From Woodins to Determinacy

Dan Saattrup Nielsen

Abstract. We provide a proof of $\mathsf{AD}^{L(\mathbb{R})}$ from a limit of Woodins with a measurable above, based on the approach in Larson (2004) using the stationary tower.

This note is dedicated to proving the following theorem.

Theorem 0.1 (The Main Theorem 5.4). *Assume there is a limit of Woodins with a measurable above. Then $\mathsf{AD}^{L(\mathbb{R})}$ holds.*

This will be done by extensive use of Woodin's stationary tower, following the proof in Larson (2004). I've split the proof into five steps, which is essentially what is also done in Larson, albeit implicitly and not in the same order. Arguments have furthermore been elaborated a bit more. I'm assuming knowledge of the stationary tower corresponding to the first two chapters of Larson (2004). The five steps of the proof are as follows, where we will assume appropriate large cardinals along the way:

(1) All homogeneously Suslin sets of reals are determined;
(2) The complement of a weakly homogeneously Suslin set of reals is homogeneously Suslin;
(3) Every universally Baire set of reals is weakly homogeneously Suslin;
(4) Every set of reals which is definable from a real, absolutely in all forcing extensions, is universally Baire;
(5) Every set of reals in $L(\mathbb{R})$ is definable from a real, absolutely in all forcing extensions.

The above steps are only morally correct, as you'll see in due time.

1 Determinacy of homogeneously Suslin sets

We start by recalling the definition of a homogeneously Suslin set of reals. For any set X let $m(X)$ be the set of countably complete measures on X.

Definition 1.1. Let X be any set, $k \leqslant n$ finite ordinals and $\mu_k \in m({}^k X)$, $\mu_n \in m({}^n X)$. Then μ_n and μ_k are **compatible** if, for all $A \subseteq {}^k X$, $A \in \mu_k$ iff $\{s \in {}^n\omega \mid s \upharpoonright k \in A\} \in \mu_n$. ○

Definition 1.2. An ω-sequence $\langle \mu_n \mid n < \omega \rangle$ is a **tower of measures** if they're all pairwise compatible and ${}^k X \in \mu_k$ for every $k < \omega$. The tower is **countably complete** if for every sequence $\vec{A} \in \Pi_{k<\omega}\mu_k$ there is a function $f : \omega \to X$ such that $f \upharpoonright k \in A_k$ for every $k < \omega$. ○

A tower of measures is countably complete iff it's associated ultrapower is wellfounded (Larson, 2004, 1.2.2).

Definition 1.3. Let κ be a cardinal and X any set. Then a tree T on $\omega \times X$ is κ-**homogeneous** if there exists a partial function $\pi : {}^{<\omega}\omega \to m({}^{<\omega}\kappa)$ such that $\pi(s)$ is a κ-complete measure with $T_s \in \pi(s)$ for every $s \in \operatorname{dom}\pi$ and furthermore for any $x \in {}^\omega\omega$ it holds that $x \in p[T]$ iff $\langle \pi(x \upharpoonright n) \mid n < \omega \rangle$ is a countably complete tower. ○

Definition 1.4. For κ an cardinal say that a set of reals A is κ-**homogeneously Suslin** (κ-hS) if $A = p[T]$ for a κ-homogeneous tree T. A is called **homogeneously Suslin** (hS) if it's κ-hS for some κ. ○

One of the basic results on hS sets of reals is that they're determined, which is the first step of our proof of Theorem 0.1.

Proposition 1.5. *Let A be a hS set of reals. Then A is determined.*

Proof. Say that $A = p[T]$ for some κ-homogeneous tree on $\omega \times X$, witnessed by $\pi : {}^{<\omega}\omega \to m({}^{<\omega}\kappa)$. We want to show that $G_\omega(A)$ is determined and towards that consider the following auxiliary game G^*:

$$\begin{array}{lccccccccc} \text{I} & \langle x_0, g_0\rangle & & \langle x_2, g_1\rangle & & \langle x_4, g_2\rangle & & \cdots & \\ \text{II} & & x_1 & & x_3 & & x_5 & & \cdots \end{array}$$

Here the x_i's are finite ordinals and $g_i \in X$ for every $i < \omega$. Then I wins iff $\langle x, g\rangle \in [T]$. Since $[T]$ is closed, the game G^* is determined. If I has a winning strategy in G^* then he also has one in $G_\omega(A)$ by just ignoring the g_i's. Assume thus that II has a winning strategy τ in G^* – we have to show that she then also has one in $G_\omega(A)$.

For $n < \omega$, $s \in {}^{2n+1}\omega$ and $t \in {}^{n+1}X$ let $\tau(s,t) \in \omega$ be the strategic response according to τ to $\langle\langle s_0, t_0\rangle, s_1, \cdots, \langle s_{2n}, t_n\rangle\rangle$. For $s \in {}^{2n+1}\omega$ and $k < \omega$ set

$$Z_{s,k} := \{t \in T_{s \restriction (n+1)} \mid \tau(s,t) = k\}$$

and define a strategy $\tilde{\tau} : \bigcup_{n<\omega} {}^{2n+1}\omega \to \omega$ by $\tilde{\tau}(s) := k$ iff $Z_{s,k} \in \pi(s \restriction (n+1))$. That this is welldefined follows from the $\aleph_1$-completeness of the measures, as it implies that there's a unique $k < \omega$ such that $Z_{s,k} \in \pi(s \restriction (n+1))$. Assume for a contradiction that $\tilde{\tau}$ is *not* winning and let $x \in A$ witness that. By homogeneity of T there is a function $f : \omega \to X$ such that

$$f \restriction (n+1) \in Z_{x \restriction (2n+1), \tilde{\tau}(x \restriction (2n+1))},$$

so by definition of the $Z_{s,k}$'s we get that $\langle x, f\rangle \in [T]$ and so corresponds to a play of G^*. But again by definition of the $Z_{s,k}$'s this play is a play according to τ and τ was winning, $\lightning$. Thus $\tilde{\tau}$ is winning. ■

2 From weakly homogeneously Suslin sets to homogeneously Suslin sets

The next notion we'll recall is that of a weakly homogeneously Suslin set of reals.

Definition 2.1. Let κ be a cardinal and X any set. Then a tree T on $\omega \times X$ is κ-**weakly homogeneous** if there exists a countable set $\sigma \subseteq m({}^{<\omega}X)$ of κ-complete measures such that for any $x \in p[T]$ there is a countably complete tower $\langle \mu_n \mid n < \omega\rangle \in {}^\omega\sigma$ with $T_{x \restriction n} \in \mu_n$ for every $n < \omega$. ○

Definition 2.2. For κ a cardinal, say that a set of reals A is κ-**weakly homogeneously Suslin** (κ-whS) if $A = p[T]$ for a κ-weakly homogeneous tree, and A is **weakly homogeneously Suslin** (whS) if it's κ-whS for some κ. ○

A result connecting the hS sets and the whS sets is the following.

Proposition 2.3. *Let κ be a cardinal. Then a set of reals A is κ-whS iff $A = pB$ for a κ-hS set B.*

Proof. See (Kanamori, 2009, Exercise 32.3). ■

Of course, for the above Proposition to make sense we have to slightly generalise the definition of whS- and hS sets to subsets of ${}^k({}^\omega\omega)$ for any $k < \omega$. The ingredient in this second step of the proof of Theorem 0.1 is then the following result.

Theorem 2.4. *Assume that δ is a Woodin cardinal and let A be a δ^+-whS set of reals. Then $\neg A$ is α-hS for every $\alpha < \delta$.* ■

This is the main theorem in Martin and Steel (1989), and I will omit it in this note.

Corollary 2.5. *If δ is a Woodin then* $\mathrm{Det}(\delta^+\text{-}whS)$ *holds.*

Proof. Directly from Theorem 2.4 and Proposition 1.5. ■

3 From universally Baire sets to weakly homogeneously Suslin sets

We now come to the notion of a universally Baire set of reals.

Definition 3.1. Let κ be a cardinal and A a set of reals. Then A is **κ-universally Baire** (κ-uB) if there exist trees T and S such that $A = p[T]$ and given any forcing notion $\mathbb{P} \in V_\kappa$ it holds that $\Vdash_\mathbb{P} p[\check{T}] = \neg p[\check{S}]$. We say that A is **universally Baire** (uB) if it's κ-uB for *all* κ. ○

Step three of the proof of Theorem 0.1 is then the following. Recall that $\mathbb{Q}_{<\delta}$ is the countable stationary tower at δ.

Theorem 3.2. *Let δ be a Woodin. Assume that T and S are trees projecting to sets of reals such that $\Vdash_{\mathbb{Q}_{<\delta}} p[\check{T}] = \neg p[\check{S}]$. Then S and T are α-weakly homogeneous for every $\alpha < \delta$. In particular, if A is a δ^+-uB set of reals then A is α-whS for every $\alpha < \delta$.*

Proof. Fix a V-generic $g \subseteq \mathbb{Q}_{<\delta}$. Let X and Y be sets such that T is a tree on $\omega \times X$ and S is a tree on $\omega \times Y$.

Claim 3.2.1. We can without loss of generality assume that $X = Y = \delta$.

Proof of claim. Let $\eta \gg \delta$ such that $T, S \in V_\eta$, and define then $T^* := T \cap \Sigma_\omega^{V_\eta}(V_\delta \cup \{T, S, \delta\})$ and $S^* := S \cap \Sigma_\omega^{V_\eta}(V_\delta \cup \{T, S, \delta\})$. Since $|T^*|, |S^*| \leqslant \delta$ is suffices to show that $p[T^*] = p[T]$ and $p[S^*] = p[S]$, as then T^* and S^* still project to complements in $V[g]$. But note that every real in $V[g]$ is the realisation of a $\mathbb{Q}_{<\gamma}$-name for a real, for some inaccessible $\gamma < \delta$, making it definable with parameters from V_δ. As we can then furthermore define the least $\alpha < \delta$ such that $\langle r, \alpha \rangle \in [T]$ for any $r \in \mathbb{R}^{V[g]}$, we get that $p[T] = p[T^*]$. The result for S is analogous. $\dashv$

Fix now a $< \delta - T-$strong $\kappa < \delta$. As there are cofinally many of such κ (using the above claim to ensure that T can be encoded as a subset of δ), it suffices to show that T is κ-weakly homogeneous. Using the $< \delta - T-$strongness of κ fix for each $\lambda \in (\kappa, \delta$ an elementary embedding $j_\lambda : V \to \mathcal{M}_\lambda$ with $\operatorname{crit} j_\lambda = \kappa$ such that $V_\lambda \subseteq \mathcal{M}_\lambda$ and $j(T) \cap V_\lambda = T \cap V_\lambda$. For every $\lambda \in (\kappa, \delta)$ we will now define a continuous function

$$\Sigma_\lambda^* : [T \cap V_\lambda] \to {}^\omega\mathscr{P}\mathscr{P}({}^{<\omega}\kappa)$$

such that $\Sigma_\lambda^*(x, f)$ is a countably complete tower of κ-complete measures. Firstly define $\Sigma_\lambda : \mathcal{T} \cap V_\lambda \to \mathscr{P}\mathscr{P}({}^{<\omega}\kappa)$ as $\Sigma_\lambda(s, u) := \{X \subseteq {}^{<\omega}\kappa \mid u \in j_\lambda(X)\}$. Note here that $u \in {}^{<\omega}\lambda$, so these measures are simply just the induced ones from the u's and j_λ's, making them κ-complete and concentrating on $T_s \cap \kappa$. It's also simple to see that these measures are compatible, so that $\Sigma_\lambda^*(x, f) := \langle \Sigma_\lambda(x \restriction n, f \restriction n) \mid n < \omega \rangle$ is indeed a tower of κ-complete measures. We now just need to show that the towers are countably complete. But if $\langle x, f \rangle \in [T \cap V_\lambda]$ then $\operatorname{Ult}(V, \Sigma_\lambda^*(x, f))$ is isomorphic to $\{j_\lambda(g)(f \restriction n) \mid g : {}^{<\omega}\kappa \to V \wedge g \in V \wedge n < \omega\}$, which is an elementary submodel of $\mathcal{M}_\lambda$, making it wellfounded.

Now let $i : V \to \mathcal{N}$ be the stationary tower embedding induced by g.

Claim 3.2.2. The set $\sigma := \{\mu \in i[m({}^{<\omega}\kappa)] \mid \mathcal{N} \models \mu \text{ is } i(\kappa)\text{-complete}\}$ witnesses that $i(T)$ is $i(\kappa)$-weakly homogeneous in $\mathcal{N}$.

Proof of claim. First of all σ is countable in $\mathcal{N}$ as $i(\aleph_1) = \delta$, so let $x \in p[i(T)] \cap \mathcal{N}$ – we have to find a countably complete tower of measures from σ with the n'th measure concentrating on $i(T)_{x \restriction n}$ for every $n < \omega$. First notice that $p[T] \subseteq p[i(T)]$,

$p[S] \subseteq p[i(S)]$ and $p[i(T)] \cap p[i(S)]$. As we've assume that $p[T] = \neg p[S]$ in $V[g]$, it then holds that $x \in p[T]$. Fix some $\lambda < \delta$ such that $x \in p[T \cap V_\lambda]$ and pick some f such that $\langle x, f \rangle \in [T \cap V_\lambda]$. Now the tower $\langle i(\Sigma_\lambda)(x \upharpoonright n, i(f \upharpoonright n)) \mid n < \omega \rangle$ is an element of $\mathcal{N}$ as $\mathcal{N}$ is closed under countable sequences in $V[g]$. By elementarity this tower is countably complete and satisfies that $i(\Sigma_\lambda)(x \upharpoonright n, i(f \upharpoonright n))$ is an $i(\kappa)$-complete measure concentrating on $i(T)_{x \upharpoonright n}$, for every $n < \omega$. $\dashv$

Using this claim, elementarity then implies that T is κ-weakly homogeneous, which is what we wanted to show. ■

Corollary 3.3. *If $\delta_0 < \delta_1$ are Woodins then $\mathrm{Det}(\delta_1^+\text{-}uB)$ holds.*

Proof. Directly from Theorem 3.2 and Corollary 2.5. ■

4 From forcing absolute definability to universally Baire sets

Theorem 4.1. *Let δ be a Woodin and let φ, ψ be binary formulas, x, y sets and assume that, for every real $r \in V^{\mathbb{Q}_{<\delta}}$,*

$$\Vdash_{\mathbb{Q}_{<\delta}} (\mathcal{M} \models \psi[r, j(\check{y})] \Leftrightarrow \check{V}[r] \models \varphi[r, \check{x}]),$$

where $j : V \to \mathcal{M}$ is the embedding induced by the countable tower $\mathbb{Q}_{<\delta}$. Then $\{r \in \mathbb{R} \mid \psi[r, y]\}$ is δ-uB.

Proof. Firstly fix some $\lambda \gg \delta$ such that $\mathrm{cof}\,\lambda > \delta$, $\{x, y\} \in V_\lambda$ and that φ and ψ reflect to $V[g]_\lambda$ in every forcing extension $V[g]$ by a δ-small forcing. For the remainder of this proof say that $Z \prec V_\lambda$ is **good** if $\{x, y, \delta\} \in Z$ and, letting $\pi : Z \to \bar{Z}$ be the transitive collapse, it holds that given any forcing $\mathbb{P} \in V_\delta \cap Z$, $\bar{Z}$-generic $g \subseteq \pi(\mathbb{P})$ and real $r \in \bar{Z}[g]$,

$$\psi[r, y] \Leftrightarrow \bar{Z}[r] \models \varphi[r, \pi(x)].$$

Of course, Z is **bad** if it's not good, witnessed by a triple $\langle \mathbb{P}, g, r \rangle$. The key point is then the following claim.

Claim 4.1.1. $\{Z \in \mathscr{P}_{\aleph_1} V_\lambda \mid Z \prec V_\lambda \text{ is good}\}$ contains a club.

Proof of claim. Set $a := \{Z \in \mathscr{P}_{\aleph_1} V_\lambda \mid Z \prec V_\lambda \text{ is bad}\}$ and assume towards a contradiction that a is stationary. To be able to use our stationary tower machinery we have to "reduce" a to a stationary subset of $\mathscr{P}_{\aleph_1} V_\kappa$ for some $\kappa < \delta$, making it an element of $\mathbb{Q}_{<\delta}$. Towards this define $f : a \to V_\lambda$ which associates to each $Z \in a$ a (code for a) pair $\langle \mathbb{P}, r \rangle$ which is a part of a badness witness of Z. As such $\mathbb{P}, r$ are elements of Z, normality of the club filter on $\mathscr{P}_{\aleph_1} V_\lambda$ implies that there is a stationary $b \subseteq a$ such that for a fixed $\mathbb{P}$ and τ, for every $Z \in b$ with $\pi : Z \to \bar{Z}$ the collapse, there's a $g \subseteq \pi(\mathbb{P}$ such that $\langle \mathbb{P}, g, \pi(\tau)_g \rangle$ is a badness witness for Z. Pick some inaccessible $\kappa < \delta$ such that $\mathbb{P} \in V_\kappa$. Assume for simplicity that $\pi(\tau)_g$ determines g.

But b is still only stationary over $\mathscr{P}_{\aleph_1} V_\lambda$, so define now $c \subseteq \mathscr{P}_{\aleph_1} V_\kappa$ as all the countable elementary submodels $Y \prec V_\kappa$ such that $Y = Z \cap V_\kappa$ for some $Z \in b$, so c is now stationary over $\mathscr{P}_{\aleph_1} V_\kappa$, and $\mathbb{P}, \tau$ are now a part of a witness to the badness of every countable $Z \prec V_\lambda$ with $Z \cap V_\kappa \in c$. Now with our stationary $c \subseteq \mathscr{P}_{\aleph_1} V_\kappa$ at hand, we're able to throw our theory of the stationary tower at our problem: let $h \subseteq \mathbb{Q}_{<\delta}$ be V-generic with $c \in h$ and $j : V \to \mathcal{M}$ the associated embedding. Now, in V, pick $X \prec V_\lambda$ of cardinality $< \delta$ with $V_\kappa \subseteq X$. As ${}^\omega \mathcal{M} \subseteq \mathcal{M}$ in $V[h]$, $j"X \prec j(V_\lambda)$ is countable in $\mathcal{M}$.

As $c \in h$ we get that $j"V_\kappa \in j(c)$, so since $j"V_\kappa = j"X \cap j(V_\kappa)$, $j"X$ is a bad elementary substructure of $j(V_\kappa)$ in $\mathcal{M}$, with $j(\mathbb{P})$ and $j(\tau)$ being a part of the witness for it. Now let $\bar{X}$ be the transitive collapse of X and let $\bar{\mathbb{P}}, \bar{\tau}, \bar{x}$ be the images of $\mathbb{P}, \tau, x$ under this collapse. Note that $\bar{X}$ is also the collapse of $j"X$ and the images of $j(\mathbb{P}), j(\tau), j(x)$ under this latter collapse is still $\bar{\mathbb{P}}, \bar{\tau}, \bar{x}$. Pick an $\bar{X}$-generic $g \subseteq \bar{\mathbb{P}}$ in $\mathcal{M}$ so that $\langle j(\mathbb{P}), g, r \rangle$, with $r := \bar{\tau}_g$, is a badness witness for $j"X$. We now get that

$$\bar{X}[r] \models \varphi[r, \bar{x}] \Leftrightarrow V[r]_\lambda \models \varphi[r, x] \Leftrightarrow V[r] \models \varphi[r, x], \tag{1}$$

where $\bar{X}[r]$ makes sense because $\mathscr{P}(\mathbb{P}) \subseteq X$ and that r determined the corresponding $\mathbb{P}$-generic filter. Here the first equivalence is then by elementarity and the second by choice of λ. The badness of $j"X$ in $\mathcal{M}$ also implies that

$$\bar{X}[r] \models \varphi[r, \bar{x}] \Leftrightarrow \mathcal{M} \models \neg\psi[r, j(y)]. \tag{2}$$

But now (1) and (2) contradicts the assumption that $\mathcal{M} \models \psi[r, j(y)]$ holds iff $V[r] \models \varphi[r, x]$. $\dashv$

The idea is now to associate to each real a pair $\langle Z, g\rangle$ of a good countable elementary substructure $Z \prec V_\lambda$ and a Z-generic $g \subseteq \mathbb{P} \cap Z$ for some forcing notion $\mathbb{P}$. This is goiing to be done in a tree-like fashion, resulting in $A := \{r \in \mathbb{R} \mid \psi[r, y]\}$ being the projection of the tree, which is the first step towards showing that A is δ-uB. Towards building the tree, the above claim implies that we may fix a function $F : [V_\lambda]^{<\omega} \to V_\lambda$ such that whenever we have a countable set closed under F, that set is a good elementary substructure of V_λ. Now, slightly abusing notation, let T be the tree of triples $\langle s, t, u\rangle \in {}^{<\omega}(\omega \times \mathscr{P}_{\aleph_0} V_\lambda \times V_\delta)$ such that

(i) $t_0 = \{\mathbb{P}, \tau\}$, where $\mathbb{P} \in V_\delta$ is a forcing notion and τ a $\mathbb{P}$-name for a real;

(ii) $t_{n+1} = F\text{"}[\operatorname{ran} u \restriction (n+1) \cup \operatorname{ran} t \restriction (n+1)]^{<\omega}$ for every $n < \omega$;

(iii) u is a $<_\mathbb{P}$-decreasing sequence of elements of $\mathbb{P}$ such that u_n is in all open dense subsets of $\mathbb{P}$ which are in t_n;

(iv) $u_0 \Vdash_\mathbb{P} \check{V}[\tau] \models \varphi[\tau, \check{x}]$.

(v) u_n determines the value of $\tau \restriction n$, which is equal to $s \restriction n$, for every $n < \omega$;

Now, whenever $\langle x, y, z\rangle \in [T]$ we have that $Z := \cup \operatorname{ran} y$ is a good elementary substructure of V_λ, $g := \{p \in \mathbb{P} \mid \exists q \in \operatorname{ran} y : q \leqslant_\mathbb{P} p\} \subseteq Z \cap \mathbb{P}$ is a Z-generic filter and $x = \tau_g$ is a real. These satisfy that, setting $\bar{Z}$ to be the transitive collapse of Z and $\bar{x}$ the image of x under this collapse, $\bar{Z}[r] \models \varphi[r, \bar{x}]$. Furthermore define $\tilde{T}$ to be the same tree, but where condition (iv) is changed to $u_0 \Vdash_\mathbb{P} \check{V}[\tau] \models \neg\varphi[\tau, \check{x}]$.

We then claim that $A = p[T] = \neg p[\tilde{T}]$. First of all, as Z is good, $\bar{Z}[r] \models \varphi[r, \bar{x}]$ holds iff $\psi[r, y]$, giving us that $p[T] \subseteq A$. Given any $r \in A$ we can also just take the trivial forcing $\mathbb{P}$ and its name for r, showing that $r \in p[T]$. The second equality is analogous. It remains to show that that T and $\tilde{T}$ still project to complements in any δ-small forcing extension, which is shown in the following claim.

Claim 4.1.2. $\Vdash_{\operatorname{Col}(\omega,<\delta)} p[\check{T}] = \neg p[\check{\tilde{T}}]$.

Proof of claim. Firstly note that $p[T] \cap p[\tilde{T}] = \varnothing$, and as we can witness this fact by the wellfoundedness of the tree of attempts to get a common branch of the two, this fact is absolute between all forcing extensions. Let now $g \subseteq \operatorname{Col}(\omega, < \delta)$ be V-generic; we want to show that given any real $r \in V[g]$, either $r \in p[T]$ or $r \in p[\tilde{T}]$. Using that $\operatorname{Col}(\omega, < \delta)$ has the δ-cc we can fix some $\kappa < \delta$ such that $h := g \cap \operatorname{Col}(\omega, < \kappa)$ is V-generic with $r \in V[h]$.

First suppose that $V[r] \models \varphi[r, x]$. Then we can build a tree T as before starting with $t_0 := \{\operatorname{Col}(\omega, < \kappa), \tau\}$ where $\tau_h = r$, so that $r \in p[T]$. As $\operatorname{Col}(\omega, < \kappa) \in V[g]$ this tree can be built within $V[g]$, so that this fact holds within $V[g]$ as well. Analogously, if $V[r] \models \neg\varphi[r, x]$ then we get that $r \in p[\tilde{T}]$ inside $V[g]$. $\dashv$

As every δ-small forcing notion is absorbed by $\mathrm{Col}(\omega, < \delta)$, we can then conclude that A is δ-uB. ■

Corollary 4.2. *If $\delta_0 < \delta_1 < \delta_2$ are Woodins and A is a set of reals such that there exists a real r satisfying that $\Vdash_{\mathbb{P}} A = \{x \in \mathbb{R}^{\check{V}} \mid \varphi[x, \check{r}]\}$ for any δ_2-small forcing notion $\mathbb{P}$, then A is determined.*

Proof. Note that the assumptions and 4.1 imply that A is δ_2-uB, so that Corollary 3.3 implies that it's determined. ■

5 From $L(\mathbb{R})$ to forcing absolute definability

Lemma 5.1. *Let $\mathcal{M}$ be a transitive model of ZFC, κ a strong limit cardinal in $\mathcal{M}$ and $\sigma \subseteq \mathbb{R}$ be a set of reals each generic over $\mathcal{M}$ such that $\sigma = \mathbb{R} \cap \mathcal{M}(\sigma)$. Then*

In some generic extension of $\mathcal{M}$ there is an $\mathcal{M}$-generic
$H \subseteq \mathrm{Col}(\omega, < \kappa)$ such that $\sigma = \bigcup_{\alpha<\kappa} \mathbb{R} \cap \mathcal{M}[H \cap \mathrm{Col}(\omega, < \alpha)]$

iff every $x \in \sigma$ is $\mathcal{M}$-generic for some κ-small forcing and $\kappa = \sup\{\aleph_1^{\mathcal{M}[x]} \mid x \in \sigma\}$.

Proof. The forward direction is clear, so assume the latter statement. Define

$$\mathbb{P} := \bigcup_{x\in\sigma} \{g \in \mathcal{M}[x] \mid \exists \alpha < \kappa : g \subseteq \mathrm{Col}(\omega, < \alpha) \text{ is } \mathcal{M}\text{-generic}\},$$

ordered by extension. Then $\mathbb{P} \in \mathcal{M}(\sigma)$ and since every $\alpha < \kappa$ is countable in $\mathcal{M}[x]$ for some $x \in \sigma$ and κ is a strong limit, $\mathbb{P} \neq \emptyset$. Now assume that $G \subseteq \mathbb{P}$ is $\mathcal{M}(\sigma)$-generic and set $H := \bigcup G$, making $H \subseteq \mathrm{Col}(\omega, < \kappa)$ a filter.

Claim 5.1.1. $H \subseteq \mathrm{Col}(\omega, < \kappa)$ is an $\mathcal{M}$-generic filter.

Proof of claim. Let $D \subseteq \mathrm{Col}(\omega, < \kappa)$ be dense in $\mathcal{M}$ and define

$$D' := \{g \in \mathbb{P} \mid \exists p \in D : p \in g\}.$$

We'll first show that D' is dense in $\mathbb{P}$. Pick $g \in \mathbb{P}$ and $\eta < \kappa$ such that g is $\mathcal{M}$-generic for $\mathrm{Col}(\omega, < \eta)$. As g is $\mathcal{M}$-generic and $\{p \cap \mathrm{Col}(\omega, < \eta) \mid p \in D\}$ is dense in $\mathrm{Col}(\omega, < \eta)$, we can find some $p \in D$ such that $p \cap \mathrm{Col}(\omega, < \eta) \in g$.

Pick some $\eta' \in [\eta, \kappa)$ such that $p \in D \cap \mathrm{Col}(\omega, <\eta')$. As $\mathrm{Col}(\omega, <\eta)$ regularly embeds into $\mathrm{Col}(\omega, <\eta')$ and $2^{\eta'} < \delta$ in $\mathcal{M}$, we can find some $y \in \sigma$ and an $\mathcal{M}$-generic $g' \subseteq \mathrm{Col}(\omega, <\eta')$ in $\mathcal{M}[y]$ extending g with $p \in g'$. This shows that D' is dense in $\mathbb{P}$. Now, genericity of G implies that there is a $g \in G$ with $g \cap D \neq \varnothing$; i.e. that there is some $p \in H \cap D$, which shows that H is $\mathcal{M}$-generic. $\dashv$

Now note that $\bigcup_{\alpha<\kappa} \mathbb{R} \cap \mathcal{M}[H \cap \mathrm{Col}(\omega, <\alpha)] \subseteq \sigma$, which follows from $H \cap \mathrm{Col}(\omega, <\alpha) \in \mathcal{M}(\sigma)$ for every $\alpha < \kappa$ and that $\mathbb{R} \cap \mathcal{M}(\sigma) = \sigma$ by assumption. It remains to show the other inclusion, i.e. that

$$\sigma \subseteq \bigcup_{\alpha<\kappa} \mathbb{R} \cap \mathcal{M}[H \cap \mathrm{Col}(\omega, <\alpha)]. \tag{1}$$

Suppose $x \in \sigma$ and define $D_x := \{g \in \mathbb{P} \mid x \in \mathcal{M}[g]\}$. We first show that D_x is dense in $\mathbb{P}$. Let $g \in \mathbb{P}$ and fix $y \in \sigma$ and $\eta < \delta$ so that $g \in \mathcal{M}[y]$ and $g \subseteq \mathrm{Col}(\omega, <\eta)$ is $\mathcal{M}$-generic. If $x \notin \mathcal{M}[g]$ then x is in some η'-small generic extension of $\mathcal{M}[g]$, for some $\eta' < \kappa$. Let $z \in \sigma$ be such that $\mathcal{M} \cap V_{\eta'+1}$ is countable in $\mathcal{M}[z]$ and $x, y \in \mathcal{M}[z]$.

Then there is an $\mathcal{M}$-generic $g' \subseteq \mathrm{Col}(\omega, <\eta')$ in $\mathcal{M}[z]$ extending g such that $x \in \mathcal{M}[g']$. This shows that D_x is dense in $\mathbb{P}$. But then we can pick some $g \in G$ such that $x \in \mathcal{M}[g]$. As $g \subseteq H \cap \mathrm{Col}(\omega, <\alpha)$ for some $\alpha < \kappa$ by definition of g and H, equality holds as both of them are generics, showing (1). This finishes the proof. ■

Say that a **successor Woodin** is a Woodin which isn't a limit of Woodins.

Lemma 5.2. *Let κ be a limit of Woodins and let a be the set of countable $X \prec V_{\kappa+1}$ such that for some $\gamma < \kappa$ it holds that given any successor Woodin $\delta \in X \cap (\gamma, \kappa)$, X captures every dense subset of $\mathbb{Q}_{<\delta}$ in X. Then given any stationary set $z \in V_\delta$ containing only countable sets, the set of $X \in a$ such that $z \in X$ and $X \cap (\cup z) \in z$ is stationary.*

Proof. Let $z \in \mathbb{Q}_{<\delta}$ and fix some function $F : [V_{\kappa+1}]^{<\omega} \to V_{\kappa+1}$. It will suffice to find a countable $Y \prec V_{\delta+2}$ such that $F, z \in Y$, $Y \cap (\cup z) \in z$ and $Y \cap V_{\kappa+1} \in a$. Fix some inaccessible $\gamma_0 < \delta$ with $z \in \mathbb{Q}_{<\gamma_0}$, and let W be the set of successor Woodins in (γ_0, κ).

We will construct a sequence $\langle Y_\alpha \mid \alpha < \omega_1 \rangle$ of countable elementary submodels of $V_{\delta+\omega}$ and a sequence $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ of elements of W such that $\{z, F, \gamma_0\} \in Y_0$, $Y_0 \cap (\cup z) \in z$ and for each $\alpha < \omega_1$,

(i) $Y_0 \subseteq Y_\alpha$ and $Y_\alpha \cap V_{\gamma_0} = Y_0 \cap V_{\gamma_0}$;
(ii) δ_α is the α'th member of $Y_\alpha \cap W$;

(iii) For every $\beta \in (\alpha, \omega_1)$, if $\chi := \gamma_0 \cup \sup(W \cap \delta_\alpha)$ then $Y_\beta \cap V_\chi = Y_\alpha \cap V_\chi$;

(iv) If δ is among the first α many members of $Y_\alpha \cap W$ then Y_α captures every predense subset of $\mathbb{Q}_{<\delta}$ in Y_α.

Stationarity of z implies that such a Y_0 exists. Also, (Larson, 2004, 2.7.12) assures the existence of $Y_{\alpha+1}$ given Y_α. Take unions at limit stages. Conditions (ii) and (iii) implies that for each $\beta < \omega_1$, $\langle \delta_\alpha \mid \alpha < \beta \rangle$ lists the first β elements of $Y_\beta \cap W$.

It remains to show that $Y_\alpha \cap V_{\kappa+1} \in a$ for some $\alpha < \omega_1$. If it wasn't the case then there is a least $\delta \in W$ such that $\delta \in Y^* := \bigcup_{\alpha<\omega_1} Y_\alpha$ but for no $\alpha < \omega_1$ is δ among the first α many members of W in Y_α. Fix some $\alpha_0 < \omega_1$ such that $\delta \in Y_{\alpha_0}$. Condition (iii) implies that $W \cap Y^* \cap \delta$ has ordertype ω_1. But a standard catch-up argument shows that there is a countable limit ordinal $\alpha > \alpha_0$ such that $W \cap Y_\alpha \cap \delta$ consists of the first α many members of $W \cap Y^* \cap \delta$. This implies that $\delta = \delta_\alpha$, $\lightning$. ■

Theorem 5.3. *Assume κ is a limit of Woodins and $\lambda > \kappa$ is a measurable. Then given any κ-small forcing $\mathbb{P}$ it holds that $\Vdash_{\mathbb{P}} (\mathbb{R}^\sharp)^{\check{V}} = \mathbb{R}^\sharp \cap \check{V}$.*

Proof. Define a as the set of countable $X \prec V_{\kappa+1}$ such that for some $\gamma \in X \cap \kappa$ it holds that given any successor Woodin $\delta \in X \cap (\gamma, \kappa)$, X captures every predense $D \subseteq \mathbb{Q}_{<\delta}$ in X. Then $a \in \mathbb{P}_{<\lambda}$ and by Lemma 5.2, defining for $z \in \mathbb{Q}_{<\delta}$ the sets $a_z := \{X \in a \mid z \in X \wedge X \cap (\cup z) \in z\}$, a_z is stationary in $\mathscr{P}_{\aleph_1} V_{\kappa+1}$.

Now let $g \subseteq \mathbb{P}_{<\lambda}$ be V-generic with $a \in g$ and $j : V \to \langle M, E \rangle$ the associated embedding. As λ is measurable and hence an inaccessible limit of completely Jónsson cardinals, $j(\lambda) = \lambda$ (Larson, 2004, 2.3.5). The idea is now to show that $(\mathbb{R}^{\mathcal{M}})^\sharp$ exists in $V[g]$, but we want to use the measurable λ to do this, and that requires that $|\mathbb{R}^{\mathcal{M}}| < \lambda$, which isn't necessarily true. The plan is then to find some suitable substructure $\mathcal{M}^*$ of $\mathcal{M}$ which *does* have this property.

Letting $f : a \to V_{\kappa+1}$ associate to each $X \in a$ the least $\gamma \in X \cap \kappa$ witnessing that $X \in a$, normality ensures that there is an $a' \subseteq a$ in g and $\gamma_g < \kappa$ such that f is constant on a' with value γ_g. Set W to be the set of successor Woodins in (γ_g, κ) and for each $\delta \in W$ let $g_\delta := g \cap \mathbb{Q}_{<\delta}$.

Now as $a' \in g$, every $g_\delta \subseteq \mathbb{Q}_{<\delta}$ is V-generic (Larson, 2004, 2.7.14). For $\delta \in W$ let $j_\delta : V \to \mathcal{M}_\delta \subseteq V[g_\delta]$ be the associated embedding, $k_\delta : \mathcal{M}_\delta \to \langle \mathcal{M}, E \rangle$ the induced factorisation and $k_{\delta_0\delta_1} : \mathcal{M}_{\delta_0} \to \mathcal{M}_{\delta_1}$ the natural map for $\delta_0 < \delta_1$.

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "j_{\delta_0}"'] \arrow[dr, "k_{\delta_0}"] & \langle \mathcal{M}, E \rangle \\
\mathcal{M}_{\delta_0} \arrow[r, "k_{\delta_0\delta_1}"'] \arrow[ur, "j_{\delta_1}"'] & \mathcal{M}_{\delta_1} \arrow[u, "k_{\delta_1}"']
\end{tikzcd}$$

Now let $\langle \mathcal{M}^*, E^* \rangle$ be the direct limit of the $\mathcal{M}_\delta$'s along the $k_{\delta_0\delta_1}$'s, and for $\delta \in W$ let $\iota_\delta : \mathcal{M}_\delta \to \langle \mathcal{M}^*, E^* \rangle$ be the coprojection. Note that $\mathcal{M}_\delta$ is the transitive collapse of

$$Z_\delta := \{j(f)(j\text{“} \cup b) \mid f \in {}^bV \cap V \wedge b \in g\},$$

and as $\delta_0 < \delta_1$ implies $Z_{\delta_0} \subseteq Z_{\delta_1}$, $k_{\delta_0\delta_1}$ is the collapse of the inclusion map for $\delta_0 < \delta_1$, so that we get a map $j^* : V \to \langle \mathcal{M}^*, E^* \rangle$ by $j^*(x) := j_\delta(x)$ for sufficiently large δ, yielding the following commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "j_\delta"'] \arrow[dr, "j^*"] & \langle \mathcal{M}, E \rangle \\
\mathcal{M}_\delta \arrow[r, "\iota_\delta"'] & \langle \mathcal{M}^*, E^* \rangle \arrow[u, "k"']
\end{tikzcd}$$

Now $\mathbb{R}^{\mathcal{M}^*} = \bigcup_{\delta \in W} \mathbb{R}^{\mathcal{M}_\delta}$ and for each δ, $\mathbb{R}^{\mathcal{M}_\delta} = \mathbb{R}^{V[g_\delta]} = \mathbb{R}^{V[g \cap V_\delta]}$. This means that every $x \in \mathbb{R}^{\mathcal{M}^*}$ is then in some κ-small forcing extension and furthermore we also have that $\kappa = \sup\{\aleph_1^{V[x]} \mid x \in \mathbb{R}^{\mathcal{M}^*}\}$ because $\aleph_1^{V[g_\delta]} = j_\delta(\aleph_1) = \delta$ for every $\delta \in W$. These two facts then imply by Lemma 5.1 that we can find a V-generic $h \subseteq \mathrm{Col}(\omega, <\kappa)$ such that

$$\mathbb{R}^{\mathcal{M}^*} = \bigcup_{\alpha<\kappa} \mathbb{R} \cap V[h \cap \mathrm{Col}(\omega, <\alpha)].$$

As λ is still measurable in $V[h]$ as it's a λ-small forcing, both $(\mathbb{R}^V)^\sharp$ and $(\mathbb{R}^{\mathcal{M}^*})^\sharp$ exists in $V[h]$, the latter being because $|\mathbb{R}^{\mathcal{M}^*}| < \lambda$. This means that every $V[h]$-regular V-cardinal $> |\mathbb{R}^V|$ is an $\mathbb{R}^V$-indiscernible and every $V[h]$-regular $V[h]$-cardinal $> |\mathbb{R}^{\mathcal{M}^*}|$ is an $\mathbb{R}^{\mathcal{M}^*}$-indiscernible.

Noting that λ is a limit of completely Jónsson cardinals (in V), cofinally many of these are fixed points of j^* and that these are still regular in $V[h]$, we get that $(\mathbb{R}^V)^\sharp \subseteq (\mathbb{R}^{\mathcal{M}^*})^\sharp$, using that $j^*(\lambda) = \lambda$ implies that $j^* \restriction L_\lambda(\mathbb{R}^V) : L_\lambda(\mathbb{R}^V) \to L_\lambda(\mathbb{R}^{\mathcal{M}^*})$. This means that $(\mathbb{R}^V)^\sharp = (\mathbb{R}^{\mathcal{M}^*})^\sharp \cap V$, so since any κ-small forcing is absorbed by

$\mathrm{Col}(\omega, <\alpha)$ for some $\alpha < \kappa$, any such forcing extension satisfies $(\mathbb{R}^V)^\sharp = \mathbb{R}^\sharp \cap V$. ■

Theorem 5.4 (The Main Theorem 0.1). *Assume there is a limit of Woodins with a measurable above. Then $\mathsf{AD}^{L(\mathbb{R})}$ holds.*

Proof. Let κ be the limit of Woodins and let $A \in L(\mathbb{R})$ be a set of reals. The above theorem implies that $\mathbb{R}^\sharp$ exists, so that there exists some $\langle \varphi, \vec{r}, \vec{\alpha} \rangle \in \mathbb{R}^\sharp$ such that $A = \{x \in \mathbb{R} \mid \varphi[x, \vec{r}, \vec{\alpha}]\}$. The theorem also implies that going to any κ-small forcing extension, A will still be defined using the same formula and parameters. By Corollary 4.2 we then get that A is determined. ■

References

Kanamori, A. (2009). *The Higher Infinite*. Springer-Verlag Berlin, second edition.

Larson, P. B. (2004). *The Stationary Tower - Notes on a Course by W. Hugh Woodin*. University Lecture Series Volume 32, American Mathematical Society.

Martin, D. and Steel, J. (1989). A proof of projective determinacy. *Journal of the American Mathematical Society*, 2(1):71–125.