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Gödel's Constructible Universe

Bachelor Thesis in Mathematics

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Abstract

This is an introduction to constructibility theory. We start off by constructing Gödel's constructible universe L rigorously, in which we also correct Devlin's errors in his book in this regard, by implementing a solution proven by Mathias; the details of this is put in the appendix. After having shown basic properties of L, we show that both Global Choice and the Generalized Continuum Hypothesis hold in L, implying that $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$. We then dedicate a chapter to giving a glimpse of the rich combinatorial structure of L, proving the negation of Suslins Hypothesis in L via. the use of the combinatorial principle $\diamondsuit$. In the last chapter we prove Scott's Theorem, stating that there exist no measurable cardinals in L.

Resumé

Dette er en introduktion til konstruérbarhedsteori. Vi starter med stringent at konstruere Gödels konstruérbare univers L, hvor vi retter Devlins fejl i hans bog vedrørende dette, ved at implementere en løsning, bevist af Mathias; detaljerne er lagt i appendikset. Efter at have vist de basale egenskaber ved L, viser vi både at det globale udvalgsaksiom samt den generaliserede kontinuumshypotese er sande i L, som medfører at $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$. Herefter dedikerer vi et kapitel til at give et indblik i L's rige kombinatoriske struktur, ved at bevise at negationen af Suslins sætning gælder i L, via. brugen af $\diamondsuit$. I det sidste kapitel beviser vi Scotts sætning, som siger at der ikke eksisterer nogen målelige kardinaltal i L.

> *"A favorite example against the pragmatic view that we accept an axiom because of its elegance (simplicity) and power (usefulness) is the constructibility hypothesis. It should be accepted according to the pragmatic view but is not generally accepted as true."*
>
> - Hao Wang [Wan83]

Preface

This text is intended as an introduction to constructibility theory, and it is assumed that readers have basic knowledge of set theory at the level of ordinals and cardinals, as well as knowledge of model theory up to the Löwenheim-Skolem Theorem; these facts will only be briefly recalled, mostly without proof. I do not assume prior knowledge about large cardinals, tree theory, ultraproducts or proof theory, and will explain what is needed from these fields slightly more detailed.

My notation used is pretty standard. I denote tuples in angled brackets $\langle x_1, \ldots, x_n \rangle$, languages in calligraphic font $\mathcal{L}$ and models in fraktur font $\mathfrak{A} = \langle A, \ldots \rangle$. If two structures $\mathfrak{A}, \mathfrak{B}$ are isomorphic, I denote this by $\mathfrak{A} \cong \mathfrak{B}$, and I write $x \approx y$ to indicate that the sets x and y are equinumerous (i.e. that there exists a bijection between them). Logical equivalence is denoted $\varphi \equiv \psi$. I use both ω_n and $\aleph_n$ to denote the n'th infinite cardinal, where ω_n will be used to emphasize ordinal properties and $\aleph_n$ cardinal properties. The identity class function is denoted $\mathrm{id} := \{\langle x, y \rangle \mid x = y\}$; the identity function on a set x is thus denoted $\mathrm{id} \upharpoonright x$. $\mathrm{dom}\, f$ and $\mathrm{ran}\, f$ denotes the domain and range of a function f, respectively, and I use the abbreviation $f``x := \mathrm{ran}(f \upharpoonright x)$. On denotes the class of ordinals and I'll write "$\alpha \in \mathrm{On}$" as an abbreviation for "α is an ordinal". I will use the notation $\vec{v}$ for $v_1, \ldots, v_n$ (so it's not an n-tuple). I work in first-order logic with the adequate set $\{\wedge, \neg, \exists\}$ of logical symbols. ZFC is assumed throughout, and if a theorem σ is proved by assuming a different set of axioms T, I will write this as $T \vdash \sigma$. I write $\lightning$ to denote a contradiction.

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Contents

INTRODUCTION

In the years 1900 and 1902, Hilbert introduced his well-known list of twenty-three problems. One of the problems, called the Continuum Hypothesis (CH), was placed at the very top of his list, which indicated the importance of the problem.

CONJECTURE 0.0.1 (CH). *Every subset of the reals $X \subseteq \mathbb{R}$ is equinumerous to either to $\mathbb{N}$ or $\mathbb{R}$.*

In 1926, Hilbert claimed in his article "On the Infinite" [Hil83] to have a solution to the problem, although the proof turned out to be flawed. In 1931, Gödel shocked the Hilbert school with his proofs of his Incompleteness Theorems, of which the first stated that there would always be unprovable true propositions in a formal system which is big enough to entail Peano's Axioms of arithmetic[1]. As the witness of the statement in the proof was seen as a pathological example of an unprovable proposition, few actually thought that this would matter to "real" mathematics. But in 1938, Gödel presented his *Constructible Universe* L, in which he showed that CH holds; which entailed that it is consistent with ZF to take the Continuum Hypothesis to be true via Gödel's Completeness Theorem. In fact, he showed that a more general statement holds in L, the *Generalized Continuum Hypothesis* (GCH).

CONJECTURE 0.0.2 (GCH). $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ *for all* $\alpha \in \mathrm{On}$.

Several years later, in 1963, Paul Cohen proved the converse: it is also consistent with ZF to take the Continuum Hypothesis to be false. This then implied that the Continuum Hypothesis was a "natural" mathematical statement which witnesses Gödel's First Incompleteness Theorem within ZFC set theory. Besides using L to prove $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZF + GCH)}$, it was also worthy of independent interest, since it turned out to have several nice properties such as the combinatorial principles $\diamondsuit$ and $\square$, making L very useful in a combinatorial setting.

As for the structure of this thesis, the first chapter will be dedicated to laying down the fundamental concepts we will need in the proceeding chapters. The second chapter is dedicated to showing Gödel's result that $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZFC + GCH)}$. Firstly we will construct the constructible universe L, which already seems like a daunting task, and most of the details are put in the appendix. After this we show some basic properties of L and move on to showing $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZFC)}$. The next task then is to show consistency of GCH as well, but this turns out not to be the hardest task after we have shown that L satisfies a strong property called *condensation*, which briefly states that any elementary substructure of a certain stage in L must be isomorphic to a previous stage.

Condensation is used again in the third chapter, which deals with L's *combinatorial* structure. We investigate *Suslin's Hypothesis* (SH for short), stating that the reals is the *unique* dense linear ordering without endpoints and which satisfies that every set of disjoint intervals is countable. It turns out that

[1] Actually, PA is a bit too strong, and a weaker subtheory known as *Robinson arithmetic* suffices.

this hypothesis is equivalent to the existence of a so-called *Suslin tree*, and that $\diamondsuit$ implies the existence of such a tree. Lastly, by using condensation, we show that L satisfies $\diamondsuit$, and thus the negation of SH as well. It turns out that both $\diamondsuit$ and SH are independent of ZFC, but like GCH we will only show half of this theorem: namely that $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \diamondsuit)$ and $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \neg\,\mathrm{SH})$.

In the last chapter we show Scott's Theorem, saying that there does not exist any *measurable cardinal* in L, which is used as an argument against the *axiom of constructibility*, stating that $V = L$, i.e. that every set is constructible. To be able to prove this result, we will delve into the machinery of *ultraproducts*, which is a rich topic in its own right. The fact that such a measurable cardinal does not exist in L raises the question though, whether there exists a universe with the same rich and canonical structure as L but being large enough to allow the existence of so-called *large cardinals* such as the measurable cardinals? This question spawned an entire mathematical field called *inner model theory*, and the question still remains unsolved (despite significant ongoing progress).

1 Preliminaries

1.1 Basic set theory

Recall that Zermelo-Fraenkel set theory with Choice, **ZFC**, is given by the following axioms:

(i) Extensionality: $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to (x = y)]$;
(ii) Union: $\forall x \exists y \forall z[z \in y \leftrightarrow (\exists u \in x)(z \in u)]$;
(iii) Infinity: $\exists x[\exists y(y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$;
(iv) Power Set: $\forall x \exists y \forall z[z \in y \leftrightarrow z \subseteq x]$;
(v) Foundation: $\forall x[\exists y(y \in x) \to (\exists y \in x)(\forall z \in y)(z \notin x)]$;
(vi) Comprehension schema: $\forall \vec{a} \forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \wedge \varphi(z, \vec{a}))]$;
(vii) Replacement schema: $\forall \vec{a}[\forall x \exists y \varphi(y, x, \vec{a}) \to \forall u \exists v(\forall x \in u)(\exists y \in v)\varphi(y, x, \vec{a})]$;
(viii) Choice: $\forall x[((\forall y \in x)(y \neq \emptyset) \wedge (\forall y \in x)(\forall y' \in x)(y \neq y' \to \forall w(w \in y \to w \notin y'))) \to \exists z[(\forall y \in x)(\exists! v \in y)(v \in z) \wedge (\forall v \in z)(\exists! y \in x)(v \in y)]]$.

A **relation** is a set of ordered pairs $\langle x, y\rangle$, and a **function** is a relation satisfying that for every $x \in \operatorname{dom} f$ there is exactly one $y \in \operatorname{ran} f$ such that $\langle x, y\rangle \in f$. A **partial order** is a relation R which is reflexive, transitive and antisymmetric. A set with a partial order is called a **poset**. A relation R is said to be **well-founded** if every non-empty subset has an R-least element, **linear** (sometimes called *total*) if it is irreflexive, transitive and satisfies trichotomy (i.e. xRy, yRx or $y = x$ holds for all x, y), and **well-ordered** if it is linear and well-founded. A set with a well-ordering is called a **well-ordered set**. If a relation R satisfies that xRy implies that there exists some z such that xRz and zRy, then R is called **dense**. A set with a dense linear ordering is abbreviated a **DLO**. A **class** is a collection $\{x \mid \varphi(x)\}$ definable by a first-order formula φ; it is not necessarily a set. A set x is **transitive** if $z \in y$ and $y \in x$ implies $z \in x$ for all sets $z \in y$ and $y \in x$.

Definition 1.1.1 (Down- and up-sets). Let R be a binary relation on a class M. Then $\downarrow x := \{y \in M \mid yRx\}$ is called the *down-set* of x, and $\uparrow x := \{y \in M \mid xRy\}$ is called the *up-set*.

Definition 1.1.2 (Set-like relation). Let R be a binary relation on a class M. Then R is *set-like* if y's down-set $\downarrow y = \{x \in M \mid xRy\}$ is a set for all $y \in M$.

Most definitions within set theory are given recursively, which is justified by the following theorems (note that they do not depend upon choice).

Theorem 1.1.3 (Well-founded recursion). *Let R be a binary, well-founded and set-like relation on a class M and let $G : V \times V \to V$ be a total class function. Then there is a unique total class function F with $\operatorname{dom} F = M$ such that*

$$\mathrm{ZF} \vdash F(x) = G(x, F \restriction (\downarrow x)).$$

Proof. See Jech [Jec03, Theorem 6.11] ■

Theorem 1.1.4 ($\in$-recursion). *Let M be a transitive class and let $G : V \to V$ be a total class function. Then there is a unique total class function F with* $\operatorname{dom} F = M$ *such that*

$$\mathrm{ZF} \vdash F(x) = G(F \restriction x).$$

Proof. See Jech [Jec03, Theorem 6.5] ■

An example of such a recursive definition is the notion of the **rank** of set, given by

$$\operatorname{rank}(x) := \bigcup\{\operatorname{rank}(y) + 1 \mid y \in x\},$$

where $\alpha = \operatorname{rank}(x)$ is equivalent to $\alpha \in \mathrm{On}$ being the least such that $x \in V_{\alpha+1}$.

The **ordinals** On is then the transitive sets which are well-ordered by the membership relation $\in$. We define the **successor of α** as $S(\alpha) := \alpha \cup \{\alpha\}$ for $\alpha \in \mathrm{On}$. The **order-type** of a well-ordered set $\langle x, R\rangle$ is the *unique* ordinal α such that $\langle x, R\rangle \cong \langle \alpha, \in\rangle$, denoted $\operatorname{ot}(\langle x, R\rangle) = \alpha$ (or just $\operatorname{ot}(x) = \alpha$ if R is understood).

Lemma 1.1.5. *Every ordinal α satisfies exactly one of the following:*
- $\alpha = 0$;
- $\alpha = S(\beta)$ *for some* $\beta \in \mathrm{On}$ *(α successor);*
- $\alpha = \bigcup \alpha$, $\alpha > 0$ *(α limit).* ■

Definition 1.1.6 (Ordinal arithmetic). Let $\alpha, \beta, \delta \in \mathrm{On}$, δ limit. Then we define ordinal addition, multiplication and exponentiation by $\in$-recursion on On:

$$\begin{array}{lll} \alpha + 0 := \alpha & \alpha \cdot 0 := 0 & \alpha^0 := 1 \\ \alpha + S(\beta) := S(\alpha + \beta) & \alpha S(\beta) := \alpha\beta + \alpha & \alpha^{S(\beta)} := \alpha^\beta \alpha. \\ \alpha + \delta := \sup_{\gamma<\delta}(\alpha + \gamma) & \alpha\delta := \sup_{\gamma<\delta} \alpha\gamma & \alpha^\delta := \sup_{\gamma<\delta} \alpha^\gamma \end{array}$$

A **cardinal** κ is an ordinal satisfying that for every $\lambda < \kappa$ we have $\lambda \not\approx \kappa$. If x is a set, then $|x|$ is the least ordinal such that $x \approx |x|$, called the **cardinality** of x. Clearly $|x|$ is a cardinal for every set x. For κ a cardinal, the **cardinal successor** of κ, denoted κ^+, is the least cardinal strictly greater than κ.

Definition 1.1.7 ($\aleph_\alpha$). Define the *Aleph hierachy* $(\aleph_\alpha)_{\alpha\in\mathrm{On}}$ by $\in$-recursion on On:
- $\aleph_0 := \omega$;
- $\aleph_{\alpha+1} := \aleph_\alpha^+$;
- $\aleph_\delta := \bigcup_{\gamma<\delta} \aleph_\gamma$ for δ limit.

Lemma 1.1.8. *For every infinite cardinal κ there exists $\alpha \in \mathrm{On}$ such that $\kappa = \aleph_\alpha$.* ■

Like for ordinals, we also have arithmetic for cardinals.

Definition 1.1.9 (Cardinal arithmetic). For cardinals κ, λ, define $\kappa + \lambda := |\kappa \coprod \lambda|$ (where $\coprod$ denotes the disjoint union), $\kappa\lambda := |\kappa \times \lambda|$ and $\kappa^\lambda := |{}^\lambda\kappa|$ (where ${}^\lambda\kappa$ denotes the set of functions from λ to κ).

As there's a notational clash between ordinal- and cardinal arithmetic, we will use the greek letters α, β, γ, δ, ξ, ζ for ordinals (and ordinal arithmetic) and κ, λ, μ, ν for cardinals (and cardinal arithmetic).

Lemma 1.1.10. *Let κ, λ be cardinals satisfying $0 < \kappa \leq \lambda$ and $\lambda \geq \aleph_0$. Then $\kappa + \lambda = \kappa\lambda = \lambda$.* ■

The **cofinality** of a limit ordinal δ, denoted $\operatorname{cf}\delta$, is the least ordinal α such that there exists a function $f : \alpha \to \delta$ satisfying $\sup(\operatorname{ran} f) = \delta$. If $\operatorname{cf}(\delta) < \delta$ then δ is **singular**; otherwise it is **regular**.

1.2 Basic model theory

Definition 1.2.1 (Language). A (first-order) *language* $\mathcal{L} = \{F_0, F_1, \ldots, R_0, R_1, \ldots, C_0, C_1, \ldots\}$ is a collection of function symbols F_i with an attached arity $a(F_i) \geq 1$, relation symbols R_j with an attached arity $a(R_i) \geq 1$ and constant symbols C_k, as well as the implicit first-order logical symbols $\wedge$, $\neg$, $\exists$, "(", ")" and v_n for $n \geq 0$. Every language furthermore includes the binary relation symbol $=$.

Definition 1.2.2 (Structure). An $\mathcal{L}$-*structure* $\mathfrak{A} = \langle A, F_0^{\mathfrak{A}}, F_1^{\mathfrak{A}}, \ldots, R_0^{\mathfrak{A}}, R_1^{\mathfrak{A}}, \ldots, C_0^{\mathfrak{A}}, C_1^{\mathfrak{A}}, \ldots\rangle$ for a language $\mathcal{L}$ is a set A along with functions $F_i^{\mathfrak{A}} : A^{a(F_i)} \to A$, relations $R_j^{\mathfrak{A}} \subseteq A^{a(R_j)}$ and constants $C^{\mathfrak{A}} \in A$ for each function symbol F_i, relation symbol R_j and constant symbol C_k in $\mathcal{L}$, respectively. We note that the interpretation of $=$ is the usual equality of sets, given by the Extensionality axiom.

For every structure $\mathfrak{A}, \mathfrak{B}, \ldots$, we will always denote the underlying set by the corresponding regular letter $A, B, \ldots$.

Definition 1.2.3 (Substructure). Let $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures. We say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ iff $A \subseteq B$, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$ for every n-ary relation symbol R in $\mathcal{L}$, $F^{\mathfrak{A}} = F^{\mathfrak{B}} \restriction A^m$ for every m-ary function symbol F in $\mathcal{L}$ and $C^{\mathfrak{A}} = C^{\mathfrak{B}}$ for every constant symbol C in $\mathcal{L}$.

Definition 1.2.4 (Terms). The $\mathcal{L}$-*terms* of a language $\mathcal{L}$ is defined recursively:

- Any variable or constant symbol is an $\mathcal{L}$-term;
- If $t_1, \ldots, t_n$ are $\mathcal{L}$-terms and F is an n-ary function symbol in $\mathcal{L}$, then $F(t_1 \ldots t_n)$ is an $\mathcal{L}$-term.

Definition 1.2.5 (Formulas). The $\mathcal{L}$-*formulas* of a language $\mathcal{L}$ is defined recursively:

- If $t_1, \ldots, t_n$ are $\mathcal{L}$-terms and R is an n-ary $\mathcal{L}$-relation symbol, then $R(t_1 \ldots t_n)$ is an *atomic* $\mathcal{L}$-formula;
- If φ, ψ are $\mathcal{L}$-formulas, then so are $(\varphi \wedge \psi)$, $(\neg\varphi)$ and $(\exists x\varphi)$.

We will usually write binary relations $R(t_1 t_2)$ in infix notation $t_1 R t_2$. We will denote interpreted constants as x for the corresponding constant symbol $\dot{x}$ for clarity, instead of writing the formally more correct constant symbol x and interpretation $x^{\mathfrak{A}}$. We will usually omit the parentheses for reading comprehension, as well as introduce the abbreviation $(\exists x \in y)\varphi$ for $\exists x(x \in y \wedge \varphi)$. When not otherwise specified, we will *always* presume our formulas to be in the language of set theory $\mathcal{L}_\in := \{\in\}$.

Definition 1.2.6 (Free variables). Let φ be an $\mathcal{L}$-formula. A *free variable* of φ is a variable which has a free occurrence in φ, i.e. does not occur within the scope of a quantifier. We usually write $\varphi(\vec{v})$ to denote the formula φ with free variables among $\vec{v}$.

A **sentence** σ is a formula with no free variables. A **theory** T is a collection of sentences, and any subset of a theory is called a **subtheory**. A theory is **satisfiable** if it has a model.

Definition 1.2.7 (Interpretation of terms). Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, $t(\vec{v})$ an $\mathcal{L}$-term and $\vec{x} \in A$. Then the *interpretation* of $t(\vec{v})$ in $\mathfrak{A}$ at $\vec{x}$, denoted $t^{\mathfrak{A}}(x_1/v_1, \ldots, x_n/v_n)$, is defined recursively:

- If t is a variable v then $t^{\mathfrak{A}}(x/v) := x$;
- If t is a constant symbol c then $t^{\mathfrak{A}}(x/v) := c^{\mathfrak{A}}$;
- If $t = F(t_1 \ldots t_k)$ where F is a k-ary function symbol of $\mathcal{L}$, then $t^{\mathfrak{A}}(x_1/v_1, \ldots, x_n/v_n) := F^{\mathfrak{A}}(t_1^{\mathfrak{A}}(x_1/v_1, \ldots, x_n/v_n), \ldots, t_k^{\mathfrak{A}}(x_1/v_1, \ldots, x_n/v_n))$.

Although $t^{\mathfrak{A}}(x_1/v_1, \ldots, x_n/v_n)$ is the more correct terminology, we will usually just write $t^{\mathfrak{A}}[x_1, \ldots, x_n]$ (or $t[x_1, \ldots, x_n]$ when the structure is understood) instead.

Definition 1.2.8 (Satisfaction relation). Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, $\varphi(\vec{v})$ an $\mathcal{L}$-formula and $\vec{x} \in A$. Then we recursively define the *satisfaction relation* $\mathfrak{A} \models \varphi[\vec{x}]$ as follows, where ψ and χ are $\mathcal{L}$-formulas:

- If $\varphi(\vec{v})$ is $R(t_1(\vec{v}) \ldots t_n(\vec{v}))$ for an n-ary relation symbol R in $\mathcal{L}$ and $t_1, \ldots, t_n$ $\mathcal{L}$-terms, then $\mathfrak{A} \models \varphi[\vec{x}]$ if
$\langle t_1^{\mathfrak{A}}[\vec{x}], \ldots, t_{n_j}^{\mathfrak{A}}[\vec{x}] \rangle \in R_j^{\mathfrak{A}}$;
- If $\varphi(\vec{v})$ is $(\neg\psi(\vec{v}))$ then $\mathfrak{A} \models \varphi[\vec{x}]$ if $\mathfrak{A} \not\models \psi[\vec{x}]$;
- If $\varphi(\vec{v})$ is $(\psi(\vec{v}) \wedge \chi(\vec{v}))$ then $\mathfrak{A} \models \varphi[\vec{x}]$ if $\mathfrak{A} \models \psi[\vec{x}]$ and $\mathfrak{A} \models \chi[\vec{x}]$;
- If $\varphi(\vec{v})$ is $(\exists y \psi(y, \vec{v}))$ then $\mathfrak{A} \models \varphi[\vec{x}]$ if $\mathfrak{A} \models \psi[y, \vec{x}]$ for some $y \in A$.

If $\mathfrak{A} \models \varphi$ (where φ has no free variables), then we say that $\mathfrak{A}$ *models* φ, $\mathfrak{A}$ *satisfies* φ or $\mathfrak{A}$ *is a model of* φ.

Definition 1.2.9 (Definability). Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, $P \subseteq A$ and $X \subseteq A^n$ for some $n \geq 1$. Then we say that X is *definable with parameters from P in* $\mathfrak{A}$ if there is an $\mathcal{L}$-formula $\varphi(\vec{v}, \vec{w})$ and $\vec{p} \in P$ such that $X = \{\langle a_1, \ldots, a_n \rangle \in A^n \mid \mathfrak{A} \models \varphi[\vec{a}, \vec{p}]\}$. If $P = \emptyset$ then we say that X is *definable without parameters*. X is *definable* in $\mathfrak{A}$ if it is definable, with possible parameters from A in $\mathfrak{A}$.

Definition 1.2.10 (Elementary embedding/substructure/extension). Let $\mathcal{L}$ be a language and $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures. Then $j : A \to B$ is an *elementary embedding*, denoted $j : \mathfrak{A} \preceq \mathfrak{B}$, if j is injective and for every $\mathcal{L}$-formula $\varphi(\vec{v})$ and $\vec{x} \in A$ it holds that

$$\mathfrak{A} \models \varphi[x_1, \ldots, x_n] \Leftrightarrow \mathfrak{B} \models \varphi[j(x_1), \ldots, j(x_n)].$$

If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the inclusion map $A \hookrightarrow B$ is an elementary embedding, then $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ ($\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$), denoted $\mathfrak{A} \preceq \mathfrak{B}$.

Note that $j : \mathfrak{A} \preceq \mathfrak{B}$ does not necessarily imply that $A \subseteq B$.

Definition 1.2.11 (Elementary equivalence). Two $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ are said to be *elementarily equivalent* if for every $\mathcal{L}$-sentence σ it holds that $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{B} \models \sigma$.

Definition 1.2.12 (Isomorphism). An isomorphism $\pi : \mathfrak{A} \cong \mathfrak{B}$ between $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, is a bijection $\pi : A \approx B$ such that for all $\vec{x} \in A$ the following holds:

(i) $\langle x_1, \ldots, x_n \rangle \in R^{\mathfrak{A}}$ iff $\langle \pi(x_1), \ldots, \pi(x_n) \rangle \in R^{\mathfrak{B}}$ for every n-ary $\mathcal{L}$-relation symbol R;
(ii) $\pi(F^{\mathfrak{A}}(x_1, \ldots, x_n)) = F^{\mathfrak{B}}(\pi(x_1), \ldots, \pi(x_n))$ for every n-ary $\mathcal{L}$-function symbol F.

Lemma 1.2.13. *Let $\mathcal{L}$ be a language and $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures. If $\pi : \mathfrak{A} \cong \mathfrak{B}$ then for every $\mathcal{L}$-formula $\varphi(\vec{v})$ and $\vec{x} \in A$ we have that*

$$\mathfrak{A} \models \varphi[x_1, \ldots, x_n] \Leftrightarrow \mathfrak{B} \models \varphi[\pi(x_1), \ldots, \pi(x_n)].$$

Proof. It suffices to show the "$\Rightarrow$"-direction, as π^{-1} is an isomorphism as well; let $\vec{x} \in A$. We prove it by induction on $\mathcal{L}$-formulas φ. If φ is atomic, then it follows by definition of being an isomorphism. If φ is $\psi \wedge \chi$ or $\neg\psi$ then it follows by definition of $\models$ and the induction hypothesis. Finally if $\varphi(\vec{v})$ is $\exists y \psi(y, \vec{v})$ then by definition of $\models$ we have $\mathfrak{A} \models \psi[y, \vec{x}]$ for some $y \in A$, implying $\mathfrak{B} \models \psi[\pi(y), \pi(x_1), \ldots, \pi(x_n)]$ by the induction hypothesis, and then $\mathfrak{B} \models \varphi[\pi(x_1), \ldots, \pi(x_n)]$ by definition of $\models$ again. ■

One of the tools that we will use on a frequent basis is the following theorem, which we will state without proof here.

Theorem 1.2.14 (Löwenheim-Skolem). *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $X \subseteq A$. Then there exists $\mathfrak{B} \preceq \mathfrak{A}$ such that $|B| \leq |X| + |\mathcal{L}| + \aleph_0$ and $X \subseteq B$. In particular, if $\mathfrak{A}$ is an infinite $\mathcal{L}$-structure and κ an infinite cardinal such that $|\mathcal{L}| \leq \kappa \leq |A|$, then $\mathfrak{A}$ has an elementary substructure of cardinality κ.*

Proof. See e.g. Marker [Mar02, Theorem 2.3.7]. ■

1.3 Absoluteness and the Lévy Hierachy

We will often abuse notation and write $X \models \varphi$ rather than the correct $\langle X, \in\rangle \models \varphi$. We begin with the notion of a relativization of a formula to a class M, which is intuitively constructed by simply replacing unbounded quantifiers with quantifiers bounded by M.

Definition 1.3.1 (Relativization). Let M be a class. Then the *relativization* φ^M for any formula φ is defined by recursion on the complexity of formulas:

- $(v_0 = v_1)^M$ is $v_0 = v_1$ and $(v_0 \in v_1)^M$ is $v_0 \in v_1$;
- $(\varphi \wedge \psi)^M$ is $\varphi^M \wedge \psi^M$ and $(\neg\varphi)^M$ is $\neg\varphi^M$;
- $(\exists x \varphi)^M$ is $(\exists x \in M)\varphi^M$.

The relativization carries along with it a notion of absoluteness of formulas. If M is a class and $\varphi(\vec{v})$ is a formula, then φ is **upwards absolute** for M if $\varphi^M[\vec{x}]$ implies $\varphi[\vec{x}]$ for all $\vec{x} \in M$. Likewise φ is **downwards absolute** for M, if $\varphi[\vec{x}]$ implies $\varphi^M[\vec{x}]$ for $\vec{x} \in M$. φ is **absolute** for M if it is both downwards and upwards absolute for M, i.e. that $\varphi^M[\vec{x}] \equiv \varphi[\vec{x}]$.

Definition 1.3.2 (Δ_0 formula). The property of being a Δ_0 formula is defined recursively on the complexity of formulas:

- $(v_0 = v_1)$ and $(v_0 \in v_1)$ are Δ_0;
- If φ and ψ are Δ_0, then so are $(\varphi \wedge \psi)$ and $(\neg\varphi)$;
- If φ is Δ_0 and y is a set, then $(\forall x \in y)\varphi$ and $(\exists x \in y)\varphi$ are Δ_0.

Intuitively, Δ_0-formulas are thus formulas with only bounded quantifiers. For notational ease, we also write that a formula is Σ_0 or Π_0 iff it is Δ_0. A lot of formulas are indeed Δ_0. Basic formulas such as $x = y$, $x \in y$, $x \subseteq y$, $x = \{y_0, \ldots, y_n\}$, $x = \langle y_1, \ldots, y_n\rangle$, $x = y \cup z$, $x = y \cap z$, $x = \bigcup y$, $x = \bigcap y$, $x = y \setminus z$ and

$x = y \times z$ are all Δ_0, as one can check by writing them out. We also have that the property of being an ordered pair, relation, function, function from x to y, injection, surjection, domain/range/image/restriction of a function, ordinal, limit ordinal, successor ordinal, successor (i.e. $x = y \cup \{y\}$), natural number and first/second element of an ordered pair (denoted fst/snd, i.e. $\mathrm{fst}\langle x, y\rangle = x$ and $\mathrm{snd}\langle x, y\rangle = y$) are Δ_0. To summarize:

Lemma 1.3.3. *The following are expressible by Δ_0 formulas:*

$$x = y, x \in y, x \subseteq y, x = \{y_1, \dots, y_n\}, x = \langle y_1, \dots, y_n\rangle, x = y \cup z, x = y \cap z, x = \bigcup y,$$
$$x = \bigcap y, x = y \backslash z, x = y \times z, \mathsf{isOrdPair}(p), \mathsf{isRel}(r), \mathsf{isFct}(f), x = \operatorname{dom} r, x = \operatorname{ran} r,$$
$$x = f(y), x = f"y, x = f \restriction y, \mathsf{on}(\alpha), \mathsf{isLimit}(\delta), \mathsf{isSucc}(\alpha), x = S(y),$$
$$\mathsf{isNat}(n), f : x \to y, \mathsf{isInj}(f), \mathsf{isSurj}(f), x = \operatorname{fst} p, x = \operatorname{snd} p.$$

Proof. They are all straight forward. For instance $\mathsf{on}(\alpha)$ is "α is transitive"$\wedge$"α is well-ordered by $\in$", $\mathsf{isLimit}(\delta)$ is $\mathsf{on}(\delta) \wedge \delta = \bigcup \delta \wedge \delta \neq \emptyset$, $\mathsf{isNat}(n)$ is $\mathsf{on}(n) \wedge \neg\mathsf{isLimit}(n) \wedge (\forall k \in n)\neg\mathsf{isLimit}(k)$, $x = \operatorname{fst} p$ is $\mathsf{isOrdPair}(p) \wedge (\exists a \in p)(x \in a \wedge (\forall b \in a)(b = x))$ and so on. ■

From the Δ_0-formulas, we proceed to construct the entirety of the so-called Lévy hierachy:

Definition 1.3.4 (Lévy hierachy). Let φ be a formula. Then
- φ is Σ_{n+1} if φ is logically equivalent to $\exists x_1 \exists x_2 \cdots \exists x_n \psi$ where ψ is Π_n;
- φ is Π_{n+1} if φ is logically equivalent to $\forall x_1 \forall x_2 \cdots \forall x_n \psi$ where ψ is Σ_n;
- φ is Δ_n if it is both Σ_n and Π_n.

If $j : \mathfrak{A} \to \mathfrak{B}$ is an elementary embedding for all Σ_n formulas $\varphi(\vec{v})$, we write $j : \mathfrak{A} \preceq_n \mathfrak{B}$ and analogously for $\mathfrak{A} \preceq_n \mathfrak{B}$. Note that if $j : \mathfrak{A} \preceq_n \mathfrak{B}$ then it implies that j is also an elementary embedding for Π_n and Δ_n formulas. Indeed, if $\varphi(\vec{v})$ is Π_n and $\vec{x} \in A$, then $\neg\varphi(\vec{v})$ is Σ_n, so $\mathfrak{A} \models \neg\varphi[\vec{x}]$ iff $\mathfrak{B} \models \neg\varphi[j(x_1), \dots, j(x_n)]$; the result now follows from the fact that $\mathfrak{A} \models \neg\varphi[\vec{x}]$ iff $\mathfrak{A} \not\models \varphi[\vec{x}]$. For transitive classes, we have the following extremely useful lemma, which will be used repeatedly throughout the thesis.

Lemma 1.3.5. *Let M be a transitive class and φ a formula. Then*
(i) If φ is Δ_0 then φ is absolute for M;
(ii) If φ is Σ_1 then φ is upwards absolute for M;
(iii) If φ is Π_1 then φ is downwards absolute for M;
(iv) If φ is Δ_1 then φ is absolute for M.

Proof. (i): We proceed by induction of the complexity of φ. If φ is an atomic, a conjunction or a negation, then it follows by definition of relativization. Suppose now $\varphi(y, \vec{v})$ is $(\exists x \in y)\psi(x, y, \vec{v})$ with ψ being absolute for M. Let $y, \vec{z} \in M$. We show $\varphi^M[y, \vec{z}] \Leftrightarrow \varphi[y, \vec{z}]$.
"$\Rightarrow$": Assume $\varphi^M(y, \vec{z})$, meaning $[(\exists x \in y)\psi(x, y, \vec{z})]^M$. Thus there exists some $x \in M$ satisfying $x \in y$ and $\psi^M[x, y, \vec{z}]$, which by the induction hypothesis entails $\psi[x, y, \vec{z}]$. This means exactly that $(\exists x \in y)\psi[x, y, \vec{z}]$, which is to say that $\varphi[y, \vec{z}]$.
"$\Leftarrow$": Assume $\varphi[y, \vec{z}]$. So for some $x \in y$ we have that $\psi[x, y, \vec{z}]$. As M is transitive, we have $x \in M$, as $y \in M$. Hence by induction we then have $\psi^M[x, y, \vec{z}]$ and thus $((\exists x \in y)\varphi[x, y, \vec{z}])^M$, meaning $\varphi^M[y, \vec{z}]$.

(ii): Let $\varphi(\vec{v})$ be $\exists x\psi(x,\vec{v})$ with ψ being Δ_0, and assume $\varphi^M[\vec{z}]$ with $\vec{z}\in M$. This means that there is some $x\in M$ satisfying $\psi^M[x,\vec{z}]$. As ψ is Δ_0, we have $\psi[x,\vec{z}]$ by (i). Thus $\exists x\psi[x,\vec{z}]$.

(iii): Let $\varphi(\vec{v})$ be $\forall x\psi(x,\vec{v})$ with ψ being Δ_0, and assume $\varphi[\vec{z}]$ with $\vec{z}\in M$. Thus for all x we have that $\psi[x,\vec{z}]$, and in particular for all $x\in M$. Hence by (i), we have that $\psi^M[x,\vec{z}]$, entailing $(\forall x\in M)\psi^M[x,\vec{z}]$, which is the same as $\varphi^M[\vec{z}]$.

(iv): Follows directly from (ii) and (iii). ■

We also have a slightly weaker analogous hierachy of formulas, with a corresponding absoluteness result.

Definition 1.3.6 (Relativized Lévy hierachy). Let φ be a formula and T a subtheory of ZF. Then
- φ is Σ_n^T if $T\vdash\varphi\leftrightarrow\psi$, where ψ is Σ_n;
- φ is Π_n^T if $T\vdash\varphi\leftrightarrow\psi$, where ψ is Π_n;
- φ is Δ_n^T if it is both Σ_n^T and Π_n^T.

Notice that, in a theory T with the pairing axiom, formulas can always be reduced to ones with alternating universal and existential quantifiers, as a formula such as $\forall x\forall y\varphi(x,y)$ with $\varphi(x,y)$ being Σ_n can be reduced to $\forall z(\mathsf{isOrdPair}(z)\to\varphi(\mathrm{fst}\,z,\mathrm{snd}\,z))$. Thus such a formula would be Π_{n+1}^T as well, and the same argument goes through for Σ-formulas.

Lemma 1.3.7. *Let T be a subtheory of* ZF*, and let M be a transitive class such that $T\vdash\sigma^M$ for every axiom σ of T. Let $\varphi(\vec{v})$ a formula. Then*
(i) If φ is Δ_0^T then φ is absolute for M;
(ii) If φ is Σ_1^T then φ is upwards absolute for M;
(iii) If φ is Π_1^T then φ is downwards absolute for M;
(iv) If φ is Δ_1^T then φ is absolute for M.

Proof. (i): By definition of φ being Δ_0^T, there is some Δ_0 formula $\psi(\vec{v})$ such that $T\vdash\varphi\leftrightarrow\psi$. As T is a subtheory of ZF, we have that $\forall x_1\cdots\forall x_n(\varphi[\vec{x}]\leftrightarrow\psi[\vec{x}])$. Since M satisfies every axiom of T, this implies $[\forall x_1\cdots\forall x_n(\varphi(\vec{x})\leftrightarrow\psi(\vec{x}))]^M$, which means that

$$(\forall x_1\in M)\cdots(\forall x_n\in M)(\varphi^M[\vec{x}]\leftrightarrow\psi^M[\vec{x}]).$$

As $\psi^M\equiv\psi$ by Lemma 1.3.5, we have that

$$(\forall x_1\in M)\cdots(\forall x_n\in M)(\varphi^M[\vec{x}]\leftrightarrow\psi^M[\vec{x}]\leftrightarrow\psi[\vec{x}]\leftrightarrow\varphi[\vec{x}]).$$

(ii)-(iv) is analogous. ■

Lemma 1.3.8. *The formula $v=\mathrm{rank}\,u$ is Δ_1^{ZF} and Σ_1.*

Proof. First let $\varphi(f)$ be the formula

$$\mathsf{isFct}(f)\wedge(\forall x\in\mathrm{dom}\,f)(x\subseteq\mathrm{dom}\,f\wedge f(x)=\bigcup\{f(y)+1\mid y\in x\}),$$

which states that f is a rank function. Now

$$(v=\mathrm{rank}\,u)\equiv\exists f(\varphi(f)\wedge\langle u,v\rangle\in f),$$

so it is clearly Σ_1, as φ is easily seen to be Δ_0. Furthermore we have that

$$(v = \operatorname{rank} u) \equiv \forall f(\varphi(f) \wedge u \in \operatorname{dom} f \to \langle u, v\rangle \in f),$$

since the rank function is defined by $\in$-recursion and is thus unique by the Recursion Theorem 1.1.4, whence we conclude that $v = \operatorname{rank} u$ is Δ_1^{ZF}. ■

1.4 Trees

This section will only be used in Chapter 3.

A **tree** $\mathfrak{T} = \langle T, <_T\rangle$ is a poset where each down-set $\downarrow x = \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. For every $x \in T$, we define the **height** of x to be the order-type of $\downarrow x$, denoted $\operatorname{ht} x$. The **α-level** of $\mathfrak{T}$ is defined as $T_\alpha := \{x \in T \mid \operatorname{ht} x = \alpha\}$. The **restriction** of (the underlying set of) a tree to an ordinal α is defined as $T \upharpoonright \alpha := \bigcup_{\gamma<\alpha} T_\gamma$, and the restriction to the tree is given by $\mathfrak{T} \upharpoonright \alpha := \langle T \upharpoonright \alpha, <_{T\upharpoonright\alpha}\rangle$, where $<_{T\upharpoonright\alpha} := <_T \upharpoonright(T \upharpoonright \alpha)^2$ is the restriction of the ordering on T to $T \upharpoonright \alpha$. A tree $\mathfrak{T}$ has **unique limits** if for every $x, y \in T$ we have that $\downarrow x = \downarrow y$ implies $x = y$.

An **α-branch** $b \subseteq T$ is a downwards closed ($\downarrow x \subseteq b$ for every $x \in b$) and linearly ordered set with respect to $<_T$, of order-type α. A **maximal branch** is a branch which is not properly contained in any other branch. An **antichain** $A \subseteq T$ is a subset in which every pair of elements $x, y \in A$ is $<_T$-incomparable, written $x \perp y$. A **maximal antichain** is an antichain which is not properly contained in another antichain. Maximal branches and antichains always exist by Zorn's lemma, which is a well-known equivalent to the axiom of choice.

Let $\alpha \in \mathrm{On}$, and κ a cardinal. Then $\mathfrak{T}$ is said to be an **(α,κ)-tree** if $0 < |T_\gamma| < \kappa$ for every $\gamma < \alpha$ and $T_\alpha = \emptyset$ (i.e. "height" α and "width" κ).

Definition 1.4.1 (Normal tree). An (α,κ)-tree $\mathfrak{T}$ is *normal* if

(i) It has unique limits;
(ii) It has one root, i.e. $|T_0| = 1$;
(iii) Every node can be split, i.e. for every $\gamma + 1 < \alpha$ and $x \in T_\gamma$ there are distinct $y_1, y_2 \in T_{\gamma+1}$ such that $x <_T y_1$ and $x <_T y_2$;
(iv) Every node can be extended, i.e. for every $\gamma < \beta < \alpha$ and $x \in T_\gamma$ there is $y \in T_\beta$ such that $x <_T y$.

A normal (κ,κ)-tree, where κ is an infinite cardinal, is called a **κ-tree**.

1.5 Clubs

This section will only be used in Chapter 3.

Let δ be a limit ordinal. A set $A \subseteq \delta$ is **unbounded** in δ if $(\forall\gamma < \delta)(\exists a \in A)(\gamma \leq a)$. A **limit point** of A is a limit ordinal $\delta \in \mathrm{On}$ such that $A \cap \delta$ is unbounded in δ. A is **closed** in δ if either of the three equivalent conditions are satisfied:

(i) $\bigcup(A \cap \gamma) \in A$ for all $\gamma < \delta$;
(ii) A contains all its limit points below δ;

(iii) If $\gamma \in \mathrm{On}$ is limit and $(\alpha_i)_{i<\gamma}$ is a strictly increasing sequence of elements of A not cofinal in δ, then $\bigcup_{i<\gamma} \alpha_i \in A$.

If a set is both closed and unbounded, it is called **club**. Let $A \subseteq \delta$. If $C \cap A \neq \emptyset$ for every club C in δ, then A is called **stationary**. As an analogy to measure theory, one can think of club sets as sets with "measure 1", stationary sets as sets with "positive measure" and non-stationary sets as "null sets".

Lemma 1.5.1. *Let κ be an uncountable cardinal. If $\lambda < \operatorname{cf} \kappa$ and A_i is a club subset of κ for all $i < \lambda$, then $\bigcap_{i<\lambda} A_i$ is a club subset of κ.* ■

The study of club and stationary sets is a part of the rich field of combinatorial set theory, and we have only included the bare minimum that we require in this text.

1.6 Filters

This section will only be used in Chapter 4.

Definition 1.6.1 (Filter). Let I be a set. Then a *filter* $D \subseteq \mathcal{P}(I)$ is a family of subsets of I satisfying that

(i) $I \in D$;
(ii) If $x \in D$, $y \in \mathcal{P}(I)$ and $y \supseteq x$, then $y \in D$;
(iii) If $x, y \in D$, then $x \cap y \in D$.

For $D \subseteq \mathcal{P}(I)$ we write D is a filter *over* I and D is a filter *on* $\mathcal{P}(I)$. A **principal filter** is a set of the form $\{y \in \mathcal{P}(I) \mid z \subseteq y\}$ for some fixed $z \in \mathcal{P}(I)$, which can be checked is indeed a filter; a non-principal filter is called a **free filter**. A **proper filter** is a filter D on I which is not $\mathcal{P}(I)$ itself, which is equivalent to $\emptyset \notin D$. An **ultrafilter** is a proper filter D such that for every $y \in \mathcal{P}(I)$, either $y \in D$ or $I \backslash y \in D$. Ultrafilters can be viewed as being maximal:

Lemma 1.6.2. *A filter D over a set I is an ultrafilter iff it is a maximal proper filter (i.e. no proper filter properly contains it).* ■

We remark that ultrafilters can always be found (using axiom of choice):

Theorem 1.6.3. *Any proper filter over a set I can be extended to an ultrafilter over I.* ■

1.7 Ultraproducts

This section will only be used in Chapter 4.

If $I \neq \emptyset$ is a set, A_i is a set for each $i \in I$ and D is a proper filter over I, then the **Cartesian product** $\prod_{i\in I} A_i$ is the set of all functions f with domain I satisfying $f(i) \in A_i$. Two $f, g \in \prod_{i\in I} A_i$ are ***D*-equivalent**, written $f =_D g$, if $\{i \in I \mid f(i) = g(i)\} \in D$.

Lemma 1.7.1. *Let D be a proper filter over a set I. Then the relation $=_D$ is an equivalence relation over $\prod_{i\in I} A_i$.* ■

Now the **reduced product of A_i modulo D**, denoted $\prod_{i\in I} A_i/D := \{(f)_D \mid f \in \prod_{i\in I} A_i\}$, is the set of equivalence classes $(f)_D$ of $=_D$. If D is an ultrafilter, the reduced product is an **ultraproduct**. If $A_i = A$ for all $i \in I$, then the construction is called a **reduced power** and **ultrapower**, respectively.

Definition 1.7.2 (Reduced product for models). Let $I \neq \emptyset$, D a proper filter over I and $\mathfrak{A}_i$ an $\mathcal{L}$-model for each $i \in I$. Then the *reduced product* $\mathfrak{P} := \prod_{i\in I} \mathfrak{A}_i/D$ is the $\mathcal{L}$-structure defined as follows:

(i) The underlying set is the reduced product $\prod_{i\in I} A_i/D$;

(ii) For every n-ary relation symbol R in $\mathcal{L}$, we have

$$R^{\mathfrak{P}}((f_1)_D, \dots, (f_n)_D) \Leftrightarrow \{i \in I \mid R^{\mathfrak{A}_i}(f_1(i), \dots, f_n(i))\} \in D;$$

(iii) For every n-ary function symbol F in $\mathcal{L}$, we have

$$F^{\mathfrak{P}}((f_1)_D, \dots, (f_n)_D) := (\{\langle i, F^{\mathfrak{A}_i}(f_1(i), \dots, f_n(i))\rangle \mid i \in I\})_D;$$

(iv) For every constant symbol C in $\mathcal{L}$, we have $C^{\mathfrak{P}} := (\{\langle i, C^{\mathfrak{A}_i}\rangle \mid i \in I\})_D$.

The **ultraproduct over models** is just the reduced product where the proper filter D is an ultrafilter. The **reduced power over models** and **ultrapower over models** are defined analogously.

2 An alternative universe

We would like to construct a transitive class model of ZFC consisting of only the *bare minimum* of sets, only allowing sets which have to be there by necessity of ZFC. So at the very least, for every X in the universe and φ a formula, our new universe has to contain the set

$$\{x \in X \mid \varphi(x)\}$$

to satisfy Comprehension. The problem arises when we want to write up this intuitive idea explicitly. An approach to constructing such a minimal universe would be to construct it analogously to V, but with a restrictive power set operation instead, which only includes the definable subsets. We thus want to define the *definable power set* $\operatorname{Def}(X)$ of a set X as

$$\{x \mid \exists\varphi(\exists t_1 \in X)\cdots(\exists t_n \in X)(\forall y \in X)(y \in x \leftrightarrow X \models \varphi[y, t_1, \dots, t_n])\}$$

but as φ is not a set, the quantifier $\exists\varphi$ does not make any sense. Our first goal is then to formally define our universe to model this intuitive idea.

2.1 Construction of L

We define a new language $\mathscr{L}$, which is to be the *internal* variant of our language of set theory. The symbols and formulas of $\mathscr{L}$ will then be *sets*, making it possible to state propositions in the language of set theory about the nature of the formulas of $\mathscr{L}$. We start off by defining our logical symbols. First of all, $\mathscr{L}$ is really going to be the internal version of an *extended* language of set theory $\mathcal{L}^+_\in := \{\in\} \cup \{\dot{x} \mid x = x\}$, where we also have constant symbols $\dot{x}$ for every set x. We encode parentheses "(", ")", variables v_n for all $n \geq 0$, constant symbols $\dot{x}$ (for the corresponding set x), the relations $\in$ and $=$, as well as the adequate set $\{\wedge, \neg, \exists\}$ of logical connectives; the rest of the logical symbols such as $\vee, \rightarrow, \forall$ will be counted as abbreviations. Formally, our encoding will be a class function $\ulcorner\cdot\urcorner : \mathcal{L}^+_\in \to V$, defined as in Figure 2.1.

Symbol	Set
(	0
)	1
v_n	$\langle 2, n\rangle$
$\dot{x}$	$\langle 3, x\rangle$
$\in$	4
$=$	5
$\wedge$	6
$\neg$	7
$\exists$	8

Figure 2.1: Encoding of symbols in $\mathscr{L}$

We define the formulas in $\mathscr{L}$ to be *sequences* of symbols. An atomic formula such as $(v_0 \in v_1)$ would thus be encoded as the sequence $04\langle 2, 0\rangle\langle 2, 1\rangle 1$. For ease of notation, set $\mathscr{L}_X$ to be $\mathscr{L}$ with only constant symbols from the set X. Note that $\mathscr{L}$ is a proper class and $\mathscr{L}_X$ is a set for all X.

Definition 2.1.1 (Gödel encoding symbol). For every formula φ, denote by $\ulcorner\varphi\urcorner$ the corresponding encoded formula in $\mathscr{L}$; i.e. $\ulcorner\varphi\urcorner$ is a set.

We are still not set for defining our universe though; we now lack an internal *uniform* notion of satisfaction, corresponding to the metamathematical $\models$, where uniformity in this context means that it should be independent of the particular choice of formula; the same satisfaction notion should work in all cases. This is worked out in detail in Appendix A, resulting in the L-absolute formula $\mathsf{sat}(M, \ulcorner\varphi(\dot{x}_1, \dots, \dot{x}_n)\urcorner)$, which expresses that $\ulcorner\varphi(\dot{x}_1, \dots, \dot{x}_n)\urcorner \in \mathscr{L}_M$ encodes a formula $\varphi(\vec{v})$ and $\varphi[\vec{x}]$ is true in M. We would thus like sat to correspond to our metamathematical satisfaction notion $\models$, and indeed, we have the following lemma.

Lemma 2.1.2. *Let $\varphi(v_1, \dots, v_n)$ be a formula. Then*

$$\mathrm{ZF} \vdash \forall M(\forall x_1 \in M) \cdots (\forall x_n \in M)(M \models \varphi[x_1, \dots, x_n] \leftrightarrow \mathsf{sat}[M, \ulcorner\varphi(\dot{x}_1, \dots \dot{x}_n)\urcorner]).$$

Proof. See Lemma A.2.9 in the appendix. ■

As this lemma implies that $\{x \in X \mid X \models \varphi[x]\} = \{x \in X \mid \mathsf{sat}[X, \ulcorner\varphi(\dot{x})\urcorner]\}$, we will often denote such a set by the former. We are now well-equipped to define our universe properly.

Definition 2.1.3 (Definable power set). For any set X, define the *definable power set* $\mathrm{Def}(X)$ as

$$\mathrm{Def}(X) := \{x \in \mathcal{P}(X) \mid (\exists\ulcorner\varphi\urcorner \in \mathscr{L}_X)(\exists\vec{p} \in X)\mathsf{sat}[X, \ulcorner\varphi(\dot{x}, \dot{p}_1, \dots, \dot{p}_n)\urcorner]\}$$

Definition 2.1.4 (Constructible universe). Define by $\in$-recursion on On the sets $L_\alpha := \bigcup_{\gamma<\alpha} \mathrm{Def}(L_\gamma)$ (i.e. $L_0 := \emptyset$, $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$ and $L_\delta = \bigcup_{\gamma<\delta} L_\gamma$ for limit $\delta \in \mathrm{On}$). Then the *constructible universe* is given as $L := \bigcup_{\alpha\in\mathrm{On}} L_\alpha$.

We start off by showing some basic properties of L.

Lemma 2.1.5. *Let $\alpha, \beta \in \mathrm{On}$. Then:*

(i) $\alpha \leq \beta \Rightarrow L_\alpha \subseteq L_\beta$;
(ii) L_α *is transitive (and* L *is thus transitive as well);*
(iii) $L_\alpha \subseteq V_\alpha$;
(iv) $\alpha \leq \omega \Rightarrow L_\alpha = V_\alpha$;
(v) $\alpha < \beta \Rightarrow \alpha, L_\alpha \in L_\beta$;
(vi) $L \cap \alpha = L_\alpha \cap \mathrm{On} = \alpha$ *(and hence* $\mathrm{On} \subseteq L$*);*
(vii) $\alpha \geq \omega \Rightarrow |L_\alpha| = |\alpha|$.

Proof. (i) and (ii): Transfinite induction on β. "$\beta = 0$": trivial. "β limit": As (i) holds for all $\gamma < \beta$ *ex hypothesi* and any $\gamma < \beta$ will belong to some L_ξ for $\xi < \beta$, (i) follows by the definition of union; (ii) arises from the fact that a union of transitive sets is transitive. "$\beta = \gamma + 1$": For (i), it suffices to show that $L_\gamma \subseteq L_\beta$. Let $x \in L_\gamma$. Then by induction hypothesis of (ii), $x \subseteq L_\gamma$. Thus $x = \{y \in L_\gamma \mid L_\gamma \models y \in x\}$ by Δ_0 absoluteness of $y \in x$ for L_γ, so $x \in \mathrm{Def}(L_\gamma) = L_\beta$. For (ii), let $x \in y \in L_\beta$. Then $y \subseteq L_\gamma$, as $\mathrm{Def}(L_\gamma) \subseteq \mathcal{P}(L_\gamma)$. Thus $x \in L_\gamma$ and hence $x \in L_\beta$, as we just proved $L_\gamma \subseteq L_\beta$.

(iii): Straightforward transfinite induction on α. (iv): Induction on α. The induction start is trivial, so let $\alpha < \omega$ and assume that $L_\alpha = V_\alpha$. As $L_{\alpha+1} \subseteq V_{\alpha+1}$ by (iii), it suffices to show that $V_{\alpha+1} \subseteq L_{\alpha+1}$. Let

$x \in V_{\alpha+1}$, so $x \subseteq V_\alpha = L_\alpha$ and as V_α is finite, we have that $x = \{a_1, \ldots, a_n\}$ for $a_1, \ldots, a_n \in L_\alpha$, and hence by Δ_0 absoluteness

$$x = \{y \in L_\alpha \mid L_\alpha \models y = a_1 \vee \ldots \vee y = a_n\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1}.$$

Thus we also have that $L_\omega = \bigcup_{\alpha<\omega} L_\alpha = \bigcup_{\alpha<\omega} V_\alpha = V_\omega$.

(v): By (i) it is sufficient to show that $\alpha, L_\alpha \in L_{\alpha+1}$. Firstly we have that

$$L_\alpha = \{x \in L_\alpha \mid L_\alpha \models x = x\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1},$$

so it remains to show that $\alpha \in L_{\alpha+1}$, which we will prove by transfinite induction on α. Assume that $\gamma \in L_{\gamma+1}$ for all $\gamma < \alpha$. As $L_{\gamma+1} \subseteq L_\alpha$ by (i), we have that $\alpha \subseteq L_\alpha$. As $\alpha \notin L_\alpha$ (an easy transfinite induction), we have by (ii) that $\alpha = L_\alpha \cap \mathrm{On}$. But since $\mathsf{on}(v_0)$ is Δ_0, it is absolute for L_α:

$$\alpha = \{x \in L_\alpha \mid L_\alpha \models \mathsf{on}[x]\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1}.$$

(vi): We proved in (v) that $L_\alpha \cap \mathrm{On} = \alpha$. Then $L \cap \alpha = L \cap (L_\alpha \cap \mathrm{On}) = L_\alpha \cap \mathrm{On} = \alpha$. (vii): (vi) entails that $|\alpha| = |L_\alpha \cap \mathrm{On}\,| \leq |L_\alpha|$ for all $\alpha \in \mathrm{On}$. We prove by transfinite induction on $\alpha \geq \omega$ that $|L_\alpha| \leq |\alpha|$. "$\alpha = \omega$": a countable union of finite sets is countable. "$\alpha = \beta + 1$": Assume $|L_\beta| \leq |\beta|$. Since $|\mathscr{L}_{L_\beta}| = \aleph_0 + |L_\beta| = |L_\beta|$, we have that

$$|L_{\beta+1}| = |\,\mathrm{Def}(L_\beta)| = |L_\beta| \leq |\beta| = |\beta + 1|.$$

"α limit": Assume $|L_\gamma| \leq |\gamma|$ for all $\gamma < \alpha$. Then we have that

$$|L_\alpha| = \left|\bigcup_{\gamma<\alpha} L_\gamma\right| \leq \sum_{\gamma<\alpha} |L_\gamma| \leq \sum_{\gamma<\alpha} |\alpha| = |\alpha| \cdot |\alpha| = |\alpha|,$$

so the theorem is proved. ■

L also satisfies a reflection principle: truths in the universe are always reflected down to some limit stage in the hierarchy.

Theorem 2.1.6 (Reflection Theorem). *Let $\varphi(\vec{v})$ be a formula. Then*

$$\mathrm{ZF} \vdash (\forall \alpha \in \mathrm{On})(\exists \delta > \alpha)(\mathsf{isLimit}(\delta) \wedge (\forall \vec{x} \in L_\delta)(\varphi^L[\vec{x}] \leftrightarrow \varphi^{L_\delta}[\vec{x}])).$$

Proof. Let $\alpha \in \mathrm{On}$ and $\varphi(\vec{v})$ a formula. We will first find a suitable $\delta > \alpha$; let $\varphi_0(\vec{v}_0), \ldots, \varphi_n(\vec{v}_n)$ be such that φ_n is φ, φ_0 is atomic, and $\varphi_{i+1}(\vec{v}_{i+1})$ is obtained from $\varphi_i(\vec{v}_i)$ by means of a single logical operation, for all $i \leq n$ (thus the φ_i's consist of φ's "logical parts"). Now construct the class functions $f_i : L \to \mathrm{On}$ for $i \leq n$ as follows:

- If $\varphi_i(\vec{v}_i)$ is atomic or of the form $\varphi_j(\vec{v}_j) \wedge \varphi_k(\vec{v}_k)$ or $\neg\varphi_j(\vec{v}_j)$ for $j, k < i$, then set $f_i(\vec{x}_i) := 0$;
- If $\varphi_i(\vec{v}_i)$ is of the form $\exists y \varphi_j(y, \vec{v}_i)$ for $j < i$, then set

$$f_i(\vec{x}_i) := \min\{\alpha \in \mathrm{On} \mid (\exists y \in L)\varphi_j^L[y, x_1, \ldots, x_n] \to (\exists y \in L_\alpha)\varphi_j^L[y, x_1, \ldots, x_n]\}.$$

Claim 2.1.6.1. There exists a limit ordinal $\delta > \alpha$ such that $(\forall i \leq n)(\forall \vec{x}_i \in L_\delta)(f_i(\vec{x}_i) < \delta)$.

Proof of claim. Define the function $G : \omega \to \mathrm{On}$ recursively as follows. Set $G(0) := \alpha + 1$. For $k < \omega$, let $G(k+1)$ be a limit ordinal satisfying that $(\forall x \in L_{G(k)})(\forall i \leq n)(f_i(\vec{x}_i) < G(k+1))$, which exists by Replacement. Finally define $\delta := \bigcup_{n<\omega} G(n)$, which is clearly a limit ordinal with the wanted properties. ♢

Now given our δ, we prove by induction that $\varphi_i^L[\vec{x}_i] \leftrightarrow \varphi_i^{L_\delta}[\vec{x}_i]$ for every $i \leq n$ and $\vec{x}_i \in L_\delta$. The atomic, conjunction and negation step are trivial, so suppose $\varphi_i(\vec{v}_i)$ is $\exists y \varphi_j(y, \vec{v}_i)$ with $j < i$. Let $\vec{x}_i \in L_\delta$. We show each implication separately.

"$\Rightarrow$": Assume $\varphi_i^L[\vec{x}_i]$, meaning $(\exists y \in L)\varphi_j^L[y, \vec{x}_i]$. Then by construction of δ we have $(\exists y \in L_\delta)\varphi_j^L[y, \vec{x}_i]$. But now by induction we have $\varphi_j^{L_\delta}[y, \vec{x}_i]$, so $\varphi_i^{L_\delta}[\vec{x}_i]$.
"$\Leftarrow$": Assume $\varphi_i^{L_\delta}[\vec{x}_i]$, meaning $(\exists y \in L_\delta)\varphi_j^{L_\delta}[y, \vec{x}_i]$. By induction we have $(\exists y \in L_\delta)\varphi_j^L[y, \vec{x}_i]$, so in particular $(\exists y \in L)\varphi_j^L[y, \vec{x}_i]$, which is to say that $\varphi_i^L[\vec{x}_i]$. ■

2.2 L as an inner model of ZF

Definition 2.2.1 (Inner model). An *inner model* M of a theory T is a transitive class containing all the ordinals and which satisfies $\mathrm{ZF} \vdash \sigma^M$ for every sentence σ in the theory T.

We will often say M is an inner model to mean an inner model of ZF.

Theorem 2.2.2. $\mathrm{ZF} \vdash \sigma^L$ *for every axiom* σ *of* ZF.

Proof. *Extensionality*: Extensionality is Π_1 and thus downwards absolute for transitive classes by Lemma 1.3.5.

Union: Let $x \in L$, i.e. $x \in L_\alpha$ for some $\alpha \in \mathrm{On}$. Define $y := \bigcup x$. Then $y \subseteq L_\alpha$ as L_α is transitive. By Union in V we have that $\forall z(z \in y \leftrightarrow (\exists w \in x)(z \in w))$, which is Π_1, so it holds in L_α as well. We thus have that

$$y = \{z \in L_\alpha \mid L_\alpha \models (\exists w \in x)(z \in w)\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1} \subseteq L.$$

Infinity: Follows from $\omega \in L_{\omega+1} \subseteq L$ by Lemma 2.1.5(v)

Power Set: Let $x \in L$. We have to construct $y \in L$, satisfying $(\forall z \in L)(z \in y \leftrightarrow z \subseteq x)$. Set $y := \{z \in \mathcal{P}(x) \mid z \in L\}$ by using Power Set and Comprehension in V; it thus remains to show that $y \in L$. Define $f(z)$ to be the least $\alpha \in \mathrm{On}$ so $z \in L_\alpha$, and use Replacement to find $\beta \in \mathrm{On}$ greater than all $f(z)$ for $z \in y$. Then clearly $y \subseteq L_\beta$, meaning that

$$y = \{z \in L_\beta \mid L_\beta \models z \subseteq x\} \in \mathrm{Def}(L_\beta) = L_{\beta+1} \subseteq L.$$

Foundation: Foundation is Π_1 and therefore downwards absolute for transitive classes by Lemma 1.3.5.

Comprehension: Let $\varphi(v_0, \ldots, v_n)$ be some formula, and $x, a_1, \ldots, a_n \in L$. We are required to construct some $y \in L$ satisfying $(\forall z \in L)[z \in y \leftrightarrow (z \in x \wedge \varphi^L[z, a_1, \ldots, a_n])]$; we start by picking $\alpha \in \mathrm{On}$ such that $x, a_1, \ldots, a_n \in L_\alpha$. By the Reflection Theorem 2.1.6 we can construct limit $\delta > \alpha$ satisfying

$$(\forall z_0 \in L_\delta) \cdots (\forall z_n \in L_\delta)(\varphi^{L_\delta}[z_0, \ldots, z_n] \leftrightarrow \varphi^L[z_0, \ldots, z_n]).$$

Now we construct

$$y := \{z \in L_\delta \mid L_\delta \models \varphi[z, a_1, \ldots, a_n] \wedge z \in x\} \in \mathrm{Def}(L_\delta) = L_{\delta+1} \subseteq L.$$

But then we have that

$$\{z \in x \mid \varphi^L[z, a_1, \dots, a_n]\} = \{z \in x \mid \varphi^{L_\delta}[z, a_1, \dots, a_n]\} = y \in L.$$

Replacement: Let $\varphi(v_0, \dots, v_n)$ be a formula, $a_2, \dots, a_n \in L$ be given and assume that $(\forall x \in L)(\exists y \in L)\varphi^L[y, x, a_2, \dots, a_n]$. We have to show that for any $u \in L$ we can construct $v \in L$ satisfying

$$[(\forall x \in u)(\exists y \in v)\varphi[y, x, a_2, \dots, a_n]]^L,$$

which is to say that

$$(\forall x \in u)(\exists y \in v)\varphi^L[y, x, a_2, \dots, a_n]$$

since L is transitive. Let $x \in u$. Denote again by $f(x)$ the least ordinal α satisfying that $(\exists y \in L_\alpha)\varphi^L[y, x, a_2, \dots, a_n]$. Using Replacement in V let β surpass all $f(x)$ for $x \in u$. Set $v := L_\beta$, which satisfies that $v \in L$. We thus have that $(\forall x \in u)(\exists y \in v)\varphi^L[y, x, a_2, \dots, a_n]$, and we are finished. ■

Lemma 2.2.3. *Let M be an inner model. Then for any $\alpha \in \mathrm{On}$ we have $L_\alpha \in M$ and $L_\alpha^M = L_\alpha$ (meaning $[v_0 = L_\alpha]^M \leftrightarrow v_0 = L_\alpha$), implying that $L^M = L$.*

Proof. Define the class function $F : \mathrm{On} \to M$ by $\in$-recursion on On:

$$F(\alpha) := \bigcup\{y \mid [y = \mathrm{Def}(z)]^M \wedge z \in F``\alpha\}.$$

Then $F(\alpha) = L_\alpha^M$ and $F \restriction \alpha \in M$, as the $\in$-recursion can be done within M due to M being an inner model of ZF. Now since $y = \mathrm{Def}(z)$ is absolute for transitive class models of ZF by Lemma 1.3.7 and Lemma A.3.4, we have that

$$F(\alpha) = \bigcup\{\mathrm{Def}(z) \mid z \in F``\alpha\} = L_\alpha$$

for all $\alpha \in \mathrm{On}$, so in particular $L_\alpha = (F \restriction S(\alpha))(\alpha) \in M$ as M is transitive and $F \restriction S(\alpha) \in M$. Then

$$L^M = \left[\bigcup_{\alpha \in \mathrm{On}} L_\alpha\right]^M = \bigcup_{\alpha \in \mathrm{On}} L_\alpha^M = \bigcup_{\alpha \in \mathrm{On}} L_\alpha = L,$$

where we used that $\mathrm{On}^M = \mathrm{On}$. ■

Theorem 2.2.4. *L is the smallest inner model.*

Proof. L is transitive and $\mathrm{On} \subseteq L$ by Lemma 2.1.5, so L is an inner model by Theorem 2.2.2. Assume M is another inner model of ZF. Then $L^M = L$ by Lemma 2.2.3, so $L \subseteq M$. ■

2.3 Axiom of Constructibility

Definition 2.3.1 ($V{=}L$). The *Axiom of Constructibility,* abbreviated $V{=}L$, is the statement

$$\forall x(\exists\alpha \in \mathrm{On})(x \in L_\alpha).$$

Theorem 2.3.2. $\mathrm{ZF} \vdash (V{=}L)^L$.

Proof. As $\mathrm{On} \subseteq L$, we have to show that $(\forall x \in L)(\exists\alpha \in \mathrm{On})[x \in L_\alpha]^L$. But as $L_\alpha^L = L_\alpha$ by Lemma 2.2.3, it remains to show that $(\forall x \in L)(\exists\alpha \in \mathrm{On})(x \in L_\alpha)$, which simply follows from the definition of L. ■

Lemma 2.3.3. *Let σ be a sentence, $T \supseteq \mathrm{ZF}$ a theory and M a class satisfying*

$$T \vdash (T + \sigma)^M.$$

Then $\mathrm{Con}(T) \Rightarrow \mathrm{Con}(T + \sigma)$.

Proof. Assume $\neg\,\mathrm{Con}(T+\sigma)$ and let $\psi_0, \dots, \psi_n$ be a (formal) proof of a contradiction in $T+\sigma$. Then ψ_i is either an axiom of $T+\sigma$ or ψ_i follows logically from $\psi_0, \dots, \psi_{i-1}$ for all $i = 0, \dots, n$, and assume without loss of generality that ψ_n is $(0 = 1)$. Now consider the sequence $\psi_0^M, \dots, \psi_{i-1}^M$. If ψ_i follows logically from $\psi_0, \dots, \psi_{i-1}$, then so does ψ_i^M from $\psi_0^M, \dots, \psi_{i-1}^M$, by using the same deduction rules. If ψ_i is an axiom of $T+\sigma$, then ψ_i^M is a theorem of T, by assumption on M. Thus ψ_n^M is a theorem of T. But as ψ_n^M is $(0 = 1)^M \equiv (0 = 1)$, we have arrived at a contradiction in T, concluding $\neg\,\mathrm{Con}(T)$. ■

Corollary 2.3.4. $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + V{=}L)$.

Proof. By Theorem 2.3.2 and Lemma 2.3.3. ■

2.4 A global well-ordering

Theorem 2.4.1. $\mathrm{ZF} + V{=}L \vdash \mathrm{AC}$.

Proof. Assume $V{=}L$, i.e that $\forall x(\exists\alpha \in \mathrm{On})(x \in L_\alpha)$. We construct the sets $<_\alpha$ where $<_\alpha$ well-orders L_α for every $\alpha \in \mathrm{On}$, and satisfy the following properties:

(i) $<_{\alpha+1} \restriction L_\alpha = <_\alpha$ for every $\alpha \in \mathrm{On}$;
(ii) If $x \in L_{\alpha+1} \backslash L_\alpha$ and $y \in L_\alpha$, then $y <_{\alpha+1} x$ for every $\alpha \in \mathrm{On}$.

The first condition will ensure that the orderings agree and the second will make sure that the $<_\alpha$'s form a well-order on all of L. To construct the well-orderings, we first enumerate all $\mathscr{L}_\emptyset$-formulas ($\mathscr{L}$-formulas without constant symbols), by e.g. listing them lexicographically; as there are countably many such formulas, we thus have a set $\{\ulcorner\varphi_n\urcorner \mid n < \omega\}$, where $\ulcorner\varphi_n\urcorner <_{\mathrm{lex}} \ulcorner\varphi_{n+1}\urcorner$ with $<_{\mathrm{lex}}$ being the lexicographic ordering (restricted to the set). Then we define the class functions $A : L \to \mathrm{On}$, $N : L \to \omega$ and $P : L \to L$ as follows:

- $A(x) := \min\{\alpha \in \mathrm{On} \mid x \in L_{\alpha+1}\}$;
- $N(x) := \min\{n < \omega \mid x = \{y \in L_{A(x)} \mid (\exists\vec{t} \in L_{A(x)})(L_{A(x)} \models \varphi_n[y, t_1, \dots, t_k])\}\}$;

- $P(x) := \{\langle t_1, \ldots, t_k\rangle \mid \vec{t} \in L_{A(x)} \wedge x = \{y \in L_{A(x)} \mid L_{A(x)} \models \varphi_{N(x)}[y, t_1, \ldots, t_k]\}\}$.

I.e. that $N(x) < \omega$ is least such that x is definable over $L_{A(x)}$ via $\varphi_{N(x)}$ with parameters in $L_{A(x)}$ and $P(x)$ is the set of possible tuples of the parameters $\vec{t} \in L_{A(x)}$ used to define x over $L_{A(x)}$ via $\varphi_{N(x)}$. Note that $A(x) \neq \emptyset$ as we assumed $V{=}L$.

We now begin the recursive construction of the $<_\alpha$'s. Set $<_0 := \emptyset$ for δ limit; clearly it retains the properties (i) and (ii), described above. Now assume $<_\gamma$ has been constructed for all $\gamma \leq \alpha$ and let $x, y \in L_{\alpha+1}$. Then we define $x <_{\alpha+1} y$ by case analysis. Set $x <_{\alpha+1} y$ iff

- Either $A(x) \in A(y)$,
- or else $A(x) = A(y)$ and $N(x) \in N(y)$,
- or else $A(x) = A(y)$ and $N(x) = N(y)$ and $\min_{<^*_{A(x)}} P(x) <^*_{A(x)} \min_{<^*_{A(x)}} P(y)$, where $<^*_{A(x)}$ is the lexicographic ordering on the tuples induced by $<_{A(x)}$.

As all three orderings are well-orderings, $<_{\alpha+1}$ is as well. Furthermore both (i) and (ii) are clearly satisfied by construction. Define lastly $<_\delta := \bigcup_{\gamma<\delta} <_\gamma$, which also retains both (i) and (ii). As the entire construction was done inside of L, we have that $<_\alpha \in L$ for all $\alpha \in \mathrm{On}$ by the Recursion Theorem 1.1.4[1]. Thus given a set x, we can find some $\alpha \in \mathrm{On}$ such that $x \in L_\alpha$ as $V{=}L$, and then $<_\alpha \restriction x \times x$ is a well-order of x. ■

For later purposes, we define the *global well-order* of L as $<_L := \bigcup\{<_\alpha \mid \alpha \in \mathrm{On}\}$. Thus for $x, y \in L$, we have that $x <_L y$ iff $x <_\alpha y$ for some $\alpha \in \mathrm{On}$ such that $x, y \in L_\alpha$ (which is well-defined by properties (i) and (ii) of $<_\alpha$, given in the proof above). Note that $<_L$ is not a set, as $\{<_\alpha \mid \alpha \in \mathrm{On}\}$ has a bijective correspondence to On, and note furthermore that $<_L$ is definable by a formula, by writing up the canonical formula expressing the definition given in the proof - see Appendix A for details.

Corollary 2.4.2. $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZFC)}$.

Proof. Follows from Theorem 2.4.1 and 2.3.4, since clearly $T \vdash \sigma$ implies $\mathrm{Con}(T) \Rightarrow \mathrm{Con}(T + \sigma)$. ■

2.5 Condensation

Definition 2.5.1 (Extensional structure). Let $\mathcal{L}$ be a language and $\mathfrak{A} := \langle A, R\rangle$ be an $\mathcal{L}$-structure where R is a binary $\mathcal{L}$-relation on A. Then $\mathfrak{A}$ is *extensional* if it satisfies the Extensionality axiom; i.e. if $(\forall x, y \in A)[(\forall z \in A)(zRx \leftrightarrow zRy) \rightarrow x = y]$ or equivalently,

$$(\forall x, y \in A)[x \neq y \rightarrow (\exists z \in A)(zRx \leftrightarrow \neg(zRy))].$$

We state here the *Mostowski collapse*, sometimes called the *transitive collapse* of a structure. It is stated here in a very general form, because it is required in Chapter 4. As for the applications in this chapter, the structure $\mathfrak{A}$ will always be $\langle A, \in\rangle$ with A being a set.

Theorem 2.5.2 (Mostowski Collapse). *Let $\mathfrak{A} := \langle A, R\rangle$ be a (possibly proper class) structure where R is a binary relation on A. If R is well-founded and set-like, and $\mathfrak{A}$ is extensional, then there is a unique*

[1]It can also easily be checked that $<_\alpha \in L_{\alpha+3}$ for every $\alpha \in \mathrm{On}$.

transitive structure $\langle T, \in\rangle$ and a unique isomorphism $\pi : \mathfrak{A} \cong \langle T, \in\rangle$. If $R = \in$, then π fixes every transitive $Y \subseteq A$; i.e. $\pi \restriction Y = \mathrm{id} \restriction Y$.

Proof. We start by proving existence. Define the function π with $\operatorname{dom} \pi = A$ by well-founded recursion on R:

$$\pi(a) := \{\pi(x) \mid xRa\}.$$

Note that $\pi(a)$ is a set for each $a \in A$, since R is set-like. Set $T := \operatorname{ran} \pi$. We show that π and T are as given in the theorem. We start by showing that $\pi : A \approx T$, by showing π is injective. Suppose not. Construct

$$x_1 := \min_R \{x \in A \mid (\exists w \in A)[x \neq w \wedge \pi(x) = \pi(w)]\}$$

which exists as R is well-founded, and let $x_2 \in A$ be such that $x_1 \neq x_2$ and $\pi(x_1) = \pi(x_2)$ which exists as π is not injective *ex hypothesi*. As $\mathfrak{A}$ is extensional, we can find some $y \in A$ such that either yRx_1 and $\neg(yRx_2)$, or $\neg(yRx_1)$ and yRx_2; assume that yRx_1 and $\neg(yRx_2)$. Then $\pi(y) \in \pi(x_1)$ by definition of π and thus $\pi(y) \in \pi(x_2)$ as well since $\pi(x_1) = \pi(x_2)$. Thus we can find some zRx_2 such that $\pi(y) = \pi(z)$ by definition of π. As zRx_2 and $\neg(yRx_2)$, we have $y \neq z$. But then y contradicts minimality of x_1, $\lightning$. The other case is the same *mutatis mutandis*. Thus we have $\pi : A \approx T$.

We now show that $\pi : \mathfrak{A} \cong \langle T, \in\rangle$. Let $x, y \in A$ and assume first that xRy, implying $\pi(x) \in \pi(y)$ by definition. Assume conversely that $\pi(x) \in \pi(y)$. Then $\pi(x) = \pi(z)$ for some zRy, but then $z = x$ due to π being injective, concluding that $\pi : \mathfrak{A} \cong \langle T, \in\rangle$.

Now to show uniqueness of π (and thus T as well, as $T = \operatorname{ran} \pi$), suppose $\tau : \mathfrak{A} \cong \langle M, \in\rangle$ with M being transitive. Let $x, y \in A$ and yRx. Then $\tau(y) \in \tau(x)$ as τ is an isomorphism, so $\{\tau(z) \mid zRx\} \subseteq \tau(x)$. Now let $w \in \tau(x)$, meaning $w \in M$ by transitivity. Thus $w = \tau(z)$ for some unique $z \in A$. Hence $\tau(z) \in \tau(x)$, entailing zRx since τ is an isomorphism. But then $\tau(x) \subseteq \{\tau(y) \mid yRx\}$, so $\tau(x) = \{\tau(y) \mid yRx\} = \pi(x)$.

Assume now that $R = \in$. To show that π fixes every transitive $Y \subseteq A$, let such a Y be given and assume that it is not the case. By definition we have $\pi(x) = \{\pi(z) \mid z \in x\}$ for all $x \in Y$. Construct $y_0 := \min_{\mathrm{rank}}\{y \in Y \mid \pi(y) \neq y\}$. Then $\pi(y) = y$ for $y \in y_0$ by minimality, so $\pi(y_0) = \{\pi(y) \mid y \in y_0\} = \{y \mid y \in y_0\} = y_0$, but $\pi(y_0) \neq y_0$ by construction, $\lightning$, which completes the proof. ■

Notice that since $\in$ is automatically set-like and well-founded as well as $\langle A, \in\rangle$ being extensional, we *always* have the unique isomorphism $\pi : \langle A, \in\rangle \cong \langle T, \in\rangle$. The constructible universe has a powerful property, called *condensation*: every elementary substructure X of a limit stage L_δ in the hierarchy will be isomorphic to a previous stage. In fact, it is not even required that X is a full-blown elementary substructure; $X \preceq_1 L_\delta$ is enough.

Theorem 2.5.3 (Condensation). *Let $\delta \in \mathrm{On}$ be a limit ordinal and $X \preceq_1 L_\delta$. Then there is a unique limit ordinal $\alpha \leq \delta$ and a unique isomorphism $\pi : \langle X, \in\rangle \cong \langle L_\alpha, \in\rangle$ which fixes every transitive $Y \subseteq X$.*

Proof. We treat the cases $\delta = \omega$ and $\delta > \omega$ separately, so assume first $\delta = \omega$.

Claim 2.5.3.1. $L_m \subseteq X$ for all $m < \omega$.

Proof of claim. Base case is trivial, so assume $L_m \subseteq X$ and let $x \in L_{m+1}$. Then $x = \{a_1, \ldots, a_k\}$ for $\vec{a} \in L_m$ and $L_\delta \models \sigma$, where σ is

$$\exists x[a_1 \in x \wedge \cdots \wedge a_k \in x \wedge (\forall z \in x)(z = a_1 \vee \cdots \vee z = a_k)].$$

As σ is Σ_1 and $X \preceq_1 L_\delta$, $X \models \sigma$, so $x \in X$; therefore $L_{m+1} \subseteq X$. $\diamondsuit$

By the claim we then also have $L_\delta = L_\omega = \bigcup_{m<\omega} L_m \subseteq X$. Thus $L_\delta = X$ and the theorem is trivial. Now assume $\delta > \omega$. By the Mostowski collapse we then get the unique transitive T and the unique $\pi : \langle X, \in\rangle \cong \langle T, \in\rangle$; it remains to show that $T = L_\alpha$ for some limit $\alpha \leq \delta$. We set $\alpha := T \cap \mathrm{On}$, which is an ordinal since T is transitive.

Claim 2.5.3.2. α is a limit ordinal.

Proof of claim. As δ is limit, we have that $L_\delta \models (\forall\zeta \in \mathrm{On})(\exists\xi \in \mathrm{On})(\zeta < \xi)$, so in particular we have that $(\forall\zeta \in \mathrm{ot}(\mathrm{On} \cap X))[L_\delta \models (\exists\xi \in \mathrm{On})(\zeta < \xi)]$. As $X \preceq_1 L_\delta$ we have $(\forall\zeta \in \mathrm{ot}(\mathrm{On} \cap X))[X \models (\exists\xi \in \mathrm{On})(\zeta < \xi)]$ and thus $X \models (\forall\zeta \in \mathrm{On})(\exists\xi \in \mathrm{On})(\zeta < \xi)$ and since $\langle X, \in\rangle \cong \langle T, \in\rangle$ we have $\langle X, \in\rangle$ is elementarily equivalent to $\langle T, \in\rangle$ by Lemma 1.2.13, so $T \models (\forall\zeta \in \mathrm{On})(\exists\xi \in \mathrm{On})(\zeta < \xi)$, which is to say that $(\forall\zeta < \alpha)(\exists\xi < \alpha)(\zeta < \xi)$ since $\mathrm{On}^T = \mathrm{On} \cap T = \alpha$ and $\zeta < \xi$ is Δ_0, making α a limit ordinal. $\diamondsuit$

Thus $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$, so we need to show that $T = \bigcup_{\gamma<\alpha} L_\gamma$. First of all note that since $\delta > \omega$, by Lemma A.3.6 we have that for all $\gamma < \delta$ and $v \in L_\delta$:

$$[L_\delta \models v = L_\gamma] \Leftrightarrow [v = L_\gamma]. \tag{†}$$

We start by showing "$\subseteq$". We have that $(\forall x \in L_\delta)(\exists v \in L_\delta)(\exists\gamma < \delta)(x \in v \wedge v = L_\gamma)$ by definition of L_δ, so by (†) it holds that $(\forall x \in L_\delta)[L_\delta \models \exists v(\exists\gamma \in \mathrm{On})(x \in v \wedge v = L_\gamma)]$ and thus in particular for all $x \in X$. As $X \preceq_1 L_\delta$, we have that $X \models \forall x \exists v(\exists\gamma \in \mathrm{On})(x \in v \wedge v = L_\gamma)$ as $v = L_\gamma$ is Σ_1 by Lemma A.3.6, so T satisfies the same by Lemma 1.2.13 since $\langle X, \in\rangle \cong \langle T, \in\rangle$. Thus $(\forall x \in T)(\exists v \in T)(\exists\gamma < \alpha)(x \in v \wedge [v = L_\gamma]^T)$, and as $v = L_\gamma$ is Σ_1 by Lemma A.3.6 and therefore upwards absolute by Lemma 1.3.5, we have that $T \subseteq L_\alpha$.

Now for "$\supseteq$". As $(\forall\gamma < \delta)(\exists v \in L_\delta)(v = L_\gamma)$ holds by definition of L_δ, by the same procedure, *mutatis mutandis*, as for "$\subseteq$", we arrive at $(\forall\gamma < \alpha)(\exists v \in T)(v = L_\gamma)$, so $L_\alpha \subseteq T$. Thus $T = L_\alpha$. This finishes the proof. ■

2.6 Generalized Continuum Hypothesis

Definition 2.6.1 (GCH). The *Generalized Continuum Hypothesis* is the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha \in \mathrm{On}$. [2]

[2] This was not how it was originally stated. The original Continuum Hypothesis postulated that every $X \subseteq \mathbb{R}$ satisfies either $X \approx \mathbb{N}$ or $X \approx \mathbb{R}$, which is equivalent to $2^{\aleph_0} = \aleph_1$.

Note that as every infinite cardinal is of the form $\aleph_\alpha$ for some $\alpha \in \mathrm{On}$ by Lemma 1.1.8, GCH says that $2^\kappa = \kappa^+$ for every infinite cardinal κ.

Theorem 2.6.2. ZFC $+V{=}L \vdash$ GCH.

Proof. Let $\kappa \geq \omega$ be a cardinal and let $A \subseteq \kappa$. We would like to find some $\beta \in \mathrm{On}$ such that $\beta < \kappa^+$ and $A \in L_\beta$, because then $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$ and thus $2^\kappa = |\mathcal{P}(\kappa)| \leq |L_{\kappa^+}| = \kappa^+$, allowing us to conclude $2^\kappa = \kappa^+$.

Let $\alpha \in \mathrm{On}$ be such that $A \in L_\alpha$, which is possible as $V{=}L$. Let $X \preceq L_\alpha$ be of size κ and such that $\kappa \cup \{A\} \subseteq X$, which exists by the Löwenheim-Skolem Theorem 1.2.14. Let $\pi : X \to L_\beta$ be the collapse by the Condensation Theorem 2.5.3. Since $|X| = \kappa$ we also have $|L_\beta| = \kappa$ as π is a bijection, and then furthermore $|\beta| = \kappa$ by Lemma 2.1.5. It remains to show that $A \in L_\beta$. As $\pi : \langle X, \in\rangle \cong \langle L_\beta, \in\rangle$, we have $a \in A \Leftrightarrow \pi(a) \in \pi(A)$, which implies that $\pi(A) = \pi" A$. But $\pi" A = A$, as $A \subseteq \kappa$ and $\pi \upharpoonright \kappa = \mathrm{id} \upharpoonright \kappa$ since κ is transitive; thus $A = \pi" A = \pi(A) \in L_\beta$ and we are done. ■

Notice that the essential bit of the argument was the use of condensation. We now arrive at Gödel's famous result, regarding the relative consistency of the generalized continuum hypothesis.

Corollary 2.6.3 (Gödel). $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZF + GCH)}$.

Proof. By Corollaries 2.3.4 and 2.4.2 we have $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con}(\mathrm{ZFC} + V{=}L)$, so by Theorem 2.6.2 we have the result, as $T \vdash \sigma$ implies that $\mathrm{Con}(T) \Rightarrow \mathrm{Con}(T + \sigma)$. ■

3 | Combinatorics in L

We now move to an entirely different aspect of L, which is its surprisingly rich combinatorial structure. We will in this section show a fraction of this combinatorial structure, by proving that L satisfies the so-called *Suslin Hypothesis* and the combinatorial principle $\diamondsuit$.

3.1 Suslin hypothesis

A result from Cantor states that the real numbers are uniquely determined (up to isomorphism) as the DLO without endpoints containing a countable dense subset. Suslin asked whether we could generalize it to stating that $\mathbb{R}$ is isomorphic to a DLO without endpoints having the *Suslin property*: that every collection of disjoint intervals is countable. Define a **Suslin line** R to be a set $\langle R, <_R\rangle \not\cong \langle \mathbb{R}, <_{\mathbb{R}}\rangle$ which is a DLO without endpoints with the Suslin property. Then the **Suslin Hypothesis** (SH) is defined as the statement saying that there does not exist any Suslin lines. First define a *Suslin tree*:

Definition 3.1.1 (Suslin tree). A *Suslin tree* is an $\aleph_1$-tree with only countable antichains.

We start by showing a simpler condition to proving the existence of a Suslin tree.

Lemma 3.1.2. *Let $\mathfrak{T}$ be an $(\omega_1, \aleph_1)$-tree having*

(i) unique limits;
(ii) no uncountable branch;
(iii) no uncountable antichain.

Then there is some $X \subseteq T$ such that $\mathfrak{X} := \langle X, <_T \restriction (X \times X)\rangle$ is a Suslin tree.

Proof. Since T_0 is countable, there is some $x_0 \in T_0$ such that $\uparrow x_0 \subseteq T$ is uncountable. Now let $\alpha_x < \omega_1$ be the ordinal such that $x \in T_{\alpha_x}$ for $x \in T$. Define

$$T' := \{x \in \uparrow x_0 \mid (\forall \gamma > \alpha_x)(\exists y \in T_\gamma \cap \uparrow x_0)(x <_T y)\},$$

i.e. the members of $\uparrow x_0$ having extensions on all higher levels of $\uparrow x_0$. Thus for every $x \in T'$ we can find $y, z \in T'$ such that $x <_T y$ and $x <_T z$ with $y \perp z$, since otherwise $\uparrow x$ would be an uncountable branch. Thus we can define the function $f : \omega_1 \to \omega_1$ by transfinite recursion:

- $f(0) := 0$;
- $f(\alpha + 1) := \min\{\beta > f(\alpha) \mid (\forall x \in T'_{f(\alpha)})(\exists y, z \in T'_\beta)(x <_T y \wedge x <_T z \wedge y \neq z)\}$;
- $f(\delta) := \sup_{\gamma<\delta} f(\gamma)$ for limit δ.

Now set $X := \bigcup_{\alpha<\omega_1} T'_{f(\alpha)}$. Then clearly $\mathfrak{X}$ is Suslin by construction. ■

An important result then reduces the existence of a Suslin line to the existence of a Suslin tree:

Theorem 3.1.3. *There exists a Suslin tree iff there exists a Suslin line.*

Proof. "$\Rightarrow$": Let $\mathfrak{T}$ be a Suslin tree; we will construct a Suslin line. First of all we may assume that every $x \in T$ has infinitely many successors, because if not, just restrict $\mathfrak{T}$ to its limit stages, which is still Suslin. Then $|T_\alpha| = \aleph_0$ for all $\alpha < \omega_1$, so there is a bijection $\mathbb{Q} \approx T_\alpha$. Now define a dense linear order without end-points $<_\alpha$ on T_α, induced by the dense linear order without end-points $<_\mathbb{Q}$ on $\mathbb{Q}$. Let B be the set of all maximal branches of $\mathfrak{T}$ and define a dense linear order $<_B$ on B as $b <_B d$ iff $b_\alpha <_\alpha d_\alpha$, where $\alpha := \min\{\gamma < \omega_1 \mid b \cap T_\gamma \neq d \cap T_\gamma\}$ and b_α (resp. d_α) denotes the unique element of $b \cap T_\alpha$ (resp. $d \cap T_\alpha$). Thus $\mathfrak{B} := \langle B, <_B \rangle$ is a DLO of cardinality $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ by simple cardinal arithmetic.

We thus first need to show that $\mathfrak{B}$ has the Suslin property. Let $I := (b, d)$ be any interval in $\mathfrak{B}$ and define $\beta := \min\{\gamma < \omega_1 \mid b_\gamma \neq d_\gamma\}$. As $<_\beta$ is dense, pick $x_I \in T_\beta$ such that $b_\beta <_\beta x_I <_\beta d_\beta$, and let $e_I \in B$ be a maximal branch of $\mathfrak{T}$ containing x_I. Thus by definition of $<_B$, $e_I \in (b, d)$. Suppose I and J are disjoint intervals in $\mathfrak{B}$; then clearly $e_I \notin J$ and $e_J \notin I$, so $x_I \perp x_J$ with respect to $<_T$. As $\mathfrak{T}$ has no uncountable antichains, we then have that $\mathfrak{B}$ has the Suslin property.

Next, we need to show that every countable subset of B is not dense in $\mathfrak{B}$, so let $A \subseteq B$ be countable. For every $b, d \in A$ satisfying $b \neq d$, define

$$\alpha(b, d) := \min\{\gamma < \omega_1 \mid b_\gamma \neq d_\gamma\} \text{ and } \delta := \sup\{\alpha(b, d) \mid b, d \in A \wedge b \neq d\}.$$

Since A is countable, $\delta < \omega_1$ (a countable union of countable sets is countable). Choose some $w \in T_\delta$ and $x, y, z \in T_{\delta+1}$ such that $w <_T x, y, z$ and $x <_{\delta+1} y <_{\delta+1} z$, which exist as we assumed every $w \in T$ had infinitely many successors. Let $b_x, b_y, b_z \in B$ be maximal branches of $\mathfrak{T}$ containing x, y, z, respectively. If A was dense in $\mathfrak{B}$, we could find $d, d' \in A$ such that $b_x <_B d <_B b_y <_B d' <_B b_z$. But since $w \in b_x \cap b_y \cap b_z$, we have that $d_\xi = d'_\xi$ for every $\xi \leq \delta$, making $\alpha(d, d') > \delta$, contradicting the definition of δ. Thus $\mathfrak{B}$ does not have a countable dense subset, making $\mathfrak{B}$ a Suslin line.

"$\Leftarrow$": Assume there exists a Suslin line $\mathfrak{X}$. We will construct a Suslin tree, and by Lemma 3.1.2 it is enough to construct an $(\omega_1, \aleph_1)$-tree which has
- (i) unique limits;
- (ii) no uncountable branches;
- (iii) no uncountable antichains.

We define by recursion on the levels a so-called *partition tree* $\mathfrak{T} := \langle T, \supseteq \rangle$ consisting of subsets of X with reverse inclusion as the order. Set $T_0 := \{X\}$. Suppose now T_α has been constructed. For every $I \in T_\alpha$ with $|I| > 1$ choose some interior point $x_I \in I$, which exists as $<_X$ is dense. Define the sets $I_0 := \{y \in I \mid y <_X x_I\}$ and $I_1 := \{y \in I \mid x_I \leq_X y\}$, and set now

$$T_{\alpha+1} := \{I_0 \mid I \in T_\alpha \wedge |I| > 1\} \cup \{I_1 \mid I \in T_\alpha \wedge |I| > 1\}.$$

For δ limit and assuming T_γ has been defined for $\gamma < \delta$, set

$$T_\delta := \{\bigcap b \mid b \text{ is a } \delta\text{-branch of } \mathfrak{T} \restriction \delta \text{ such that } |\bigcap b| > 1\}.$$

We now need to show that $\mathfrak{T}$ satisfies conditions (i)-(iii). (i) is trivially satisfied by construction of T_α on limit steps. For (ii), suppose B was an uncountable branch of $\mathfrak{T}$ and let $(I_\alpha)_{\alpha<\omega_1}$ be the canonical enumeration of the first ω_1 elements of B. Construct now

$$A_0 := \{\alpha < \omega_1 \mid (\forall y \in I_{\alpha+1})(y <_X x_{I_\alpha})\}$$
$$A_1 := \{\alpha < \omega_1 \mid (\forall y \in I_{\alpha+1})(x_{I_\alpha} \leq_X y)\},$$

which is a disjoint partition of ω_1, visualized in Figure 3.1. As ω_1 is uncountable, at least one of A_0, A_1 has to be as well; say A_0. For $\alpha \in A_0$, define J_α as the $\mathfrak{X}$-interval $J_\alpha := (x_{I_\beta}, x_{I_\alpha})$, where $\beta := \min\{\gamma \in A_0 \mid \alpha < \gamma\}$, which exists due to A_0 being uncountable. If $\alpha \in A_0$ and $\alpha < \beta$, then $x_{I_\beta} <_X x_{I_\alpha}$. Thus $\{J_\alpha \mid \alpha \in A_0\}$ is an uncountable set of pairwise disjoint intervals of $\mathfrak{X}$, but X has the Suslin property, $\lightning$. Hence (ii) holds.

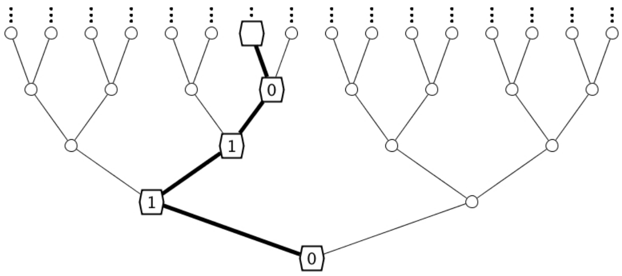

Figure 3.1: Bold edges represent B and a node $x \in T_\alpha$ is labelled $i \in 2$ if $\alpha \in A_i$.

Now for (iii). Suppose $\{I_\alpha \mid \alpha < \omega_1\}$ was an uncountable antichain in $\mathfrak{T}$. Then we could pick $x_\alpha, y_\alpha \in I_\alpha$ such that $x_\alpha <_X y_\alpha$ for every $\alpha < \omega_1$, which would imply that $\{(x_\alpha, y_\alpha) \mid \alpha < \omega_1\}$ is an uncountable set of pairwise disjoint intervals of $\mathfrak{X}$, $\lightning$. Hence $\mathfrak{T}$ satisfies (iii) as well.

It remains to check that $\mathfrak{T}$ is an $(\omega_1, \aleph_1)$-tree, and since $\mathfrak{T}$ is an $(\alpha, \aleph_1)$-tree for some $\alpha \leq \omega_1$ by (ii) and (iii), it suffices to show that T is uncountable.

Claim 3.1.3.1. $\{x_I \in X \mid I \in T\}$ is dense in $\mathfrak{X}$.

Proof of claim. Let x_I, x_J for $I, J \in T$ be such that $x_I <_X x_J$ and let $\alpha_I, \alpha_J < \omega_1$ be such that $I \in T_{\alpha_I}$ and $J \in T_{\alpha_J}$. If $\alpha_I \leq \alpha_J$ then $x_I <_X x_{J_0} <_X x_J$ and if $\alpha_J \leq \alpha_I$ then $x_I <_X x_{I_1} <_X x_J$; hence the set is dense in $\mathfrak{X}$. $\diamondsuit$

Since $\mathfrak{X}$ contains no countable dense subset *ex hypothesi*, $\{x_I \in X \mid I \in T\}$ is uncountable and hence T is as well. The proof is complete. ■

3.2 Diamond principle

Definition 3.2.1 ($\diamondsuit$). $\diamondsuit$ is the statement that there exists a sequence $(A_\alpha)_{\alpha<\omega_1}$, called a $\diamondsuit$-sequence, such that for every $A \subseteq \omega_1$ the set $\{\alpha \in \mathrm{On} \mid A \cap \alpha = A_\alpha\}$ is stationary.

Such a $\diamondsuit$-sequence can intuitively be seen as a sequence which miraculously can "estimate" any given subset $X \subseteq \omega_1$, since every club subset $C \subseteq \omega_1$ (read: a large subset) contain some $\alpha < \omega_1$, such that the α-initial segment of X is *equal* to A_α.

Theorem 3.2.2. *$\diamondsuit$ implies the existence of a Suslin tree.*

Proof. Let $(A_\alpha)_{\alpha<\omega_1}$ be a $\diamondsuit$-sequence. We will recursively construct a tree $\mathfrak{T}$ satisfying

(i) $x \in T \restriction \omega$ implies $x < \omega$;

(ii) $T_\alpha = \{\xi \in \mathrm{On} \mid \omega\alpha \leq \xi < \omega(\alpha+1)\}$ for $\alpha \geq \omega$;
(iii) $\mathfrak{T} \restriction \alpha$ is a normal $(\alpha, \aleph_1)$-tree for every $\alpha < \omega_1$;
(iv) If A_δ is a maximal antichain of $\mathfrak{T} \restriction \delta$ for $\delta \in \mathrm{On}$ limit, then $(\forall x \in T_\delta)(\exists y \in A_\delta)(y <_T x)$.

We start by showing why these conditions imply that $\mathfrak{T}$ is Suslin. First of all, (iii) implies $\mathfrak{T}$ is an $\aleph_1$-tree, so we need to show that every antichain is countable; clearly it suffices to check that every *maximal* antichain is countable. Let $A \subseteq \omega_1$ be a maximal antichain of $\mathfrak{T}$, and construct

$$C := \{\alpha < \omega_1 \mid \omega\alpha = \alpha \wedge \text{“}A \cap \alpha \text{ is a maximal antichain of } \mathfrak{T} \restriction \alpha\text{”}\}.$$

Claim 3.2.2.1. C is club in ω_1.

Proof of claim. Observe first that $\omega\alpha = \alpha$ implies that $T \restriction \alpha = T \cap \alpha$ by (i) and (ii).

Closure: Let $\delta \in \mathrm{On}$ be limit. If $(x_i)_{i<\delta}$ is a sequence in C not cofinal in ω_1, then $\bigcup_{i<\delta} x_i \in C$. Indeed, $\omega \bigcup_{i<\delta} x_i = \bigcup_{i<\delta}(\omega x_i) = \bigcup_{i<\delta} x_i$ and since $x_i \cap A$ is a maximal antichain in $\mathfrak{T} \restriction x_i$ for all $i < \delta$ we have that $\bigcup_{i<\delta}(A \cap x_i)$ is a maximal antichain in

$$\bigcup_{i<\delta}(\mathfrak{T} \restriction x_i) = \bigcup_{i<\delta}(\mathfrak{T} \cap x_i) = \mathfrak{T} \cap \bigcup_{i<\delta} x_i = \mathfrak{T} \restriction \bigcup_{i<\delta} x_i.$$

Unboundedness: Let $\alpha < \omega_1$. Define the sequence $(\alpha_n)_{n<\omega}$ recursively by setting $\alpha_0 := \omega^\alpha$ (ordinal exponentiation) and

$$\alpha_{n+1} := \min\{\gamma > \alpha_n \mid (\forall x \in T \restriction \alpha_n)(\exists y \in A \cap T \restriction \gamma)(x <_T y) \wedge (\exists \delta < \omega_1)(\mathsf{isLimit}(\delta) \wedge \gamma = \omega^\delta)\}.$$

Set $\alpha_\omega := \bigcup_{n<\omega} \alpha_n$. Then $\omega\alpha_\omega = \alpha_\omega$ and $A \cap \alpha_\omega$ is a maximal antichain in $\mathfrak{T} \cap \alpha_\omega = \mathfrak{T} \restriction \alpha_\omega$, so $\alpha_\omega \in C$. $\diamondsuit$

As $\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary in ω_1 by $\diamondsuit$, there is some $\alpha \in C$ with $\alpha \cap A = A_\alpha$. Then A_α is a maximal antichain of $\mathfrak{T} \restriction \alpha$, and as α is limit due to $\omega\alpha = \alpha$, every element of T_α is above some element of A_α by (iv). Thus $A \cap \alpha = A_\alpha$ is a maximal antichain of $\mathfrak{T}$, and hence $A = A \cap \alpha$, making A countable. Thus $\mathfrak{T}$ is Suslin.

It thus remains to construct $\mathfrak{T}$ satisfying the four conditions (i)-(iv). We construct it recursively on the levels. Start by setting $T_0 := 1$. Now assume $\alpha = \beta + 1$ for some $\beta \in \mathrm{On}$, and that $\mathfrak{T} \restriction \alpha$ has been defined. We split it into two cases. Assume first that $\beta < \omega$. Then define $\mathfrak{T} \restriction (\alpha+1)$ by setting the successors of each $x \in T_\beta$ to be the next two unused finite ordinals. Formally, define a function $\mathrm{succ} : T_\beta \times 2 \to \omega$ recursively by

$$\begin{aligned}
\mathrm{succ}(k, 0) &:= \min\{n < \omega \mid (\forall x \in T_\beta)[x < n \wedge (x < k \to \mathrm{succ}(x, 1) < n)]\} \\
\mathrm{succ}(k, 1) &:= \mathrm{succ}(k, 0) + 1
\end{aligned}$$

and then setting $T_\alpha := \{\mathrm{succ}(k, i) \mid k \in T_\beta \wedge i \in 2\}$ and $x <_T y$ iff $(\exists i \in 2)(y = \mathrm{succ}(x, i))$ for all $x \in T_\beta$, $y \in T_\alpha$. If $\beta \geq \omega$ define $\mathfrak{T} \restriction (\alpha+1)$ analogously by setting the successors of each $x \in T_\beta$ to be the next two unused ordinals in the set $\{\xi \in \mathrm{On} \mid \omega\alpha \leq \xi < \omega(\alpha+1)\}$, which is possible since T_β is countable.

Assume now that δ is limit and $\mathfrak{T} \restriction \delta$ has been defined. We will construct a 1-1 correspondence between the δ-branches of $\mathfrak{T} \restriction \delta$ and $\mathfrak{T}_\delta$. For each $x \in \mathfrak{T} \restriction \delta$, set b_x to be a δ-branch of $\mathfrak{T} \restriction \delta$ containing x, and if

A_δ is a maximal antichain of $\mathfrak{T}_\delta$ then $b_x \cap A_\delta \neq \emptyset$. The first condition is possible as $\mathfrak{T} \restriction \delta$ is normal *ex hypothesi* (in particular every node can be extended), and the second is due to maximality of A_δ. Now fix some well-ordering $<_b$ of the b_x's, and define for each b_x the ordinal

$$p_{b_x} := \min\{\xi \in \mathrm{On} \mid \omega\delta \leq \xi < \omega(\delta+1) \wedge (\forall b_y <_b b_x)(p_{b_y} \neq \xi)\}.$$

Set $x <_T p_{b_x}$ for each $x \in \mathfrak{T} \restriction \delta$, i.e. that p_{b_x} is the one-point extension of each b_x. We see immediately that (i)-(iv) is satisfied by how we constructed $\mathfrak{T}$. The proof is complete. ■

3.3 Suslin tree in L

We now show that a Suslin tree exists in L, thereby showing that $\neg\,\mathrm{SH}$ holds in L. We do this by showing that $\diamondsuit$ holds in L. First a useful lemma.

Lemma 3.3.1. *Assume $V{=}L$ and let $\kappa > \omega_1$ be a cardinal. If $X \preceq L_\kappa$ then $X \cap L_{\omega_1} = L_\alpha$ for some $\alpha \leq \omega_1$.*

Proof. Since $X \preceq L_\kappa$, we have that $L_{\omega_1} \in X$ as $L_{\omega_1} \in L_\kappa$ is definable by a formula. Thus for any formula $\varphi(\vec{v})$ and $\vec{x} \in X \cap L_{\omega_1}$, we have

$$L_{\omega_1} \models \varphi[\vec{x}] \Leftrightarrow L_\kappa \models \varphi^{L_{\omega_1}}[\vec{x}] \Leftrightarrow X \models \varphi^{L_{\omega_1}}[\vec{x}] \Leftrightarrow X \cap L_{\omega_1} \models \varphi[\vec{x}],$$

so $X \cap L_{\omega_1} \preceq L_{\omega_1}$. By condensation we have $\pi : X \cap L_{\omega_1} \cong L_\alpha$ for some $\alpha \leq \omega_1$ which fixes every transitive subset. Thus, if $X \cap L_{\omega_1}$ is transitive, we will be done. Let $x \in X \cap L_{\omega_1}$; we will show $x \subseteq X \cap L_{\omega_1}$. As $x \in L_{\omega_1}$, $x \in L_\gamma$ for some $\gamma < \omega_1$. Then $x \subseteq L_\gamma$ since L_γ is transitive, whence x is countable because L_γ is countable. Thus there exists a surjection $f : \omega \to x$. Let f be the $<_L$-least such function, which is an element of L_{ω_1} as $x, \omega \in L_{\omega_1}$, and f is definable by formula, since $<_L$ is definable (see Appendix A). Then $f \in X \cap L_{\omega_1}$ because $X \cap L_{\omega_1} \preceq L_{\omega_1}$, and since we also have $\omega \subseteq X \cap L_{\omega_1}$ (natural numbers are definable), it holds that $f(n) \in X \cap L_{\omega_1}$ for every $n < \omega$ because $z = f(n) \Leftrightarrow \varphi_f[n, z]$, where $f = \{\langle n, z\rangle \in \omega \times x \mid \varphi_f[n, z]\}$ and thus $f(n) = \bigcup\{z \in L_\gamma \mid \varphi_f[n, z]\} \in X \cap L_{\omega_1}$. Hence we conclude $x = f``\omega \subseteq X$. ■

Lemma 3.3.2. *Assume $V{=}L$. Let κ be an infinite cardinal. If $x \subseteq L_\alpha$ for some $\alpha < \kappa$, then $x \in L_\kappa$; hence $\mathcal{P}(L_\alpha) \subseteq L_\kappa$. In particular, $\mathcal{P}(L_\alpha) \in L_{\kappa^+}$.*

Proof. If $\kappa = \omega$ then it is trivial, due to $V_\omega = L_\omega$, so assume $\kappa > \omega$. Pick $\alpha \in \mathrm{On}$ such that $\omega \leq \alpha < \kappa$ and $x \subseteq L_\alpha$. Fix limit $\delta \geq \kappa$ such that $x \in L_\delta$, which exists as $V{=}L$. Use the Löwenheim-Skolem Theorem 1.2.14 to find $M \preceq L_\delta$ such that $L_\alpha \cup \{x\} \subseteq M$ and $|M| = |L_\alpha|$. Use condensation to get $\pi : M \cong L_\gamma$ for some $\gamma \leq \delta$. As $L_\alpha \cup \{x\}$ is transitive, $\pi \restriction (L_\alpha \cup \{x\}) = \mathrm{id} \restriction (L_\alpha \cup \{x\})$, and in particular $\pi(x) = x$, so $x \in L_\gamma$. Now

$$|\gamma| = |L_\gamma| = |M| = |L_\alpha| = |\alpha| \leq \alpha < \kappa,$$

so $\gamma < \kappa$, entailing $L_\gamma \subseteq L_\kappa$ and thus $x \in L_\kappa$. ■

Theorem 3.3.3. *$V{=}L$ implies $\diamondsuit$.*

Proof. We start by recursively constructing a sequence $(\langle A_\alpha, C_\alpha\rangle)_{\alpha<\omega_1}$ of sets $A_\alpha \subseteq \alpha$ and $C_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$. Set $A_0 = C_0 = \emptyset$, $A_{\alpha+1} = C_{\alpha+1} = \alpha + 1$ and for limit δ set $\langle A_\delta, C_\delta\rangle$ to be the $<_L$-least pair of subsets of δ such that C_δ is club in δ and $(\forall\gamma \in C_\delta)(A_\delta \cap \gamma \neq A_\gamma)$ if such sets exist, and otherwise set $A_\delta = C_\delta = \delta$.

Notice that $(\langle A_\gamma, C_\gamma\rangle)_{\gamma<\omega_1}$ is definable in L_{ω_2} as $V{=}L$ implies that $\mathcal{P}(\gamma) \in L_{\omega_2}$ for every $\gamma < \omega_1$ by Lemma 3.3.2, as well as $<_L$ being definable, cf. Appendix A. We show that $(A_\gamma)_{\gamma<\omega_1}$ is a $\diamondsuit$-sequence. Suppose it is not, meaning that there is a set $A \subseteq \omega_1$ such that $\{\alpha \in \omega_1 \mid A \cap \alpha = A_\alpha\}$ is not stationary in ω_1; i.e. there is a club $C \subseteq \omega_1$ such that $A \cap \gamma \neq A_\gamma$ for every $\gamma \in C$. Let $\langle A, C\rangle$ be the $<_L$-least such pair, which will be definable in L_{ω_2} for the same reason as before.

Use the Löwenheim-Skolem Theorem 1.2.14 to find some countable $X \preceq L_{\omega_2}$ and use condensation to find a unique $\pi : X \cong L_\delta$. By Lemma 3.3.1, $X \cap L_{\omega_1} = L_\gamma$ for some $\gamma \leq \omega_1$. In particular $X \cap L_{\omega_1}$ is transitive, so $\alpha := X \cap \omega_1 = X \cap (L_{\omega_1} \cap \omega_1) = (X \cap L_{\omega_1}) \cap \omega_1$ is an ordinal. We have that $\omega_1 \in X$, $\langle A, C\rangle \in X$ and $(\langle A_\gamma, C_\gamma\rangle)_{\gamma<\omega_1} \in X$, as all these are definable in L_{ω_2} and $X \preceq L_{\omega_2}$; hence by absoluteness we have that

$$\begin{aligned} X \models \text{“}&\langle A, C\rangle \text{ is the } <_L\text{-least pair of subsets of } \omega_1 \text{ such that } C \text{ is club in } \omega_1 \\ &\text{and } (\forall\gamma \in C)(A \cap \gamma \neq A_\gamma)\text{”}. \end{aligned}$$

By definition of π we then have that $\pi(\omega_1) = \alpha$ and $\pi \restriction L_\alpha = \mathrm{id} \restriction L_\alpha$, so we furthermore have that

$$\pi(A) = A \cap \alpha \qquad \pi(C) = C \cap \alpha \qquad \pi((\langle A_\gamma, C_\gamma\rangle)_{\gamma<\omega_1}) = (\langle A_\gamma, C_\gamma\rangle)_{\gamma<\alpha},$$

meaning by Lemma 1.2.13 that

$$\begin{aligned} L_\delta \models \text{“}&\langle A \cap \alpha, C \cap \alpha\rangle \text{ is the } <_L\text{-least pair of subsets of } \alpha \text{ such that } C \cap \alpha \text{ is club in } \alpha \\ &\text{and } (\forall\gamma \in C \cap \alpha)(A \cap \alpha \cap \gamma \neq A_\gamma)\text{”}. \end{aligned}$$

Thus by absoluteness this holds in V as well[1]; hence $A_\alpha = A \cap \alpha$ and $C_\alpha = C \cap \alpha$ by definition of $\langle A_\alpha, C_\alpha\rangle$. But as $C \cap \alpha$ is club in α, it is in particular unbounded in α, so as C is closed in ω_1 and $\alpha < \omega_1$, we have $\alpha \in C$; thus $A \cap \alpha \neq A_\alpha$ by definition of C, $\lightning$. Hence $(S_\gamma)_{\gamma<\omega_1}$ *is* in fact a $\diamondsuit$-sequence. ■

Corollary 3.3.4. Con(ZF) $\Rightarrow$ Con(ZF $+\diamondsuit + \neg$SH).

Proof. By Theorems 3.1.3, 3.2.2 and 3.3.3 we have that ZFC $+V{=}L \vdash \diamondsuit \wedge \neg$SH. By Corollaries 2.4.2 and 2.3.4 we have Con(ZF) $\Rightarrow$ Con(ZFC $+V{=}L$), so the result follows from $T \vdash \sigma \Rightarrow (\mathrm{Con}(T) \Rightarrow \mathrm{Con}(T + \sigma))$. ■

[1]For a proof of the fact that $<_L$ is L_δ-absolute for limit $\delta > \omega$, see the appendix.

4 | Why not $V{=}L$?

> *"L takes on the character of a very thin inner model indeed, bare ruined choirs appended to the slender life-giving spine which is the class of ordinals."*
>
> - K. Kanamori and M. Magidor [KM78]

After having proved a variety of theorems in L in the last chapters, one might wonder why the general mathematical community are convinced that $V{\neq}L$. Their arguments are either intrinsic, arguing that since the construction of L is restrictive, it cannot be equal to the universe as our universe should be *maximal*. Other arguments builds on extrinsic evidence concerning the non-existence of *measurable cardinals* in L, which we will investigate in this chapter.

4.1 κ-completeness

A filter D is $\boldsymbol{\kappa}$**-complete** for some infinite cardinal κ if every subset $X \subseteq D$ with $|X| < \kappa$ satisfies $\bigcap X \in D$.

Lemma 4.1.1. *(i) Every filter is ω-complete;*
(ii) A filter D is κ-complete for all cardinals κ iff D is principal;
(iii) If $\kappa < \lambda$, then every λ-complete filter is κ-complete.

Proof. (i): Definition of filter. (ii): " $\Rightarrow$ ": $\bigcap D \in D$. " $\Leftarrow$ ": Every intersection contains the generator. (iii): Trivial. ■

We will be interested in free ultrafilters, so by (ii) in the above lemma we cannot assume the filter to be κ-complete for all κ. We can restrict this result even further, by the following lemma.

Lemma 4.1.2. *Let D be a filter over a set I with $|I| = \kappa$. If D is κ^+-complete, then D is principal.*

Proof. Define $E := \{I\backslash\{i\} \mid i \in I \wedge I\backslash\{i\} \in D\}$. Since $|I| = \kappa$, we have $|E| \leq \kappa < \kappa^+$, so $\bigcap E \in D$ by κ^+-completeness. Now, if $X \in D$ then $\bigcap E \subseteq X$. Indeed, if $X = I$ it is trivial, so assume $X \neq I$; let $x \in I\backslash X$. Then $X \subseteq I\backslash\{x\}$, so $I\backslash\{x\} \in D$, $I\backslash\{x\} \in E$ and thus $x \notin \bigcap E$. Thus D is principal, generated from $\bigcap E$. ■

We therefore have an upper bound on completeness. The cardinals κ equipped with a free ultrafilter that is maximally complete, i.e. κ-complete, are called *measurable*:

Definition 4.1.3 (Measurable cardinal). A *measurable cardinal* is a cardinal κ on which there exists a free κ-complete ultrafilter.

Measurable cardinals are an example of a general notion of being a *large cardinal*. Before we delve into these measurables however, we need some more basic results; firstly we have a handy equivalent way of stating κ-completeness.

Lemma 4.1.4. *Let U be an ultrafilter over a set I. Then U is κ-complete iff for every partition $\bigcup P = I$ of I with $|P| < \kappa$ there is some $x \in P$ such that $x \in U$.*

Proof. " $\Rightarrow$ ": Assume U is κ-complete, and let $\bigcup P = I$ be a partition of I; write $P = \{p_\gamma \mid \gamma < \alpha\}$ for some $\alpha < \kappa$. Since $\bigcap_{\gamma<\alpha}(I \backslash p_\gamma) = \emptyset$, there have to be some $\gamma < \alpha$ such that $I \backslash p_\gamma \notin U$, due to U being κ-complete. By the ultra property, $p_\gamma \in U$.

" $\Leftarrow$ ": Assume that for every partition of I in less than κ parts, one of the parts is in U. Let $E \subseteq U$ with $|E| < \kappa$; write $E = \{e_\gamma \mid \gamma < \alpha\}$ for some $\alpha < \kappa$. Define the function $f : I \to \alpha + 1$ as

$$f(i) := \begin{cases} \alpha & , i \in \bigcap E \\ \min\{\gamma < \alpha \mid i \notin e_\gamma\} & , i \in I \backslash \bigcap E \end{cases}$$

Now $\bigcup_{\gamma<\alpha+1} f^{-1}(\gamma)$ is a partition of I in $\alpha + 1 < \kappa$ parts (κ is limit), so there is some $\gamma < \alpha + 1$ such that $f^{-1}(\gamma) \in U$ by assumption. But $f^{-1}(\gamma) \cap e_\gamma = \emptyset$ for all $\gamma < \alpha$ by construction of f, so $f^{-1}(\gamma) \notin U$ for all $\gamma < \alpha$. Hence $\bigcap E = f^{-1}(\alpha) \in U$. ■

4.2 Elementary embeddings

There is a deep connection between elementary embeddings and large cardinals, and we will here investigate some of the correlations between measurable cardinals and the so-called *natural embedding* from a set (or even proper class) to its ultrapower. We will start off with some basic properties of elementary embeddings.

Lemma 4.2.1. *Let $\mathfrak{A}, \mathfrak{B}$ be inner models and $j : \mathfrak{A} \preceq_1 \mathfrak{B}$. Then*

(i) For any $\alpha \in \mathrm{On}$, $j(\alpha) \in \mathrm{On}$ and $j(\alpha) \geq \alpha$;

(ii) If $j \neq \mathrm{id}_A$ and $B \subseteq A$, then $j(\alpha) > \alpha$ for some $\alpha \in \mathrm{On}$.

Proof. (i): As $\mathsf{on}(v)$ is Δ_0, we have $\mathrm{On}^{\mathfrak{A}} = \mathrm{On}^{\mathfrak{B}} = \mathrm{On}$, and since $j : \mathfrak{A} \preceq_1 \mathfrak{B}$ we have $j(\alpha) \in \mathrm{On}$ for every $\alpha \in \mathrm{On}$. We show $j(\alpha) \geq \alpha$ for all $\alpha \in \mathrm{On}$ by transfinite induction. Clearly $j(0) \geq 0$. Assume $\alpha = \gamma + 1$ and $j(\alpha) < \alpha$. Then $j(\alpha) \leq \gamma \leq j(\gamma)$, so $j(\alpha) \subseteq j(\gamma)$ and thus $\alpha \subseteq \gamma$ since subsets are Δ_0-definable, $\lightning$. Assume now α is limit and let $x \in \alpha = \bigcup \alpha$. Then there is some $\gamma \in \alpha$ such that $x \in \gamma$. But $\gamma \leq j(\gamma)$ by assumption, so $x \in \gamma \subseteq j(\gamma) \in j(\alpha)$, so $x \in \bigcup j(\alpha) = j(\bigcup \alpha) = j(\alpha)$. Thus $\alpha \subseteq j(\alpha)$, whence $\alpha \leq j(\alpha)$.

(ii): As $j \neq \mathrm{id}_A$, let x be of least rank such that $j(x) \neq x$; set $\alpha := \mathrm{rank}\, x$. Then $x \subseteq j(x)$ since for every $y \in x$ we have $y = j(y) \in j(x)$. It also holds that $j(\alpha) > \alpha$: First of all $\mathrm{rank}(j(x)) \neq \alpha$, since assuming $\mathrm{rank}(j(x)) = \alpha$ we can pick some $z \in j(x) \backslash x \subseteq B \subseteq A$ because $x \subsetneq j(x)$ and thus $j(z) = z \in j(x)$ by minimality of α, since $\alpha = \mathrm{rank}(j(x)) > \mathrm{rank}\, z$, so $z \in x$, $\lightning$. Since $\mathrm{rank}(j(x)) = j(\alpha)$ by Δ_1^{ZF}-definability of rank cf. Lemma 1.3.8, we have $\mathrm{rank}(j(x)) = j(\alpha) \geq \alpha$, so $j(\alpha) > \alpha$. ■

If $j \neq \mathrm{id}_A$, denote the **critical point of j**, written $\mathrm{crit}\, j$, as the least ordinal α such that $j(\alpha) > \alpha$, which exists by (ii) in the above lemma.

We mentioned before that we would be dealing with ultrapowers of (potentially proper) *classes*. To make sense of this, we have to slightly modify our notion of an ultrapower. With our usual ultrapower construction, we would have equivalence classes $(f)_U$, which would become *proper* classes when the codomain of the functions is a proper class, and the ultrapower definition ceases to make any sense (only sets can be members of classes). Scott came up with a trick to avoid this, restricting the equivalence classes to only consist of the functions with minimal rank:

Definition 4.2.2 (Class ultrapowers). Let M be a class, I a set and U an ultrafilter over I. Then the *class ultrapower* of M is defined as

$$\prod_{i\in I} M/U^{\times} := \{(f)_U^{\times} \mid f : I \to V\},$$

where $(f)_U^{\times} := \{f \in (f)_U \mid (\forall g \in (f)_U)(\operatorname{rank} f \leq \operatorname{rank} g)\}$.

As all $g \in (f)_U^{\times}$ then have the same rank, $(f)_U^{\times} \subseteq V_\gamma$ for some ordinal γ. Hence $(f)_U^{\times} \in V_{\gamma+1} \subseteq V$, making it into a set and the class ultrapower definition is then well-defined. On top of the $=_U$ relation as an analogue to our usual $=$ relation, we also introduce the relation $\in_U$ as the analogue to $\in$. Let $(f)_U, (g)_U \in \prod_{i\in I} M/U$ for some set M. Then

$$(f)_U \in_U (g)_U \Leftrightarrow \{i \in I \mid f(i) \in g(i)\} \in U.$$

Of course, analogous definitions hold for the class ultrapowers. This relation then induces a *class ultrapower structure*, $\mathrm{Ult}(M, U) := \langle \prod_{i\in I} M/U^{\times}, \in_U\rangle$, and as we are mostly concerned with ultrapowers of V, we abbreviate $\mathrm{Ult} := \mathrm{Ult}(V, U)$. The following theorem states that the truths in V is in correspondence with the truths in Ult. The theorem is not stated in its usual form, but this version suits our needs.

Theorem 4.2.3 (Łoś). *Let U be an ultrafilter over a set I, $\varphi(\vec{v})$ a formula and $f_1, \ldots, f_n : I \to V$ class functions. Then*

$$\mathrm{Ult} \models \varphi[(f_1)_U^{\times}, \ldots, (f_n)_U^{\times}] \Leftrightarrow \{i \in I \mid \varphi[f_1(i), \ldots, f_n(i)]\} \in U.$$

Proof. We prove it by induction on the complexity of φ. *Atomic step:* Assume φ is $t_1 = t_2$. As there are no constant symbols in our language, we can assume that t_1 and t_2 are variables u and v, respectively. We then have that

$$\begin{aligned}\mathrm{Ult} \models \varphi[(f_1)_U^{\times}, (f_2)_U^{\times}] &\Leftrightarrow \mathrm{Ult} \models u =_U v\\ &\Leftrightarrow \{i \in I \mid u = v\} \in U\\ &\Leftrightarrow \{i \in I \mid \varphi[f_1(i), f_2(i)]\} \in U.\end{aligned}$$

Assume now φ is $t_1 \in t_2$. Recall that $\in^{\mathrm{Ult}}$ is the relation $\in_U$:

$$\begin{aligned}\mathrm{Ult} \models \varphi[(f_1)_U^{\times}, (f_2)_U^{\times}] &\Leftrightarrow \mathrm{Ult} \models u \in_U v\\ &\Leftrightarrow \{i \in I \mid u \in v\} \in U\\ &\Leftrightarrow \{i \in I \mid \varphi[f_1(i), f_2(i)]\} \in U.\end{aligned}$$

Sentential step: Assume φ is $\psi \wedge \chi$. Then

$$\begin{aligned}&\mathrm{Ult} \models \varphi[(f_1)_U^{\times}, \ldots, (f_n)_U^{\times}]\\ \Leftrightarrow\ &\mathrm{Ult} \models \psi[(f_1)_U^{\times}, \ldots, (f_n)_U^{\times}] \text{ and } \mathrm{Ult} \models \chi[(f_1)_U^{\times}, \ldots, (f_n)_U^{\times}]\\ \Leftrightarrow\ &\{i \in I \mid \psi[f_1(i), \ldots, f_n(i)]\} \in U \text{ and } \{i \in I \mid \chi[f_1(i), \ldots, f_n(i)]\} \in U\\ \Leftrightarrow\ &\{i \in I \mid \psi[f_1(i), \ldots, f_n(i)] \wedge \chi[f_1(i), \ldots, f_n(i)]\} \in U\\ \Leftrightarrow\ &\{i \in I \mid \varphi[f_1(i), \ldots, f_n(i)]\} \in U,\end{aligned}$$

where we used the induction hypothesis and that $x, y \in U \Leftrightarrow x \cap y \in U$ along the way. Now assume φ is $\neg\psi$. Then

$$\begin{aligned}
&\mathrm{Ult} \models \varphi[(f_1)_U^\times, \ldots, (f_n)_U^\times] \\
\Leftrightarrow\, &\mathrm{Ult} \not\models \psi[(f_1)_U^\times, \ldots, (f_n)_U^\times] \\
\Leftrightarrow\, &\{i \in I \mid \psi[f_1(i), \ldots, f_n(i)]\} \notin U \\
\Leftrightarrow\, &I \backslash \{i \in I \mid \psi[f_1(i), \ldots, f_n(i)]\} \in U \\
\Leftrightarrow\, &\{i \in I \mid \neg\psi[f_1(i), \ldots, f_n(i)]\} \in U \\
\Leftrightarrow\, &\{i \in I \mid \varphi[f_1(i), \ldots, f_n(i)]\} \in U,
\end{aligned}$$

where we used the induction hypothesis and the ultra property. *Quantifier step:* Assume φ is $\exists y\psi$. Then

$$\begin{aligned}
&\mathrm{Ult} \models \varphi[(f_1)_U^\times, \ldots, (f_n)_U^\times] \\
\Leftrightarrow\, &\mathrm{Ult} \models \exists (g)_U^\times \psi[(g)_U^\times, (f_1)_U^\times, \ldots, (f_n)_U^\times] \\
\Leftrightarrow\, &\text{There is a } (g)_U^\times \in \prod_{i \in I} V/U^\times \text{ such that } \mathrm{Ult} \models \psi[(g)_U^\times, (f_1)_U^\times, \ldots, (f_n)_U^\times] \\
\Leftrightarrow\, &\text{There is a } (g)_U^\times \in \prod_{i \in I} V/U^\times \text{ such that } \{i \in I \mid \psi[(g)_U^\times, (f_1)_U^\times, \ldots, (f_n)_U^\times]\} \in U \\
\Leftrightarrow\, &\{i \in I \mid (\exists (g)_U^\times \in \prod_{i \in I} V/U^\times)\psi[g(i), f_1(i), \ldots, f_n(i)]\} \in U \\
\Leftrightarrow\, &\{i \in I \mid \varphi[f_1(i), \ldots, f_n(i)]\} \in U,
\end{aligned}$$

where we used the induction hypothesis. ■

Note that as Ult is a proper class, Łoś' Theorem as stated here is really a schema, for each formula φ.

Corollary 4.2.4. Ult *is elementarily equivalent to* V.

Proof. By Łoś' theorem 4.2.3 we have that for every sentence σ:

$$\mathrm{Ult} \models \sigma \Leftrightarrow \{i \in I \mid \sigma\} \in U \Leftrightarrow \sigma,$$

where it was used that $\emptyset \notin U$. ■

The **natural embedding** is defined as the function $d : \langle X, \in \rangle \to \mathrm{Ult}(X, U)$, given by $d(x) := (f_x)_U^\times$, where $f_x : I \to X$ is given by $f_x(i) := x$.

Corollary 4.2.5. *The natural embedding $d : \langle V, \in\rangle \to \mathrm{Ult}$ is an elementary embedding.*

Proof. Let $\varphi(\vec{v})$ be a formula and let $\vec{x}$ be sets. Then by Łoś' theorem 4.2.3 we have

$$\mathrm{Ult} \models \varphi[d(x_1), \ldots, d(x_n)] \Leftrightarrow \{i \in I \mid \varphi[x_1, \ldots, x_n]\} \in U \Leftrightarrow \varphi[x_1, \ldots, x_n],$$

where it was used that $\emptyset \notin U$. ■

We are now getting closer to have our desired knowledge of the ultrapowers we need to prove Scott's result on measurable cardinals in L. Our next goal is to show that Ult can be subject to the Mostowski collapse 2.5.2, granting us an inner model; we thus need to check that $\in_U$ is both well-founded and set-like, as well as checking that Ult is extensional. Extensionality is due to Ult being elementarily equivalent to V, cf. Corollary 4.2.4.

Lemma 4.2.6. *Let U be an ultrafilter. Then U is ω_1-complete iff $\in_U$ is well-founded.*

Proof. " $\Rightarrow$ ": Let M be a class and assume $\in_U$ is ill-founded. Let $\cdots \in_U (f_1)^\times_U \in_U (f_0)^\times_U$ be an infinitely descending sequence in $\prod_{i\in I} M/U^\times$. If U is ω_1-complete, we would have

$$X := \bigcap_{n<\omega} \{i \in I \mid f_{n+1}(i) \in f_n(i)\} \in U$$

by definition of $\in_U$ and ω_1-completeness. As U is proper, we have that $\emptyset \notin U$, so X is non-empty; let $z \in X$. Then $\cdots \in f_1(z) \in f_0(z)$ is an infinitely descending $\in$-sequence, contradicting Foundation.

" $\Leftarrow$ ": Assume U is ω_1-incomplete. We show that $\in_U$ is ill-founded. Let $\{X_n \mid n < \omega\} \subseteq U$ but $\bigcap\{X_n \mid n < \omega\} \notin U$ witness the ω_1-incompleteness. Define the function $g_k : I \to V$ given by

$$g_k(i) := \begin{cases} n-k & , i \in (\bigcap_{m<n} X_m)\backslash X_n, n \geq k \\ 0 & , \text{otherwise} \end{cases}$$

Then $(\bigcap_{m\leq k} X_m)\backslash(\bigcap_{n<\omega} X_n) \in U$. Indeed, $\bigcap_{m\leq k} X_m \in U$ as it is a finite intersection, and $I\backslash\bigcap_{n<\omega} X_n \in U$ since $\bigcap_{n<\omega} X_n \notin U$ and U is an ultrafilter; hence

$$\left(\bigcap_{m\leq k} X_m\right)\backslash\left(\bigcap_{n<\omega} X_n\right) = \left(\bigcap_{m\leq k} X_m\right) \cap \left(I\backslash\bigcap_{n<\omega} X_n\right) \in U.$$

Claim 4.2.6.1. $\{i \in I \mid g_{k+1}(i) \in g_k(i)\} \supseteq \bigcap_{m\leq k} X_m \backslash \bigcap_{n<\omega} X_n$

Proof of claim. Let $x \in (\bigcap_{n\leq k} X_n)\backslash(\bigcap_{n<\omega} X_n)$. As $x \notin \bigcap_{n<\omega} X_n$, there is some $l > k$ such that $x \notin X_l$; let l be the least such. Then $x \in \bigcap_{n<l} X_n$ and $x \notin X_l$, so $g_{k+1}(x) = l-(k+1) \in l-k = g_k(x)$ and hence $x \in \{i \in I \mid g_{k+1}(i) \in g_k(i)\}$, whence the inclusion holds. ♢

But now $\{i \in I \mid g_{k+1}(i) \in g_k(i)\} \in U$ for every $k < \omega$, so $\cdots \in_U (g_1)^\times_U \in_U (g_0)^\times_U$ is infinitely descending, and $\in_U$ is thus ill-founded. ■

Lemma 4.2.7. *$\in_U$ is set-like for every ultrafilter U.*

Proof. Let $(g)^\times_U \in_U (f)^\times_U$ and $g_0 \in (g)^\times_U$. Define $g_1 : I \to V$ by

$$g_1(i) := \begin{cases} g_0(i) & , g_0(i) \in f(i) \\ \emptyset & , \text{otherwise} \end{cases}$$

Then $g_1 \in (g)^\times_U$ by definition and $\operatorname{rank} g_1 \leq \operatorname{rank} f$. Since all elements of $(g)^\times_U$ have the same rank, we have that $\operatorname{rank}((g)^\times_U) \leq \operatorname{rank}(f) + 1$, so $(g)^\times_U \in V_{\operatorname{rank}(f)+2}$ and thus $\{(g)^\times_U \mid (g)^\times_U \in_U (f)^\times_U\} \in V_{\operatorname{rank}(f)+3}$, making it a set. ■

Now that we have shown Ult is subject to the Mostowski collapse (if U is ω_1-complete), we have an isomorphism $\pi_U : \text{Ult} \cong \langle M, \in \rangle$, where M is transitive. Hence we have that $d : V \preceq \text{Ult}$ and $\pi_U : \text{Ult} \cong \langle M, \in \rangle$, making M an inner model. Defining the composite $j := \pi_U \circ d : V \to M$, we usually write up this scenario as $j : V \preceq M \cong \text{Ult}$, due to the fact that π_U being an isomorphism and d an elementary embedding implies that $j : V \preceq M$, cf. Lemma 1.2.13. We denote by $[f]_U$ the set $\pi_U((f)^\times_U) \in M$, which means that $j(\alpha) = [f_\alpha]_U$.

4.3 Measurable cardinals

We now apply the previous section to the study of measurable cardinals. As every measurable cardinal κ comes along with a free κ-complete ultrafilter U, it also has a corresponding elementary embedding $j : V \preceq M \cong \text{Ult}$. Since M is an inner model, $j \neq \text{id}$ implies that $\operatorname{crit} j$ exists, cf. Lemma 4.2.1(ii). It turns out that the κ-completeness of the ultrafilter determines exactly what $\operatorname{crit} j$ is.

Lemma 4.3.1. *Let U be a κ-complete ultrafilter over a measurable cardinal κ with corresponding embedding $j : V \preceq M \cong \text{Ult}$. Then $j \neq \text{id}$ and $\operatorname{crit} j = \kappa$.*

Proof.

Claim 4.3.1.1. $j(\alpha) = \alpha$ for all $\alpha < \kappa$.

Proof of claim. Assume not, and let $\alpha < \kappa$ be the least ordinal such that $j(\alpha) > \alpha$. As $\alpha \in M$ and $\pi_U : \prod_{i \in I} V/U^\times \approx M$, there is some unique $(f)^\times_U \in \prod_{i \in I} V/U^\times$ such that $[f]_U = \alpha$. Then $[f]_U = \alpha \in j(\alpha) \stackrel{\text{def}}{=} [f_\alpha]_U$, so $(f)^\times_U \in_U (f_\alpha)^\times_U$ by application of π_U^{-1}. Thus $\{\gamma < \kappa \mid f(\gamma) \in f_\alpha(\gamma) \stackrel{\text{def}}{=} \alpha\} \in U$ by definition of $\in_U$. So since

$$\{\gamma < \kappa \mid f(\gamma) \notin \alpha\} \cup \bigcup_{\xi < \alpha} \{\gamma < \kappa \mid f(\gamma) = \xi\}$$

is a partition of κ in $\alpha + 1 < \kappa$ parts, one of the parts belongs to U by Lemma 4.1.4. As $\{\gamma < \kappa \mid f(\gamma) \notin \alpha\} \notin U$ since U is an ultrafilter, we must have that for some $\beta < \alpha$,

$$\{\gamma < \kappa \mid f(\gamma) = \beta\} \in U.$$

But then $(f)^\times_U = (f_\beta)^\times_U$ and hence $\alpha = [f]_U = [f_\beta]_U = j(\beta) = \beta < \alpha$ by minimality of α, $\lightning$. ◇

Now, U contains no bounded subsets, because say $S \in U$ is bounded by $\alpha < \kappa$. Then $\bigcup_{\xi \in S}\{\xi\} \cup \kappa \backslash S$ is a partition of κ in less than κ parts, so $\{\beta\} \in U$ for some $\beta < \alpha$ by Lemma 4.1.4 because $\kappa \backslash S \notin U$

since U is proper. But then either $\emptyset \in U$ or U is principal, both causing a contradiction. Thus for all $\alpha < \kappa$ we have that

$$\{\gamma < \kappa \mid \alpha < \gamma < \kappa\} \in U$$

by the ultra property. But then by definition of $\in_U$, we have $\alpha = j(\alpha) \stackrel{\text{def}}{=} [f_\alpha]_U \in [\mathrm{id}_\kappa]_U \in [f_\kappa]_U \stackrel{\text{def}}{=} j(\kappa)$, so $\alpha < [\mathrm{id}_\kappa]_U < j(\kappa)$ for all $\alpha < \kappa$, implying $\kappa \leq [\mathrm{id}_\kappa]_U < j(\kappa)$, so $\mathrm{crit}(j) = \kappa$. ■

We finally arrive at our aforementioned extrinsic evidence for $V{\neq}L$; the fact that *no* measurable cardinals exists in L.

Theorem 4.3.2 (Scott). *There exists no measurable cardinal in L.*

Proof. Suppose κ is the least measurable cardinal with corresponding embedding $j : V \preceq M \cong \mathrm{Ult}$, and assume $V{=}L$. As M is an inner model, we have $L \subseteq M \subseteq V = L$, so $M = L$ and hence

$$L = V \models \text{“}\kappa \text{ is the least measurable cardinal”}.$$

But we also have

$$L = M \models \text{“}j(\kappa) \text{ is the least measurable cardinal”}$$

by elementarity, contradicting $j(\kappa) > \kappa$ from Lemma 4.3.1. ■

As no measurable cardinal is in L, we say that L is too *thin*: even though the cardinal itself is in L since $\mathrm{On} \subseteq L$, the associated ultrafilter is not. This is due to our restriction of the power set, not being powerful enough to include such a complicated subset of $\mathcal{P}(\kappa)$ as a free κ-complete ultrafilter.[1]

[1] For more information about measurable cardinals, elementary embeddings and other large cardinals, see Kanamori [Kan09].

A | A treatment of $\mathscr{L}$

As $\mathscr{L}$ is the language on which L is founded, we find it necessary to include here a proper treatment of it. Our main goal in this chapter is to show that the formula $\mathsf{sat}(M, \ulcorner\varphi(\vec{x})\urcorner)$ is absolute for L, and also proving that Def and $v = L_\alpha$ are Δ_1^{ZF} and L_δ-absolute for limit $\delta > \omega$. To help reading comprehension, we will in this chapter use the notation $\varphi :\equiv \psi$ to say that we *define* φ to be ψ and (henceforth) write φ, ψ, χ instead of $\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner, \ulcorner\chi\urcorner$ due to them always being sets throughout the chapter. Furthermore if s is a sequence, we define $s_k := s(k)$.

A.1 The theory DS

To be able to achieve the above mentioned absoluteness result, we need to work in a weaker theory than ZF, as the limit L_δ's do not satisfy things like Replacement. This was done in Devlin [Dev84] using a theory called Basic Set Theory, abbreviated BS. Later on, Stanley [Sta87] pointed out errors in this theory in a review of Devlin's book, causing a few of the fundamental lemmas to be outright wrong (counter-examples were found by Solovay). After some time, Mathias [Mat04] did a thorough analysis of the situation and came up with multiple possible solutions; i.e. theories which are stronger than Devlin's BS theory, but still weak enough for the L_δ's to satisfy it, for limit $\delta > \omega$. Here we introduce one of Mathias' solutions, namely the theory Devlin Strengthened, abbreviated DS:

Definition A.1.1 (DS).
- (i) *Extensionality*: $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to (x = y)]$;
- (ii) Π_1 *foundation*: $\forall\vec{a}\forall x[(\forall y(y \in x \leftrightarrow \varphi(\vec{a}, y)) \wedge x \neq \emptyset) \to (\exists z \in x)(z \cap x = \emptyset)]$ with $\varphi(\vec{a}, y)$ being a Π_1 formula;
- (iii) *Pairing*: $\forall x \forall y \exists z \forall w[w \in z \leftrightarrow (w = x \vee w = y)]$;
- (iv) *Union*: $\forall x \exists y \forall z[z \in y \leftrightarrow (\exists u \in x)(z \in u)]$;
- (v) *Infinity*: $\exists x[\mathsf{on}(x) \wedge x \neq \emptyset \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$;
- (vi) *Finite Power Set*: $\forall x \exists y \forall w[w \in y \leftrightarrow (w \subseteq x \wedge \mathsf{isFin}(w)]$;
- (vii) Δ_0 *comprehension*: $\forall\vec{a}\forall x \exists y \forall z[z \in y \leftrightarrow z \in x \wedge \varphi(\vec{a}, z))]$, with $\varphi(\vec{a}, z)$ being a Δ_0 formula.

The Finite Power Set axiom in other words states that $\mathcal{S}(x) := \{y \subseteq x \mid \mathsf{isFin}(y)\}$ is a set for all sets x. Notice the change in the Infinity Axiom, now stating that an infinite *ordinal* exists, rather than just any infinite set. It is easily checked that DS is a subtheory of ZF, as every axiom in DS is a theorem or an axiom of ZF. We will need a few basic complexity results regarding DS, where this first one will be stated without proof.

Lemma A.1.2.
- *(i) "$x \in \mathcal{S}(y)$" is Δ_1^{DS};*
- *(ii) "$x = \mathcal{S}(y)$" is Δ_1^{DS};*
- *(iii) "$x = y + z$" is Δ_1^{DS}.*

Proof. See Corollaries 8.11, 8.13 and 8.16 in [Mat04], where the results follow after a thorough analysis of DS. ∎

Lemma A.1.3. *"$x = y \times z$" is Δ_1^{DS}.*

Proof. We have that "$x = y \times z$" is logically equivalent to the formula

$$x \subseteq \mathcal{S}(\mathcal{S}(y \cup z)) \wedge (\forall w \in x)(\mathsf{isOrdPair}(w) \wedge \mathrm{fst}\, w \in y \wedge \mathrm{snd}\, w \in z) \wedge$$
$$(\forall w \in y)(\forall w' \in z)(\langle w, w' \rangle \in x),$$

which is Δ_1^{DS} by Lemma A.1.2(i). ■

DS also has a recursion principle:

Lemma A.1.4 (Finite Δ_0-recursion). *(DS) Let $G : V \to V$ be a Δ_0 class function and $n < \omega$. Then there is a unique function F with $\operatorname{dom} F = n$ and $F(k) = G(F \restriction k)$.*

Proof. "Existence": As $G(x)$ is a set for all sets x, any finite collection $\{G(x_1), \ldots, G(x_n)\}$ is a set by Pairing. Thus $F \subseteq n \times X$ for some finite set X, where $n \times X$ is a set by Lemma A.1.3. As both n and X are Δ_0-definable, F is a set by Δ_0-Comprehension.

"Uniqueness": Let F' be another such function. As $F \approx n \approx F'$, $F \approx F'$. But $F(0) = G(0) = F'(0)$, $F(1) = G(\{\langle 0, G(0)\rangle\}) = F'(1)$ and so on, finitely many times. Hence by Extensionality, $F = F'$. ■

A.2 Analysis of sat

Starting from scratch, we slowly build up our arsenal of $\mathscr{L}$-formulas. Recall the correspondence between $\mathscr{L}$ and our regular language of set theory with constants for each set, denoted $\mathcal{L}_\in^+$:

$\mathcal{L}_\in^+$	$\mathscr{L}$
(	0
)	1
v_n	$\langle 2, n \rangle$
$\dot{x}$	$\langle 3, x \rangle$
$\in$	4
$=$	5
$\wedge$	6
$\neg$	7
$\exists$	8

We start with the basics and construct Δ_0 formulas $\mathsf{isVar}(v_0)$, $\mathsf{isConst}(v_0)$, $\mathsf{isPForm}(v_0)$ and $\mathsf{isFinSeq}(v_0)$ stating that v_0 is a variable, constant, primitive formula, sequence and finite sequence, respectively:

$$\begin{aligned}
\mathsf{isVar}(x) &:\equiv \mathsf{isOrdPair}(x) \wedge \mathrm{fst}\, x = 2 \wedge \mathsf{isNat}(\mathrm{snd}\, x) \\
\mathsf{isConst}(x) &:\equiv \mathsf{isOrdPair}(x) \wedge \mathrm{fst}\, x = 3 \\
\mathsf{isSeq}(s) &:\equiv \mathsf{isFct}(s) \wedge (\forall n \in \operatorname{dom} s)\mathsf{isNat}(n) \\
\mathsf{isFinSeq}(s) &:\equiv \mathsf{isSeq}(s) \wedge (\exists n \in \operatorname{dom} s)(\forall m \in \operatorname{dom} s)(m \in n \vee m = n) \\
\mathsf{isPForm}(s) &:\equiv \mathsf{isFinSeq}(s) \wedge \operatorname{dom} s = 5 \wedge s_0 = 0 \wedge (s_1 = 4 \vee s_1 = 5) \wedge \\
&\qquad (\mathsf{isVar}(s_2) \vee \mathsf{isConst}(s_2)) \wedge (\mathsf{isVar}(s_3) \vee \mathsf{isConst}(s_3)) \wedge s_4 = 1
\end{aligned}$$

We then begin building our basic $\mathscr{L}$-formulas:

- $\mathsf{F}_\in(\varphi, x, y)$, saying that φ is of the form $(x \in y)$:
$$\begin{aligned}\mathsf{F}_\in(\varphi, x, y) :\equiv &\mathsf{isFinSeq}(\varphi) \wedge \operatorname{dom}\varphi = 5 \wedge \varphi_0 = 0 \wedge \varphi_1 = 4 \wedge \varphi_2 = x \wedge \\ &\varphi_3 = y \wedge \varphi_4 = 1\end{aligned}$$
- $\mathsf{F}_=(\varphi, x, y)$, saying that φ is of the form $(x = y)$:
$$\begin{aligned}\mathsf{F}_=(\varphi, x, y) :\equiv &\mathsf{isFinSeq}(\varphi) \wedge \operatorname{dom}\varphi = 5 \wedge \varphi_0 = 0 \wedge \varphi_1 = 5 \wedge \varphi_2 = x \wedge \\ &\varphi_3 = y \wedge \varphi_4 = 1\end{aligned}$$
- $\mathsf{F}_\wedge(\varphi, \psi, \chi)$, saying that φ is of the form $(\psi \wedge \chi)$:
$$\begin{aligned}\mathsf{F}_\wedge(\varphi, \psi, \chi) :\equiv &\mathsf{isFinSeq}(\varphi) \wedge \mathsf{isFinSeq}(\psi) \wedge \mathsf{isFinSeq}(\chi) \wedge \operatorname{dom}\varphi = \operatorname{dom}\psi + \operatorname{dom}\chi + 3 \wedge \\ &\varphi_0 = 0 \wedge \varphi_1 = 6 \wedge \varphi(\operatorname{dom}\varphi - 1) = 1 \wedge (\forall i \in \operatorname{dom}\varphi)(\varphi_{i+2} = \psi_i) \wedge \\ &(\forall i \in \operatorname{dom}\chi)(\varphi_{\operatorname{dom}\psi + i + 2} = \chi_i)\end{aligned}$$
- $\mathsf{F}_\neg(\varphi, \psi)$, saying that φ is of the form $(\neg\psi)$:
$$\begin{aligned}\mathsf{F}_\neg(\varphi, \psi) :\equiv &\mathsf{isFinSeq}(\varphi) \wedge \mathsf{isFinSeq}(\psi) \wedge \operatorname{dom}\varphi = \operatorname{dom}\psi + 3 \wedge \varphi_0 = 0 \wedge \varphi_1 = 7 \wedge \\ &\varphi_{\operatorname{dom}\varphi - 1} = 1 \wedge (\forall i \in \operatorname{dom}\psi)(\varphi_{i+2} = \psi_i)\end{aligned}$$
- $\mathsf{F}_\exists(\varphi, x, \psi)$, saying that φ is of the form $(\exists x\psi)$:
$$\begin{aligned}\mathsf{F}_\exists(\varphi, x, \psi) :\equiv &\mathsf{isFinSeq}(\varphi) \wedge \mathsf{isFinSeq}(\psi) \wedge \operatorname{dom}\varphi = \operatorname{dom}\psi + 4 \wedge \varphi_0 = 0 \wedge \varphi_1 = 8 \wedge \\ &\varphi_2 = x \wedge \varphi_{\operatorname{dom}\varphi - 1} = 1 \wedge (\forall i \in \operatorname{dom}\psi)(\varphi_{i+1} = \psi_i).\end{aligned}$$

These are all Δ_0 except $\mathsf{F}_\wedge(\varphi, \psi, \chi)$, which will be Δ_1^{DS} due to its use of addition.

Lemma A.2.1. *$\mathsf{F}_\wedge(\varphi, \psi, \chi)$ is Δ_1^{DS}.*

Proof. The troublesome clause here is "$\operatorname{dom}\varphi = \operatorname{dom}\psi + \operatorname{dom}\chi + 3$", as both $\operatorname{dom}\psi$ and $\operatorname{dom}\chi$ are arbitrary. But as this clause is Δ_1^{DS} by Lemma A.1.2, the result follows. ■

Now, to be able to construct our internalized formulas in $\mathscr{L}$, we will need two formulas. The first one will be $\mathsf{build}(\varphi, \psi)$, expressing that φ is "built" from the finite sequence of formulas $\psi_0, \ldots, \psi_n$, where by "built" we mean that $\psi_n = \varphi$ and every ψ_i is either atomic or is constructed from one of two of the ψ_j's by means of a logical connective or quantifier. It is defined thus, and is clearly Δ_1^{DS} due to its use of $\mathsf{F}_\wedge$:
$$\begin{aligned}\mathsf{build}(\varphi, \psi) :\equiv &\mathsf{isFinSeq}(\psi) \wedge \psi_{\operatorname{dom}\psi - 1} = \varphi \wedge (\forall i \in \operatorname{dom}\psi)[\mathsf{isPForm}(\psi_i) \vee \\ &(\exists j, k \in i)\mathsf{F}_\wedge(\psi_i, \psi_j, \psi_k) \vee (\exists j \in i)\mathsf{F}_\neg(\psi_i, \psi_j) \vee \\ &(\exists j \in i)(\exists u \in \operatorname{ran}\varphi)(\mathsf{isVar}(u) \wedge \mathsf{F}_\exists(\psi_i, u, \psi_j))].\end{aligned}$$

Now we can construct the formula $\mathsf{isForm}(\varphi) :\equiv \exists f \mathsf{build}(\varphi, f)$, stating that φ is an $\mathscr{L}$-formula. To prove the complexity of isForm, we first need to construct a formula $\mathsf{isSeqSet}(u, a, n)$ saying "$u = {}^{<n}a$", where ${}^{<n}a := \bigcup_{k<n} {}^k a$. We do this by the following formula:
$$\begin{aligned}\mathsf{isSeqSet}(u, a, n) :\equiv &\exists s[\mathsf{isFinSeq}(s) \wedge \mathsf{isNat}(n) \wedge \operatorname{dom}s = n \wedge u = \bigcup \operatorname{ran}s \wedge \\ &(\forall i \in \operatorname{dom}s)(\forall x \in s_i)(\mathsf{isFinSeq}(x) \wedge \operatorname{dom}x = i \wedge (\forall j \in i)(x_j \in a)) \wedge \\ &(\forall i \in \operatorname{dom}s)(\forall j \in i)(\forall x \in s_j)(\forall p \in a)(i = j + 1 \to x \cup \{\langle i, p\rangle\} \in s_i)].\end{aligned}$$

Lemma A.2.2. DS $\vdash \forall x(\forall n < \omega)\exists y(y = {}^{<n}x)$.

Proof. Given a set x and $n < \omega$, ${}^{<n}x \subseteq \mathcal{S}(\omega \times x)$, since $\mathcal{S}(\omega \times x)$ is a set by Lemma A.1.3. Thus by Δ_0 comprehension we get that

$$^{<n}x = \{f \in \mathcal{S}(\omega \times x) \mid \mathsf{isFct}[f] \wedge (\exists m < n)(\operatorname{dom} f = m \wedge (\forall k < m)(f(k) \in x))\},$$

so DS proves that it is a set, due to the formula clearly being Δ_0. ■

Lemma A.2.3. $\mathsf{isSeqSet}(u, a, n)$ *is* Δ_1^{DS}.

Proof. Clearly $\mathsf{isSeqSet}(u, a, n)$ is Σ_1. We have that DS $\vdash \forall a(\forall n \in \omega)\exists u(\mathsf{isSeqSet}(u, a, n))$ by Lemma A.2.2. But we furthermore have that DS proves that ${}^{<n}a$ is unique by Lemma A.1.4, so

$$\mathrm{DS} \vdash \mathsf{isSeqSet}(u, a, n) \leftrightarrow (\mathsf{isNat}(n) \wedge \forall z(\mathsf{isSeqSet}(z, a, n) \rightarrow z = u)),$$

which proves that $\mathsf{isSeqSet}(u, a, n)$ is Π_1^{DS}; thus we have the result. ■

Lemma A.2.4. $\mathsf{isForm}(\varphi)$ *is* Δ_1^{DS}.

Proof. $\mathsf{isForm}(\varphi)$ is clearly Σ_1. The f in $\exists f \mathsf{build}(\varphi, f)$ is an element of the collection

$$A(\varphi) :={}^{<\operatorname{dom}\varphi+1}\left({}^{<\operatorname{dom}\varphi+1}\operatorname{ran}\varphi\right),$$

which is a set by Lemma A.2.2, so f exists. Furthermore we clearly have DS $\vdash \forall x\exists y[y = A(x)]$, so

$$\begin{aligned}\mathrm{DS} \vdash \mathsf{isForm}(\varphi) \leftrightarrow &\mathsf{isFinSeq}(\varphi) \wedge \forall u\forall v((\mathsf{isSeqSet}(u, \operatorname{ran}\varphi, \operatorname{dom}\varphi + 1) \wedge \\ &\mathsf{isSeqSet}(v, u, \operatorname{dom}\varphi + 1)) \rightarrow (\exists f \in v)\mathsf{build}(\varphi, f)),\end{aligned}$$

and we thus have that $\mathsf{isForm}(\varphi)$ is Π_1^{DS}, making it Δ_1^{DS}. ■

For future reference, we construct a formula where the constants is bounded by a given set as $\mathsf{isBConst}(x, u) :\equiv \mathsf{isConst}(x) \wedge \operatorname{fst} x \in u$, and by replacing each instance of $\mathsf{isConst}(x)$ with $\mathsf{isBConst}(x, u)$ in the formulas $\mathsf{isPForm}(\varphi)$ and $\mathsf{isForm}(\varphi)$, we obtain the formulas $\mathsf{isBPForm}(\varphi, u)$ and $\mathsf{isBForm}(\varphi, u)$. Then clearly $\mathsf{isBForm}(\varphi, u)$ iff φ is a formula of $\mathscr{L}_u$.

Now, we need to be able to treat free variables of formulas as well as substitutions in formulas. We start with constructing a formula $\mathsf{isFree}(\varphi, x)$, stating that x is the set of free variables occurring in the $\mathscr{L}$-formula φ. We do this by "capturing" the free variables in each of the stages occurring in $\mathsf{build}(\varphi, \psi)$:

$$\begin{aligned}\mathsf{isFree}(\varphi, x) :\equiv &\exists\psi\exists f[\mathsf{build}(\varphi, \psi) \wedge \mathsf{isFinSeq}(f) \wedge \operatorname{dom} f = \operatorname{dom}\psi \wedge x = f_{\operatorname{dom} f-1} \wedge \\ &(\forall i \in \operatorname{dom} f)[(\exists j, k \in i)[\mathsf{F}_\wedge(\psi_i, \psi_j, \psi_k) \wedge f_i = f_j \cup f_k] \vee \\ &(\exists j \in i)[\mathsf{F}_\neg(\psi_i, \psi_j) \wedge f_i = f_j] \vee \\ &(\exists j \in i)(\exists u \in \operatorname{ran}\varphi)[\mathsf{isVar}(u) \wedge \mathsf{F}_\exists(\psi_i, u, \psi_j) \wedge (f_i = f_j \backslash \{u\})] \vee \\ &[\mathsf{isPForm}(\psi_i) \wedge [[\mathsf{isVar}((\psi_i)_2) \wedge \mathsf{isVar}((\psi_i)_3) \wedge f_i = \{(\psi_i)_2, (\psi_i)_3\}] \vee \\ &[\mathsf{isVar}((\psi_i)_2) \wedge \mathsf{isConst}((\psi_i)_3) \wedge f_i = \{(\psi_i)_2\}] \vee \\ &[\mathsf{isConst}((\psi_i)_2) \wedge \mathsf{isVar}((\psi_i)_3) \wedge f_i = \{(\psi_i)_3\}] \vee \\ &[\mathsf{isConst}((\psi_i)_2) \wedge \mathsf{isConst}((\psi_i)_3) \wedge f_i = \emptyset]]]].\end{aligned}$$

Lemma A.2.5. *$\mathsf{isFree}(\varphi, x)$ is Δ_1^{DS}.*

Proof. Clearly it is Σ_1. As the set x is furthermore unique by Lemma A.1.4, we have that

$$\mathrm{DS} \vdash \mathsf{isFree}(\varphi, x) \leftrightarrow [\mathsf{isForm}(\varphi) \wedge \forall z(\mathsf{isFree}(\varphi, z) \to z = x)],$$

so $\mathsf{isFree}(\varphi, x)$ is Π_1^{DS}, and thus Δ_1^{DS}. ■

Now, for the substitution we construct a formula $\mathsf{sub}(\varphi', \varphi, v, t)$, saying that φ' is the result of substituting each occurrence of the variable v in φ with the constant t. We do this again by fixing a "build-sequence", and substituting every occurrence of v with t at each step. If we reach a quantifier, then we need to cancel any substitutions made within its scope. To increase legibility, we split the formula up and treat the case where φ is atomic separately, resulting in the formula $\mathsf{subAtom}(\varphi', \varphi, v, t)$:

$$\begin{aligned}
\mathsf{subAtom}(\varphi', \varphi, v, t) :\equiv\, & \mathsf{isPForm}(\varphi') \wedge \mathsf{isPForm}(\varphi) \wedge \mathsf{isVar}(v) \wedge \mathsf{isConst}(t) \wedge [(\mathsf{F}_=(\varphi, \varphi_2, \varphi_3) \wedge \\
& [(\varphi_2 \neq v \wedge \varphi_3 \neq v \wedge \varphi' = \varphi) \vee (\varphi_2 = v \wedge \varphi_3 \neq v \wedge \mathsf{F}_=(\varphi', t, \varphi_3)) \vee \\
& (\varphi_2 \neq v \wedge \varphi_3 = v \wedge \mathsf{F}_=(\varphi', \varphi_2, t)) \vee (\varphi_2 = v \wedge \varphi_3 = v \wedge \mathsf{F}_=(\varphi', t, t))]) \vee \\
& (\mathsf{F}_\in(\varphi, \varphi_2, \varphi_3) \wedge [(\varphi_2 \neq v \wedge \varphi_3 \neq v \wedge \varphi' = \varphi) \vee \\
& (\varphi_2 = v \wedge \varphi_3 \neq v \wedge \mathsf{F}_\in(\varphi', t, \varphi_3)) \vee (\varphi_2 \neq v \wedge \varphi_3 = v \wedge \mathsf{F}_\in(\varphi', \varphi_2, t)) \vee \\
& (\varphi_2 = v \wedge \varphi_3 = v \wedge \mathsf{F}_\in(\varphi', t, t))])]
\end{aligned}$$

We can now define the substitution formula $\mathsf{sub}(\varphi', \varphi, v, t)$ as follows:

$$\begin{aligned}
\mathsf{sub}(\varphi', \varphi, v, t) :\equiv\, & \mathsf{isForm}(\varphi') \wedge \mathsf{isForm}(\varphi) \wedge \mathsf{isVar}(v) \wedge \mathsf{isConst}(t) \wedge \exists\psi\exists\chi[\mathsf{build}(\varphi, \psi) \wedge \\
& \mathsf{isFinSeq}(\chi) \wedge \operatorname{dom}\chi = \operatorname{dom}\psi \wedge \chi_{\operatorname{dom}\chi - 1} = \varphi' \wedge \\
& (\forall i \in \operatorname{dom}\psi)[(\exists j, k \in i)(\mathsf{F}_\wedge(\psi_i, \psi_j, \psi_k) \wedge \mathsf{F}_\wedge(\chi_i, \chi_j, \chi_k)) \vee \\
& (\exists j \in i)(\mathsf{F}_\neg(\psi_i, \psi_j) \wedge \mathsf{F}_\neg(\chi_i, \chi_j)) \vee \\
& (\exists j \in i)(\exists u \in \operatorname{ran}\varphi)(\mathsf{isVar}(u) \wedge u \neq v \wedge \mathsf{F}_\exists(\psi_i, u, \psi_j) \wedge \mathsf{F}_\exists(\chi_i, u, \chi_j)) \vee \\
& (\exists j \in i)(\mathsf{F}_\exists(\psi_i, v, \psi_j) \wedge \chi_i = \psi_i) \vee \\
& \mathsf{subAtom}(\chi_i, \psi_i, v, t)]].
\end{aligned}$$

Here the clause "$\chi_i = \psi_i$" on the penultimate line expresses the previously mentioned idea to cancel any substitutions made within a quantifiers scope.

Lemma A.2.6. *$\mathsf{sub}(\varphi', \varphi, v, t)$ is Δ_1^{DS}.*

Proof. As the substituted formula is unique by Lemma A.1.4, we have that

$$\mathrm{DS} \vdash \mathsf{sub}(\varphi', \varphi, v, t) \leftrightarrow \mathsf{isForm}(\varphi) \wedge \mathsf{isVar}(v) \wedge \mathsf{isConst}(t) \wedge \forall\psi[\mathsf{sub}(\psi, \varphi, v, t) \to \psi = \varphi'],$$

granting a Π_1^{DS} formula, and the lemma follows. ■

We are finally ready to define the satisfaction relation $\mathsf{sat}(u, \varphi)$, which we want to express that $\varphi \in \mathscr{L}_u$ and φ is true within u, with each constant symbol $\dot{x}$ replaced by the corresponding set x. The idea behind the construction is to define two functions f, g with domain ω, such that $f(0)$ is the set of all atomic formulas of $\mathscr{L}_u$, and $f(i+1)$ is the set of $\mathscr{L}_u$-formulas, constructed from the formulas of $f(i)$ via the use of a single logical connective or quantifier. The function g is then defined as the function taking all formulas from f which have no free variables and which are true in u. The reason for even having f is to treat the negation case. Even though f, g has an infinite domain, what we will use in practice is $f \restriction n, g \restriction n$ for some sufficiently large $n < \omega$, which is sufficient as our formulas are finite. We start again by considering the atomic case separately:

$$\mathsf{satAtom}(u, \varphi) :\equiv (\exists x, y \in u)(x \in y \wedge \mathsf{F}_{\in}(\varphi, \langle 3, x\rangle, \langle 3, y\rangle)) \vee (\exists x \in u)\mathsf{F}_{=}(\varphi, \langle 3, x\rangle, \langle 3, x\rangle).$$

Lemma A.2.7. $\mathsf{satAtom}(u, \varphi)$ *is* Δ_0.

Proof. The only troublesome clauses are the ones such as $(\exists x, y \in u)\mathsf{F}_{\in}(\varphi, \langle 3, x\rangle, \langle 3, y\rangle)$ due to presence of the pairs, but observe that we can write out $\mathsf{satAtom}(u, \varphi)$ as

$$\begin{aligned}&(\exists x, y \in u)(\exists x', y' \in \operatorname{ran}\varphi)(x \in y \wedge x' = \langle 3, x\rangle \wedge y' = \langle 3, y\rangle \wedge \mathsf{F}_{\in}(\varphi, x', y'))\vee\\ &(\exists x \in u)(\exists x' \in \operatorname{ran}\varphi)(x' = \langle 3, x\rangle \wedge \mathsf{F}_{=}(\varphi, x', x')),\end{aligned}$$

from which it is easily seen. ∎

We can now write out a formula, $\mathsf{almostSat}(u, \varphi)$, which does what we want it to do, but we are going to do some changes to make sure it is Δ_1^{DS}, which, after all, is our prime goal in this section. The formula is given thus:

$$\begin{aligned}&\mathsf{almostSat}(u, \varphi) :\equiv\\ &\quad u \neq \emptyset \wedge \mathsf{isBForm}(\varphi, u) \wedge \exists f \exists g[\mathsf{isFinSeq}(f) \wedge \mathsf{isFinSeq}(g) \wedge \operatorname{dom} f = \operatorname{dom} g \wedge \varphi \in g_{\operatorname{dom} g - 1}\wedge\\ &\quad \forall\psi(\psi \in f_0 \leftrightarrow \mathsf{isBPForm}(\psi, u)) \wedge \forall\psi(\psi \in g_0 \leftrightarrow \mathsf{satAtom}(\psi, u))\wedge\\ &\quad (\forall j \in \operatorname{dom} f)(\forall i \in j)(\forall\psi)[\psi \in f_{i+1} \leftrightarrow (\psi \in f_i \vee (\exists\chi, \chi' \in f_i)\mathsf{F}_{\wedge}(\psi, \chi, \chi')\vee\\ &\quad (\exists\chi \in f_i)\mathsf{F}_{\neg}(\psi, \chi) \vee (\exists\chi \in f_i)(\exists v \in \operatorname{ran}\psi)(\mathsf{isVar}(v) \wedge \mathsf{F}_{\exists}(\psi, v, \chi))]\wedge\\ &\quad (\forall j \in \operatorname{dom} g)(\forall i \in j)(\forall\psi)[\psi \in g_{i+1} \leftrightarrow (\psi \in g_i \vee (\exists\chi, \chi' \in g_i)\mathsf{F}_{\wedge}(\psi, \chi, \chi')\vee\\ &\quad (\exists\chi \in f_i)(\chi \notin g_i \wedge \mathsf{F}_{\neg}(\psi, \chi)) \vee (\exists\chi \in f_i)(\exists v \in \operatorname{ran}\psi))(\exists x \in u)(\exists\chi' \in g_i)[\mathsf{isVar}(v)\wedge\\ &\quad \mathsf{F}_{\exists}(\psi, v, \chi) \wedge \mathsf{sub}(\chi', \chi, v, \langle 3, x\rangle)]]].\end{aligned}$$

It is a bit of a mouthful, but by near inspection this in fact captures the aforementioned idea. Now, the problem with this formula is the unbounded quantifiers $\forall\psi$, $\forall f$, $\forall g$ as well as the ones occurring in $\mathsf{sub}(\chi', \chi, v, \langle 3, x\rangle)$. Define $X := 9 \cup \{v_i \mid i < \omega\} \cup \{\langle 3, x\rangle \mid x \in u\}$ and

$$w(u, \varphi) := {}^{\operatorname{dom}\varphi+1}X \cup {}^{\operatorname{dom}\varphi+1}\left({}^{\operatorname{dom}\varphi+1}X\right).$$

It is easily seen that if we bound every unbounded quantifier in $\mathsf{almostSat}(u, \varphi)$ by $w(u, \varphi)$, we get a logically equivalent Δ_0 formula $\mathsf{bAlmostSat}(u, \varphi, w)$. This requires DS to prove that $w(u, \varphi)$ is a set for all u and φ though, but this follows from Lemma A.2.2. We can now define $\mathsf{sat}(u, \varphi)$ as

$$\begin{aligned}\mathsf{sat}(u, \varphi) :\equiv\ &\exists w \exists x \exists y \exists a \exists b \exists t[a = \{\langle 3, x\rangle \mid x \in u\} \wedge b = \{v_i \mid i \in t\}\wedge\\ &[(\mathsf{on}(t) \wedge \mathsf{isLimit}(t) \wedge (\forall i \in t)((\exists j \in i)(i = j + 1) \vee i = \emptyset)]\wedge\\ &\mathsf{isSeqSet}(x, 9 \cup a \cup b, \operatorname{dom}\varphi + 1) \wedge \mathsf{isSeqSet}(y, x, \operatorname{dom}\varphi + 1)\wedge\\ &w = x \cup y \wedge \mathsf{bAlmostSat}(u, \varphi, w)].\end{aligned}$$

Lemma A.2.8. $\mathsf{sat}(u, \varphi)$ *is* Δ_1^{DS} *and* Σ_1.

Proof. It is clearly Σ_1. As we furthermore have that

$$\mathrm{DS} \vdash \mathsf{sat}(u, \varphi) \leftrightarrow \mathsf{isBForm}(\varphi, u) \wedge \mathsf{isFree}(\varphi, \emptyset) \wedge \forall\psi(\neg[\mathsf{F}_{\neg}(\psi, \varphi) \wedge \mathsf{sat}(u, \psi)]),$$

it follows that $\mathsf{sat}(u, \varphi)$ is Π_1^{DS}, and thus Δ_1^{DS}, since it is also Σ_1^{DS}. ■

We see that this definition of sat is the "correct" one, as it coincides with our usual metamathematical satisfaction relation $\models$.

Lemma A.2.9. *Let $\varphi(v_1, \ldots, v_n)$ be a formula. Then*

$$\mathrm{ZF} \vdash \forall M (\forall x_1 \in M) \cdots (\forall x_n \in M)(M \models \varphi[\vec{x}] \leftrightarrow \mathsf{sat}[M, \ulcorner \varphi(\dot{x}_1, \ldots \dot{x}_n)\urcorner]).$$

Proof. Induction on the complexity of φ. *Atomic step:* " $\Leftarrow$ ": Directly from the definition of sat. " $\Rightarrow$ ": If φ is $(v_0 \in v_1)$ and $M \models (x \in y)$ then $\ulcorner \varphi(\dot{x}, \dot{y})\urcorner = 04\langle 3, x\rangle\langle 3, y\rangle 1$, so $\mathsf{F}_\in[\ulcorner\varphi\urcorner, \langle 3, x\rangle, \langle 3, y\rangle]$ holds; hence $\mathsf{sat}[M, \ulcorner\varphi\urcorner]$ holds. If φ is $(v_0 = v_1)$ and $M \models (x = y)$ then $\ulcorner \varphi(\dot{x}, \dot{y})\urcorner = 05\langle 3, x\rangle\langle 3, y\rangle 1 = 05\langle 3, x\rangle\langle 3, x\rangle 1$ and hence $\mathsf{F}_=[\ulcorner\varphi\urcorner, \langle x, 3\rangle, \langle x, 3\rangle]$ holds; hence $\mathsf{sat}[M, \ulcorner\varphi\urcorner]$ holds.

Sentential step: " $\Leftarrow$ ": Since $\mathsf{sat}[M, \ulcorner\varphi\urcorner]$ holds, there is some $i < \omega$ such that $\ulcorner\varphi\urcorner \in g(i+1)$ by definition of sat. If φ is $(\psi \wedge \chi)$, then $\mathsf{F}_\wedge[\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner, \ulcorner\chi\urcorner]$ holds, so $\ulcorner\psi\urcorner, \ulcorner\chi\urcorner \in g(i)$. By induction hypothesis, $M \models \psi[\vec{x}]$ and $M \models \chi[\vec{x}]$, so $M \models \varphi[\vec{x}]$ by definition of $\models$. If φ is $(\neg\psi)$, then $\mathsf{F}_\neg[\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner]$ holds, meaning $\ulcorner\psi\urcorner \notin g(i)$ and thus by induction hypothesis, $M \not\models \psi[\vec{x}]$ and thus $M \models \varphi[\vec{x}]$.

" $\Rightarrow$ ": If φ is $(\psi \wedge \chi)$, then $M \models \psi[\vec{x}]$ and $M \models \chi[\vec{x}]$ by definition of $\models$, so by induction hypothesis $\mathsf{sat}[M, \ulcorner\psi\urcorner]$ and $\mathsf{sat}[M, \ulcorner\chi\urcorner]$ holds; thus there is some $i, j < \omega$ with $\ulcorner\psi\urcorner \in g(i)$ and $\ulcorner\chi\urcorner \in g(j)$ and thus $\ulcorner\varphi\urcorner \in g(\max\{i, j\} + 1)$, implying $\mathsf{sat}[M, \ulcorner\varphi\urcorner]$. If φ is $(\neg\psi)$ then $M \not\models \psi[\vec{x}]$, so $\ulcorner\psi\urcorner \in f(i)$ and $\ulcorner\psi\urcorner \notin g(i)$ for some $i < \omega$ by induction assumption; thus $\mathsf{sat}[M, \ulcorner\varphi\urcorner]$ again.

Quantifier step: Assume φ is $(\exists v_0 \psi)$. " $\Leftarrow$ ": There is some $y \in M$ such that $\mathsf{sat}[M, \ulcorner\psi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n)\urcorner]$ holds, which by induction hypothesis means $M \models \psi[y, \vec{x}]$; thus $M \models \varphi[\vec{x}]$ by definition of $\models$. " $\Rightarrow$ ": By definition of $\models$, $M \models \psi[y, \vec{x}]$ for some $y \in M$. Then $\mathsf{sat}[M, \ulcorner\psi\urcorner]$ holds by induction assumption, so there is some $i < \omega$ with $\ulcorner\psi\urcorner \in g(i)$. Then by definition, $\mathsf{sat}[M, \ulcorner\varphi\urcorner]$ holds. ■

A.3 Analysis of $\mathrm{Def}(x)$ and $v = L_\alpha$

Theorem A.3.1. *Let $\delta \in \mathrm{On}$ be a limit ordinal satisfying $\delta > \omega$. Then $L_\delta \models \mathrm{DS}$.*

Proof. L_δ is transitive by Lemma 2.1.5, and Extensionality and Π_1-Foundation are satisfied by transitivity and Π_1 downwards absoluteness. We show that L_δ satisfies conditions (iii)-(vii). (iii): Let $x, y \in L_\delta$. Find $\alpha < \delta$ such that $x, y \in L_\alpha$. Then

$$\{x, y\} = \{z \in L_\alpha \mid L_\alpha \models z = x \vee z = y\} \in L_{\alpha+1} \subseteq L_\delta.$$

(iv): Let $x \in L_\delta$ and find $\alpha < \delta$ such that $x \in L_\alpha$. As L_α is transitive, we have $\bigcup x \subseteq L_\alpha$ and furthermore

$$\bigcup x = \{z \in L_\alpha \mid L_\alpha \models (\exists u \in x)(z \in u)\} \in L_{\alpha+1} \subseteq L_\delta.$$

(v): $\omega \in L_\delta$ since $\delta > \omega$.
(vi): Let $x \in L_\delta$, find $\alpha < \delta$ such that $x \in L_\alpha$. Since every finite subset of L_α is definable with parameters in L_α and thus belong to $L_{\alpha+1}$, we have that

$$\mathcal{S}(x) = \{z \in L_{\alpha+1} \mid L_{\alpha+1} \models \mathsf{isFin}[z] \wedge (\forall y \in z)(y \in x)\} \in L_{\alpha+2} \subseteq L_\delta.$$

(vii): Let $R \subseteq L_\delta$ be Δ_0-definable with parameters in L_δ, witnessed by the Δ_0 formula $\varphi(\vec{v})$ and parameters $\vec{a} \in L_\delta$. Let $u \in L_\delta$; we need to show that $R \cap u \in L_\delta$. Pick $\alpha < \delta$ such that $u, \vec{a} \in L_\alpha$. By transitivity $u \subseteq L_\alpha$, so

$$R \cap u = \{x \mid x \in u \wedge x \in R\} = \{x \in L_\alpha \mid x \in u \wedge x \in R\}.$$

As φ is Δ_0, we have that $L_\delta \models \varphi[\vec{a}] \leftrightarrow L_\alpha \models \varphi[\vec{a}]$, as $\vec{a} \in L_\alpha$. Thus

$$R \cap u = \{x \in L_\alpha \mid L_\alpha \models x \in u \wedge \varphi[x, \vec{a}]\} \in L_{\alpha+1} \subseteq L_\delta.$$

■

Notice that as L_δ for limit $\delta > \omega$ is a model of DS, we have that every Δ_1^{DS}-formula is absolute for such L_δ by Lemma 1.3.7. We want to construct a formula expressing that $v = \mathrm{Def}(u)$. Ideally this should be Δ_1^{DS} from which we could conclude L_δ-absoluteness for limit $\delta > \omega$, but DS is simply too weak a theory to allow such a construction. Instead we will construct a Δ_1^{ZF} formula, and we will use the Δ_1^{DS} results that we have proved so far (in conjunction with the above Theorem A.3.1) to show that it actually *is* the case that it is L_δ-absolute for limit $\delta > \omega$. We start off with a first attempt.

$$\begin{aligned}\mathsf{almostDef}(v, u) :\equiv & (\forall x \in v)(\exists \varphi)[\mathsf{isBForm}(\varphi, u) \wedge \mathsf{isFree}(\varphi, \{v_0\}) \wedge x \subseteq u \wedge \\ & (\forall z \in u)(z \in x \leftrightarrow \exists \psi(\mathsf{sub}(\psi, \varphi, v_0, \dot{z}) \wedge \mathsf{almostSat}(u, \psi)))] \wedge \\ & \forall \varphi[(\mathsf{isBForm}(\varphi, u) \wedge \mathsf{isFree}(\varphi, \{v_0\})) \rightarrow \\ & (\exists x \in v)[x \subseteq u \wedge (\forall z \in u)(z \in x \leftrightarrow \exists \psi(\mathsf{sub}(\psi, \varphi, v_0, \dot{z}) \wedge \mathsf{almostSat}(u, \psi)))]]\end{aligned}$$

The formula is equivalent to $v = \mathrm{Def}(u)$, as can be easily seen. But it is clearly not Δ_1^{ZF}, so we proceed as we have done before, by trying to bound all the quantifiers in the formula. The reason for using $\mathsf{almostSat}(u, \psi)$ over $\mathsf{sat}(u, \psi)$ was to make it easier to construct such a bound - as the two formulas are equivalent, the meaning remains the same. We would like to use the bound

$$\mathrm{Bound}(u) := {}^{<\omega}f(u) \cup {}^{<\omega}({}^{<\omega}f(u)) \cup {}^{<\omega}\mathcal{S}(\{v_i \mid i < \omega\}),$$

where $f(u) := 9 \cup \{v_i \mid i < \omega\} \cup \{\langle 3, x\rangle \mid x \in u\}$. But to be able to use this bound, we need to make sure that $u = \mathrm{Bound}(v)$ is Δ_1^{ZF}. We start by constructing the formula $y = {}^{<\omega}x$.

$$\begin{aligned} y = {}^{<\omega}x :\equiv & \exists f[\mathsf{isFct}(f) \wedge \mathrm{dom}\, f = \omega \wedge f(0) = \{\emptyset\} \wedge y = \bigcup \mathrm{ran}\, f \wedge \\ & (\forall n < \omega)(\forall s \in f(n+1))(\exists t \in f(n))(\exists a \in x)(s = t \cup \{\langle n, a\rangle\}) \wedge \\ & (\forall n < \omega)(\forall s \in f(n))(\forall a \in x)(\exists t \in f(n+1))(t = s \cup \{\langle n, a\rangle\})]\end{aligned}$$

Lemma A.3.2. *$y = {}^{<\omega}x$ is Δ_1^{ZF} and L_δ-absolute for limit $\delta > \omega$.*

Proof. The potential trouble comes from the use of ω. But by first appending the clause

$$\exists w[\emptyset \in w \wedge \mathsf{on}(w) \wedge (\forall u \in w)\mathsf{isNat}(u) \wedge (\forall u \in w)(\exists v \in w)(u \in v)]$$

to the formula, and thereafter replacing each instance of ω with w, we arrive at a Σ_1 formula. By $\in$-recursion, we can construct the unique recursive function f appearing in the formula $y = {}^{<\omega}x$, which means that we have

$$\mathrm{ZF} \vdash y = {}^{<\omega}x \leftrightarrow \forall z(z = {}^{<\omega}x \rightarrow z = y),$$

so $y = {}^{<\omega}x$ is Δ_1^{ZF}. Now, to show that it is L_δ-absolute we only need to show downwards absoluteness, as upwards absoluteness follows from the formula being Σ_1. This is to say that assuming $y = {}^{<\omega}x$ for $x \in L_\delta$, we need to show that $y \in L_\delta$ and $f \in L_\delta$ as well, where f is the recursive function defining y. Pick $\alpha < \delta$ such that $x \in L_\alpha$. For any $a \in x$ we have $\langle n, a\rangle = \{\{n\}, \{n, a\}\} \in L_{\alpha+1}$ for all $n < \omega$, so every finite sequence from x is in $L_{\alpha+2}$, making ${}^{<\omega}x \in L_{\alpha+3} \subseteq L_\delta$. Since we furthermore have that

$$y = {}^{<\omega}x \Leftrightarrow (\exists n < \omega)(s \in {}^{<n}x),$$

then $L_\delta \models y = {}^{<\omega}x$ as well, due to $z = {}^{<n}x$ being absolute for L_δ cf. Theorem A.3.1 since it is Δ_1^{DS} by Lemma A.2.3. For f, we have that

$$f = \{\langle s, n\rangle \mid s = {}^{n}x \wedge n < \omega\},$$

where it is clear that ${}^{n}x \in L_{\alpha+3}$, making $\langle {}^{n}x, n\rangle \in L_{\alpha+5}$ for all $n < \omega$, concluding $f \in L_{\alpha+6} \subseteq L_\delta$. Note that $y = {}^{n}x \Leftrightarrow y = \{z \in {}^{<n+1}x \mid L_\delta \models \operatorname{dom} z = n\}$, so we again have that $L_\delta \models f = \{\langle s, n\rangle \mid s = {}^{n}x \wedge n < \omega\}$ by Theorem A.3.1, since ${}^{<n}x = y$ is Δ_1^{DS} by Lemma A.2.3. ■

We also need the formula $y = \mathcal{S}(x)$, given by

$$y = \mathcal{S}(x) :\equiv \exists z(z = {}^{<\omega}x \wedge y = \{\operatorname{ran} u \mid u \in z\}).$$

Lemma A.3.3. *$y = \mathcal{S}(x)$ is Δ_1^{ZF} and L_δ-absolute for limit $\delta > \omega$.*

Proof. It is clearly Σ_1 and like in Lemma A.3.2 by $\in$-recursion we get $\mathrm{ZF} \vdash \forall x \exists! y(y = \mathcal{S}(x))$, so it is Δ_1^{ZF}. We showed in Theorem A.3.1 that $\mathcal{S}(x) \in L_\delta$ for $x \in L_\delta$, and by definition of DS and Theorem A.3.1 we have that $y = \mathcal{S}(x) \Leftrightarrow L_\delta \models y = \mathcal{S}(x)$. ■

We can now write down the formula for $w = \mathrm{Bound}(u)$:

$$\begin{aligned} w = \mathrm{Bound}(u) :\equiv &\exists a \exists b \exists c \exists d \exists e \exists f[(\forall z \in d)\mathsf{isVar}(z) \wedge (\forall i < \omega)(v_i \in d) \wedge \\ &(\forall z \in e)\mathsf{isBConst}(z, u) \wedge (\forall z \in u)(\langle 3, z\rangle \in e) \wedge a = {}^{<\omega}(9 \cup d \cup e) \wedge \\ &b = {}^{<\omega}a \wedge f = \mathcal{S}(d) \wedge c = {}^{<\omega}f \wedge w = a \cup b \cup c]. \end{aligned}$$

By letting $\mathsf{bAlmostDef}(w, v, u)$ be the formula gotten by replacing each unbounded quantifier with w, we can define $v = \mathrm{Def}(u)$ as

$$v = \mathrm{Def}(u) :\equiv \exists w[w = \mathrm{Bound}(u) \wedge \mathsf{bAlmostDef}(w, v, u)].$$

By the same procedure, *mutatis mutandis*, as in the previous two lemmas, we achieve our first goal in this section.

Lemma A.3.4. *$y = \mathrm{Def}(x)$ is Δ_1^{ZF} and L_δ-absolute for limit $\delta > \omega$.*

Proof. Clearly $y = \mathrm{Def}(x)$ is Σ_1. As $\mathrm{Def}(x)$ is furthermore unique as it is defined by $\in$-recursion, we have that

$$\mathrm{ZF} \vdash y = \mathrm{Def}(x) \leftrightarrow \forall z(z = \mathrm{Def}(x) \to z = y),$$

making $y = \mathrm{Def}(x)$ Δ_1^{ZF}. Furthermore for $x \in L_\delta$ we can find $\alpha < \delta$ such that $x \in L_\alpha$, meaning that $\mathrm{Def}(x) \in L_{\alpha+2}$. As $\mathrm{Def}(x)$ consists of formulas absolute for L_δ, $\delta > \omega$ ($\mathsf{almostSat}$, isFree, sub and so on), it will also be absolute for such L_δ. ■

Now we continue towards defining the formula $v = L_\alpha$. We start by defining the formula $f = \{\langle\gamma, L_\gamma\rangle \mid \gamma \leq \alpha\}$, which is equivalent to the following.

$$\begin{aligned}\mathsf{almostPred}(f,\alpha) :\equiv &\mathsf{on}(\alpha) \wedge \mathsf{isFct}(f) \wedge \operatorname{dom} f = \alpha + 1 \wedge f(0) = \emptyset \wedge \\ &(\forall\gamma \in \operatorname{dom} f)[((\mathsf{isLimit}(\gamma) \wedge \gamma > 0) \to f(\gamma) = \bigcup\{f(\xi) \mid \xi < \gamma\}) \wedge \\ &(\mathsf{isSucc}(\gamma) \to \mathsf{almostDef}(f(\gamma), f(\gamma - 1)))].\end{aligned}$$

Again we did not use $v = \operatorname{Def}(u)$, as we want to bound this formula. In this case, the bound $\operatorname{Bound}(\bigcup \operatorname{ran} f)$ is sufficient, as only the function values are arguments in $\mathsf{almostDef}$; we thus in the same way as previously obtain the formula $\mathsf{bAlmostPred}(w, f, \alpha)$ and we define

$$f = \{\langle\gamma, L_\gamma\rangle \mid \gamma \leq \alpha\} :\equiv \exists w[w = \operatorname{Bound}(\bigcup \operatorname{ran} f) \wedge \mathsf{bAlmostPred}(w, f, \alpha)].$$

Lemma A.3.5. *$f = \{\langle\gamma, L_\gamma\rangle \mid \gamma \leq \alpha\}$ and L_δ-absolute for limit $\delta > \omega$.*

Proof. By writing out $f(\gamma) = \bigcup\{f(\xi) \mid \xi < \gamma\}$ to

$$(\forall x \in f(\gamma))(\exists\xi \in \gamma)(x \in f(\xi)) \wedge (\forall\xi \in \gamma)(f(\xi) \subseteq f(\gamma)),$$

the formula is clearly seen to be Σ_1. Furthermore we have that

$$\mathrm{ZF} \vdash (\forall\alpha \in \mathrm{On})\exists f(f = \{\langle\gamma, L_\gamma\rangle \mid \gamma \leq \alpha\},$$

as we can construct it by $\in$-recursion. The second part of the lemma is trivial, as it can easily be shown that $\{\langle\gamma, L_\gamma\rangle \mid \gamma \leq \alpha\} \in L_{\alpha+4}$ for all $\alpha > \omega$ and it is built up from formulas which we have shown are Δ_1^{DS} and hence absolute for L_δ, $\delta > \omega$. ■

Now we arrive at the goal of this section: we define

$$v = L_\alpha :\equiv \exists f[f = \{\langle\gamma, L_\gamma\rangle \mid \gamma \leq \alpha\} \wedge x = f(\alpha)],$$

and by essentially the same arguments as in the previous lemmas, we get the following.

Lemma A.3.6. *$v = L_\alpha$ is Δ_1^{ZF}, Σ_1 and L_δ-absolute for limit $\delta > \omega$.* ■

A.4 Analysis of $<_L$

In this section we will show that $<_L$ is definable by a Δ_1 formula as well as being L_δ-absolute for limit $\delta > \omega$. We start off by analysing the formula $x = \ulcorner\varphi_n\urcorner$.

Lemma A.4.1. *$x = \ulcorner\varphi_n\urcorner$ is Δ_1^{ZF} and Σ_1.*

Proof. We start off by defining the lexicographic ordering $<_{\mathrm{lex}}$ on the set $\{\ulcorner\varphi_n\urcorner \mid n < \omega\}$:

$$\begin{aligned}\ulcorner\varphi_k\urcorner <_{\mathrm{lex}} \ulcorner\varphi_n\urcorner :\equiv &\operatorname{dom}\ulcorner\varphi_k\urcorner < \operatorname{dom}\ulcorner\varphi_n\urcorner \vee (\exists m < \omega)(\forall l < n)(\ulcorner\varphi_k\urcorner(l) = \ulcorner\varphi_n\urcorner(l) \wedge \\ &\ulcorner\varphi_k\urcorner(m) <_{\mathrm{rank}} \ulcorner\varphi_n\urcorner(m)),\end{aligned}$$

which is seen to be Δ_1^{ZF} and Σ_1, as rank is Δ_1^{ZF}- and Σ_1-definable cf. Lemma 1.3.8. Now define

$$x = \ulcorner\varphi_n\urcorner :\equiv \mathsf{isBForm}(x, \emptyset) \wedge (\forall k < n)(\ulcorner\varphi_k\urcorner <_{\mathrm{lex}} x) \wedge (\forall m > n)(x <_{\mathrm{lex}} \ulcorner\varphi_m\urcorner),$$

which is then seen to be both Δ_1^{ZF} and Σ_1. ■

We now move on to defining the class functions A, N and P, and in this regard we will use abbreviations such as $x = \{y \in z \mid \mathsf{sat}(z, \ulcorner\varphi(\dot{y})\urcorner)\}$ for their obvious formal counterparts.

Lemma A.4.2. *The following class functions $A : L \to \mathrm{On}$, $N : L \to \omega$, $P : L \to L$ are Δ_1^{ZF}- and Σ_1-definable:*
- $A(x) := \min\{\alpha \in \mathrm{On} \mid x \in L_{\alpha+1}\}$;
- $N(x) := \min\{n < \omega \mid x = \{y \in L_{A(x)} \mid (\exists \vec{t} \in L_{A(x)})(L_{A(x)} \models \varphi_n[y, t_1, \dots, t_n])\}\}$;
- $P(x) := \{\langle t_1, \dots, t_n\rangle \mid \vec{t} \in L_{A(x)} \wedge x = \{y \in L_{A(x)} \mid L_{A(x)} \models \varphi_{N(x)}[y, t_1, \dots, t_n]\}\}$.

Proof. We have that

$$\alpha = A(x) :\equiv \mathsf{on}(\alpha) \wedge x \in L_{\alpha+1} \wedge (\forall \gamma < \alpha)(x \notin L_{\gamma+1}),$$

which is seen to be Δ_1^{ZF} and Σ_1 as well, due to $x = L_\alpha$ being both Δ_1^{ZF} and Σ_1, cf. Lemma A.3.6. For N, we can define it as

$$\begin{aligned} n = N(x) :\equiv &\mathsf{isNat}(n) \wedge x = \bigcup\{\{y \in L_{A(x)} \mid \mathsf{sat}(L_{A(x)}, \ulcorner\varphi_n(\dot{y}, \dot{t}_1, \dots, \dot{t}_m)\urcorner)\} \mid t_1, \dots, t_m \in L_{A(x)}\} \wedge \\ &(\forall k < n)(x \neq \bigcup\{\{y \in L_{A(x)} \mid \mathsf{sat}(L_{A(x)}, \ulcorner\varphi_k(\dot{y}, \dot{t}_1, \dots, \dot{t}_m)\urcorner)\} \mid t_1, \dots, t_m \in L_{A(x)}\}), \end{aligned}$$

which is Δ_1^{ZF} and Σ_1 as well (sat is Δ_1^{ZF} and Σ_1 cf. Lemma A.2.8). Lastly we define P as

$$\begin{aligned} x = P(x) :\equiv &(\forall y \in x)(\exists \vec{t} \in L_{A(x)})(y = \langle t_1, \dots, t_n\rangle \wedge \\ &x = \{z \in L_{A(x)} \mid \mathsf{sat}(L_{A(x)}, \ulcorner\varphi_{N(x)}(\dot{z}, \dot{t}_1, \dots, \dot{t}_n)\urcorner)\} \wedge \\ &(\forall \vec{t} \in L_{A(x)})(x = \{z \in L_{A(x)} \mid \mathsf{sat}(L_{A(x)}, \ulcorner\varphi_{N(x)}(\dot{z}, \dot{t}_1, \dots, \dot{t}_n)\urcorner)\} \\ &\to \langle t_1, \dots, t_n\rangle \in x), \end{aligned}$$

which again is seen to be Δ_1^{ZF} and Σ_1. ■

Lemma A.4.3. *The global well-ordering on L, $<_L$, is Δ_1^{ZF}- and Σ_1-definable.*

Proof. $<_0 = \emptyset$ is clearly Δ_0. For successor ordinals we have that

$$\begin{aligned} x <_{\alpha+1} y :\equiv &A(x) \in A(y) \vee (A(x) = A(y) \wedge N(x) \in N(y)) \vee \\ &(A(x) = A(y) \wedge N(x) = N(y) \wedge \min_{<^*_{A(x)}} P(x) <^*_{A(x)} \min_{<^*_{A(x)}} P(y)), \end{aligned}$$

which is Δ_1^{ZF} and Σ_1 if we can show that the ordering $<^*_\alpha$ is Δ_1^{ZF} and Σ_1, where $<^*_\alpha$ is the lexicographic ordering induced by $<_\alpha$ on the n-tuples $\langle t_1, \dots, t_n\rangle$ with $t_1, \dots, t_n \in L_\alpha$. But we have that

$$\begin{aligned} \langle t_1, \dots, t_n\rangle <^*_\alpha \langle s_1, \dots, s_m\rangle :\equiv &n < m \vee (n = m \wedge (\exists k < n)(\forall l < k)(t_l = s_l) \wedge t_k <_\alpha s_k) \wedge \\ &t_1, \dots, t_n, s_1, \dots, s_m \in L_\alpha), \end{aligned}$$

which is clearly Δ_1^{ZF} and Σ_1. Thus $<_\alpha$ is Δ_1^{ZF} and Σ_1 for every $\alpha \in \mathrm{On}$, whence $<_L$ can be defined as

$$x <_L y :\equiv \exists\alpha(\mathsf{on}(\alpha) \wedge x <_\alpha y),$$

making it Σ_1. But as $<_\alpha$ is clearly a linear order by construction for every $\alpha \in \mathrm{On}$, we also have that

$$x <_L y \equiv \forall\alpha(\mathsf{on}(\alpha) \to \neg(y <_\alpha x \vee y = x)),$$

so it is Π_1^{ZF} as well, making it Δ_1^{ZF}. ■

Lemma A.4.4. *The formula defining $<_L$ is L_δ-absolute for limit $\delta > \omega$.*

Proof. Let $x, y \in L_\delta$. Then $L_\delta \models x <_L y \Rightarrow L \models x <_L y$ by upwards absoluteness since $<_L$ is Σ_1; assume thus $L \models x <_L y$. We need to show that $L_\delta \models x <_L y$. Set $\beta \in \mathrm{On}$ to be the least such that $x, y \in L_\beta$. Then $L \models x <_\beta y$, since the $<_\alpha$'s have to agree by construction. By minimality of β we have that $<_\beta \in L_{\beta+3} \subseteq L_\delta$. Furthermore $<_\beta$ is L_δ-absolute for limit $\delta > \omega$ by the previous absoluteness results in this appendix and the definition of the formula defining $<_\beta$. Thus $(\exists\beta \in L_\delta)(x <_\beta y)$ and $L_\delta \models \exists\beta(x <_\beta y)$, which by definition means that $L_\delta \models x <_L y$. ■

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