Classification of covering spaces

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ABSTRACT. asd

1 Basic results

We start off with recalling the basic definitions and the two lifting theorems regarding covering spaces.

DEFINITION 1.1. Let X be a topological space. Then $p: \tilde{X} \to X$ is a **covering** map and \tilde{X} a covering space if p is a surjective continuous map such that for every $x \in X$ there exists some open neighborhood $U \subseteq X$ around x which satisfies that $p^{-1}(U) = \coprod_i \tilde{U}_i$, where $\tilde{U}_i \subseteq \tilde{X}$ are open sets and $p \upharpoonright \tilde{U}_i$ is a homeomorphism onto U.

Proposition 1.2. Every covering map is a quotient map.

THEOREM 1.3. Let $p: \tilde{X} \to X$ be a covering map, A a topological space and $h: A \times I \to X$ a homotopy. Given a diagram

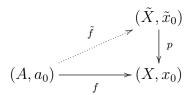
$$A \times \{0\} \xrightarrow{\tilde{h}_0} \tilde{X} \\ \downarrow p \\ A \times I \xrightarrow{\tilde{h}} X$$

there is a unique $\tilde{h}: A \times I \to \tilde{X}$ such that both above triangles commute.

For the special case where A is the trivial space, we get the *unique path lifting theorem*: given any $\tilde{x} \in \tilde{X}$ and any path $u: I \to X$ in X starting at $u(\tilde{x})$, there's a unique lift of u to the path $\tilde{x} \cdot u: I \to \tilde{X}$, starting at \tilde{x} .

DEFINITION 1.4. A topological space X is **path connected** (abbreviated pc) if there exists a path between every two points in X. X is **locally path connected** (abbreviated lpc) if X has a basis of path connected sets.

THEOREM 1.5. Let $p: \tilde{X} \to X$ be a covering map, A a topological space and $f: (A, a_0) \to (X, x_0)$ a pointed map. If A is path connected and locally path connected then $f_*\pi_1(A, a_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ iff there is a unique map $\tilde{f}: (A, a_b) \to (\tilde{X}, \tilde{x}_0)$ where $\tilde{x}_0 \in p^{-1}(\{x_0\})$ such that the following diagram commutes:



2 Category of covering spaces

DEFINITION 2.1. Let X be a topological space. Then Cov(X) is the category with covering maps over X as objects and a map between $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ is a continuous map $h: \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ h$.

Thus, Cov(X) is a full subcategory of the slice category Top/X. Recall that the fundamental groupoid $\pi(X)$ of a topological space X is the category with objects the elements of X and arrows $f: x \to y$ being path homotopy classes of paths between x and y.

DEFINITION 2.2. Let $p: \tilde{X} \to X$ be a covering map. Then the **monodromy functor** of p is the functor $F_p: \pi(X) \to \mathsf{Set}$ defined as

- $F_p(x) := p^{-1}(\{x\});$
- $F_p(u:x\to y):=(\tilde{x}\mapsto (\tilde{x}\cdot u)(1)).$

Proposition 2.3. The monodromy functor F_p is a functor for every covering map $p: \tilde{X} \to X$.

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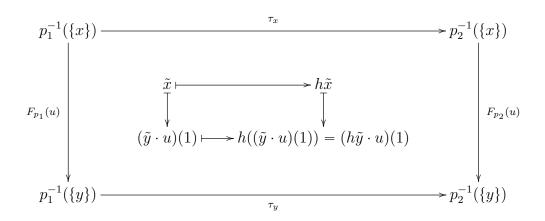
PROOF. Follows from the uniqueness of the lifted paths.

Definition 2.4. For X a topological space, define $\Phi_X: \mathsf{Cov}(X) \to \mathsf{Set}^{\pi(X)}$ as

- $\Phi_X(p: \tilde{X} \to X) := F_p;$
- $\Phi_X(h:p\to p'):=(\tau_x:\tilde{x}\mapsto h\tilde{x}).$

Proposition 2.5. Φ_X is a well-defined functor for every topological space X.

PROOF. To show that Φ_X is well-defined, we only need to show that $\tau:=\Phi_X(h:p_1\to p_2)$ is in fact a natural transformation $\tau:F_{p_1}\to F_{p_2}$. But we have the commutative diagram



The equality is due to uniqueness of path lifting. Hence Φ_X is well-defined, and it's clear that it's a functor.

DEFINITION 2.6. A topological space X is **semi-locally simply connected** (abbreviated slsc) if any neighborhood around any point $x \in X$ contains a neighborhood U of x such that any loop at x in U is contractible in X. Equivalently, $i_*: \pi_1(U,x) \to \pi_1(X,x)$ is the zero map

Definition 2.7. Given a functor $F: \pi(X) \to \mathsf{Set}$, define

- $\tilde{X}_F := \prod_{x \in X} F(x);$
- $p_F: \tilde{X}_F \to X$, where $p_F"F(x) := \{x\}$.

Lemma 2.8. Let X be a topological space and $F: \pi(X) \to \mathsf{Set}$ a functor. If X is slsc and lpc then there is a topology on \tilde{X}_F such that p_F is a covering map.

PROOF. Let $x \in X$ and $U \subseteq X$ an open pc neighborhood of x such that any loop in U on x is nullhomotopic in X. This implies that there's a unique path homotopy class u_y from x to any $y \in U$. Define $f: U \times F(x) \to p_F^{-1}(U)$ as $f(y,z) := F(u_y)(z)$. Then f is bijective as $f \upharpoonright (\{y\} \times F(x))$ for every $y \in U$ and $f = \coprod_{y \in U} f \upharpoonright (\{y\} \times F(x))$.

For each $z \in F(x)$ set $(U,z) := f''(U \times \{z\})$. By assumption, X has a basis of such U as described above. Now the sets (U,z) form a basis for a topology on \tilde{X} . Indeed, assume $w \in (U,z) \cap (V,z')$. Then there is some $y \in U$ and $y' \in V$ such that $F(u_y)(z) = w = F(u_{y'})(z')$. But then $w \in F(y) \cap F(y')$, meaning $y = y' \in U \cap V$ as the fibers under F are disjoint. Now $F(u_y)(z) = F(u_y)(z')$, so z = z' as $F(u_y)$ is a bijection. Thus $w \in (U \cap V, z) \subseteq (U, z) \cap (V, z')$.

Clearly $p_F^{-1}(U) = \coprod_{y \in U} \tilde{U}_y$ with $p_F \upharpoonright \tilde{U}_y$ a homeomorphism, so p_F is also continuous and hence a covering map.

DEFINITION 2.9. Let X be a topological space which is both slsc and lpc. Then define $\Psi_X : \mathsf{Set}^{\pi(X)} \to \mathsf{Cov}(X)$ given by

- $\bullet \ \Psi_X(F) := p_F;$
- $\Psi_X(\tau: F \Rightarrow G) := (\prod_{x \in X} \tau_x : \prod_{x \in X} F(x) \rightarrow \prod_{x \in X} G(x)).$

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THEOREM 2.10. Assume X is a topological space which is both slsc and lpc. Then

$$\Phi_X : Cov(X) \cong Set^{\pi(X)}.$$

Proof. We have that

- $\Phi_X \Psi_X(F) = \Phi_X(p_F) = F_{p_F} = F;$
- $\Phi_X \Psi_X(\tau : F \Rightarrow G) = \Phi_X(\coprod_{x \in X} \tau_x) = (\tau'_x : \tilde{x} \mapsto \coprod_{x \in X} \tau_x(\tilde{x})) = \tau;$
- $\Psi_X \Phi_X(p) = \Psi_X(F_p) = p_{F_p} = p;$
- $\Psi_X \Phi_X(h:p \to p') = \Psi_X(\tau_x: \tilde{x} \mapsto h\tilde{x}) = \prod_{x \in X} \tau_x = h.$

Hence it only remains to check that the topologies on \tilde{X} and $\coprod_{x \in X} F_p(x) = \coprod_{x \in X} p^{-1}(\{x\})$ are the same. missing!

DEFINITION 2.11. Let G be a group. Then GSet is the category of G-sets and G-maps between them.

Proposition 2.12. Let X be a topological space which is pc, lpc and slsc. Then

$$Cov(X) \simeq \pi_1(X, x_0) Set$$

 $(p: \tilde{x} \to x) \mapsto p^{-1}(\{x_0\})$

is an equivalence of categories.

PROOF. The inclusion $\pi_1(X, x_0) \to \pi(X)$ is an equivalence since it's essentially surjective as X is pc and clearly fully faithful. Then the induced functor

$$\mathsf{Set}^{\pi(X)} \to \mathsf{Set}^{\pi_1(X,x_0)}$$

is also an equivalence as functors preserve equivalences. But notice that

$$\mathsf{Set}^{\pi_1(X,x_0)} \cong \pi_1(X,x_0)\mathsf{Set}$$

via the functors $G: \mathsf{Set}^{\pi_1(X,x_0)} \to \pi_1(X,x_0) \mathsf{Set} : H$ given by

- $G(F) := F(x_0);$
- $G(\tau : F \Rightarrow F') := (\tau_{x_0} : F(x_0) \to F'(x_0));$
- $H(A)(x_0) := A \text{ and } H(A)(g : x_0 \to x_0) := (a \mapsto a.g);$
- $H(f: A \to B) := (\tau_{x_0} := f).$

missing details!

Hence we now have that

$$\mathsf{Cov}(X) \cong \mathsf{Set}^{\pi(X)} \simeq \mathsf{Set}^{\pi_1(X,x_0)} \cong \pi_1(X,x_0)\mathsf{Set}.$$

DEFINITION 2.13. Let G be a group. Let $\operatorname{Conj}(G)$ be the category consisting of subgroups of G and arrows $\operatorname{Inn}(H_2n): H_1 \to H_2$, where $nH_1n^{-1} \subseteq H_2$ and $\operatorname{Inn}(H_2n)$ is conjugation with the unique element in $\{x \in H_2n \mid \forall h \in H_2 - \{1\}: h \nmid x\}$.

DEFINITION 2.14. Let $Cov_0(X)$ be the full subcategory of Cov(X) generated by all the pc covering spaces over X.

We aim to prove the following:

THEOREM 2.15. Let X be a topological space which is pc, lpc and slsc. Then

$$Cov_0(X) \simeq Conj(\pi_1(X, x_0))$$

is an equivalence of categories.

To get there, however, requires working through some more categories.

DEFINITION 2.16. The **orbit category** \mathcal{O}_G of a group G is the full subcategory of GSet, generated by the transistive G-sets, i.e. the ones with a single orbit.

Lemma 2.17. Let X be a topological space which is pc, lpc and slsc. Then

$$\operatorname{Cov}_0(X) \simeq \mathcal{O}_{\pi_1(X,x_0)}$$
 $(p: \tilde{X} \to X) \mapsto p^{-1}(\{x_0\})$

is an equivalence of categories.

PROOF. We show that \tilde{X} is pc iff $p^{-1}(\{x_0\})$ is a transitive $\pi_1(X,x_0)$ -set with action $\tilde{x}.u:=(\tilde{x}\cdot u)(1)$, because then we can just restrict the equivalence from Proposition 2.12 to $\mathsf{Cov}_0(X)$ and we're done.