

Jónsson Covering below a Woodin

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ABSTRACT. We modify the proof in Welch (2000) of the weak covering lemma at Jónsson cardinals below a Woodin cardinal to the measurable-free context from Jensen and Steel (2013).

This note is dedicated to proving the following theorem.

THEOREM 0.1. *Assume that there is no inner model with a Woodin cardinal. Let K be the Jensen-Steel core model and κ a Jónsson cardinal. Then $\kappa^+ = \kappa^{+K}$.*

And immediate consequence of the theorem is then the following.

COROLLARY 0.2. *Assume that there is no inner model with a Woodin cardinal and let κ be a Jónsson cardinal. Then \square_κ holds.*

1 Setting the scene

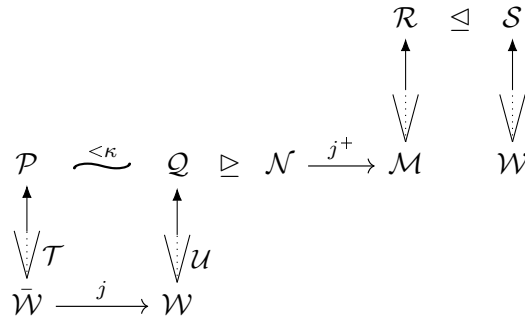
We start by setting up the scene and giving an overview of what is to come. By the weak covering lemma in K , we can assume that κ is regular. Set $\lambda := \kappa^{+K}$ and assume towards a contradiction that $\lambda < \kappa^+$, so that the weak covering lemma in K implies that $\text{cof } \lambda = |\lambda| = \kappa$. Fix some monotone cofinal map $D : \kappa \rightarrow \lambda$.

For some suitably large $\Omega > \kappa$ we can ensure that $\tilde{K}(\kappa^+, \Omega)$ contains κ^+ , so fix a “very soundness” witness \mathcal{W} for $\tilde{K}(\kappa^+, \Omega) \restriction \kappa^+$, i.e. a weasel such that $S(\mathcal{W})$ is \mathcal{W} -thick and $\kappa^+ \subseteq \mathcal{W}$ – note that $\kappa^{+K} = \kappa^{+\mathcal{W}}$. Say \vec{E} is \mathcal{W} ’s extender sequence and let $X < \langle V_\eta, \epsilon, \vec{E}, D \rangle$ for some sufficiently large η and with $|X| = \kappa$ and $X \cap \kappa \neq \kappa$; X exists by the Jónsson property.

Let $\pi : \langle \mathcal{H}, \epsilon, \vec{\bar{E}}, \bar{D} \rangle \cong \langle X, \epsilon, \vec{E}, D \rangle$ be the uncollapse. Set $\bar{x} := \pi^{-1}(x)$ for every $x \in X$. Then by virtue of D we get that $\text{cof } \bar{\lambda} = \kappa > \text{crit } \pi$, so that π is continuous at $\bar{\lambda}$. Note that $\vec{\bar{E}}$ is the extender sequence of $\bar{\mathcal{W}}$ and define $j := \pi \upharpoonright \bar{\mathcal{W}} : \bar{\mathcal{W}} \rightarrow \mathcal{W}$. We now want to compare $\bar{\mathcal{W}}$ with \mathcal{W} . Our proof strategy will be the following.

- (i) Compare $\bar{\mathcal{W}}$ with \mathcal{W} via a non-standard coiteration, yielding trees \mathcal{T}, \mathcal{U} with last models \mathcal{P}, \mathcal{Q} , such that \mathcal{P} agrees with \mathcal{Q} below κ ;
- (ii) Pick the least $\mathcal{N} \triangleleft \mathcal{Q}$ such that $\bar{\lambda}$ is definably collapsed over \mathcal{N} ;
- (iii) Use that \mathcal{N} agrees with \mathcal{P} below κ to lift $j : \bar{\mathcal{W}} \rightarrow \mathcal{W}$ to some $j^+ : \mathcal{N} \rightarrow \mathcal{M}$;
- (iv) Show that $\Psi_{\mathcal{M}} := \Phi(j \mathcal{T})^{\wedge} \langle \mathcal{M}, \kappa \rangle$ is a stable iterable phalanx, which allows us to compare \mathcal{M} with \mathcal{W} ;
- (v) λ is definably collapsed over \mathcal{M} , so by an elementarity argument this is also true over \mathcal{W} , but $\lambda = \kappa^+ \mathcal{W}$, \nless .

The hardest part will be constructing the tree \mathcal{U} in part (i). This is very reminiscent of the proof of the weak covering lemma for K , but to construct \mathcal{U} in that scenario they use that κ is singular and a strong limit. In our case we could tweak the argument so that the Jónsson property achieves the result that would normally use the singularity of κ , but the argument using that κ is a strong limit does not *prima facie* admit such a tweak. Instead we will follow Mitchell's idea of “special drops”, also described in Welch (2000). Here is an illustration of the proof to come.



2 Constructing \mathcal{U}

We now provide the definition of \mathcal{U} . Simultaneously we will define trees \mathcal{T} on $\bar{\mathcal{W}}$ and $\bar{\mathcal{U}}$ on \mathcal{W} , and $\deg^{\mathcal{U}}(\alpha)$ -embeddings $\pi_\alpha : \mathcal{M}_\alpha^{\mathcal{U}} \rightarrow \mathcal{M}_\alpha^{\bar{\mathcal{U}}}$. The tree $\bar{\mathcal{U}}$ and the π_α 's are only there to make sure that the models in \mathcal{U} are wellfounded.

Let us now say that λ is a limit and $\mathcal{T} \upharpoonright \lambda$, $\mathcal{U} \upharpoonright \lambda$ and $\bar{\mathcal{U}} \upharpoonright \lambda$ have been constructed. Then $\bar{\mathcal{U}} \upharpoonright \lambda$ is an ω -iteration tree and \mathcal{W} 's iteration strategy picks the unique cofinal wellfounded branch \bar{b} of $\bar{\mathcal{U}}$. Fix some $\gamma \in \bar{b}$ such that $\bar{b} - \gamma$ has no drops, and set $\mathcal{M}_\lambda^{\bar{\mathcal{U}}}$ to be the direct limit of the models along \bar{b} . Then set $\mathcal{M}_\lambda^{\mathcal{U}}$ to be the direct limit of all the $\mathcal{M}_\alpha^{\mathcal{U}}$ such that $\alpha \in \bar{b}$.

If $\alpha = \beta + 1$ and $\mathcal{T} \upharpoonright \alpha$, $\bar{\mathcal{U}} \upharpoonright \alpha$ and $\mathcal{U} \upharpoonright \alpha$ are defined then we have models $\mathcal{M}_\alpha^{\mathcal{T}}$, $\mathcal{M}_\beta^{\bar{\mathcal{U}}}$ and $\mathcal{M}_\beta^{\mathcal{U}}$. Let $j_\alpha : \mathcal{M}_\alpha^{\mathcal{T}} \rightarrow \mathcal{M}_\alpha^{j_\alpha^{\mathcal{T}}}$ be the copy maps and define the set

$$\Sigma := \{\delta \leq \kappa \mid \delta \text{ is a regular } V\text{-cardinal} \wedge j_\delta''\delta \subseteq \delta \wedge \delta \geq \sup_{\gamma < \delta} \nu_\gamma^{\mathcal{U}}\}.$$

Note that Σ is stationary below κ since κ is Mahlo by Shelah (1994). Furthermore define for $\alpha < \theta$ the phalanxes

$$\Psi_\alpha := \Phi(j_\alpha \mathcal{T} \upharpoonright \alpha) \hat{\smallfrown} \langle \text{Ult}(\mathcal{M}_\alpha^{\mathcal{U}}, E_{j_\alpha} \upharpoonright \alpha), \alpha \rangle.$$

Case 1: $\beta \notin \Sigma$.

If $\mathcal{M}_\beta^{\mathcal{T}}$ agrees with $\mathcal{M}_\beta^{\mathcal{U}}$ below κ then pad one more step, meaning that we set $\mathcal{M}_\alpha^{\mathcal{T}} := \mathcal{M}_\beta^{\mathcal{T}}$, $\mathcal{M}_\alpha^{\bar{\mathcal{U}}} := \mathcal{M}_\beta^{\bar{\mathcal{U}}}$, $\mathcal{M}_\alpha^{\mathcal{U}} := \mathcal{M}_\beta^{\mathcal{U}}$ and $\pi_\alpha := \pi_\beta$. Otherwise let η be the least disagreement between $\mathcal{M}_\beta^{\mathcal{T}}$ and $\mathcal{M}_\beta^{\mathcal{U}}$. If $E_\beta^{\mathcal{U}} \neq \emptyset$ then set $E_\beta^{\bar{\mathcal{U}}} := \dot{E}_{\pi_\beta(\eta)}^{\mathcal{M}_\beta^{\bar{\mathcal{U}}}}$ and note that $(\mathcal{M}_\alpha^{\mathcal{U}})^* \leq \mathcal{M}_{\text{pred}^{\mathcal{U}}(\alpha)}^{\mathcal{U}}$ implies that $(\mathcal{M}_\alpha^{\bar{\mathcal{U}}})^* \leq \mathcal{M}_{\text{pred}^{\bar{\mathcal{U}}}(\alpha)}^{\bar{\mathcal{U}}}$. Set $\pi_\alpha[a, f] := [\pi_\beta(a), \pi_{\text{pred}^{\mathcal{U}}(\alpha)}(f)]$. Then π_α is a $\deg^{\mathcal{U}}(\alpha)$ -embedding. If $E_\beta^{\mathcal{U}} = \emptyset$ then set $\pi_\alpha := \pi_\beta$ and follow the rules of the iteration game.

Case 2: $\mathcal{M}_\beta^{\mathcal{U}}$ is a stable weasel and Ψ_β is not iterable.

Here we call $\beta + 1$ a **special drop**. Let \mathcal{V} be a bad tree on Ψ_β and let

$$\pi_+ : \text{cHull}^{V_{\Omega+1}}(\alpha \cup \{j_\beta, \mathcal{V}, \mathcal{T} \upharpoonright \alpha, \mathcal{U} \upharpoonright \alpha, \mathcal{M}(\mathcal{V}), \mathcal{Q}(\mathcal{V})\}) \rightarrow V_{\Omega+1}$$

be the uncollapse. Then set $\mathcal{M}_\alpha^\mathcal{U} := \pi_+^{-1}(\mathcal{M}_\beta^\mathcal{U})$, $D^\mathcal{U} \upharpoonright^{\alpha+1} := D^\mathcal{U} \upharpoonright^\alpha \cup \{\alpha\}$, $\nu_\beta^\mathcal{U} := 0$, $\deg^\mathcal{U}(\alpha) = \omega$, $\beta <_\mathcal{U} \alpha$, $\pi_\alpha := \pi_\beta \circ \pi_+$ and pad \mathcal{T} and $\bar{\mathcal{U}}$. Note that we still have that $\mathcal{M}_\alpha^\mathcal{U}$ agrees with $\mathcal{M}_\alpha^\mathcal{T}$ below $\sup\{\max(\nu_\gamma^\mathcal{U}, \nu_\gamma^\mathcal{U}) \mid \gamma < \beta\}$ since this supremum is $\leq \beta$ (and thus $< \alpha$) by case hypothesis.

Case 3: otherwise.

If the $\mathcal{M}_\beta^\mathcal{T}$ agrees with $\mathcal{M}_\beta^\mathcal{U}$ below κ then halt. Otherwise pick the least disagreement and proceed as in case 1.

LEMMA 2.1. *If $\theta \geq \kappa$ and Ψ_κ is not iterable then there is some $\delta \in [0, \kappa)_\mathcal{U} \cap \Sigma$ such that Ψ_δ is not iterable.*

PROOF. Let \mathcal{V} be a bad tree on Ψ_κ . As there is no inner model with a Woodin, we can without loss of generality assume that Ψ_κ is properly 1-small. Use that Σ is stationary in κ to pick some $\delta \in [0, \kappa)_\mathcal{U} \cap \Sigma$, which is possible as $[0, \kappa)_\mathcal{U}$ is club in κ . Now let

$$\mathcal{H} := \text{cHull}^{\mathcal{V}_{\Omega+1}}(\{j_\kappa, \mathcal{V}, \mathcal{T} \upharpoonright \kappa + 1, \mathcal{U} \upharpoonright \kappa + 1, \delta, \mathcal{M}(\mathcal{V}), \mathcal{Q}_\mathcal{V}\})$$

with $\mathcal{Q}_\mathcal{V}$ the \mathcal{Q} -structure looking for branches in \mathcal{V} . Then by the absoluteness argument in (Nielsen, 2016, Theorem 6.21), using that \mathcal{H} is countable and also again that no inner model with a Woodin exists if $\bar{\mathcal{V}}$ had a cofinal wellfounded branch (in V) then so would \mathcal{V} – therefore $\bar{\mathcal{V}}$ is truly a bad tree on $\bar{\Psi}_\kappa$. Now define

$$\mathcal{H}^+ := \text{cHull}^{\mathcal{V}_{\Omega+1}}(\delta \cup \{j_\kappa, \mathcal{V}, \mathcal{T} \upharpoonright \kappa + 1, \mathcal{U} \upharpoonright \kappa + 1, \mathcal{M}(\mathcal{V}), \mathcal{Q}_\mathcal{V}\})$$

and let $\psi : \mathcal{H} \rightarrow \mathcal{H}^+$ be the canonical embedding. Then $\psi\bar{\mathcal{V}}$ is a bad tree on the phalanx

$$\begin{aligned} \psi(\bar{\Psi}_\kappa) &= \psi(\overline{\Phi(j\mathcal{T} \upharpoonright \kappa)}) \frown \langle \psi(\overline{\text{Ult}(M_\kappa^\mathcal{U}, E_{j_\kappa} \upharpoonright \kappa)}), \psi(\bar{\kappa}) \rangle \\ &= \pi_+^{-1}(\Phi(j\mathcal{T} \upharpoonright \kappa)) \frown \langle \text{Ult}(\pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}), E_{j_{\pi_+^{-1}(\kappa)}} \upharpoonright \pi_+^{-1}(\kappa)), \pi_+^{-1}(\kappa) \rangle, \\ &= \Phi(j\mathcal{T} \upharpoonright \delta) \frown \langle \text{Ult}(\pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}), E_{j_\delta} \upharpoonright \delta), \delta \rangle, \end{aligned}$$

where $\pi_+ : \mathcal{H}^+ \rightarrow V_{\Omega+1}$ is the uncollapse. But now define $\tau : \pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}) \rightarrow \mathcal{M}_\delta^\mathcal{U}$ as $\tau := (i_{\delta\kappa}^\mathcal{U})^{-1} \circ \pi_+$. To show that this is well-defined we have to show that $\pi_+'' \pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}) \subseteq \text{ran } i_{\delta\kappa}^\mathcal{U}$, so let $x = \pi_+(\bar{x})$ for $\bar{x} \in \pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U})$. By elementarity we get that $\bar{x} \in \mathcal{M}_\delta^\mathcal{U} \cap \mathcal{H}^+$, so since $\delta \in [0, \kappa)_\mathcal{U}$ there is some $\eta <_\mathcal{U} \delta$ and $z \in \mathcal{M}_\eta^\mathcal{U}$ such that $\bar{x} = i_{\eta\delta}^\mathcal{U}(z)$. But then

$$x = \pi_+(\bar{x}) = \pi_+(i_{\eta\delta}^\mathcal{U}(z)) = i_{\eta\kappa}^\mathcal{U}(z) = i_{\delta\kappa}^\mathcal{U}(i_{\eta\delta}^\mathcal{U}(z)) \in \text{ran } i_{\delta\kappa}^\mathcal{U}$$

and τ is well-defined. Lift τ to $\tilde{\tau} : \text{Ult}(\pi_+^{-1}(\mathcal{M}_\kappa^\mathcal{U}), E_{j_\delta} \restriction \delta) \rightarrow \text{Ult}(\mathcal{M}_\delta^\mathcal{U}, E_{j_\delta} \restriction \delta)$, so that $\tilde{\tau}\psi\bar{\mathcal{V}}$ is then a bad tree on $\Phi(j\mathcal{T} \restriction \delta) \frown \langle \text{Ult}(\mathcal{M}_\delta^\mathcal{U}, E_{j_\delta} \restriction \delta), \delta \rangle = \Psi_\delta$. ■

PROPOSITION 2.2. *The above procedure halts at some $\theta \leq \kappa$.*

PROOF. It suffices to show that $\kappa + 1$ does not satisfy cases 1 and 2. But $\kappa \in \Sigma$ since κ is regular and $j(\kappa) = \kappa$, and $\sup_{\gamma < \kappa} \nu_\gamma^\mathcal{U} \leq \kappa$ by definition of \mathcal{U} . Assume thus towards a contradiction that $\kappa + 1$ is a special drop. In particular $\mathcal{M}_\kappa^\mathcal{U}$ is a weasel (since κ is a regular V -cardinal), and note that for this to happen we cannot have had any drops on the \mathcal{W} -to- $\mathcal{M}_\kappa^\mathcal{U}$ branch, special or not. This is purely for cardinality reasons, as otherwise we would have dropped to a structure of cardinality $\leq \kappa < \Omega$. But Lemma 2.1 implies that there is some $\delta \in [0, \kappa)_\mathcal{U} \cap \Sigma$ such that Ψ_δ is not iterable, and as there are no drops on the branch, $\mathcal{M}_\delta^\mathcal{U}$ is a weasel (since δ is a regular V -cardinal), making $\delta + 1 <_\mathcal{U} \kappa$ a special drop, \nless . ■

This finishes the construction of \mathcal{T} , \mathcal{U} and $\bar{\mathcal{U}}$. Let $\theta \leq \kappa$ be the length of this comparison process and set $\mathcal{P} := \mathcal{M}_\theta^\mathcal{T}$, $\mathcal{Q} := \mathcal{M}_\theta^\mathcal{U}$ and $j^+ := j_\theta$.

3 Iterability

LEMMA 3.1. *If \mathcal{Q} is a stable weasel then $\Psi := \Phi(j\mathcal{T}) \frown \langle \text{Ult}(\mathcal{Q}, E_{j^+} \restriction \kappa), \kappa \rangle$ is a stable iterable phalanx.*

PROOF. Note first that \mathcal{Q} being a weasel implies that there are no drops on the \mathcal{W} -to- \mathcal{Q} branch, special or not, just as before. Assume that the lemma fails, which then forces $\theta = \kappa$, as there are cofinally many disagreements below κ due to $\text{crit } j < \kappa$.

This means that $\kappa + 1$ is a special drop, but then Lemma 2.1 implies that there is a special drop $\delta + 1 \in [0, \kappa)_{\mathcal{U}}$, \nless . \blacksquare

LEMMA 3.2. *If \mathcal{Q} is a set premouse then for any proper initial segment $\mathcal{N} \triangleleft \mathcal{Q}$ such that $n := \min\{k < \omega \mid \rho_{k+1}(\mathcal{N}) \leq \theta\}$ exists it holds that*

$$\Psi := \Phi(j \mathcal{T})^{\frown} \langle (\text{Ult}_n(\mathcal{N}, E_{j+} \upharpoonright \theta), n), \theta \rangle$$

is a stable and iterable phalanx.

PROOF. Since \mathcal{Q} is a set premouse there is a drop on the \mathcal{W} -to- \mathcal{Q} branch, so that $\mathcal{P} \trianglelefteq \mathcal{Q}$. As there are no drops on the $\bar{\mathcal{W}}$ -to- \mathcal{P} branch, $|\mathcal{P}| = \kappa$. If $\theta < \kappa$ then regularity of κ would entail that $|\mathcal{Q}| < \kappa$, \nless , so $\kappa = \theta$.

Suppose the claim fails, so that we have a bad tree \mathcal{V} on Ψ . Let b be the \mathcal{W} -to- \mathcal{Q} branch, $\gamma \in b$ least above all drops and $\delta \in b \cap \Sigma$ least above γ , which is possible as b is club in κ . Then just as in the proof of Lemma 2.1 we get that

$$\Phi(j \mathcal{T})^{\frown} \langle (\text{Ult}_n(\pi_+^{-1}(\mathcal{N}), E_{j_\delta} \upharpoonright \delta), n), \delta \rangle$$

is not iterable. Since $\delta \in b - \gamma$, $\mathcal{M}_\delta^{\mathcal{U}}$ is the direct limit of previous models, so that elementarity of π_+ implies that $\pi_+^{-1}(\mathcal{M}_\kappa^{\mathcal{U}}) = \mathcal{M}_\delta^{\mathcal{U}}$. Thus $\mathcal{N}_0 := \pi_+^{-1}(\mathcal{N}) \triangleleft \mathcal{M}_\delta^{\mathcal{U}}$.

We will show that $\varepsilon + 1 := \text{succ}_{\mathcal{U}}(\delta)$ is a drop, which will contradict that $\delta > \gamma$ and γ being above all drops in b . The extender $E_\varepsilon^{\mathcal{U}}$ is the first one used on $[\delta, \kappa)_{\mathcal{U}}$. We claim that $\text{lh } E_\varepsilon^{\mathcal{U}} \leq o(\mathcal{N}_0)$. If this was not the case then $\mathcal{N}_0 \trianglelefteq \mathcal{M}_\varepsilon^{\mathcal{T}}$ and thus also $\mathcal{N}_0 \trianglelefteq \mathcal{P}$. But then $\text{Ult}_n(\mathcal{N}_0, E_{j_\delta} \upharpoonright \delta)$ can be embedded into $\text{Ult}_n(\mathcal{P}, E_{j+} \upharpoonright \kappa)$, so that $\Psi' := \Phi(j \mathcal{T})^{\frown} \langle (\text{Ult}_n(\mathcal{P}, E_{j+} \upharpoonright \kappa), n), \kappa \rangle$ is not iterable, but Ψ' is the derived phalanx of an iteration tree on the iterable \mathcal{W} , \nless . Thus $\text{lh } E_\varepsilon^{\mathcal{U}} \leq o(\mathcal{N}_0)$.

Note also that since δ is a regular V -cardinal it holds that for every extender E used on \mathcal{U} , if $\text{crit } E < \delta$ then $\text{lh } E < \delta$. In particular this means that $\mu := \text{crit } E_\varepsilon^{\mathcal{U}} \geq \delta$, so that $\mu^{+\mathcal{M}_{\varepsilon+1}^{\mathcal{U}}} > \delta$. Say furthermore that $\mu^{+\mathcal{M}_{\varepsilon+1}^{\mathcal{U}}} \geq o(\mathcal{N}_0)$. Then by the result in the above paragraph, $\mu^{+\mathcal{M}_{\varepsilon+1}^{\mathcal{U}}} \geq \text{lh } E_\varepsilon^{\mathcal{U}} > \mu^{+\mathcal{M}_\varepsilon^{\mathcal{U}}}$, which is a contradiction since $\mu^{+\mathcal{M}_{\varepsilon+1}^{\mathcal{U}}} = \mu^{+(\mathcal{M}_{\varepsilon+1}^{\mathcal{U}})^*} \leq \mu^{+\mathcal{M}_\varepsilon^{\mathcal{U}}}$ by definition of $(\mathcal{M}_{\varepsilon+1}^{\mathcal{U}})^*$. Thus $\mu^{+\mathcal{M}_{\varepsilon+1}^{\mathcal{U}}} \in (\delta, o(\mathcal{N}_0))$.

But now by definition of \mathcal{N}_0 , $\rho(\mathcal{N}_0) = \rho(\mathcal{N}) \leq \delta$, so that since $\mathcal{N}_0 \triangleleft \mathcal{M}_\delta^\mathcal{U}$, \mathcal{N}_0 is sound and $\mu^+ \mathcal{M}_{\varepsilon+1}^\mathcal{U}$ is collapsed in $\mathcal{M}_\delta^\mathcal{U}$. This means that $(\mathcal{M}_{\varepsilon+1}^\mathcal{U})^* \triangleleft \mathcal{M}_\delta^\mathcal{U}$, making $\varepsilon + 1$ a drop, \nless . \blacksquare

4 The action

LEMMA 4.1. $i_\mathcal{U}''\kappa \not\subseteq \kappa$.

PROOF. Suppose that $i_\mathcal{U}''\kappa \subseteq \kappa$. We claim that there are no drops on the \mathcal{W} -to- \mathcal{Q} branch, special or not. If there were only standard drops on the branch then there would not be any drops on the $\bar{\mathcal{W}}$ -to- \mathcal{P} branch, so universality of \mathcal{W} implies that $|\mathcal{Q}| = \kappa$ which in turn requires that $\theta = \kappa$ by regularity of κ . But then we also get that $i_\mathcal{U}''\kappa \not\subseteq \kappa$, \nless .

Suppose now $\varepsilon + 1$ is a special drop on the \mathcal{W} -to- \mathcal{Q} branch, entailing that $|\mathcal{M}_{\varepsilon+1}^\mathcal{U}| < \kappa$ and $\Phi(j\mathcal{T})^\frown \langle \text{Ult}_n(\mathcal{M}_\varepsilon^\mathcal{U}, n), E_{j_\varepsilon} \restriction \varepsilon, \varepsilon \rangle$ is not iterable, by definition of special drop. As $|\mathcal{M}_{\varepsilon+1}^\mathcal{U}| < \kappa$ and $i_\mathcal{U}''\kappa \subseteq \kappa$ we get that $\theta < \kappa$, so the coiteration procedure ends with $\mathcal{Q} \leq \mathcal{P}$. This means that $\text{Ult}_n(\mathcal{M}_\varepsilon^\mathcal{U}, E_{j_\varepsilon} \restriction \varepsilon)$ embeds into $\text{Ult}_n(\mathcal{P}, E_{j+} \restriction \kappa)$, and then $\Phi(j\mathcal{T})^\frown \langle \text{Ult}_n(\mathcal{P}, E_{j+} \restriction \kappa), n, \kappa \rangle$ is not iterable – but this phalanx is the derived phalanx of an iteration tree on the iterable \mathcal{W} , \nless .

Thus there are no drops on the \mathcal{W} -to- \mathcal{Q} branch and \mathcal{Q} is thus a weasel. Now by Lemma 3.1, setting $\Psi_\mathcal{W} := \Phi(j\mathcal{T})$, $\mathcal{M} := \text{Ult}(\mathcal{Q}, E_{j+} \restriction \kappa)$ and $\Psi_\mathcal{M} := \Phi(j\mathcal{T})^\frown \langle \mathcal{M}, \kappa \rangle$, we can compare $\Psi_\mathcal{W}$ with $\Psi_\mathcal{M}$:

$$\begin{array}{ccc} \mathcal{R} & \leq & \mathcal{S} \\ \uparrow & & \uparrow \\ \bigvee \mathcal{V} & & \bigvee \mathcal{V}' \\ \Psi_\mathcal{M} & & \Psi_\mathcal{W} \end{array}$$

The thick trick (Nielsen, 2016, Lemma 8.6) implies that \mathcal{M} is below \mathcal{R} and letting $\iota : \mathcal{Q} \rightarrow \mathcal{M}$ be the ultrapower map we have that $\varphi := i_\mathcal{V} \circ \iota \circ i_\mathcal{U} : \mathcal{W} \rightarrow \mathcal{R} \leq \mathcal{S}$, so Dodd-Jensen implies that $\mathcal{R} = \mathcal{S}$. Lastly, we can view \mathcal{V}' as an iteration tree \mathcal{V}'' on \mathcal{W} . All in all, we now have the coiteration

$$\begin{array}{ccc}
\mathcal{R} & = & \mathcal{S} \\
\uparrow & & \uparrow \\
\bigvee \mathcal{V} & & \bigvee \mathcal{V}'' \\
\mathcal{M} & & \mathcal{W}
\end{array}$$

This implies that $\varphi'' \text{Def}^{\mathcal{W}} = \text{Def}^{\mathcal{S}} = i_{\mathcal{V}''}'' \text{Def}^{\mathcal{W}}$, and as $\kappa^+ \subseteq \text{Def}^{\mathcal{W}}$ we also get $\varphi \restriction \kappa = i_{\mathcal{V}''} \restriction \kappa$, so that $\text{crit } i_{\mathcal{V}''} = \text{crit } \varphi \leq \text{crit } \iota = \text{crit } j < \kappa$. We will now finish the proof by showing that $i_{\mathcal{U}}(\text{crit } i_{\mathcal{V}''}) \geq \kappa$, contradicting $i_{\mathcal{U}}'' \kappa \subseteq \kappa$.

As $\text{crit } i_{\mathcal{V}} \geq \kappa$ (as \mathcal{M} is below \mathcal{R}) and $i'' \kappa \subseteq \kappa$, it suffices to show that $\varphi(\text{crit } i_{\mathcal{V}''}) \geq \kappa$, or equivalently $i_{\mathcal{V}}(\text{crit } i_{\mathcal{V}''}) \geq \kappa$. If $i_{\mathcal{V}}(\text{crit } i_{\mathcal{V}''}) < \kappa$ and E is the first extender on the \mathcal{W} -to- \mathcal{S} branch then $\text{lh } E < i_{\mathcal{V}}(\text{crit } i_{\mathcal{V}''}) < \kappa$, but \mathcal{M} agrees with \mathcal{W} up to κ , $\not\leq$. \blacksquare

Note that Lemma 4.1 implies that $\theta = \kappa$: indeed, it implies that either

$$\sup_{\alpha < \theta} \text{crit } E_{\alpha}^{\mathcal{U}} \geq \kappa \quad (\dagger)$$

or there being an extender $E_{\alpha}^{\mathcal{U}}$ such that $\text{crit } E_{\alpha}^{\mathcal{U}} < \kappa$ and $\text{lh } E_{\alpha}^{\mathcal{U}} \geq \kappa$. But \mathcal{U} has only extenders of length $< \kappa$, so only (\dagger) is possible. Regularity of κ then ensures that $\theta = \kappa$. Furthermore, a standard regressive set argument shows that there is a club $C \subseteq \kappa$ such that for $\alpha, \beta \in C$ with $\alpha < \beta$, it holds that $i_{\alpha\beta}^{\mathcal{U}}(\kappa_{\alpha}) = \kappa_{\beta} = \beta$, where $\kappa_{\xi} := \text{crit } i_{\xi\kappa}^{\mathcal{U}}$.

Now for $\alpha \in C$ we have $i_{\alpha\kappa}(\kappa_{\alpha}) = \kappa$ and moreover $\sup(i_{\alpha\kappa}^{\mathcal{U}})'' \kappa_{\alpha}^+ \mathcal{M}_{\alpha}^{\mathcal{U}} = \kappa^+ \mathcal{Q}$. Then $\text{cof } \kappa^+ \mathcal{Q} < \kappa$ while $\text{cof } \bar{\lambda} = \kappa$, so that $\kappa^+ \mathcal{Q} \neq \bar{\lambda}$. Assume for a contradiction that $\kappa^+ \mathcal{Q} < \bar{\lambda}$. Then as $\bar{\lambda} = \kappa^+ \mathcal{R}$ and no extenders on \mathcal{T} overlap κ , there would be a drop on \mathcal{T} on the next step in the coiteration, contradicting the universality of \mathcal{W} . Thus $\kappa^+ \mathcal{Q} > \bar{\lambda}$. This means that there is some least $\mathcal{N} \triangleleft \mathcal{Q}$ and some $n < \omega$ such that

$$\rho_{n+1}(\mathcal{N}) = \kappa < \bar{\lambda} \leq \rho_n(\mathcal{N}).$$

Setting $\mathcal{M} := \text{Ult}_n(\mathcal{N}, E_{j+} \restriction \kappa)$, Lemma 3.2 entails that $\Psi_{\mathcal{M}} := \Phi(j \mathcal{T}) \frown \langle \mathcal{M}, \kappa \rangle$ is stable and iterable, so that applying the thick trick (Nielsen, 2016, Lemma 8.6) to $\langle \Psi_{\mathcal{M}}, \Phi(j \mathcal{T}) \rangle$ and using that trees on $\Phi(j \mathcal{T})$ are really trees on \mathcal{W} gives us the coiteration

$$\begin{array}{ccc}
\mathcal{R} & \triangleleft & \mathcal{S} \\
\uparrow & & \uparrow \\
\mathcal{M} & & \mathcal{W}
\end{array}$$

But now $\sup(j^+)''\bar{\lambda} = j^+(\bar{\lambda}) = \lambda$ is definably collapsed in \mathcal{M} and the $(n+1)$ 'st mastercode $A_{\mathcal{M}}^{n+1}$ is definable over \mathcal{R} so that $\lambda < \kappa^{+\mathcal{S}}$. As it furthermore holds that $\mathcal{P}^{\mathcal{S}}(\kappa) = \mathcal{P}^{\mathcal{W}}(\kappa)$ we get $\lambda < \kappa^{+\mathcal{W}} = \kappa^{+K} = \lambda$, $\not\leq$. This finishes the proof of Theorem 0.1.

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