Jónsson Covering below a Woodin

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Abstract. We modify the proof in Welch (2000) of the weak covering lemma at Jónsson cardinals below a Woodin cardinal to the measurable-free context from Jensen and Steel (2013).

This note is dedicated to proving the following theorem.

Theorem 0.1. *Assume that there is no inner model with a Woodin cardinal. Let K be the Jensen-Steel core model and κ a Jónsson cardinal. Then $\kappa^+ = \kappa^{+K}$.*

And immediate consequence of the theorem is then the following.

Corollary 0.2. *Assume that there is no inner model with a Woodin cardinal and let κ be a Jónsson cardinal. Then $\square_\kappa$ holds.*

1 Setting the scene

We start by setting up the scene and giving an overview of what is to come. By the weak covering lemma in K, we can assume that κ is regular. Set $\lambda := \kappa^{+K}$ and assume towards a contradiction that $\lambda < \kappa^+$, so that the weak covering lemma in K implies that $\operatorname{cof} \lambda = |\lambda| = \kappa$. Fix some monotone cofinal map $D : \kappa \to \lambda$.

For some suitably large $\Omega > \kappa$ we can ensure that $\tilde{K}(\kappa^+, \Omega)$ contains κ^+, so fix a "very soundness" witness $\mathcal{W}$ for $\tilde{K}(\kappa^+, \Omega)|\kappa^+$, i.e. a weasel such that $S(\mathcal{W})$ is $\mathcal{W}$-thick and $\kappa^+ \subseteq \mathcal{W}$ – note that $\kappa^{+K} = \kappa^{+\mathcal{W}}$. Say $\vec{E}$ is $\mathcal{W}$'s extender sequence and let $X \prec \langle V_\eta, \in, \vec{E}, D\rangle$ for some sufficiently large η and with $|X| = \kappa$ and $X \cap \kappa \neq \kappa$; X exists by the Jónsson property.

Let $\pi : \langle \mathcal{H}, \in, \vec{\bar{E}}, \bar{D} \rangle \cong \langle X, \in, \vec{E}, D \rangle$ be the uncollapse. Set $\bar{x} := \pi^{-1}(x)$ for every $x \in X$. Then by virtue of D we get that $\operatorname{cof} \bar{\lambda} = \kappa > \operatorname{crit} \pi$, so that π is continuous at $\bar{\lambda}$. Note that $\vec{\bar{E}}$ is the extender sequence of $\bar{\mathcal{W}}$ and define $j := \pi \restriction \bar{\mathcal{W}} : \bar{\mathcal{W}} \to \mathcal{W}$. We now want to compare $\bar{\mathcal{W}}$ with $\mathcal{W}$. Our proof strategy will be the following.

(i) Compare $\bar{\mathcal{W}}$ with $\mathcal{W}$ via a non-standard coiteration, yielding trees $\mathcal{T}, \mathcal{U}$ with last models $\mathcal{P}, \mathcal{Q}$, such that $\mathcal{P}$ agrees with $\mathcal{Q}$ below κ;

(ii) Pick the least $\mathcal{N} \lhd \mathcal{Q}$ such that $\bar{\lambda}$ is definably collapsed over $\mathcal{N}$;

(iii) Use that $\mathcal{N}$ agrees with $\mathcal{P}$ below κ to lift $j : \bar{\mathcal{W}} \to \mathcal{W}$ to some $j^+ : \mathcal{N} \to \mathcal{M}$;

(iv) Show that $\Psi_{\mathcal{M}} := \Phi(j\,\mathcal{T})^\frown\langle \mathcal{M}, \kappa \rangle$ is a stable iterable phalanx, which allows us to compare $\mathcal{M}$ with $\mathcal{W}$;

(v) λ is definably collapsed over $\mathcal{M}$, so by an elementarity argument this is also true over $\mathcal{W}$, but $\lambda = \kappa^{+\mathcal{W}}$, $\lightning$.

The hardest part will be constructing the tree $\mathcal{U}$ in part (i). This is very reminiscent of the proof of the weak covering lemma for K, but to construct $\mathcal{U}$ in that scenario they use that κ is singular and a strong limit. In our case we could tweak the argument so that the Jónsson property achieves the result that would normally use the singularity of κ, but the argument using that κ is a strong limit does not prima facie admit such a tweak. Instead we will follow Mitchell's idea of "special drops", also described in Welch (2000). Here is an illustration of the proof to come.

$$\begin{tikzcd}
 & & & \mathcal{R} \arrow[r, phantom, "\trianglelefteq"] & \mathcal{S} \\
\mathcal{P} \arrow[r, phantom, "\overset{<\kappa}{\sim}"] & \mathcal{Q} \arrow[r, phantom, "\trianglerighteq"] & \mathcal{N} \arrow[r, "j^+"] & \mathcal{M} \arrow[u] & \mathcal{W} \arrow[u] \\
\bar{\mathcal{W}} \arrow[u, "\mathcal{T}"'] \arrow[r, "j"] & \mathcal{W} \arrow[u, "\mathcal{U}"'] & & &
\end{tikzcd}$$

2 Constructing $\mathcal{U}$

We now provide the definition of $\mathcal{U}$. Simultaneously we will define trees $\mathcal{T}$ on $\bar{\mathcal{W}}$ and $\bar{\mathcal{U}}$ on $\mathcal{W}$, and $\deg^{\mathcal{U}}(\alpha)$-embeddings $\pi_\alpha : \mathcal{M}^{\mathcal{U}}_\alpha \to \mathcal{M}^{\bar{\mathcal{U}}}_\alpha$. The tree $\bar{\mathcal{U}}$ and the π_α's are only there to make sure that the models in $\mathcal{U}$ are wellfounded.

Let us now say that λ is a limit and $\mathcal{T} \restriction \lambda$, $\mathcal{U} \restriction \lambda$ and $\bar{\mathcal{U}} \restriction \lambda$ have been constructed. Then $\bar{\mathcal{U}} \restriction \lambda$ is an ω-iteration tree and $\mathcal{W}$'s iteration strategy picks the unique cofinal wellfounded branch $\bar{b}$ of $\bar{\mathcal{U}}$. Fix some $\gamma \in \bar{b}$ such that $\bar{b} - \gamma$ has no drops, and set $\mathcal{M}^{\bar{\mathcal{U}}}_\lambda$ to be the direct limit of the models along $\bar{b}$. Then set $\mathcal{M}^{\mathcal{U}}_\lambda$ to be the direct limit of all the $\mathcal{M}^{\mathcal{U}}_\alpha$ such that $\alpha \in \bar{b}$.

If $\alpha = \beta + 1$ and $\mathcal{T} \restriction \alpha, \bar{\mathcal{U}} \restriction \alpha$ and $\mathcal{U} \restriction \alpha$ are defined then we have models $\mathcal{M}^{\mathcal{T}}_\alpha$, $\mathcal{M}^{\bar{\mathcal{U}}}_\beta$ and $\mathcal{M}^{\mathcal{U}}_\beta$. Let $j_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{j\mathcal{T}}_\alpha$ be the copy maps and define the set

$$\Sigma := \{\delta \leqslant \kappa \mid \delta \text{ is a regular } V\text{-cardinal} \wedge j_\delta\text{"}\delta \subseteq \delta \wedge \delta \geqslant \sup_{\gamma<\delta} \nu^{\mathcal{U}}_\gamma\}.$$

Note that Σ is stationary below κ since κ is Mahlo by Shelah (1994). Furthermore define for $\alpha < \theta$ the phalanxes

$$\Psi_\alpha := \Phi(j\mathcal{T} \restriction \alpha)^\frown\langle \mathrm{Ult}(\mathcal{M}^{\mathcal{U}}_\alpha, E_{j_\alpha} \restriction \alpha), \alpha\rangle.$$

$\boxed{\textbf{Case 1}\text{: } \beta \notin \Sigma.}$

If $\mathcal{M}^{\mathcal{T}}_\beta$ agrees with $\mathcal{M}^{\mathcal{U}}_\beta$ below κ then pad one more step, meaning that we set $\mathcal{M}^{\mathcal{T}}_\alpha := \mathcal{M}^{\mathcal{T}}_\beta$, $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha := \mathcal{M}^{\bar{\mathcal{U}}}_\beta$, $\mathcal{M}^{\mathcal{U}}_\alpha := \mathcal{M}^{\mathcal{U}}_\beta$ and $\pi_\alpha := \pi_\beta$. Otherwise let η be the least disagreement between $\mathcal{M}^{\mathcal{T}}_\beta$ and $\mathcal{M}^{\mathcal{U}}_\beta$. If $E^{\mathcal{U}}_\beta \neq \varnothing$ then set $E^{\bar{\mathcal{U}}}_\beta := \dot{E}^{\mathcal{M}^{\bar{\mathcal{U}}}_\beta}_{\pi_\beta(\eta)}$ and note that $(\mathcal{M}^{\mathcal{U}}_\alpha)^* \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\mathrm{pred}^{\mathcal{U}}(\alpha)}$ implies that $(\mathcal{M}^{\bar{\mathcal{U}}}_\alpha)^* \trianglelefteq \mathcal{M}^{\bar{\mathcal{U}}}_{\mathrm{pred}^{\bar{\mathcal{U}}}(\alpha)}$. Set $\pi_\alpha[a, f] := [\pi_\beta(a), \pi_{\mathrm{pred}^{\mathcal{U}}(\alpha)}(f)]$. Then π_α is a $\deg^{\mathcal{U}}(\alpha)$-embedding. If $E^{\mathcal{U}}_\beta = \varnothing$ then set $\pi_\alpha := \pi_\beta$ and follow the rules of the iteration game.

$\boxed{\textbf{Case 2}\text{: } \mathcal{M}^{\mathcal{U}}_\beta \text{ is a stable weasel and } \Psi_\beta \text{ is not iterable.}}$

Here we call $\beta + 1$ a **special drop**. Let $\mathcal{V}$ be a bad tree on Ψ_β and let

$$\pi_+ : \mathrm{cHull}^{V_{\Omega+1}}(\alpha \cup \{j_\beta, \mathcal{V}, \mathcal{T} \restriction \alpha, \mathcal{U} \restriction \alpha, \mathcal{M}(\mathcal{V}), \mathcal{Q}(\mathcal{V})\}) \to V_{\Omega+1}$$

be the uncollapse. Then set $\mathcal{M}^{\mathcal{U}}_\alpha := \pi_+^{-1}(\mathcal{M}^{\mathcal{U}}_\beta)$, $D^{\mathcal{U}\restriction\alpha+1} := D^{\mathcal{U}\restriction\alpha} \cup \{\alpha\}$, $\nu^{\mathcal{U}}_\beta := 0$, $\deg^{\mathcal{U}}(\alpha) = \omega$, $\beta <_{\mathcal{U}} \alpha$, $\pi_\alpha := \pi_\beta \circ \pi_+$ and pad $\mathcal{T}$ and $\bar{\mathcal{U}}$. Note that we still have that $\mathcal{M}^{\mathcal{U}}_\alpha$ agrees with $\mathcal{M}^{\mathcal{T}}_\alpha$ below $\sup\{\max(\nu^{\mathcal{U}}_\gamma, \nu^{\mathcal{U}}_\gamma) \mid \gamma < \beta\}$ since this supremum is $\leqslant \beta$ (and thus $< \alpha$) by case hypothesis.

$\boxed{\textbf{Case 3}\text{: otherwise.}}$

If the $\mathcal{M}^{\mathcal{T}}_\beta$ agrees with $\mathcal{M}^{\mathcal{U}}_\beta$ below κ then halt. Otherwise pick the least disagreement and proceed as in case 1.

Lemma 2.1. *If $\theta \geqslant \kappa$ and Ψ_κ is not iterable then there is some $\delta \in [0,\kappa)_{\mathcal{U}} \cap \Sigma$ such that Ψ_δ is not iterable.*

Proof. Let $\mathcal{V}$ be a bad tree on Ψ_κ. As there is no inner model with a Woodin, we can without loss of generality assume that Ψ_κ is properly 1-small. Use that Σ is stationary in κ to pick some $\delta \in [0,\kappa)_{\mathcal{U}} \cap \Sigma$, which is possible as $[0,\kappa)_{\mathcal{U}}$ is club in κ. Now let

$$\mathcal{H} := \mathrm{cHull}^{\mathcal{V}_{\Omega+1}}(\{j_\kappa, \mathcal{V}, \mathcal{T}\restriction\kappa+1, \mathcal{U}\restriction\kappa+1, \delta, \mathcal{M}(\mathcal{V}), \mathcal{Q}_{\mathcal{V}}\})$$

with $\mathcal{Q}_{\mathcal{V}}$ the $\mathcal{Q}$-structure looking for branches in $\mathcal{V}$. Then by the absoluteness argument in (Nielsen, 2016, Theorem 6.21), using that $\mathcal{H}$ is countable and also again that no inner model with a Woodin exists if $\bar{\mathcal{V}}$ had a cofinal wellfounded branch (in V) then so would $\mathcal{V}$ – therefore $\bar{\mathcal{V}}$ is truly a bad tree on $\bar{\Psi}_\kappa$. Now define

$$\mathcal{H}^+ := \mathrm{cHull}^{V_{\Omega+1}}(\delta \cup \{j_\kappa, \mathcal{V}, \mathcal{T}\restriction\kappa+1, \mathcal{U}\restriction\kappa+1, \mathcal{M}(\mathcal{V}), \mathcal{Q}_{\mathcal{V}}\})$$

and let $\psi : \mathcal{H} \to \mathcal{H}^+$ be the canonical embedding. Then $\psi\bar{\mathcal{V}}$ is a bad tree on the phalanx

$$\begin{aligned}\psi(\bar{\Psi}_\kappa) &= \psi(\overline{\Phi(j\mathcal{T}\restriction\kappa)})^\frown\langle\psi(\overline{\mathrm{Ult}(M^{\mathcal{U}}_\kappa, E_{j_\kappa}\restriction\kappa)}), \psi(\bar{\kappa})\rangle \\ &= \pi_+^{-1}(\Phi(j\mathcal{T}\restriction\kappa))^\frown\langle\mathrm{Ult}(\pi_+^{-1}(\mathcal{M}^{\mathcal{U}}_\kappa), E_{j_{\pi_+^{-1}(\kappa)}}\restriction\pi_+^{-1}(\kappa)), \pi_+^{-1}(\kappa)\rangle, \\ &= \Phi(j\mathcal{T}\restriction\delta)^\frown\langle\mathrm{Ult}(\pi_+^{-1}(\mathcal{M}^{\mathcal{U}}_\kappa), E_{j_\delta}\restriction\delta), \delta\rangle,\end{aligned}$$

where $\pi_+ : \mathcal{H}^+ \to V_{\Omega+1}$ is the uncollapse. But now define $\tau : \pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}) \to \mathcal{M}_\delta^\mathcal{U}$ as $\tau := (i_{\delta\kappa}^\mathcal{U})^{-1} \circ \pi_+$. To show that this is well-defined we have to show that π_+"$\pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}) \subseteq \operatorname{ran} i_{\delta\kappa}^\mathcal{U}$, so let $x = \pi_+(\bar{x})$ for $\bar{x} \in \pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U})$. By elementarity we get that $\bar{x} \in \mathcal{M}_\delta^\mathcal{U} \cap \mathcal{H}^+$, so since $\delta \in [0,\kappa)_\mathcal{U}$ there is some $\eta <_\mathcal{U} \delta$ and $z \in \mathcal{M}_\eta^\mathcal{U}$ such that $\bar{x} = i_{\eta\delta}^\mathcal{U}(z)$. But then

$$x = \pi_+(\bar{x}) = \pi_+(i_{\eta\delta}^\mathcal{U}(z)) = i_{\eta\kappa}^\mathcal{U}(z) = i_{\delta\kappa}^\mathcal{U}(i_{\eta\delta}^\mathcal{U}(z)) \in \operatorname{ran} i_{\delta\kappa}^\mathcal{U}$$

and τ is well-defined. Lift τ to $\tilde{\tau} : \operatorname{Ult}(\pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}), E_{j_\delta} \restriction \delta) \to \operatorname{Ult}(\mathcal{M}_\delta^\mathcal{U}, E_{j_\delta} \restriction \delta)$, so that $\tilde{\tau}\psi\bar{\mathcal{V}}$ is then a bad tree on $\Phi(j\,\mathcal{T} \restriction \delta)^\frown\langle \operatorname{Ult}(\mathcal{M}_\delta^\mathcal{U}, E_{j_\delta} \restriction \delta), \delta\rangle = \Psi_\delta$. ■

Proposition 2.2. *The above procedure halts at some $\theta \leqslant \kappa$.*

Proof. It suffices to show that $\kappa + 1$ does not satisfy cases 1 and 2. But $\kappa \in \Sigma$ since κ is regular and $j(\kappa) = \kappa$, and $\sup_{\gamma<\kappa} \nu_\gamma^\mathcal{U} \leqslant \kappa$ by definition of $\mathcal{U}$. Assume thus towards a contradiction that $\kappa + 1$ is a special drop. In particular $\mathcal{M}_\kappa^\mathcal{U}$ is a weasel (since κ is a regular V-cardinal), and note that for this to happen we cannot have had any drops on the $\mathcal{W}$-to-$\mathcal{M}_\kappa^\mathcal{U}$ branch, special or not. This is purely for cardinality reasons, as otherwise we would have dropped to a structure of cardinality $\leqslant \kappa < \Omega$. But Lemma 2.1 implies that there is some $\delta \in [0,\kappa)_\mathcal{U} \cap \Sigma$ such that Ψ_δ is not iterable, and as there are no drops on the branch, $\mathcal{M}_\delta^\mathcal{U}$ is a weasel (since δ is a regular V-cardinal), making $\delta + 1 <_\mathcal{U} \kappa$ a special drop, $\lightning$. ■

This finishes the construction of $\mathcal{T}$, $\mathcal{U}$ and $\bar{\mathcal{U}}$. Let $\theta \leqslant \kappa$ be the length of this comparison process and set $\mathcal{P} := \mathcal{M}_\theta^\mathcal{T}$, $\mathcal{Q} := \mathcal{M}_\theta^\mathcal{U}$ and $j^+ := j_\theta$.

3 Iterability

Lemma 3.1. *If $\mathcal{Q}$ is a stable weasel then $\Psi := \Phi(j\,\mathcal{T})^\frown\langle \operatorname{Ult}(\mathcal{Q}, E_{j^+} \restriction \kappa), \kappa\rangle$ is a stable iterable phalanx.*

Proof. Note first that $\mathcal{Q}$ being a weasel implies that there are no drops on the $\mathcal{W}$-to-$\mathcal{Q}$ branch, special or not, just as before. Assume that the lemma fails, which then forces $\theta = \kappa$, as there are cofinally many disagreements below κ due to $\operatorname{crit} j < \kappa$.

This means that $\kappa + 1$ is a special drop, but then Lemma 2.1 implies that there is a special drop $\delta + 1 \in [0, \kappa)_{\mathcal{U}}$, $\lightning$. ■

Lemma 3.2. *If $\mathcal{Q}$ is a set premouse then for any proper initial segment $\mathcal{N} \lhd \mathcal{Q}$ such that $n := \min\{k < \omega \mid \rho_{k+1}(\mathcal{N}) \leqslant \theta\}$ exists it holds that*

$$\Psi := \Phi(j\,\mathcal{T})^\frown\langle(\mathrm{Ult}_n(\mathcal{N}, E_{j^+} \upharpoonright \theta), n), \theta\rangle$$

is a stable and iterable phalanx.

Proof. Since $\mathcal{Q}$ is a set premouse there is a drop on the $\mathcal{W}$-to-$\mathcal{Q}$ branch, so that $\mathcal{P} \trianglelefteq \mathcal{Q}$. As there are no drops on the $\bar{\mathcal{W}}$-to-$\mathcal{P}$ branch, $|\mathcal{P}| = \kappa$. If $\theta < \kappa$ then regularity of κ would entail that $|\,\mathcal{Q}\,| < \kappa$, $\lightning$, so $\kappa = \theta$.

Suppose the claim fails, so that we have a bad tree $\mathcal{V}$ on Ψ. Let b be the $\mathcal{W}$-to-$\mathcal{Q}$ branch, $\gamma \in b$ least above all drops and $\delta \in b \cap \Sigma$ least above γ, which is possible as b is club in κ. Then just as in the proof of Lemma 2.1 we get that

$$\Phi(j\,\mathcal{T})^\frown\langle(\mathrm{Ult}_n(\pi_+^{-1}(\mathcal{N}), E_{j_\delta} \upharpoonright \delta), n), \delta\rangle$$

is not iterable. Since $\delta \in b - \gamma$, $\mathcal{M}^{\mathcal{U}}_\delta$ is the direct limit of previous models, so that elementarity of π_+ implies that $\pi_+^{-1}(\mathcal{M}^{\mathcal{U}}_\kappa) = \mathcal{M}^{\mathcal{U}}_\delta$. Thus $\mathcal{N}_0 := \pi_+^{-1}(\mathcal{N}) \lhd \mathcal{M}^{\mathcal{U}}_\delta$.

We will show that $\varepsilon + 1 := \mathrm{succ}_{\mathcal{U}}(\delta)$ is a drop, which will contradict that $\delta > \gamma$ and γ being above all drops in b. The extender $E^{\mathcal{U}}_\varepsilon$ is the first one used on $[\delta, \kappa)_{\mathcal{U}}$. We claim that $\mathrm{lh}\, E^{\mathcal{U}}_\varepsilon \leqslant o(\mathcal{N}_0)$. If this was not the case then $\mathcal{N}_0 \trianglelefteq \mathcal{M}^{\mathcal{T}}_\varepsilon$ and thus also $\mathcal{N}_0 \trianglelefteq \mathcal{P}$. But then $\mathrm{Ult}_n(\mathcal{N}_0, E_{j_\delta} \upharpoonright \delta)$ can be embedded into $\mathrm{Ult}_n(\mathcal{P}, E_{j^+} \upharpoonright \kappa)$, so that $\Psi' := \Phi(j\,\mathcal{T})^\frown\langle(\mathrm{Ult}_n(\mathcal{P}, E_{j^+} \upharpoonright \kappa), n), \kappa\rangle$ is not iterable, but Ψ' is the derived phalanx of an iteration tree on the iterable $\mathcal{W}$, $\lightning$. Thus $\mathrm{lh}\, E^{\mathcal{U}}_\varepsilon \leqslant o(\mathcal{N}_0)$.

Note also that since δ is a regular V-cardinal it holds that for every extender E used on $\mathcal{U}$, if $\mathrm{crit}\, E < \delta$ then $\mathrm{lh}\, E < \delta$. In particular this means that $\mu := \mathrm{crit}\, E^{\mathcal{U}}_\varepsilon \geqslant \delta$, so that $\mu^{+\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}} > \delta$. Say furthermore that $\mu^{+\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}} \geqslant o(\mathcal{N}_0)$. Then by the result in the above paragraph, $\mu^{+\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}} \geqslant \mathrm{lh}\, E^{\mathcal{U}}_\varepsilon > \mu^{+\mathcal{M}^{\mathcal{U}}_\varepsilon}$, which is a contradiction since $\mu^{+\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}} = \mu^{+(\mathcal{M}^{\mathcal{U}}_{\varepsilon+1})^*} \leqslant \mu^{+\mathcal{M}^{\mathcal{U}}_\varepsilon}$ by definition of $(\mathcal{M}^{\mathcal{U}}_{\varepsilon+1})^*$. Thus $\mu^{+\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}} \in (\delta, o(\mathcal{N}_0))$.

But now by definition of $\mathcal{N}_0$, $\rho(\mathcal{N}_0) = \rho(\mathcal{N}) \leqslant \delta$, so that since $\mathcal{N}_0 \lhd \mathcal{M}^{\mathcal{U}}_\delta$, $\mathcal{N}_0$ is sound and $\mu^{+\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}}$ is collapsed in $\mathcal{M}^{\mathcal{U}}_\delta$. This means that $(\mathcal{M}^{\mathcal{U}}_{\varepsilon+1})^* \lhd \mathcal{M}^{\mathcal{U}}_\delta$, making $\varepsilon+1$ a drop, $\lightning$. ■

4 The action

Lemma 4.1. *$i_{\mathcal{U}}``\kappa \nsubseteq \kappa$.*

Proof. Suppose that $i_{\mathcal{U}}``\kappa \subseteq \kappa$. We claim that there are no drops on the $\mathcal{W}$-to-$\mathcal{Q}$ branch, special or not. If there were only standard drops on the branch then there would not be any drops on the $\bar{\mathcal{W}}$-to-$\mathcal{P}$ branch, so universality of $\mathcal{W}$ implies that $|\mathcal{Q}| = \kappa$ which in turn requires that $\theta = \kappa$ by regularity of κ. But then we also get that $i_{\mathcal{U}}``\kappa \nsubseteq \kappa$, $\lightning$.

Suppose now $\varepsilon+1$ is a special drop on the $\mathcal{W}$-to-$\mathcal{Q}$ branch, entailing that $|\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}| < \kappa$ and $\Phi(j\mathcal{T})^\frown\langle(\mathrm{Ult}_n(\mathcal{M}^{\mathcal{U}}_\varepsilon, n), E_{j_\varepsilon} \restriction \varepsilon), \varepsilon\rangle$ is not iterable, by definition of special drop. As $|\mathcal{M}^{\mathcal{U}}_{\varepsilon+1}| < \kappa$ and $i_{\mathcal{U}}``\kappa \subseteq \kappa$ we get that $\theta < \kappa$, so the coiteration procedure ends with $\mathcal{Q} \trianglelefteq \mathcal{P}$. This means that $\mathrm{Ult}_n(\mathcal{M}^{\mathcal{U}}_\varepsilon, E_{j_\varepsilon} \restriction \varepsilon)$ embeds into $\mathrm{Ult}_n(\mathcal{P}, E_{j^+} \restriction \kappa)$, and then $\Phi(j\mathcal{T})^\frown\langle(\mathrm{Ult}_n(\mathcal{P}, E_{j^+} \restriction \kappa), n), \kappa\rangle$ is not iterable – but this phalanx is the derived phalanx of an iteration tree on the iterable $\mathcal{W}$, $\lightning$.

Thus there are no drops on the $\mathcal{W}$-to-$\mathcal{Q}$ branch and $\mathcal{Q}$ is thus a weasel. Now by Lemma 3.1, setting $\Psi_{\mathcal{W}} := \Phi(j\mathcal{T})$, $\mathcal{M} := \mathrm{Ult}(\mathcal{Q}, E_{j^+} \restriction \kappa)$ and $\Psi_{\mathcal{M}} := \Phi(j\mathcal{T})^\frown\langle\mathcal{M}, \kappa\rangle$, we can compare $\Psi_{\mathcal{W}}$ with $\Psi_{\mathcal{M}}$:

$$\begin{array}{ccc} \mathcal{R} & \trianglelefteq & \mathcal{S} \\ \uparrow \mathcal{V} & & \uparrow \mathcal{V}' \\ \Psi_{\mathcal{M}} & & \Psi_{\mathcal{W}} \end{array}$$

The thick trick (Nielsen, 2016, Lemma 8.6) implies that $\mathcal{M}$ is below $\mathcal{R}$ and letting $\iota : \mathcal{Q} \to \mathcal{M}$ be the ultrapower map we have that $\varphi := i_{\mathcal{V}} \circ \iota \circ i_{\mathcal{U}} : \mathcal{W} \to \mathcal{R} \trianglelefteq \mathcal{S}$, so Dodd-Jensen implies that $\mathcal{R} = \mathcal{S}$. Lastly, we can view $\mathcal{V}'$ as an iteration tree $\mathcal{V}''$ on $\mathcal{W}$. All in all, we now have the coiteration

$$\begin{tikzcd} \mathcal{R} \arrow[r, phantom, "="] & \mathcal{S} \\ \mathcal{M} \arrow[u, "\mathcal{V}"] & \mathcal{W} \arrow[u, "\mathcal{V}''"] \end{tikzcd}$$

This implies that φ" $\mathrm{Def}^{\mathcal{W}} = \mathrm{Def}^{\mathcal{S}} = i_{\mathcal{V}''}$" $\mathrm{Def}^{\mathcal{W}}$, and as $\kappa^+ \subseteq \mathrm{Def}^{\mathcal{W}}$ we also get $\varphi \restriction \kappa = i_{\mathcal{V}''} \restriction \kappa$, so that $\operatorname{crit} i_{\mathcal{V}''} = \operatorname{crit} \varphi \leqslant \operatorname{crit} \iota = \operatorname{crit} j < \kappa$. We will now finish the proof by showing that $i_{\mathcal{U}}(\operatorname{crit} i_{\mathcal{V}''}) \geqslant \kappa$, contradicting $i_{\mathcal{U}}$"$\kappa \subseteq \kappa$.

As $\operatorname{crit} i_{\mathcal{V}} \geqslant \kappa$ (as $\mathcal{M}$ is below $\mathcal{R}$) and ι"$\kappa \subseteq \kappa$, it suffices to show that $\varphi(\operatorname{crit} i_{\mathcal{V}''}) \geqslant \kappa$, or equivalently $i_{\mathcal{V}}(\operatorname{crit} i_{\mathcal{V}''}) \geqslant \kappa$. If $i_{\mathcal{V}}(\operatorname{crit} i_{\mathcal{V}''}) < \kappa$ and E is the first extender on the $\mathcal{W}$-to-$\mathcal{S}$ branch then $\operatorname{lh} E < i_{\mathcal{V}}(\operatorname{crit} i_{\mathcal{V}''}) < \kappa$, but $\mathcal{M}$ agrees with $\mathcal{W}$ up to κ, $\lightning$. ■

Note that Lemma 4.1 implies that $\theta = \kappa$: indeed, it implies that either

$$\sup_{\alpha<\theta} \operatorname{crit} E^{\mathcal{U}}_{\alpha} \geqslant \kappa \tag{†}$$

or there being an extender $E^{\mathcal{U}}_{\alpha}$ such that $\operatorname{crit} E^{\mathcal{U}}_{\alpha} < \kappa$ and $\operatorname{lh} E^{\mathcal{U}}_{\alpha} \geqslant \kappa$. But $\mathcal{U}$ has only extenders of length $< \kappa$, so only (†) is possible. Regularity of κ then ensures that $\theta = \kappa$. Furthermore, a standard regressive set argument shows that there is a club $C \subseteq \kappa$ such that for $\alpha, \beta \in C$ with $\alpha < \beta$, it holds that $i^{\mathcal{U}}_{\alpha\beta}(\kappa_\alpha) = \kappa_\beta = \beta$, where $\kappa_\xi := \operatorname{crit} i^{\mathcal{U}}_{\xi\kappa}$.

Now for $\alpha \in C$ we have $i_{\alpha\kappa}(\kappa_\alpha) = \kappa$ and moreover $\sup(i^{\mathcal{U}}_{\alpha\kappa})$"$\kappa_\alpha^{+\mathcal{M}^{\mathcal{U}}_\alpha} = \kappa^{+\mathcal{Q}}$. Then $\operatorname{cof} \kappa^{+\mathcal{Q}} < \kappa$ while $\operatorname{cof} \bar\lambda = \kappa$, so that $\kappa^{+\mathcal{Q}} \neq \bar\lambda$. Assume for a contradiction that $\kappa^{+\mathcal{Q}} < \bar\lambda$. Then as $\bar\lambda = \kappa^{+\mathcal{R}}$ and no extenders on $\mathcal{T}$ overlap κ, there would be a drop on $\mathcal{T}$ on the next step in the coiteration, contradicting the universality of $\mathcal{W}$. Thus $\kappa^{+\mathcal{Q}} > \bar\lambda$. This means that there is some least $\mathcal{N} \lhd \mathcal{Q}$ and some $n < \omega$ such that

$$\rho_{n+1}(\mathcal{N}) = \kappa < \bar\lambda \leqslant \rho_n(\mathcal{N}).$$

Setting $\mathcal{M} := \mathrm{Ult}_n(\mathcal{N}, E_{j^+} \restriction \kappa)$, Lemma 3.2 entails that $\Psi_{\mathcal{M}} := \Phi(j\,\mathcal{T})^\frown\langle\mathcal{M}, \kappa\rangle$ is stable and iterable, so that applying the thick trick (Nielsen, 2016, Lemma 8.6) to $\langle\Psi_{\mathcal{M}}, \Phi(j\,\mathcal{T})\rangle$ and using that trees on $\Phi(j\,\mathcal{T})$ are really trees on $\mathcal{W}$ gives us the coiteration

$$\begin{tikzcd} \mathcal{R} & \lhd & \mathcal{S} \\ \mathcal{M} \arrow[u] & & \mathcal{W} \arrow[u] \end{tikzcd}$$

But now $\sup(j^+)"\bar{\lambda} = j^+(\bar{\lambda}) = \lambda$ is definably collapsed in $\mathcal{M}$ and the $(n+1)$'st mastercode $A^{n+1}_{\mathcal{M}}$ is definable over $\mathcal{R}$ so that $\lambda < \kappa^{+\mathcal{S}}$. As it furthermore holds that $\mathscr{P}^{\mathcal{S}}(\kappa) = \mathscr{P}^{\mathcal{W}}(\kappa)$ we get $\lambda < \kappa^{+\mathcal{W}} = \kappa^{+K} = \lambda$, $\lightning$. This finishes the proof of Theorem 0.1.

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