Chap 3.5

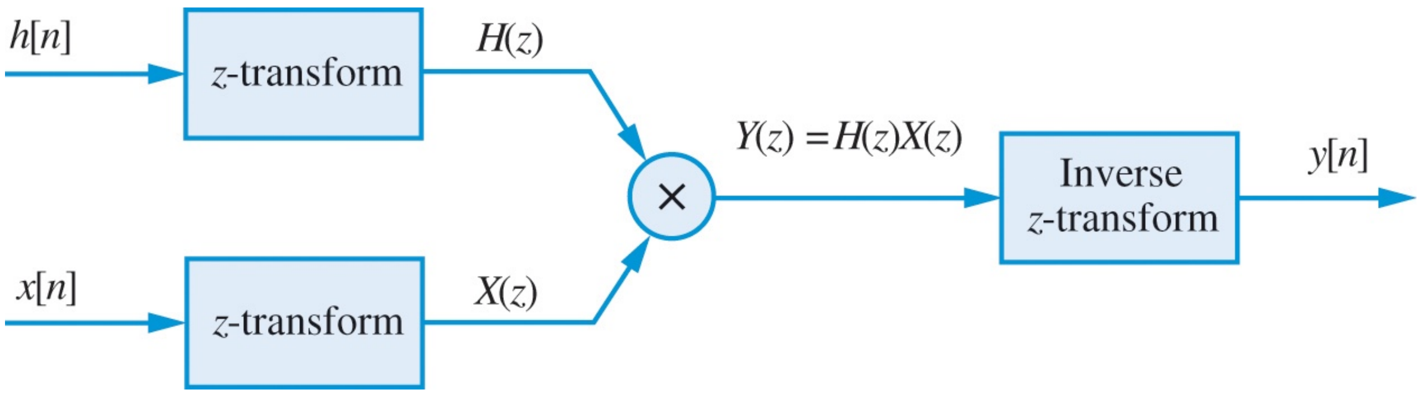
Recall that every LTI system is characterized by the impulse response $h[n]$. The output $y[n]$ of an LTI system to any input $x[n]$ is

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

If we find the z-transform of $h[n]$, $H(z)$ and input $x[n]$, $X(z)$, then

$$Y(z) = X(z)H(z)$$

To find $y[n]$, we can do the inverse z-transform on $Y(z)$. Note for $Y(z)$ to make sense, the ROC of $H(z)$ and $H(z)$ need to overlap. Otherwise $Y(z) = \infty$



$H(z)$ is known as the system function or the transfer function of the system.

Ex 3.13

What is the output of a system with impulse response $h[n] = a^n u[n]$, $|a| < 1$ to input $x[n] = u[n]$

$$H(z) = \frac{1}{1 - az^{-1}}, \quad ROC: |z| > |a|, \ |a| < 1$$

$$X(z) = \frac{1}{1 - z^{-1}}, \quad ROC: |z| > 1$$

Note that the ROC of $H(z)$ and $X(z)$ always overlap

$$Y(z) = H(z) X(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

$$ROC: |z| > 1$$

Use inverse Z-transform to find $y[n]$

$$Y(z) = \frac{A_1}{1 - az^{-1}} + \frac{A_2}{1 - z^{-1}} = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

$$A_1(1-z^{-1}) + A_2(1-az^{-1}) = 1$$

$$A_1 + A_2 = 1$$

$$-A_1 - A_2 a = 0$$

$$A_2 - A_2 a = 1$$

$$A_2(1-a) = 1$$

$$A_2 = \frac{1}{1-a}$$

$$A_1 = 1 - \frac{1}{1-a}$$

$$y[n] = \left(1 - \frac{1}{1-a}\right)(a)^n u[n] + \left(\frac{1}{1-a}\right)(1)^n u[n]$$

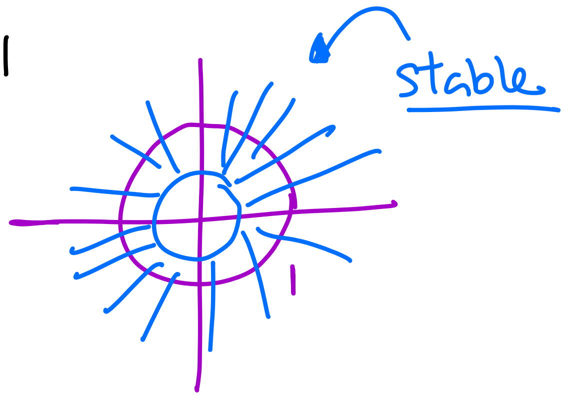
$$= \left(\frac{-a}{1-a}\right) a^n u[n] + \left(\frac{1}{1-a}\right) u[n]$$

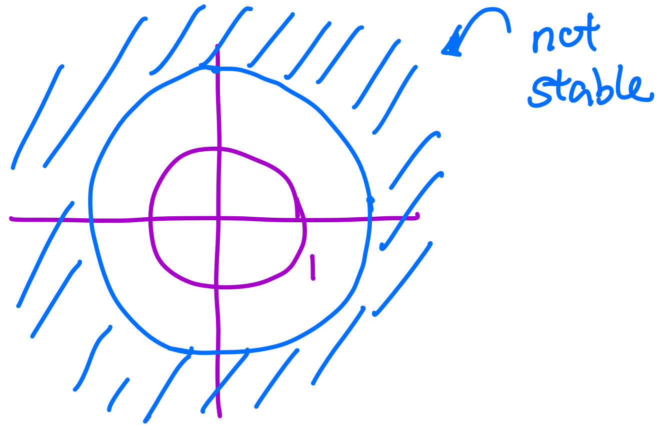
$u[n]$ to make $y[n]$ a right-sided function because we know ROC of $Y(z)$ is $|z| > 1$

Since there is a unique relation between $h[n]$ and $H(z)$. System property like causality and stability can also be inferred from $H(z)$

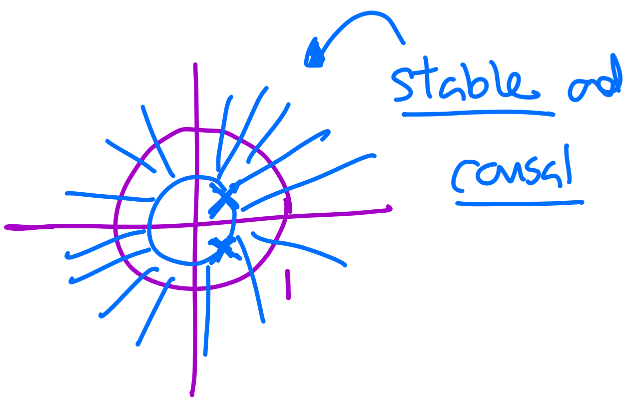
1) A LTI system is stable if and only if the ROC of $H(z)$ includes the unit circle

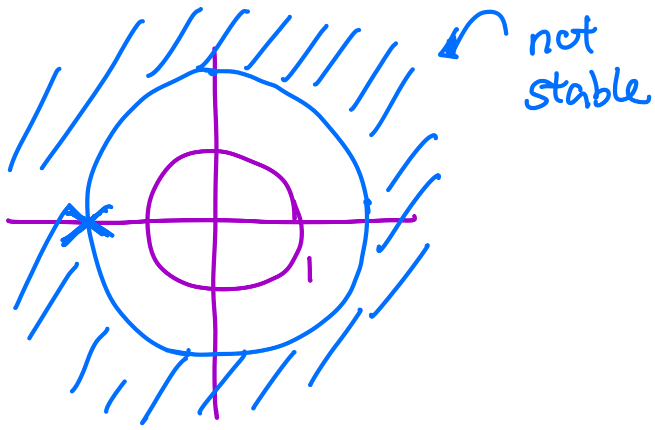
$|z| = 1$



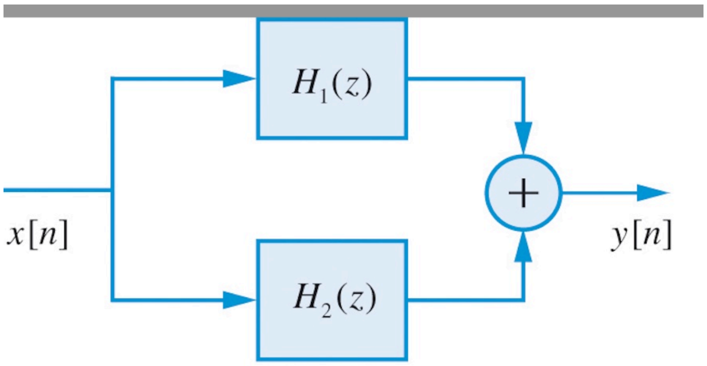


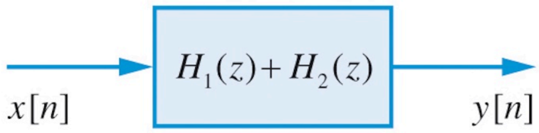
2) An LTI system with rational $H(z)$ is both causal and stable if and only if all the poles of $H(z)$ are inside the unit circle and ROC of $H(z)$ is on the exterior of a circle extending to infinity





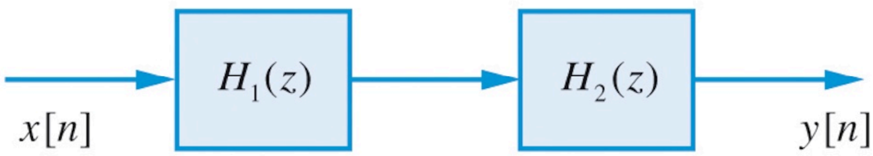
System in parallel

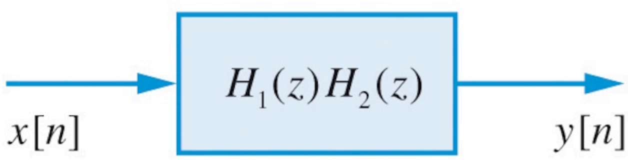




System in series

- note that we can just do multiplication instead of convolution





Chap 3.6

The reason we focus on rational $H(z)$ is because the LTI system we are interested in are linear constant coefficient difference equations

note this is a causal system

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

The z-transform is

using linearity and time-shift properties

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

If a transfer function is given in the rational form, we can find the difference equation without computation

Ex.

$$H(z) = \frac{6 - 10z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \frac{Y(z)}{X(z)}$$

time shift

$$(6 - 10z^{-1} + 2z^{-2})X(z) = (1 - 3z^{-1} + 2z^{-2})Y(z)$$

$$6x[n] - 10x[n-1] + 2x[n-2] = y[n] - 3y[n-1] + 2y[n-2]$$

If we assume the system is causal:

$$y[n] = \underbrace{3y[n-1] - 2y[n-2]}_{\text{feedback terms}} + \underbrace{6x[n] - 10x[n-1] + 2x[n-2]}_{\text{input terms}}$$

If there are feedback terms. The system is a recursive system.

If there are no feedback terms, then it is a nonrecursive system

Another way of classifying LTI system is by the length of the impulse response. If $h[n]$ has infinite nonzero values such as $h[n] = \left(\frac{1}{2}\right)^n u[n]$, the system is called Infinite Impulse Response (IIR) system.

If $h[n]$ has finite non zero value such as $h[n] = 3\delta[n] + 2\delta[n-1] + 4\delta[n-2]$, the system is called Finite Impulse Response (FIR) System

It is possible for an impulse response to contain both FIR and IIR components.

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}},$$

we can write it as

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - P_k z^{-1}}$$

The inverse z-transform is

$$h[n] = \underbrace{\sum_{k=0}^{M-N} C_k \delta[n-k]}_{\text{FIR}} + \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u[n]}_{\text{IIR}}$$

Note that FIR has no poles. So you don't have to worry about stability since ROC will always contain the unit circle