Chap 3.2

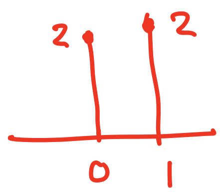
z-transform of a sequence (can be input, output, impulse response, etc.) $x[n]$ is a function $X(z)$ defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

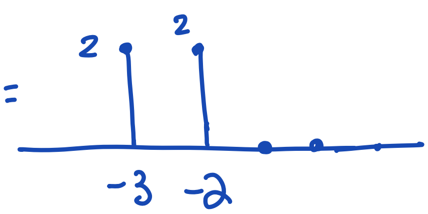
z-transform that sum from $-\infty$ to ∞ is known as two-sided z-transform

where the variable z is an complex number. The z-transform is also a complex number. So be sure to do your review on complex numbers!

Ex :

$h[n] =$ 

$$H(z) = h[0] z^{-0} + h[1] z^{-1} = 2 + 2z^{-1}$$

$x[n] =$ 

$$X(z) = x[-3] z^{-(-3)} + x[-2] z^{-(-2)}$$

$$= 2z^{3} + 2z^{2}$$

So you can see that finding the z-transform for $x[n]$ with finite number of nonzero is trivial

Finding the z-transform when $x[n]$ is infinite length (i.e. $x[n] = \frac{1}{2}^n$, $\cos(\omega_o n)$, $e^{j\omega_o n}$, ...) means we need to be a bit more careful

1) Since the z-transform requires summation from $-\infty$ to ∞, it may be that $X(z = 3+j2) = \infty$. In this case the z-transform does not exist for $z = 3+j2$.

For any given sequence $x[n]$, the set of values of z for which $X(z)$ converges (i.e. $X(z) \neq \infty$) is known as the <u>region of convergence</u> (ROC)

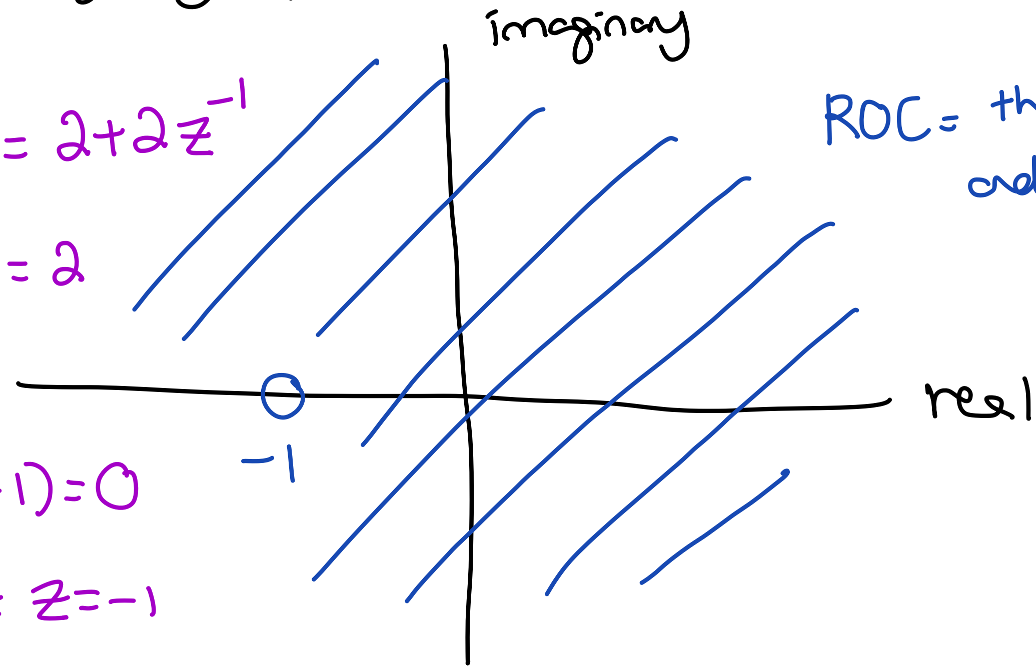
Values of z for which $X(z) = 0$ are called the <u>zeros of $X(z)$</u>

Values of z for which $X(z) = \infty$ are called the poles of $X(z)$. ROC <u>can not</u> contain any poles

If poles and zeros are complex numbers, then they appear in complex conjugate pairs

Since z and $H(z)$ are complex numbers, we visualize the region of convergence in the real imaginary plane

ex) $H(z) = 2 + 2z^{-1}$



ROC = the entire real and imaginary plane

$H(z=0) = 2$

$H(z=-1) = 0$

zero at $z=-1$

there is no z for which $H(z) = \infty$

2) For common signals, we have z-transform pairs

z-transform needs ROC!!

1) $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^{0} = 1 \quad , \quad \boxed{\text{ROC: all } z}$$

2) $x[n] = u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = z^{0} + z^{-1} + z^{-2} + \dots$$

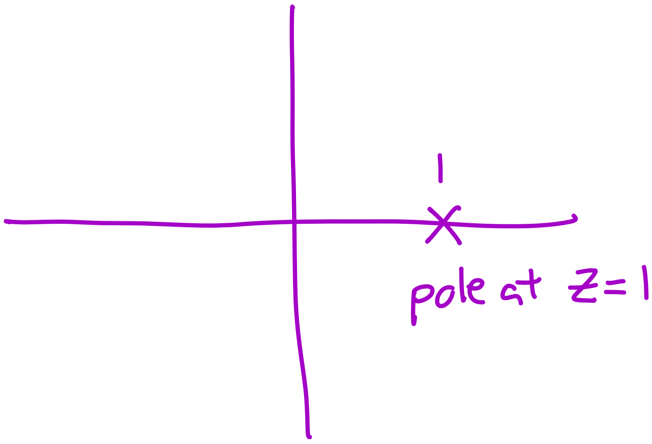
$$= \frac{1}{1 - z^{-1}}$$

$$X(z) = \frac{1}{1 - z^{-1}}$$

If $z = 1$, $X(z) = \frac{1}{0}$ = world ending!!, nonconvergence!!

$= \infty$

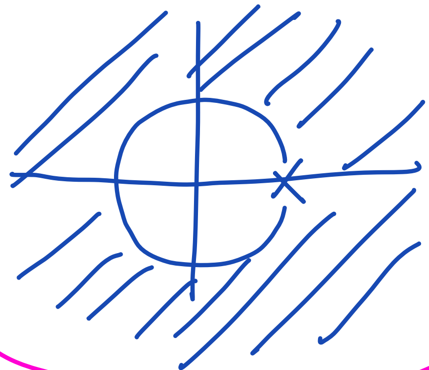
$X(z = 1) = \infty$. There is a pole at $z = 1$



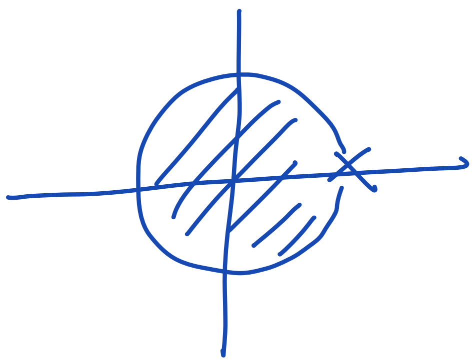
The region of convergence can not contain $z = 1$.

So ROC can either be

1) $|z| > 1$ or



2) $|z| < 1$



We need one addition piece of information. That is to realize that $u[n]$ is an infinite duration, right-sided sequence

If the infinite duration $x[n]$ is

		ROC
1) right-sided	$(x[n]=0,\ n<n_0) \Rightarrow$	$\lvert z\rvert > r$
2) left-sided	$(x[n]=0,\ n>n_0) \Rightarrow$	$\lvert z\rvert < r$
3) two-sided	$\Rightarrow$	$a < \lvert z\rvert < b$

3) $x[n] = a^n u[n]$ — $x[n]$ is right-sided

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

By geometric series, we know the summation converges to if $|z|>|a|$

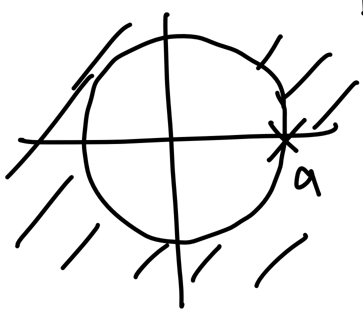
$$= \frac{1}{1-az^{-1}}$$

Zeros : there is no zero in the system

poles : one pole at $z = a$

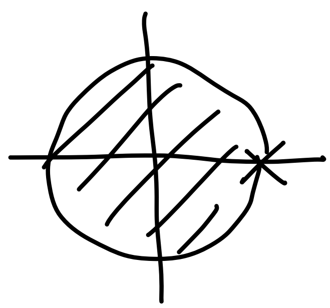
Therefore

ROC: $|z|>|a|$



or

ROC: $|z|<|a|$



Since $x[n] = a^n u[n]$ is a right-sided sequence, we know ROC is $|z| > |a|$

4) $x[n] = -a^n u[-n-1]$ x[n] is left-sided

$$= \begin{cases} 0, & n \geq 0 \\ -a^n, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (a z^{-1})^n$$

$$= -a^{-1} z (1 + a^{-1} z + a^{-2} z^2 + \dots)$$

geometric series converges if $|z| < |a|$

$$= \frac{1}{1 - a z^{-1}}$$ $\boxed{\text{ROC: } |z| < |a|}$

We also know ROC: $|z| < |a|$ since x[n] is a left sided sequence

Table 3.1 Some common z-transform pairs

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$(r^n\cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
10.	$(r^n\sin\omega_0 n)u[n]$	$\dfrac{(r\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

A pole and a zero can cancel each other other if they are in the same location. This is called a pole-zero cancellation. In this case, the region may be in the ROC since the pole is cancelled.

Ex: $$X(z) = \frac{1-z^{-(M+1)}}{1-z^{-1}}$$

Poles at $z = 1$

$M+1$ zeros at $z = 1$. What are they?

$$z^{M+1} = 1 = e^{j0} = e^{j2\pi} = e^{j4\pi} = e^{j6\pi} = \cdots$$

$$z = 1, \; e^{j\frac{2\pi}{M+1}}, \; e^{j\frac{4\pi}{M+1}}, \; e^{j\frac{6\pi}{M+1}}, \; \cdots$$

The zero at $z=1$ cancels the pole at $z=1$