

# STRUCTURING THE MATHEMATICAL UNIVERSE

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*Draft version 0.6*



# Preface

This booklet is not designed to teach the reader about a subject from start to finish. It is not intended as an article, providing new results. It is designed for people who seek an overview - people who have learned about various mathematical subjects and want to know how these fit together. It also doubles as an encyclopaedia through a wide range of examples of mathematical structures.

To make it as easy as possible to look up definitions and links, I have refrained myself from writing unnecessary explanatory text and have tried to be as concise as possible while still giving the necessary information for understanding the structure at hand.

The map, which illustrates all the links described in this booklet, has been categorized within the same structure as this booklet's chapters. Furthermore, the arrows in the map indicate if a structure implies another structure. If the arrow is based on a definition, it is coloured black. If it is based on a theorem, remark or anything else not a definition, it is coloured green.

Throughout this booklet I will use the bold notation  $\mathbf{A}$  to denote a structure, and the corresponding non-bold letter  $A$  to denote the underlying set. Furthermore, when substructures are mentioned, then algebraic substructures of the structures in this booklet are closed under their operations, metric substructures have metrics restricted to the underlying subset and the pavings of paving substructures are defined by intersecting the open sets with the underlying subset.

New versions will be available at:

[https://www.dropbox.com/sh/2ypp1f13aptp7ww/2QZ\\_DjBlxK](https://www.dropbox.com/sh/2ypp1f13aptp7ww/2QZ_DjBlxK)

# Change log

## Changes in versions 0.5 and 0.6

Fixed some more minor errors.

## Changes in version 0.4

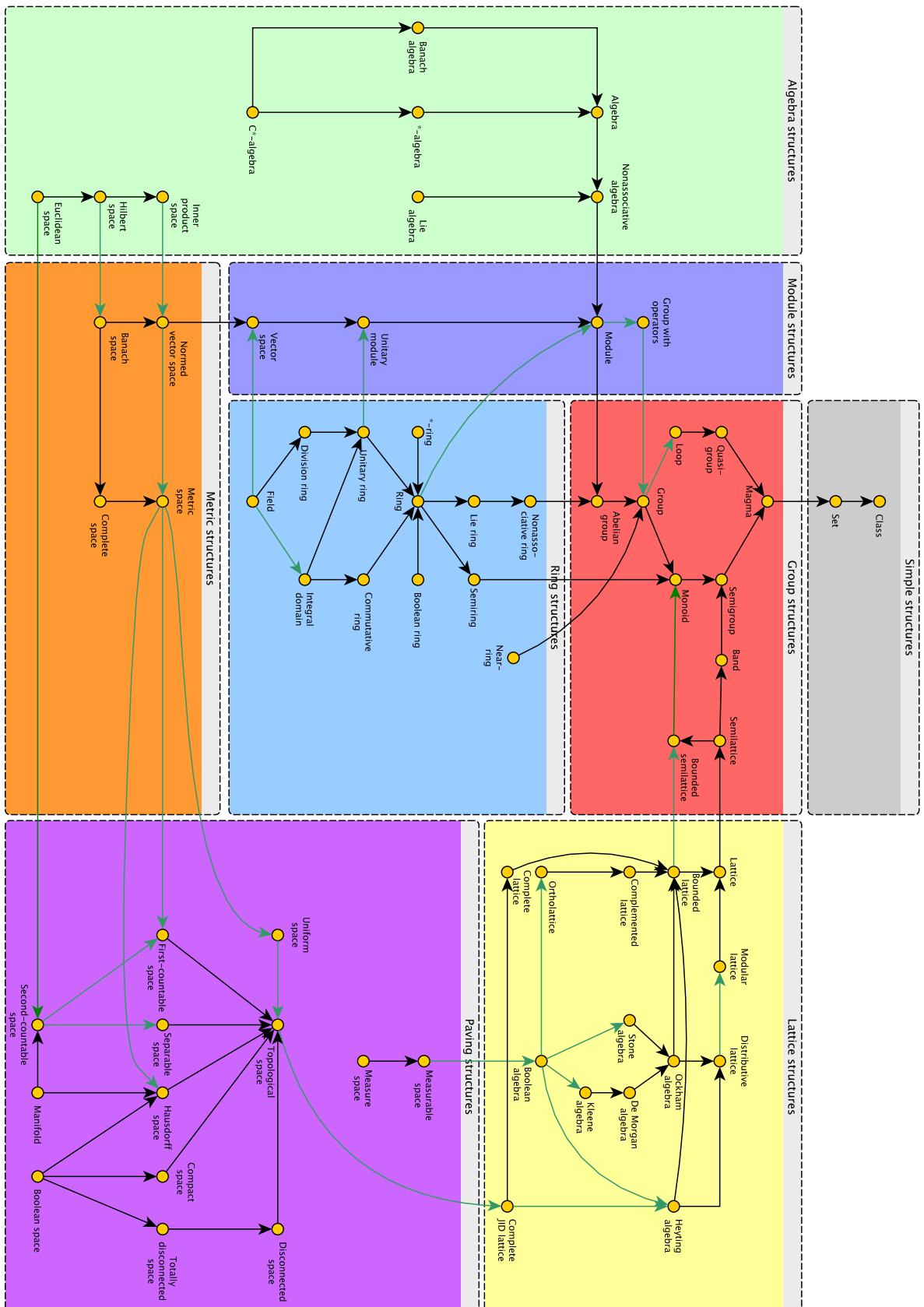
Fixed minor errors, switched order of some sections. Added that metric spaces are Hausdorff spaces and rings are Lie rings. Added some examples. Made things a bit prettier.

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# The map



# Chapter 1

## Simple structures

A simple structure is a structure with no operations equipped to it.

### 1.1 Classes

**Definition 1.1.1.** A class is a collection of elements. If an object  $x$  is an element of  $A$ , then this is denoted  $x \in A$ . Classes have the equivalence relation “ $=$ ” defined as:

$$(x \in A \Leftrightarrow x \in B) \Leftrightarrow A = B$$

**Definition 1.1.2.** A class  $A$  is called a *subclass* of a class  $B$ , denoted  $A \subseteq B$ , if

$$\forall x \in A : x \in A \Rightarrow x \in B$$

**Example 1.1.3.** Examples of proper classes (that is, a class which is not a set) can usually be given by the collection of every all algebraic objects of a given type. Examples of this is the class of all sets, the class of all groups and the class of all vector spaces.

### 1.2 Sets

**Definition 1.2.1.** A *set* is a class  $A$  for which there exist a class  $B$ , such that  $A \in B$ .

**Definition 1.2.2.** A *subclass*  $A$  of a class  $B$  is a subset of  $B$  if and only if  $A$  is a set.

**Definition 1.2.3.** The *empty set* or null set, denoted  $\emptyset$ , is the set with no elements:

$$\forall x : x \notin \emptyset$$



# Chapter 2

## Group structures

A group structure is a structure defined on a set, equipped with a single binary operation.

**Definition 2.0.4.** A *binary operation* on a set  $X \neq \emptyset$  is a map  $\cdot : X \times X \rightarrow X$ , often denoted with infix notation  $a \cdot b$  instead of with function notation  $\cdot(a, b)$ .

### 2.1 Magmas

**Definition 2.1.1.** A *magma*  $(A, \cdot)$  is a set  $A$  along with a binary operation  $\cdot$ .

### 2.2 Semigroups

**Definition 2.2.1.** A *semigroup*  $(A, \cdot)$  is a magma  $(A, \cdot)$  satisfying  $A \neq \emptyset$  and that  $\cdot$  is associative:

$$\forall a, b, c \in A : a(bc) = (ab)c$$

### 2.3 Monoids

**Definition 2.3.1.** A *monoid*  $(A, \cdot, e)$  is a semigroup  $(A, \cdot)$  with an identity element:

$$\exists e \in A \forall a \in A : ae = ea = a$$

### 2.4 Quasigroups

**Definition 2.4.1.** A *quasigroup*  $(A, \cdot, \backslash, /)$  is a magma  $(A, \cdot)$ , satisfying:

- (i)  $A \neq \emptyset$
- (ii)  $\forall a, b \in A \exists x, y \in A : ax = b \wedge ya = b$

The unique solutions to these equations are  $x = a \backslash b$  and  $y = b / a$ , called resp. left and right division.

**Example 2.4.2.** The nonzero rationals with division  $(\mathbb{Q} \setminus \{0\}, \div)$  form a quasigroup  $(\mathbb{Q} \setminus \{0\}, \div, \backslash, /)$  with  $a \backslash b = b \div a$  and  $a / b = a \div b$ .

## 2.5 Loops

**Definition 2.5.1.** A *loop*  $(A, \cdot, \backslash, /, e)$  is a quasigroup  $(A, \cdot, \backslash, /)$  with an identity element:

$$\exists e \in A \forall x \in A : xe = ex = x$$

**Example 2.5.2.** The nonzero *octonions*  $\mathbb{O}$ , which are an extension of the quaternions  $\mathbb{H}$  (which again are an extension of the complex numbers  $\mathbb{C}$ ), form a nonassociative loop under multiplication - this loop is known as the *Moufang loop*.

## 2.6 Groups

**Definition 2.6.1.** A *group*  $(A, \cdot, e, {}^{-1})$  is a monoid  $(A, \cdot, e)$ , in which every element has an inverse:

$$\forall a \in G \exists a^{-1} \in G : aa^{-1} = a^{-1}a = e$$

**Theorem 2.6.2.** Every group  $(A, \cdot, e, {}^{-1})$  induces a loop  $(A, \cdot, e, {}^{-1}, \backslash, /)$ .

*Proof.* Let  $\mathbf{A} := (A, \cdot, e, {}^{-1})$  be a group. Since every loop is a magma, it has to be shown that  $\cdot$  has an identity element and that left and right division are well-defined on  $\mathbf{A}$ .  $\mathbf{A}$  has an identity element since it is a monoid. For every  $a, b \in A$ , define  $x := a^{-1}b$  and  $y := ba^{-1}$ . Then:

$$\begin{aligned} ax &= a(a^{-1}b) = (aa^{-1})b = eb = b \\ ya &= (ba^{-1})a = b(a^{-1}a) = be = b \end{aligned}$$

And thus  $\mathbf{A}$  induces a loop. ■

## 2.7 Abelian groups

**Definition 2.7.1.** An *Abelian group*  $(A, \cdot, e, {}^{-1})$  is a group in which  $\cdot$  is commutative:

$$\forall a, b \in A : ab = ba$$

## 2.8 Bands

**Definition 2.8.1.** A *band*  $(A, \cdot)$  is a semigroup  $(A, \cdot)$ , in which every element is idempotent:

$$\forall a \in A : aa = a$$

## 2.9 Semilattices

**Definition 2.9.1.** A *semilattice*  $(A, \cdot)$  is a commutative band. The binary operation is often either  $\wedge$ , called “meet”, or  $\vee$ , called “join”.

## 2.10 Bounded semilattices

**Definition 2.10.1.** A *bounded semilattice*  $(A, \cdot, e)$  is a semilattice  $(A, \cdot)$  with an identity element:

$$\forall a \in A : ae = a$$

*Remark 2.10.2.* A bounded semilattice is equivalent to an idempotent commutative monoid.

## Chapter 3

### Lattice structures

A lattice structure is a structure defined on a set, equipped with two or more binary operations, including binary operations called *meet* and *join*.

#### 3.1 Lattices

**Definition 3.1.1.** A *lattice*  $(A, \vee, \wedge)$  is a structure in which  $(A, \vee)$  and  $(A, \wedge)$  are both semilattices, as well as satisfying the absorption laws:

- (i)  $\forall a, b \in A : a \vee (a \wedge b) = a$
- (ii)  $\forall a, b \in A : a \wedge (a \vee b) = a$

*Remark 3.1.2.* A poset  $(A, \leq)$  is a lattice if and only if for all  $a, b \in A$  both  $\sup\{a, b\}$  and  $\inf\{a, b\}$  exist in  $A$ . This is called the *dual definition* of lattices. If a lattice  $\mathbf{A}$  is defined algebraically, it can be defined dually by defining  $\leq$  as  $a \leq b \Leftrightarrow a = a \wedge b \Leftrightarrow b = a \vee b$ . If  $\mathbf{A}$  is defined as a poset, then the two operations  $\wedge$  and  $\vee$  can be defined as  $a \wedge b := \inf\{a, b\}$  and  $a \vee b := \sup\{a, b\}$ .

**Example 3.1.3.** The natural numbers  $\mathbb{N}$  form a lattice  $(\mathbb{N}, \vee, \wedge)$  with  $a \vee b := \max\{a, b\}$  and  $a \wedge b := \min\{a, b\}$ .

#### 3.2 Bounded lattices

**Definition 3.2.1.** A *bounded lattice*  $(A, \vee, \wedge, \perp, \top)$  is a lattice, which satisfies the identity laws:

- (i)  $\forall a \in A : a \vee \perp = a$
- (ii)  $\forall a \in A : a \wedge \top = a$

*Remark 3.2.2.* Every bounded lattice is a bounded semilattice as well, since these satisfy their respective (join/meet-)identity law as well.

**Example 3.2.3.** Let  $A$  be a set. Then the power set  $P(A)$  form a bounded lattice  $(P(A), \cup, \cap, \emptyset, A)$ .

### 3.3 Modular lattices

**Definition 3.3.1.** A *modular lattice*  $(A, \vee, \wedge)$  is a lattice, which satisfies the modular law:

$$\forall a, b, c \in A : (a \wedge b) \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge b$$

**Example 3.3.2.** Let  $\mathbf{A}$  be a group. Then the normal subgroups  $\mathcal{N}(\mathbf{A})$  of  $\mathbf{A}$  form a modular lattice  $(\mathcal{N}(\mathbf{A}), \vee, \wedge)$  with  $\mathbf{A} \wedge \mathbf{B} := \mathbf{A} \cap \mathbf{B}$  and  $\mathbf{A} \vee \mathbf{B} := \mathbf{AB} = \{ab \mid a \in A, b \in B\}$ .

### 3.4 Distributive lattices

**Definition 3.4.1.** A *distributive lattice*  $(A, \vee, \wedge)$  is a lattice, which is distributive:

$$\forall a, b, c \in A : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

**Example 3.4.2.** The natural numbers  $\mathbb{N}$  form a distributive lattice  $(\mathbb{N}, \vee, \wedge)$  with  $a \wedge b := \gcd\{a, b\}$  and  $a \vee b := \text{lcm}\{a, b\}$ .

**Theorem 3.4.3.** *Every distributive lattice is a modular lattice.*

*Proof.* Let  $\mathbf{A}$  be a distributive lattice. It has to be checked that it satisfies the modular law. Let  $a, b, c \in A$ :

$$(a \vee (b \wedge c)) \wedge b = (a \wedge b) \vee (b \wedge (b \wedge c)) = (a \wedge b) \vee ((b \wedge b) \wedge c) = (a \wedge b) \vee (b \wedge c)$$

where the distributive, associative, commutative and idempotent laws were used. ■

### 3.5 Complemented lattices

**Definition 3.5.1.** A *complemented lattice*  $(A, \vee, \wedge, \perp, \top)$  is a bounded lattice which satisfies that every element has atleast one complement b:

$$(i) \quad \forall a \in A \exists b \in A : a \vee b = \top$$

$$(ii) \quad \forall a \in A \exists b \in A : a \wedge b = \perp$$

**Theorem 3.5.2.** A *complemented distributive lattice*  $(A, \vee, \wedge, \perp, \top)$  has, for each  $a \in A$ , a *unique complement*  $\neg a$ .

*Proof.* Let  $a \in A$  and assume  $b, c \in A$  are both complements to  $a$ , meaning:

$$(i) \quad a \vee b = a \vee c = \top$$

$$(ii) \quad a \wedge b = a \wedge c = \perp$$

Then:

$$b = b \wedge (b \vee a) = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c) = c \wedge (a \vee b) = c \quad \blacksquare$$

### 3.6 Ortholattices

**Definition 3.6.1.** An *ortholattice*  $(A, \vee, \wedge, \perp, \top, \neg)$  is a complemented lattice equipped with the unary operation  $\neg : A \rightarrow C$ , where  $C \subseteq A$  is the set of complements of  $A$ . Furthermore,  $\neg a \in C_a$  for each  $a \in A$ , where  $C_a \subseteq C$  is the set of complements to  $a$ . Lastly,  $\neg$  is idempotent and satisfies De Morgans laws:

$$(i) \quad \forall a \in A : \neg \neg a = a$$

$$(ii) \quad \forall a, b \in A : \neg(a \vee b) = \neg a \wedge \neg b$$

$$(iii) \quad \forall a, b \in A : \neg(a \wedge b) = \neg a \vee \neg b$$

**Example 3.6.2.** Let  $\mathbf{X}$  be an inner product space. Then  $(P(X), \cup, \cap, \emptyset, X, \perp)$  is an ortholattice, where  $\perp : P(X) \rightarrow P(X)$  is defined as:

$$A^\perp := \{x \in X \mid \forall a \in A : \langle a, x \rangle = 0\}$$

### 3.7 Complete lattices

**Definition 3.7.1.** Let  $\mathbf{A}$  be a lattice and  $S \subseteq A$  a set. The *supremum* of  $S$ ,  $\bigvee S$ , is defined as an element  $b \in A$ , satisfying:

$$\bigvee S := \min\{b \in A \mid \forall a \in S : a \vee b = b\}$$

Analogously, the *infimum* of  $S$ ,  $\bigwedge S$ , is defined as an element  $c \in A$ , satisfying:

$$\bigwedge S := \max\{c \in A \mid \forall a \in S : a \wedge c = c\}$$

**Definition 3.7.2.** A *complete lattice*  $(A, \vee, \wedge, \perp, \top)$  is a bounded lattice, in which every subset  $S \subseteq A$  has a supremum  $\bigvee S$  and an infimum  $\bigwedge S$ .

### 3.8 Heyting algebras

**Definition 3.8.1.** A *Heyting algebra*  $(A, \vee, \wedge, \perp, \top, \rightarrow)$  is a bounded distributive lattice equipped with the binary operation  $\rightarrow$ , which satisfies that  $a \rightarrow b$  is the largest element  $x$  which satisfies  $x \wedge a \leq b$ :

$$\forall a, b, c \in A : (a \wedge b \leq c) \Leftrightarrow (a \leq (b \rightarrow c))$$

*Remark 3.8.2.* An equivalent definition for a Heyting algebra is if it holds that:

$$\forall a, b, c \in A : \bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}$$

### 3.9 Complete JID lattices

**Definition 3.9.1.** A *complete JID lattice*  $(A, \vee, \wedge, \perp, \top)$  is a complete distributive lattice, which satisfies the join-infinite distributive law:

$$\forall a \in A, S \subseteq A : a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

**Theorem 3.9.2.** Every complete JID lattice is a Heyting algebra.

*Proof.* Let  $\mathbf{A}$  be a complete JID lattice. By Remark 3.8.2 and completeness, it suffices to show that for  $a, b \in A$ :

$$\bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}$$

Notice for any  $u \in \{c \in A \mid a \wedge c \leq b\}$  trivially  $a \wedge u \leq b$ , so by JID:

$$a \wedge \bigvee \{c \in A \mid a \wedge c \leq b\} = \bigvee \{a \wedge c \in A \mid a \wedge c \leq b\} \leq b$$

which means that  $\bigvee \{c \in A \mid a \wedge c \leq b\} \in \{c \in A \mid a \wedge c \leq b\}$ . ■

### 3.10 Ockham algebras

**Definition 3.10.1.** An *Ockham algebra*  $(A, \vee, \wedge, \perp, \top, \neg)$  is a bounded distributive lattice with a unary operation  $\neg$ , such that:

- (i)  $\forall a, b \in A : \neg(a \wedge b) = \neg a \vee \neg b$
- (ii)  $\forall a, b \in A : \neg(a \vee b) = \neg a \wedge \neg b$
- (iii)  $\neg \top = \perp$
- (iv)  $\neg \perp = \top$

### 3.11 Stone algebras

**Definition 3.11.1.** A *Stone algebra*  $(A, \vee, \wedge, \perp, \top, \neg)$  is an Ockham algebra  $(A, \vee, \wedge, \perp, \top, \neg)$  satisfying:

- (i)  $\forall a \in A : \neg \neg a \vee \neg a = \top$
- (ii)  $\forall a \in A : a \wedge \neg a = \perp$

### 3.12 De Morgan algebras

**Definition 3.12.1.** A *De Morgan algebra*  $(A, \vee, \wedge, \perp, \top, \neg)$  is an Ockham algebra  $(A, \vee, \wedge, \perp, \top, \neg)$ , which is idempotent:

$$\forall a \in A : \neg \neg a = a$$

**Example 3.12.2.** The standard *fuzzy algebra*  $([0, 1], \max\{a, b\}, \min\{a, b\}, 0, 1, 1 - a)$ , used in the study of fuzzy logic, is a De Morgan algebra.



### 3.13 Kleene algebras

**Definition 3.13.1.** A *Kleene algebra*  $(A, \vee, \wedge, \perp, \top, \neg)$  is a De Morgan algebra, which in addition satisfies:

$$\forall a, b \in A : (a \wedge \neg a) \vee (b \vee \neg b) = b \vee \neg b$$

**Example 3.13.2.** Let  $\Sigma$  be a finite set of symbols, called an alphabet. Then the set of all formal languages  $\mathfrak{L}_\Sigma$  over  $\Sigma$  forms a Kleene algebra  $(\mathfrak{L}_\Sigma, +, \cdot, \emptyset, \{\varepsilon\}, *)$ , where  $A + B := A \cup B$ ,  $A \cdot B$  is concatenation and  $A^*$  is the *Kleene star* operation.

### 3.14 Boolean algebras

**Definition 3.14.1.** A *Boolean algebra*  $(A, \vee, \wedge, \perp, \top, \neg)$  is a complemented distributive lattice.

*Remark 3.14.2.* Boolean algebras complement is unique, cf. Theorem 3.5.2.

**Example 3.14.3.** In logic, the two-element Boolean algebra  $(\{0, 1\}, \vee, \wedge, 0, 1, \neg)$  is used where 0 denotes false, 1 denotes true,  $\vee$  means “or”,  $\wedge$  means “and” and  $\neg$  means “not”.

**Theorem 3.14.4.** *Every Boolean algebra is an ortholattice.*

*Proof.* Let  $(A, \vee, \wedge, \perp, \top, \neg)$  be a Boolean algebra. Since both Boolean algebras and ortholattices are complemented lattices, it has to be shown that ortholattices additional requirements for  $\neg$  hold.

(i):  $\neg\neg a = \neg\neg a \vee \perp = \neg\neg a \vee (a \wedge \neg a) = (\neg\neg a \vee a) \wedge (\neg\neg a \vee \neg a) = (\neg\neg a \vee a) \vee \top = (\neg\neg a \vee a) \wedge (a \vee \neg a) = a \vee (\neg a \wedge \neg\neg a) = a \vee \perp = a$ , where the identity, distributive, commutative and complement laws were used.

(ii): Since complements in Boolean algebras are unique cf. Theorem 3.5.2, it suffices to show that  $\neg a \wedge \neg b$  is the complement to  $a \vee b$ , which is equivalent to:

$$1.) (a \vee b) \vee (\neg a \wedge \neg b) = \top$$

$$2.) (a \vee b) \wedge (\neg a \wedge \neg b) = \perp$$

1.):  $(a \vee b) \vee (\neg a \wedge \neg b) = ((a \vee b) \vee \neg a) \wedge ((a \vee b) \vee \neg b) = (b \vee (a \vee \neg a)) \wedge (a \vee (b \vee \neg b)) = (b \vee \top) \wedge (a \vee \top) = \top \wedge \top = \top$ , where the distributive, associative, complement, identity and idempotent laws were used.

**2.):**  $(a \vee b) \wedge (\neg a \wedge \neg b) = ((\neg a \wedge \neg b) \wedge a) \vee ((\neg a \wedge \neg b) \wedge b) = (\neg b \wedge (a \vee \neg a)) \vee (\neg a \wedge (b \wedge \neg b)) = (\neg b \wedge \perp) \vee (\neg a \wedge \perp) = \perp \vee \perp = \perp$ , where the distributive, associative, complement, identity, idempotent and commutative laws were used. **(iii)** follows analogously from **(ii)**. ■

**Theorem 3.14.5.** *Every Boolean algebra is a Kleene algebra.*

*Proof.* Let  $\mathbf{A}$  be a Boolean algebra and  $a, b \in A$ . It has to be shown that:

$$(i) \quad \neg(a \vee b) = \neg a \wedge \neg b$$

$$(ii) \quad \neg(a \wedge b) = \neg a \vee \neg b$$

$$(iii) \quad \neg \top = \perp$$

$$(iv) \quad \neg \perp = \top$$

$$(v) \quad \neg \neg a = a$$

$$(vi) \quad (a \wedge \neg a) \vee (b \vee \neg b) = b \vee \neg b$$

**(i)**, **(ii)** and **(v)** follows from  $\mathbf{A}$  inducing an ortholattice cf. Theorem 3.14.4. **(iv)** will not be shown, since it follows analogously from **(iii)**. **(iii)**:  $\neg \top = \neg(a \vee \neg a) = \neg a \wedge \neg \neg a = a \wedge \neg a = \perp$ , where the complement law and Theorem 3.14.4 were used. **(vi)**:  $(a \wedge \neg a) \vee (b \vee \neg b) = \perp \vee \top = \top = b \vee \neg b$ , where the complement and identity laws were used. ■

**Theorem 3.14.6.** *Every Boolean algebra is a Stone algebra.*

*Proof.* Let  $\mathbf{A}$  be a Boolean algebra. By Theorem 3.14.5,  $\mathbf{A}$  is a Kleene algebra as well, and therefore also an Ockham algebra. It thus suffices to show that the Stone identities are satisfied. **(i)** follows from  $\mathbf{A}$  being an ortholattice by Theorem 3.14.4 and complement. **(ii)** follows directly from the complement. ■

**Theorem 3.14.7.** *Every Boolean algebra is a Heyting algebra.*

*Proof.* Let  $(A, \vee, \wedge, \perp, \top, \neg)$  be a Boolean algebra. Define the map  $\rightarrow: A \times A \rightarrow A$ , given by:

$$a \rightarrow b := \neg a \vee b$$

It has to be checked that the map  $\rightarrow$  satisfies the requirements given in Heyting algebras. Let  $a, b, c \in A$ .

“ $\Rightarrow$ ”: Assume  $a \wedge b \leq c$ . Then  $b \rightarrow c = \neg b \vee c \geq \neg b \vee (a \wedge b) = (\neg b \vee a) \wedge (\neg b \vee b) = (\neg b \vee a) \wedge \top = \neg b \vee a \geq a$ , where the assumption, distributive law, complement, identity

law and the dual lattice definition to conclude  $\neg b \vee a \geq a$ .

“ $\Leftarrow$ ”: Assume  $a \leq b \rightarrow c$ . Then  $a \wedge b \leq (b \rightarrow c) \wedge b = (\neg b \vee c) \wedge b = (\neg b \wedge b) \vee (b \vee c) = \perp \vee (b \vee c) = b \vee c \geq c$ , where the assumption, distributive law, complement, identity law and the dual lattice definition was used to conclude  $b \vee c \geq c$ . ■

## Chapter 4

### Ring structures

A ring structure is a structure defined on a set, with two binary operations.

#### 4.1 Semirings

**Definition 4.1.1.** A *semiring*  $(A, +, \cdot, e_1, e_2)$  is a set  $A \neq \emptyset$  together with two binary operations  $+$  and  $\cdot$ , such that:

- (i)  $(A, +, e_1)$  is a commutative monoid
- (ii)  $(A, \cdot, e_2)$  is a monoid
- (iii)  $\forall a, b, c \in A : a(b + c) = ab + ac$
- (iv)  $\forall a, b, c \in A : (a + b)c = ac + bc$
- (v)  $\forall a \in A : e_1a = ae_1 = e_1$

**Example 4.1.2.** The natural numbers including zero  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  under ordinary addition and multiplication form a semiring  $(\mathbb{N}_0, +, \cdot, 0, 1)$ .

#### 4.2 Near-rings

**Definition 4.2.1.** A *near-ring*  $(A, +, \cdot, e, -)$  is a set  $A \neq \emptyset$  together with two binary operations  $+$  and  $\cdot$  such that:

- (i)  $(A, +, e, -)$  is a group
- (ii)  $(A, \cdot)$  is a semigroup
- (iii)  $\forall a, b, c \in A : a(b + c) = ab + ac$
- (iv)  $\forall a, b, c \in A : (a + b)c = ac + bc$

**Example 4.2.2.** Near-rings occur naturally in studying maps on a group  $\mathbf{A}$  onto itself, where the set of all those maps is denoted  $M(A)$ . Such maps  $f : A \rightarrow A$  with the point-wise addition and composition form near-ring  $(M(A), +, \circ, 0, -)$ , which is not a ring.

### 4.3 Nonassociative rings

**Definition 4.3.1.** A *nonassociative ring*  $(A, +, \cdot, e, -)$  is a set  $A \neq \emptyset$  together with two binary operations  $+$  and  $\cdot$  such that:

- (i)  $(A, +, e, -)$  is an Abelian group
- (ii)  $(A, \cdot)$  is a magma
- (iii)  $\forall a, b, c \in A : a(b + c) = ab + ac$
- (iv)  $\forall a, b, c \in A : (a + b)c = ac + bc$

### 4.4 Rings

**Definition 4.4.1.** A *ring*  $(A, +, \cdot, e, -)$  is a set  $A \neq \emptyset$  together with two binary operations  $+$  and  $\cdot$  such that:

- $(R_1)$   $(A, +, e, -)$  is an Abelian group
- $(R_2)$   $(A, \cdot)$  is a semigroup
- $(R_3)$   $\forall a, b, c \in A : a(b + c) = ab + ac$
- $\forall a, b, c \in A : (a + b)c = ac + bc$

*Remark 4.4.2.* Every ring reduces to a near-ring, since they only differ on  $(A, +, e, -)$  being a group or an Abelian group.

*Remark 4.4.3.* Every ring reduces to a nonassociative ring, since they only differ on  $(A, \cdot)$  being associative.

### 4.5 Commutative rings

**Definition 4.5.1.** A ring  $(A, +, \cdot, e, -)$  is called a *commutative ring* if  $\cdot$  commutes:

$$\forall a, b \in A : ab = ba$$

## 4.6 Unitary rings

**Definition 4.6.1.** A ring  $(A, +, \cdot, e_1, e_2, -)$  is called a *unitary ring* if a multiplicative identity element exists:

$$\exists e_2 \in A \forall a \in A : e_2 a = a e_2 = a$$

*Remark 4.6.2.* Every unitary ring is a unitary module, since every ring is a module cf. Theorem 5.2.3.

## 4.7 Division rings

**Definition 4.7.1.** An element  $a$  in a unitary ring  $(A, +, \cdot, e_1, e_2, -)$  is said to be *invertible*, or to be a *unit*, if

$$\exists a^{-1} \in A : a^{-1} a = a a^{-1} = e_2$$

**Definition 4.7.2.** A *unitary ring*  $(A, +, \cdot, e_1, e_2, -, ^{-1})$  with  $e_1 \neq e_2$ , in which every element  $a \neq e_1$  is a unit, is called a *division ring*.

**Example 4.7.3.** The real quaternions  $\mathbb{H}$ , which extends the complex numbers  $\mathbb{C}$ , forms a noncommutative division ring  $(\mathbb{H}, +, \cdot, 0, 1, -)$ .

## 4.8 Integral domains

**Definition 4.8.1.** An element  $a \neq e$  in a ring  $(A, +, \cdot, e, -)$  is said to be a *zero divisor* if

$$\exists e \neq b \in A : ab = ba = e$$

**Definition 4.8.2.** An *integral domain* is a commutative unitary ring  $(A, +, \cdot, e_1, e_2, -)$  with  $e_1 \neq e_2$  and no zero divisors.

*Remark 4.8.3.* An equivalent definition of an integral domain  $(A, +, \cdot, e_1, e_2, -)$  is a commutative unitary ring, where the zero rule applies:

$$\forall a, b \in A : ab = e_1 \Rightarrow (a = e_1) \vee (b = e_1)$$

**Example 4.8.4.** The integers  $(\mathbb{Z}, +, \cdot, 0, 1, -)$  is an example of an integral domain.

## 4.9 Fields

**Definition 4.9.1.** A *field*  $(A, +, \cdot, e_1, e_2, -, {}^{-1})$  is a commutative division ring.

*Remark 4.9.2.* Every field is a vector space, since every ring is a module cf. Theorem 5.2.3.

**Example 4.9.3.** The real numbers  $(\mathbb{R}, +, \cdot, 0, 1, -, {}^{-1})$ , the rational numbers  $(\mathbb{Q}, +, \cdot, 0, 1, -, {}^{-1})$  and the complex numbers  $(\mathbb{C}, +, \cdot, 0, 1, -, {}^{-1})$  are examples of fields.

**Theorem 4.9.4.** *Every field is an integral domain.*

*Proof.* Let  $(A, +, \cdot, e_1, e_2, -, {}^{-1})$  be a field. It has to be checked that the zero rule applies. Let  $a, b \in A$  and assume  $ab = e_1$  and  $a \neq e_1$ :  $b = e_2b = (a^{-1}a)b = a^{-1}(ab) = a^{-1}e_1 = e_1$ . ■

## 4.10 Lie rings

**Definition 4.10.1.** A *Lie ring*  $(A, +, \cdot, e, -)$  is a nonassociative ring which satisfies:

- $\forall a, b, c \in A : a(bc) + b(ca) + c(ab) = e$
- $\forall a \in A : aa = e$

**Theorem 4.10.2.** *Every ring  $(A, +, \cdot, e, -)$  induces a Lie ring  $(A, +, \cdot, e, -, \odot)$ .*

*Proof.* Let  $\mathbf{A}$  be a ring and let  $a, b, c \in A$ . Then define  $\odot : A \times A \rightarrow A$ , given by  $a \odot b := ab - ba$ . It has to be shown that  $\odot$  satisfies the two properties of a Lie ring. The first is shown:

$$\begin{aligned}
 & a \odot (b \odot c) + b \odot (c \odot a) + c \odot (a \odot b) \\
 &= a \odot (bc - cb) + b \odot (ca - ac) + c \odot (ab - ba) \\
 &= a(bc - cb) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c \\
 &= abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\
 &= (abc - abc) + (acb - acb) + (bca - bca) + (cba - cba) + (bac - bac) + (cab - cab) \\
 &= e + e + e + e + e + e = e
 \end{aligned}$$

The second is shown:

$$a \odot a = aa - aa = e$$

Thus is  $(A, +, \cdot, e, -, \odot)$  a Lie ring. ■

## 4.11 Boolean ring

**Definition 4.11.1.** A *Boolean ring*  $(A, +, \cdot, e, -)$  is a ring, in which  $\cdot$  is idempotent:

$$\forall a \in A : aa = a$$

**Example 4.11.2.** Let  $A$  be a set and  $P(A)$  the powerset. Then  $(P(A), \Delta, \cap, \emptyset, Id)$  is a Boolean ring, where  $Id$  is the identity map (every element is its own inverse) and  $\Delta$  is the *symmetric difference*:

$$A\Delta B := (A \cup B) \setminus (A \cap B)$$

## 4.12 \*-rings

**Definition 4.12.1.** A *\*-ring*  $(A, +, \cdot, e, -, *)$  is a ring  $(A, +, \cdot, e, -)$  equipped with the unary operation  $*$ , which satisfies for all  $a, b \in A$ :

- $(a + b)^* = a^* + b^*$
- $(ab)^* = b^*a^*$
- $a^{**} = a$



## Chapter 5

### Module structures

A module structure is a structure defined on two sets with at least two binary operations.

#### 5.1 Groups with operators

**Definition 5.1.1.** A *group with operators*  $(A, \cdot, e, {}^{-1}, \Omega, f)$  is a group  $(A, \cdot, e, {}^{-1})$  along with a set  $\Omega$  and the map  $f : \Omega \times A \rightarrow A$ , given by  $f(\omega, a) = a^\omega$  and satisfies:

$$\forall a, b \in A, \omega \in \Omega : f(\omega, ab) = f(\omega, a)f(\omega, b)$$

*Remark 5.1.2.* Every group  $\mathbf{A}$  trivially induces a group with operators  $(\mathbf{A}, \emptyset, f)$ .

#### 5.2 Modules

**Definition 5.2.1.** Let  $(R, \oplus, \odot, e_r, \ominus)$  be a ring. An  $(\mathbf{R}\text{-})$ module  $(A, +, e_m, -, R, \oplus, \odot, e_r, \ominus, \star)$  is an Abelian group  $(A, +, e_m, -)$  together with a map  $\star : R \times A \rightarrow A$ , such that for all  $r, s \in R$  and  $a, b \in A$ :

- (i)  $r \star (a + b) = r \star a + r \star b$
- (ii)  $(r \oplus s) \star a = r \star a + s \star a$
- (iii)  $r \star (s \star a) = (r \odot s) \star a$

**Theorem 5.2.2.** *Every module induces a group with operators.*

*Proof.* Set  $\Omega = R$  and  $f = \star$ . Then it follows from  $\star$ 's first property. ■

**Theorem 5.2.3.** *Every ring  $(R, +, \cdot, e_1, -)$  induces a module  $(R, +, e_1, -, R, +, \cdot, e_1, -)$ .*

*Proof.* Let  $\mathbf{R} := (R, +, \cdot, e_1, -)$  be a ring. Since  $\mathbf{G} := (R, +, e_1, -)$  is an Abelian group, a module  $\mathbf{A}$  can be constructed from  $\mathbf{R}$  and  $\mathbf{G}$ . Thus can a module be created directly from an arbitrary ring. ■

### 5.3 Unitary modules

**Definition 5.3.1.** Let  $(R, +, \cdot, e_1, e_2, -)$  be a unitary ring. Then a module  $\mathbf{A}$  over  $\mathbf{R}$  is called a *unitary module*.

### 5.4 Vector spaces

**Definition 5.4.1.** Let  $(\mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1})$  be a field. Then a module  $\mathbf{A}$  over  $\mathbb{F}$  is called an  $(\mathbb{F}-)$ vector space, denoted  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star)$ .

## Chapter 6

### Metric structures

A metric structure is a structure defined on 2 sets with a single (unary/binary)-operation.

#### 6.1 Metric spaces

**Definition 6.1.1.** Let  $A \neq \emptyset$  be a set. A map  $d : A \times A \rightarrow \mathbb{R}$  is called a *metric*, if it satisfies the following for arbitrary  $a, b, c \in A$ :

$$(M_1) \quad d(a, b) \geq 0, \quad d(a, b) = 0 \Leftrightarrow a = b$$

$$(M_2) \quad d(a, b) = d(b, a)$$

$$(M_3) \quad d(a, b) \leq d(a, c) + d(c, b)$$

**Definition 6.1.2.** A *metric space*  $(A, d)$  is a set  $A \neq \emptyset$  equipped with a metric  $d$ .

#### 6.2 Complete spaces

**Definition 6.2.1.** Let  $\mathbf{A}$  be a metric space, and let  $(m_i)_{i \in \mathbb{N}} \subseteq A$  be a sequence in  $A$ . Then  $(m_i)_{i \in \mathbb{N}}$  is called a *Cauchy sequence* if:

$$\forall j, k \in \mathbb{N} : \lim_{j, k \rightarrow \infty} d(m_j, m_k) = 0$$

**Definition 6.2.2.** A *complete space*  $(A, d)$  is a metric space  $(A, d)$  in which all Cauchy sequences in  $A$  converge.

#### 6.3 Normed vector spaces

**Definition 6.3.1.** A *norm* on a vector space  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star)$  is a map  $\|\cdot\| : A \rightarrow \mathbb{R}$ , which satisfies the following:

$$(N_1) \quad \forall a \in A : \|a\| \geq e_{\mathbb{R}^n,1}, \|a\| = e_{\mathbb{R}^n,1} \Leftrightarrow a = e_A$$

$$(N_2) \quad \forall a \in A, \lambda \in \mathbb{R}^n : \|\lambda \star a\| = |\lambda| \odot \|a\|$$

$$(N_3) \quad \forall a, b \in A : \|a + b\| \leq \|a\| \oplus \|b\|$$

**Definition 6.3.2.** A *normed vector space*  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \|\cdot\|)$  is a vector space over a field  $\mathbb{F}$ , which is either  $\mathbb{R}$  or  $\mathbb{C}$ , equipped with the norm  $\|\cdot\|$ .

**Theorem 6.3.3.** Every *normed vector space*  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \|\cdot\|)$  induces a *metric space*  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \|\cdot\|, d)$ .

*Proof.* Let  $\mathbf{A} = (A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \|\cdot\|)$  be a normed vector space. Then define the map  $d : A \times A \rightarrow \mathbb{F}$  given by

$$d(a, b) := \|b - a\|$$

It has to be checked that  $d$  is a metric.  $(M_1)$  follows directly from  $(N_1)$ . Let  $\lambda = -1$ . Then by  $(N_2)$ , with  $a, b \in A$  and  $\lambda \in \mathbb{F}$ :

$$\|b - a\| = \|\lambda \star (a - b)\| = |\lambda| \odot \|a - b\| = \|a - b\|$$

which shows  $(M_2)$ . Lastly, by  $(N_3)$ , with  $a, b, c \in A$ :

$$d(a, b) = \|a - b\| = \|(a - c) + (c - b)\| \stackrel{(N_3)}{\leq} \|a - c\| + \|c - b\| = d(a, c) + d(c, b)$$

which then proves that  $d$  is a metric and  $\mathbf{A}$  a metric space. ■

## 6.4 Banach spaces

**Definition 6.4.1.** A *Banach space*  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \|\cdot\|)$  is a complete normed vector space.

## Chapter 7

### Algebra structures

An algebra structure is a structure defined on two sets: a ring  $\mathbf{R}$  and an  $\mathbf{R}$ -module  $\mathbf{A}$ .

#### 7.1 Nonassociative algebras

**Definition 7.1.1.** A *nonassociative (bR-)algebra*  $\mathbf{A} = (A, +, e_A, -, R, \oplus, \odot, e_R, \ominus, \star, [\cdot, \cdot])$  is an  $\mathbf{R}$ -module  $(A, +, e_A, -, R, \oplus, \odot, e_R, \ominus, \star)$  over a commutative ring  $\mathbf{R}$  equipped with a binary operation  $[\cdot, \cdot]$  on  $A$ , called **A-multiplication**, which satisfies:

- (i)  $\forall r, s \in R, a, b, c \in A : [r \star a + s \star b, c] = r \star [a, c] + s \star [b, c]$
- (ii)  $\forall r, s \in R, a, b, c \in A : [c, r \star a + s \star b] = r \star [c, a] + s \star [c, b]$

#### 7.2 Algebras

**Definition 7.2.1.** An *algebra*  $\mathbf{A}$  is a nonassociative algebra, in which its binary operation  $[\cdot, \cdot] : A \times A \rightarrow A$  is associative:

$$\forall a, b, c \in A : [[a, b], c] = [a, [b, c]]$$

#### 7.3 Banach algebras

**Definition 7.3.1.** A *Banach algebra*  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star, \|\cdot\|, [\cdot, \cdot])$  is an algebra over a Banach space  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star, \|\cdot\|)$ , which satisfies:

$$\forall a, b \in A : \|ab\| \leq \|a\| \|b\|$$

## 7.4 \*-algebras

**Definition 7.4.1.** A *\*-algebra*  $(A, +, e_A, -, R, \oplus, \odot, e_R, \ominus, \star, [\cdot, \cdot], *)$  is an algebra, in which the ring  $\mathbf{R}$  in addition is a \*-ring and  $R \subseteq A$ , as well as satisfying:

$$\forall r, s \in R, a, b \in A : (r \star a + s \star b)^* = r^* \star a^* + s^* \star b^*$$

## 7.5 C\*-algebras

**Definition 7.5.1.** A *C\*-algebra*  $(A, +, e_A, -, \mathbb{C}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star, \|\cdot\|, [\cdot, \cdot], *)$  is a complex Banach algebra and a \*-algebra, which satisfies:

$$\forall a \in A : \|a^* a\| = \|a\|^2$$

## 7.6 Lie algebras

**Definition 7.6.1.** A *Lie algebra*  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, \star, [\cdot, \cdot])$  is a nonassociative  $\mathbb{F}$ -algebra over a field  $\mathbb{F}$ , in which the binary operation  $[\cdot, \cdot]$  satisfies alternation and the Jacobi identity:

- (i)  $\forall a \in A : [a, a] = e_A$
- (ii)  $\forall a, b, c \in A : [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = e_A$

## 7.7 Inner product spaces

**Definition 7.7.1.** Let  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star)$ , be an  $\mathbb{F}$ -vector space. Then an *inner product* on  $A$  is a map  $\langle \cdot, \cdot \rangle : A \times A \rightarrow \mathbb{F}$  with satisfies the following for all  $a, b, c \in A$  and  $\alpha, \beta \in \mathbb{F}$ :

- (IP<sub>1</sub>)  $\langle a, a \rangle > e_{\mathbb{F},1} \Leftrightarrow a \neq e_{\mathbb{F},1}$
- (IP<sub>2</sub>)  $\langle a, b \rangle = \overline{\langle b, a \rangle}$
- (IP<sub>3</sub>)  $\langle \alpha \star a + \beta \star b, c \rangle = \alpha \odot \langle a, c \rangle + \beta \odot \langle b, c \rangle$

**Definition 7.7.2.** A vector space  $\mathbf{A}$  equipped with an inner product  $\langle \cdot, \cdot \rangle$  is called an *inner product space*, denoted by  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, {}^{-1}, \star, \langle \cdot, \cdot \rangle)$ .

**Theorem 7.7.3.** *Every inner product space  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle)$  induces a normed vector space  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle, ||\cdot||)$ .*

*Proof.* Let  $(A, +, e_A, -, \mathbb{F}, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle)$  be an inner product space. Define the map  $||\cdot|| : A \rightarrow [0, \infty)$ , given by  $||a|| := \sqrt{\langle a, a \rangle}$ . It has to be shown that  $||\cdot||$  is a norm. Because of  $(IP_1)$ ,  $\sqrt{\langle a, a \rangle}$  is well-defined.  $(N_1)$  follows from  $(IP_1)$ .  $(N_2)$  is shown:

$$||\alpha a||^2 \stackrel{\text{def}}{=} \langle \alpha \star a, \alpha \star a \rangle \stackrel{(IP_3), (IP_2)}{=} \alpha \odot \bar{\alpha} \langle a, a \rangle = |\alpha|^2 ||a||^2$$

The last  $(N_3)$  is shown:

$$\begin{aligned} ||a + b||^2 &= \langle a + b, a + b \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= ||a||^2 + 2\text{Re} \langle a, b \rangle + ||b||^2 \\ &\leq ||a||^2 + 2|\langle a, b \rangle| + ||b||^2 \\ &\stackrel{(*)}{\leq} ||a||^2 + 2||a|| ||b|| + ||b||^2 \\ &= (||a|| + ||b||)^2 \end{aligned}$$

At  $(*)$ , the Cauchy-Schwarz' inequality is used. By taking the square root of both sides,  $(N_3)$  is shown, and thus is  $(A, +, e_A, -, \mathbb{R}^n, \oplus, \odot, e_{\mathbb{F},1}, e_{\mathbb{F},2}, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle, ||\cdot||)$  a normed space. ■

## 7.8 Hilbert spaces

**Definition 7.8.1.** A *Hilbert space* is a complete inner product space.

**Corollary 7.8.2.** *Every Hilbert space induces a Banach space.*

*Proof.* It follows from the fact that every inner product space induces a normed space cf. Theorem 7.7.3. ■

## 7.9 Euclidean spaces

**Definition 7.9.1.** A *Euclidean space*  $(A, +, \underline{0}, -, \mathbb{R}^n, \oplus, \odot, \underline{0}, 1, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle)$  is a Hilbert space  $(A, +, e_A, -, \mathbb{R}^n, \oplus, \odot, e_{\mathbb{R}^n,1}, e_{\mathbb{R}^n,2}, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle)$  over the field  $\mathbb{R}$  which satisfies that the inner product is the dot product,  $e_A$  is the zero vector  $\underline{0}$ ,  $e_{\mathbb{R}^n,1} = \underline{0}$  and  $e_{\mathbb{R}^n,2} = 1$ . Furthermore,  $\star$  is often denoted with  $\cdot$ ,  $\oplus$  often denoted with  $+$  and  $\odot$  with  $\cdot$ , without confusion.

## Chapter 8

### Paving structures

A paving structure is a structure which is defined on a family of subsets of a set.

**Definition 8.0.2.** A *family of sets* indexed by the class  $I \neq \emptyset$  is a class of sets  $(A_i)_{i \in I} := \{A_i \mid i \in I\}$ . For such a family, its union and intersection are defined to be respectively the classes

$$\bigcup_{i \in I} A_i := \{a \mid \exists i \in I : a \in A_i\}$$

$$\bigcap_{i \in I} A_i := \{a \mid \forall i \in I : a \in A_i\}$$

#### 8.1 Topological spaces

**Definition 8.1.1.** A *topology* on a set  $A$  is a family of subsets of  $A$  denoted  $\mathcal{O}$ , which satisfies the following:

$$(T_1) \quad \emptyset, A \in \mathcal{O}.$$

$$(T_2) \quad \forall G_1, \dots, G_n \in \mathcal{O} : G_1 \cap \dots \cap G_n \in \mathcal{O}$$

$$(T_3) \quad (A_i)_{i \in I} \in \mathcal{O} \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{O}.$$

**Definition 8.1.2.** A *topological space*  $(A, \mathcal{O})$  is a set  $A$  equipped with a topology  $\mathcal{O}$ .

**Definition 8.1.3.** Let  $A$  be a set. Then a *basis* for a topology  $\mathcal{O}$  on  $A$  is a collection  $\mathcal{B}$  of subsets of  $A$  (called basis elements), such that:

$$(i) \quad \forall a \in A \exists B \in \mathcal{B} : a \in B$$

$$(ii) \quad \forall a \in A \forall B_1, B_2 \in \mathcal{B} : a \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B} : a \in B_3 \subseteq B_1 \cap B_2$$



**Lemma 8.1.4.** *Let  $(A, \mathcal{O})$  be a topological space and let  $\mathcal{B}$  be a basis for  $\mathcal{O}$ . Then  $\mathcal{O}$  equals the collection of all unions of elements of  $\mathcal{B}$ .*

*Proof.* Given a collection of elements of  $\mathcal{B}$ , they are also elements of  $\mathcal{O}$ . Because  $\mathcal{O}$  is a topology, their union is in  $\mathcal{O}$ . Conversely, given  $U \in \mathcal{O}$ , choose for each  $a \in U$  an element  $B_a \in \mathcal{B}$  such that  $a \in B_a \subseteq U$ . Then  $U = \bigcup_{a \in U} B_a$ , so  $U$  equals a union of elements of  $\mathcal{B}$ . ■

**Lemma 8.1.5.** *Let  $(A, \mathcal{O})$  be a topological space. If  $\mathcal{C} \subseteq \mathcal{O}$  satisfies:*

$$\forall S \in \mathcal{O}, a \in S \exists C \in \mathcal{C} : a \in C \subseteq S$$

*then  $\mathcal{C}$  is a basis for  $\mathcal{O}$ .*

*Proof.* We must show that  $\mathcal{C}$  is a basis. The first condition for a basis is shown: Given  $a \in A$ , since  $A \in \mathcal{O}$ , there is by assumption a  $C \in \mathcal{C}$  such that  $a \in C \subseteq A$ . To check the second condition, let  $a \in C_1 \cap C_2$ , where  $C_1, C_2 \in \mathcal{C}$ . Since  $C_1, C_2 \in \mathcal{O}$ , then  $C_1 \cap C_2 \in \mathcal{O}$ . Therefore, there exists by assumption an element  $C_3 \in \mathcal{C}$  such that  $a \in C_3 \subseteq C_1 \cap C_2$ .

We must now show that the topology  $\mathcal{O}'$  generated by  $\mathcal{C}$  equals  $\mathcal{O}$ . First, note that if  $U \in \mathcal{O}$  and if  $x \in U$ , then there is by assumption an element  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . It follows that  $U \in \mathcal{O}'$  by definition. Conversely, if  $W \in \mathcal{O}'$ , then  $W$  is a union of elements of  $\mathcal{C}$ , by Lemma 8.1.4. Since each element of  $\mathcal{C}$  belongs to  $\mathcal{O}$  and  $\mathcal{O}$  is a topology,  $W \in \mathcal{O}$ . ■

**Theorem 8.1.6.** *Every metric space  $(A, d)$  induces a topological space  $(A, d, \mathcal{O})$ .*

*Proof.* Let  $(A, d)$  be a metric space and denote  $B_d(a, \varepsilon) := \{b \in A \mid d(a, b) < \varepsilon\}$  the  $\varepsilon$ -ball centered on  $b \in A$ . Now define  $\mathcal{G}$  as the family of all  $\varepsilon$ -balls  $B_d(x, \varepsilon)$  for  $a \in A$  and  $\varepsilon > 0$ . It is clear that  $\mathcal{G}$  satisfies the condition given in Lemma 8.1.5, meaning that  $\mathcal{G}$  is a basis for a topological space. Then by Lemma 8.1.4, the collection of all unions of  $\mathcal{G}$  is a topology  $\mathcal{O}$  on  $A$ . Thus is  $(A, d, \mathcal{O})$  an induced topological space. ■

**Theorem 8.1.7.** *Every topological space  $(A, \mathcal{O})$  induces monoids  $(A, \mathcal{O}, \cup, \emptyset)$  and  $(A, \mathcal{O}, \cap, A)$ .*

*Proof.* Let  $(A, \mathcal{O})$  be a topological space. It has to be shown that the induced  $(A, \mathcal{O}, \cup)$  is a semigroup and has an identity element  $e$ , since the proof for inducing  $(A, \mathcal{O}, \cap)$  follows analogously. To show that it is a semigroup, it is firstly noted that it is a magma, since  $\cup$  is a binary operation by  $(T_3)$ . It has to be shown that  $\cup$  is associative:

$$\begin{aligned} a \in A \cup (B \cup C) &\Leftrightarrow a \in A \vee (a \in B \vee a \in C) \\ &\Leftrightarrow (a \in A \vee a \in B) \vee a \in C \\ &\Leftrightarrow a \in (A \cup B) \cup C \end{aligned}$$

Thus is  $(A, \mathcal{O}, \cup)$  a semigroup, and it has to be shown that it has an identity. Let  $S \in \mathcal{O}$  and consider the empty set  $\emptyset$ , which exists by  $(T_1)$ :

$$\begin{aligned}\emptyset \cup S &= \{a \in A \mid a \in S \vee a \in \emptyset\} \stackrel{1.2.3}{=} \{a \in A \mid a \in S\} = S \\ S \cup \emptyset &= \{a \in A \mid a \in \emptyset \vee a \in S\} \stackrel{1.2.3}{=} \{a \in A \mid a \in S\} = S\end{aligned}$$

Thus is  $e = \emptyset \in \mathcal{O}$ , making  $(A, \mathcal{O}, \cup, \emptyset)$  a monoid. Similarly it can be shown that  $(A, \mathcal{O}, \cap, A)$  is a monoid as well by  $(T_1)$  and  $(T_2)$ . ■

**Theorem 8.1.8.** *Every topological space  $(A, \mathcal{O})$  induces a complete JID lattice  $(A, \mathcal{O}, \cup, \cap, \emptyset, A)$ .*

*Proof.* Let  $(A, \mathcal{O})$  be a topological space. It was shown in Theorem 8.1.7 that both  $(A, \mathcal{O}, \cup, \emptyset)$  and  $(A, \mathcal{O}, \cap, A)$  were monoids. Since  $\cup$  and  $\cap$  are well known to be both idempotent and commutative, it has to be shown that  $(A, \mathcal{O}, \cup, \cap, \emptyset, A)$  satisfies the absorption laws, distributive laws and completeness. The absorption laws and distributive laws follows analogously from the proof in Theorem 8.12.5, so it has to be shown that every set  $S_i \subseteq A$  has both a supremum  $\bigvee S_i := \bigcup S_i$  and an infimum  $\bigwedge S_i := \bigcap S_i$ .

By  $(T_2)$ ,  $A$  is closed under arbitrary unions, meaning  $\bigvee S_i$  exists. Since topologies aren't closed under arbitrary intersections, the infimum can instead be constructed by taking the interior of arbitrary intersections. Since interiors are always open, the infimum exists. Moreover, the join-infinite distribution law is shown. Let  $a \in A, S_0 \in \mathcal{O}, (S_i)_{i \in I} \subseteq \mathcal{O}$ . Then:

$$\begin{aligned}a \in S_0 \cap \bigcup_{i \in I} S_i &\Leftrightarrow (a \in S_0) \wedge (a \in \bigcup_{i \in I} S_i) \Leftrightarrow (a \in S_0) \wedge (\exists i \in I : a \in S_i) \\ &\Leftrightarrow \exists i \in I : (a \in S_i) \wedge (a \in S_0) \Leftrightarrow a \in \bigcup_{i \in I} (S_i \cap S_0)\end{aligned}$$

■

## 8.2 First-countable spaces

**Definition 8.2.1.** Let  $(A, \mathcal{O})$  be a topological space and let  $a \in A$ . Then a *neighbourhood* of  $a$  is a subset  $S \subseteq A$  which satisfies that:

$$\exists O \in \mathcal{O} : a \in S \subseteq O$$

**Definition 8.2.2.** A *first-countable space*  $(A, \mathcal{O})$  is a topological space  $(A, \mathcal{O})$  with a countable neighbourhood basis - that is, for every  $a \in A$  there exists a sequence of open neighbourhoods  $(S_i)_{i \in \mathbb{N}}$  of  $a$ , such that:

$$\forall U \in \mathcal{O} \exists i \in \mathbb{N} : a \in U \Rightarrow a \in S_i \subseteq U$$

**Theorem 8.2.3.** *Every metric space  $(A, d)$  induces a first-countable space  $(A, d, \mathcal{O})$ .*

*Proof.* Let  $(A, d)$  be a metric space and  $a \in A$ . Since every metric space induces a topological space by Theorem 8.1.6, it has to be shown that  $\mathbf{A}$  has a countable local basis. Define such a candidate  $\mathcal{B} := \{B_{1/n}(a) \mid n \in \mathbb{N}\}$ , where  $B_\varepsilon(a)$  denotes the open  $\varepsilon$ -ball in  $A$  with center in  $a$  and radius  $\varepsilon$ .

First of all, since a surjection  $f : \mathbb{N} \rightarrow \mathcal{B}$  can be created,  $\mathcal{B}$  is countable. It is clear that every element  $b \in \mathcal{B}$  is an open neighbourhood of  $a$ , since the balls are open with  $a$  as center. Now let  $U \subseteq A$  be an open neighbourhood of  $a$ . By definition of open sets, an open ball  $B_\varepsilon(a) \subseteq U$ . By the Archimedean principle, there exist an  $n \in \mathbb{N}$  such that  $n > \frac{1}{\varepsilon}$  which implies  $\varepsilon > \frac{1}{n}$ , meaning  $B_{1/n}(a) \subseteq B_\varepsilon(a) \subseteq U$ . This proves that  $\mathcal{B}$  is a countable local basis for  $A$ . ■

### 8.3 Separable spaces

**Definition 8.3.1.** A *separable space*  $(A, \mathcal{O})$  is a topological space  $(A, \mathcal{O})$  in which there exist a countable dense subset - that is, a sequence  $(a_n)_{n \in \mathbb{N}} \subseteq A$ , which satisfies:

$$\forall U \in \mathcal{O} \exists i \in \mathbb{N} : a_i \in U$$

### 8.4 Second-countable spaces

**Definition 8.4.1.** A *second-countable space*  $(A, \mathcal{O}, \mathcal{B})$  is a topological space  $(A, \mathcal{O})$  with a countable collection of sets  $\mathcal{B} = (B_i)_{i \in \mathbb{N}} \subseteq \mathcal{O}$ , called a countable basis, such that:

$$\forall V \in \mathcal{O} \exists (B_k)_{k \in \mathbb{N}} \subseteq \mathcal{B} : V = \bigcup_{k \in \mathbb{N}} B_k$$

**Theorem 8.4.2.** *Every second-countable space reduces to both a first-countable space and a separable space.*

*Proof.* Let  $(A, \mathcal{O}, \mathcal{B})$  be a second-countable space. For each  $a \in A$ , the set  $\{U \in \mathcal{B} \mid a \in U\}$  is a countable neighbourhood basis. Therefore it is a first-countable space. Let  $S$  consist of one point from each member of  $\mathcal{B}$ . Then  $S$  is a countable dense subset of  $A$ . Therefore it is a separable space. ■

**Theorem 8.4.3.** *Every separable metric space  $(A, d, \mathcal{O})$  induces a second-countable space  $(A, d, \mathcal{O}, \mathcal{B})$ .*

*Proof.* Let  $(A, d)$  be a metric space with a countable dense subset  $S$ . We will show that the set  $\mathcal{B}$  of open balls with rational radii around points in  $S$  form a countable basis for  $A$ . Consider the basic open set  $B_r(a)$ , for some  $a \in A$  and  $r > 0$ . Let  $b \in B_r(a)$ . There

is an  $s > 0$  such that  $B_s(b) \subseteq B_r(x)$ . Let  $c \in S \cap B_{s/3}(b)$  and let  $t \in (s/3, 2s/3) \cap \mathbb{Q}$ . Then  $b \in B_t(c)$ , and it follows from the triangle inequality that  $B_t(c) \subseteq B_s(b) \subseteq B_r(a)$ . So we have shown that every point in  $B_r(a)$  is inside a set in  $\mathcal{B}$  that is contained in  $B_r(a)$ . Therefore  $B_r(a)$  is a union of members of  $\mathcal{B}$ , and therefore  $\mathcal{B}$  is a basis. Since  $\mathcal{B}$  is in one-to-one correspondance with the product  $S \times ((0, \infty) \cap \mathbb{Q})$  of countable sets,  $\mathcal{B}$  is countable. ■

**Theorem 8.4.4.** *Every Euclidean space  $(A, +, \underline{0}, -, \mathbb{R}^n, \oplus, \odot, 0, 1, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle)$  induces a second-countable space  $(A, +, \underline{0}, -, \mathbb{R}^n, \oplus, \odot, 0, 1, \ominus, ^{-1}, \star, \langle \cdot, \cdot \rangle, \mathcal{O})$ .*

*Proof.* Let  $\mathbf{A}$  be a Euclidean space. By definition,  $\mathbf{A}$  is a Hilbert space and an inner product space. Then by Theorem 7.7.3,  $\mathbf{A}$  induces a normed space. Then by Theorem 6.3.3,  $\mathbf{A}$  induces a metric space. By Theorem 8.4.3, it thus suffices to show that  $\mathbf{A}$  is separable, ie. that it has a countable dense subset. Consider  $\mathbb{Q}^n$ . It is known that  $\mathbb{Q}^n$  is countable for all  $n \in \mathbb{N}$ . Furthermore, it is clear that  $\forall n \in \mathbb{N} : \mathbb{Q}^n \subseteq \mathbb{R}^n$ . Lastly, it is also known that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , making  $\mathbb{Q}^n$  dense in  $\mathbb{R}^n$  by applying it to each coordinate. Thus by Theorem 8.4.3,  $\mathbf{A}$  is a second-countable space. ■

## 8.5 Hausdorff spaces

**Definition 8.5.1.** A *Hausdorff space*  $(A, \mathcal{O})$  is a topological space  $(A, \mathcal{O})$ , which satisfies that there for each  $a, b \in A$  exist disjoint neighbourhoods  $U, V$  around  $a$  and  $b$ , respectively.

**Theorem 8.5.2.** *Every metric space  $(A, d)$  induces a Hausdorff space  $(A, d, \mathcal{O})$ .*

*Proof.* Let  $(A, d)$  be a metric space and  $a, b \in A$  with  $a \neq b$ . Then by  $(M_1)$ ,  $d(a, b) \neq 0$ . By Theorem 8.1.6,  $(A, d, \mathcal{O})$  is a topological space where  $\mathcal{O}$  is the topology generated from the open  $\varepsilon$ -balls. Construct two open  $\varepsilon$ -balls  $B_a := B_d(a, d(a, b)/2)$  and  $B_b := B_d(b, d(a, b)/2)$ , which contain  $a$  and  $b$ , respectively. If it is shown that  $B_a$  and  $B_b$  are disjoint, then we are done. Assume  $c \in B_a$  and  $c \in B_b$ . Then  $d(c, a) < d(a, b)/2$  and  $d(c, b) < d(a, b)/2$ . But then  $d(c, a) + d(c, b) < d(a, b)$ , which contradicts  $(M_3)$ . Thus is the induced topological space  $(A, d, \mathcal{O})$  a Hausdorff space. ■

## 8.6 Manifolds

**Definition 8.6.1.** Let  $\mathbf{A} = (A, \mathcal{O}_A)$  and  $\mathbf{B} = (B, \mathcal{O}_B)$  be topological spaces. A map  $f : A \rightarrow B$  is then called *continuous* if it for every  $S_B \in \mathcal{O}_B$  holds that  $f^{-1}(S_B) \in \mathcal{O}_A$ .

**Definition 8.6.2.** A *homeomorphism*  $f : \mathbf{A} \rightarrow \mathbf{B}$  is a continuous map between topological spaces  $\mathbf{A}$  and  $\mathbf{B}$ , that has a continuous inverse map. If such a map exists,  $\mathbf{A}$  and  $\mathbf{B}$  are said to be homeomorphic.

**Definition 8.6.3.** A *local homeomorphism*  $f : \mathbf{A} \rightarrow \mathbf{B}$  is a map between topological spaces  $\mathbf{A} = (A, \mathcal{O}_A)$  and  $\mathbf{B} = (B, \mathcal{O}_B)$ , which satisfies that for each  $a \in A$  exist a neighbourhood  $S \in \mathcal{O}_A$ , which satisfies  $f(S) \in \mathcal{O}_B$  and that  $f|_S : S \rightarrow f(S)$  is a homeomorphism. If such a map exists,  $\mathbf{A}$  and  $\mathbf{B}$  are said to be locally homeomorphic.

**Definition 8.6.4.** An  $n$ -dimensional *manifold*  $(A, \mathcal{O}, \mathbb{R}^n)$  is a second-countable Hausdorff space that is locally homeomorphic to a Euclidean space  $\mathbb{R}^n$ .

## 8.7 Disconnected spaces

**Definition 8.7.1.** A *disconnected space*  $(A, \mathcal{O})$  is a topological space  $(A, \mathcal{O})$ , in which there exist a collection of disjoint subspaces  $(A_i)_{i \in I}$ , which satisfies:

$$A = \bigcup_{i \in I} A_i$$

## 8.8 Totally disconnected spaces

**Definition 8.8.1.** A *totally disconnected space*  $(A, \mathcal{O})$  is a disconnected space, in which the connected subspaces are one-point sets.

## 8.9 Compact spaces

**Definition 8.9.1.** A *compact space*  $(A, \mathcal{O})$  is a topological space  $(A, \mathcal{O})$ , in which each of its open covers has a finite subcover. E.g., for every collection  $(U_i)_{i \in I} \subseteq \mathcal{O}$  with  $A = \bigcup_{i \in I} U_i$  exists a finite set  $J \subseteq I$ , such that:

$$A = \bigcup_{i \in J} U_i$$

## 8.10 Boolean spaces

**Definition 8.10.1.** A *Boolean space*  $(A, \mathcal{O})$  is a totally disconnected compact Hausdorff space.

## 8.11 Uniform spaces

**Definition 8.11.1.** A *uniformity*  $\phi$  on a set  $A$  is a nonempty family of subsets of  $A \times A$ , that satisfies the following:

$$(U_1) \quad \forall U \in \phi : \{(a, a) \mid a \in A\} \in U$$

$$(U_2) \quad \forall U \in \phi, V \subseteq A \times A : U \subseteq V \Rightarrow V \in \phi$$

$$(U_3) \quad \forall U, V \in \phi : U \cap V \in \phi$$

$$(U_4) \quad \forall U \in \phi \exists V \in \phi : (a, b), (b, c) \in V \Rightarrow (a, c) \in U$$

$$(U_5) \quad \forall U \in \phi : \{(b, a) \mid (a, b) \in U\} \in \phi$$

**Definition 8.11.2.** A *uniform space*  $(A, \phi)$  is a set  $A$  equipped with a uniformity  $\phi$ .

**Theorem 8.11.3.** Every uniform space  $(A, \phi)$  induces a topological space  $(A, \phi, \mathcal{O})$ .

*Proof.* Let  $(A, \phi)$  be a uniform space. Define a subset  $S \subseteq A$  to be open if and only if  $\forall a \in S \exists U \in \phi : \{b \in A \mid (a, b) \in U\} \subseteq S$ . Define  $\mathcal{O} := \{S \subseteq A \mid S \text{ open}\}$ . It has to be shown that  $\mathcal{O}$  is a topology.

(**T<sub>1</sub>**): It holds that  $\emptyset \in \mathcal{O}$  trivially, since it doesn't have any elements.  $A$  is open as well, by picking  $U := \{(a, a) \mid a \in A\}$ , which is an element in  $\phi$  by  $(U_1)$ .

(**T<sub>2</sub>**): Let  $G_1, \dots, G_n \in \mathcal{O}$ . It has to be shown that  $G_1 \cap \dots \cap G_n \in \mathcal{O}$ . By definition there exist  $U_1, \dots, U_n \in \phi$  such that  $\forall 1 \leq i \leq n \forall a \in G_i : \{b \in A \mid (a, b) \in U_i\} \subseteq G_i$ . It has to be shown that there exist  $U_0 \in \phi$  such that  $\forall a \in G_1 \cap \dots \cap G_n : \{b \in A \mid (a, b) \in U_0\} \subseteq G_1 \cap \dots \cap G_n$ . Define  $U_x := U_1 \cap \dots \cap U_n$ . It thus has to be shown that  $U_x = U_0$ . Let  $a_0 \in G_1 \cap \dots \cap G_n$ . Then by definition  $a_0 \in G_1 \wedge \dots \wedge a_0 \in G_n$ , meaning that  $\forall 1 \leq i \leq n : \{b \in A \mid (a_0, b) \in U_i\} \subseteq G_i$ . But this means that  $\{b \in A \mid (a_0, b) \in U_x\} \subseteq G_1 \cap \dots \cap G_n$ . Since the choice of  $a_0$  was arbitrary, it means that  $U_x = U_0$ .

(**T<sub>3</sub>**): Let  $(A_i)_{i \in \mathbb{N}} \in \mathcal{O}$ . It has to be shown that  $A_0 := \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{O}$ . By definition, there exist  $(U_i)_{i \in \mathbb{N}} \in \phi$  such that  $\forall i \in \mathbb{N} \forall a \in A_i : \{b \in A \mid (a, b) \in U_i\} \subseteq A_i$ . It has to be shown that there exist  $U_0$  such that  $\forall a \in A_0 : \{b \in A \mid (a, b) \in U_0\} \subseteq A_0$ . Define  $U_x := \bigcup_{i \in \mathbb{N}} U_i$ . It has to be shown that  $U_x = U_0$ . Let  $a_0 \in A_0$ . Then  $\exists i \in \mathbb{N} : a_0 \in A_i$  and thus  $\{b \in A \mid (a_0, b) \in U_i\} \subseteq A_i \subseteq A_0$ . Since this holds for all  $i \in \mathbb{N}$ , it is seen that  $\{b \in A \mid (a, b) \in U_x\} \subseteq A_0$ . Since the choice of  $a_0$  was arbitrary, it means that  $U_x = U_0$ . ■

**Theorem 8.11.4.** Every metric space  $(A, d)$  induces a uniform space  $(A, d, \phi)$ .

*Proof.* Let  $(A, d)$  be a metric space. Define  $\delta_\varepsilon := \{(a, b) \in A \times A \mid d(a, b) < \varepsilon\}$  and  $\phi := \{\delta_\varepsilon \mid \varepsilon > 0\}$ . It has to be shown that  $\phi$  is a uniformity on  $A$ .

(U<sub>1</sub>): Since  $d(a, a) = 0$  by  $(M_1)$  and since  $\varepsilon > 0$ ,  $(a, a)$  will be included in every  $d_\varepsilon$  for all  $a \in A$ .

(U<sub>2</sub>): Let  $V \subseteq A \times A$ ,  $U \in \phi$  and  $U \subseteq V$ . It has to be shown that  $V \in \phi$ . Since  $U \in \phi$ , there exist an  $\varepsilon > 0$  such that  $U = \delta_\varepsilon$ . Since  $U \subseteq V$  then  $\delta_\varepsilon \subseteq V$ . If  $\max\{d(a, b) \mid (a, b) \in V\} < \varepsilon$ , then trivially  $V \in \phi$ . If on the other hand  $\max\{d(a, b) \mid (a, b) \in V\} \geq \varepsilon$ , then set  $\varepsilon_0 := \max\{d(a, b) \mid (a, b) \in V\}$ , meaning  $V = \delta_{\varepsilon_0} \Rightarrow V \in \phi$ .

(U<sub>3</sub>): Let  $U, V \in \phi$ . Then there exist  $\varepsilon_1 > 0$  and  $\varepsilon_2 > 0$  such that  $U = \delta_{\varepsilon_1}$  and  $V = \delta_{\varepsilon_2}$ . Then it is clear that  $U \cap V = \delta_{\varepsilon_1} \cap \delta_{\varepsilon_2} := \min\{\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\} \in \phi$ .

(U<sub>4</sub>): Let  $U \in \phi$ . Then there exist an  $\varepsilon > 0$  such that  $U = \delta_\varepsilon$ . Define  $V := \delta_{\varepsilon/2}$  and let  $(a, b), (b, c) \in V$ . Then  $d(a, b), d(b, c) < \varepsilon/2$  by definition, and then  $d(a, c) \leq d(a, b) + d(b, c) < \varepsilon/2 + \varepsilon/2 = \varepsilon$  by  $(M_3)$ , meaning  $d(a, c) < \varepsilon$  and then  $(a, c) \in U$ .

(U<sub>5</sub>) follows from  $(M_2)$ . ■

## 8.12 Measurable spaces

**Definition 8.12.1.** A  $\sigma$ -algebra  $\mathcal{A}$  on a set  $A$  is a family of subsets of  $A$  with the following properties:

$$(\Sigma_1) \quad A \in \mathcal{A}$$

$$(\Sigma_2) \quad S \in \mathcal{A} \Rightarrow S^c \in \mathcal{A}$$

$$(\Sigma_3) \quad (S_i)_{i \in \mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} S_i \in \mathcal{A}$$

**Definition 8.12.2.** A *measurable space*  $(A, \mathcal{A})$  is a set  $A$ , equipped with a  $\sigma$ -algebra  $\mathcal{A}$ .

*Remark 8.12.3.* Every measurable space doesn't induce a topological space, and vice versa.

*Proof.* That every measurable space can't induce a topological space is due to the fact that topologies requires closure of arbitrarily many unions  $(T_3)$  and  $\sigma$ -algebras only required closure of countably many unions  $(\Sigma_3)$ . The reverse is true since  $\sigma$ -algebras require closure under complements  $(\Sigma_2)$ , and this is only always true for  $X$  and  $\emptyset$  in topologies  $(T_1)$ . ■

**Theorem 8.12.4.** Every measurable space  $(A, \mathcal{A})$  induces monoids  $(A, \mathcal{A}, \cup, \emptyset)$  and  $(A, \mathcal{A}, \cap, A)$ .

*Proof.* The proof follows analogously from Theorem 8.1.7, since  $A \in \mathcal{A}$  by  $(\Sigma_1)$ ,  $\emptyset \in \mathcal{A}$  by  $(\Sigma_1)$  and  $(\Sigma_2)$ ,  $\cup$  is closed by  $(\Sigma_3)$  and  $\cap$  is closed by  $(\Sigma_2)$  and  $(\Sigma_3)$ , since  $a \cap b = (a^c \cup b^c)^c$ . ■

**Theorem 8.12.5.** *Every measurable space  $(A, \mathcal{A})$  induces a Boolean algebra  $(A, \mathcal{A}, \cup, \cap, \emptyset, A, {}^c)$ .*

*Proof.* Let  $(A, \mathcal{A})$  be a measurable space. Since  $\mathbf{A}$  is a monoid by Theorem 8.12.4, as well as it is well known that both  $\cup$  and  $\cap$  are idempotent and commutative, both  $(A, \mathcal{A}, \cup, \emptyset)$  and  $(A, \mathcal{A}, \cap, A)$  are bounded semilattices. It has to be shown that  $(A, \mathcal{A}, \cup, \cap, \emptyset, A)$  satisfies the absorption laws, distributive laws and that each element has a complement - ie. that for all  $a, b, c \in \mathcal{A}$ :

$$(i) \quad a \cup (a \cap b) = a$$

$$(ii) \quad a \cap (a \cup b) = a$$

$$(iii) \quad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

$$(iv) \quad a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

$$(v) \quad \forall a \in \mathcal{A} \exists a^c \in \mathcal{A} : (a \cup a^c = A) \wedge (a \cap a^c = \emptyset)$$

(i): Since  $a \cap b \subseteq a$ , it follows trivially. (ii): Since  $a \subseteq a \cup b$ , it follows trivially. (iii): Let  $x \in a \cup (b \cap c)$ . Then  $(x \in a) \vee (x \in b \cap c) \Leftrightarrow (x \in a) \vee ((x \in b) \wedge (x \in c)) \Leftrightarrow ((x \in a) \vee (x \in b)) \wedge ((x \in a) \vee (x \in c)) \Leftrightarrow (x \in a \cup b) \wedge (x \in a \cup c) \Leftrightarrow x \in (a \cup b) \cap (a \cup c)$ . (iv) follows analogously as (iii). (v) follows from  $(\Sigma_2)$ . ■

### 8.13 Measure spaces

**Definition 8.13.1.** Let  $\mathcal{A}$  be a  $\sigma$ -algebra defined on the set  $A$ . Then a *measure* is a map  $\mu : \mathcal{A} \rightarrow [0, \infty]$ , satisfying:

$$(\mu_1) \quad \mu(\emptyset) = 0$$

$$(\mu_2) \quad \text{Given a family of pairwise disjoint sets } (S_i)_{i \in \mathbb{N}} \subseteq \mathcal{A}, \text{ then}$$

$$\mu \left( \bigcup_{i \in \mathbb{N}} S_i \right) = \sum_{i \in \mathbb{N}} \mu(S_i)$$

**Definition 8.13.2.** A  $\sigma$ -algebra  $\mathcal{A}$ , defined on a set  $A$  and equipped with a measure  $\mu$  is called a *measure space*, denoted  $(A, \mathcal{A}, \mu)$ .