Přikrý forcing theory I

Let κ be measurable and let $\mathcal{U}$ be a κ-complete ultrafilter on κ. Then the following are equivalent:

- $\mathcal{U}$ is normal;
- $\mathcal{U} = \{X \subseteq \kappa \mid \kappa \in j(X)\}$;
- $\mathcal{U}$ is closed under diagonal intersections;
- If $S \subseteq V_\kappa$ and $\vec{A} \in \prod_{s\in S} \mathcal{U}$ then $\triangle \vec{A} := \{\alpha < \kappa \mid \forall s \in V_\alpha \cap S (\alpha \in A_s)\} \in \mathcal{U}$;
- $\forall A \in \mathcal{U}\, \forall f : A \to \kappa$ regressive $\exists A' \subseteq A$ $(A' \in \mathcal{U} \wedge |f'' A'| = 1)$;
- $\kappa = [\mathrm{id}]_{\mathcal{U}}$.

Definition. Let κ be measurable with $\mathcal{U}$ the corresponding measure. Then Přikrý forcing is the poset $\mathbb{P}_{\mathcal{U}}$ having conditions (s, A) with $s \in (\kappa)^{<\omega}$ and $A \in \mathcal{U}$, where $(\kappa)^{<\omega} \subseteq \kappa^{<\omega}$ is the increasing sequences of elements of κ, and we say $(t, B) \leq (s, A)$ iff t end-extends s, $B \subseteq A$ and $\mathrm{ran}(t \restriction \mathrm{dom}(t) - \mathrm{dom}(s)) \subseteq A$; I.e. that new stuff in the sequence are elements of A. $\dashv$

Looking at the first coordinate of the generic we get an ω-sequence cofinal in κ by a simple density argument — call it $\langle \kappa_n \mid n < \omega \rangle$. Then

$$g = \{(s, A) \in \mathbb{P} \mid s \trianglelefteq \vec{\kappa} \wedge \vec{\kappa} - s \subseteq A\},$$

so $V[g] = V[\vec{\kappa}]$.

For $p := (s, A) \in \mathbb{P}_\mathcal{U}$ we say that s is the lower part or the stem of p, and A is the measure one set of p. Often it's required that $\max(\operatorname{ran} s) \leq \min A$, which makes the $\mathbb{P}_\mathcal{U}$-ordering into an actual partial order.

The proof of the following is straight-forward.

Lemma. $\mathbb{P}_\mathcal{U}$ has the κ^+-cc. $\dashv$

Definition. If $s \in (\kappa)^{<\omega}$ and $A, B \in \mathcal{U}$ then (s, B) is a direct extension of (s, A) if $B \subseteq A$, and we write $(s, B) \leq^* (s, A)$. $\dashv$

Přikrý's Lemma. Let $p \in \mathbb{P}_\mathcal{U}$ and φ a sentence in the forcing language. Then there's a condition $q \in \mathbb{P}_\mathcal{U}$ such that $q \leq^* p$ and $q \parallel \varphi$, meaning either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

Proof. Let $p = (s, A)$. For each $t \trianglerighteq s$, if possible choose A^1_t such that $(t, A^1_t) \leq (s, A)$ and $(t, A^1_t) \parallel \varphi$ — otherwise set $A^1_t := A$. So in any case, $(t, A^1_t) \in \mathbb{P}_\mathcal{U}$. Let now $A^1 := \triangle \langle A^1_t \mid t \trianglerighteq s \rangle$.

Claim. If $(t, B) \leq (s, A)$ then

- if $(t, B) \Vdash \varphi$ then $(t, A^1) \Vdash \varphi$, and
- if $(t, B) \Vdash \neg\varphi$ then $(t, A^1) \Vdash \neg\varphi$.

Proof of claim. If $(t,B) \| \varphi$ then we chose A^1_t such that $(t, A^1_t) \| \varphi$. Every extension of (t, A^1) is of the form $(t^\frown u, C)$ where $u \subseteq A' - t$ and so, by definition of A^1, $u \subseteq A^1_t$. Hence

$$(t^\frown u, C \cap A^1_t) \leq (t^\frown u, C), (t, A^1_t).$$

Now note that $(t, B \cap A^1_t) \leq (t,B), (t, A^1_t)$. $\dashv$

For each stem $t \trianglerighteq s$ ask whether

① $(t^\frown \langle \kappa \rangle, j(A^1)) \Vdash j(\varphi)$; or
② $(t^\frown \langle \kappa \rangle, j(A^1)) \Vdash \neg j(\varphi)$; or
③ $(t^\frown \langle \kappa \rangle, j(A^1)) \nparallel j(\varphi)$.

In V, for each t choose a set $A^2_t \in \mathcal{U}$ with $A^2_t \subseteq A^1$ and $\forall \alpha \in A^2_t ((t^\frown \langle \alpha \rangle, A^1)$ behaves the same way), i.e. all fall in the same case ①, ②, ③ above.

Let now $A^2 := \Delta_t A^2_t$. Then (s, A^2) is as required. $\square$

Corollary. 1) $\mathbb{P}_{\mathcal{U}}$ adds no bounded subsets of V_κ;

2) All cardinals are preserved except for κ, and the same for cofinalities. $\dashv$

Some "easy" applications

(1) Failure of SCH.

GCH can fail at a measurable (this is due to Silver modulo a supercompact, and due to Woodin-Gitik modulo a measurable κ with $o(\kappa)=\kappa^{++}$). Then apply Příkrý forcing to get a failure of SCH at κ.

(2) Failure of covering.

If $g \subseteq \mathbb{P}_u$ is V-generic then V fails (badly) to cover $V[g]$.

(3) Failure of reflection.

We need a couple of results first.

<u>Lemma</u>. If λ is regular uncountable and $\mathbb{P}$ is λ-cc and $p \in \mathbb{P}$ forces $\dot{C} \subseteq \lambda$ to be club then there's $D \subseteq \lambda$ such that $D \subseteq \lambda$ is club and $p \Vdash \ulcorner \check{D} \subseteq \dot{C} \urcorner$. $\dashv$

<u>Corollary</u>. λ-cc forcings preserve the stationarity in λ. $\dashv$

In V, $S^{\kappa^+}_\kappa$ is non-reflecting, since for any $\alpha < \kappa^+$ with $\operatorname{cof} \alpha = \kappa$ there's a club $C \subseteq \alpha$ with $\operatorname{ot}(C) = \kappa$ and $\operatorname{cof} \beta < \kappa$ for every $\beta \in C$. Just pick successor points of C to avoid $S^{\kappa^+}_\kappa$.

In $V[g]$, $(S^{\kappa^+}_{\kappa})^V \subseteq S^{\kappa^+}_{\omega}$ is then non-reflecting, and still stationary as $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc.

④ Strongly compacts.
In $V[g]$ there are no strongly compacts below κ, as SCH holds above strongly compacts.

⑤ Trees.
In $V[g]$ there's a special κ^+-tree.