Some facts about Jónsson cardinals

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1 A plethora of characterisations

Theorem 1.1. *Let κ be a cardinal. Then the following are equivalent:*

(i) *(Jónsson '62) Every algebra of size κ has a proper subalgebra of the same size;*

(ii) *(Keisler-Rowbottom '65) Every transitive structure $\mathcal{M}$ of size κ in a countable language has a proper elementary substructure $X \prec \mathcal{M}$ of the same size;*

(iii) *(Erdős-Hajnal '66) $\kappa \to [\kappa]^{<\omega}_{\kappa}$; i.e. for every $c : [\kappa]^{<\omega} \to \kappa$ there exists $H \in [\kappa]^{\kappa}$ such that $c``[H]^n \neq \kappa$ for every $n < \omega$.*

(iv) *(Tryba '84) There exists $\theta > \kappa$ and an elementary embedding $j : M \to H_\theta$ such that $j(\kappa) = \kappa$ and $\operatorname{crit} j < \kappa$;*

(v) *(Tryba '84) For every limit cardinal $\lambda \leqslant \kappa$, $\delta < \lambda$, $\theta > \kappa$ and $A \subseteq \kappa$ there exists an elementary embedding $j : M \to H_\theta$ such that $j(\kappa) = \kappa$, $\operatorname{crit} j \in (\delta, \lambda)$ and $A \in \operatorname{ran} j$;*

(vi) *For every $A \subseteq \kappa$ there exists a proper $\langle X, \in, A \cap X\rangle \prec \langle V_\kappa, \in, A\rangle$ of size κ;*

(vii) *For every $A \subseteq \kappa$ there exists a proper $\langle X, \in, A \cap X\rangle \prec \langle L_\kappa[A], \in, A\rangle$ of size κ;*

Proof. $\blacksquare$ Missing

Definition 1.2. A cardinal κ is a **Jónsson cardinal** if it satisfies any of the equivalent properties in Theorem 1.1. $\dashv$

Theorem 1.3 (Rowbottom-Devlin '73). *The least Jónsson cardinal is either singular of countable cofinality or weakly inaccessible.* $\dashv$

2 Singular Jónssons

Theorem 2.1 (Přikrý '70). *Every singular limit of measurables is Jónsson.* $\dashv$

Theorem 2.2 (Přikrý '70). *Adding a Přikrý sequence to a measurable cardinal makes it singular Jónsson of countably cofinality in the generic extension.* $\dashv$

Theorem 2.3 (Mitchell '79). *Assuming there is no inner model with a measurable cardinal, every Jónsson cardinal is Ramsey in K. Consequently, if there exists a singular Jónsson cardinal then there exists an inner model with a measurable cardinal.* $\dashv$

Corollary 2.4. *The following theories are equiconsistent:*

- *There exists a singular Jónsson cardinal of cofinality ω;*
- *There exists a measurable cardinal.*

$\dashv$

Theorem 2.5 (Tryba '84). *If κ is a singular Jónsson cardinal of uncountable cofinality then there is a club $C \subseteq \kappa$ consisting of Jónsson cardinals.* $\dashv$

Theorem 2.6 (Koepke '88). *The following theories are equiconsistent:*

- *There exists a singular Jónsson of uncountable cofinality δ;*
- *There exist δ measurable cardinals.*

$\dashv$

Definition 2.7. Say a cardinal κ is an **small Jónsson cardinal** if it's a singular Jónsson cardinal satisfying $\kappa < \aleph_\kappa$. $\dashv$

Theorem 2.8 (Koepke '88). *If there exists a small Jónsson cardinal then 0^{long} exists. In particular, there exists an inner model with α measurables for every $\alpha \in \mathrm{On}$.* $\dashv$

Theorem 2.9 (Todorčević '85). *If $\aleph_\omega$ is Jónsson then $\Box_{\aleph_n}$ fails for some $n < \omega$.* $\dashv$

3 Regular Jónssons

3.1 Stationary reflection

Definition 3.2. $\mathrm{Refl}(\lambda, \kappa)$ is the statement that given any $\leqslant \lambda$-sized collection

$$\{S_\alpha \subseteq \kappa \mid \alpha \leqslant \lambda\}$$

of stationary sets of κ, there exists $\gamma < \kappa$ such that $S_\alpha \cap \gamma$ is stationary in γ for every $\alpha < \kappa$. Such a γ is called a **reflection point**. $\dashv$

Definition 3.3. Say a cardinal κ is α-**reflecting** if $\mathrm{Refl}(\alpha, \kappa)$ holds. $\dashv$

Theorem 3.4 (Tryba-Woodin '84). *Every regular Jónsson cardinal κ is 1-reflecting.*

PROOF. Let $S \subseteq \kappa$ be stationary and fix an elementary $j : \mathcal{M} \to H_\theta$ with $\theta \gg \kappa$, $S \in \operatorname{ran} j$, $j(\kappa) = \kappa$ and $\operatorname{crit} j < \kappa$.

Claim 3.4.1. $\{\alpha \in S \mid j(\alpha) > \alpha\}$ is stationary in κ.

PROOF OF CLAIM. Assume not and pick a club $C \subseteq \kappa$ such that

$$C \cap S \subseteq \{\alpha \in S \mid j(\alpha) = \alpha\}.$$

In $\mathcal{M}$ partition $\bar{S}$ into $\bar{S}_\alpha$ for $\alpha < \kappa$ (using the regularity of κ) and let $\bar{f} \in \mathcal{M} \cap {}^{\bar{S}}\kappa$ be such that $\bar{f}^{-1}[\{\alpha\}] = \bar{S}_\alpha$ for every $\alpha < \kappa$. Fix α such that $j(\alpha) > \alpha$ and note that $C \cap S_\alpha \neq \varnothing$. As $S_\alpha \subseteq S$ we get that $C \cap S_\alpha \subseteq \{\xi \in S \mid j(\xi) = \xi\}$. Fix $\beta \in C \cap S_\alpha$ (note that $j(\beta) = \beta$), so that $j(\alpha) = j(\bar{f}(\beta)) = f(\beta) = \alpha$, $\lightning$. ⊣

Use now the claim to pick $\delta \in S$ such that $j(\delta) > \delta$ which also without loss of generality satisfies $j[\delta] \subseteq \delta$, as there are club many such. We then claim that $j(\delta)$ is a reflection point for S.

Assume not, so there's a club $C \subseteq j(\delta)$ disjoint from S. By elementarity we may assume that $C \in \operatorname{ran} j$. For any $\alpha < \delta$ we get $j(\alpha) < \delta$ as well, and since $\bar{C}$ is club in δ we can find $\gamma \in \bar{C}$ above $j(\alpha)$, so that $\alpha < j(\gamma) < \delta$. As α was arbitrary, δ is a limit point of C and thus $\delta \in C$. But $\delta \in S$ and $C \cap S = \varnothing$, $\lightning$. ■

Note that the above proof shows that there are stationarily many reflection points to each stationary $S \subseteq \kappa$. Jónsson successors satisfy a much stronger reflection principle.

THEOREM 3.5 (Eisworth '12). *Every successor cardinal ρ^+ is $< \operatorname{cof}(\rho)$-reflecting.* ⊣

3.6 Squares and threadables

DEFINITION 3.7. Say a cardinal κ is α-**threadable** for $\alpha < \kappa$ if $\square(\kappa, \alpha)$ fails, and **threadable** if it's 1-threadable. ⊣

THEOREM 3.8 (Rinot '14). *Every regular Jónsson cardinal is threadable.* ⊣

THEOREM 3.9 (Hayut, Lambie-Hanson '16). *For a cardinal λ, every $< \lambda$-reflecting regular cardinal is $< \lambda$-threadable.* ⊣

COROLLARY 3.10. *Every successor Jónsson ρ^+ is $< \operatorname{cof}(\rho)$-threadable.* ⊣

Theorem 3.11 (Welch '98). *Assume there's no inner model with a Woodin cardinal. Then*

(i) $\kappa^{+K} = \kappa^+$ *holds for every Jónsson cardinal* κ*;*

(ii) $\lambda^{+K} = \lambda^+$ *holds for stationarily many regular* $\lambda < \kappa$*.*

$\dashv$

Corollary 3.12. *If there's no inner model with a Woodin cardinal and* κ *is regular Jónsson, then*

(i) $\square_\kappa$ *holds;*

(ii) $\square_\lambda$ *holds for stationarily many regular* $\lambda < \kappa$*.*

$\dashv$

3.13 Diamonds

Theorem 3.14 (Shelah '10). *Let* κ *be uncountable. Then* $\diamondsuit_{\kappa^+}$ *holds iff* $2^\kappa = \kappa^+$*.*

Theorem 3.15 (Erdős-Hajnal-Rado). *If* $\kappa^+ = 2^\kappa$ *then* κ^+ *is not Jónsson.* $\dashv$

These above two results then gives us some more information about the Jónsson successors.

Corollary 3.16. *If* κ *is a Jónsson successor then* $\diamondsuit_\kappa$ *fails.*

3.17 Sharps

Theorem 3.18 (Rinot-Steel-Schindler '14). *Let* κ *be regular Jónsson. Then* $A^\sharp$ *exists for every* $A \subseteq \kappa$*.*

Proof. As every bounded subset of κ has a sharp, it suffices to show that mouse reflection holds at κ. But by Lemma 2.1.6 in Schindler and Steel (2014) it suffices to show that $\kappa \geqslant \aleph_2$ is threadable. But every Jónsson satisfies $\kappa \geqslant \aleph_2$ and by Rinot (2014) every regular Jónsson is threadable. ■

3.19 Successor Jónssons

Theorem 3.20 (Mitchell-Steel-Schimmerling '94). *If* $\square_\rho$ *fails for a singular* ρ *then there exists an inner model with a Woodin cardinal.* $\dashv$

Theorem 3.21 (Adolf '17). *If $\square_\rho$ fails for a singular strong limit ρ then there exists a transitive model containing all the ordinals and reals, and which satisfies $\ulcorner \mathsf{ZF} + \mathsf{AD}_\mathbb{R} + \Theta$ is regular$\urcorner$.* $\dashv$

Corollary 3.22. *Assume there exists a successor Jónsson ρ^+. Then there exists an inner model with a Woodin cardinal. If furthermore ρ is a strong limit then there exists a transitive model containing all the ordinals and reals, and which satisfies $\ulcorner \mathsf{ZF} + \mathsf{AD}_\mathbb{R} + \Theta$ is regular$\urcorner$.* $\dashv$

We have the following restrictions to an eventual Jónsson successor.

Theorem 3.23. *Assume ρ^+ is Jónsson. Then*

(i) *(Chang-Rowbottom-Erdős-Hajnal '66) ρ is Jónsson;*

(ii) *(Shelah '78) ρ is a limit of weakly inaccessible Jónssons, so ρ is not small;*

(iii) *(Tryba-Woodin '84) ρ is singular;*

(iv) *(Shelah '94) $\rho^+ \not\twoheadrightarrow [\operatorname{cof}\rho]^2_{\operatorname{cof}\rho}$;*

(v) *(Shelah-Abraham-Magidor '10) If $\operatorname{cof}(\rho) > \omega$ then there are club many $\lambda < \rho$ such that λ^+ is Jónsson;*

(vi) *(Eisworth '12) ρ^+ is $< \operatorname{cof}(\rho)$-reflecting;*

(vii) *(Hayut, Lambie-Hanson '16) ρ^+ is $< \operatorname{cof}(\rho)$-threadable.*

$\dashv$

The Shelah-Abraham-Magidor ('10) result above implies that the first Jónsson successor ρ^+ satisfies $\operatorname{cof}\rho = \omega$, so since $\square(\rho)$ holds iff $\square(\operatorname{cof}\rho)$ holds (Lemma 2.1 in Schimmerling '07), we get the following when also coupling it with the result of Hayut and Lambie-Hanson ('16) above.

Corollary 3.24. *If ρ^+ is the first Jónsson successor cardinal then ρ is threadable and ρ^+ is $< \omega$-reflecting (in particular also $< \omega$-threadable).* $\dashv$

3.25 Weakly inaccessible Jónssons

Theorem 3.26 (Shelah '98). *Weakly inaccessible Jónsson cardinals are weakly hyper-Mahlo.* $\dashv$

Theorem 3.27 (Mitchell '99). *Assume there's no inner model with a Woodin cardinal. Then every weakly inaccessible Jónsson is Ramsey in K.* $\dashv$

The above theorem is usually stated for all regular Jónsson cardinals. But given that a successor Jónsson has a lot higher consistency strength than a Woodin, this formulation is equivalent.

Corollary 3.28. *The following theories are equiconsistent.*

- *⌜ZFC + there exists a Jónsson cardinal⌝;*
- *⌜ZFC + there exists a weakly inaccessible Jónsson cardinal⌝;*
- *⌜ZFC + there exists an inaccessible Jónsson cardinal⌝;*
- *⌜ZFC + there exists a Ramsey cardinal⌝.*

$\dashv$

3.29 Structural Reflection

Definition 3.30 (Bagaria). $\mathrm{SR}_\alpha(A)$ holds if every $\Sigma_1(A)$-definable class $\mathcal{C}$ of structures has the property that whenever $\mathcal{M} \in \mathcal{C}$ then there exists $\mathcal{N} \in \mathcal{C} \cap V_\alpha$ and an elementary $j : \mathcal{N} \to \mathcal{M}$. $\dashv$

Theorem 3.31 (N.). *A cardinal κ is Jónsson if and only if $SR_{\kappa^+}(\kappa)$ holds.*

Proof. ($\Leftarrow$): Assume $\mathrm{SR}_{\kappa^+}(\kappa)$ and define the class

$$\mathcal{C} := \{(L_\kappa[x], \in, x, \kappa) \mid x \subseteq \kappa\},$$

which is $\Sigma_1(\kappa)$-definable. Fix any $x \subseteq \kappa$ and consider $\mathcal{M} := (L_\kappa[x], \in, x)$. By $\mathrm{SR}_{\kappa^+}(\kappa)$ we get an elementary

$$j : (L_\kappa[\bar{x}], \in, \bar{x}, \kappa) \to (L_\kappa[x], \in, x, \kappa),$$

so $\mathrm{crit}\, j < \kappa$, which then witnesses Jónssonness.

($\Rightarrow$): Let κ be Jónsson and $\mathcal{C}$ a $\Sigma_1(\kappa)$-definable class of structures. Let $\mathcal{M} \in \mathcal{C}$, $\theta \gg 0$ and $j : \mathcal{H} \to H_\theta$ a Jónsson embedding with $\mathcal{M} \in \mathrm{ran}\, j$; i.e. that $\mathrm{crit}\, j < \kappa$ and $j(\kappa) = \kappa$. Since κ is fixed we get that $\overline{\mathcal{M}} \in \mathcal{C}$, but we don't know if $|\overline{\mathcal{M}}| < \kappa^+$ yet. Set

$$\mathcal{N} := \mathrm{cHull}^{\mathcal{H}}((V_\kappa \cap \mathcal{H}) \cup \{\kappa\})$$

and let $\pi : \overline{\overline{\mathcal{M}}} \to \overline{\mathcal{M}}$ be the uncollapse, which again fixes κ, so by elementarity $\overline{\overline{\mathcal{M}}} \in \mathcal{C}$, and we now also get that $\left|\overline{\overline{\mathcal{M}}}\right| < \kappa^+$, obtaining $\mathrm{SR}_{\kappa^+}(\kappa)$. ■

4 Woodin cardinals

Proposition 4.1. *If there exists a threadable Woodin δ then every $A \subseteq \delta$ has a sharp, so that $M_1^\sharp(A)$ exists for all $A \subseteq \delta$.*

Proof. Woodins are stationary limits of measurables, so every bounded subset of δ immediately has a sharp. But as regular threadable cardinals satisfy mouse reflection, this holds for all subsets of δ. As δ being Woodin is witnessed by extenders $\vec{E} \subseteq \delta$, we immediately get that $M_1^\sharp(A)$ exists for every $A \subseteq \delta$. ■

5 Weakly n-Jónssons

Definition 5.1. Say a cardinal κ is **weakly n-Jónsson** if κ is uncountable and $\kappa \to [\kappa]^n_\kappa$. Say κ is **weakly Jónsson** if it's weakly 2-Jónsson. ⊣

Remark 5.2. Every uncountable cardinal κ is weakly 1-Jónsson. Indeed, let $f : \kappa \to \kappa$ be a colouring and define $A := f^{-1}(\{0\})$. If $|A| = \kappa$ we're done, so assume not. Then $|\neg A| = \kappa$ and $0 \notin f``\neg A$, so $\neg A$ works.

Proposition 5.3. *Every Jónsson cardinal is weakly n-Jónsson for every $n < \omega$.* ⊣

Proposition 5.4. *Every weakly compact cardinal is weakly n-Jónsson for every $n < \omega$.* ⊣

Theorem 5.5 (Todorčević '87). *$\aleph_1$ is not weakly Jónsson.* ⊣

A lot of the results concerning Jónssons are really about weakly Jónssons.

Theorem 5.6. *Let κ be a regular weakly Jónsson cardinal. Then*

(i) *(Todorčević '81) There exists no κ-Souslin tree, so κ is fully $\square$-inaccessible;*
(ii) *(Todorčević '87) κ is 1-reflecting;*
(iii) *(Shelah '94) κ is ω-Mahlo;*
(iv) *(Rinot '14) κ is threadable;*
(v) *(Schindler-Steel '14) Mouse reflection holds at κ.* ⊣

Theorem 5.7 (Todorčević '87). *Every uncountable threadable cardinal is weakly compact in L.* ⊣

Corollary 5.8. *Regular weakly Jónssons are weakly compact in L, so that, for every $n < \omega$,*

$$\mathrm{Con}(\exists \textit{weakly compact}) \Leftrightarrow \mathrm{Con}(\exists \textit{regular weakly } n\textit{-Jónsson}). \qquad \dashv$$

Theorem 5.9. *If ρ^+ is weakly Jónsson then*

(i) *(Erdős-Hajnal-Rado '65) CH_κ fails;*
(ii) *(Todorčević '90) ρ is a singular limit of regular weakly Jónssons;*
(iii) *(Shelah '94) $Pr_1(\rho^+, \rho^+, \operatorname{cof}\rho, \operatorname{cof}\rho)$ fails;*
(iv) *(Eisworth '12) ρ^+ is $< \operatorname{cof}(\rho)$-reflecting;*
(v) *(Eisworth '13) $\operatorname{pp}(\rho) > \rho^+$, i.e. $\rho^{\operatorname{cof}\rho} > \rho^+$;*
(vi) *(Hayut, Lambie-Hanson '16) ρ^+ is $< \operatorname{cof}(\rho)$-threadable.* $\dashv$

We also get a few slightly different results. The following is by tweaking the proof the above Chang-Rowbottom-Erdős-Hajnal result.

Theorem 5.10 (Chang-Rowbottom-Erdős-Hajnal '65). *If ρ^+ is weakly $(n+1)$-Jónsson then ρ is weakly n-Jónsson.* $\dashv$

6 Open questions

6.1 Regular Jónssons

Question 6.2. Does every regular Jónsson provably have the tree property? As a special case, is every inaccessible Jónsson provably weakly compact? (Conjecture: no)

Question 6.3. Is every regular Jónsson 1.5-reflecting? I.e. given a stationary-co-stationary $A \subseteq \kappa$ for κ a regular Jónsson, do A and $\neg A$ simultaneously reflect?

A negative answer to question 6.2 would motivate a lot of the investigations of properties of regular Jónsson cardinals, as they all follow from the tree property. A positive answer to question 6.3 would unify reflection and threadability of regular Jónssons, as 1.5-reflecting implies both 1-reflecting and threadable, whereas 1-reflecting does not imply threadability.

6.4 Successor Jónssons

Question 6.5. Is the existence of a successor Jónsson consistent, relative to large cardinals?

Question 6.6. Does the existence of a successor Jónsson ρ^+ imply $\mathsf{AD}^{L(\mathbb{R})}$? (Conjecture: yes. It already implies PD, and if ρ is a strong limit then it implies $\mathrm{Con}(\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular))

Question 6.5 is a major open problem and a positive answer to question 6.6 would shed some more light on question 6.5.

6.7 Small Jónssons

Question 6.8. Is the existence of a small Jónsson consistent, relative to large cardinals?

Question 6.9. Does the existence of a small Jónsson imply $0^{\P}$? (Conjecture: yes)

Question 6.8 is a major open problem, since $\aleph_\omega$ is small, and a positive answer to question 6.9 would shed some more light on question 6.8.

6.10 Woodins

Question 6.11. Does Con(Jónsson Woodin) $\Rightarrow$ Con(Weakly compact Woodin)? (Conjecture: yes)

An answer to question 6.11 would give some more information about the Jónsson Woodins.