Chap 4.3

The principle of Fourier representation of signals is to break up all signals ($x(t)$ or $x[n]$) into sums scaled complex exponentials

If $x[n]$ is a discrete-time periodic signal with period N such as $x[n] = A\cos(\omega n + \theta)$, $x[n]$ is equal to a sum of harmonically related complex exponentials. This is called the discrete-time Fourier series (DTFS)

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} \xleftrightarrow{\text{DTFS}} c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

sum of harmonically related complex exponentials

To know the Fourier series, we just need to know the coefficients c_k

In practice however, most signals of interest are not periodic. We can say that aperiodic signal $x[n]$ has period $N = \infty$. $x[n]$ is equal to an infinite sum of complex exponentials. This is called the discrete-time Fourier transform (DTFT) coefficients

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \quad \xleftrightarrow{\text{DTFT}} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}\,d\omega$$

Since $x[n]$ is aperiodic, we don't have harmonically related complex exponentials

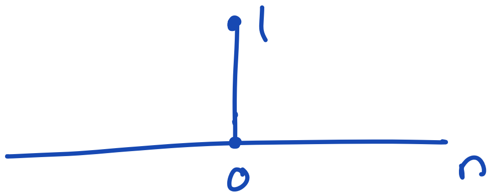
$X(e^{j\omega})$ = spectrum

$|X(e^{j\omega})|$ = magnitude spectrum

$\angle X(e^{j\omega})$ = phase spectrum

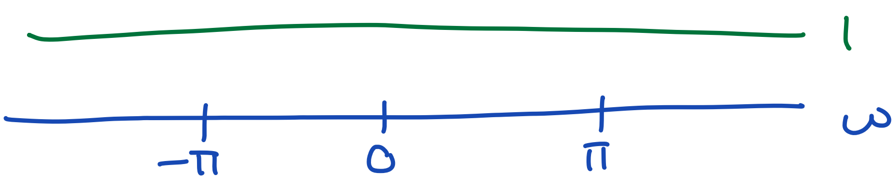
ex:

$x[n] = \delta[n]$

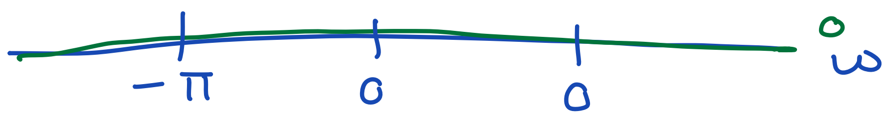


$X(e^{j\omega}) = x[0]e^{-j\omega(0)} = 1$ for all ω (the Fourier transform is all real)

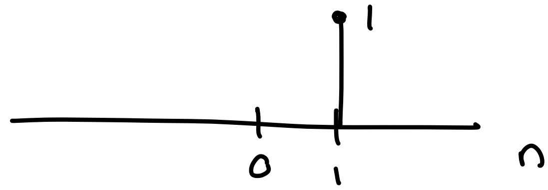
$|X(e^{j\omega})| =$



$\angle X(e^{j\omega}) =$

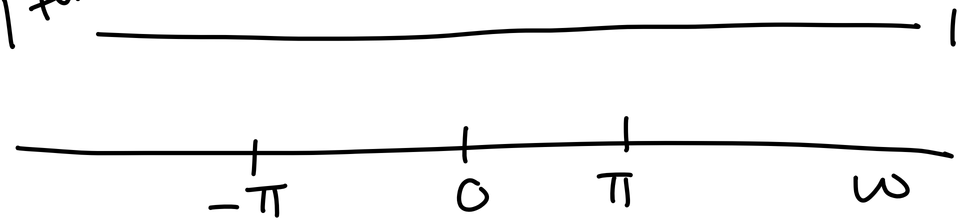


ex $x[n] = \delta[n-1]$

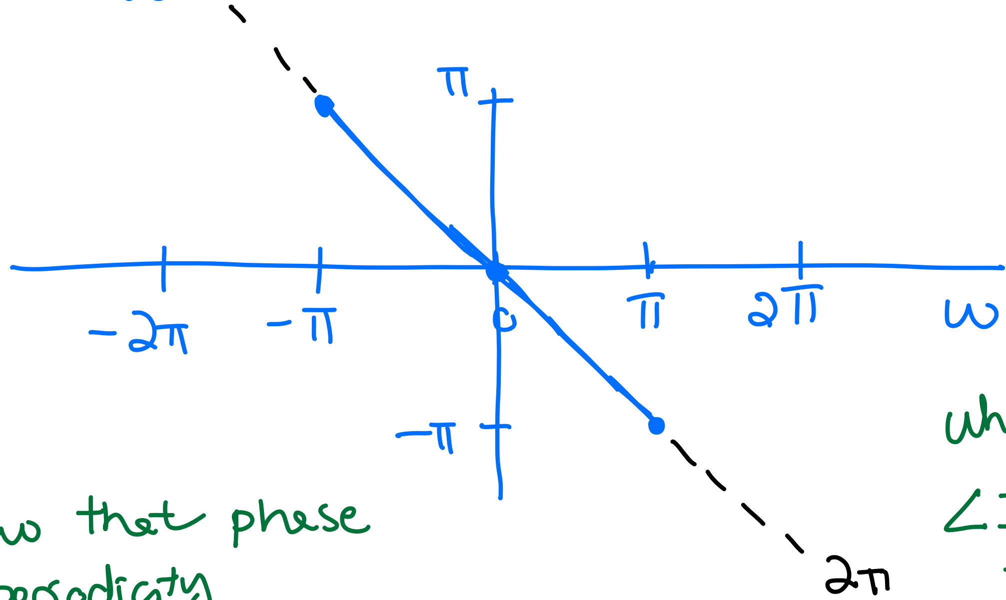


$$X(e^{j\omega}) = x[1]e^{-j\omega(1)} = e^{-j\omega}$$

$|X(e^{j\omega})| = 1$ for all ω



$\angle X(e^{j\omega}) = -\omega$ for all ω



when $\omega = \frac{3\pi}{2}$

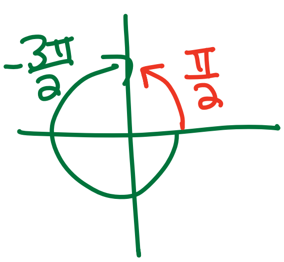
$\angle X(e^{j\omega})$

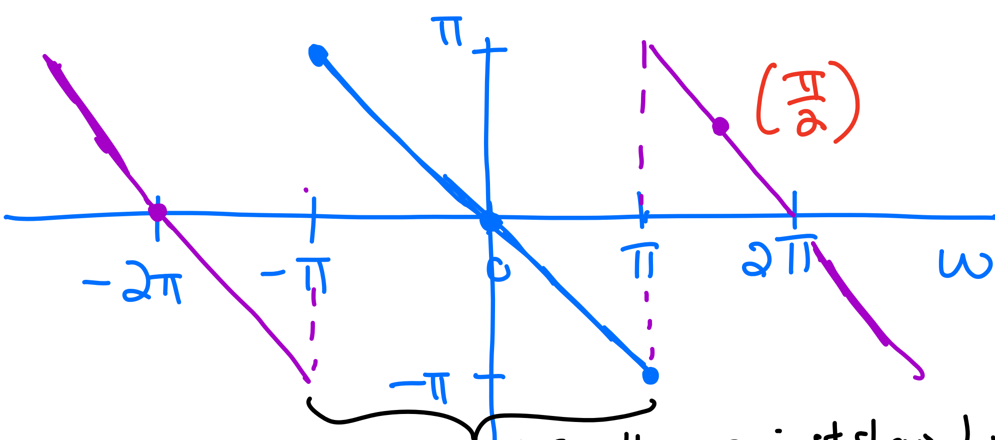
$= -\frac{3\pi}{2}$

$= \frac{\pi}{2}$

but we know that phase has 2π periodicity

$\angle X(e^{j\omega}) = -\omega$ for all ω





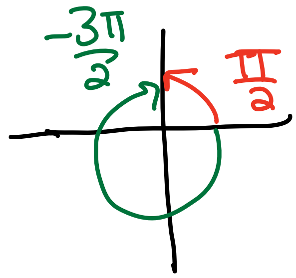
usually we just show 1 period

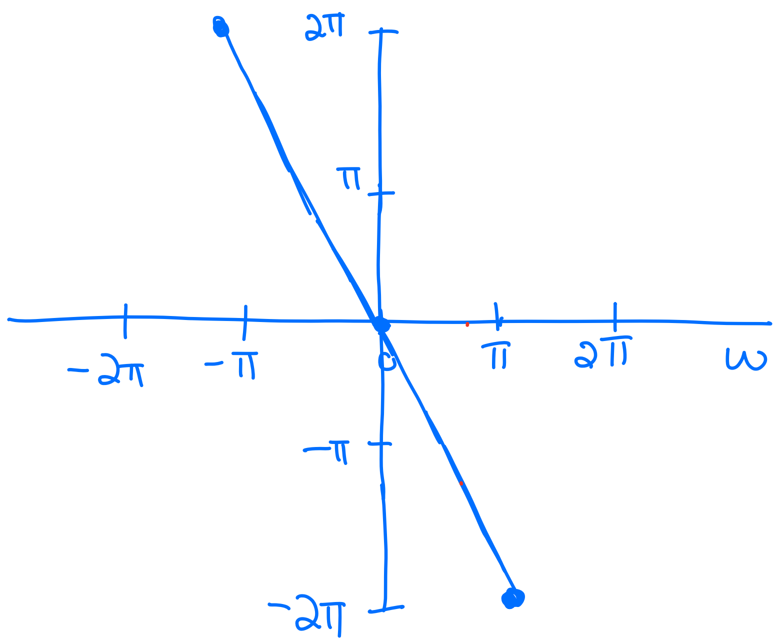
ex: $x[n] = \delta[n-2]$

$$X(e^{j\omega}) = x[2]\, e^{-j\omega 2} = e^{-j2\omega}$$

$$|X(e^{j\omega})| = 1 \quad \text{for all } \omega$$

$$\angle X(e^{j\omega}) = -2\omega \quad \text{for all } \omega$$



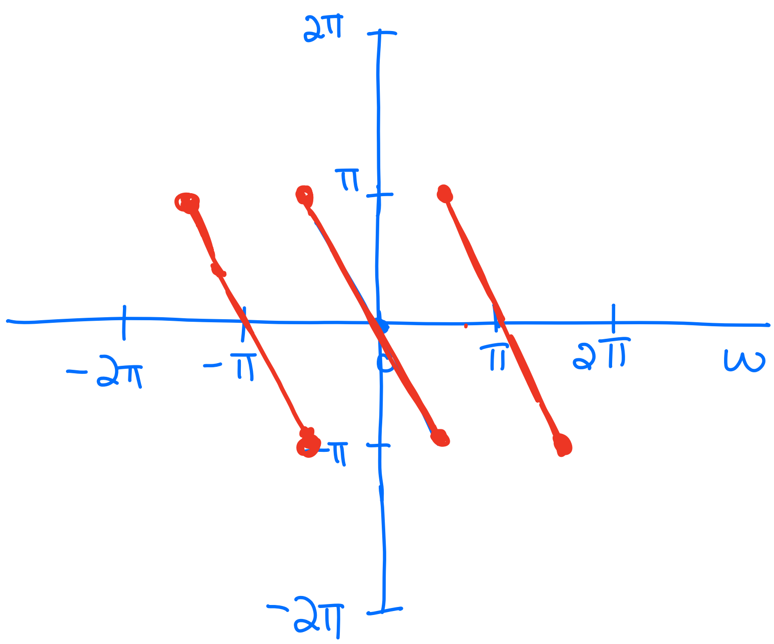


when $\omega = \frac{3\pi}{4}$,

$$\angle X(e^{j\omega}) = -2\left(\frac{3\pi}{4}\right)$$

$$= \frac{-6\pi}{4} = \frac{-3\pi}{2}$$

$$= \frac{\pi}{2}$$



y-axis of phase plot is always from $-\pi$ to π

When $x[n]$ is not a causal signal, you have to be careful using Matlab function freqz

Ex:

$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

$$X(e^{j\omega}) = \sum_{n=-1}^{1} x[n] e^{-j\omega n} = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos(\omega)$$

$$|X(e^{j\omega})| = |1 + 2\cos(\omega)|$$

$$\angle X(e^{j\omega}) = \begin{cases} 0, & X(e^{j\omega}) > 0 \\ \pi, & X(e^{j\omega}) < 0 \end{cases}$$

← why?

If you use freqz, you will get the wrong phase because it only works with causal signal

Ex

$$x[n] = \cos(\omega_0 n) = \cos(2\pi f_0 n)$$

so I know this is periodic. So I can find the Fourier series.

But wait! We know that by Euler's formula

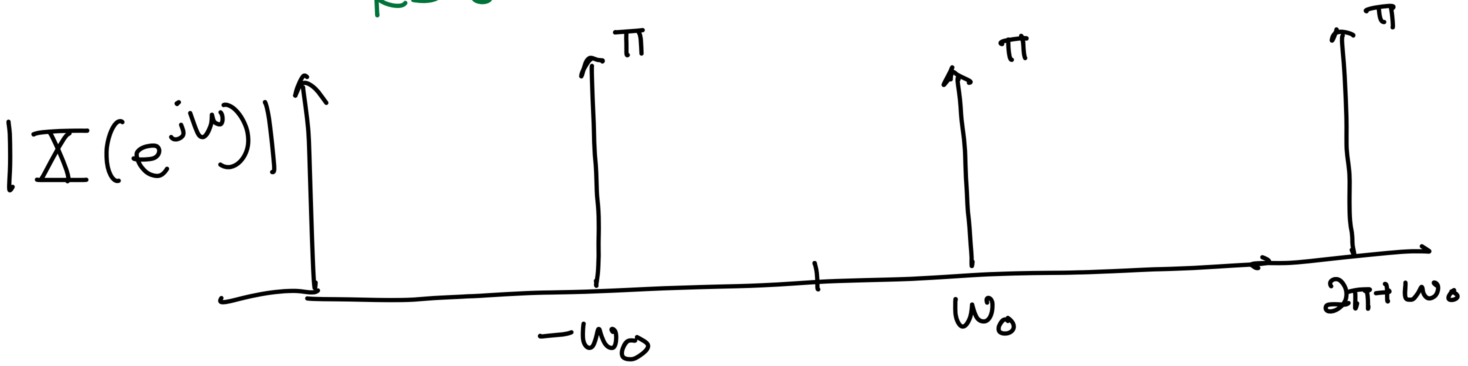
$$x[n] = \cos(\omega_0 n) = \underbrace{\frac{1}{2}e^{j\omega_0 n}}_{x_1[n]} + \underbrace{\frac{1}{2}e^{-j\omega_0 n}}_{x_2[n]}$$

$$\text{DTFT}(x[n]) = \text{DTFT}(x_1[n]) + \text{DTFT}(x_2[n]$$

$$\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi k)\left(\tfrac{1}{2}\right)$$

$$+ \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega+\omega_0-2\pi k)\left(\tfrac{1}{2}\right)$$

$$X(e^{jw}) = \sum_{k=-\infty}^{\infty} \pi\left[\delta(\omega+\omega_0-2\pi k) + \delta(\omega-\omega_0-2\pi k)\right]$$



$\angle X(e^{jw}) = 0$ for all w since $X(e^{jw})$ are all real

Energy of signal

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw})|^2 dw$$