Chap 3.3

We can recover the discrete-time sequence $x[n]$ from the z-transform using

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

But we don't need to do complex integration in DSP since we have simple / finite-duration discrete-time sequences

We just need to remember basics of <u>partial fraction expansion</u> and

$$x[n] = \sum_{k=1}^{N} A_k (p_k)^n \longleftrightarrow X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

with ROC of $X(z)$ being the intersection of the ROC of individual exponential sequences

By combining the summation term, we have

$$X(z)=\sum_{k=1}^{N}\frac{A_k}{1-p_k z^{-1}} = \boxed{\frac{\sum_{k=1}^{N} A_k \prod_{\substack{m=1\\ m\neq l}}^{N}(1-p_m z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})}}$$

this tells us the poles and zeros

$$=\frac{b_0+b_1 z^{-1}+\dots b_{N-1}z^{-(N-1)}}{1+a_1 z^{-1}+\dots a_N z^{-N}}\Bigg\}\text{ a rational function}$$

A proper rational function is one where the degree of the numerator is less than the degree of the denominator

Ex: $x[n]=\sum_{k=1}^{N} A_k (p_k)^n$

a specific example is

$$x[n] = 4u[n] - 3\left(\tfrac{1}{2}\right)^n u[n]$$

$x[n]$ is a right-sided sequence, this helps us with ROC

$A_1 = 4 \qquad A_2 = -3$

$p_1 = 1 \qquad p_2 = \frac{1}{2}$

I know then

$$X(z)=\frac{4}{1-z^{-1}}+\frac{(-3)}{1-\frac{1}{2}z^{-1}}$$

$$X(z)=\sum_{k=1}^{N}\frac{A_k}{1-p_k z^{-1}}$$

If I am given $X(z)$, you can find $x[n]$ from the pairs table

$X(z)$ in rational form is then

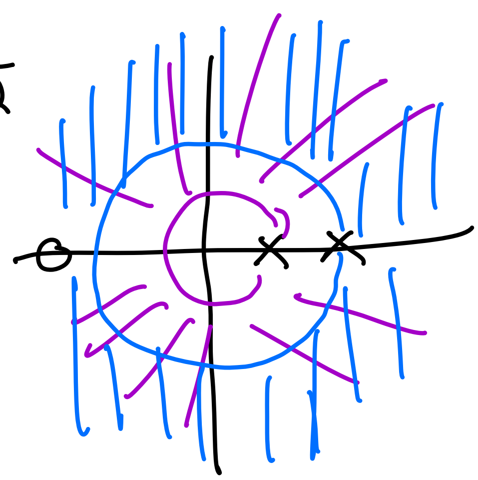
$$X(z) = \frac{4}{1-z^{-1}} + \frac{(-3)}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{4(1-\frac{1}{2}z^{-1}) + (-3)(1-z^{-1})}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{4-2z^{-1}-3+3z^{-1}}{1-\frac{1}{2}z^{-1}-z^{-1}+\frac{1}{2}z^{-2}}$$

$$= \frac{1+z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

In rational form, we see that there is a zero at $z=-1$ and two poles at $z=1$ and $z=\frac{1}{2}$



From the properties of linearity

$$ROC = \{|z| > \tfrac{1}{2}\} \cap \{|z| > 1\}$$

$$= |z| > 1$$

In DSP, finding $x[n]$ given $X(z)$ is about transforming $X(z)$ to and from rational form so we can use the pairs table

Ex 3.9 $\quad X(z) = \dfrac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$

We are given the rational form of the Z-transform. We need to do partial fraction expansion to see the exponential functions.

We have a zero at $z=-1$

Where are the poles? We can find by finding the roots of $1-z^{-1}+0.5z^{-2}$

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\qquad \dfrac{1 \pm \sqrt{1-4(\frac{1}{2})}}{2} = \dfrac{1 \pm \sqrt{-1}}{2}$

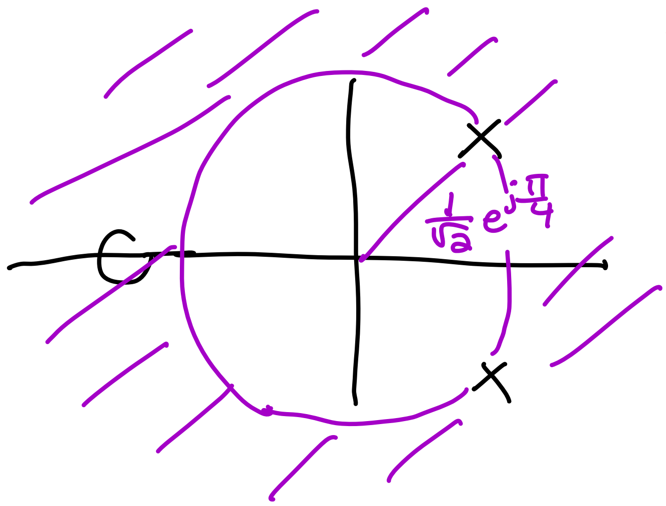
the poles are at $\dfrac{1 \pm j}{2}$

$$\frac{1+j}{2} = \frac{1}{\sqrt{2}} e^{j\pi/4}$$

$$\frac{1-j}{2} = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$$

We are told that $x[n]$ is a causal (right-sided) sequence. So we know

ROC : $|z| > \frac{1}{\sqrt{2}}$



By knowing the poles, we also know

$$X(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}z^{-1}\right)\left(1-\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

we need to do partial fraction so we have

$$X(z) = \frac{A_1}{\left(1-\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

we know

$$A_1\left(1-\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}z^{-1}\right) + A_2\left(1-\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}z^{-1}\right) = 1+z^{-1}$$

so $A_1 + A_2 = 1$

$$-A_1\left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right) - A_2\left(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\right) = 1$$

$$A_1 = \frac{1-j3}{2}, \quad A_2 = \frac{1+j3}{2}$$

therefore

$$X(z) = \frac{\frac{1-j3}{2}}{1-\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}z^{-1}} + \frac{\frac{1+j3}{2}}{1-\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}z^{-1}}$$

$$\boxed{A_1} = \frac{1-j3}{2}, \quad \boxed{A_2} = \frac{1+j3}{2}$$

$$\boxed{P_1} = \frac{1+j}{2}, \quad \boxed{P_2} = \frac{1-j}{2}$$

We are also told that $x[n]$ is a right-sided (causal sequence)

$$x[n] = \sum_{k=1}^{N} A_k (P_k)^n$$

$$x[n] = \frac{1-j3}{2}\left(\frac{1+j}{2}\right)^n u[n] + \left(\frac{1-j3}{2}\right)\left(\frac{1-j}{2}\right)^n u[n]$$

to make $x[n]$ causal

Note that A_1 and A_2 are complex conjugates and P_1 and P_2 are complex conjugates

If we have rational function with distinct poles

$$\underline{X}(z) = \frac{b_0 + b_1 z^{-1} + \dots b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

partial fraction will give

$$= \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - P_k z^{-1}}$$

$C_k = 0$ when $M < N$ (i.e. $\underline{X}(z)$ is a proper rational function)

C_k, P_k, and A_k can be computed using Matlab. But you need to know what to plug in.