Remarkables are strongly unfoldable

DAN SAATTRUP NIELSEN

DEFINITION 0.1. A cardinal κ is **remarkable** if for all regular cardinals $\theta > \kappa$ there exists countable transitive M and N with elementary embeddings $\pi : M \to H_\theta$ and $j : M \to N$ such that, letting $\bar{\kappa} := \pi^{-1}(\kappa)$,

(i) $\operatorname{crit} j = \bar{\kappa}$;
(ii) $o(M)$ is a regular cardinal in N;
(iii) $j(\bar{\kappa}) > o(M)$;
(iv) $M = H^N_{o(M)}$.

$\dashv$

DEFINITION 0.2. A cardinal κ is **strongly unfoldable** if every $\lambda > \kappa$ and $A \subseteq \kappa$ there exists a transitive κ-model M with $A \in M$ and an elementary embedding $j : M \to N$ with N transitive, and such that $\operatorname{crit} j = \kappa$, $j(\kappa) \geqslant \lambda$ and $V_\lambda \subseteq N$. $\dashv$

THEOREM 0.3. *Every remarkable cardinal is strongly unfoldable.*

PROOF. Let κ be remarkable, $\theta \gg \kappa$ and let $j : M \to N$ and $\pi : M \to H_\theta$ witness the remarkability of $\kappa \in \operatorname{ran} \pi$. It suffices to show that $\bar{\kappa} := \pi^{-1}(\kappa)$ is strongly unfoldable in M. Let $A \subseteq \bar{\kappa}$ be fixed, where $A \in M$. Let $\alpha \in (\bar{\kappa}, \bar{\kappa}^{+M})$ be such that $M|\alpha \models \mathsf{ZFC}^-$, $A \in M|\alpha$ and

$$M \models (M|\alpha)^{<\kappa} \subseteq M|\alpha.$$

Let $\lambda < o(M)$ and note that $j \restriction (M|\alpha) \in H^N_{\bar{\kappa}^{+M}} \subseteq M$, using that we could've picked θ such that $\bar{\kappa}$ is *not* the largest cardinal in M. Let $E \in M$ be the $(\bar{\kappa}, \lambda)$-extender derived from $j \restriction (M|\alpha)$ over M and let $k : \operatorname{Ult}(M|\alpha, E) \to N$ be the factor map. Then, in M, i_E and $M|\alpha$ witness unfoldability of $\bar{\kappa}$, and as $k \restriction \lambda = \operatorname{id}$,

$$V^M_\lambda = V^N_\lambda \subseteq \operatorname{Ult}(M|\alpha, E),$$

showing that $\bar{\kappa}$ is strongly unfoldable in M, which is what we wanted to show. ■