The consistency strength of blunt Jónsson cardinals

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Abstract. We introduce the concept of a *blunt* Jónsson cardinal, being a regular Jónsson cardinal with a subset having no sharp, and show that the existence of a blunt Jónsson cardinal implies the existence of an inner model with a Woodin cardinal. This improves a result of Welch (1998).

Definition 0.1. A cardinal κ is a **blunt Jónsson cardinal** if it's a regular Jónsson cardinal such that $A^\sharp$ doesn't exist for some $A \subseteq \kappa$. $\dashv$

Theorem 0.2 (Welch (1998)). *If there is a blunt Jónsson cardinal then* $0^\P$ *exists.* $\dashv$

Theorem 0.3 (N.). *If there is a blunt Jónsson cardinal then there exists an inner model with a Woodin cardinal.*

Proof. Let κ be blunt Jónsson. If κ is a successor cardinal then it's already known that we get an inner model with a Woodin cardinal, so we may assume that κ is weakly inaccessible and that there is no inner model with a Woodin cardinal. Let $A \subseteq \kappa$ witness the bluntness of κ and assume without loss of generality that $K|\kappa \subseteq L[A]$, using the weak inaccessibility of κ. Let $\theta \gg \kappa$ be regular and let $i : M \to H_\theta$ be a Jónsson embedding with $A \in \operatorname{ran} i$. Restrict i to $L_\kappa[\bar{A}]$ to get an embedding $j : L_\kappa[\bar{A}] \to L_\kappa[A]$. Using the regularity of κ we can lift j to $j^+ : L[\bar{A}] \to L[A]$. Now $\bar{A}^\sharp$ doesn't exist either, as otherwise we would be able to transfer the indiscernibles of $L[\bar{A}]$ to $L[A]$ via j^+.

As there's no inner model with a Woodin we can form K as well as internal variants $K^{L[\bar{A}]}$ and $K^{L[A]}$. Let $\pi := j^+ \upharpoonright (K^{L[\bar{A}]}|\lambda^+)$, where $\lambda := \operatorname{crit} j$, so that $\pi : K^{L[\bar{A}]}|\lambda^+ \to K^{L[A]}|j(\lambda^+) = K|j(\lambda^+)$, where we used that $j(\lambda^+) < \kappa$ and $K|\kappa \subseteq L[A]$. Now compare $K|\lambda^+$ with $K^{L[\bar{A}]}|\lambda^+$, noting that $K|\lambda^+$ is universal for premice of height $\leqslant \lambda^+$ by Schimmerling and Steel (1999) since λ^+ is a V-cardinal. This means that we get iteration trees $\mathcal{T}$ and $\mathcal{U}$ with last models $\mathcal{P}$ and $\mathcal{Q}$ on $K|\lambda^+$ and $K^{L[\bar{A}]}|\lambda^+$, respectively, and that $\mathcal{Q} \trianglelefteq \mathcal{P}$.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, phantom, "\trianglerighteq"] & \mathcal{Q} \\
K|\lambda^+ \arrow[u, squiggly, "\mathcal{T}"'] & K^{L[\bar{A}]}|\lambda^+ \arrow[u, squiggly, "\mathcal{U}"']
\end{tikzcd}$$

Claim 0.3.1. Both $\mathcal{T}$ and $\mathcal{U}$ have no drops.

Proof of claim. First note that both K and $K^{L[\bar{A}]}$ are universal weasels, as they both compute the successors of stationarily many ordinals correctly (this is where we use that $\bar{A}^\sharp$ doesn't exist). This means that the trees in the comparison between K and $K^{L[\bar{A}]}$ can't drop, and as $\mathcal{T}$ and $\mathcal{U}$ are subtrees of these trees, they can't drop either. $\dashv$

Let α be the least disagreement between K and $K^{L[\bar{A}]}$.

Claim 0.3.2. $\alpha \in (\lambda, \lambda^{+K})$.

Proof of claim. Firstly $\alpha > \lambda$ simply since it's the least disagreement and λ being the critical point of j. Assume that $\alpha \geqslant \lambda^{+K}$, so that π gives rise to an embedding $\pi \restriction (K^{L[\bar{A}]}|\lambda^{+K}) : K||\lambda^{+K} \to K||j(\lambda^{+K})$. We can form the (λ, λ^{+K})-pre-extender E over K and form the ultrapower embedding $i_E : K \to \text{Ult}(K, E)$. We claim that the ultrapower is wellfounded. To see this, first note μ^+ can't be Jónsson for any regular cardinal $\mu > \lambda$, so $j(\mu^+) > \mu^+$ and thus $j(\xi) > \xi$ for any $\xi < \kappa$ with $\text{cof}^V \xi = \mu^+$. We could therefore have picked λ to be one of these ξ instead, ensuring that $\text{cof}^V \lambda > \aleph_0$, making the ultrapower wellfounded. But then $i_E : K \to K$, $\lightning$. $\dashv$

Now perform the copy construction to π to get a non-dropping tree $\pi\mathcal{U}$ on $K|j(\lambda^+)$ with a last model $\mathcal{R}$, and with an elementary embedding $\tilde{\pi} : \mathcal{Q} \to \mathcal{R}$ such that $\tilde{\pi} \circ i^{\mathcal{U}} = i^{\pi\mathcal{U}} \circ \pi$. By definition of the copy construction all extenders used in $\pi\mathcal{U}$ are indexed $\geqslant j(\alpha) > j(\lambda) \geqslant \lambda^+$, where the last inequality is because $j(\lambda) = i(\lambda) \geqslant \lambda^+$ since $i : M \to H_\theta$.

$$\begin{tikzcd}
\mathcal{N} & \mathcal{P} \arrow[l, "i_E"'] & \trianglerighteq & \mathcal{Q} \arrow[r, "\tilde{\pi}"] & \mathcal{R} \\
& K|\lambda^+ \arrow[u, squiggly, "\mathcal{T}"'] & & K^{L[\bar{A}]}|\lambda^+ \arrow[u, squiggly, "\mathcal{U}"'] \arrow[r, "\pi"] & K|\pi(\lambda^+) \arrow[u, squiggly, "\pi\mathcal{U}"']
\end{tikzcd}$$

By agreement of extenders in iteration trees we then get that $\mathcal{R}$ agrees with K below $j(\alpha)$. But if we then let E be the $(\lambda, j(\alpha))$-extender over $L_{j(\alpha)}(\mathcal{P})$ derived from $\tilde{\pi}$, we get an ultrapower embedding $i_E : L_{j(\alpha)}(\mathcal{P}) \to \mathrm{Ult}(L_{j(\alpha)}(\mathcal{P}), E)$, which we can restrict to $i_E \upharpoonright \mathcal{P} : \mathcal{P} \to \mathcal{N}$. Since the ultrapower agrees with K below $j(\alpha)$, $\mathcal{N}$ agrees with K below $j(\alpha)$ as well. But now we get the composite map $i_E \circ i^{\mathcal{T}} : K|\lambda^+ \to \mathcal{N}$.

Claim 0.3.3. $\mathcal{T}$ is trivial.

Proof of claim. Assume not and let F be the first extender used in $\mathcal{T}$. By coherence and normality, no extenders on the $\mathcal{P}$-sequence are indexed at index F. But after applying i_E we get that very same F back, as $\mathcal{N}$ agrees with K below $j(\alpha)$. This means that there is some extender G on the $\mathcal{P}$-sequence with index $G <$ crit F and $i_E(\text{index } G) = \text{index } F$ and $j(\operatorname{crit} G) = i_E(\operatorname{crit} G) = \operatorname{crit} F$. Thus $\lambda = \operatorname{crit} i_E \leqslant \operatorname{crit} G$. But then index $F > j(\lambda) \geqslant \lambda^+$ and F is on the $(K|\lambda^+)$-sequence, $\lightning$. $\dashv$

Claim 0.3.4. $\mathcal{U}$ is trivial.

Proof of claim. Assume not and let F be the first extender used in $\mathcal{U}$. Then F is necessarily indexed at α since $\mathcal{T}$ is trivial, making α a cardinal in $\mathcal{Q}$ by properties of iteration trees. This implies that α is also a cardinal in $\mathcal{P} = K|\lambda^+$, using that $\mathscr{P}^K(\alpha) \subseteq K|\alpha^+ \subseteq K|\lambda^+$. But $\alpha \in (\lambda, \lambda^{+K})$, $\lightning$. $\dashv$

But now $\pi : K|\lambda^+ \to K|j(\lambda^+)$, which we can then lift to an embedding $K \to K$ using the V-regularity of λ^+, $\lightning$. Hence there is an inner model with a Woodin cardinal. ■

References

Schimmerling, E. and Steel, J. (1999). The maximality of the core model. *Transactions of the American Mathematical Society*, 351(8):3119–3141.

Welch, P. D. (1998). Some remarks on the maximality of inner models. *Logical Colloquium '98*.