Chap 12.1

In real-world system, we may need to convert a signal from one sampling rate to another (for example, store 10 instead of 20 points to save memory)

Computation of a sequence $x_0[n] \equiv x_c(nT_0)$ from the known sequence $x[n] = x_c(nT)$ for $T_0 \neq T$ <u>without</u> reconstructing $x_c(t)$ is called resampling or sampling rate change

T = original sampling period

T_0 = new sampling period

3 cases to consider

1) $T_0 = DT$, D is an integer ($T_0 = 3T$)

new sampling period is longer and integer multiple of T

$\Updownarrow$

new sampling frequency is slower

= This is known as downsampling or sampling rate compression

2) $T_0 = \frac{T}{I}$, I is an integer $(T_0 = \frac{T}{5})$

new sampling period is shorter

$\Updownarrow$

new sampling frequency is faster

= This is known as interpolation

3) $T_0 = T\left(\frac{D}{I}\right)$, D and I are integer, $\frac{D}{I}$ is a rational number

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<u>Case 1</u>, sampling frequency decrease by an integer factor

$T_0 = DT$

$x[n] \equiv x_c(nT)$ original signal sampled at T period

$x_D[n] \equiv x_c(nDT)$

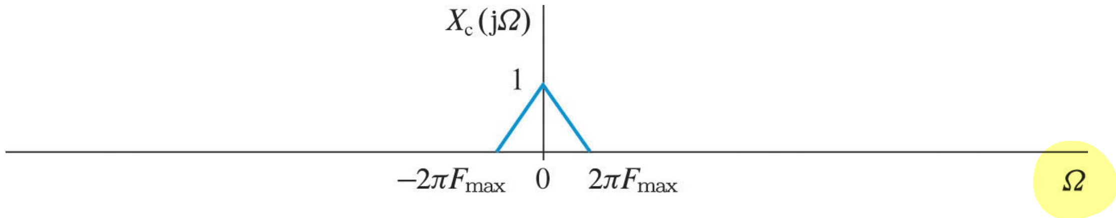
This means $x_D[n] \equiv x[nD]$. This is called downsampling. We will see that often times, just downsampling is <u>not</u> enough.

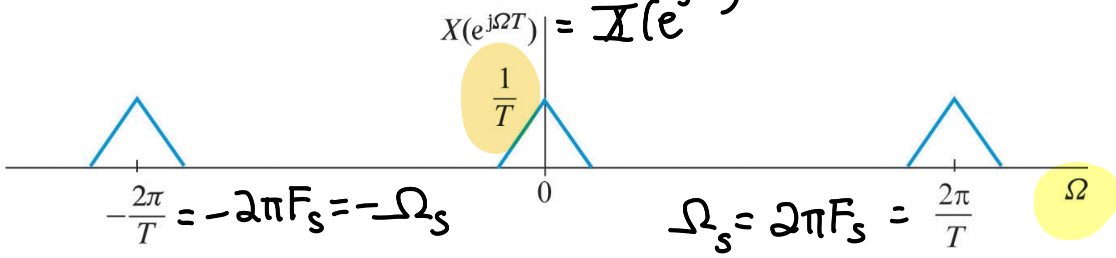
We want to understand how the down sampling affects the frequency domain

We know that for $x[n] = x_c(nT)$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

repetition every Ω_s





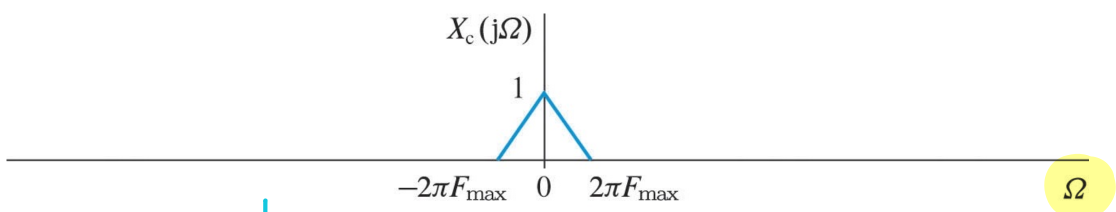
$X(e^{j\Omega T}) = X(e^{j\omega})$

$-\frac{2\pi}{T} = -2\pi F_s = -\Omega_s$

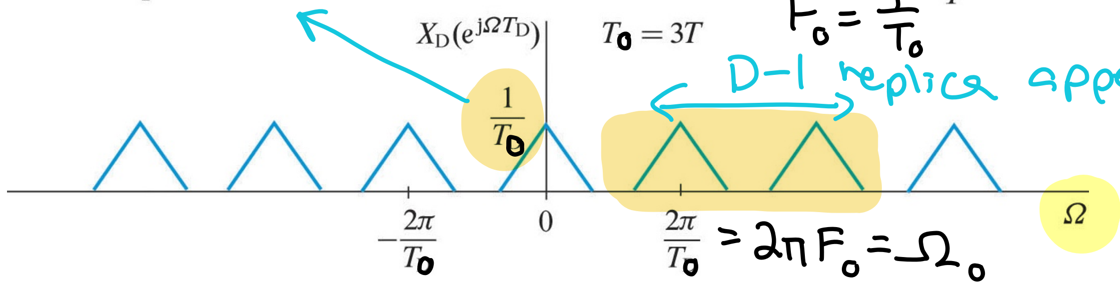
$\Omega_s = 2\pi F_s = \frac{2\pi}{T}$

And for $x_D[n] = x_c(nDT)$

$$X_D(e^{j\omega}) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_0)) = \frac{1}{DT}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - k\frac{\Omega_s}{D}\right)\right)$$



Scaling is $\frac{1}{DT}$



$F_0 = \frac{1}{T_0}$

D−1 replica appeared

$\frac{2\pi}{T_0} = 2\pi F_0 = \Omega_0$

$$\boxed{\Omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{DT} = \frac{1}{D}(\Omega_s)}$$

$$F_0 = \frac{1}{T_0} = \frac{1}{DT} = \frac{1}{D}(F_s)$$

However, things get a bit confusing if we want to plot the DTFT with a normalized frequency axes. Which sampling frequency do we normalize by F_s or F_0?

normalized frequency normalized by F_s

$$f = \frac{F}{F_s}$$

normalized frequency normalized by F_0

$$f = \frac{F}{F_0} = \frac{F}{\frac{1}{D}(F_s)} = D\frac{F}{F_s}$$

Let us choose to define the normalized frequency as

$$\omega = 2\pi f = 2\pi\left(\frac{F}{F_0}\right) = \Omega T_0$$

Then

$$X_D(e^{j\omega}) = \frac{1}{D}\sum_{m=0}^{D-1} X\left(e^{j\frac{(\omega - 2\pi m)}{D}}\right)$$

DTFT of $x_D[n]$

DTFT of $x[n]$

How are the 2 DTFT related

1) $X_D(e^{jw})$ has a $\frac{1}{D}$ scaling

2) $X_D(e^{jw})$ has $D-1$ replicants

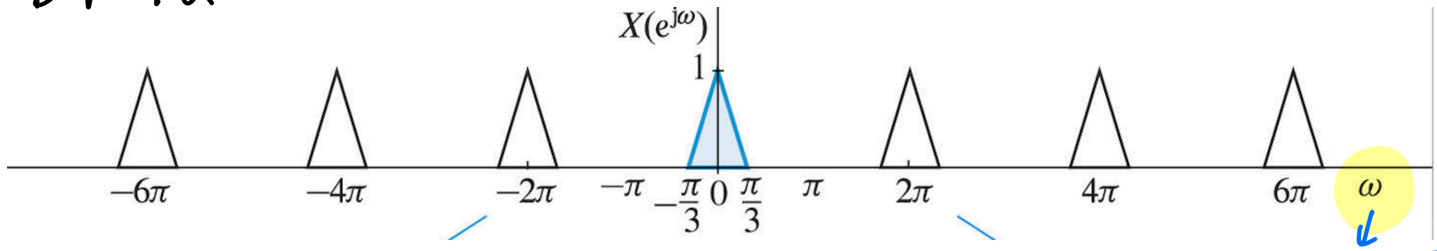
2) suppose $w = \Omega T_0 = \pi$, then

$$X_D(e^{j\pi}) = \frac{1}{D}\sum_{m=0}^{D-1} X\left(e^{j\left(\frac{\pi - 2\pi m}{D}\right)}\right)$$

DTFT of $X_D(e^{jw})$ at $w=\pi$ $\neq$ DTFT of $X(e^{jw})$ at $w=\pi$. It equals the DTFT of $X(e^{jw})$ at $w = \frac{\pi}{D}, \frac{\pi-2\pi}{D}, \frac{\pi-3\pi}{D}, \ldots$

one way to think of this is that $X_D(e^{jw})$ stretches $X(e^{jw})$

DTFT of $x[n]$

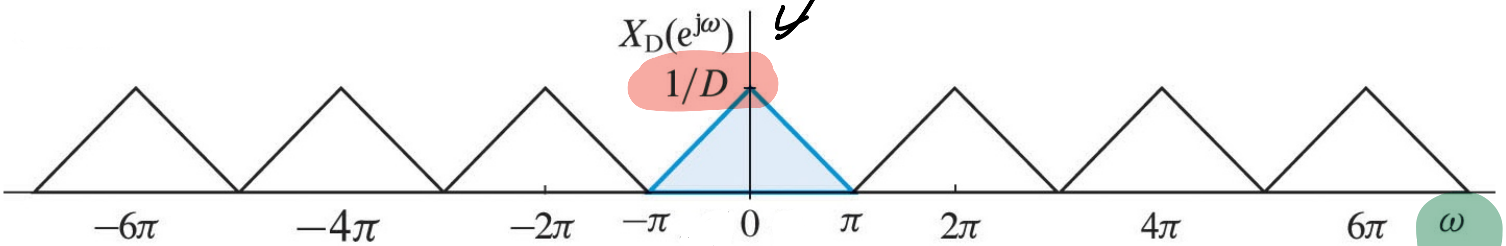


$w = \frac{2\pi F}{F_s}$

DTFT of $x_D[n] = x[3n]$

amplitude scaling

$D = 3$



stretched by D $\left(\text{i.e.} \left(\frac{\pi}{3}\right)3 = \pi\right)$

$w = \frac{2\pi F}{F_0}$

$$X_D(e^{jw}) = \frac{1}{3}\sum_{m=0}^{3-1} X\left(e^{j\left(\frac{w-2\pi m}{3}\right)}\right)$$

when $w = \pi$

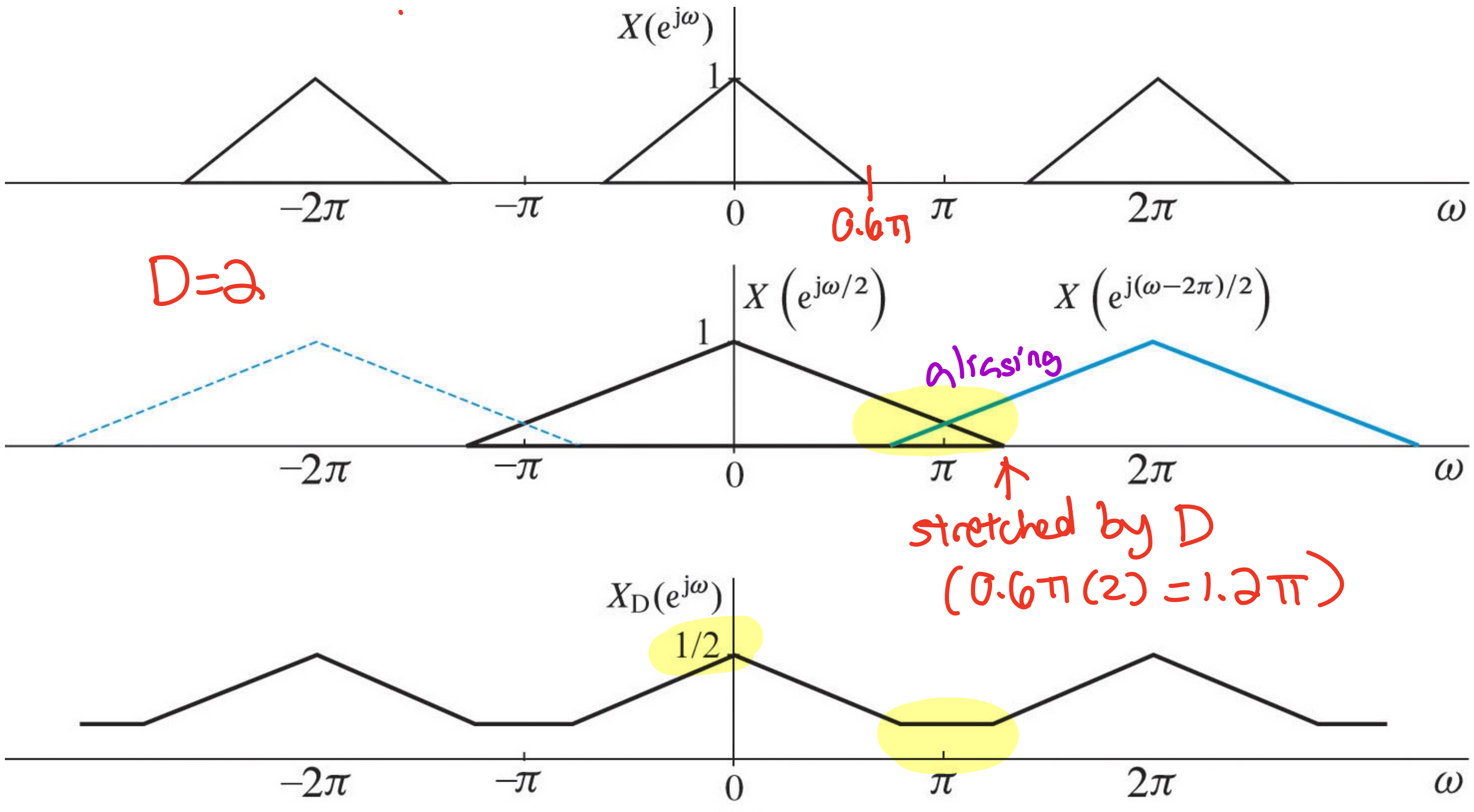
$$X_D(e^{j\pi}) = \frac{1}{3}\sum_{m=0}^{3-1} X\left(e^{j\left(\frac{\pi-2\pi m}{3}\right)}\right)$$

$$= \frac{1}{3} X\left(e^{j\left(\frac{\pi}{3}\right)}\right) + \frac{1}{3} X\left(e^{j\left(\frac{\pi-2\pi}{3}\right)}\right)$$

$$+ \frac{1}{3} X\left(e^{j\left(\frac{\pi-4\pi}{3}\right)}\right)$$

$$= \frac{1}{3} X\left(e^{j\frac{\pi}{3}}\right) + \frac{1}{3} X\left(e^{-j\frac{\pi}{3}}\right) + \frac{1}{3} X\left(e^{-j\pi}\right)$$

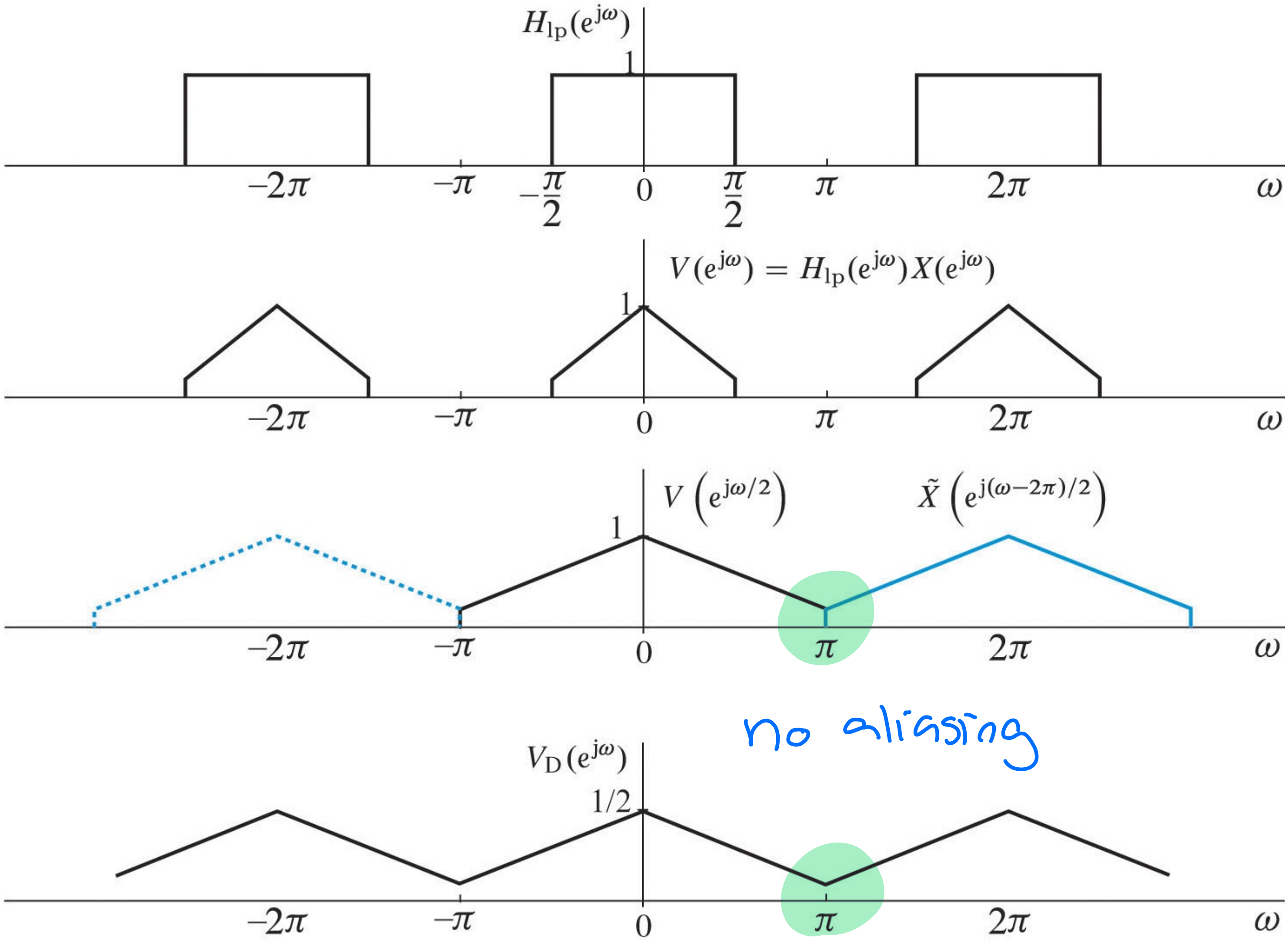
Because of the "stretching" and replicants, integer factor D has to be chosen carefully to avoid aliasing



(a)

To avoid alising the maximum frequency component of x[n] has to be $\omega \le \frac{\pi}{D}$

If this is not true, the x[n] need to be passed through a low pass filter

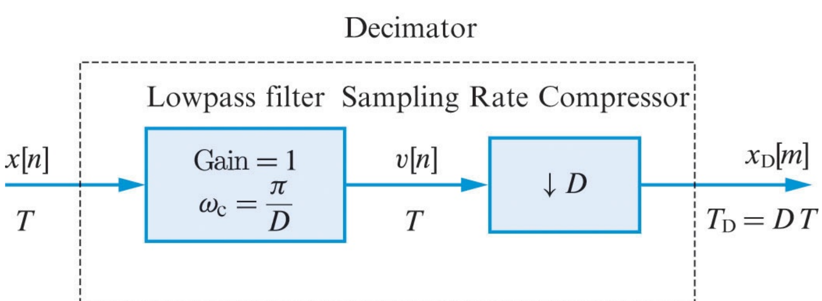


no aliasing

(b)

In reality, to avoid aliasing, we need to lowpass filter x[n] before downsampling.

lowpass filter + downsampling = decimation



Of course in practice, we can not use an ideal lowpass filter. Therefore, unlike downsampling, decimator output $x_D[m]$ do not perfectly reproduce the values of $x[n]$

When do you known down sampling is sufficient?

Aliasing can be avoided if

$$X(e^{j\omega}) = 0, \quad \omega_H \le |\omega| \le \pi,$$

maximum frequency component of $x[n]$

then $$\omega_s = \frac{2\pi}{D} \ge 2\omega_H$$

In the previous example $\omega_H = 0.6\pi$,

so

$$\frac{2\pi}{D} \ge 2(0.6\pi) = 1.2\pi$$

If $D=2$, we see that $\pi \ngeq 1.2\pi$.

Therefore, any downsampling of previous example will introduce aliasing! We need to decimate!