Structuring the Mathematical Universe

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Draft version 0.7

Preface

This booklet is not designed to teach the reader about a subject from start to finish. It is not intended as an article, providing new results. It is designed for people who seek an overview - people who have learned about various mathematical subjects and want to know how these fit together. It also doubles as an encyclopaedia through a wide range of examples of mathematical structures.

To make it as easy as possible to look up definitions and links, I have refrained myself from writing unnecessary explanatory text and have tried to be as concise as possible while still giving the necessary information for understanding the structure at hand.

The map, which illustrates all the links described in this booklet, has been categorized within the same structure as this booklet's chapters. Furthermore, the arrows in the map indicate if a structure implies another structure. If the arrow is based on a definition, it is coloured black. If it is based on a theorem, remark or anything else not a definition, it is coloured green.

Substructures are used in the usual sense, being that algebraic substructures are closed under their operations, relational substructures have their relation restricted to the underlying set and pavings of substructures are the intersection between the elements of the paving and the underlying subset. I assume ETCS; an (informal) overview of the axioms is given in the first chapter.

My notation used is pretty standard. I use the bold notation $\mathbf{A}$ to denote a structure, and the corresponding non-bold letter A to denote the underlying set. I use $|A|$ to denote the cardinality of A, ω as the first infinite ordinal and $\aleph_0 = |\omega|$. Structures will be denoted with angled brackets $\langle A, f, g, h, \ldots \rangle$, where A is a set and $\{f, g, h, \ldots\}$ is the language of the structure.

New versions will be available at

`bit.ly/mathuniverse`.

Change log

Changes in version 0.7

Added the lattice duality, smooth manifolds, Lie groups and redefined sets to be based on ETCS. Revamped a lot of the notation.

Changes in versions 0.5 and 0.6

Fixed some more minor errors.

Changes in version 0.4

Fixed minor errors, switched order of some sections. Added that metric spaces are Hausdorff spaces and rings are Lie rings. Added some examples. Made things a bit prettier.

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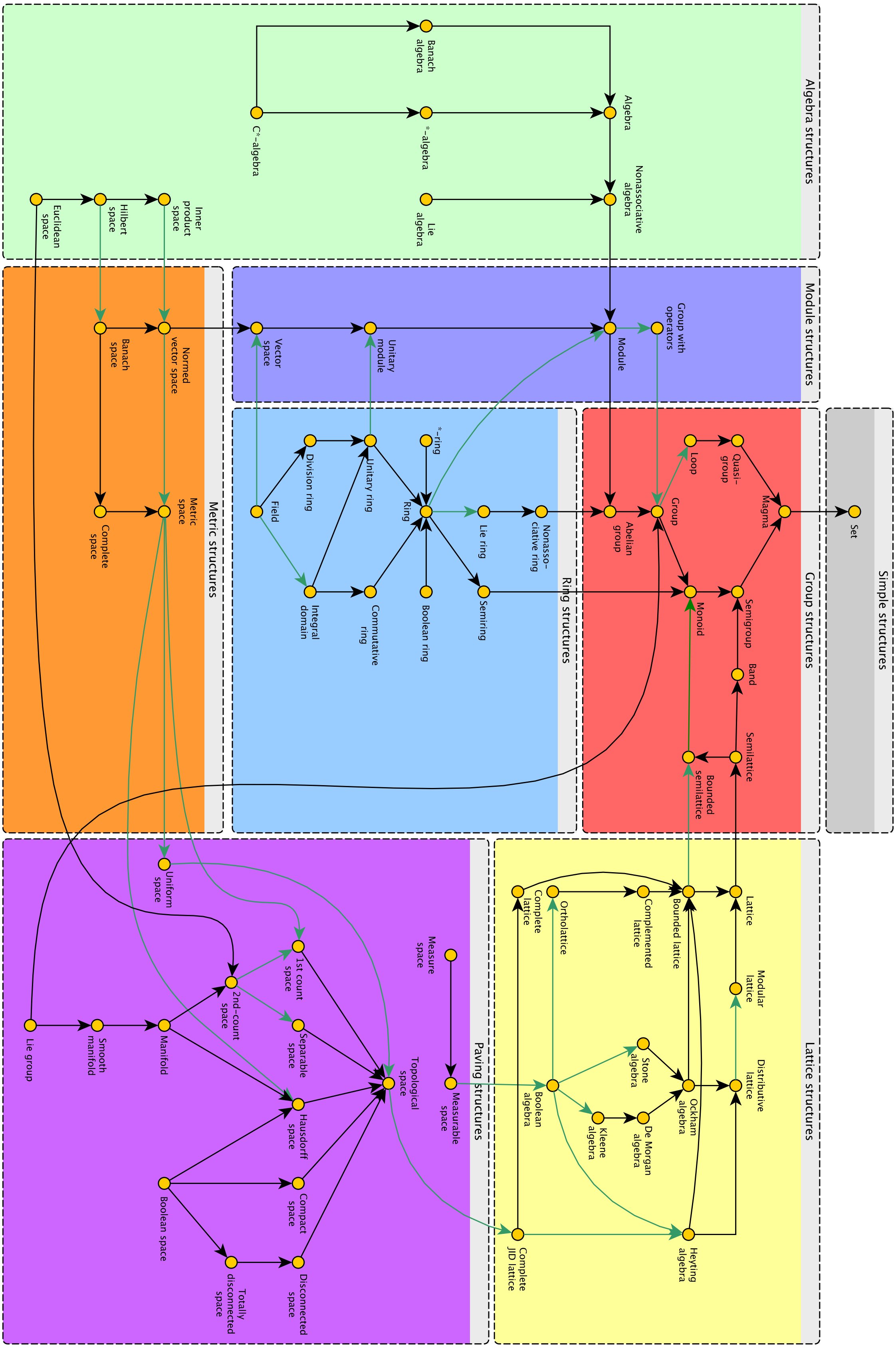
The map


Algebra structures
Banach algebra
C*-algebra
*-algebra
Algebra
Nonassociative algebra
Lie algebra
Euclidean space
Hilbert space
Inner product space
Module structures
Vector space
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Module
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Stone algebra
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Kleene algebra
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Complete JID lattice
Heyting algebra

Chapter 1

Simple structures

1.1 Sets

Our primitive objects in our domain of discourse will be *sets* x and *functions* $f : x \to y$ *between two given sets* x, y.

Axiom 1.1.1 (Composition)**.** If f, g is a function, then $f \circ g$ is a function as well. Furthermore $\circ$ is associative.

Axiom 1.1.2 (Identity function)**.** Identity functions $1_x : x \to x$ exist and satisfies $1_x \circ f = f$ for $f : y \to x$ and $g \circ 1_x = g$ for $g : x \to z$.

Axiom 1.1.3 (Singleton)**.** There exists a singleton set.

Axiom 1.1.4 (Empty set)**.** There exists a set $\emptyset$ with no elements.

Axiom 1.1.5 (Function equality)**.** Two functions $f, g : x \to y$ are equal if they have the same function values.

Axiom 1.1.6 (Products)**.** Every pair of sets x, y has a product set $x \times y$.

Axiom 1.1.7 (Function sets)**.** Every pair of sets x, y has a set of functions y^x from x to y.

Axiom 1.1.8 (Inverse images)**.** For every function $f : x \to y$ and element $z \in y$, the inverse image $f^{-1}(\{z\})$ exists.

Axiom 1.1.9 (Characteristic function)**.** Given a set x, every subset $y \subseteq x$ has a characteristic function $\chi_y : x \to \{\top, \bot\}$.

Axiom 1.1.10 (Natural numbers)**.** The natural numbers exist.

Axiom 1.1.11 (Choice)**.** Every surjection as a right inverse.

Chapter 2

Group structures

Definition 2.0.12. A *binary operation* on a set $X \neq \emptyset$ is a map $\cdot : X \times X \to X$, often denoted with infix notation $a \cdot b$ instead of the usual prefix $\cdot(a, b)$.

2.1 Magmas

Definition 2.1.1. A *magma* $\langle A, \cdot \rangle$ is a set A along with a binary operation $\cdot$.

2.2 Semigroups

Definition 2.2.1. A *semigroup* $\langle A, \cdot \rangle$ is a magma satisfying $A \neq \emptyset$ and that $\cdot$ is associative:

$$\forall a, b, c \in A : a(bc) = (ab)c.$$

2.3 Monoids

Definition 2.3.1. A *monoid* $\langle A, \cdot, e \rangle$ is a semigroup $\langle A, \cdot \rangle$ with an identity element e:

$$\exists e \in A \forall a \in A : ae = ea = a.$$

2.4 Quasigroups

Definition 2.4.1. A *quasigroup* $\langle A, \cdot, \backslash, / \rangle$ is a magma $\langle A, \cdot \rangle$, satisfying:

(i) $A \neq \emptyset$

(ii) $\forall a, b \in A \exists x, y \in A : ax = b \wedge ya = b$

The unique solutions to these equations are $x = a \backslash b$ and $y = b/a$, called resp. left and right division.

Example 2.4.2. The nonzero rationals with division $\langle \mathbb{Q}\backslash\{0\}, \div \rangle$ form a quasigroup $\langle \mathbb{Q}\backslash\{0\}, \div, \backslash, / \rangle$ with $a\backslash b = b \div a$ and $a/b = a \div b$.

2.5 Loops

Definition 2.5.1. A *loop* $\langle A, \cdot, \backslash, /, e\rangle$ is a quasigroup $\langle A, \cdot, \backslash, / \rangle$ with an identity element e:

$$\exists e \in A \forall x \in A : xe = ex = x.$$

Example 2.5.2. The nonzero *octonions* $\mathbb{O}$, which are an extension of the quaternions $\mathbb{H}$ (which again are an extension of the complex numbers $\mathbb{C}$), form a nonassociative loop under multiplication - this loop is known as the *Moufang loop*.

2.6 Groups

Definition 2.6.1. A *group* $\langle A, \cdot, e, ^{-1} \rangle$ is a monoid $\langle A, \cdot, e\rangle$, in which every element x has an inverse x^{-1}:

$$\forall a \in G \exists a^{-1} \in G : aa^{-1} = a^{-1}a = e.$$

Theorem 2.6.2. *Every group $\langle A, \cdot, e, ^{-1} \rangle$ induces a loop $\langle A, \cdot, e, ^{-1}, \backslash, / \rangle$.*

Proof. Let $\mathbf{A} := \langle A, \cdot, e, ^{-1} \rangle$ be a group. Since every loop is a magma, it has to be shown that $\cdot$ has an identity element and that left and right division are well-defined on $\mathbf{A}$. $\mathbf{A}$ has an identity element since it is a monoid. For every $a, b \in A$, define $x := a^{-1}b$ and $y := ba^{-1}$. Then

$$\begin{aligned} ax &= a(a^{-1}b) = (aa^{-1})b = eb = b \\ ya &= (ba^{-1})a = b(a^{-1}a) = be = b, \end{aligned}$$

making $\mathbf{A}$ induce a loop. ∎

2.7 Abelian groups

Definition 2.7.1. An *Abelian group* $\langle A, \cdot, e, ^{-1} \rangle$ is a group in which $\cdot$ is commutative:

$$\forall a, b \in A : ab = ba.$$

2.8 Bands

Definition 2.8.1. A *band* $\langle A, \cdot \rangle$ is a semigroup in which every element is idempotent:

$$\forall a \in A : aa = a.$$

2.9 Semilattices

Definition 2.9.1. A *semilattice* $\langle A, \cdot \rangle$ is a commutative band. The binary operation is often either $\wedge$, called "meet", or $\vee$, called "join".

2.10 Bounded semilattices

Definition 2.10.1. A *bounded semilattice* $\langle A, \cdot, e \rangle$ is a semilattice $\langle A, \cdot \rangle$ with an identity element e:

$$\forall a \in A : ae = a.$$

Remark 2.10.2. A bounded semilattice is equivalent to an idempotent commutative monoid.

Chapter 3

Lattice structures

3.1 Lattices

Definition 3.1.1. A *lattice* $\langle A, \vee, \wedge \rangle$ is a structure in which $\langle A, \vee \rangle$ and $\langle A, \wedge \rangle$ are both semilattices, as well as satisfying the absorption laws:

(i) $\forall a, b \in A : a \vee (a \wedge b) = a$

(ii) $\forall a, b \in A : a \wedge (a \vee b) = a$

Example 3.1.2. The ordered natural numbers $\langle \mathbb{N}, \leq \rangle$ form a lattice $\langle \mathbb{N}, \leq, \vee, \wedge \rangle$ with $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

Definition 3.1.3. A *partial order* on a set A is a binary relation $\leq$ satisfying:

- $\forall a \in A : a \leq a$
- $\forall a, b \in A : (a \leq b) \wedge (b \leq a) \Rightarrow a = b$
- $\forall a, b, c \in A : (a \leq b) \wedge (b \leq c) \Rightarrow a \leq c$

Definition 3.1.4. A *poset*, or a partially ordered set, $\langle A, \leq \rangle$ is a set A equipped with a partial order $\leq$.

Theorem 3.1.5 (Lattice duality)**.** *A lattice $\langle A, \vee, \wedge \rangle$ is equivalent to a poset $\langle A, \leq \rangle$ in which there exist both* $\sup\{a, b\}$ *and* $\inf\{a, b\}$ *for all $a, b \in A$.*

Proof. Let $\langle A, \leq \rangle$ be a poset and assume that for all $a, b \in A$ exist both $\sup\{a, b\}$ and $\inf\{a, b\}$. Then define

$$a \vee b := \sup\{a, b\} \qquad a \wedge b := \inf\{a, b\}.$$

Both the sup and inf are easily seen to be associative, idempotent and commutative. The absorption laws are shown:

$$\begin{aligned} a \vee (b \wedge a) &= \sup\{a, \inf\{b, a\}\} = a, \\ a \wedge (b \vee a) &= \inf\{a, \sup\{b, a\}\} = a. \end{aligned}$$

Since $a \geq \inf\{b, a\}$ and $a \leq \sup\{b, a\}$. Now let $\langle A, \vee, \wedge \rangle$ be a lattice and define the binary relation $\leq$ as

$$a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

The requirements for partial orders shall be shown. (i): Follows from idempotency. (ii): Assume $a \vee b = b$ and $b \wedge a = b$. Then $a = a \wedge (b \vee a) = a \wedge (a \vee b) = a \wedge b = b \wedge a = b$, where the absorption law, commutativity and assumptions were used. (iii): Assume $a \vee b = b$ and $b \vee c = c$. Then $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$, where associativity and assumptions were used. It is thus a partial order, where the infimum is given as the meet and the supremum as the join. ■

3.2 Bounded lattices

Definition 3.2.1. A *bounded lattice* $\langle A, \vee, \wedge, \perp, \top \rangle$ is a lattice, which satisfies the identity laws:

(i) $\forall a \in A : a \vee \perp = a$

(ii) $\forall a \in A : a \wedge \top = a$

Remark 3.2.2. Every bounded lattice is a bounded semilattice as well, since these satisfy their respective (join/meet-)identity law as well.

Example 3.2.3. Let A be a set. Then the power set $\mathcal{P}(A)$ form a bounded lattice $\langle \mathcal{P}(A), \cup, \cap, \emptyset, A \rangle$.

3.3 Modular lattices

Definition 3.3.1. A *modular lattice* $\langle A, \vee, \wedge \rangle$ is a lattice, which satisfies the modular law:

$$\forall a, b, c \in A : (a \wedge b) \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge b.$$

Example 3.3.2. Let $\mathbf{A}$ be a group. Then the normal subgroups $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$ form a modular lattice $\langle \mathcal{N}(\mathbf{A}), \vee, \wedge \rangle$ with $\mathbf{A} \wedge \mathbf{B} := \mathbf{A} \cap \mathbf{B}$ and $\mathbf{A} \vee \mathbf{B} := \mathbf{AB} = \{ab \mid a \in A, b \in B\}$.

3.4 Distributive lattices

Definition 3.4.1. A *distributive lattice* $\langle A, \vee, \wedge \rangle$ is a lattice, which is distributive:

$$\forall a, b, c \in A : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Example 3.4.2. The natural numbers with multiplication $\langle \mathbb{N}, \cdot \rangle$ form a distributive lattice $\langle \mathbb{N}, \cdot, \vee, \wedge \rangle$ with $a \wedge b := \gcd\{a, b\}$ and $a \vee b := \operatorname{lcm}\{a, b\}$.

Theorem 3.4.3. *Every distributive lattice is a modular lattice.*

Proof. Let $\mathbf{A}$ be a distributive lattice. It has to be checked that it satisfies the modular law. Let $a, b, c \in A$:

$$(a \vee (b \wedge c)) \wedge b = (a \wedge b) \vee (b \wedge (b \wedge c)) = (a \wedge b) \vee ((b \wedge b) \wedge c) = (a \wedge b) \vee (b \wedge c),$$

where the distributive, associative, commutative and idempotent laws were used. ∎

3.5 Complemented lattices

Definition 3.5.1. A *complemented lattice* $\langle A, \vee, \wedge, \bot, \top \rangle$ is a bounded lattice which satisfies that every element has atleast one complement b:

(i) $\forall a \in A \exists b \in A : a \vee b = \top$

(ii) $\forall a \in A \exists b \in A : a \wedge b = \bot$

Theorem 3.5.2. *A complemented distributive lattice $\langle A, \vee, \wedge, \perp, \top \rangle$ has, for each $a \in A$, a unique complement $\neg a$.*

Proof. Let $a \in A$ and assume $b, c \in A$ are both complements to a, meaning

(i) $a \vee b = a \vee c = \top$

(ii) $a \wedge b = a \wedge c = \perp$

Then

$$b = b \wedge (b \vee a) = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c) = c \wedge (a \vee b) = c. \quad \blacksquare$$

3.6 Ortholattices

Definition 3.6.1. An *ortholattice* $\langle A, \vee, \wedge, \perp, \top, \neg \rangle$ is a complemented lattice equipped with the unary operation $\neg : A \to C$, where $C \subseteq A$ is the set of complements of A. Furthermore, $\neg a \in C_a$ for each $a \in A$, where $C_a \subseteq C$ is the set of complements to a. Lastly, $\neg$ is idempotent and satisfies De Morgans laws:

(i) $\forall a \in A : \neg\neg a = a$

(ii) $\forall a, b \in A : \neg(a \vee b) = \neg a \wedge \neg b$

(iii) $\forall a, b \in A : \neg(a \wedge b) = \neg a \vee \neg b$

Example 3.6.2. Let $\mathbf{X}$ be an inner product space. Then $\langle \mathcal{P}(X), \cup, \cap, \emptyset, X, \perp \rangle$ is an ortholattice, where $\perp : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as

$$A^{\perp} := \{x \in X \mid \forall a \in A : \langle a, x \rangle = 0\}.$$

3.7 Complete lattices

Definition 3.7.1. Let $\mathbf{A}$ be a lattice and $S \subseteq A$ a set. The *supremum* of S, $\bigvee S$, is defined as an element $b \in A$, satisfying

$$\bigvee A := \min\{b \in A \mid \forall a \in A : a \vee b = b\}.$$

Analogously, the *infimum* of S, $\bigwedge S$, is defined as an element $c \in A$, satisfying

$$\bigwedge A := \max\{c \in A \mid \forall a \in A : a \wedge c = c\}.$$

Definition 3.7.2. A *complete lattice* $\langle A, \vee, \wedge, \bot, \top\rangle$ is a bounded lattice, in which every subset $S \subseteq A$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$.

3.8 Heyting algebras

Definition 3.8.1. A *Heyting algebra* $\langle A, \vee, \wedge, \bot, \top, \rightarrow\rangle$ is a bounded distributive lattice equipped with the binary operation $\rightarrow$, which satisfies that $a \rightarrow b$ is the largest element x which satisfies $x \wedge a \leq b$:

$$\forall a, b, c \in A : (a \wedge b \leq c) \Leftrightarrow (a \leq (b \rightarrow c))$$

Remark 3.8.2. An equivalent definition is for a Heyting algebra is if it holds that

$$\forall a, b, c \in A : \bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}.$$

3.9 Complete JID lattices

Definition 3.9.1. A *complete JID lattice* $\langle A, \vee, \wedge, \bot, \top\rangle$ is a complete distributive lattice, which satisfies the join-infinite distributive law:

$$\forall a \in A, S \subseteq A : a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

Theorem 3.9.2. *Every complete JID lattice is a Heyting algebra.*

Proof. Let $\mathbf{A}$ be a complete JID lattice. By Remark 3.8.2 and completeness, it suffices to show that for $a, b \in A$:

$$\bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}.$$

Notice for any $u \in \{c \in A \mid a \wedge c \leq b\}$ trivially $a \wedge u \leq b$, so by JID:

$$a \wedge \bigvee \{c \in A \mid a \wedge c \leq b\} = \bigvee \{a \wedge c \in A \mid a \wedge c \leq b\} \leq b,$$

which means that $\bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}$. ∎

3.10 Ockham algebras

Definition 3.10.1. An *Ockham algebra* $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ is a bounded distributive lattice with a unary operation $\neg$, such that:

(i) $\forall a, b \in A : \neg(a \wedge b) = \neg a \vee \neg b$

(ii) $\forall a, b \in A : \neg(a \vee b) = \neg a \wedge \neg b$

(iii) $\neg\top = \bot$

(iv) $\neg\bot = \top$

3.11 Stone algebras

Definition 3.11.1. A *Stone algebra* $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ is an Ockham algebra satisfying

(i) $\forall a \in A : \neg\neg a \vee \neg a = \top$

(ii) $\forall a \in A : a \wedge \neg a = \bot$

3.12 De Morgan algebras

Definition 3.12.1. A *De Morgan algebra* $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ is an Ockham algebra in which $\neg$ is idempotent:

$$\forall a \in A : \neg\neg a = a.$$

Example 3.12.2. The standard *fuzzy algebra* $\langle [0, 1], \max\{a, b\}, \min\{a, b\}, 0, 1, 1 - a\rangle$, used in the study of fuzzy logic, is a De Morgan algebra.

3.13 Kleene algebras

Definition 3.13.1. A *Kleene algebra* $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ is a De Morgan algebra, which in addition satisfies

$$\forall a, b \in A : (a \wedge \neg a) \vee (b \vee \neg b) = b \vee \neg b.$$

Example 3.13.2. Let Σ be a finite set of symbols, called an alphabet. Then the set of all formal languages $\mathfrak{L}_\Sigma$ over Σ forms a Kleene algebra $\langle \mathfrak{L}_\Sigma, +, \cdot, \emptyset, \{\varepsilon\}, ^*\rangle$, where $A + B := A \cup B$, $A \cdot B$ is concatenation and A^* is the *Kleene star* operation.

3.14 Boolean algebras

Definition 3.14.1. A *Boolean algebra* $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ is a complemented distributive lattice.

Remark 3.14.2. Boolean algebras complement is unique, cf. Theorem 3.5.2.

Example 3.14.3. In logic, the two-element Boolean algebra $\langle\{0,1\}, \vee, \wedge, 0, 1, \neg\rangle$ is used where 0 denotes false, 1 denotes true, $\vee$ means "or", $\wedge$ means "and" and $\neg$ means "not".

Theorem 3.14.4. *Every Boolean algebra is an ortholattice.*

Proof. Let $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ be a Boolean algebra. Since both Boolean algebras and ortholattices are complemented lattices, it has to be shown that ortholattices additional requirements for $\neg$ hold.

(**i**): $\neg\neg a = \neg\neg a \vee \bot = \neg\neg a \vee (a \wedge \neg a) = (\neg\neg a \vee a) \wedge (\neg\neg a \vee \neg a) = (\neg\neg a \vee a) \vee \top = (\neg\neg a \vee a) \wedge (a \vee \neg a) = a \vee (\neg a \wedge \neg\neg a) = a \vee \bot = a$, where the identity, distributive, commutative and complement laws were used.

(**ii**): Since complements in Boolean algebras are unique cf. Theorem 3.5.2, it suffices to show that $\neg a \wedge \neg b$ is the complement to $a \vee b$, which is equivalent to

1.) $(a \vee b) \vee (\neg a \wedge \neg b) = \top$

2.) $(a \vee b) \wedge (\neg a \wedge \neg b) = \bot$

1.): $(a \vee b) \vee (\neg a \wedge \neg b) = ((a \vee b) \vee \neg a) \wedge ((a \vee b) \vee \neg b) = (b \vee (a \vee \neg a)) \wedge (a \vee (b \vee \neg b)) = (b \vee \top) \wedge (a \vee \top) = \top \wedge \top = \top$, where the distributive, associative, complement, identity and idempotent laws were used.

2.): $(a \vee b) \wedge (\neg a \wedge \neg b) = ((\neg a \wedge \neg b) \wedge a) \vee ((\neg a \wedge \neg b) \wedge b) = (\neg b \wedge (a \vee \neg a)) \vee (\neg a \wedge (b \wedge \neg b)) = (\neg b \wedge \bot) \vee (\neg a \wedge \bot) = \bot \vee \bot = \bot$, where the distributive, associative, complement, identity, idempotent and commutative laws were used. (**iii**) follows analogously from (**ii**). ∎

Theorem 3.14.5. *Every Boolean algebra is a Kleene algebra.*

Proof. Let $\mathbf{A}$ be a Boolean algebra and $a, b \in A$. It has to be shown that

(i) $\neg(a \vee b) = \neg a \wedge \neg b$

(ii) $\neg(a \wedge b) = \neg a \vee \neg b$

(iii) $\neg\top = \bot$

(iv) $\neg\bot = \top$

(v) $\neg\neg a = a$

(vi) $(a \wedge \neg a) \vee (b \vee \neg b) = b \vee \neg b$

(**i**), (**ii**) and (**v**) follows from **A** inducing an ortholattice cf. Theorem 3.14.4. (**iv**) will not be shown, since it follows analagously from (**iii**). (**iii**): $\neg\top = \neg(a \vee \neg a) = \neg a \wedge \neg\neg a = a \wedge \neg a = \bot$, where the complement law and Theorem 3.14.4 were used. (**vi**): $(a \wedge \neg a) \vee (b \vee \neg b) = \bot \vee \top = \top = b \vee \neg b$, where the complement and identity laws were used. ■

Theorem 3.14.6. *Every Boolean algebra is a Stone algebra.*

Proof. Let **A** be a Boolean algebra. By Theorem 3.14.5, **A** is a Kleene algebra as well, and therefore also an Ockham algebra. It thus suffices to show that the Stone identities are satisfied. (i) follows from **A** being an ortholattice by Theorem 3.14.4 and complement. (ii) follows directly from the complement. ■

Theorem 3.14.7. *Every Boolean algebra is a Heyting algebra.*

Proof. Let $\langle A, \vee, \wedge, \bot, \top, \neg\rangle$ be a Boolean algebra. Define the map $\rightarrow: A \times A \rightarrow A$, given by

$$a \rightarrow b := \neg a \vee b.$$

It has to be checked that the map $\rightarrow$ satisfies the requirements given in Heyting algebras. Let $a, b, c \in A$.

" $\Rightarrow$ ": Assume $a \wedge b \leq c$. Then $b \rightarrow c = \neg b \vee c \geq \neg b \vee (a \wedge b) = (\neg b \vee a) \wedge (\neg b \vee b) = (\neg b \vee a) \wedge \top = \neg b \vee a \geq a$, where the assumption, distributive law, complement, identity law and the duality of lattices to conclude $\neg b \vee a \geq a$.

" $\Leftarrow$ ": Assume $a \leq b \rightarrow c$. Then $a \wedge b \leq (b \rightarrow c) \wedge b = (\neg b \vee c) \wedge b = (\neg b \wedge b) \vee (b \vee c) = \bot \vee (b \vee c) = b \vee c \geq c$, where the assumption, distributive law, complement, identity law and the duality of lattices was used to conclude $b \vee c \geq c$. ■

Chapter 4

Ring structures

4.1 Semirings

Definition 4.1.1. A *semiring* $\langle A, +, \cdot, e_1, e_2 \rangle$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$, such that

(i) $\langle A, +, e_1 \rangle$ is a commutative monoid

(ii) $\langle A, \cdot, e_2 \rangle$ is a monoid

(iii) $\forall a, b, c \in A : a(b + c) = ab + ac$

(iv) $\forall a, b, c \in A : (a + b)c = ac + bc$

(v) $\forall a \in A : e_1 a = a e_1 = e_1$

Example 4.1.2. The natural numbers ω under ordinary addition and multiplication form a semiring $\langle \omega, +, \cdot, 0, 1 \rangle$.

4.2 Nonassociative rings

Definition 4.2.1. A *nonassociative ring* $\langle A, +, \cdot, e, - \rangle$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$ such that

(i) $\langle A, +, e, - \rangle$ is an Abelian group

(ii) $\langle A, \cdot \rangle$ is a magma

(iii) $\forall a, b, c \in A : a(b + c) = ab + ac$

(iv) $\forall a, b, c \in A : (a + b)c = ac + bc$

4.3 Rings

Definition 4.3.1. A *ring* $\langle A, +, \cdot, e, - \rangle$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$ such that

(R_1) $\langle A, +, e, - \rangle$ is an Abelian group

(R_2) $\langle A, \cdot \rangle$ is a semigroup

(R_3) $\forall a, b, c \in A : a(b + c) = ab + ac$

$\forall a, b, c \in A : (a + b)c = ac + bc$

Remark 4.3.2. Every ring reduces to a near-ring, since they only differ on $\langle A, +, e, - \rangle$ being a group or an Abelian group.

Remark 4.3.3. Every ring reduces to a nonassociative ring, since they only differ on $\langle A, \cdot \rangle$ being associative.

4.4 Commutative rings

Definition 4.4.1. A *commutative ring* $\langle A, +, \cdot, e, - \rangle$ is a ring in which $\cdot$ is commutative:

$$\forall a, b \in A : ab = ba.$$

4.5 Unitary rings

Definition 4.5.1. A *unitary ring* $\langle A, +, \cdot, e_1, e_2, - \rangle$ is a ring with a multiplicative identity element e_2:

$$\forall a \in A : e_2 a = a e_2 = a.$$

Remark 4.5.2. Every unitary ring is a unitary module, since every ring is a module cf. Theorem 5.2.3.

4.6 Division rings

Definition 4.6.1. An element a in a unitary ring $\langle A, +, \cdot, e_1, e_2, - \rangle$ is said to be *invertible*, or to be a *unit*, if

$$\exists a^{-1} \in A : a^{-1} a = a a^{-1} = e_2.$$

Definition 4.6.2. A *division ring* $\langle A, +, \cdot, e_1, e_2, -, ^{-1} \rangle$ is a unitary ring satisfying $e_1 \neq e_2$ and in which every element $a \neq e_1$ is a unit.

Example 4.6.3. The real quaternions $\mathbb{H}$, which extends the complex numbers $\mathbb{C}$, forms a noncommutative division ring $\langle \mathbb{H}, +, \cdot, 0, 1, - \rangle$.

4.7 Integral domains

Definition 4.7.1. An element $a \neq e$ in a ring $\langle A, +, \cdot, e, - \rangle$ is said to be a *zero divisor* if

$$\exists e \neq b \in A : ab = ba = e.$$

Definition 4.7.2. An *integral domain* $\langle A, +, \cdot, e_1, e_2, - \rangle$ is a commutative unitary ring with $e_1 \neq e_2$ and no zero divisors.

Remark 4.7.3. An equivalent definition of an integral domain is a commutative unitary ring, where the zero rule applies:

$$\forall a, b \in A : ab = e_1 \Rightarrow (a = e_1) \vee (b = e_1).$$

Example 4.7.4. The integers $\langle \mathbb{Z}, +, \cdot, 0, 1, - \rangle$ is an example of an integral domain.

4.8 Fields

Definition 4.8.1. A *field* $\langle A, +, \cdot, e_1, e_2, -, ^{-1} \rangle$ is a commutative division ring.

Remark 4.8.2. Every field is a vector space, since every ring is a module cf. Theorem 5.2.3.

Example 4.8.3. The real numbers $\langle \mathbb{R}, +, \cdot, 0, 1, -, ^{-1} \rangle$, the rational numbers $\langle \mathbb{Q}, +, \cdot, 0, 1, -, ^{-1} \rangle$ and the complex numbers $\langle \mathbb{C}, +, \cdot, 0, 1, -, ^{-1} \rangle$ are examples of fields.

Theorem 4.8.4. *Every field is an integral domain.*

Proof. Let $\langle A, +, \cdot, e_1, e_2, -, ^{-1} \rangle$ be a field. It has to be checked that the zero rule applies. Let $a, b \in A$ and assume $ab = e_1$ and $a \neq e_1$: $b = e_2 b = (a^{-1}a)b = a^{-1}(ab) = a^{-1}e_1 = e_1$. ■

4.9 Lie rings

Definition 4.9.1. A *Lie ring* $\langle A, +, \cdot, e, - \rangle$ is a nonassociative ring which satisfies

- $\forall a, b, c \in A : a(bc) + b(ca) + c(ab) = e$
- $\forall a \in A : aa = e$

Theorem 4.9.2. *Every ring* $\langle A, +, \cdot, e, - \rangle$ *induces a Lie ring* $\langle A, +, \cdot, e, -, \odot \rangle$.

Proof. Let $\mathbf{A}$ be a ring and let $a, b, c \in A$. Then define $\odot : A \times A \to A$, given by $a \odot b := ab - ba$. It has to be shown that $\odot$ satisfies the two properties of a Lie ring. The first is shown:

$$\begin{aligned}
& a \odot (b \odot c) + b \odot (c \odot a) + c \odot (a \odot b) \\
=& a \odot (bc - cb) + b \odot (ca - ac) + c \odot (ab - ba) \\
=& a(bc - cb) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c \\
=& abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\
=& (abc - abc) + (acb - acb) + (bca - bca) + (cba - cba) + (bac - bac) + (cab - cab) \\
=& e + e + e + e + e + e = e
\end{aligned}$$

The second is shown:

$$a \odot a = aa - aa = e.$$

Thus $\langle A, +, \cdot, e, -, \odot \rangle$ is a Lie ring. ■

4.10 Boolean ring

Definition 4.10.1. A *Boolean ring* $\langle A, +, \cdot, e, - \rangle$ is a ring, in which $\cdot$ is idempotent:

$$\forall a \in A : aa = a.$$

Example 4.10.2. Let A be a set and $\mathcal{P}(A)$ the powerset. Then $\langle \mathcal{P}(A), \Delta, \cap, \emptyset, Id \rangle$ is a Boolean ring, where Id is the identity map (every element is its own inverse) and Δ is the *symmetric difference*:

$$A \Delta B := (A \cup B) \backslash (A \cap B).$$

4.11 *-rings

Definition 4.11.1. A **-ring* $\langle A, +, \cdot, e, -, *\rangle$ is a ring $\langle A, +, \cdot, e, -\rangle$ equipped with the unary operation $*$, which satisfies for all $a, b \in A$:

- $(a + b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $a^{**} = a$

Chapter 5

Module structures

5.1 Groups with operators

Definition 5.1.1. A *group with operators* $\langle \mathbf{A}, \Omega, f\rangle$ is a group $\mathbf{A}$ along with a set Ω and the action $f : \Omega \times A \to A$ which satisfies

$$\forall a, b \in A, \gamma \in \Omega : f(\gamma, ab) = f(\gamma, a)f(\gamma, b).$$

Remark 5.1.2. Every group $\mathbf{A}$ trivially induces a group with operators $\langle \mathbf{A}, \emptyset, f\rangle$.

5.2 Modules

Definition 5.2.1. Let $\mathbf{R}$ be a ring. An ($\mathbf{R}$-)*module* $\langle A, +, e, -, \lambda\rangle$ is an Abelian group $\langle A, +, e, -\rangle$ together with a map $\lambda : R \to A^A$, such that for all $r, s \in R$ and $a, b \in A$:

(i) $\lambda_r(a + b) = \lambda_r(a) + \lambda_r(b)$

(ii) $\lambda_{rs}(a) = \lambda_r(a) + \lambda_s(a)$

(iii) $\lambda_r(\lambda_s(a)) = \lambda_{rs}(a)$

Theorem 5.2.2. *Every module induces a group with operators.*

Proof. Let $\Omega = R$ and $f(r, a) = \lambda_r(a)$. Then it follows from λ's first property. ∎

Theorem 5.2.3. *Every ring* $\mathbf{R} := \langle A, +, e, -, \cdot\rangle$ *induces a module* $\langle \mathbf{R}, \lambda\rangle$*.*

Proof. Set $\lambda : R \to R^R$ be given by $\lambda_r(s) := r + s$. ∎

5.3 Unitary modules

Definition 5.3.1. Let $\mathbf{R}$ be a unitary ring. Then a module $\mathbf{A}$ over $\mathbf{R}$ is called a *unitary module.*

5.4 Vector spaces

Definition 5.4.1. An ($\mathbb{F}$)-*vector space* $\mathbf{A}$ is an ($\mathbb{F}$)-module, where $\mathbb{F}$ is a field.

5.5 Nonassociative algebras

Definition 5.5.1. A *nonassociative* ($\mathbf{R}$-)*algebra* $\mathbf{A} = \langle \mathbf{M}, [\cdot,\cdot] \rangle$ is an $\mathbf{R}$-module $\mathbf{M}$ over a commutative ring $\mathbf{R}$ equipped with a binary operation $[\cdot,\cdot]$ on A, called $\mathbf{A}$-multiplication, which satisfies for all $r, s \in R$ and $a, b, c \in A$:

(i) $[\lambda_r(a) + \lambda_s(b), c] = \lambda_r([a, c]) + \lambda_s([b, c])$

(ii) $[c, \lambda_r(a) + \lambda_s(b)] = \lambda_r([c, a]) + \lambda_s([c, b])$

5.6 Algebras

Definition 5.6.1. An $\mathbf{R}-$*algebra* $\mathbf{A}$ is a nonassociative $\mathbf{R}-$algebra, in which its binary operation $[\cdot,\cdot] : A \times A \to A$ is associative:

$$\forall a, b, c \in A : [[a, b], c] = [a, [b, c]].$$

5.7 Banach algebras

Definition 5.7.1. A *Banach algebra* $\langle \mathbf{A}, [\cdot,\cdot] \rangle$ is an $\mathbf{R}-$algebra over a Banach space $\mathbf{A}$, which satisfies

$$\forall a, b \in A : ||ab|| \leq ||a|| \, ||b|| \, .$$

5.8 *-algebras

Definition 5.8.1. A *-*algebra* $\mathbf{A}$ is an $\mathbf{R}-$algebra $\mathbf{A}$, in which the ring $\mathbf{R}$ in addition is a *-ring, $R \subseteq A$ and

$$\forall r, s \in R, a, b \in A : (\lambda_r(a) + \lambda_s(b))^* = \lambda_{r^*}(a^*) + \lambda_{s^*}(b^*).$$

5.9 C*-algebras

Definition 5.9.1. A C^*-*algebra* $\mathbf{A}$ is a complex Banach algebra and a *-algebra, which satisfies

$$\forall a \in A : ||a^* a|| = ||a||^2 \, .$$

5.10 Lie algebras

Definition 5.10.1. A *Lie algebra* $\mathbf{A}$ is a nonassociative $\mathbb{F}$-algebra over a field $\mathbb{F}$, in which the binary operation $[\cdot,\cdot]$ satisfies alternation and the Jacobi identity:

(i) $\forall a \in A : [a,a] = e$

(ii) $\forall a,b,c \in A : [a,[b,c]] + [b,[c,a]] + [c,[a,b]] = e$

Chapter 6

Metric structures

6.1 Metric spaces

Definition 6.1.1. Let $A \neq \emptyset$ be a set. A map $d : A \times A \to \mathbb{R}$ is called a *metric*, if it satisfies the following for arbitrary $a, b, c \in A$:

(M_1) $d(a, b) \geq 0$, $d(a, b) = 0 \Leftrightarrow a = b$

(M_2) $d(a, b) = d(b, a)$

(M_3) $d(a, b) \leq d(a, c) + d(c, b)$

Definition 6.1.2. A *metric space* $\langle A, d\rangle$ is a set $A \neq \emptyset$ equipped with a metric d.

6.2 Complete spaces

Definition 6.2.1. Let $\mathbf{A}$ be a metric space, and let $(m_i)_{i\in\mathbb{N}} \subseteq A$ be a sequence in A. Then $(m_i)_{i\in\mathbb{N}}$ is called a *Cauchy sequence* if

$$\forall j, k \in \mathbb{N} : \lim_{j,k\to\omega} d(m_j, m_k) = 0.$$

Definition 6.2.2. A *complete space* $\langle A, d\rangle$ is a metric space in which all Cauchy sequences in A converge.

6.3 Normed vector spaces

Definition 6.3.1. A *norm* on a vector space $\mathbf{A}$ is a map $||\cdot|| : A \to \mathbb{R}$, which satisfies the following:

(N_1) $\forall a \in A : ||a|| \geq 0$, $||a|| = 0 \Leftrightarrow a = e_A$

(N_2) $\forall a \in A, r \in \mathbb{R}^n : ||\lambda_r(a)|| = |r| \times ||a||$

(N_3) $\forall a, b \in A : ||a + b|| \leq ||a|| + ||b||$

Definition 6.3.2. A *normed* $\mathbb{F}-$*vector space* $\mathbf{A}$ is a vector space where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $||\cdot||$.

Theorem 6.3.3. *Every normed vector space* $\mathbf{A}$ *induces a metric space* $\langle \mathbf{A}, d \rangle$.

Proof. Define the map $d : A \times A \to \mathbb{F}$ given by

$$d(a, b) := ||b - a||.$$

It has to be checked that d is a metric. (M_1) follows directly from (N_1). Let $r = -1$. Then by (N_2), with $a, b \in A$:

$$||b - a|| = ||\lambda_r(a - b)|| = |r| \cdot ||a - b|| = ||a - b||,$$

which shows (M_2). Lastly, by (N_3), with $a, b, c \in A$:

$$d(a, b) = ||a - b|| = ||(a - c) + (c - b)|| \overset{(N_3)}{\leq} ||a - c|| + ||c - b|| = d(a, c) + d(c, b),$$

which then proves that d is a metric and $\mathbf{A}$ a metric space. ■

6.4 Banach spaces

Definition 6.4.1. A *Banach space* is a complete normed vector space.

6.5 Inner product spaces

Definition 6.5.1. Let bA, be an $\mathbb{F}$-vector space. Then an *inner product* on A is a map $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{F}$ with satisfies the following for all $a, b, c \in A$ and $\alpha, \beta \in \mathbb{F}$:

(IP_1) $\langle a, a \rangle > e \Leftrightarrow a \neq e$

(IP_2) $\langle a, b \rangle = \overline{\langle b, a \rangle}$

(IP_3) $\langle \lambda_\alpha(a) + \lambda_\beta(b), c \rangle = \alpha \langle a, c \rangle + \beta \langle b, c \rangle$

Definition 6.5.2. An *inner product space* $\langle \mathbf{A}, \langle \cdot, \cdot \rangle \rangle$ is a vector space bA equipped with an inner product $\langle \cdot, \cdot \rangle$.

Theorem 6.5.3. *Every inner product space* $\mathbf{A}$ *induces a normed vector space* $\langle \mathbf{A}, ||\cdot|| \rangle$.

Proof. Let $\mathbf{A}$ be an inner product space. Define the map $||\cdot|| : A \to [0,\omega)$, given by $||a|| := \sqrt{\langle a, a\rangle}$. It has to be shown that $||\cdot||$ is a norm. Because of (IP_1), $\sqrt{\langle a, a\rangle}$ is well-defined. (N_1) follows from (IP_1). (N_2) is shown:

$$||\alpha a||^2 \overset{def}{=} \langle \lambda_\alpha(a), \lambda_\alpha(a)\rangle \overset{(IP_3),(IP_2)}{=} \alpha\bar{\alpha}\,\langle a, a\rangle = |\alpha|^2\,||a||^2\,.$$

The last (N_3) is shown:

$$\begin{aligned} ||a+b||^2 = \langle a+b, a+b\rangle &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= ||a||^2 + 2\mathrm{Re}\,\langle a, b\rangle + ||b||^2 \\ &\leq ||a||^2 + 2\,|\langle a, b\rangle| + ||b||^2 \\ &\overset{(*)}{\leq} ||a||^2 + 2\,||a||\,||b|| + ||b||^2 \\ &= (||a|| + ||b||)^2 \end{aligned}$$

At (*), the Cauchy-Schwarz' inequality is used. By taking the square root of both sides, (N_3) is shown, and thus $\langle \mathbf{A}, ||\cdot|| \rangle$ is a normed space. ■

6.6 Hilbert spaces

Definition 6.6.1. A *Hilbert space* is a complete inner product space.

Corollary 6.6.2. *Every Hilbert space* $\mathbf{A}$ *induces a Banach space* $\langle \mathbf{A}, ||\cdot|| \rangle$.

Proof. It follows from the fact that every inner product space induces a normed space cf. Theorem 6.5.3. ■

6.7 Euclidean spaces

Definition 6.7.1. A *Euclidean space* $\mathbf{A}$ is a Hilbert space $\mathbf{A}$ over the field $\mathbb{R}$ where the inner product is the dot product and e is the zero vector $\underline{0}$. Furthermore, $\lambda_r(s)$ is often denoted as $r \cdot s$ without confusion.

Chapter 7

Paving structures

Definition 7.0.2. A *family of sets* indexed by the class $I \neq \emptyset$ is a class of sets $(A_i)_{i\in I} := \{A_i \mid i \in I\}$. For such a family, its union and intersection are defined to be respectively the classes

$$\bigcup_{i\in I} A_i := \{a \mid \exists i \in I : a \in A_i\}$$
$$\bigcap_{i\in I} A_i := \{a \mid \forall i \in I : a \in A_i\}$$

7.1 Topological spaces

Definition 7.1.1. A *topology* on a set A is a family of subsets of A denoted $\mathcal{O}$, which satisfies

(T_1) $\emptyset, A \in \mathcal{O}$.

(T_2) $\forall G_1, \ldots, G_n \in \mathcal{O} : G_1 \cap \ldots \cap G_n \in \mathcal{O}$

(T_3) $(A_i)_{i\in I} \in \mathcal{O} \Rightarrow \bigcup_{i\in I} A_i \in \mathcal{O}$.

Definition 7.1.2. A *topological space* $\langle A, \mathcal{O}\rangle$ is a set A equipped with a topology $\mathcal{O}$.

Definition 7.1.3. Let A be a set. Then a *basis* for a topology $\mathcal{O}$ on A is a collection $\mathcal{B}$ of subsets of A (called basis elements), such that

(i) $\forall a \in A \exists B \in \mathcal{B} : x \in B$

(ii) $\forall a \in A \forall B_1, B_2 \in \mathcal{B} : a \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$

Lemma 7.1.4. *Let $\langle A, \mathcal{O}\rangle$ be a topological space and let $\mathcal{B}$ be a basis for $\mathcal{O}$. Then $\mathcal{O}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{O}$. Because $\mathcal{O}$ is a topology, their union is in $\mathcal{O}$. Conversely, given $U \in \mathcal{O}$, choose for each $a \in U$ an element $B_a \in \mathcal{B}$ such that $a \in B_a \subseteq U$. Then $U = \bigcup_{a\in U} B_a$, so U equals a union of elements of $\mathcal{B}$.■

Lemma 7.1.5. *Let $\langle A, \mathcal{O}\rangle$ be a topological space. If $\mathcal{C} \subseteq \mathcal{O}$ satisfies*

$$\forall S \in \mathcal{O}, a \in S \exists C \in \mathcal{C} : a \in C \subseteq S,$$

then $\mathcal{C}$ is a basis for $\mathcal{O}$.

Proof. We must show that $\mathcal{C}$ is a basis. The first condition for a basis is shown: Given $a \in A$, since $A \in \mathcal{O}$, there is by assumption a $C \in \mathcal{C}$ such that $x \in C \subseteq A$. To check the second condition, let $a \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$. Since $C_1, C_2 \in \mathcal{O}$, then $C_1 \cap C_2 \in \mathcal{O}$.Therefore, there exists by assumption an element $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$.

We must now show that the topology $\mathcal{O}'$ generated by $\mathcal{C}$ equals $\mathcal{O}$. First, note that if $U \in \mathcal{O}$ and if $x \in U$, then there is by assumption an element $C \in \mathcal{C}$ such that $x \in C \subseteq U$. It follows that $U \in \mathcal{O}'$ by definition. Conversely, if $W \in \mathcal{O}'$, then W is a union of elements of $\mathcal{C}$, by Lemma 7.1.4. Since each element of $\mathcal{C}$ belongs to $\mathcal{O}$ and $\mathcal{O}$ is a topology, $W \in \mathcal{O}$. ■.

Theorem 7.1.6. *Every metric space $\langle A, d\rangle$ induces a topological space $\langle A, d, \mathcal{O}\rangle$.*

Proof. Let $\langle A, d\rangle$ be a metric space and denote $B_d(a, \varepsilon) := \{b \in A \mid d(a, b) < \varepsilon\}$ the ε-ball centered on $b \in A$. Now define $\mathcal{G}$ as the family of all ε-balls $B_d(x, \varepsilon)$ for $a \in A$ and $\varepsilon > 0$. It is clear that $\mathcal{G}$ satisfies the condition given in Lemma 7.1.5, meaning that $\mathcal{G}$ is a basis for a topological space. Then by Lemma 7.1.4, the collection of all unions of $\mathcal{G}$ is a topology $\mathcal{O}$ on A. Thus is $\langle A, d, \mathcal{O}\rangle$ an induced topological space. ■

Theorem 7.1.7. *Every topological space $\langle A, \mathcal{O}\rangle$ induces monoids $\langle A, \mathcal{O}, \cup, \emptyset\rangle$ and $\langle A, \mathcal{O}, \cap, A\rangle$.*

Proof. Let $\langle A, \mathcal{O}\rangle$ be a topological space. It has to be shown that the induced $\langle A, \mathcal{O}, \cup\rangle$ is a semigroup and has an identity element e, since the proof for inducing $\langle A, \mathcal{O}, \cap\rangle$ follows analogously. To show that it is a semigroup, it is firstly noted that it is a magma, since $\cup$ is a binary operation by (T_3). It has to be shown that $\cup$ is associative:

$$\begin{aligned} a \in A \cup (B \cup C) &\Leftrightarrow a \in A \vee (a \in B \vee a \in C) \\ &\Leftrightarrow (a \in A \vee a \in B) \vee a \in C \\ &\Leftrightarrow a \in (A \cup B) \cup C \end{aligned}$$

Thus is $\langle A, \mathcal{O}, \cup \rangle$ a semigroup, and it has to be shown that it has an identity. Let $S \in \mathcal{O}$ and consider the empty set $\emptyset$, which exists by (T_1):

$$\emptyset \cup S = \{a \in A \mid a \in S \vee a \in \emptyset\} \overset{??}{=} \{a \in A \mid a \in S\} = S$$
$$S \cup \emptyset = \{a \in A \mid a \in \emptyset \vee a \in S\} \overset{??}{=} \{a \in A \mid a \in S\} = S$$

Thus is $e = \emptyset \in \mathcal{O}$, making $\langle A, \mathcal{O}, \cup, \emptyset \rangle$ a monoid. Similarly it can be shown that $\langle A, \mathcal{O}, \cap, A \rangle$ is a monoid as well by (T_1) and (T_2). ■

Theorem 7.1.8. *Every topological space $\langle A, \mathcal{O} \rangle$ induces a complete JID lattice $\langle A, \mathcal{O}, \cup, \cap, \emptyset, A \rangle$.*

Proof. Let $\langle A, \mathcal{O} \rangle$ be a topological space. It was shown in Theorem 7.1.7 that both $\langle A, \mathcal{O}, \cup, \emptyset \rangle$ and $\langle A, \mathcal{O}, \cap, A \rangle$ were monoids. Since $\cup$ and $\cap$ are well known to be both idempotent and commutative, it has to be shown that $\langle A, \mathcal{O}, \cup, \cap, \emptyset, A \rangle$ satisfies the absorption laws, distributive laws and completeness. The absorption laws and distributive laws follows analogously from the proof in Theorem 7.14.5, so it has to be shown that every set $S_i \subseteq A$ has both a supremum $\bigvee S_i := \bigcup S_i$ and an infimum $\bigwedge S_i := \bigcap S_i$.

By (T_2), $\mathbf{A}$ is closed under arbitrary unions, meaning $\bigvee S_i$ exists. Since topologies aren't closed under arbitrary intersections, the infimum can instead be constructed by taking the interior of arbitrary intersections. Since interiors are always open, the infimum exists. Moreover, the join-infinite distribution law is shown. Let $a \in A, S_0 \in \mathcal{O}, (S_i)_{i \in I} \subseteq \mathcal{O}$. Then

$$\begin{aligned} a \in S_0 \cap \bigcup_{i \in I} S_i &\Leftrightarrow (a \in S_0) \wedge (a \in \bigcup_{i \in I} S_i \Leftrightarrow (a \in S_0) \wedge (\exists i \in I : a \in S_i) \\ &\Leftrightarrow \exists i \in I : (a \in S_i) \wedge (a \in S_0) \Leftrightarrow a \in \bigcup_{i \in I} (S_i \cap S_0). \end{aligned}$$ ■

7.2 First-countable spaces

Definition 7.2.1. Let $\langle A, \mathcal{O} \rangle$ be a topological space and let $a \in A$. Then a *neighbourhood* of a is a subset $S \subseteq A$ which satisfies that

$$\exists O \in \mathcal{O} : a \in S \subseteq O.$$

Definition 7.2.2. A *first-countable space* $\langle A, \mathcal{O} \rangle$ is a topological space with a countable neighbourhood basis - that is, for every $a \in A$ there exists a sequence of open neighbourhoods $(S_i)_{i \in \mathbb{N}}$ of a, such that

$$\forall U \in \mathcal{O} \exists i \in \mathbb{N} : x \in U \Rightarrow x \in S_i \subseteq U.$$

Theorem 7.2.3. *Every metric space $\langle A, d\rangle$ induces a first-countable space $\langle A, d, \mathcal{O}\rangle$.*

Proof. Let $\langle A, d\rangle$ be a metric space and $a \in A$. Since every metric space induces a topological space by Theorem 7.1.6, it has to be shown that $\mathbf{A}$ has a countable local basis. Define such a candidate $\mathcal{B} := \{B_{1/n}(a) \mid n \in \mathbb{N}\}$, where $B_\varepsilon(a)$ denotes the open ε-ball in A with center in a and radius ε.

First of all, since a surjection $f : \mathbb{N} \to \mathcal{B}$ can be created, $\mathcal{B}$ is countable. It is clear that every element $b \in \mathcal{B}$ is an open neighbourhood of a, since the balls are open with a as center. Now let $U \subseteq A$ be an open neighbourhood of a. By definition of open sets, an open ball $B_\varepsilon(a) \subseteq U$. By the Archimedean principle, there exist an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$ which implies $\varepsilon > \frac{1}{\varepsilon}$, meaning $B_{1/n}(a) \subseteq B_\varepsilon(a) \subseteq U$. This proves that $\mathcal{B}$ is a countable local basis for A. ■

7.3 Separable spaces

Definition 7.3.1. A *separable space* $\langle A, \mathcal{O}\rangle$ is a topological space in which there exist a countable dense subset - that is, a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, which satisfies

$$\forall U \in \mathcal{O} \exists i \in \mathbb{N} : a_i \in U.$$

7.4 Second-countable spaces

Definition 7.4.1. A *second-countable space* $\langle A, \mathcal{O}, \mathcal{B}\rangle$ is a topological space $\langle A, \mathcal{O}\rangle$ with a countable collection of sets $\mathcal{B} = (B_i)_{i\in\mathbb{N}} \subseteq \mathcal{O}$, called a countable basis, such that

$$\forall V \in \mathcal{O} \exists (B_k)_{k\in\mathbb{N}} \subseteq \mathcal{B} : V = \bigcup_{k\in\mathbb{N}} B_k.$$

Theorem 7.4.2. *Every second-countable space reduces to both a first-countable space and a separable space.*

Proof. Let $\langle A, \mathcal{O}, \mathcal{B}\rangle$ be a second-countable space. For each $a \in A$, the set $\{U \in \mathcal{B} \mid a \in U\}$ is a countable neighbourhood basis. Therefore it is a first-countable space. Let S consist of one point from each member of $\mathcal{B}$. Then S is a countable dense subset of A. Therefore it is a separable space. ■

Theorem 7.4.3. *Every separable metric space $\langle A, d, \mathcal{O}\rangle$ induces a second-countable space $\langle A, d, \mathcal{O}, \mathcal{B}\rangle$.*

Proof. Let $\langle A, d\rangle$ be a metric space with a countable dense subset S. We will show that the set $\mathcal{B}$ of open balls with rational radii around points in S form a countable basis for A. Consider the basic open set $B_r(a)$, for some $a \in A$ and $r > 0$. Let $b \in B_r(a)$. There is an $s > 0$ such that $B_s(b) \subseteq B_r(x)$. Let $c \in S \cap B_{s/3}(b)$ and let $t \in (s/3, 2s/3) \cap \mathbb{Q}$. Then $b \in B_t(c)$, and it follows from the triangle inequality that $B_t(c) \subseteq B_s(b) \subseteq B_r(a)$. So we have shown that every point in $B_r(a)$ is inside a set in $\mathcal{B}$ that is contained in $B_r(a)$. Therefore $B_r(a)$ is a union of members of $\mathcal{B}$, and therefore $\mathcal{B}$ is a basis. Since $\mathcal{B}$ is in one-to-one correspondance with the product $S \times ((0, \omega) \cap \mathbb{Q})$ of countable sets, $\mathcal{B}$ is countable. ■

Theorem 7.4.4. *Every Euclidean space* $\mathbf{A}$ *induces a second-countable space* $\langle \mathbf{A}, \mathcal{O}, \mathcal{B} \rangle$.

Proof. Let $\mathbf{A}$ be a Euclidean space. By definition, $\mathbf{A}$ is a Hilbert space and an inner product space. Then by Theorem 6.5.3, $\mathbf{A}$ induces a normed space. Then by Theorem 6.3.3, $\mathbf{A}$ induces a metric space. By Theorem 7.4.3, it thus suffices to show that $\mathbf{A}$ is separable, ie. that it has a countable dense subset. Consider $\mathbb{Q}^n$. It is known that $\mathbb{Q}^n$ is countable for all $n \in \mathbb{N}$. Furthermore, it is clear that $\forall n \in \mathbb{N} : \mathbb{Q}^n \subseteq \mathbb{R}^n$. Lastly, it is also known that $\mathbb{Q}$ is dense in $\mathbb{R}$, making $\mathbb{Q}^n$ dense in $\mathbb{R}^n$ by applying it to each coordinate. Thus by Theorem 7.4.3, $\mathbf{A}$ is a second-countable space. ■

7.5 Hausdorff spaces

Definition 7.5.1. A *Hausdorff space* $\langle A, \mathcal{O} \rangle$ is a topological space, which satisfies that there for each $a, b \in A$ exist disjoint neighbourhoods U, V around a and b, respectively.

Theorem 7.5.2. *Every metric space* $\langle A, d \rangle$ *induces a Hausdorff space* $\langle A, d, \mathcal{O} \rangle$.

Proof. Let $\langle A, d \rangle$ be a metric space and $a, b \in A$ with $a \neq b$. Then by (M_1), $d(a, b) \neq 0$. By Theorem 7.1.6, $\langle A, d, \mathcal{O} \rangle$ is a topological space where $\mathcal{O}$ is the topology generated from the open ε-balls. Construct two open ε-balls $B_a := B_d(a, d(x, y)/2)$ and $B_b := B_d(b, d(x, y)/2)$, which contain a and b, respectively. If it is shown that B_a and B_b are disjoint, then we are done. Assume $c \in B_a$ and $c \in B_b$. Then $d(c, a) < d(a, b)/2$ and $d(c, b) < d(a, b)/2$. But then $d(c, a) + d(c, b) < d(a, b)$, which contradicts (M_3). Thus is the induced topological space $\langle A, d, \mathcal{O} \rangle$ a Hausdorff space. ■

7.6 Manifolds

Definition 7.6.1. Let $\mathbf{A} = \langle A, \mathcal{O}_A \rangle$ and $\mathbf{B} = \langle B, \mathcal{O}_B \rangle$ be topological spaces. A map $f : A \to B$ is then called *continuous* if it for every $S_B \in \mathcal{O}_B$ holds that $f^{-1}(S_B) \in \mathcal{O}_A$.

Definition 7.6.2. A *homeomorphism* $f : \mathbf{A} \to \mathbf{B}$ is a continuous map between topological spaces $\mathbf{A}$ and $\mathbf{B}$, that has a continuous inverse map. If such a map exists, $\mathbf{A}$ and $\mathbf{B}$ are said to be homeomorphic.

Definition 7.6.3. A *local homeomorphism* $f : \mathbf{A} \to \mathbf{B}$ is a map between topological spaces $\mathbf{A} = \langle A, \mathcal{O}_A \rangle$ and $\mathbf{B} = \langle B, \mathcal{O}_B \rangle$, which satisfies that for each $a \in A$ exist a neighbourhood $S \in \mathcal{O}_A$, which satisfies $f(S) \in \mathcal{O}_B$ and that $f|_S : S \to f(S)$ is a homeomorphism. If such a map exists, $\mathbf{A}$ and $\mathbf{B}$ are said to be locally homeomorphic.

Definition 7.6.4. An n-dimensional (real) *manifold* $\langle A, \mathcal{O}, \mathbb{R}^n \rangle$ is a second-countable Hausdorff space that is locally homeomorphic to a Euclidean space $\mathbb{R}^n$ by a collection of homeomorphisms $(\varphi_i)_{i \in I}$, called *charts* - such a collection is called an *atlas*.

7.7 Smooth manifolds

Definition 7.7.1. Let $\langle A, \mathcal{O}, \mathbb{R}^n \rangle$ be a manifold with atlas $(\varphi_i)_{i \in I}$. Then a *transition map* is a map $\tau : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\tau = \varphi_i \circ \varphi_j^{-1} \qquad , \varphi_i, \varphi_j \in (\varphi_i)_{i \in I}.$$

Definition 7.7.2. A map f is called *smooth* if derivatives of all orders exist.

Definition 7.7.3. A *smooth manifold* $\langle A, \mathcal{O}, \mathbb{R}^n \rangle$ is a manifold $\langle A, \mathcal{O}, \mathbb{R}^n \rangle$, where all the transition maps in the associated atlas $(\varphi_i)_{i \in I}$ are smooth.

7.8 Lie groups

Definition 7.8.1. A (real) *Lie group* $\langle A, \mathcal{O}, \mathbb{R}^n, \cdot, e, ^{-1} \rangle$ is a smooth manifold $\langle A, \mathcal{O}, \mathbb{R}^n \rangle$ and a group $\langle A, \cdot, e, ^{-1} \rangle$ in which the operations $\cdot$ and $^{-1}$ are smooth maps.

Example 7.8.2. The *general linear group* $GL(n, \mathbb{R}) := \langle A, \mathcal{O}, \mathbb{R}^{n^2}, \cdot, I_n, ^{-1} \rangle$ where $A \subseteq \mathbb{R}^{n^2}$ is the set of invertible $n \times n$ matrices with real entries, $\cdot$ is the matrix product and I_n is the $n \times n$ identity matrix, is a Lie group.

7.9 Disconnected spaces

Definition 7.9.1. A *disconnected space* $\langle A, \mathcal{O} \rangle$ is a topological space in which there exist a collection of disjoint subspaces $(A_i)_{i \in I}$, which satisfies

$$A = \bigcup_{i \in I} A_i.$$

7.10 Totally disconnected spaces

Definition 7.10.1. A *totally disconnected space* $\langle A, \mathcal{O} \rangle$ is a disconnected space, in which the connected subspaces are one-point sets.

7.11 Compact spaces

Definition 7.11.1. A *compact space* $\langle A, \mathcal{O} \rangle$ is a topological space in which each of its open covers has a finite subcover. E.g., for every collection $(U_i)_{i \in I} \subseteq \mathcal{O}$ with $A = \bigcup_{i \in I} U_i$ exists a finite set $J \subseteq I$, such that

$$A = \bigcup_{i \in J} U_i.$$

7.12 Boolean spaces

Definition 7.12.1. A *Boolean space* $\langle A, \mathcal{O} \rangle$ is a totally disconnected compact Hausdorff space.

7.13 Uniform spaces

Definition 7.13.1. A *uniformity* ϕ on a set A is a nonempty family of subsets of $A \times A$, that satisfies the following:

(U_1) $\forall U \in \phi : \{(a, a) \mid a \in A\} \in U$

(U_2) $\forall U \in \phi, V \subseteq A \times A : U \subseteq V \Rightarrow V \in \phi$

(U_3) $\forall U, V \in \phi : U \cap V \in \phi$

(U_4) $\forall U \in \phi \exists V \in \phi : (a, b), (b, c) \in V \Rightarrow (a, c) \in U$

(U_5) $\forall U \in \phi : \{(b, a) \mid (a, b) \in U\} \in \phi$

Definition 7.13.2. A *uniform space* $\langle A, \phi \rangle$ is a set A equipped with a uniformity ϕ.

Theorem 7.13.3. *Every uniform space $\langle A, \phi \rangle$ induces a topological space $\langle A, \phi, \mathcal{O} \rangle$.*

Proof. Let $\langle A, \phi \rangle$ be a uniform space. Define a subset $S \subseteq A$ to be open if and only if $\forall a \in S \exists U \in \phi : \{b \in A \mid (a, b) \in U\} \subseteq S$. Define $\mathcal{O} := \{S \subseteq A \mid S \text{ open}\}$. It has to be shown that $\mathcal{O}$ is a topology.

($\mathbf{T_1}$): It holds that $\emptyset \in \mathcal{O}$ trivially, since it doesn't have any elements. A is open as well, by picking $U := \{(a,a) \mid a \in A\}$, which is an element in ϕ by (U_1).

($\mathbf{T_2}$): Let $G_1, \ldots, G_n \in \mathcal{O}$. It has to be shown that $G_1 \cap \ldots \cap G_n \in \mathcal{O}$. By definition there exist $U_1, \ldots, U_n \in \phi$ such that $\forall 1 \leq i \leq n \forall a \in G_i : \{b \in A \mid (a,b) \in U_i\} \subseteq G_i$. It has to be shown that there exist $U_0 \in \phi$ such that $\forall a \in G_1 \cap \ldots \cap G_n : \{b \in A \mid (a,b) \in U_0\} \subseteq G_1 \cap \ldots \cap G_n$. Define $U_x := U_1 \cap \ldots \cap U_n$. It thus has to be shown that $U_x = U_0$. Let $a_0 \in G_1 \cap \ldots \cap G_n$. Then by definition $a_0 \in G_1 \wedge \ldots \wedge a_0 \in G_n$, meaning that $\forall 1 \leq i \leq n : \{b \in A \mid (a_0,b) \in U_i\} \subseteq G_i$. But this means that $\{b \in A \mid (a_0,b) \in U_x\} \subseteq G_1 \cap \ldots \cap G_n$. Since the choice of a_0 was arbitrary, it means that $U_x = U_0$.

($\mathbf{T_3}$): Let $(A_i)_{i\in\mathbb{N}} \in \mathcal{O}$. It has to be shown that $A_0 := \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{O}$. By definition, there exist $(U_i)_{i\in\mathbb{N}} \in \phi$ such that $\forall i \in \mathbb{N} \forall a \in A_i : \{b \in A \mid (a,b) \in U_i\} \subseteq A_i$. It has to be shown that there exist U_0 such that $\forall a \in A_0 : \{b \in A \mid (a,b) \in U_0\} \subseteq A_0$. Define $U_x := \bigcup_{i\in\mathbb{N}} U_i$. It has to be shown that $U_x = U_0$. Let $a_0 \in A_0$. Then $\exists i \in \mathbb{N} : a_0 \in A_i$ and thus $\{b \in A \mid (a_0,b) \in U_i\} \subseteq A_i \subseteq A_0$. Since this holds for all $i \in \mathbb{N}$, it is seen that $\{b \in A \mid (a,b) \in U_x\} \subseteq A_0$. Since the choice of a_0 was arbitrary, it means that $U_x = U_0$.■

Theorem 7.13.4. *Every metric space $\langle A, d\rangle$ induces a uniform space $\langle A, d, \phi\rangle$.*

Proof. Let $\langle A, d\rangle$ be a metric space. Define $\delta_\varepsilon := \{(a,b) \in A \times A \mid d(a,b) < \varepsilon\}$ and $\phi := \{\delta_\varepsilon \mid \varepsilon > 0\}$. It has to be shown that ϕ is a uniformity on A.

($\mathbf{U_1}$): Since $d(a,a) = 0$ by (M_1) and since $\varepsilon > 0$, (a,a) will be included in every d_ε for all $a \in A$.

($\mathbf{U_2}$): Let $V \subseteq A \times A$, $U \in \phi$ and $U \subseteq V$. It has to be shown that $V \in \phi$. Since $U \in \phi$, there exist an $\varepsilon > 0$ such that $U = \delta_\varepsilon$. Since $U \subseteq V$ then $\delta_\varepsilon \subseteq V$. If $\max\{d(a,b) \mid (a,b) \in V\} < \varepsilon$, then trivially $V \in \phi$. If on the other hand $\max\{d(a,b) \mid (a,b) \in V\} \geq \varepsilon$, then set $\varepsilon_0 := \max\{d(a,b) \mid (a,b) \in V\}$, meaning $V = \delta_{\varepsilon_0} \Rightarrow V \in \phi$.

($\mathbf{U_3}$): Let $U, V \in \phi$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U = \delta_{\varepsilon_1}$ and $V = \delta_{\varepsilon_2}$. Then it is clear that $U \cap V = \delta_{\varepsilon_1} \cap \delta_{\varepsilon_2} := \min\{\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\} \in \phi$.

($\mathbf{U_4}$): Let $U \in \phi$. Then there exist an $\varepsilon > 0$ such that $U = \delta_\varepsilon$. Define $V := \delta_{\varepsilon/2}$ and let $(a,b), (b,c) \in V$. Then $d(a,b), d(b,c) < \varepsilon/2$ by definition, and then $d(a,c) \leq d(a,b) + d(b,c) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (M_3), meaning $d(a,c) < \varepsilon$ and then $(a,c) \in U$.

($\mathbf{U_5}$) follows from (M_2). ■

7.14 Measurable spaces

Definition 7.14.1. A *σ-algebra* $\mathcal{A}$ on a set A is a family of subsets of A with the following properties:

(Σ_1) $A \in \mathcal{A}$

(Σ_2) $S \in \mathcal{A} \Rightarrow S^c \in \mathcal{A}$

(Σ_3) $(S_i)_{i\in\mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcup_{i\in\mathbb{N}} S_i \in \mathcal{A}$

Definition 7.14.2. A *measurable space* $\langle A, \mathcal{A}\rangle$ is a set A, equipped with a σ-algebra $\mathcal{A}$.

Remark 7.14.3. Every measurable space doesn't induce a topological space, and vice versa.

Proof. That every measurable space can't induce a topological space is due to the fact that topologies requires closure of arbitrarily many unions (T_3) and σ-algebras only required closure of countably many unions (Σ_3). The reverse is true since σ-algebras require closure under complements (Σ_2), and this is only always true for X and $\emptyset$ in topologies (T_1). ■

Theorem 7.14.4. *Every measurable space $\langle A, \mathcal{A}\rangle$ induces monoids $\langle A, \mathcal{A}, \cup, \emptyset\rangle$ and $\langle A, \mathcal{A}, \cap, A\rangle$.*

Proof. The proof follows analogously from Theorem 7.1.7, since $A \in \mathcal{A}$ by (Σ_1), $\emptyset \in \mathcal{A}$ by (Σ_1) and (Σ_2), $\cup$ is closed by (Σ_3) and $\cap$ is closed by (Σ_2) and (Σ_3), since $a\cap b = (a^c \cup b^c)^c$. ■

Theorem 7.14.5. *Every measurable space $\langle A, \mathcal{A}\rangle$ induces a Boolean algebra $\langle A, \mathcal{A}, \cup, \cap, \emptyset, A, ^c\rangle$.*

Proof. Let $\langle A, \mathcal{A}\rangle$ be a measurable space. Since $\mathbf{A}$ is a monoid by Theorem 7.14.4, as well as it is well known that both $\cup$ and $\cap$ are idempotent and commutative, both $\langle A, \mathcal{A}, \cup, \emptyset\rangle$ and $\langle A, \mathcal{A}, \cap, A\rangle$ are bounded semilattices. It has to be shown that $\langle A, \mathcal{A}, \cup, \cap, \emptyset, A\rangle$ satisfies the absorption laws, distributive laws and that each element has a complement - ie. that for all $a, b, c \in \mathcal{A}$:

(i) $a \cup (a \cap b) = a$

(ii) $a \cap (a \cup b) = a$

(iii) $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

(iv) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

(v) $\forall a \in A \exists a^c \in A : (a \cup a^c = A) \wedge (a \cap a^c = \emptyset)$

(**i**): Since $a \cap b \subseteq a$, it follows trivially. (**ii**): Since $a \subseteq a \cup b$, it follows trivially. (**iii**): Let $x \in a \cup (b \cap c)$. Then $(x \in a) \vee (x \in b \cap c) \Leftrightarrow (x \in a) \vee ((x \in b) \wedge (x \in c)) \Leftrightarrow ((x \in a) \vee (x \in b)) \wedge ((x \in a) \vee (x \in c)) \Leftrightarrow (x \in a \cup b) \wedge (x \in a \cup c) \vee x \in (a \cup b) \cap (a \cup c)$. (**iv**) follows analogously as (iii). (**v**) follows from (Σ_2). ■

7.15 Measure spaces

Definition 7.15.1. Let $\mathcal{A}$ be a σ-algebra defined on the set A. Then a *measure* is a map $\mu : \mathcal{A} \to [0, \omega]$, satisfying:

(μ_1) $\mu(\emptyset) = 0$

(μ_2) Given a family of pairwise disjoint sets $(S_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$, then

$$\mu\left(\bigcup_{i\in\mathbb{N}} S_i\right) = \sum_{i\in\mathbb{N}} \mu(S_i)$$

Definition 7.15.2. A *measure space* $\langle A, \mathcal{A}, \mu\rangle$ is a measurable space $\langle A, \mathcal{A}\rangle$ equipped with a measure μ.