Chap 5.1

Recall that LTI system behavior can be completely characterized by the impulse response $h[n]$

$$h[n] \xleftrightarrow{\text{z-transform}} H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

← transfer function

$$\Big\downarrow \; z = e^{jw}$$

$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$$

← frequency response

frequency response exist if the ROC of $H(z)$ contains the unit circle

In the frequency domain

$$Y(e^{jw}) = X(e^{jw}) H(e^{jw})$$

↑ output ↑ input

$$|Y(e^{jw})| = |H(e^{jw})| \, |X(e^{jw})|$$

magnitude spectrum of output

$$\angle Y(e^{jw}) = \angle H(e^{jw}) + \angle X(e^{jw})$$

phase spectrum of output

we can see if say at $w=\frac{\pi}{4}$

$|H(e^{jw})|$ at $w=\frac{\pi}{4}$ is 0, then

$|Y(e^{jw})|$ at $w=\frac{\pi}{4}$ is 0 regardless of

what $|X(e^{jw})|$ is a $w=\frac{\pi}{4}$. This is principle we use to filter unwanted signals at specific frequency

$|H(e^{jw})|^2$ is called the gain of the system

usually this value is reported in dB

$$\text{Gain in dB} = 10\log_{10}|H(e^{jw})|^2$$

if $|H(e^{jw})|<1$, then the gain in dB is negative

Ex 1

Given a system

$$y[n] = a\,y[n-1] + b\,x[n] \quad , \quad -1<a<1$$

what is the frequency response?

Just as with the z-transform, we know that in the z-domain:

$$Y(z) = a z^{-1} Y(z) + b X(z)$$

Let $z = e^{jw}$, then in the frequency domain

$$Y(e^{jw}) = a(e^{-jw}) Y(e^{jw}) + b X(e^{jw})$$

$$Y(e^{jw})(1 - a e^{-jw}) = b X(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{b}{1 - a e^{-jw}}$$

$$|H(e^{jw})| = \frac{|Y(e^{jw})|}{|X(e^{jw})|}$$

$$= \left| \frac{b}{1 - a e^{-jw}} \right| = \frac{|b|}{|1 - a e^{-jw}|}$$

yes this is a bit tricky

$$\angle H(e^{jw}) = \angle Y(e^{jw}) - \angle X(e^{jw})$$

why is this a minus??

$$1 - a(\cos(-\omega) + j\sin(-\omega))$$

$$= 1 - a(\cos(\omega) - j\sin(\omega))$$

$$= \underbrace{1 - a\cos(\omega)}_{\text{real part}} + \underbrace{j\,a\sin(\omega)}_{\text{imaginary part}}$$

$$|1 - ae^{-j\omega}| = \sqrt{(1 - a\cos(\omega))^2 + (a\sin(\omega))^2}$$

$$= \sqrt{1 + a^2 - 2a\cos(\omega)}$$

$$\angle\, 1 - ae^{-j\omega} = \tan^{-1}\left(\frac{a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

Therefore

$$|H(e^{j\omega})| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos(\omega)}}$$

$$\angle H(e^{j\omega}) = \angle b - \tan^{-1}\left(\frac{a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

It is customary to choose b so $|H(e^{j\omega})|$ has a maximum value of 1.

If $a > 0$, the denominator is smallest when $\omega = 0$,

$$|H(e^{j0})| = \frac{|b|}{1 - a} = 1 \qquad b = \pm(1 - a)$$

If $a<0$, the denominator is smallest when $\omega=\pi$,

$$|H(e^{j\pi})| = \frac{|b|}{1+a} = 1 \quad , \quad b = \pm 1 + a$$

$$b = 1 - |a|$$

Ex 2

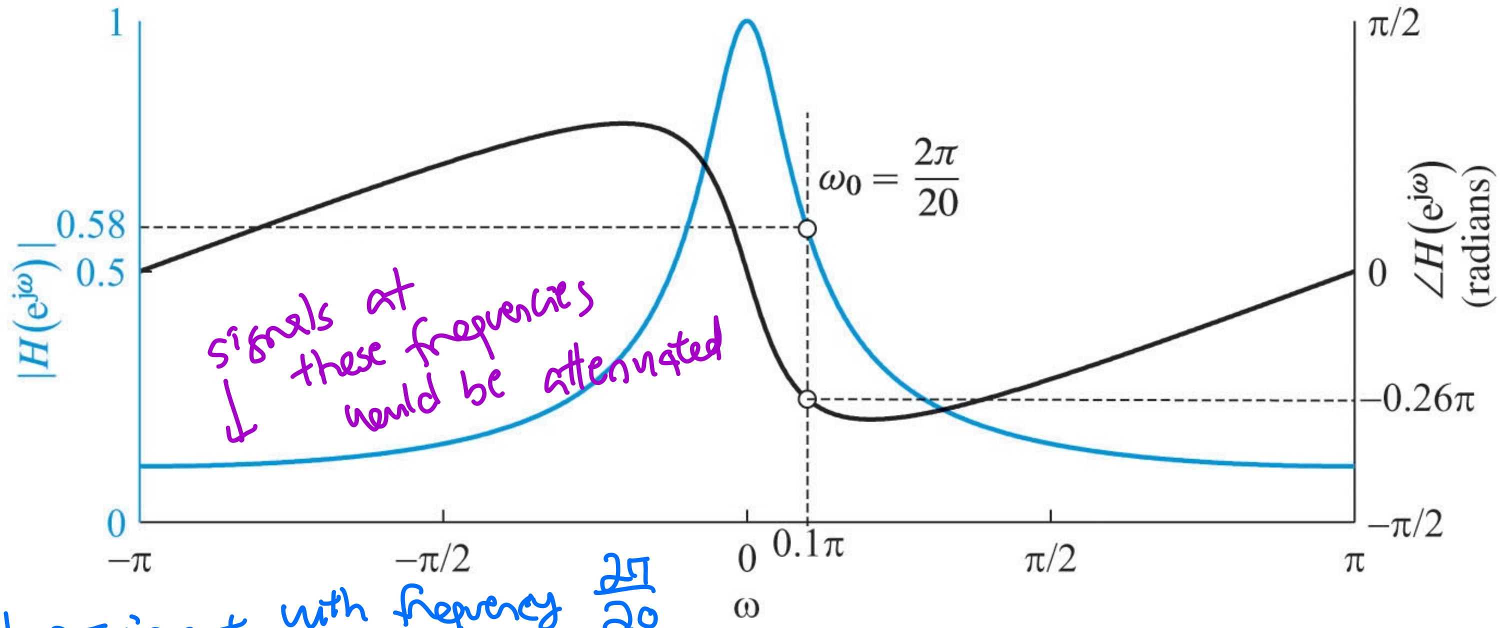
$$y[n] = \overset{a}{0.8}\, y[n-1] + \overset{b}{0.2}\, x[n]$$

We know

$$|H(e^{jw})| = \frac{0.2}{\sqrt{1+(0.8)^2 - 2(0.8)\cos(w)}}$$

$$\angle H(e^{jw}) = 0 - \tan^{-1}\left(\frac{0.8\sin(w)}{1-0.8\cos(w)}\right)$$

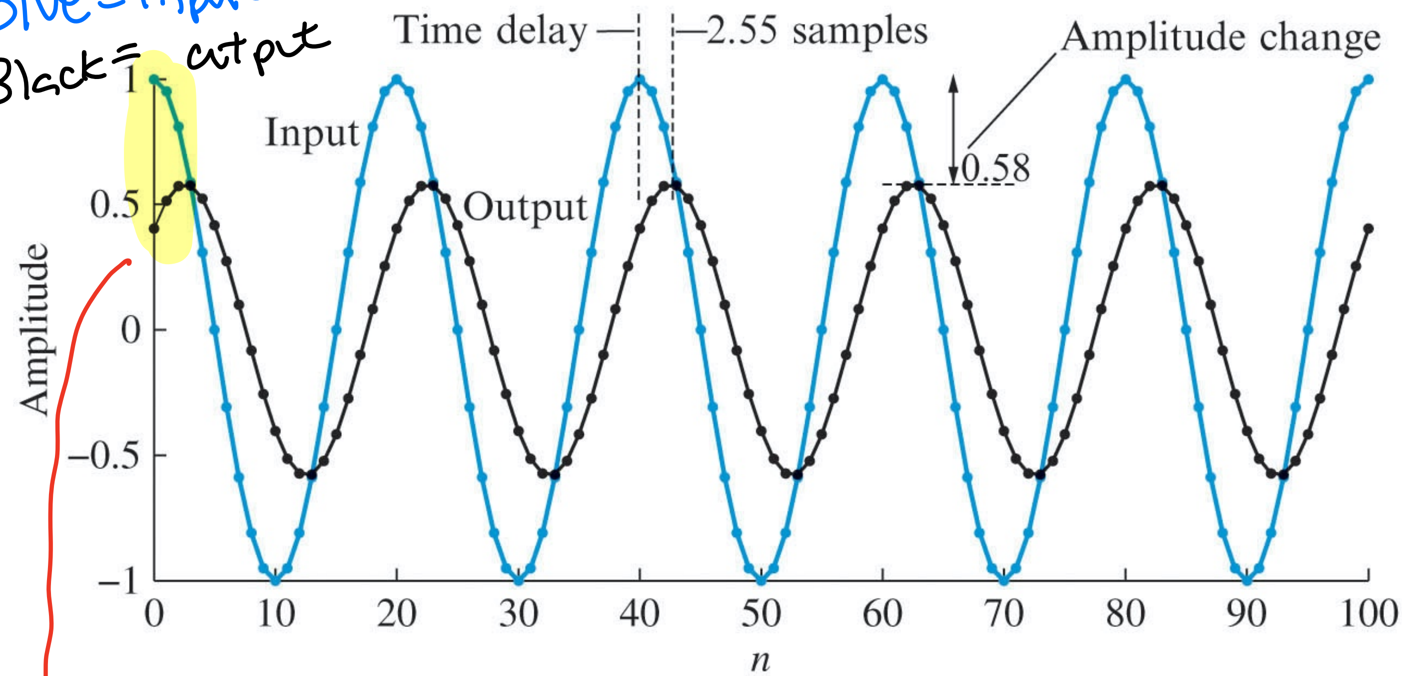
why is this 0?



signals at these frequencies would be attenuated

Blue = input with frequency $\frac{2\pi}{20}$

Black = output



notice that there is a time delay between input and output. This time delay is $\left.\frac{\angle H(e^{j\omega})}{\omega}\right|_{\omega=\frac{2\pi}{20}} = \frac{0.26\pi}{\frac{2\pi}{20}} \approx 2.6$ samples