Bachelor Defense

Gödel's Constructible Universe

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Outline

- Construction of the construcible universe L
- Relative consistency of AC and GCH
- Combinatorics in L
- Measurable cardinals are not constructible

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- Internalise formulas *within* set theory
- Truth: construct $\mathrm{sat}(M, \ulcorner \varphi \urcorner)$, corresponding to $M \models \varphi$

Definition

Let X be a set. Then the *definable power set* is given as

$$\begin{aligned}\mathrm{Def}(X) := \{x \in X \mid & (\exists \ulcorner\varphi\urcorner \in \mathscr{L}_X)(\exists p_1, \ldots, p_n \in X) \\ & \mathrm{sat}[X, \ulcorner\varphi(\dot{x}, \dot{p}_1, \ldots, \dot{p}_n)\urcorner]\}\end{aligned}$$

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Definition

Define $L_0 := \emptyset$, $L_{\alpha+1} := \mathrm{Def}(L_\alpha)$ and $L_\delta := \bigcup_{\gamma<\delta} L_\gamma$ for δ limit.
Then the *constructible universe* is given by $L := \bigcup_{\alpha \in \mathrm{On}} L_\alpha$.

Axiom of Constructibility

$V = L$ is an abbreviation for $\forall x (\exists \alpha \in \mathrm{On})(x \in L_\alpha)$.

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Lemma

$V = L$ holds *inside* L itself.

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Definition

A class M is an *inner model* (of ZF) if

1. M is transitive;
2. $\mathrm{On} \subseteq M$;
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Theorem

L is the *smallest* inner model.

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Axiom of Choice holds in L. In fact, *Global Choice* holds in L.

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Condensation Lemma (Gödel)

Let δ be a limit ordinal and $X \preceq_1 L_\delta$. Then there is a unique limit ordinal $\alpha \leq \delta$ and a unique isomorphism $\pi : \langle X, \in \rangle \cong \langle L_\alpha, \in \rangle$, which fixes every transitive $Y \subseteq X$.

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GCH is the statement that $2^\kappa = \kappa^+$ for all infinite cardinals κ.

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Theorem (Gödel)

$V{=}L$ implies GCH.

Proof.

- Let κ be an infinite cardinal and $A \subseteq \kappa$. We find $\beta \in \mathrm{On}$ such that $\beta < \kappa^+$ and $A \in L_\beta$, because then $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$ and thus $2^\kappa = |\mathcal{P}(\kappa)| \leq |L_{\kappa^+}| = \kappa^+$

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Corollary

Con(ZF) $\Rightarrow$ Con(ZFC + GCH).

Combinatorics in L

Theorem (Cantor)

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Suslin's Hypothesis (SH)

$\langle \mathbb{R}, < \rangle$ is the (up to isomorphism) unique complete dense linear ordering without endpoints, where every set of disjoint intervals is countable.

Definition

$\diamondsuit$ is the statement that there exists a sequence $(A_\alpha)_{\alpha<\omega_1}$, such that for every $A \subseteq \omega_1$, the set $\{\alpha \in \mathrm{On} \mid A \cap \alpha = A_\alpha\}$ is stationary.

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Corollary

$\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \diamondsuit + \neg\,\mathrm{SH})$.

Measurable cardinals are not constructible

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Theorem (Scott)

$V{=}L$ implies that there exists no measurable cardinal.

Proof sketch.

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- For non-trivial elementary $j : \mathfrak{A} \preceq_1 \mathfrak{B}$ with $B \subseteq A$, there is least $\alpha \in \mathrm{On}$ such that $j(\alpha) > \alpha$; define $\operatorname{crit} j := \alpha$

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- $M \models$ "$j(\kappa)$ is the least measurable cardinal"
- L is the least inner model, so $M = L$ and hence $j(\kappa)$ is the least measurable, but $j(\kappa) > \kappa$, $\lightning$.

$\square$

Perspective

The inner model problem (1960's)

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This problem spawned an entire mathematical field called *inner model theory*. There is ongoing progress, but it remains unsolved.

Thank you for listening.