EE416: Introduction to Image Processing and Computer Vision

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4 Resolution conversion

1D rate conversion.

- Decimation
  - Reduce the sampling rate of a discrete-time signal.
  - Low sampling rate reduces storage and computational requirements.
- Interpolation
  - Increase the sampling rate of a discrete-time signal.
  - Higher sampling rate preserves fidelity.

4.1 Image decimation

4.1.1 Decimation in 1D

1D periodic subsampling.

- Time domain subsampling of $x[n]$ with period D:

$$y[n] = x[Dn]$$
$$x[n] \rightarrow \boxed{\downarrow D} \rightarrow y[n]$$

- Representation in frequency domain:

$$Y(e^{\mathrm{i}\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{\mathrm{i}(\omega - 2\pi k)/D})$$

- Problem: Frequency components above π/D will alias.
  Sketch: $\boxed{?}$

- Solution: Remove frequencies above π/D.

Decimation system. $x[n] \rightarrow \boxed{H(e^{\mathrm{i}\omega})} \rightarrow \boxed{\downarrow D} \rightarrow y[n]$

- Apply filter $H(e^{\mathrm{i}\omega})$ to remove high frequencies

$$H(e^{\mathrm{i}\omega}) = \sum_{k=-\infty}^{\infty} \mathrm{rect}\left(D \cdot \frac{\omega - 2\pi k}{2\pi} \right).$$

Its impulse response is given by

$$h[n] = \frac{1}{D}\mathrm{sinc}\left(\frac{n}{D}\right).$$

where we have the DTFT pair $\mathrm{sinc}(Tn) \overset{\text{DTFT}}{\Longleftrightarrow} \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathrm{rect}\left(\frac{\omega-2\pi k}{2\pi T}\right)$.

- The representation in frequency domain is given by

$$Y(e^{\mathrm{i}\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} H(e^{\mathrm{i}(\omega-2\pi k)/D})X(e^{\mathrm{i}(\omega-2\pi k)/D})$$

Sketch: $\boxed{?}$

4.1.2 Decimation for images

Direction extension of 1D decimation to images

- Space domain subsampling of $x[m,n]$ with period (D,D):

$$y[m,n] = x[Dm, Dn]$$
$$x[m,n] \to \boxed{\downarrow (D,D)} \to y[m,n]$$

- $x[m,n] \to \boxed{\circledast h[m,n]} \to \boxed{\downarrow (D,D)} \to y[m,n]$
- Ideal choice of filter (optimal if signal is band limited):

$$h[m,n] = \frac{1}{D^2}\mathrm{sinc}\left(\frac{m}{D}\right)\mathrm{sinc}\left(\frac{n}{D}\right)$$

- Problems: The filter above has infinite extend and is not strictly positive. In particular, infinite extends of filters require high computations.

Alternative filters for image decimation

- Direct subsampling:
  - $h[m,n] = \delta(m,n)$
  - Advantage: low computation
  - Disadvantage: excessive aliasing
- Block averaging:
  - $h[m,n] = \frac{1}{4}\left(\delta(m,n) + \delta(m+1,n) + \delta(m,n+1) + \delta(m+1,n+1)\right)$
  - Advantage: low computation
  - Disadvantage: some aliasing

DSFT property for 2D down-sampling.

Remind that we have the following equality for geometric series for complex exponentials:

$$\sum_{n=0}^{N-1} e^{\mathrm{i}\omega n} = \sum_{n=0}^{N-1}\left(e^{\mathrm{i}\omega}\right)^n = \begin{cases} N, & e^{\mathrm{i}\omega} = 1 \\ \frac{1-\left(e^{\mathrm{i}\omega}\right)^N}{1-e^{\mathrm{i}\omega}}, & e^{\mathrm{i}\omega} \neq 1 \end{cases} = \begin{cases} N, & \omega = 0, \pm 2\pi, \pm 4\pi, \ldots \\ \frac{1-e^{\mathrm{i}\omega N}}{1-e^{\mathrm{i}\omega}}, & \text{otherwise.} \end{cases}$$

A particularly important special case of this general formula is when $\omega = 2\pi \frac{k}{K}$ for $k \in \mathbb{Z}$ for which we have:

$$\sum_{n=0}^{N-1} e^{\mathrm{i}\frac{2\pi}{N}kn} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases} = N \sum_{l=-\infty}^{\infty} \delta[k - lN] = N\delta[k \mod N].$$

An earlier exercise problem examined down-sampling by two and how that affects the spectrum of a signal. Here we generalize this to down-sampling by arbitrary (integer) factors. Suppose that $f[m,n] \overset{\text{DSFT}}{\Longleftrightarrow} F(e^{\mathrm{i}\mu}, e^{\mathrm{i}\nu})$ and we define the down-sampled version $f_{\downarrow D}[m,n] \triangleq f[Dm, Dn]$ for $D \in \mathbb{N}$. How does $F_{\downarrow D}(e^{\mathrm{i}\mu}, e^{\mathrm{i}\nu})$ relate to $F(e^{\mathrm{i}\mu}, e^{\mathrm{i}\nu})$?

$$\begin{aligned}
F_{\downarrow D}(e^{\mathrm{i}\mu}, e^{\mathrm{i}\nu}) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[Dm, Dn] e^{-\mathrm{i}(\mu m + \nu n)} \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta[m \mod D]\delta[n \mod D] f[m,n] e^{-\mathrm{i}(\mu m/D + \nu n/D)} \\
&= \frac{1}{D^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{D-1} \sum_{l=0}^{D-1} e^{\mathrm{i}\frac{2\pi}{D}(km+ln)} f[m,n] e^{-\mathrm{i}(\mu m/D + \nu n/D)} \\
&= \frac{1}{D^2} \sum_{k=0}^{D-1} \sum_{l=0}^{D-1} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] \exp\left(-\mathrm{i}\left(\frac{\mu - 2\pi k}{D} m + \frac{\nu - 2\pi k}{D} n\right)\right) \\
&= \frac{1}{D^2} \sum_{k=0}^{D-1} \sum_{l=0}^{D-1} F\left(e^{\mathrm{i}\frac{\mu - 2\pi k}{D}}, e^{\mathrm{i}\frac{\nu - 2\pi l}{D}}\right)
\end{aligned}$$

In particular, for $D = 2$,

$$f_{\downarrow 2}[m,n] \overset{\text{DSFT}}{\Longleftrightarrow} \frac{1}{4}\left\{F\left(e^{\mathrm{i}\frac{\mu}{2}}, e^{\mathrm{i}\frac{\nu}{2}}\right) + F\left(e^{\mathrm{i}(\frac{\mu}{2}-\pi)}, e^{\mathrm{i}\frac{\nu}{2}}\right) + F\left(e^{\mathrm{i}\frac{\mu}{2}}, e^{\mathrm{i}(\frac{\nu}{2}-\pi)}\right) + F\left(e^{\mathrm{i}(\frac{\mu}{2}-\pi)}, e^{\mathrm{i}(\frac{\nu}{2}-\pi)}\right)\right\}.$$

4.2 Image interpolation

4.2.1 Interpolation in 1D

1D upsampling.

- Time domain upsampling of $x[n]$ by U:

  $$y[n] = \begin{cases} x[n/U], & \text{if } n = KU \text{ for some } K \\ 0, & \text{otherwise.} \end{cases}$$

  Alternatively,

  $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kU]$$

- Representation in frequency domain:

  $$Y(e^{\mathrm{i}\omega}) = X(e^{\mathrm{i}\omega U})$$

- Problem: Repeating frequency components by squeezing specturm
  Sketch: $\boxed{?}$

- Solution: Remove unnecessary spectrum

Interpolation system. $x[n] \to \boxed{\uparrow U} \to \boxed{H(e^{\mathrm{i}\omega})} \to y[n]$

- Apply interpolating filter $H(e^{\mathrm{i}\omega})$ to remove unnecessary spectrum

$$H(e^{\mathrm{i}\omega}) = U \sum_{k=-\infty}^{\infty} \mathrm{rect}\left(U \cdot \frac{\omega - 2\pi k}{2\pi}\right)$$

  Its impulse response is given by

$$h[n] = \mathrm{sinc}\left(\frac{n}{U}\right).$$

- Interpolation filter has the form

$$Y(e^{\mathrm{i}\omega}) = H(e^{\mathrm{i}\omega})X(e^{\mathrm{i}\omega U})$$

  Sketch: $\boxed{?}$

4.2.2 Interpolation for images

Direction extension of 1D interpolation to images

- Space domain upsampling of $x[m,n]$ by (U,U):

$$y[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x[k,l]\delta[m-kU, n-lU]$$

$$x[m,n] \to \boxed{\uparrow (U,U)} \to y[m,n]$$

- $x[m,n] \to \boxed{\uparrow (U,U)} \to \boxed{\circledast h[m,n]} \to y[m,n]$
- Ideal choice of filter:

$$H(e^{\mathrm{i}\mu}, e^{\mathrm{i}\nu}) = U^2 \sum_{k=-\infty}^{\infty} \mathrm{rect}\left(U \cdot \frac{\mu - 2\pi k}{2\pi}\right) \sum_{l=-\infty}^{\infty} \mathrm{rect}\left(U \cdot \frac{\nu - 2\pi l}{2\pi}\right)$$

$$h[m,n] = \mathrm{sinc}\left(\frac{m}{U}, \frac{n}{U}\right)$$

- Problems: Impulse response of the sinc function.
  - Infinite support; infinite computation
  - Negative side-lobes; ringing artifacts at edges

Alternative filters for image interpolation

- Pixel replication:
  - $h[m,n] = \begin{cases} 1, & \text{for } 0 \le m \le U-1,\ 0 \le n \le U-1 \\ 0, & \text{otherwise.} \end{cases}$
  - Replicates each pixel U^2 times.
- Bilinear interpolation
  - Impulse response of filter: $h[m,n] = \Lambda(m/U)\Lambda(n/U)$
  - Results in linear interpolation of intermediate pixels.

To read: See advanced interpolation methods in [T1, §4.7] and [T2, §4.8.2-D Spline Interpolation].

Homework (due by 10/27, 11:55 PM; Upload your solution to Laulima/Assignments)

Prob. 1. Let the image $y[m, n]$ be formed by applying a 2D interpolation by a factor of $U = 2$ to the signal $x[m, n]$ with an interpolation filter of the form

$$\begin{aligned} h[m,n] = & 0.25\delta[m-1,n-1] + 0.5\delta[m,n-1] + 0.25\delta[m+1,n-1] + \\ & 0.5\delta[m-1,n] + \delta[m,n] + 0.5\delta[m+1,n] + \\ & 0.25\delta[m-1,n+1] + 0.5\delta[m,n+1] + 0.25\delta[m+1,n+1]. \end{aligned}$$

(a) Use the zero boundary condition to compute $y[m, n]$ for the input $x[m, n]$ given by

$$\begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

(b) Compute $H(e^{i\mu}, e^{i\nu})$, the DSFT of the filter $h[m, n]$.
(c) Write an expression for $Y(e^{i\mu}, e^{i\nu})$ in terms of $X(e^{i\mu}, e^{i\nu})$ and $H(e^{i\mu}, e^{i\nu})$.
(d) Describe the advantages and disadvantages of this interpolation method.