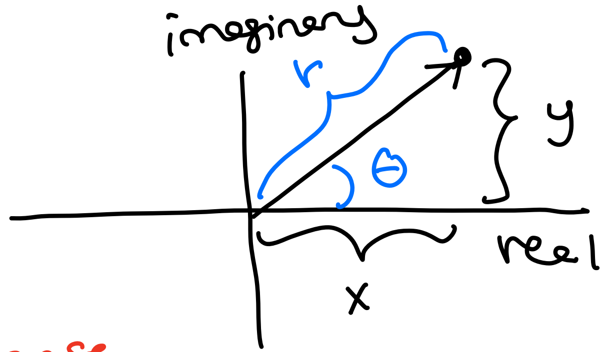
Complex numbers is used in DSP. A complex number can be represented in <u>Cartesian form</u>

$$z = \underset{\uparrow \text{ real part}}{x} + j\underbrace{y}_{\text{imaginary part}} = Re\{z\} + j\,Im\{z\}$$



or

<u>Polar form</u>

angle or phase

$$z = \underset{\uparrow \text{ magnitude}}{r}\, e^{j\theta} = |z|\, e^{j\arg(z)}$$

magnitude

θ, the phase, is usually represented in radians instead of degrees. $(\pi = 180°,\ \frac{\pi}{2} = 90°,\ -\frac{\pi}{4} = -25°)$

A full period is 2π so θ can be from 0 to 2π <u>or</u> from $-\pi$ to π

Polar form and Cartesian form can be converted from one another

$$r^2 = (Re\{z\})^2 + (Im\{z\})^2 = x^2 + y^2$$

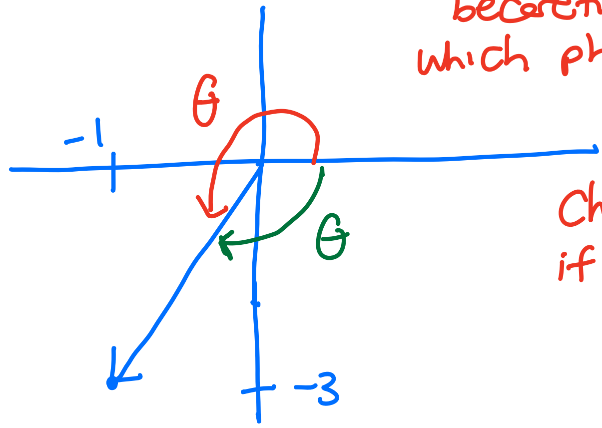
$$\theta = \tan^{-1} \frac{Im\{z\}}{Re\{z\}} = \frac{y}{x}$$

Ex.

$$z = -1 - j3$$

$$r^2 = 1 + (3)^2 = 10 \,,$$

$$r = \sqrt{10}$$



becareful which phase you give.

Check visually if you're not sure

$$\theta = \tan^{-1} \frac{-3}{-1} = \tan^{-1} 3$$

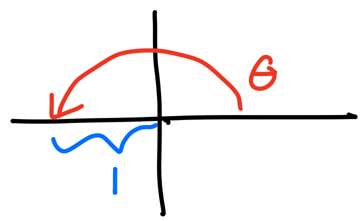
$$\theta = 1.249\pi \qquad\qquad \theta = -0.751\pi$$

$$z = \sqrt{10}\, e^{j1.249\pi} \quad \underline{\underline{\text{or}}} \quad z = \sqrt{10}\, e^{-j0.751\pi}$$

Ex.

$$z = e^{j\pi}$$

$$\left.\begin{aligned} r^2 = 1 &= x^2 + y^2 \\ \pi &= \tan^{-1} \frac{y}{x} \end{aligned}\right\} \text{solve or draw}$$



$$z = 0 - j = -j$$

<u>Complex cojugate</u> of a complex number z is denoted as z^*

$z = x + jy$, then $z^* = x - jy$

imaginary component changes signs

j is the imaginary unit. In physics, they use 'i'

$$j = \sqrt{-1}$$

$$j^2 = (\sqrt{-1})(\sqrt{-1}) = -1$$

$$j^3 = j^2 \cdot j = (-1)j = -j$$

$$j^4 = j^2 \cdot j^2 = (-1)(-1) = 1$$

<u>Euler's formula</u> relates sinusoids to complex numbers

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Then we know

$$\cos\theta = \frac{\overbrace{(\cos\theta + j\sin\theta)}^{e^{j\theta}} + \overbrace{(\cos\theta - j\sin\theta)}^{e^{-j\theta}}}{2}$$

$$= \frac{e^{j\theta} + e^{-j\theta}}{2} = \mathrm{Re}\{e^{j\theta}\}$$

$$\sin\theta = \frac{\cos\theta + j\sin\theta - (\cos\theta - j\sin\theta)}{2j}$$

$$= \frac{e^{j\theta} - e^{-j\theta}}{2j} = \mathrm{Im}\{e^{j\theta}\}$$

Roots of Complex Numbers

We want to find $x^3 = 8$. Yes we know $x = 2$ is a solution but $x^3 - 8 = 0$ tells us that there are $\underline{\underline{3}}$ roots of the equation. The other 2 roots are complex numbers!

The easiest way to find all the roots is to use complex exponential.

We know that

why?

$$8 = 8e^{j0} = 8e^{j2\pi} = 8e^{j4\pi}$$

Therefore we have

$$x^3 = 8e^{j0} \qquad x^3 = 8e^{j2\pi} \qquad x^3 = 8e^{j4\pi}$$

$$(x^3)^{\frac{1}{3}} = (8e^{j0})^{\frac{1}{3}} \qquad (x^3)^{\frac{1}{3}} = (8e^{j2\pi})^{\frac{1}{3}} \qquad (x^3)^{\frac{1}{3}} = (8e^{j4\pi})^{\frac{1}{3}}$$

$$x = 2e^{j0} = 2 \qquad x = 2e^{j\frac{2\pi}{3}} \qquad x = 2e^{j\frac{4\pi}{3}}$$

where did this come from?

We see the roots are

$$x = 2,\ 2e^{j\frac{2\pi}{3}},\ 2e^{j\frac{4\pi}{3}}$$