The forcing category

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ABSTRACT. I'm an abstract

DEFINITION 0.1. Let $\mathbb{F}$ be the category of separative forcing posets and complete embeddings between them. $\dashv$

Note that when we're dealing with *separative* posets then complete embeddings are really embeddings; i.e. they're injective.

PROPOSITION 0.2. *Define $\mathcal{D} := \{f \in Mor(\mathbb{F}) \mid f \text{ is dense}\}$. Then $\mathcal{D}$ is a right multiplicative system for $\mathbb{F}$.*

PROOF. (S1) For any forcing poset $\mathbb{P}$ it trivally holds that $\mathrm{id}_{\mathbb{P}} : \mathbb{P} \to \mathbb{P}$ is dense.

(S2) Let $f : \mathbb{P} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{R}$ be dense, and let $r \in \mathbb{R}$. Use density of g to pick $q \in \mathbb{Q}$ with $g(q) \leqslant r$, and then use density of f to pick $p \in \mathbb{P}$ such that $f(p) \leqslant q$. Then $(g \circ f)(p) = g(f(p)) \leqslant g(q) \leqslant r$ by order-preservation of g, making $g \circ f$ dense.

(S3) Let $f : \mathbb{P} \to \mathbb{Q}$ be complete and $i : \mathbb{P} \to \hat{\mathbb{P}}$ dense. We have to find a forcing $\hat{\mathbb{Q}}$, a complete $g : \hat{\mathbb{P}} \to \hat{\mathbb{Q}}$ and a dense $j : \mathbb{Q} \to \hat{\mathbb{Q}}$ such that

$$\begin{tikzcd} \hat{\mathbb{P}} \arrow[r, "g"] & \hat{\mathbb{Q}} \\ \mathbb{P} \arrow[u, "i"] \arrow[r, "f"'] & \mathbb{Q} \arrow[u, "j"'] \end{tikzcd}$$

commutes. ■