Structuring the Mathematical Universe

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Preface

This booklet is not designed to teach the reader about a subject from start to finish. It is not intended as an article, providing new results. It is designed for people who seek an overview - people who have learned about various mathematical subjects and want to know how these fit together. It also doubles as an encyclopaedia through a wide range of examples of mathematical structures.

To make it as easy as possible to look up definitions and links, I have refrained myself from writing unnecessary explanatory text and have tried to be as concise as possible while still giving the necessary information for understanding the structure at hand.

The map, which illustrates all the links described in this booklet, has been categorized within the same structure as this booklet's chapters. Furthermore, the arrows in the map indicate if a structure implies another structure. If this arrow is based on a definition, it is coloured black. If it is based on a theorem, remark or anything else not a definition, it is coloured green.

Contents

1 The map **4**

2 Simple structures **5**
2.1 Classes . . . 5
2.2 Sets . . . 5

3 Group structures **6**
3.1 Magmas . . . 6
3.2 Semigroups . . . 6
3.3 Monoids . . . 6
3.4 Quasigroups . . . 6
3.5 Loops . . . 6
3.6 Groups . . . 7
3.7 Abelian groups . . . 7
3.8 Bands . . . 7
3.9 Semilattices . . . 7
3.10 Bounded semilattices . . . 7

4 Lattice structures **9**
4.1 Lattices . . . 9
4.2 Bounded lattices . . . 9
4.3 Modular lattices . . . 9
4.4 Distributive lattices . . . 10
4.5 Complemented lattices . . . 10
4.6 Ortholattices . . . 11
4.7 Complete lattices . . . 11
4.8 Heyting algebras . . . 11
4.9 Complete JID lattices . . . 12
4.10 Ockham algebras . . . 12
4.11 Stone algebras . . . 12
4.12 De Morgan algebras . . . 13
4.13 Kleene algebras . . . 13
4.14 Boolean algebras . . . 13

5 Ring structures **15**
5.1 Semirings . . . 15
5.2 Near-rings . . . 15
5.3 Nonassociative rings . . . 15
5.4 Rings . . . 16
5.5 Commutative rings . . . 16
5.6 Unitary rings . . . 16
5.7 Division rings . . . 17
5.8 Integral domains . . . 17
5.9 Fields . . . 17
5.10 Lie rings . . . 17

5.11 Boolean ring . . . 18
5.12 *-rings . . . 18

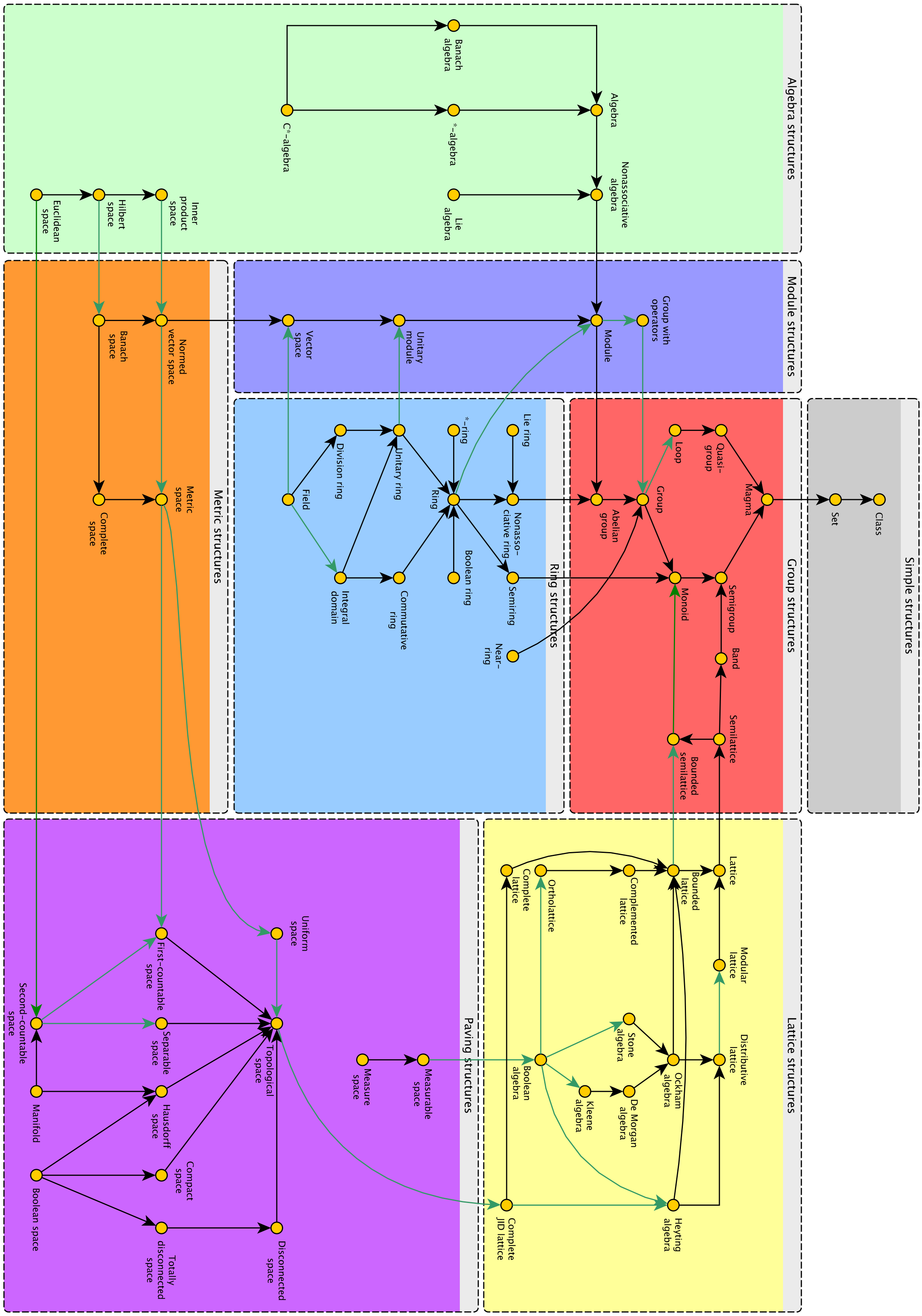
6 Module structures 19
6.1 Groups with operators . . . 19
6.2 Modules . . . 19
6.3 Unitary modules . . . 19
6.4 Vector spaces . . . 19

7 Algebra structures 20
7.1 Nonassociative algebras . . . 20
7.2 Algebras . . . 20
7.3 Banach algebras . . . 20
7.4 *-algebras . . . 20
7.5 C*-algebras . . . 21
7.6 Lie algebras . . . 21
7.7 Inner product spaces . . . 21
7.8 Hilbert spaces . . . 22
7.9 Euclidean spaces . . . 22

8 Metric structures 24
8.1 Metric spaces . . . 24
8.2 Complete spaces . . . 25
8.3 Normed vector spaces . . . 25
8.4 Banach spaces . . . 26

9 Paving structures 27
9.1 Measurable spaces . . . 27
9.2 Measure spaces . . . 28
9.3 Topological spaces . . . 28
9.4 First-countable spaces . . . 30
9.5 Separable spaces . . . 30
9.6 Second-countable spaces . . . 30
9.7 Hausdorff spaces . . . 31
9.8 Manifolds . . . 31
9.9 Disconnected spaces . . . 31
9.10 Totally disconnected spaces . . . 31
9.11 Compact spaces . . . 31
9.12 Boolean spaces . . . 32
9.13 Uniform spaces . . . 32

1 The map


Algebra structures
Banach algebra
Algebra
C*-algebra
*-algebra
Nonassociative algebra
Lie algebra
Euclidean space
Hilbert space
Inner product space
Module structures
Group with operators
Module
Unitary module
Vector space
Normed vector space
Banach space
Metric structures
Metric space
Complete space
Ring structures
Lie ring
*-ring
Division ring
Unitary ring
Field
Ring
Nonasso-ciative ring
Boolean ring
Semiring
Integral domain
Commutative ring
Near-ring
Group structures
Loop
Quasi-group
Magma
Group
Abelian group
Monoid
Semigroup
Band
Semilattice
Bounded semilattice
Simple structures
Set
Class
Paving structures
Uniform space
First-countable space
Second-countable space
Separable space
Topological space
Hausdorff space
Manifold
Compact space
Boolean space
Totally disconnected space
Disconnected space
Measure space
Measurable space
Lattice structures
Lattice
Bounded lattice
Complete lattice
Ortholattice
Complemented lattice
Modular lattice
Stone algebra
Distributive lattice
Ockham algebra
Boolean algebra
Kleene algebra
De Morgan algebra
Heyting algebra
Complete JID lattice

2 Simple structures

A simple structure is a structure with no operations equipped to it.

2.1 Classes

Definition 2.1.1. *A class is a collection of elements. If an object x is an element of A, then this is denoted $x \in A$. Classes have the equivalence relation '$=$' defined as:*

$$(x \in A \Leftrightarrow x \in B) \Leftrightarrow A = B$$

Definition 2.1.2. *A class A is called a subclass of a class B, denoted $A \subset B$, if*

$$\forall x \in A : x \in A \Rightarrow x \in B$$

2.2 Sets

Definition 2.2.1. *A set is a class A for which there exist a class B, such that $A \in B$.*

Definition 2.2.2. *A subclass A of a class B is a subset of B if, and only if, A is a set.*

Definition 2.2.3. *The empty set or null set, denoted $\emptyset$, is the set with no elements:*

$$\forall x : x \notin \emptyset$$

3 Group structures

A group structure is a structure defined on a set, equipped with a single binary operation.

Definition 3.0.4. *A binary operation on a set $X \neq \emptyset$ is a map $\cdot : X \times X \to X$, often denoted with infix notation $a \cdot b$ instead of with function notation $\cdot(a, b)$.*

3.1 Magmas

Definition 3.1.1. *A magma $(A, \cdot)$ is a set A along with a binary operation $\cdot$.*

3.2 Semigroups

Definition 3.2.1. *A semigroup $(A, \cdot)$ is a magma $(A, \cdot)$ satisfying $M \neq \emptyset$ and that $\cdot$ is associative:*

$$\forall a, b, c \in A : a(bc) = (ab)c$$

3.3 Monoids

Definition 3.3.1. *A monoid $(A, \cdot, e)$ is a semigroup $(A, \cdot)$ with an identity element:*

$$\exists e \in A \forall a \in A : ae = ea = a$$

3.4 Quasigroups

Definition 3.4.1. *A quasigroup $(A, \cdot, \backslash, /)$ is a magma $(A, \cdot)$, satisfying:*

(i) $A \neq \emptyset$

(ii) $\forall a, b \in A \exists x, y \in A : ax = b \wedge ya = b$

The unique solutions to these equations are $x = a\backslash b$ and $y = b/a$, called resp. left and right division.

3.5 Loops

Definition 3.5.1. *A loop $(A, \cdot, \backslash, /, e)$ is a quasigroup $(A, \cdot, \backslash, /)$ with an identity element:*

$$\exists e \in A \forall x \in A : xe = ex = x$$

3.6 Groups

Definition 3.6.1. *A group* $(A, \cdot, e, ^{-1})$ *is a monoid* $(A, \cdot, e)$*, in which every element has an inverse:*

$$\forall a \in G \exists a^{-1} \in G : aa^{-1} = a^{-1}a = e$$

Theorem 3.6.2. *Every group induces a loop.*

Proof. Let $(A, \cdot, e, ^{-1})$ be a group. Since every loop is a magma, it has to be shown that $\cdot$ has an identity element and that left and right division are well-defined on **A**. **A** has an identity element since it is a monoid. For every $a, b \in A$, define $x := a^{-1}b$ and $y := ba^{-1}$. Then:

$$\begin{aligned} ax &= a(a^{-1}b) = (aa^{-1})b = eb = b \\ ya &= (ba^{-1})a = b(a^{-1}a) = be = b \end{aligned}$$

And thus **A** induces a loop. ■

3.7 Abelian groups

Definition 3.7.1. *A group* $(A, \cdot, e, ^{-1})$ *is called Abelian, if* $\cdot$ *is commutative:*

$$\forall a, b \in A : ab = ba$$

3.8 Bands

Definition 3.8.1. *A band* $(A, \cdot)$ *is a semigroup* $(A, \cdot)$*, in which every element is idempotent:*

$$\forall a \in A : aa = a$$

3.9 Semilattices

Definition 3.9.1. *A semilattice* $(A, \cdot)$ *is a commutative band. The binary operation is often either* $\wedge$*, called 'meet', or* $\vee$*, called 'join'.*

3.10 Bounded semilattices

Definition 3.10.1. *A bounded semilattice* $(A, \cdot, e)$ *is a semilattice* $(A, \cdot)$ *with an identity element:*

$$\forall a \in A : ae = a$$

Remark 3.10.2. *A bounded semilattice is equivalent to an idempotent commutative monoid.*

4 Lattice structures

A lattice structure is a structure defined on a set, equipped with two or more binary operations, including binary operations called *meet* and *join*.

4.1 Lattices

Definition 4.1.1. *A lattice* $(A, \vee, \wedge)$ *is an algebraic structure, in which* $(A, \vee)$ *and* $(A, \wedge)$ *are both semilattices, as well as satisfying the absorption laws:*

(i) $\forall a, b \in A : a \vee (a \wedge b) = a$

(ii) $\forall a, b \in A : a \wedge (a \vee b) = a$

Remark 4.1.2. *A poset* $(A, \leq)$ *is a lattice if and only if for all* $a, b \in A$ *both* $\sup\{a, b\}$ *and* $\inf\{a, b\}$ *exist in* A*. This is called the dual definition of lattices. If a lattice* $\mathbf{A}$ *is defined algebraically, it can be defined dually by defining* $\leq$ *as* $a \leq b \Leftrightarrow a = a \wedge b \Leftrightarrow b = a \vee b$*. If* $\mathbf{A}$ *is defined as a poset, then the two operations* $\wedge$ *and* $\vee$ *can be defined as* $a \wedge b := \inf\{a, b\}$ *and* $a \vee b := \sup\{a, b\}$*.*

4.2 Bounded lattices

Definition 4.2.1. *A bounded lattice* $(A, \vee, \wedge, \bot, \top)$ *is a lattice, which satisfies the identity laws:*

(i) $\forall a \in A : a \vee \bot = a$

(ii) $\forall a \in A : a \wedge \top = a$

Remark 4.2.2. *Every bounded lattice is a bounded semilattice as well, since these satisfy their respective (join/meet-)identity law as well.*

4.3 Modular lattices

Definition 4.3.1. *A modular lattice* $(A, \vee, \wedge)$ *is a lattice, which satisfies the modular law:*

$$\forall a, b, c \in A : (a \wedge b) \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge b$$

4.4 Distributive lattices

Definition 4.4.1. *A distributive lattice* $(A, \vee, \wedge)$ *is a lattice, which is distributive:*

$$\forall a, b, c \in A : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Theorem 4.4.2. *Every distributive lattice is a modular lattice.*

Proof. Let **A** be a distributive lattice. It has to be checked that it satisfies the modular law. Let $a, b, c \in A$:

$$(a \vee (b \wedge c)) \wedge b = (a \wedge b) \vee (b \wedge (b \wedge c)) = (a \wedge b) \vee ((b \wedge b) \wedge c) = (a \wedge b) \vee (b \wedge c)$$

where the distributive, associative, commutative and idempotent laws were used. ■

4.5 Complemented lattices

Definition 4.5.1. *A complemented lattice* $(A, \vee, \wedge, \perp, \top)$ *is a bounded lattice which satisfies that every element has atleast one complement* b*:*

(i) $\forall a \in A \exists b \in A : a \vee b = \top$

(ii) $\forall a \in A \exists b \in A : a \wedge b = \perp$

Theorem 4.5.2. *A complemented distributive lattice* $(A, \vee, \wedge, \perp, \top)$ *has, for each* $a \in A$*, a unique complement* $\neg a$*.*

Proof. Let $a \in A$ and assume $b, c \in A$ are both complements to a, meaning:

(i) $a \vee b = a \vee c = \top$

(ii) $a \wedge b = a \wedge c = \perp$

Then:

$$b = b \wedge (b \vee a) = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c) = c \wedge (a \vee b) = c$$ ■

4.6 Ortholattices

Definition 4.6.1. *An ortholattice $(A, \vee, \wedge, \bot, \top, \neg)$ is a complemented lattice equipped with the unary operation $\neg : A \to C$, where $C \subset A$ is the set of complements of A. Furthermore, $\neg a \in C_a$ for each $a \in A$, where $C_a \subset C$ is the set of complements to a. Lastly, $\neg$ is idempotent and satisfies De Morgans laws:*

(i) $\forall a \in A : \neg\neg a = a$

(ii) $\forall a, b \in A : \neg(a \vee b) = \neg a \wedge \neg b$

(iii) $\forall a, b \in A : \neg(a \wedge b) = \neg a \vee \neg b$

Example 4.6.2. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then $(P(X), \cup, \cap, \emptyset, X, \bot)$ is an ortholattice, where $\bot : P(X) \to P(X)$ is defined as:*

$$A^{\bot} := \{x \in X \mid \forall a \in A : \langle a, x \rangle = 0\}$$

4.7 Complete lattices

Definition 4.7.1. *Let $\mathbf{A}$ be a lattice and $S \subset A$ a set. The supremum of S, $\bigvee S$, is defined as an element $b \in A$, satisfying:*

$$\bigvee A := \min\{b \in A \mid \forall a \in A : a \vee b = b\}$$

Analogously, the infimum of S, $\bigwedge S$, is defined as an element $c \in A$, satisfying:

$$\bigwedge A := \max\{c \in A \mid \forall a \in A : a \wedge c = c\}$$

Definition 4.7.2. *A complete lattice $(A, \vee, \wedge, \bot, \top)$ is a bounded lattice, in which every subset $S \subset A$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$.*

4.8 Heyting algebras

Definition 4.8.1. *A Heyting algebra $(A, \vee, \wedge, \bot, \top, \to)$ is a bounded distributive lattice equipped with the binary operation $\to$, which satisfies that $a \to b$ is the largest element x which satisfies $x \wedge a \leq b$:*

$$\forall a, b, c \in A : (a \wedge b \leq c) \Leftrightarrow (a \leq (b \to c))$$

Remark 4.8.2. *An equivalent definition is for a Heyting algebra is if it holds that:*

$$\forall a, b, c \in A : \bigvee\{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}$$

4.9 Complete JID lattices

Definition 4.9.1. *A complete JID lattice $(A, \vee, \wedge)$ is a complete distributive lattice, which satisfies the join-infinite distributive law:*

$$\forall a \in A, S \subset A : a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

Theorem 4.9.2. *Every complete JID lattice is a Heyting algebra.*

Proof. Let $\mathbf{A}$ be a complete JID lattice. By Remark 4.8.2 and completeness, it suffices to show that for $a, b \in A$:

$$\bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}$$

Notice for any $u \in \{c \in A \mid a \wedge c \leq b\}$ trivially $a \wedge u \leq b$, so by JID:

$$a \wedge \bigvee \{c \in A \mid a \wedge c \leq b\} = \bigvee \{a \wedge c \in A \mid a \wedge c \leq y\} \leq y$$

which means that $\bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq y\}$. ■

4.10 Ockham algebras

Definition 4.10.1. *An Ockham algebra $(A, \vee, \wedge, \perp, \top, \neg)$ is a bounded distributive lattice with a unary operation $\neg$, such that:*

(i) $\forall a, b \in A : \neg(a \wedge b) = \neg a \vee \neg b$

(ii) $\forall a, b \in A : \neg(a \vee b) = \neg a \wedge \neg b$

(iii) $\neg \top = \perp$

(iv) $\neg \perp = \top$

4.11 Stone algebras

Definition 4.11.1. *A Stone algebra $(A, \vee, \wedge, \perp, \top, \neg)$ is an Ockham algebra $(A, \vee, \wedge, \perp, \top, \neg)$ satisfying:*

(i) $\forall a \in A : \neg\neg a \vee \neg a = \top$

(ii) $\forall a \in A : a \wedge \neg a = \perp$

4.12 De Morgan algebras

Definition 4.12.1. *A De Morgan algebra* $(A, \vee, \wedge, \bot, \top, \neg)$ *is an Ockham algebra* $(A, \vee, \wedge, \bot, \top, \neg)$*, which is idempotent:*

$$\forall a \in A : \neg\neg a = a$$

4.13 Kleene algebras

Definition 4.13.1. *A Kleene algebra* $(A, \vee, \wedge, \bot, \top, \neg)$ *is a De Morgan algebra, which in addition satisfies:*

$$\forall x, y \in A : (x \wedge \neg x) \vee (y \vee \neg y) = y \vee \neg y$$

4.14 Boolean algebras

Definition 4.14.1. *A Boolean algebra* $(A, \vee, \wedge, \bot, \top, \neg)$ *is a complemented distributive lattice.*

Remark 4.14.2. *Boolean algebras complement is unique, cf. Theorem 4.5.2.*

Theorem 4.14.3. *Every Boolean algebra induces an ortholattice.*

Proof. Let $(A, \vee, \wedge, \bot, \top, \neg)$ be a Boolean algebra. Since both Boolean algebras and ortholattices are complemented lattices, it has to be shown that ortholattices additional requirements for $\neg$ hold.

(**i**): $\neg\neg a = \neg\neg a \vee \bot = \neg\neg a \vee (a \wedge \neg a) = (\neg\neg a \vee a) \wedge (\neg\neg a \vee \neg a) = (\neg\neg a \vee a) \vee \top = (\neg\neg a \vee a) \wedge (a \vee \neg a) = a \vee (\neg a \wedge \neg\neg a) = a \vee \bot = a$, where the identity, distributive, commutative and complement laws were used.

(**ii**): Since complements in Boolean algebras are unique cf. Theorem 4.5.2, it suffices to show that $\neg a \wedge \neg b$ is the complement to $a \vee b$, which is equivalent to:

1.) $(a \vee b) \vee (\neg a \wedge \neg b) = \top$

2.) $(a \vee b) \wedge (\neg a \wedge \neg b) = \bot$

1.): $(a \vee b) \vee (\neg a \wedge \neg b) = ((a \vee b) \vee \neg a) \wedge ((a \vee b) \vee \neg b) = (b \vee (a \vee \neg a)) \wedge (a \vee (b \vee \neg b)) = (b \vee \top) \wedge (a \vee \top) = \top \wedge \top = \top$, where the distributive, associative, complement, identity and idempotent laws were used.

2.): $(a \vee b) \wedge (\neg a \wedge \neg b) = ((\neg a \wedge \neg b) \wedge a) \vee ((\neg a \wedge \neg b) \wedge b) = (\neg b \wedge (a \vee \neg a)) \vee (\neg a \wedge (b \wedge \neg b)) = (\neg b \wedge \bot) \vee (\neg a \wedge \bot) = \bot \vee \bot = \bot$, where the distributive, associative, complement, identity, idempotent and commutative laws were used. (**iii**) follows analogously from (**ii**). ■

Theorem 4.14.4. *Every Boolean algebra induces a Kleene algebra.*

Proof. Let **A** be a Boolean algebra and $a, b \in A$. It has to be shown that:

(i) $\neg(a \vee b) = \neg a \wedge \neg b$

(ii) $\neg(a \wedge b) = \neg a \vee \neg b$

(iii) $\neg\top = \bot$

(iv) $\neg\bot = \top$

(v) $\neg\neg a = a$

(vi) $(a \wedge \neg a) \vee (b \vee \neg b) = b \vee \neg b$

(**i**), (**ii**) and (**v**) follows from **A** inducing an ortholattice cf. Theorem 4.14.3. (**iv**) will not be shown, since it follows analagously from (**iii**). (**iii**): $\neg\top = \neg(a \vee \neg a) = \neg a \wedge \neg\neg a = a \wedge \neg a = \bot$, where the complement law and Theorem 4.14.3 were used. (**vi**): $(a \wedge \neg a) \vee (b \vee \neg b) = \bot \vee \top = \top = b \vee \neg b$, where the complement and identity laws were used. ■

Theorem 4.14.5. *Every Boolean algebra induces a Stone algebra.*

Proof. Let **A** be a Boolean algebra. By Theorem 4.14.4, **A** is a Kleene algebra as well, and therefore also an Ockham algebra. It thus suffices to show that the Stone identities are satisfied. (i) follows from **A** being an ortholattice by Theorem 4.14.3 and complement. (ii) follows directly from the complement. ■

Theorem 4.14.6. *Every Boolean algebra induces a Heyting algebra.*

Proof. Let $(A, \vee, \wedge, \bot, \top, \neg)$ be a Boolean algebra. Define the map $\to : A \times A \to A$, given by:

$$a \to b := \neg a \vee b$$

It has to be checked that the map $\to$ satisfies the requirements given in Heyting algebras. Let $a, b, c \in A$.

$'\Rightarrow'$: Assume $a \wedge b \leq c$. Then $b \to c = \neg b \vee c \geq \neg b \vee (a \wedge b) = (\neg b \vee a) \wedge (\neg b \vee b) = (\neg b \vee a) \wedge \top = \neg b \vee a \geq a$, where the assumption, distributive law, complement, identity law and the dual lattice definition to conclude $\neg b \vee a \geq a$.

$'\Leftarrow'$: Assume $a \leq b \to c$. Then $a \wedge b \leq (b \to c) \wedge b = (\neg b \vee c) \wedge b = (\neg b \wedge b) \vee (b \vee c) = \bot \vee (b \vee c) = b \vee c \geq c$, where the assumption, distributive law, complement, identity law and the dual lattice definition was used to conclude $b \vee c \geq c$. ■

5 Ring structures

A ring structure is a structure defined on a set, with two binary operations.

5.1 Semirings

Definition 5.1.1. *A semiring $(A, +, \cdot, e_1, e_2)$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$, such that:*

(i) $(A, +, e_1)$ *is a commutative monoid*

(ii) $(A, \cdot, e_2)$ *is a monoid*

(iii) $\forall a, b, c \in A : a(b + c) = ab + ac$

(iv) $\forall a, b, c \in A : (a + b)c = ac + bc$

(v) $\forall a \in A : e_1 a = a e_1 = e_1$

5.2 Near-rings

Definition 5.2.1. *A near-ring $(A, +, \cdot, e, -)$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$ such that:*

(i) $(A, +, e, -)$ *is a group*

(ii) $(A, \cdot)$ *is a semigroup*

(iii) $\forall a, b, c \in A : a(b + c) = ab + ac$

(iv) $\forall a, b, c \in A : (a + b)c = ac + bc$

5.3 Nonassociative rings

Definition 5.3.1. *A nonassociative ring $(A, +, \cdot, e, -)$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$ such that:*

(i) $(A, +, e, -)$ *is an Abelian group*

(ii) $(A, \cdot)$ *is a magma*

(iii) $\forall a, b, c \in A : a(b + c) = ab + ac$

(iv) $\forall a, b, c \in A : (a + b)c = ac + bc$

5.4 Rings

Definition 5.4.1. *A ring $(A, +, \cdot, e, -)$ is a set $A \neq \emptyset$ together with two binary operations $+$ and $\cdot$ such that:*

(R_1) $(A, +, e, -)$ *is an Abelian group*

(R_2) $(A, \cdot)$ *is a semigroup*

(R_3) $\forall a, b, c \in A : a(b + c) = ab + ac$

$\forall a, b, c \in A : (a + b)c = ac + bc$

Remark 5.4.2. *Every ring reduces to a near-ring, since they only differ on $(R, +, e, -)$ being a group or an Abelian group.*

Remark 5.4.3. *Every ring reduces to a nonassociative ring, since they only differ on $(R, \cdot)$ being associative.*

Theorem 5.4.4. *Every ring induces a module.*

Proof. Let $\mathbf{R} := (R, +, \cdot, e_1, -)$ be a ring. Since $\mathbf{G} := (R, +, e_1, -)$ is an Abelian group, a module $\mathbf{A}$ can be constructed from $\mathbf{R}$ and $\mathbf{G}$. Thus can a module be created directly from an arbitrary ring. ∎

5.5 Commutative rings

Definition 5.5.1. *A ring $(A, +, \cdot, e, -)$ is called a commutative ring if $\cdot$ commutes:*

$$\forall a, b \in A : ab = ba$$

5.6 Unitary rings

Definition 5.6.1. *A ring $(A, +, \cdot, e_1, e_2, -)$ is called a unitary ring if a multiplicative identity element exists:*

$$\exists e_2 \in A \forall a \in A : e_2 a = a e_2 = a$$

Remark 5.6.2. *Every unitary ring is a unitary module, since every ring is a module cf. Theorem 5.4.4.*

5.7 Division rings

Definition 5.7.1. *An element a in a unitary ring $(A, +, \cdot, e_1, e_2, -)$ is said to be invertible, or to be a unit, if*

$$\exists a^{-1} \in A : a^{-1}a = aa^{-1} = e_2$$

Definition 5.7.2. *A unitary ring $(A, +, \cdot, e_1, e_2, -, ^{-1})$ with $e_1 \neq e_2$, in which every element $a \neq e_1$ is a unit, is called a division ring.*

5.8 Integral domains

Definition 5.8.1. *An element $a \neq e$ in a ring $(A, +, \cdot, e, -)$ is said to be a zero divisor if*

$$\exists e \neq b \in A : ab = ba = e$$

Definition 5.8.2. *An integral domain is a commutative unitary ring $(A, +, \cdot, e_1, e_2, -)$ with $e_1 \neq e_2$ and no zero divisors.*

Remark 5.8.3. *An equivalent definition of an integral domain $(A, +, \cdot, e_1, e_2, -)$ is a commutative unitary ring, where the zero rule applies:*

$$\forall a, b \in A : ab = e_1 \Rightarrow (a = e_1) \vee (b = e_1)$$

5.9 Fields

Definition 5.9.1. *A field $(A, +, \cdot, e_1, e_2, -, ^{-1})$ is a commutative division ring.*

Remark 5.9.2. *Every field is a vector space, since every ring is a module cf. Theorem 5.4.4.*

Theorem 5.9.3. *Every field is an integral domain.*

Proof. Let $(A, +, \cdot, e_1, e_2, -, ^{-1})$ be a field. It has to be checked that the zero rule applies. Let $a, b \in A$ and assume $ab = e_1$ and $a \neq e_1$: $b = e_2 b = (a^{-1}a)b = a^{-1}(ab) = a^{-1}e_1 = e_1$. ■

5.10 Lie rings

Definition 5.10.1. *A Lie ring $(A, +, \cdot, e, -)$ is a nonassociative ring which satisfies:*

- $\forall a, b, c \in A : a(bc) + b(ca) + c(ab) = e$
- $\forall a \in A : aa = e$

5.11 Boolean ring

Definition 5.11.1. *A Boolean ring $(A, +, \cdot, e, -)$ is a ring, in which $\cdot$ is idempotent:*

$$\forall a \in A : aa = a$$

5.12 *-rings

Definition 5.12.1. *A *-ring $(A, +, \cdot, e, -, *)$ is a ring $(A, +, \cdot, e, -)$ equipped with the unary operation $*$, which satisfies for all $a, b \in A$:*

- $(a + b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $a^{**} = a$

6 Module structures

A module structure is a structure defined on two sets with at least two binary operations.

6.1 Groups with operators

Definition 6.1.1. *A group with operators* $(A, \cdot, e, ^{-1}, \Omega, f)$ *is a group* $(A, \cdot, e, ^{-1})$ *along with a set* Ω *and the function* $f : \Omega \times A \to A$*, given by* $f(\omega, a) = a^\omega$ *and satisfies:*

$$\forall a, b \in A, \omega \in \Omega : f(\omega, ab) = f(\omega, a)f(\omega, b)$$

Remark 6.1.2. *Every group* $\mathbf{A}$ *trivially induces a group with operators* $(\mathbf{A}, \emptyset, f)$*.*

6.2 Modules

Definition 6.2.1. *Let* $(R, \oplus, \odot, e_r, \ominus)$ *be a ring. An (R-)module* $(A, +, e_m, -, R, \oplus, \odot, e_r, \ominus, \star)$ *is an Abelian group* $(A, +, e_m, -)$ *together with a function* $\star : R \times A \to A$*, such that for all* $r, s \in R$ *and* $a, b \in A$*:*

(i) $r \star (a + b) = r \star a + r \star b$

(ii) $(r \oplus s) \star a = r \star a + s \star a$

(iii) $r \star (s \star a) = (r \odot s) \star a$

Theorem 6.2.2. *Every module induces a group with operators.*

Proof. Set $\Omega = R$ and $f = \star$. Then it follows from $\star$'s first property. ■

6.3 Unitary modules

Definition 6.3.1. *Let* $(R, +, \cdot, e_1, e_2, -)$ *be a unitary ring. Then a module* $\boldsymbol{A}$ *over* R *is called a unitary module.*

6.4 Vector spaces

Definition 6.4.1. *Let* $(\mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1})$ *be a field. Then a module* $\boldsymbol{A}$ *over* $\mathbb{F}$ *is called an* $(\mathbb{F}-)$*vector space, denoted* $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star)$*.*

7 Algebra structures

An algebra structure is a structure defined on two sets: a ring $\mathbf{R}$ and an R-module $\mathbf{A}$.

7.1 Nonassociative algebras

Definition 7.1.1. *A nonassociative (R-)algebra* $\mathbf{A} = (A, +, e_A, -, R, \oplus, \odot, e_R, \ominus, \star, [\cdot, \cdot])$ *is an R-module* $(A, +, e_A, -, R, \oplus, \odot, e_R, \ominus, \star)$ *over a commutative ring* $\mathbf{R}$ *equipped with a binary operation* $[\cdot, \cdot]$ *on A, called* $\mathbf{A}$*-multiplication, which satisfies:*

(i) $\forall r, s \in R, a, b, c \in A : [r \star a + s \star b, c] = r \star [a, c] + s \star [b, c]$

(ii) $\forall r, s \in R, a, b, c \in A : [c, r \star a + s \star b] = r \star [c, a] + s \star [c, b]$

7.2 Algebras

Definition 7.2.1. *An algebra* $\mathbf{A}$ *is a nonassociative algebra, in which its binary operation* $[\cdot, \cdot] : A \times A \to A$ *is associative:*

$$\forall a, b, c \in A : [[a, b], c] = [a, [b, c]]$$

7.3 Banach algebras

Definition 7.3.1. *A Banach algebra* $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, ||\cdot||, [\cdot, \cdot])$ *is an algebra over a Banach space* $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, ||\cdot||)$*, which satisfies:*

$$\forall a, b \in A : ||ab|| \leq ||a|| \, ||b||$$

7.4 *-algebras

Definition 7.4.1. *A* **-algebra* $(A, +, e_A, -, R, \oplus, \odot, e_R, \ominus, \star, [\cdot, \cdot], *)$ *is an algebra, in which the ring* $\mathbf{R}$ *in addition is a* **-ring and* $R \subset A$*, as well as satisfying:*

$$\forall r, s \in R, a, b \in A : (r \star a + s \star b)^* = r^* \star a^* + s^* \star b^*$$

7.5 C*-algebras

Definition 7.5.1. *A C^*-algebra $(A, +, e_A, -, \mathbb{C}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, ||\cdot||, [\cdot,\cdot], *)$ is a complex Banach algebra and a *-algebra, which satisfies:*

$$\forall a \in A : ||a^* a|| = ||a||^2$$

7.6 Lie algebras

Definition 7.6.1. *A Lie algebra $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F}}, \ominus, \star, [\cdot,\cdot])$ is a nonassociative $\mathbb{F}$-algebra over a field $\mathbb{F}$, in which the binary operation $[\cdot,\cdot]$ satisfies alternation and the Jacobi identity:*

(i) $\forall a \in A : [a, a] = e_A$

(ii) $\forall a, b, c \in A : [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = e_A$

7.7 Inner product spaces

Definition 7.7.1. *Let $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star)$, be an $\mathbb{F}$-vector space. Then an inner product on A is a map $\langle\cdot,\cdot\rangle : A \times A \to \mathbb{F}$ with satisfies the following for all $a, b, c \in A$ and $\alpha, \beta \in \mathbb{F}$:*

(IP_1) $\langle a, a\rangle > e_{\mathbb{F},1} \Leftrightarrow a \neq e_{\mathbb{F},1}$

(IP_2) $\langle a, b\rangle = \overline{\langle b, a\rangle}$

(IP_3) $\langle \alpha \star a + \beta \star b, c\rangle = \alpha \odot \langle a, c\rangle + \beta \odot \langle b, c\rangle$

Definition 7.7.2. *A vector space **A** equipped with an inner product $\langle\cdot,\cdot\rangle$ is called an inner product space, denoted by $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \langle\cdot,\cdot\rangle)$.*

Theorem 7.7.3. *Every inner product space induces a normed vector space.*

Proof. Let $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \langle\cdot,\cdot\rangle)$ be an inner product space. Define the map $||\cdot|| : A \to [0,\infty)$, given by $||a|| := \sqrt{\langle a, a\rangle}$. It has to be shown that $||\cdot||$ is a norm. Because of (IP_1), $\sqrt{\langle a, a\rangle}$ is well-defined. (N_1) follows from (IP_1). (N_2) is shown:

$$||\alpha a||^2 \overset{def}{=} \langle \alpha \star a, \alpha \star a\rangle \overset{(IP_3),(IP_2)}{=} \alpha \odot \bar{\alpha} \langle a, a\rangle = |\alpha|^2 ||a||^2$$

The last (N_3) is shown:

$$\begin{aligned} ||a+b||^2 = \langle a+b, a+b\rangle &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= ||a||^2 + 2\mathrm{Re}\,\langle a, b\rangle + ||b||^2 \\ &\leq ||a||^2 + 2\,|\langle a, b\rangle| + ||b||^2 \\ &\overset{(*)}{\leq} ||a||^2 + 2||a||||b|| + ||b||^2 \\ &= (||a|| + ||b||)^2 \end{aligned}$$

At (*), the Cauchy-Schwarz' inequality is used. By taking the square root of both sides, (N_3) is shown, and thus is $(A, +, e_A, -, \mathbb{R}^n, \oplus, \odot, e_{\mathbb{R}^n,1}, e_{\mathbb{R}^n,2}, \ominus, ^{-1}, \star, \langle\cdot,\cdot\rangle, ||\cdot||)$ a normed space. ■

7.8 Hilbert spaces

Definition 7.8.1. *A Hilbert space is a complete inner product space.*

Corollary 7.8.2. *Every Hilbert space induces a Banach space.*

Proof. It follows from the fact that every inner product space induces a normed space cf. Theorem 7.7.3. ■

7.9 Euclidean spaces

Definition 7.9.1. *A Euclidean space $(A, +, \underline{0}, -, \mathbb{R}^n, \oplus, \odot, 0, 1, \ominus, ^{-1}, \star, \langle\cdot,\cdot\rangle)$ is a Hilbert space $(A, +, e_A, -, \mathbb{R}^n, \oplus, \odot, e_{\mathbb{R}^n,1}, e_{\mathbb{R}^n,2}, \ominus, ^{-1}, \star, \langle\cdot,\cdot\rangle)$ over the field $\mathbb{R}^n$ which satisfies that the inner product is the dot product, e_A is the zero vector $\underline{0}$, $e_{\mathbb{R}^n,1} = 0$ and $e_{\mathbb{R}^n,2} = 1$. Furthermore, $\star$ is often denoted with $\cdot$, $\oplus$ often denoted with $+$ and $\odot$ with $\cdot$, without confusion.*

Theorem 7.9.2. *Every Euclidean space induces a second-countable space.*

Proof. Let $\mathbf{A}$ be a Euclidean space. By definition, $\mathbf{A}$ is a Hilbert space and an inner product space. Then by Theorem 7.7.3, $\mathbf{A}$ induces a normed space. Then by Theorem 8.3.3, $\mathbf{A}$ induces a metric space. By Theorem 9.6.3, it thus suffices to show that $\mathbf{A}$ is separable, ie. that it has a countable dense subset. Consider $\mathbb{Q}^n$. It is known that $\mathbb{Q}^n$ is countable for all $n \in \mathbb{N}$. Furthermore, it is clear that $\forall n \in \mathbb{N} : \mathbb{Q}^n \subset \mathbb{R}^n$. Lastly, it is also known that $\mathbb{Q}$ is dense in $\mathbb{R}$, making $\mathbb{Q}^n$ dense in $\mathbb{R}^n$ by applying it to each coordinate. Thus by Theorem 9.6.3, $\mathbf{A}$ is a second-countable space. ■

8 Metric structures

A metric structure is a structure defined on 2 sets with a single (unary/binary)-operation.

8.1 Metric spaces

Definition 8.1.1. *Let $A \neq \emptyset$ be a set. A map $d : A \times A \to \mathbb{R}$ is called a metric, if it satisfies the following for arbitrary $a, b, c \in A$:*

(M_1) $d(a,b) \geq 0$, $d(a,b) = 0 \Leftrightarrow a = b$

(M_2) $d(a,b) = d(b,a)$

(M_3) $d(a,b) \leq d(a,c) + d(c,b)$

Definition 8.1.2. *A metric space (A,d) is a set $A \neq \emptyset$ equipped with a metric d.*

Theorem 8.1.3. *Every metric space induces a topological space.*

Proof. Let (A,d) be a metric space and denote $\mathcal{G}(A) = \mathcal{G}$ the family of all open subsets of A. It has to be shown that $\mathcal{G}$ is in fact a topology on A, and it thus has to satisfy the three conditions of a topology given in Definition 9.3.1. (T_1): Trivial.

(T_2): Let $a \in G_1 \cap \ldots \cap G_n$. We have to show that $a \in \mathcal{G}$, or in other words, that a is an inner point in A. For all $i \in \{1, \ldots, n\}$ it holds that $a \in G_i$ and since G_i is open, there exist an $r_i > 0$ such that $B_{r_i}(a) \subset G_i$. Now set $r := \min(r_1, \ldots, r_n)$. Then $B_r(a) \subset G_1 \cap \ldots \cap G_n$, which shows that a is an inner point.

(T_3): Let $a \in \bigcup_{i \in I} G_i$. We have to show that a is an inner point in A. There exist an $i_0 \in I$ such that $a \in G_{i_0}$, but since G_{i_0} is open, there exist an $r > 0$ such that $B_r(a) \subset G_{i_0}$. But then $B_r(a) \subset \bigcup_{i \in I} G_i$, which shows that a is an inner point in $\bigcup_{i \in I} G_i$. ■

Theorem 8.1.4. *Every metric space induces a first-countable space.*

Proof. Let (A,d) be a metric space and $a \in A$. Since every metric space induces a topological space by Theorem 8.1.3, it has to be shown that $\mathbf{A}$ has a countable local basis. Define such a candidate $\mathcal{B} := \{B_{1/n}(a) \mid n \in \mathbb{N}\}$, where $B_\varepsilon(a)$ denotes the open ε-ball in A with center in a and radius ε.

First of all, since a surjection $f : \mathbb{N} \to \mathcal{B}$ can be created, $\mathcal{B}$ is countable. It is clear that every element $b \in \mathcal{B}$ is an open neighbourhood of a, since the balls are open with a as center. Now let $U \subset A$ be an open neighbourhood of a. By definition of open sets, an open ball $B_\varepsilon(a) \subset U$. By the Archimedean principle, there exist an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$ which implies $\varepsilon > \frac{1}{\varepsilon}$, meaning $B_{1/n}(a) \subset B_\varepsilon(a) \subset U$. This proves that $\mathcal{B}$ is a countable local basis for A. ■

Theorem 8.1.5. *Every metric space induces a uniform space.*

Proof. Let (A, d) be a metric space. Define $\delta_\varepsilon := \{(a, b) \in A \times A \mid d(a, b) < \varepsilon\}$ and $\Delta := \{\delta_\varepsilon \mid \varepsilon > 0\}$. It has to be shown that Δ is a uniformity on A, since every metric space is a topological space by Theorem 8.1.3.

($\mathbf{U_1}$): Since $d(a, a) = 0$ by (M_1) and since $\varepsilon > 0$, (a, a) will be included in every d_ε for all $a \in A$.

($\mathbf{U_2}$): Let $V \subset A \times A$, $U \in \Delta$ and $U \subset V$. It has to be shown that $V \in \Delta$. Since $U \in \Delta$, there exist an $\varepsilon > 0$ such that $U = \delta_\varepsilon$. Since $U \subset V$ then $\delta_\varepsilon \subset V$. If $\max\{d(a, b) \mid (a, b) \in V\} < \varepsilon$, then trivially $V \in \Delta$. If on the other hand $\max\{d(a, b) \mid (a, b) \in V\} \geq \varepsilon$, then set $\varepsilon_0 := \max\{d(a, b) \mid (a, b) \in V\}$, meaning $V = \delta_{\varepsilon_0} \Rightarrow V \in \Delta$.

($\mathbf{U_3}$): Let $U, V \in \Delta$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U = \delta_{\varepsilon_1}$ and $V = \delta_{\varepsilon_2}$. Then it is clear that $U \cap V = \delta_{\varepsilon_1} \cap \delta_{\varepsilon_2} := \min\{\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\} \in \Delta$.

($\mathbf{U_4}$): Let $U \in \Delta$. Then there exist an $\varepsilon > 0$ such that $U = \delta_\varepsilon$. Define $V := \delta_{\varepsilon/2}$ and let $(a, b), (b, c) \in V$. Then $d(a, b), d(b, c) < \varepsilon/2$ by definition, and then $d(a, c) \leq d(a, b) + d(b, c) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (M_3), meaning $d(a, c) < \varepsilon$ and then $(a, c) \in U$.

($\mathbf{U_5}$) follows from (M_2). ■

8.2 Complete spaces

Definition 8.2.1. *Let* **A** *be a metric space, and let* $(m_i)_{i\in\mathbb{N}} \subset A$ *be a sequence in* A. *Then* $(m_i)_{i\in\mathbb{N}}$ *is called a Cauchy sequence if:*

$$\forall j, k \in \mathbb{N} : \lim_{j,k\to\infty} d(m_j - m_k) = 0$$

Definition 8.2.2. *A complete space* (A, d) *is a metric space* (A, d) *if all Cauchy sequences in* A *converge.*

8.3 Normed vector spaces

Definition 8.3.1. *A norm on a vector space* $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star)$ *is a map* $||\cdot|| : A \to \mathbb{R}$, *which satisfies the following:*

(N_1) $\forall a \in A : ||a|| \geq e_{\mathbb{R}^n,1},\ ||a|| = e_{\mathbb{R}^n,1} \Leftrightarrow a = e_A$

(N_2) $\forall a \in A, \lambda \in \mathbb{R}^n : ||\lambda \star a|| = |\lambda| \odot ||a||$

(N_3) $\forall a, b \in A : ||a + b|| \leq ||a|| \oplus ||b||$

Definition 8.3.2. *A normed vector space* $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, ||\cdot||)$ *is a vector space over a field* $\mathbb{F}$*, which is either* $\mathbb{R}$ *or* $\mathbb{C}$*, equipped with the norm* $||\cdot||$*.*

Theorem 8.3.3. *Every normed vector space induces a metric space.*

Proof. Let $\mathbf{A} = (A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, ||\cdot||)$ be a normed vector space. Then define the map $d : A \times A \to \mathbb{F}$ given by

$$d(a, b) := ||b - a||$$

It has to be checked that d is a metric. (M_1) follows directly from (N_1). Let $\lambda = -1$. Then by (N_2), with $a, b \in A$ and $\lambda \in \mathbb{F}$:

$$||b - a|| = ||\lambda \star (a - b)|| = |\lambda| \odot ||a - b|| = ||a - b||$$

which shows (M_2). Lastly, by (N_3), with $a, b, c \in A$:

$$d(a, b) = ||a - b|| = ||(a - c) + (c - b)|| \overset{(N_3)}{\leq} ||a - c|| + ||c - b|| = d(a, c) + d(c, b)$$

which then proves that d is a metric and $\mathbf{A}$ a metric space. ∎

8.4 Banach spaces

Definition 8.4.1. *A Banach space* $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, ||\cdot||)$ *is a complete normed vector space.*

9 Paving structures

A paving structure is a structure which is defined on a family of subsets of a set.

Definition 9.0.2. *A family of sets indexed by the class $I \neq \emptyset$ is a class of sets $(A_i)_{i\in I} := \{A_i \mid i \in I\}$. For such a family, its union and intersection are defined to be respectively the classes*

$$\bigcup_{i\in I} A_i := \{a \mid \exists i \in I : a \in A_i\}$$
$$\bigcap_{i\in I} A_i := \{a \mid \forall i \in I : a \in A_i\}$$

9.1 Measurable spaces

Definition 9.1.1. *A σ-algebra $\mathcal{A}$ on a set A is a family of subsets of A with the following properties:*

(Σ_1) $A \in \mathcal{A}$

(Σ_2) $S \in \mathcal{A} \Rightarrow S^c \in \mathcal{A}$

(Σ_3) $(S_i)_{i\in\mathbb{N}} \subset \mathcal{A} \Rightarrow \bigcup_{i\in\mathbb{N}} S_i \in \mathcal{A}$

Definition 9.1.2. *A measurable space $(A, \mathcal{A})$ is a set A, equipped with a σ-algebra $\mathcal{A}$.*

Remark 9.1.3. *Every measurable space doesn't induce a topological space, and vice versa.*

Proof. That every measurable space can't induce a topological space is due to the fact that topologies requires closure of arbitrarily many unions (T_3) and σ-algebras only required closure of countably many unions (Σ_3). The reverse is true since σ-algebras require closure under complements (Σ_2), and this is only always true for X and $\emptyset$ in topologies (T_1). ■

Theorem 9.1.4. *Every measurable space induces a monoid.*

Proof. The proof follows analogously from Theorem 9.3.3, since $A \in \mathcal{A}$ by (Σ_1), $\emptyset \in \mathcal{A}$ by (Σ_1) and (Σ_2), $\cup$ is closed by (Σ_3) and $\cap$ is closed by (Σ_2) and (Σ_3), since $a \cap b = (a^c \cup b^c)^c$. ■

Theorem 9.1.5. *Every measurable space induces a Boolean algebra.*

Proof. Let $(A, \mathcal{A})$ be a measurable space. Since $\mathbf{A}$ is a monoid by Theorem 9.1.4, as well as it is well known that both $\cup$ and $\cap$ are idempotent and commutative, both $(A, \mathcal{A}, \cup, \emptyset)$ and $(A, \mathcal{A}, \cap, A)$ are bounded semilattices. It has to be shown that $(A, \mathcal{A}, \cup, \cap, \emptyset, A)$ satisfies the absorption laws, distributive laws and that each element has a complement - ie. that for all $a, b, c \in \mathcal{A}$:

(i) $a \cup (a \cap b) = a$

(ii) $a \cap (a \cup b) = a$

(iii) $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

(iv) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

(v) $\forall a \in A \exists a^c \in A : (a \cup a^c = A) \wedge (a \cap a^c = \emptyset)$

(**i**): Since $a \cap b \subset a$, it follows trivially. (**ii**): Since $a \subset a \cup b$, it follows trivially. (**iii**): Let $x \in a \cup (b \cap c)$. Then $(x \in a) \vee (x \in b \cap c) \Leftrightarrow (x \in a) \vee ((x \in b) \wedge (x \in c)) \Leftrightarrow ((x \in a) \vee (x \in b)) \wedge ((x \in a) \vee (x \in c)) \Leftrightarrow (x \in a \cup b) \wedge (x \in a \cup c) \vee x \in (a \cup b) \cap (a \cup c)$. (**iv**) follows analogously as (iii). (**v**) follows from (Σ_2). ■

9.2 Measure spaces

Definition 9.2.1. *Let $\mathcal{A}$ be a σ-algebra defined on the set A. Then a measure is a map $\mu : \mathcal{A} \to [0, \infty]$, satisfying:*

(μ_1) $\mu(\emptyset) = 0$

(μ_2) Given a family of pairwise disjoint sets $(S_i)_{i \in \mathbb{N}} \subset \mathcal{A}$, then

$$\mu\left(\bigcup_{i \in \mathbb{N}} S_i\right) = \sum_{i \in \mathbb{N}} \mu(S_i)$$

Definition 9.2.2. *A σ-algebra $\mathcal{A}$, defined on a set A and equipped with a measure μ is called a measure space, denoted $(A, \mathcal{A}, \mu)$.*

9.3 Topological spaces

Definition 9.3.1. *A topology on a set A is a family of subsets of A denoted $\mathcal{O}$, which satisfies the following:*

(T_1) $\emptyset, A \in \mathcal{O}$.

(T_2) $\forall G_1, \ldots, G_n \in \mathcal{O} : G_1 \cap \ldots \cap G_n \in \mathcal{O}$

(T_3) $(A_i)_{i \in I} \in \mathcal{O} \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{O}$.

Definition 9.3.2. *A topological space $(A, \mathcal{O})$ set A equipped with a topology $\mathcal{O}$.*

Theorem 9.3.3. *Every topological space induces a monoid.*

Proof. Let $(A, \mathcal{O})$ be a topological space. It has to be shown that the induced $(A, \mathcal{O}, \cup)$ is a semigroup, and has an identity element e. To show that it is a semigroup, it is firstly noted that it is a magma, since $\cup$ is a binary operation by (T_3). It has to be shown that $\cup$ is associative:

$$\begin{aligned} a \in A \cup (B \cup C) &\Leftrightarrow a \in A \vee (a \in B \vee a \in C) \\ &\Leftrightarrow (a \in A \vee a \in B) \vee a \in C \\ &\Leftrightarrow a \in (A \cup B) \cup C \end{aligned}$$

Thus is $(A, \mathcal{O}, \cup)$ a semigroup, and it has to be shown that it has an identity. Let $S \in \mathcal{O}$ and consider the empty set $\emptyset$, which exists by (T_1):

$$\begin{aligned} \emptyset \cup S = \{a \in A \mid a \in S \vee a \in \emptyset\} &\overset{2.2.3}{=} \{a \in A \mid a \in S\} = S \\ S \cup \emptyset = \{a \in A \mid a \in \emptyset \vee a \in S\} &\overset{2.2.3}{=} \{a \in A \mid a \in S\} = S \end{aligned}$$

Thus is $e = \emptyset \in \mathcal{O}$, making $(A, \mathcal{O}, \cup, \emptyset)$ a monoid. Similarly it can be shown that $(A, \mathcal{O}, \cap, A)$ is a monoid as well by (T_1) and (T_2). ■

Theorem 9.3.4. *Every topological space induces a complete JID lattice.*

Proof. Let $(A, \mathcal{O})$ be a topological space. It was shown in Theorem 9.3.3 that both $(A, \mathcal{O}, \cup, \emptyset)$ and $(A, \mathcal{O}, \cap, A)$ were monoids. Since $\cup$ and $\cap$ are well known to be both idempotent and commutative, it has to be shown that $(A, \mathcal{O}, \cup, \cap, \emptyset, A)$ satisfies the absorption laws, distributive laws and completeness. The absorption laws and distributive laws follows analogously from the proof in Theorem 9.1.5, so it has to be shown that every set $S_i \subset A$ has both a supremum $\bigvee S_i := \bigcup S_i$ and an infimum $\bigwedge S_i := \bigcap S_i$.

By (T_2), $\mathbf{A}$ is closed under arbitrary unions, meaning $\bigvee S_i$ exists. Since topologies aren't closed under arbitrary intersections, the infimum can instead be constructed by taking the interior of arbitrary intersections. Since interiors are always open, the infimum exists. Moreover, the join-infinite distribution law is shown. Let $a \in A, S_0 \in \mathcal{O}, (S_i)_{i \in I} \subset \mathcal{O}$. Then:

$$\begin{aligned} a \in S_0 \cap \bigcup_{i \in I} S_i \Leftrightarrow (a \in S_0) \wedge (a \in \bigcup_{i \in I} S_i \Leftrightarrow (a \in S_0) \wedge (\exists i \in I : a \in S_i) \\ \Leftrightarrow \exists i \in I : (a \in S_i) \wedge (a \in S_0) \Leftrightarrow a \in \bigcup_{i \in I} (S_i \cap S_0) \end{aligned}$$

■

9.4 First-countable spaces

Definition 9.4.1. *A first-countable space $(A, \mathcal{O})$ is a topological space $(A, \mathcal{O})$ with a countable neighbourhood basis - that is, for every $a \in A$ there exists a sequence of open neighbourhoods $(S_i)_{i \in \mathbb{N}}$ of a, such that:*

$$\forall U \in \mathcal{O} \exists i \in \mathbb{N} : x \in U \Rightarrow x \in S_i \subset U$$

9.5 Separable spaces

Definition 9.5.1. *A separable space $(A, \mathcal{O})$ is a topological space $(A, \mathcal{O})$ in which there exist a countable dense subset - that is, a sequence $(a_n)_{n \in \mathbb{N}} \subset A$, which satisfies:*

$$\forall U \in \mathcal{O} \exists i \in \mathbb{N} : a_i \in U$$

9.6 Second-countable spaces

Definition 9.6.1. *A second-countable space $(A, \mathcal{O}, \mathcal{B})$ is a topological space $(A, \mathcal{O})$ with a countable collection of sets $\mathcal{B} = (B_i)_{i \in \mathbb{N}} \subset \mathcal{O}$, called a countable basis, such that:*

$$\forall V \in \mathcal{O} \exists (B_k)_{k \in \mathbb{N}} \subset \mathcal{B} : V = \bigcup_{k \in \mathbb{N}} B_k$$

Theorem 9.6.2. *Every second-countable space reduces to both a first-countable space and a separable space.*

Proof. Let $(A, \mathcal{O}, \mathcal{B})$ be a second-countable space. For each $a \in A$, the set $\{U \in \mathcal{B} \mid a \in U\}$ is a countable neighbourhood basis. Therefore it is a first-countable space. Let S consist of one point from each member of $\mathcal{B}$. Then S is a countable dense subset of A. Therefore it is a separable space. ■

Theorem 9.6.3. *Every separable metric space induces a second-countable space.*

Proof. Let (A, d) be a metric space with a countable dense subset S. We will show that the set $\mathcal{B}$ of open balls with rational radii around points in S form a countable basis for A. Consider the basic open set $B_r(a)$, for some $a \in A$ and $r > 0$. Let $b \in B_r(a)$. There is an $s > 0$ such that $B_s(b) \subset B_r(x)$. Let $c \in S \cap B_{s/3}(b)$ and let $t \in (s/3, 2s/3) \cap \mathbb{Q}$. Then $b \in B_t(c)$, and it follows from the triangle inequality that $B_t(c) \subset B_s(b) \subset B_r(a)$. So we have shown that every point in $B_r(a)$ is inside a set in $\mathcal{B}$ that is contained in $B_r(a)$. Therefore $B_r(a)$ is a union of members of $\mathcal{B}$, and therefore $\mathcal{B}$ is a basis. Since $\mathcal{B}$ is in one-to-one correspondance with the product $S \times ((0, \infty) \cap \mathbb{Q})$ of countable sets, $\mathcal{B}$ is countable. ■

9.7 Hausdorff spaces

Definition 9.7.1. *A Hausdorff space $(A, \mathcal{O})$ is a topological space $(A, \mathcal{O})$, which satisfies:*

$$\forall a, b \in A \exists \varepsilon > 0 : B(a, \varepsilon) \cap B(b, \varepsilon) = \emptyset$$

9.8 Manifolds

Definition 9.8.1. *A homeomorphism $f : X \to Y$ is a continuous function between topological spaces X and Y, that has a continuous inverse function. If such a function exist, X and Y are said to be homeomorphic.*

Definition 9.8.2. *A topological space $(A, \mathcal{O})$ is said to be locally homeomorphic to a Euclidean space $\mathbb{E}^n$ if every $a \in A$ has a neighbourhood which is homeomorphic to an open Euclidean n-ball:*

$$B^n(a, \varepsilon) := \{a := (a_1, a_2, \ldots, a_n) \in \mathbb{E}^n \mid a_1^2 + a_2^2 + \ldots + a_n^2 < \varepsilon\}$$

Definition 9.8.3. *An n-dimensional manifold $(A, \mathcal{O}, \mathbb{E}^n)$ is a second-countable Hausdorff space that is locally homeomorphic to a Euclidean space $\mathbb{E}^n$.*

9.9 Disconnected spaces

Definition 9.9.1. *A disconnected space $(A, \mathcal{O})$ is a topological space $(A, \mathcal{O})$, in which there exist a collection of disjoint subspaces $(A_i)_{i \in I}$, which satisfies:*

$$A = \bigcup_{i \in I} A_i$$

9.10 Totally disconnected spaces

Definition 9.10.1. *A totally disconnected space $(A, \mathcal{O})$ is a disconnected space, in which the connected subspaces are one-point sets.*

9.11 Compact spaces

Definition 9.11.1. *A compact space $(A, \mathcal{O})$ is a topological space $(A, \mathcal{O})$, in which each of its open covers has a finite subcover. E.g., for every collection $(U_i)_{i \in I}$ with $A = \bigcup_{i \in I} U_i$ exists a finite set $J \subset I$, such that:*

$$A = \bigcup_{i \in J} U_i$$

9.12 Boolean spaces

Definition 9.12.1. *A Boolean space $(A, \mathcal{O})$ is a totally disconnected compact Hausdorff space.*

9.13 Uniform spaces

Definition 9.13.1. *A uniformity ϕ on a set A is a nonempty family of subsets of $A \times A$, that satisfies the following:*

(U_1) $\forall U \in \phi : \{(a, a) \mid a \in A\} \in U$

(U_2) $\forall U \in \phi, V \subset A \times A : U \subset V \Rightarrow V \in \phi$

(U_3) $\forall U, V \in \phi : U \cap V \in \phi$

(U_4) $\forall U \in \phi \exists V \in \phi : (a, b), (b, c) \in V \Rightarrow (a, c) \in U$

(U_5) $\forall U \in \phi : \{(b, a) \mid (a, b) \in U\} \in \phi$

Definition 9.13.2. *A uniform space (A, ϕ) is a set A equipped with a uniformity ϕ.*

Theorem 9.13.3. *Every uniform space induces a topological space.*

Proof. Let (A, ϕ) be a uniform space. Define a subset $S \subset A$ to be open if, and only if, $\forall a \in S \exists U \in \phi : \{b \in A \mid (a, b) \in U\} \subset S$. Define $\mathcal{O} := \{S \subset A \mid S \text{ open}\}$. It has to be shown that $\mathcal{O}$ is a topology.

($\mathbf{T_1}$): It holds that $\emptyset \in \mathcal{O}$ trivially, since it doesn't have any elements. A is open as well, by picking $U := \{(a, a) \mid a \in A\}$, which is an element in ϕ by (U_1).

($\mathbf{T_2}$): Let $G_1, \ldots, G_n \in \mathcal{O}$. It has to be shown that $G_1 \cap \ldots \cap G_n \in \mathcal{O}$. By definition there exist $U_1, \ldots, U_n \in \phi$ such that $\forall 1 \leq i \leq n \forall a \in G_i : \{b \in A \mid (a, b) \in U_i\} \subset G_i$. It has to be shown that there exist $U_0 \in \phi$ such that $\forall a \in G_1 \cap \ldots \cap G_n : \{b \in A \mid (a, b) \in U_0\} \subset G_1 \cap \ldots \cap G_n$. Define $U_x := U_1 \cap \ldots \cap U_n$. It thus has to be shown that $U_x = U_0$. Let $a_0 \in G_1 \cap \ldots \cap G_n$. Then by definition $a_0 \in G_1 \wedge \ldots \wedge a_0 \in G_n$, meaning that $\forall 1 \leq i \leq n : \{b \in A \mid (a_0, b) \in U_i\} \subset G_i$. But this means that $\{b \in A \mid (a_0, b) \in U_x\} \subset G_1 \cap \ldots \cap G_n$. Since the choice of a_0 was arbitrary, it means that $U_x = U_0$.

($\mathbf{T_3}$): Let $(A_i)_{i \in \mathbb{N}} \in \mathcal{O}$. It has to be shown that $A_0 := \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{O}$. By definition, there exist $(U_i)_{i \in \mathbb{N}} \in \phi$ such that $\forall i \in \mathbb{N} \forall a \in A_i : \{b \in A \mid (a, b) \in U_i\} \subset A_i$. It has to be shown that there exist U_0 such that $\forall a \in A_0 : \{b \in A \mid (a, b) \in U_0\} \subset A_0$. Define $U_x := \bigcup_{i \in \mathbb{N}} U_i$. It has to be shown that $U_x = U_0$. Let $a_0 \in A_0$. Then $\exists i \in \mathbb{N} : a_0 \in A_i$ and thus $\{b \in A \mid (a_0, b) \in U_i\} \subset A_i \subset A_0$. Since this holds for all $i \in \mathbb{N}$, it is seen that $\{b \in A \mid (a, b) \in U_x\} \subset A_0$. Since the choice of a_0 was arbitrary, it means that $U_x = U_0$.■