EE416: Introduction to Image Processing and Computer Vision

Il Yong Chun
Department of Electrical and Computer Engineering, the University of Hawai‘i, Mānoa

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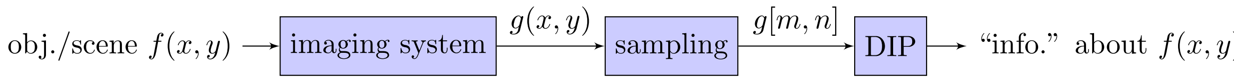
1 Continuous "space" signals and systems

1.0 Definitions

Ex. From imaging to image analysis, etc.:



1.0.1 Definition of signals and images

Mathematically, a **signal** is a function of one or more independent variables. An **image** is a function of two or three variables.

We focus on two-dimensional (2D) signals, i.e., images $g(x,y)$; in general, the independent variables (x,y) denote spatial position.

1.0.2 Continuous-space vs. discrete-space images

- Continuous-space images
  Example: $g(x,y) = e^{-(x^2+y^2)}$, $-\infty < x, y < \infty$.
  Defined for all locations/coordinates $(x,y) \in \mathbb{R}^2 = (-\infty,\infty)^2$, where $\mathbb{R}^M$ denotes the set of all M-tuples of real numbers. The default domain is $\mathbb{R}^2$ unless stated otherwise.
- Discrete-space images
  Example: $g[m,n] = (-1)^{m+n}$ (somewhat similar to a checkerboard)
  Defined only for integer locations/coordinates $(m,n) \in \mathbb{Z}^2$. Note that $\mathbb{Z} \triangleq \{\ldots,-2,-1,0,1,2,\ldots\}$.
  - Often, $g[m,n]$ denotes the value of some continuous-space image $g(x,y)$ at some discrete location, i.e., $g[m,n] = g(x_m, y_n)$, but not always.
  - Often $g[m,n]$ is called the **pixel value** at index $[m,n]$ or location (x_m, y_n), particularly when the discrete locations are spaced equally on rectangular grid, e.g., when $x_m = m\Delta_x$ and $y_n = n\Delta_y$, where Δ_x and Δ_y are horizontal and vertical pixel spacings.
  - It is *incorrect* to think of $g[m,n]$ as being zero for non-integer values of (m,n).

1.0.3 Value characteristics

- For a **continuous-valued image**, each value $g(x,y)$ lies in an uncountable set, typically an interval of real numbers, e.g., $[g_{\min}, g_{\max}]$.
- For a **discrete-valued image**, each value $g[m,n]$ lies in an *countable* set, e.g, a finite set as in $0,1,2,\ldots,255$ for an 8-bit image, or a countably infinite set, as in $\mathbb{Z}$.
  Discrete-valued images typically arise from **quantization** of continuous-valued images (e.g., A-to-D conversion), or from counting (e.g., nuclear imaging).

- A **binary** image has pixels that take only two values, typically 0 or 1.
  Binary images arise from **thresholding** continuous-valued or discrete-valued images, or forming **halftone** grayscale images.
- For generality, we will consider both **real-valued** and **complex-valued** images. A real-valued image has $g(x, y) \in \mathbb{R}$. A complex-valued image has $g(x, y) \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of all complex numbers of the form $u + \mathrm{i}v$, with $u, v \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$.
- Some image values have meaningful physical units, such as lumens. But often the image values are just relative "intensities".

1.0.4 Analog and digital images

- An **analog image** is typically a continuous-space and continuous-valued image.
- An **digital image** is typically a discrete-space and discrete-valued image.

1.0.5 Notation

- We will (usually) use parentheses for the arguments of continuous-space images, e.g., $g(x, y)$, and square brackets for the arguments of discrete-space images, e.g., $g[m, n]$.
- Note: In matrix notation rows and columns are usually transposed: $f_{n,m} = f[m, n]$, i.e., row$\to n$; column$\to m$.
- In my notes, as throughout much of engineering, when I write $g(x, y)$, there are two possible interpretations: $g(x, y)$ might mean the value of the image at location (x, y) or it might refer to the entire image. The correct interpretation will usually be clear from context. Another approach is to simply write g or $g(,)$ when referring to the entire image.

1.1 Continuous-time Fourier transform (CTFT)

1.1.1 CTFT and inverse CTFT

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}2\pi f t}\,\mathrm{d}t \qquad \text{(forward)}$$
$$f(t) = \int_{-\infty}^{\infty} F(f) e^{\mathrm{i}2\pi f t}\,\mathrm{d}f \qquad \text{(inverse)}$$

- $f(t)$ is a continuous function of continuous time $\infty < t < \infty$.
- $F(f)$ is a continuous function of continuous frequency $\infty < f < \infty$.
- Interpretation with Euler's formula ($e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$):

$$F(f) = \int_{-\infty}^{\infty} f(t) \left\{\cos(2\pi f t) - \mathrm{i}\sin(2\pi f t)\right\}\,\mathrm{d}t,$$

  i.e., the Fourier transform is an expansion of $f(t)$ multiplied by sinusoidal terms whose frequencies are determined by t.

To read: [T1, §Appendix A], [T2, §4.2.Complex Numbers].

1.1.2 Useful continuous-time signals

- Rect function: $\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| \le 1/2 \\ 0 & \text{otherwise.} \end{cases}$ Graph: $\boxed{?}$

- Step function: $\text{step}(t) \triangleq \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$ Graph: $\boxed{?}$

- Sign function: $\text{sign}(t) \triangleq \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0. \end{cases}$ Graph: $\boxed{?}$

- Sinc function: $\text{sinc}(t) \triangleq \dfrac{\sin(\pi t)}{\pi t}.$ Graph: $\boxed{?}$

$$\text{sinc}(0) \triangleq \lim_{t \to 0} \frac{\sin(\pi t)}{\pi t} = 1, \forall a \neq 0.$$

- Lambda function: $\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1. \end{cases}$ Graph: $\boxed{?}$

1.1.3 Continuous-time delta function: 1D Dirac impulse

- The **1D Dirac impulse** $\delta(t)$ has a central role in the analysis of 1D systems.
- Although the Dirac impulse is sometimes called the "Dirac delta function" in the engineering literature, it is not really a function. In the branch of mathematics that deals rigorously with such entities, they are called distributions.
- We "define" the 1D Dirac impulse by specifying its *sifting* properties: for all continuous functions $g(t)$,

$$g(0) = \int_{-\infty}^{\infty} \delta(t) g(t) \, dt.$$

More generally,

$$g(t_0) = \int_{-\infty}^{\infty} \delta(t - t_0) g(t) \, dt.$$

- Intuitively speaking, we may think of $\delta(t)$ as a very short pulse with unit area:

$$g(0) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \left\{ \frac{1}{\epsilon} \text{rect}(t/\epsilon) \right\} g(t) \, dt,$$

i.e., intuitively,

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{rect}(t/\epsilon)$$

To read: [T1, §2-1.1.C], [T2, §4.2.Impluses and Their Sifting Properties] (see Examples, in particular).

1.1.4 Useful CTFT relations

Time domain function	CTFT
$\delta(t)$	1
1	$\delta(f)$
$\mathrm{rect}(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\mathrm{rect}(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$

To read: [T1, §2-3.2, §2-3.3], [T2, §4.2.The Fourier Transform of Functions of One Continuous Variable].

1.1.5 CTFT properties

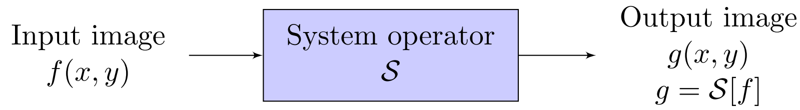
Property	Time domain function	CTFT
Linearity	$af(t) + bg(t)$	$aF(f) + bG(f)$
Conjugation	$f^*(t)$	$F^*(-f)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F(f/a)$
Shifting	$f(t - t_0)$	$e^{-\mathrm{i}2\pi f t_0}F(f)$
Modulation	$e^{\mathrm{i}2\pi f_0 t}f(t)$	$F(f - f_0)$
Convolution	$f(t) \ast g(t)$	$F(f)G(f)$
Cross correlation	$f(t) \star g(t)$	$F^*(f)G(f)$
Multiplication	$f(t)g(t)$	$F(f) \ast G(f)$
Duality	$F(t)$	$f(-f)$
Parseval's Theorem	$\int_{-\infty}^{\infty} f(t)g^*(t)\,\mathrm{d}t =$	$\int_{-\infty}^{\infty} F(f)G^*(f)\,\mathrm{d}f$

- Convolution: $f(t) \ast g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau$ (for reminder, see graphical illustration)
- Cross correlation: $f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)g(t+\tau)\,\mathrm{d}\tau$ (for reminder, see graphical illustration)
- Based on the convolution property and Parseval's Theorem above, one can deduce the following additional properties:
  Autocorrelation: $f(t) \star f(t) \overset{\text{CTFT}}{\Longleftrightarrow} \boxed{?}$
  Rayleigh's theorem: $\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \boxed{?}$

To read: [T1, §2-2.3], [T2, §4.2.Convolution].

1.2 Systems in 2D, i.e., imaging systems

An **imaging system**, or just **system**, operates on a 2D signal, called the input image, and produces a 2D signal, called the output image. For continuous-space systems, the input signal $f(x, y)$ is transformed by the system into the output signal $g(x, y)$:



Roughly speaking, there are two broad types of imaging systems: image capture systems and image processing systems. A camera and an X-ray scanner are examples of image capture systems. Zoom lenses, optical color filters, photographic film, and analog photocopiers are examples of continuous-space image processing systems. A scanner/digitizer, a contrast enhancer, a denoiser, an image compressor and an edge detector are examples of image processing systems where the output is a discrete-space image.

In an image capture system, $f(x,y)$ is typically the **true image**, or true spatial distribution of some physical quantity of interest, often called the **object**, whereas $g(x,y)$ is the **observed image** or **recorded image**. The goal is often for $g(x,y)$ to be as similar as possible to $f(x,y)$. (In image restoration problems the goal is to recover $f(x,y)$ from $g(x,y)$.)

The **input-output relationship** for an imaging system is described by a **system operator** or **system function** $\mathcal{S}$ and is written

$$g = \mathcal{S}[f], \quad \mathcal{S} : \mathcal{L}_2(\mathbb{R}^2) \to \mathcal{L}_2(\mathbb{R}^2),$$

where $\mathcal{L}_2(\mathbb{R}^2)$ denotes the collection of all finite energy images over the Cartesian plane, meaning that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)|^2 < \infty$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y)|^2 < \infty$.

In other words, $\mathcal{S}$ maps one function to another function, i.e., an image to another image. (The relationship is very often expressed using the following somewhat dangerously imprecise notation: "$g(x,y) = \mathcal{S}[f(x,y)]$.") We will often simply write one of the following:

$$f \xrightarrow{\mathcal{S}} g \quad \text{or} \quad f(x,y) \xrightarrow{\mathcal{S}} g(x,y) \quad \text{or} \quad f(x,y) \to \boxed{\mathcal{S}} \to g(x,y).$$

What is the system function $\mathcal{S}$? For most image capture systems, it is a blurring function. Often we can model the image capture system operator as the convolution of the object with the 2D (or 3D) point spread function.

1.2.1 Shift-invariance

A system $\mathcal{S}$ is called **shift invariant** iff

$$f(x,y) \xrightarrow{\mathcal{S}} g(x,y) \quad \text{implies that} \quad f(x-x_0, y-y_0) \xrightarrow{\mathcal{S}} g(x-x_0, y-y_0),$$

for every input image $f(x,y)$ and shift (x_0, y_0). That is to say, if we shift the input in space then the output is shifted by the same amount. Otherwise the system is called **shift variant** or **space variant**.
Fact: A system $\mathcal{S}$ is shift invariant iff the following two operations produce the same result (i.e., $g_1 = g_2$) for any input image f and any image shift (x_0, y_0):

$$f(x,y) \to \boxed{\mathcal{S}} \xrightarrow{g(x,y)} \boxed{\text{shift } (x_0, y_0)} \to g_1(x,y)$$
$$f(x,y) \to \boxed{\text{shift } (x_0, y_0)} \xrightarrow{f_2(x,y)} \boxed{\mathcal{S}} \to g_2(x,y)$$

Ex. Is the imaging zooming system defined by $g(x,y) = f(x/2, y/2)$ shift invariant? $\boxed{?}$
Ex. Is the ideal mirror system defined by $g(x,y) = f(-x,-y)$ shift invariant? $\boxed{?}$

1.2.2 Rotation-invariance

Usually the object being imaged has an orientation that is unknown relative to the recording device's coordinate system. Often we would like both the imaging system and the image processing operations applied to the recorded image to be independent of the orientation of the object.

A system $\mathcal{S}$ is called **rotationally invariant** iff

$$f(x,y) \xrightarrow{\mathcal{S}} g(x,y) \quad \text{implies that} \quad f_\theta(x,y) \xrightarrow{\mathcal{S}} g_\theta(x,y),$$

where

$$f_\theta(x,y) \triangleq f(x\cos\theta + y\sin\theta, -x\cos\theta + y\sin\theta)$$

and $g_\theta(x,y)$ is defined similarly, for *every* input image $f(x,y)$ and rotation θ.

Fact: A system $\mathcal{S}$ is rotation invariant iff the following two operations produce the same result (i.e., $g_1 = g_2$) for any input image f and any rotation angle θ:

$$f(x,y) \to \boxed{\mathcal{S}} \xrightarrow{g(x,y)} \boxed{\text{rotate } \theta} \to g_1(x,y)$$
$$f(x,y) \to \boxed{\text{rotate } \theta} \xrightarrow{f_2(x,y)} \boxed{\mathcal{S}} \to g_2(x,y)$$

1.2.3 Linear system

In the context of 2D continuous-space systems, we will say $\mathcal{S}$ is a **linear system** (function) iff it is a linear operator on the vector space of functions $\mathcal{L}_2(\mathbb{R}^2)$, i.e., iff

$$\mathcal{S}[\alpha f_1 + \beta f_2] = \alpha \mathcal{S}[f_1] + \beta \mathcal{S}[f_2] \quad \textbf{superposition property}$$

for all images f_1, f_2 and all constants α, β, considering both real and complex constants.

Two special cases of the above condition for linearity are particularly important in the context of linear systems.

- If the input to a linear system is scaled by a constant, then the output is scaled by that same constant:
$$\mathcal{S}[\alpha f] = \alpha \mathcal{S}[f] \quad \textbf{scaling property}.$$
- If the sum of two inputs is presented to the system, then the output will be simply the sum of the outputs that would have resulted from the inputs individually:
$$\mathcal{S}[f_1 + f_2] = \mathcal{S}[f_1] + \mathcal{S}[f_2] \quad \textbf{additive property}.$$

We can extend the superposition property to finite sums: by applying proof-by-induction:

$$\boxed{\mathcal{S}\left[\sum_{k=1}^{K} \alpha_k f_k\right] = \sum_{k=1}^{K} \mathcal{S}[f_k].}$$

For a system to be called linear, we require the superposition property to hold even for complex input signals and complex scaling constants. (This presumes that the system is defined to permit both real and complex inputs and to produce both real and complex outputs.)
Ex. Is $g(x,y) = \text{real}\{f(x,y)\}$ a linear system? $\boxed{?}$
Thus, both the additivity and scaling properties are required for a system to be linear.

1.2.4 Impulse response (point spread function, PSF) of a linear system

The **point spread function** (**PSF**) is perhaps the most important tool in the analysis of imaging systems. It is analogous to the impulse response used in 1D signals and systems.

The superposition property of a linear system enables a very simple method to characterize the system function:

- Decompose the input into some elementary functions.
- Compute the response (output) of the system to each elementary function.
- Determine the total response of the system to the desired (complex) input by simply summing the individual outputs.

A "simple" and powerful function for decomposition is the **Dirac impulse** $\delta(x, y)$; see the definition and properties of $\delta(x, y)$ in §1.3.4. The key property of the Dirac impulse is the **sifting property**:

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') \delta(x - x', y - y') \, \mathrm{d}x' \, \mathrm{d}y'$$

where the form expresses the image as a decomposition of an image into (weighted) elementary functions, namely off-center Dirac impulses $\delta(x - x', y - y')$.

PSF. Suppose the input to a system $\mathcal{S}$ is an impulse centered at (x', y'), i.e., the input is $\delta(x - x', y - y')$ where x' and y' are considered "fixed," and x, y are the independent variables of the input image. Let $h(x, y; x', y')$ denote the corresponding output signal:

$$\delta(x - x', y - y') \xrightarrow{\mathcal{S}} h(x, y; x', y').$$

The function $h(x, y; x', y')$, the output of the system $\mathcal{S}$ when the input is a Dirac impulse located at (x', y'), is called the **point spread function** (**PSF**) of the system, or (less frequently in the context of imaging systems) the **impulse response** of the system.

Ex. What is the PSF of an ideal mirror? The input-output relationship is $g(x, y) = f(-x, -y)$, so using the Dirac symmetry property (see §1.3.4):

$$\delta(x - x', y - y') \xrightarrow{\mathcal{S}} h(x, y; x', y') = \delta(-x - x', -y - y') = \delta(x + x', y + y').$$

Ex. What is the PSF of an ideal magnifying lens? For a magnification of 2, the ideal input-output relationship is $g(x, y) = \alpha f(x/2, y/2)$, for some $0 < \alpha < 1/4$ related to the absorption of the lens and the energy spreading. Thus

$$\delta(x - x', y - y') \xrightarrow{\mathcal{S}} h(x, y; x', y') = \boxed{?}.$$

More generally, the impulse response of the magnifying (or minifying) system $f(x, y) \xrightarrow{\mathcal{S}} g(x, y) = \alpha f(x/2, y/2)$ is $\boxed{?}$.

The two examples above also illustrate that linearity is distinct from shift-invariance!

1.2.5 Superposition integral

Consider a linear system with system function $\mathcal{S}$ operating on input $f(x, y)$. We want to find a simple expression for the output image $g(x, y)$ in terms of the input image $f(x, y)$. We would like an expression that is more informative than $g = \mathcal{S}[f]$. The PSF, the impulse decomposition, and the superposition property are the key ingredients.

To aid interpretation, consider a finite weighted sum of input impulses:

$$f(x; y) = \sum_n \alpha_n \delta(x - x_n, y - x_n) \xrightarrow{\mathcal{S}} g(x, y) = \sum_n \alpha_n h(x, y; x_n, y_n).$$

Generalizing to a continumm (using the "strong" superposition property[1] and changing the order of integration and linear operators freely, we obtain the following input-output relationship for *linear*

[1] We assume hereafter that for a linear operator A, the superposition property holds even for infinite summations, including integrals. That is, if for each Ω in **Euclidean n-dimensional space** or **n-tuple space** $\mathbb{R}^n$, the function f_Ω is square integrable (i.e., $\int_\Omega |f_\Omega|^2 \, \mathrm{d}\Omega$), then we assume

$$A\left[\int_\Omega f_\Omega \, \mathrm{d}\Omega\right] = \int_\Omega a(\Omega) A[f_\Omega] \, \mathrm{d}\Omega, \quad \textbf{“strong” superposition property}.$$

Further assumptions about "smoothness" or "regularity" or "continuity" of A are needed for that; this is usually no problem for real imaging systems.

systems, known as the **superposition integral**:

$$\boxed{g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x,y;x',y')\,\mathrm{d}x'\,\mathrm{d}y'.}$$

This is the general superposition integral for the case where we have decomposed the input into Dirac impulses.

The superposition integral tells us that once we have found the PSF of the imaging system for all input coordinates, then the output is *fully* determined for *any* input. Thus, a linear system function $\mathcal{S}$ is characterized fully by its PSF h. Thus we will rarely (if ever) explicitly work with $\mathcal{S}$, for linear systems.

Analysis of imaging systems is often firstly concerned with:

- finding h for the system,
- and then understanding how h is influenced by the various system parameters (such as detector spacing, etc.).
- Then we can design system parameters to "improve" $h(x,y)$.
- Particularly useful as a rule-of-thumb for examining h is the width of the PSF, often specified by the **full-width at half maximum** (**FWHM**).

1.2.6 Shift invariance or space invariance of linear systems

In any *linear* system where the PSF is the same for all input points (except for a shift), we can greatly simplify general superposition integral.

Fact: A linear system satisfies the **shift invariant** or **space invariant** condition, $f(x,y) \xrightarrow{\mathcal{S}} g(x,y)$ implies that $f(x-x_0, y-y_0) \xrightarrow{\mathcal{S}} g(x-x_0, y-y_0)$, iff the point spread function satisfies

$$\boxed{h(x,y;x',y') = h(x-x', y-y'; 0, 0), \quad \forall x, y, x', y'.}$$

(Proofs omitted.)

For shift-invariant systems, the $(0,0)$ notation above is superfluous. Therefore, to simplify notation we define

$$h(x,y) \triangleq h(x,y;0,0),$$

so that (with a little notation recycling) we can simply write

$$h(x,y;x',y') = h(x-x', y-y').$$

Because $h(x,y)$ is the response of the system to an impulse at the origin, if the system is linear and shift invariant, then we can determine the PSF by measuring the response to a single point-source input.

For LSI systems, the input-output relationship simplifies from the general superposition integral for linear systems to the 2D convolution integral.

1.3 Continuous-space Fourier transform (CSFT)

There are many transforms that are useful for image analysis, such as wavelets and cosine transforms, but of all of these, the 2D Fourier transform (i.e., CSFT) is particularly important for analyzing imaging systems. Why?

- Convolution property: A shift-invariant PSF reduces the entire system equation to a convolution integral. Such a system is easier to analyze in the Fourier domain, because convolutions in the space domain become simply multiplications in the Fourier domain. That is, the transform of the output function is the transform of the input function multiplied by the system transfer function H, the transform of the PSF.

The preceding concept is the most important property of Fourier transforms for our purposes.

- Eigenfunctions of linear shift-invariant (LSI) systems: Complex exponential signals are the eigenfunctions of LSI systems, and the eigenvalues corresponding to those eigenfunctions are exactly the values of the CSFT of the impulse response at the frequency of the signal:

$$e^{\mathrm{i}2\pi(ux+vy)} \longrightarrow \boxed{\text{LSI } h(x,y)} \longrightarrow H(u,v)e^{\mathrm{i}2\pi(ux+vy)}$$

  where LSI system $h(x,y)$ has the **frequency response** or **transfer function** $H(u,v)$, i.e., $h(x,y) \overset{\text{CTFT}}{\Longleftrightarrow} H(u,v)$ (see derivation preceding **Ex. (Eigenfunctions of LSI systems)**). The fact that the FT formula fell out of the convolution with the complex exponential signal is by itself a compelling motivation to study that integral further.

1.3.1 LSI systems and convolution

For linear shift-invariant (LSI) systems, the input-output relationship simplifies from the general superposition integral in §1.2.5 for linear systems to the following 2D convolution integral:

$$\boxed{g(x,y) = f(x,y) * h(x,y) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'.}$$

(The convolution notation is imprecise, but we abuse it.) See this link for an applet that illustrates convolution.

LSI systems are easier to describe than linear shift-variant systems, because we only need the two-argument PSF $h(x,y)$, rather than the general PSF $h(x,y;x',y')$ with its 4 arguments. (See §1.2.6.) Furthermore, for LSI systems we can simplify the convolution using Fourier methods.

Unfortunately, in *most image capture systems, the PSF is not exactly shift invariant!* For example, in phased-array ultrasound the PSF in the region of the transmit focus is vastly different from the PSF near the face of the array. However, the PSF of most imaging systems varies *slowly* over space. (If this were not the case then the images might be very difficulty to interpret visually.) Consequently, for the purposes of system design and analysis, the PSF can often be considered shift invariant over local regions of the image.

Linearity alone does not imply shift-invariance, as in the 1D pinhole example in **Prob. 1** attached at the end of §1.

1.3.2 CSFT and inverse CSFT

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-\mathrm{i}2\pi(ux+vy)}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{(forward)}$$
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{\mathrm{i}2\pi(ux+vy)}\,\mathrm{d}u\,\mathrm{d}v \qquad \text{(inverse)}$$

- Space coordinates:
  - Usually, x is horizontal coordinate and y is vertical coordinate
  - Usually, y points down.
- Frequency coordinates:
  - u corresponds to horizontal frequency components (vertical strips).
  - v corresponds to vertical frequency components (horizontal strips).

To read: [T1, §3-4.1], [T2, §4.5.The 2-D Continuous Fourier Transform Pair].

1.3.3 Useful continuous-space signals

- Box image (or 2D rect function): $\mathrm{rect}(x,y) \triangleq \mathrm{rect}(x)\mathrm{rect}(y)$ Graph: $\boxed{?}$

- 2D step function: $\mathrm{step}(x,y) \triangleq \mathrm{step}(x)\mathrm{step}(y)$ Graph: $\boxed{?}$

- 2D sinc function: $\mathrm{sinc}(x,y) \triangleq \mathrm{sinc}(x)\mathrm{sinc}(y)$ Graph: $\boxed{?}$

- Pillbox (or disk) image: $\mathrm{disk}(x,y) \triangleq \mathrm{rect}(\sqrt{x^2+y^2}) = \mathrm{rect}(r)$ Graph: $\boxed{?}$

- Unit disk image: $\mathrm{udisk}(x,y) \triangleq \mathrm{rect}\left(\frac{\sqrt{x^2+y^2}}{2}\right) = \mathrm{rect}(r/2)$ Graph: $\boxed{?}$

- Gaussian image: $e^{-\pi x^2} e^{-\pi y^2} = e^{-\pi r^2}$ Graph: $\boxed{?}$

A 2D function $g(x,y)$ is called **separable** (in Cartesian coordinates) if it can be written by the product of two 1D functions $g_X(\cdot)$ and $g_Y(\cdot)$:

$$g(x,y) = g_X(x) g_Y(y)$$

Ex. $\mathrm{rect}(x,y)$, $\mathrm{step}(x,y)$, $\mathrm{sinc}(x,y)$, etc.
Pillbox (or disk) image is a separable function $\boxed{?}$

1.3.4 Continuous-space delta function: 2D Dirac impulse

- The **2D Dirac impulse** $\delta(x,y)$ has a central role in the analysis of 2D systems.
- We "define" the 2D Dirac impulse by specifying its *sifting* properties:

$$g(x_0,y_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0, y-y_0) g(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

  if $g(x,y)$ is continuous at (x_0,y_0).
  Visualization of $\delta(x-x_0, y-y_0)$: $\boxed{?}$
- Properties:
  - Sampling property: $g(x,y)\delta(x-x_0,y-y_0) = g(x_0,y_0)$
  - Unit area property: $\int\int \delta(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$
  - Scaling property: $\delta(ax,by) = \boxed{?}$
  - Symmetry property: $\delta(-x,-y) = \delta(x,y)$
  - Support property: $\delta(x-x_0,y-y_0) = 0$ if $x \neq x_0$ or $y \neq y_0$

To read: [T2, §4.5.The 2-D Impulse and Its Sifting Property].

1.3.5 CSFT of separable functions

Let $g(t) \overset{\text{CTFT}}{\Longleftrightarrow} G(f)$ and $h(t) \overset{\text{CTFT}}{\Longleftrightarrow} H(f)$. Then, $g(x)h(y) \overset{\text{CSFT}}{\Longleftrightarrow} G(u)H(v)$.

Proof. Observe that

$$\begin{aligned}
\text{CSFT}\{g(x)h(y)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)e^{-\mathrm{i}2\pi(ux+vy)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)e^{-\mathrm{i}2\pi ux}e^{-\mathrm{i}2\pi vy}\,\mathrm{d}x\,\mathrm{d}y \\
&= \boxed{?} \\
&= G(u)H(v).
\end{aligned}$$

□

1.3.6 Useful CSFT relations

Space domain function	CSFT
1	$\delta(u,v)$
$\delta(x,y)$	1
$\text{rect}(x,y)$	$\text{sinc}(u,v)$
$\text{step}(x,y)$	$\left(\frac{1}{\mathrm{i}2\pi u}+\delta(u)\right)\left(\frac{1}{\mathrm{i}2\pi v}+\delta(v)\right)$
$\underline{\text{disk}(x,y)}$	$\underline{\text{jinc}(u,v)}$

- 2D Dirac impulse: $\text{CSFT}\{\delta(x,y)\} = \text{CSFT}\{\delta(x)\delta(y)\} = \text{CSFT}\{\delta(x)\}\text{CSFT}\{\delta(y)\} = 1\cdot 1 = 1$
- 2D rect function: $\boxed{?}$
- $\text{jinc}(u,v) \triangleq \begin{cases} \dfrac{J_1(\pi\sqrt{u^2+v^2})}{2\sqrt{u^2+v^2}}, & u\neq 0, v\neq 0, \\ \pi/4, & u=0, v=0, \end{cases}$
  where $J_k(\cdot)$ is the kth-order Bessel function (of the first kind).
- Both pillbox and jinc functions are circularly symmetric.

Circular symmetry. An image $g(x,y)$ is circularly symmetric iff it is invariant to any rotation, or equivalently, iff

$$g(x,y) = g(s,t) \quad \text{whenever} \sqrt{x^2+y^2} = \sqrt{s^2+t^2}.$$

1.3.7 CSFT properties

Properties that are analogous to 1D CTFT properties.

Property	Space domain function	CSFT
Linearity	$af(x,y)+bg(x,y)$	$aF(u,v)+bG(u,v)$
Conjugation	$f^*(x,y)$	$F^*(-u,-v)$
Scaling	$f(ax,by)$	$\frac{1}{\lvert ab\rvert}F(u/a,v/b)$
Shifting	$f(x-x_0,y-y_0)$	$e^{-\mathrm{i}2\pi(ux_0+vy_0)}F(u,v)$
Modulation	$e^{\mathrm{i}2\pi(u_0x+v_0y)}f(x,y)$	$F(u-u_0,v-v_0)$
Convolution	$f(x,y)\ast g(x,y)$	$F(u,v)G(u,v)$
Cross correlation	$f(x,y)\star g(x,y)$	$F^*(u,v)G(u,v)$
Multiplication	$f(x,y)g(x,y)$	$F(u,v)\ast G(u,v)$
Duality	$F(x,y)$	$f(-u,-v)$
Parseval's Theorem	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g^*(x,y)\,\mathrm{d}x\,\mathrm{d}y =$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)G^*(u,v)\,\mathrm{d}u\,\mathrm{d}v$

- 2D convolution: $f(x,y)\ast g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')g(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'$
- 2D cross correlation: $f(x,y)\star g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')g(x+x',y+y')\,\mathrm{d}x'\,\mathrm{d}y'$

Ex. (Eigenfunctions of LSI systems). Of all the many orthogonal bases, the set of harmonic complex exponentials is particularly important. The reason is simple: complex exponentials are eigenfunctions of LSI systems:

$$e^{\mathrm{i}2\pi(ux+vy)} \longrightarrow \boxed{\text{LSI } h(x,y)} \longrightarrow g(x,y)=e^{\mathrm{i}2\pi(ux+vy)}\ast h(x,y)=e^{\mathrm{i}2\pi(ux+vy)}H(u,v)$$

where $h(x,y)\overset{\text{CSFT}}{\Longleftrightarrow}H(u,v)$. One can prove this by the 2D convolution definition,

$$\begin{aligned} g(x,y) &= e^{\mathrm{i}2\pi(ux+vy)}\ast h(x,y)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathrm{i}2\pi\{(x-x')u+(y-y')v\}}h(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\\ &= \boxed{?}\\ &= e^{\mathrm{i}2\pi(ux+vy)}H(u,v). \end{aligned}$$

So, the response due to a complex exponential input signal is just that same complex exponential input signal scaled by $H(u,v)$, which is why the complex exponentials are called eigenfunctions of LSI systems. So if we can decompose a signal into complex exponential components, it is easy to determine the output of an LSI system by applying superposition.

Properties that are new for the 2D CSFT.

Property	Space domain function	CSFT
Separability (in Cartesian coordinates)	$f(x)g(y)$	$F(u)G(v)$
Coordinate transform (e.g., rotation)	$f\left(\mathbf{A}\begin{bmatrix} x\\ y\end{bmatrix}\right)$	$\lvert\det(\mathbf{A})\rvert^{-1}F\left([u,v]\mathbf{A}^{-1}\right)$

Ex. Suppose that $g(t)\overset{\text{CTFT}}{\Longleftrightarrow}G(f)$. Then, $g(x)\overset{\text{CSFT}}{\Longleftrightarrow}\boxed{?}$

Properties that are new for the 2D CTFT: Image rotation $\leftrightarrow$ CSFT rotation.

- Let $\mathbf{A}$ be the orthogonal (clockwise) rotation matrix:

$$\mathbf{A}=\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}.$$

- Because $\mathbf{A}$ is an orthogonal matrix, $|\det(\mathbf{A})| = 1$[2] and $\mathbf{A}^{-1} = \mathbf{A}^T$.
- Using the properties above and the coordinate transform property of CSFT, a CSFT of the function $g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is given by

$$\text{CSFT}\left\{g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right\} = |\det(\mathbf{A})|^{-1} G([u,v]\mathbf{A}^{-1}) = |\det(\mathbf{A})|^{-1} G([u,v]\mathbf{A}^{T}) = G\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right).$$

Ex. Let $\mathbf{A}$ be the counter-clockwise rotation matrix: $\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Then, $g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right) \overset{\text{CSFT}}{\Longleftrightarrow}$ $\boxed{?}$

Ex. $\text{rect}\left(\dfrac{x+y}{\sqrt{2}}, \dfrac{y-x}{\sqrt{2}}\right) \overset{\text{CSFT}}{\Longleftrightarrow} \boxed{?}$ (Hint: use the rotation matrix property.)

1.3.8 Rep and Comb operators

1D Rep and Comb operators.

- The *rep* operator periodically replicates a function with some specified period T:

$$\text{rep}_T\{f(t)\} = \sum_{k=-\infty}^{\infty} f(t-kT).$$

Sketch: $\boxed{?}$

- The *comb* operator multiplies a function by a period train of impluses:

$$\text{comb}_T\{f(t)\} = \sum_{k=-\infty}^{\infty} \delta(t-kT)f(t) = f(t)\sum_{k=-\infty}^{\infty}\delta(t-kT).$$

Sketch: $\boxed{?}$

2D Rep and Comb operators.

- 2D Rep operator:

$$\text{rep}_{X,Y}\{f(x,y)\} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(x-mX, y-nY)$$

Sketch: $\boxed{?}$

[2] Note that the determinant of *any* orthogonal matrix is either $+1$ or -1. One can show this easily:

$$1 = \det(I) = \det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = (\det(A))^2,$$

where A is an orthogonal matrix, and some properties of the determinant are used.

- 2D Comb operator:

$$\mathrm{comb}_{X,Y}\left\{f(x,y)\right\} = f(x,y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-mX, y-nY)$$

Sketch: $\boxed{?}$

CSFT properties of 2D Rep and Comb Operators.

Assume that $f(x,y) \overset{\text{CSFT}}{\Longleftrightarrow} F(u,v)$. Then, we have the following CSFT relationship:

$$\mathrm{comb}_{X,Y}\left\{f(x,y)\right\} \overset{\text{CSFT}}{\Longleftrightarrow} \frac{1}{XY}\mathrm{rep}_{\frac{1}{X},\frac{1}{Y}}\left\{F(u,v)\right\}$$
$$\mathrm{rep}_{X,Y}\left\{f(x,y)\right\} \overset{\text{CSFT}}{\Longleftrightarrow} \frac{1}{XY}\mathrm{comb}_{\frac{1}{X},\frac{1}{Y}}\left\{F(u,v)\right\}.$$

1.4 Optical imaging systems: From geometric optics to 2D convolution

1.4.1 Lens

In many optical imaging systems, the light usually propagates through at least one lens. To analyze such systems, we must be able to determine how an optical field changes as it passes through a lens. We assume monochromatic illumination of light throughout §1.4 (monochromatic wave refer to a wave with single wavelength and frequency).

Imaging properties of lenses:

- Focal length: The **focal length** of an optical system, d_f, is a measure of how strongly the system converges (or diverges) light. A lens focuses incoming parallel rays of light to a focal point, and the focal length is the distance from the center of the lens to the focal points of the lens, under a **thin lens model** – if the maximum thickness of the lens is sufficiently small, then rays entering at coordinates (x,y) will also exit approximately at the same coordinates – that is assumed hereafter.
  Sketch: $\boxed{?}$

- Image formation: Let d_o and d_i are the distances from lens to object and image planes, respectivel. To make good images, the distances must satisfy the **lens law** or **thin lens formula**:

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{d_f},$$

  so that rays from a point in the object plane cross at a point on the image plane.
  Sketch: $\boxed{?}$

- Magnification: From geometric optics, the **source magnification factor** is given as follows:

$$M = -\frac{d_i}{d_o}$$

where a negative sign indicates that image at the image plane is inverted. Besides, the **object magnification factor** (might better be called the **aperture magnification factor**) is given by

$$m = \frac{d_o + d_i}{d_0} = 1 - M.$$

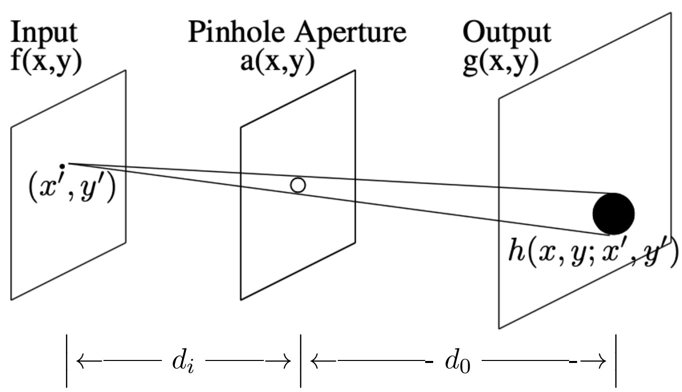
Sketch: $\boxed{?}$

- Typical imaging scenarios:
  - Typical case for photography: $d_o \gg d_f$, $d_i \approx d_f$ (but in addition, $d_i > d_f$), and $M \ll 1$
  - Typical case for microscopy: $d_i \gg d_f$, $d_o \approx d_f$ (but in addition, $d_o > d_f$), and $M \gg 1$

1.4.2 Optical imaging systems

1D pinhole models illustrate that any system with magnification or mirroring is, strictly speaking, a shift-variant system, because the PSF does not depend solely on the difference between input and output coordinates (see Prob. 1 in homework). However, we can recast such systems into equivalent shift-invariant systems by suitable coordinate changes for the input signal. This is reasonable in the context of imaging systems because the coordinates that we use are somewhat arbitrary. (In contrast, in the context of time-domain systems, "coordinate changes" are nontrivial.)

Space-domain model and PSF. We will show the required coordinate change for the example of a 2D pinhole camera. Among other things, this example will illustrate the type of manipulations that are useful for imaging systems analysis.



Consider the simple pinhole camera system shown above, assumed linear, with real world scene $f(x, y)$, focal plane image $g(x, y)$, source magnification factor M, aperture magnification factor m, and a pinhole opening described by a real aperture function $0 \leq a(x, y) \leq 1$, where $a(x, y)$ denotes that fraction of light incident on the point (x, y) that passes through.

- For a hole of diameter r_0, the aperture is $a(x, y) = a(r) = \mathrm{rect}(r/r_0)$.
- If the thickness of the aperture plane is infinitesimal (and ignoring $1/r^2$ intensity falloff), then by geometry, the PSF is

$$h(x, y; x', y') = h(x - Mx', y - My'), \quad \text{where } h(x, y) = a(x/m, y/m).$$

- Assuming this is a linear system, the output intensity is written in terms of the PSF by the superposition integral:

$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x,y;x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-Mx',y-My')\,\mathrm{d}x'\,\mathrm{d}y' \end{aligned}$$

- We can now rewrite the system equation using the simple change of variables $(x_M, y_M) = (Mx', My')$, leading to:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_M(x_M,y_M)h(x-x_M,y-y_M)\,\mathrm{d}x_M\,\mathrm{d}y_M = f_M(x,y) * h(x,y),$$

  where the modified image is defined by $f_M(x,y) \triangleq \frac{1}{M^2} f(x/M, y/M)$.
- This is now a simple 2D convolution of the modified image $f_M(x,y)$, which is a magnified, scaled and mirrored version of the input image, obtained via the coordinate change above, and the PSF $h(x,y)$, which is a magnified version of the aperture function.

To understand the $1/M^2$ in the definition of f_M, consider that for an infinitesimal pinhole, the system magnifies the image by the factor M in each dimension. Thus the spatial extent of the input image is extended by M^2 in the output. But because the total radiation striking the output plane does not change (i.e., passive camera system), the intensity of the output image is reduced by the factor $1/M^2$.

The function $h(x,y)$ is PSF of the imaging system, i.e., when $f(x,y) = \delta(x,y)$, then $g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x',y')h(x-Mx',y-My')\,\mathrm{d}x'\,\mathrm{d}y' = \boxed{?}$. The modified image $f_M(x,y)$ is what the system output would be for an ideal pinhole aperture, i.e., when $a(x,y) = \delta(x,y)$.

Transfer function.

- In the frequency domain,

$$G(u,v) = F_M(u,v)H(u,v),$$

  where $G(u,v)$, $H(u,v)$, and $F_M(u,v)$ are CSFTs of $g(x,y)$, $h(x,y)$, and $f_M(x,y)$, respectively.
- **Optical transfer function** (OFT):

$$\frac{H(u,v)}{H(0,0)},$$

  i.e., the normalized frequency response.
- **Modulation transfer function** (MFT):

$$\left|\frac{H(u,v)}{H(0,0)}\right|,$$

  i.e., magnitude of OFT.

1.5 Tomography and the Fourier slice theorem

1.5.1 Tomography

Many medical imaging systems only measure projections through an object with density $f(x,y)$.

- Projections must be collected at every angle θ and displacement r.
- Forward projections $p_\theta(r)$ are known as a Radon transform.

Aim: To reverse this process to form the original image $f(x, y)$:

- Fourier slice theorem is the basis of inverse.
- Inverse can be computed using filtered back projection (FBP)

Medical imaging modalities.

- Anatomical imaging
    - Computed tomography (CT)
    - Magnetic resonance imaging (MRI)
    - Tomosynthesis
- Functional imaging
    - Positron emission tomography (PET)
    - Single photon emission tomography (SPECT)
    - Functional magnetic resonance imaging (fMRI)
- Molecular imaging

1.5.2 Estimates of the projection integral

Photon attenuation.

- X-ray transmission sketch: $\boxed{?}$

- The number of received photons at the depth x, Y_x, can be modeled by a Poisson random variable:

$$\mathcal{P}\{Y_x = k\} = \frac{e^{-\lambda_x} \lambda_x^k}{k!}$$

with the mean $\lambda_x = \mathbb{E} Y_x$, where x is the depth into material measured in cm.

- Photon attenuation behavior:
    - As photon pass through object, they are absorbed.
    - The rate of absorption is proportional to the number of photons and the density of the object.

Differential equation for photon attenuation.

- Photon scattering reduces the number of photons:

$$\begin{aligned} \mathrm{d}\lambda_x &\propto -\lambda_x \, \mathrm{d}x \\ \mathrm{d}\lambda_x &= -\mu(x) \lambda_x \, \mathrm{d}x \end{aligned}$$

where $\mu(x)$ is the density in units of cm^{-1}.
sketch: $\boxed{?}$

- The attenuation of photons obeys the following equation:

$$\frac{\mathrm{d}\lambda_x}{\mathrm{d}x} = -\mu(x)\lambda_x$$

where $\mathrm{d}\lambda_x / \mathrm{d}x$ implies the change in the number of photons.
- The solution to this equation is given by

$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t)\,\mathrm{d}t}.$$

In other words, the number of photons at depth x exponentially decreases.
- So, we observe that

$$\int_0^x \mu(t)\,\mathrm{d}t = -\log\left(\frac{\lambda_x}{\lambda_0}\right) \approx -\log\left(\frac{Y_x}{\lambda_0}\right).$$

Estimates of the projection integral. A commonly used estimate of the projection integral is

$$\int_0^T \mu(t)\,\mathrm{d}t \approx -\log\left(\frac{Y_T}{\lambda_0}\right),$$

where λ_0 is the dosage and Y_T is the photon count at detector.

1.5.3 Radon transform

Geometric interpretation of coordinate rotation.

- Remind that counter-clockwise rotation matrix is given by $\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
- Define the new coordinate system (r, z): $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$
- Geometric interpretation: $\boxed{?}$

- Inverse transform: $\begin{bmatrix} r \\ x \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$

The Radon transform.

- For the function $f(x, y)$, we compute projections by integrating along z for each r.
- The projection integral for each r and θ is given by:

$$p_\theta(r) = \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) \mathrm{d}z = \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\sin(\theta))\,\mathrm{d}z.$$

This is called the Radon transform of $f(x, y)$.
- Geometric interpretation: $\boxed{?}$

- Observe that the projection corresponding to $r = 0$ goes through the point $(x, y) = (0, 0)$.

1.5.4 Fourier slice theorem

- The Radon transform of $f(x, y)$ is given by $p_\theta(r) = \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) \mathrm{d}z$.
- The CTFT of $p_\theta(r)$, $P_\theta(\rho)$, is then given by

$$\begin{aligned} P_\theta(\rho) &= \int_{-\infty}^{\infty} p_\theta(r) e^{-\mathrm{i}2\pi\rho r} \,\mathrm{d}r \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) \mathrm{d}z \right\} e^{-\mathrm{i}2\pi\rho r} \,\mathrm{d}r \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) e^{-\mathrm{i}2\pi\rho r} \,\mathrm{d}z \,\mathrm{d}r \end{aligned}$$

- By making the change or variables, $\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$, we obtain $r = x\cos(\theta) + y\sin(\theta)$.
  In addition, note that the Jacobian determinant is $\det(\mathbf{A}_\theta) = 1$, i.e., $|\det(\mathbf{A}_\theta)| = 1$.
- Combining all the results above gives[3]

$$P_\theta(\rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-\mathrm{i}2\pi\rho(x\cos(\theta) + y\sin(\theta))} \,\mathrm{d}x \,\mathrm{d}y = \boxed{?}$$

  where $f(x, y) \overset{\text{CSFT}}{\Longleftrightarrow} F(u, v)$.
- $P_\theta(\rho)$ is $F(u, v)$ in polar coordinates!

Inverse Radon transform.

- We compute $P_\theta(\rho) = \text{CTFT}\{p_\theta(r)\}$, where the projection measurement $p_\theta(r)$ is obtained by physical imaging systems.
- Geometric interpretation: $\boxed{?}$
- We take an inverse CSFT to form $f(x, y)$.
- **Problem:** This requires polar-to-rectangular coordinate conversion that is practically difficult.
- **Solution:** FBP

1.5.5 Filtered back projection (FBP) algorithm

- To compute the inverse CSFT of $F(u, v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation:

$$\mathrm{d}u \,\mathrm{d}v = |\rho| \,\mathrm{d}\theta \,\mathrm{d}\rho.$$

[3] The following result from multivariable calculus was used [wiki]. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following:

- T is continuously differentiable.
- T is one-to-on on a set $\mathcal{R} \subset \mathbb{R}^n$, where $\mathcal{R}$ has a boundary consisting of finitely many smooth sets and where $\mathcal{R}$ and its boundary are contained in the interior of the domain of T.
- $\det\{T\}$, the Jacobian determinant of T, is nonzero on $\mathcal{R}$.

Then if f is bounded and continuous on $T(\mathcal{R})$, the following holds:

$$\int_{T(\mathcal{R})} f(x_1, \ldots, x_n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{\mathcal{R}} f(T(x_1, \ldots, x_n)) |\det\{T(x_1, \cdots, x_n)\}| \,\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

- This gives the following expression:

$$
\begin{aligned}
f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{2\mathrm{i}\pi(xu+yv)}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\infty}^{\infty}\int_{0}^{\pi} P_\theta(\rho)e^{2\mathrm{i}\pi(x\rho\cos(\theta)+y\rho\sin(\theta))}|\rho|\,\mathrm{d}\theta\,\mathrm{d}\rho \\
&= \int_{0}^{\pi} \underbrace{\left\{\int_{-\infty}^{\infty} |\rho|P_\theta(\rho)e^{2\mathrm{i}\pi\rho(x\cos(\theta)+y\sin(\theta))}\,\mathrm{d}\rho\right\}}_{=g_\theta(x\cos(\theta)+y\sin(\theta))}\,\mathrm{d}\theta
\end{aligned}
$$

- Then $g_\theta(r)$ is rewritten by

$$
g_\theta(r) = \int_{-\infty}^{\infty} |\rho|P_\theta(\rho)e^{2\mathrm{i}\pi\rho r}\,\mathrm{d}\rho = \mathrm{CTFT}^{-1}\{|\rho|P_\theta(\rho)\} = \boxed{?}
$$

where $h(r)$ is the inverse CTFT of $|\rho|$.
- Finally, we obtain

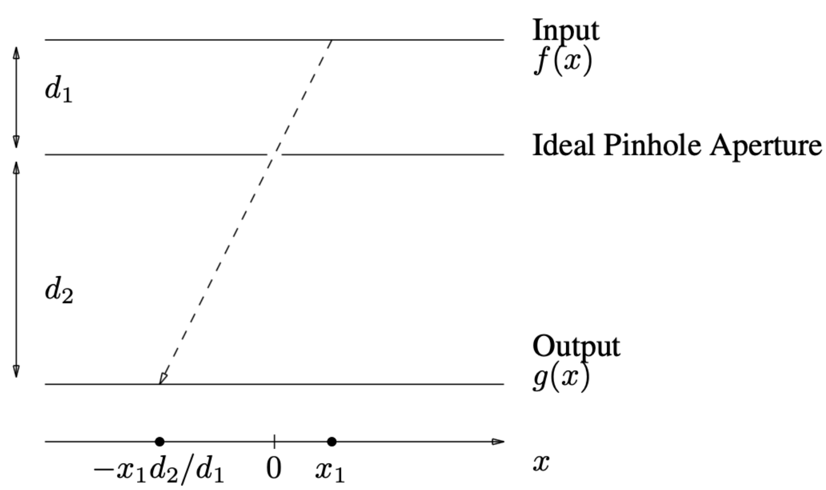
$$
f(x,y) = \int_0^\pi g_\theta(x\cos(\theta)+y\sin(\theta))\,\mathrm{d}\theta.
$$

Summary of FBP method.

- Given the projection measurements, filter the projection, i.e., $g_\theta(r) = h(r) * p_\theta(r)$.
- Back project filtered projections: $f(x,y) = \int_0^\pi g_\theta(x\cos(\theta)+y\sin(\theta))\,\mathrm{d}\theta$.
- **Limitation:** In practice, we consider a band-limited filter for $h(r)$. If one makes $h(r)$ band-limited, filtering with band-limited $h(r)$ "smears" values of $g_\theta(r)$ and one needs to back-project smeared $g_\theta(r)$ back over an image. The FBP result with band-limited $h(r)$ will be a filtered version of $f(x,y)$.

Homework (due by 9/24 11:55 PM; Upload your solution to Laulima/Assignments)

Prob. 1. Consider the following 1D pinhole camera:



Using simple geometric reasoning, our aim is to find the PSF and understand the properties of the 1D pinhole camera assuming incoherent illumination. Note that for incoherent illumination, this system is linear (in intensity). If the input were a Dirac impulse at $x = x_1$, then for an ideal (infinitesimal)

pinhole the output would be a Dirac impulse at $x = -x_1 d_2/d_1 = Mx_1$, where $M \triangleq -d_2/d_1$ is called the source magnification factor. (For now we ignore the fact that light intensity decreases with propagation distance.)
(a) Using simple ray-tracing, derive the PSF $h(x; x')$.
(b) Find the input-output relation. (Hint: Substitute the PSF into the 1D superposition integral, $g(x) = \int_{-\infty}^{\infty} f(x')h(x; x')\,\mathrm{d}x'$, and use the sifting property.)
(c) Justify that the system is highly shift-variant.

Prob. 2. Show the followings:
(a) If $g(t)$ has a CTFT of $G(f)$, then $g(t-a)$ has a CTFT of $e^{-2\imath\pi af}G(f)$.
(b) If $g(x, y)$ has a CSFT of $G(u, v)$, then $g(x/a, y/b)$ has a CSFT of $|ab|G(au, bv)$.
(c) If $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ has a CSFT of $G\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$, then $g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$ has a CSFT of $|\det(\mathbf{A})|^{-1}G\left(\left(\mathbf{A}^{-1}\right)^T \begin{bmatrix} u \\ v \end{bmatrix}\right)$.
(Hint: Use the Jacobian determinant in integration by substitution.)

Prob. 3. Let $f(x, y)$ be a 2D image. The the forward projection of $f(x, y)$ is given by

$$p_\theta(r) = \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\sin(\theta))\,\mathrm{d}z,$$

and the CTFT of $p_\theta(r)$ is given by

$$P_\theta(\rho) = \int_{-\infty}^{\infty} p_\theta(r)e^{-j2\pi\rho r}\,\mathrm{d}r.$$

To reconstruct the image $f(x, y)$ from the projections $p_\theta(r)$, we will perform filtered back projection.
(a) Specify the filter $H(\rho)$ so that

$$G_\theta(\rho) = H(\rho)P_\theta(\rho)$$

where $G_\theta(\rho)$ is the CTFT of the filtered projections required for perfect reconstruction.
(b) Is the filter obtained in (a) practical? Justify your answer.
(c) Let $\tilde{H}(\rho) = \mathrm{rect}(\rho/2)H(\rho)$. Sketch $\tilde{H}(\rho)$ and discuss its practicability and limitation over $H(\rho)$ in (a).