulse response $h[n]$ for any input

$$y[n] = \sum_{k=-\infty}^{\infty} x[n]h[n-k] \quad , -\infty < n < \infty$$

$$y[n] = x[n] * h[n]$$

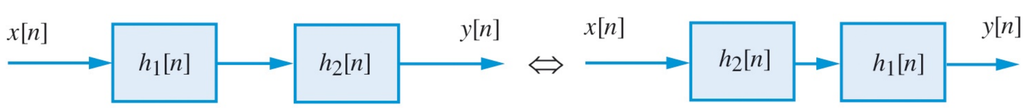
Convolution properties

Identity : $x[n] * \delta[n] = x[n]$

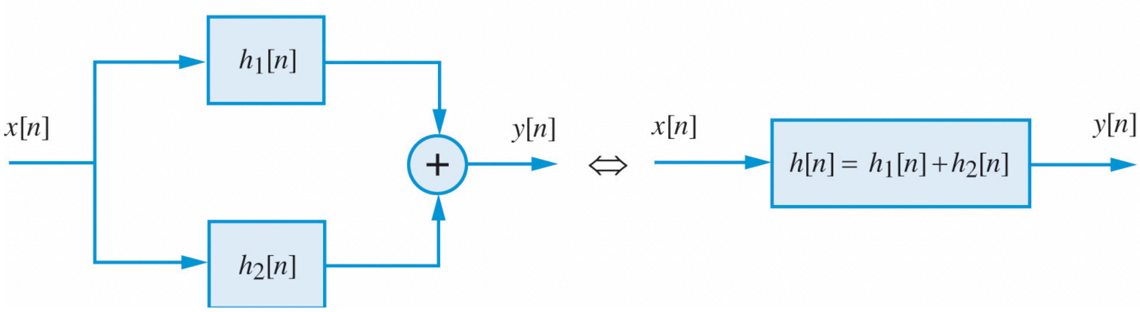
Delay : $x[n] * \delta[n-n_0] = x[n-n_0]$

Commutative : $x[n] * h[n] = h[n] * x[n]$

Associative : $(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$

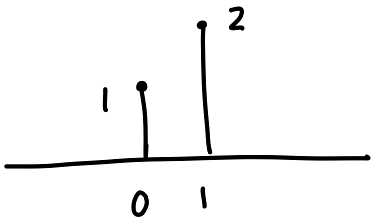


Distributative : $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

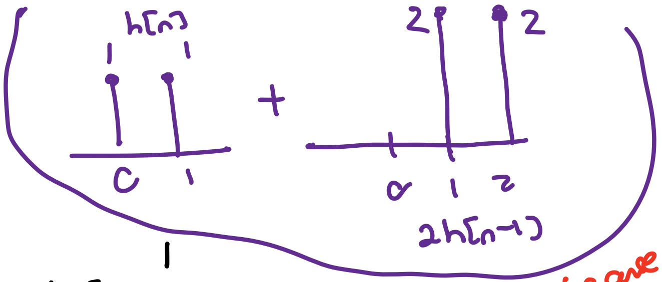


E[x]

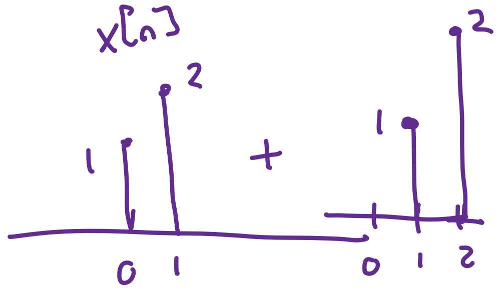
$x[n] =$



$h[n] =$



Same because of commutative



$y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[0] = \underbrace{x[0]h[0]}_{k=0} + \underbrace{x[1]h[-1]}_{k=1} = 1$$

$$y[1] = \underbrace{x[0]h[1-0]}_{k=0} + \underbrace{x[1]h[1-1]}_{k=1} + \underbrace{x[2]h[1-2]}_{k=2}$$

$$= (1)(1) + (2)(1) + 0$$

$$= 3$$

$$y[2] = \underbrace{x[0]h[2]}_{k=0} + \underbrace{x[1]h[1]}_{k=1} + \underbrace{x[2]h[0]}_{k=2}$$

$$= 0 + (2)(1) + 0$$

$$= 2$$

$$y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

$$y = \{\underset{\uparrow}{1}, 3, 2\}$$

denotes $n=0$

When $x[n]$ and $h[n]$ have a finite # of nonzero values, we can do convolution easily (esp with a computer)

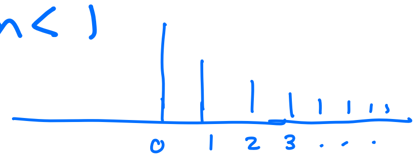
But what if $x[n]$ or $h[n]$ is $\cos(\omega_0 n)$, $\frac{1}{2}^n$, $e^{j\omega_0 n}$? very hard to compute convolution by hand. We need to go to the frequency domain via the Z-transform

First lets stay in the time-domain a little bit longer to see another perspective. $\boxed{\text{Chap 2.10}}$

We can analyze the time-domain response of an infinite-length $h[n]$:

$$h[n] = b a^n u[n] \quad , -1 < a < 1$$

We know this is stable where $-1 < a < 1$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \ldots$$

$$= bx[n] + bax[n-1] + ba^2x[n-2] + ba^3x[n-3] \ldots$$

$$= bx[n] + a(bx[n-1] + bax[n-2] + ba^2x[n-3] \ldots)$$

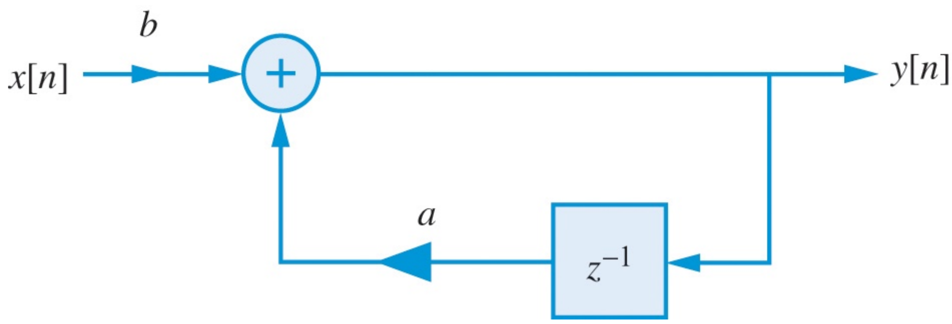
$$= bx[n] + a\,y[n-1]$$

So $h[n] = b a^n u[n], \quad -1 < a < 1$

corresponds to the system

$$y[n] = a\,y[n-1] + b\,x[n], \quad -1 < a < 1$$

Block-diagram representation of IIR system



There are several ways to study the behavior of the system in the time-domain. (Dynamical systems have been studied for centuries!)

$$y[n] = y_{zero\text{-}input}[n] + y_{zero\text{-}state}[n]$$

$y_{zero\text{-}input}[n]$ = output of the system when $x[n]=0$

$$y[n] = a\,y[n-1] + b\,x[n]:$$

$$y[0] = a\,y[-1] + b\,x[0]$$

$$y[1] = a\,y[0] + b\,x[1]$$
$$= a^2 y[-1] + b a\,x[0] + b\,x[1]$$

$$y[2] = a y[1] + b x[2]$$
$$= a^3 y[-1] + b a^2 x[0] + b a x[1] + b x[2]$$
$$\vdots$$

So we see

$$y_{zero\text{-}input}[n] = a^{n+1} \underbrace{y[-1]}_{\text{initial condition}} , \quad n \geq 0$$

$y_{zero\text{-}state}[n]$ is the response of the system assuming $y[-1] = 0$. We see that

$$y_{zero\text{-}state}[n] = b a^n x[0] + b a^{n-1} x[1] + \dots \quad b x[n]$$
$$= \sum_{k=0}^{n} h[k] x[n-k]$$

← the convolution output always assume zero initial condition

$$y[n] = y_{zero\text{-}input}[n] + y_{zero\text{-}state}[n]$$
$$= a^{n+1} y[-1] + \underbrace{\sum_{k=0}^{n} h[k] x[n-k]}$$

in DSP, we can always set the initial condition to zero, so we only study the zero-state response (system is initially at rest)

Another way to study $y[n]$ is

$$y[n] = y_{transient}[n] + y_{steady\text{-}state}[n]$$

In a stable system, $y_{transient}[n] \to 0$ as $n \to \infty$. Therefore, in DSP, we only care about the steady-state response

The system output we will derive in the frequency domain is the zero-state (system initially at rest), steady-state solution

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

are known as linear, costant-coefficient difference equation.

- N = order of system
- If a_k and b_k are constants, the system is time-invariant

For $N=0$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

is a nonrecursive system with finite duration impulse response $h[n] = b_n$ for $0 \le n \le M$