HW Ch. 5 Solution

Prob. 1 a) We aim to have that y has a uniform pdf,

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$$P_Y(y) = \begin{cases} \frac{1}{L-1}, & y \in [0, L-1] \\ 0, & o/w \end{cases}$$

Applying the first result in Prop 5.1,

$$y = T(x) = \begin{cases} (L-1)\int_0^x P_X(w)\,dw = \frac{x^2}{(L-1)}, & x \in [0, L-1]. \\ 0, & o/w \end{cases}$$

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b) CDF of Z, $$P_Z(z) = \begin{cases} \frac{z^3}{(L-1)^2}, & z \in [0, L-1]. \\ 0, & o/w. \end{cases}$$

Applying the second result in Prop. 5.1,

$$z = T'(y) = P_Z^{-1}(y) = \left((L-1)^2 y\right)^{1/3}$$

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c) $$z = \tilde{T}(x) = T'(T(x)) = \begin{cases} \left((L-1)^2 x\right)^{1/3}, & x \in [0, L-1]. \\ 0, & o/w. \end{cases}$$

Prob. 2. a)

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$$\theta^* = \arg\min_\theta \sum_{n=1}^N (x_n - \theta)^2.$$

$$\frac{d}{d\theta}\sum_{n=1}^N (x_n-\theta)^2 = \sum_{n=1}^N (-2)(x_n-\theta) = 0 \qquad \therefore \theta^* = f(\vec{x}) = \frac{1}{N}\sum_{n=1}^N x_n.$$

(i.e., mean value).

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b) $$\theta^* = \arg\min_\theta \sum_{n=1}^N |x_n - \theta|$$

$$\frac{d}{d\theta}\sum_{n=1}^N |x_n-\theta| = \sum_{n=1}^N (-1)\,\mathrm{sign}(x_n-\theta) = 0 \qquad \therefore \theta^* = f(\vec{x}) = \mathrm{median}\{x_1, \dots, x_N\}.$$

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c) $$\theta^* = \arg\min_\theta \sum_{n=1}^N |x_n - \theta|^{0.5}$$

$$\frac{d}{d\theta}\sum_{n=1}^N |x_n-\theta|^{0.5} = \sum_{n=1}^N (-0.5)|x_n-\theta|^{-0.5}\,\mathrm{sign}(x_n-\theta) = 0$$

$$\therefore \theta^* = f(\vec{x}) = \mathrm{root}_\theta\left\{\sum_{n=1}^N |x_n-\theta|^{-0.5}\,\mathrm{sign}(x_n-\theta)\right\}$$

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d) f in c) is not linear.

Consider $\begin{cases} (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (0,1,0) \\ (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}) = (1,0,0). \end{cases}$ Then, $\begin{cases} \theta^{*(1)} = 0 \\ \theta^{*(2)} = 0. \end{cases}$

$(x_1^{(1)}+x_1^{(2)}, x_2^{(1)}+x_2^{(2)}, x_3^{(1)}+x_3^{(2)}) = (1,1,0)$. Thus, $\theta^{*(3)} = 1 \neq \theta^{*(1)} + \theta^{*(2)} = 0$

f in c) is homogeneous. For $\alpha > 0$,

$$\theta^{*\prime} = \arg\min_\theta \sum_{n=1}^N |x_n' - \theta|^{0.5} = \arg\min_\theta \sum_{n=1}^N |\alpha x_n - \theta|^{0.5} = \arg\min_\theta \sum_{n=1}^N \alpha^{0.5}|x_n - \theta/\alpha|^{0.5}$$

$$= \arg\min_\theta \sum_{n=1}^N |x_n - \theta/\alpha|^{0.5} = \alpha \cdot \arg\min_\theta \sum_{n=1}^N |x_n-\theta|^{0.5} = \alpha\,\theta^*$$

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