Solutions

January 21, 2022

1) First, consider the case of adiabatic expansion where $L = \dot{\epsilon} = 0$ (i.e., no heat is entering or leaving the system). Show that temperature of a homologous expanding, radiation dominated cloud cools adiabatically as $T(t) = T_0(t/t_0)^{-1}$.

First we need to solve equation 1 and find $u(t)$. When $L = \dot{\epsilon} = 0$, equation 1 simplifies to:

$$\frac{dE_{\rm int}(t)}{dt} = -p\frac{dV(t)}{dt}. \tag{1}$$

Now we put in $E_{\rm int} = uV$ to expand the left hand side

$$\frac{dE_{\rm int}(t)}{dt} = \frac{d(uV)}{dt} = u\frac{d(V)}{dt} + V\frac{d(u)}{dt} \tag{2}$$

substitute in $p = \frac{1}{3}u$ and $\frac{dV(t)}{dt} \sim 4\pi v^3 t^2 = 3V/t$

$$u\frac{d(V)}{dt} + V\frac{d(u)}{dt} = -\frac{1}{3}u\ 3\frac{V}{t} \tag{3}$$

$$=> \frac{1}{u}\frac{du}{dt} = -\frac{4}{t}. \tag{4}$$

To solve the equation, we integrate on both sides

$$\int \frac{1}{u}du = -\int \frac{4}{t}dt \tag{5}$$

$$=> ln(u) = -4\ ln(t) + C \tag{6}$$

$$=> u = t^{-4}e^{C}. \tag{7}$$

To find the integration constant, we use the boundary condition $u(t_0) = u_0$ and find that $u(t) = u_0 t^{-4}$.
Now we know $T = (u/a)^{1/4}$, so using that $T_0 = (u_0/a)^{1/4}$, it follows that $T(t) = T_0(t)^{-1}$.

2) Next, consider the case where there is no heating ($\dot{\epsilon} = 0$) but some radiation escapes the remnant. Solve equation 1 for the energy density $u(t)$ as a function of time. Then use the diffusion equation to derive an analytic formula for the light curve, L(t) [1]. Find simple expressions (in terms of the basic physical parameters E_{int}, R_0, κ, and M) for the characteristic luminosity, $L_{\rm lc}$, of the transient and the characteristic time scale, $t_{\rm lc}$, on which the luminosity declines. Give rough values of these quantities for a neutron star merger outflow with $M = 10^{-2} M_\odot$, $E_0 = 10^{50}$ ergs.

Comment: The timescale $t_{\rm lc}$ you have derived gives the effective diffusion time in a (homologously) expanding medium, which is in general useful for optically thick outflows. It is different than the familiar static diffusion time because the density (and hence optical depth) drop as the remnant expands, making it easier for photons to escape. Note that the diffusion time in a static medium can be written $t_d \sim \kappa M / R_0 c$. You have therefore shown that $t_{\rm sn} \propto \sqrt{t_d t_e}$, where $t_e \sim R_0/v$ is the the characteristic expansion time of the cloud. In other words, the time it takes photons to diffuse out of an expanding medium is given by the geometric mean of t_d and t_e.

Finding $t_{\rm lc}$:
The diffusion of light happens when the light moves faster through the gas that the cloud is expanding.
The diffusion time from random scattering is $t_d \sim \tau \frac{R}{c}$, where tau is the opacity $\tau = R\kappa\rho$.
Comparing to the expansion time of the cloud, which is $t_e \sim \frac{R}{v}$. The light escapes when the two timescales are equal:

$$t_d = t_e \tag{8}$$

$$=> R\kappa\rho \frac{R}{c} = \frac{R}{v}. \tag{9}$$

Add in that we at this time can approxmate $r \sim vt$ and that $\rho = \frac{M}{4/3\pi R^3}$

$$\frac{R\kappa\rho}{c} = \frac{1}{v} \tag{10}$$

$$=> t_{\rm lc} = \sqrt{\frac{3\kappa M}{4\pi v}}. \tag{11}$$

Finding $L(t)$ from diffusion equation:
The diffusion equation is

$$L(r) = -4\pi r^2 \frac{c}{3\kappa r} \frac{du(r)}{dr} \tag{12}$$

[1] You can continue to approximate the spatial derivative by equation 5. Note that the radius and density in this equation also evolve with time. You can replace them with $R(t) = vt$ and $\rho(t) = M/V(t)$.

where the radial derivative of $u(r)$ can be approximated as

$$\frac{du}{dr} \sim -\frac{u(t)}{R(t)} = -\frac{E_{\rm int}(t)/V}{R(t)} \tag{13}$$

by using again that $R \sim vt$ for this stage and simplifying with our expression for $t_{\rm lc}$, we get that

$$L(r) = \frac{tE_{\rm int}(t)}{t_{\rm lc}^2} \tag{14}$$

Finding $u(t)$ from equation 1
Similar to problem 1 we start with

$$\frac{dE_{\rm int}(t)}{dt} = -p\frac{dV(t)}{dt} - L(t) \tag{15}$$

and put in $E_{\rm int} = uV$ to expand the left hand side, and substitute in $p = \frac{1}{3}u$ and $\frac{dV(t)}{dt} \sim 4\pi v^3 t^2 = 3V/t$ to arrive at

$$V\left[\frac{du}{dt} + \frac{4u}{t}\right] = -L(t). \tag{16}$$

Now we could solve this simply by integrating before, but this time we need to use a trick and simplify the expression by scaling out the adiabatic behavior. We showed above that under pure adiabatic expansion, $ut^4 = constant$. We thus we a new variable

$$w = ut^4 => u = w/t^4 \tag{17}$$

Plugging this in to the differential equation gives

$$V\left[\frac{d}{dt}\left(\frac{w}{t^4}\right) + \frac{4w}{t^5}\right] = -L(t) \tag{18}$$

when we then use the chain rule to expand the left side and make the expression much simpler

$$V\left[\frac{dw}{dt}\frac{1}{t^4} - \frac{4w}{t^5} + \frac{4w}{t^5}\right] = -L(t) \tag{19}$$

$$=> V\frac{1}{t^4}\frac{dw}{dt} = -L(t). \tag{20}$$

When we now plug back in $u(t)$ and V we get

$$\frac{4\pi v^3 t^3}{3}\frac{1}{t^4}\frac{d(ut^4)}{dt} = -L(t) \tag{21}$$

$$=> \frac{1}{t}\frac{d}{dt}\left(\frac{4\pi v^3}{3}(ut^4)\right) = -L(t) \tag{22}$$

$$=> \frac{1}{t}\frac{d}{dt}(Vut) = -L(t) \tag{23}$$

$$=> \frac{1}{t}\frac{d}{dt}(E_{\rm int}t) = -L(t). \tag{24}$$

Now we include our derivation of $L(r) = \frac{tE_{\rm int}(t)}{t_{\rm lc}^2}$ from above

$$\frac{1}{t}\frac{d}{dt}(E_{\rm int}t) = -\frac{tE_{\rm int}}{t_{\rm lc}^2}. \tag{25}$$

This is an differential equation we can solve! Move around and integrate

$$\frac{1}{tE_{\rm int}}d(E_{\rm int}t) = -dt\frac{t}{t_{\rm lc}^2} \tag{26}$$

$$ln(E_{\rm int}t) = -\frac{t^2}{2t_{\rm lc}^2} + ln(C) \tag{27}$$

where C is an integration constant. Exponentiating

$$E_{\rm int}t = Cexp\left[-\frac{t^2}{2t_{\rm lc}^2}\right] \tag{28}$$

$$=> E_{\rm int} = \frac{C}{t}exp\left[-\frac{t^2}{2t_{\rm lc}^2}\right]. \tag{29}$$

Using the boundary condition that at $E(t = t_0) = E_0$

$$E_{\rm int} = \frac{E_0t_0}{t+t_0}exp\left[-\frac{t^2}{2t_{\rm lc}^2}\right]. \tag{30}$$

Now that we have the evolution for $E_{\rm int}$, we can also have the luminosity

$$L(t) = \frac{E_0t_0}{t_{lc}^2}exp\left[-\frac{t^2}{2t_{\rm lc}^2}\right] \tag{31}$$

or using the definition of $t_0 = R_0/v$

$$L(t) = \frac{E_0R_0}{vt_{lc}^2}exp\left[-\frac{t^2}{2t_{\rm lc}^2}\right] \tag{32}$$

Finding characteristic $L_{\rm lc}$: The coefficient is the value of $L(t)$ at $t = 0$, which is a characteristic luminosity of the transient

$$L_0 = \frac{E_0R_0}{vt_{lc}^2} \tag{33}$$

where writing out $t_{\rm lc}$ gives

$$L_0 = \frac{4\pi}{3}\frac{cE_0R_0}{M\kappa} \tag{34}$$

which when plugging on values give

$$L_0 \sim 10^{39} R_1 E_{51} M_1^1 \kappa^1 \text{erg s}^{-1} \tag{35}$$

where $R_1 = R/R_\odot$, $E_{51} = E/10^{51}$, $M_1 = M/M_\odot$ and $\kappa_1 = \kappa/1\text{cm}^2\text{g}^{-1}$.

3) You'll see that the luminosity you predicted in part 2) is very dim. This is because most if the initial internal energy of the cloud is lost to expansion before it has time to be radiated away. We therefore need radioactivity to reheat the cloud at later times. Consider now the case where $\dot{\epsilon} \neq 0$. Show that at the maximum of the light curve (i.e., $dL/dt = 0$), the luminosity of the event is equal to the instantaneous radioactive energy deposition, i.e., $L(t_{\text{peak}}) = \dot{\epsilon}(t_{\text{peak}})$. This is known as Arnett's law, and is very useful for estimating the amount of radioactive material present based only on the observed peak luminosity.

From following the same steps as above, , we get to a similar expression as equation 24, but with $\dot{\epsilon} - L(t)$ on the right hand side:

$$\frac{1}{t}\frac{d}{dt}\left(E_{\text{int}} t\right) = \dot{\epsilon} - L(t) \tag{36}$$

if we plug in $L = \frac{tE_{\text{int}}(t)}{t_{\text{lc}}^2}$ from the diffusion equation, this becomes

$$\frac{t_{\text{lc}}^2}{t'}\frac{dL}{dt} = \dot{\epsilon} - L(t). \tag{37}$$

This shows that there is an extremum/maximum $\frac{dL}{dt} = 0$, when $L(t) = \dot{\epsilon}$. This is known as Arnett's law, and is very useful for estimating the amount of radioactive material present based only on the observed peak luminosity.

4) There is no simple, general solution for the case of $\dot{\epsilon} \neq 0$. By solving integrating the equation found in problem 4, we can get a function for L as an integral over $\dot{\epsilon}$:

$$L(t) = \exp\left[-\frac{t^2}{2t_{\text{lc}}^2}\right]\left(\frac{E_0 R_0}{v t_{\text{lc}}^2} + \int_0^t \dot{\epsilon}\left(t'\right)\left(\frac{t'}{t_{\text{lc}}^2}\right)\exp\left[\frac{t'^2}{2t_{\text{lc}}^2}\right] dt'\right) \tag{38}$$

(Hint: To arrive at this function it is easiest to use an *integrating factor*, here we used $\exp\left[\frac{t^2}{2t_{\text{lc}}^2}\right]$).

Getting the $L(t)$: We start from

$$\frac{t_{\text{lc}}^2}{t}\frac{dL}{dt} = \dot{\epsilon} - L(t). \tag{39}$$

We then rearrange to discover our integrating factor

$$\frac{dL}{dt} + \frac{t_{\text{lc}}^2}{t}L(t) = \frac{t_{\text{lc}}^2}{t}\dot{\epsilon}. \tag{40}$$

The integrating factor here is $\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]$, and by multiplying with it in both sides, so we can simplify the left hand side

$$\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]\frac{dL}{dt}+\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]\frac{t_{\rm lc}^2}{t}L(t)=\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]\frac{t_{\rm lc}^2}{t}\dot{\epsilon} \tag{41}$$

$$=>\frac{d}{dt}\left(\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]L\right)=\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]\frac{t_{\rm lc}^2}{t}\dot{\epsilon}. \tag{42}$$

Now we integrate on both sides

$$\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]L+C=\int\exp\left[\frac{t^2}{2t_{\rm lc}^2}\right]\frac{t_{\rm lc}^2}{t}\dot{\epsilon}dt \tag{43}$$

where C is an integration constant. Now plugging in the boundary condition that at $L(t=0)=\frac{E_0R_0}{vt_{lc}^2}$, we get the sought equation.

5) An approximate expression for the radioactive decay energy rate for r-process ejecta per particle is

$$\dot{q}\sim 1eVs^{-1}\left(\frac{t}{1\mathrm{day}}\right)^{-1.5}. \tag{44}$$

To get the the total heating rate $\dot{\epsilon}$ we can assume that most of the ejecta is in the first iron peak, Se, and then that all the ejecta contributes to the heating. Write a code (or use an existing integration package) to do the integral found in part 4). You have now calculated your own kilonova light curve!

See the notebook for the solution to problem 5 and 6.