Classification of covering spaces

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Abstract. asd

1 Basic results

We start off with recalling the basic definitions and the two lifting theorems regarding covering spaces.

Definition 1.1. Let X be a topological space. Then $p : \tilde{X} \to X$ is a **covering map** and $\tilde{X}$ a **covering space** if p is a surjective continuous map such that for every $x \in X$ there exists some open neighborhood $U \subseteq X$ around x which satisfies that $p^{-1}(U) = \coprod_i \tilde{U}_i$, where $\tilde{U}_i \subseteq \tilde{X}$ are open sets and $p \restriction \tilde{U}_i$ is a homeomorphism onto U. $\circ$

Proposition 1.2. *Every covering map is a quotient map.* $\blacksquare$

Theorem 1.3. *Let $p : \tilde{X} \to X$ be a covering map, A a topological space and $h : A \times I \to X$ a homotopy. Given a diagram*

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\
A \times I \arrow[r, "h"'] \arrow[ur, dotted, "\tilde{h}"'] & X
\end{tikzcd}$$

there is a unique $\tilde{h} : A \times I \to \tilde{X}$ such that both above triangles commute. $\blacksquare$

For the special case where A is the trivial space, we get the *unique path lifting theorem*: given any $\tilde{x} \in \tilde{X}$ and any path $u : I \to X$ in X starting at $u(\tilde{x})$, there's a unique lift of u to the path $\tilde{x} \cdot u : I \to \tilde{X}$, starting at $\tilde{x}$.

Definition 1.4. A topological space X is **path connected** (abbreviated pc) if there exists a path between every two points in X. X is **locally path connected** (abbreviated lpc) if X has a basis of path connected sets. $\circ$

Theorem 1.5. *Let $p : \tilde{X} \to X$ be a covering map, A a topological space and $f : (A, a_0) \to (X, x_0)$ a pointed map. If A is path connected and locally path connected then $f_*\pi_1(A, a_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ iff there is a unique map $\tilde{f} : (A, a_b) \to (\tilde{X}, \tilde{x}_0)$ where $\tilde{x}_0 \in p^{-1}(\{x_0\})$ such that the following diagram commutes:*

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (A, a_0) \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

■

2 Category of covering spaces

Definition 2.1. Let X be a topological space. Then $\mathsf{Cov}(X)$ is the category with covering maps over X as objects and a map between $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ is a continuous map $h : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ h$. $\circ$

Thus, $\mathsf{Cov}(X)$ is a full subcategory of the slice category Top/X. Recall that the **fundamental groupoid** $\pi(X)$ of a topological space X is the category with objects the elements of X and arrows $f : x \to y$ being path homotopy classes of paths between x and y.

Definition 2.2. Let $p : \tilde{X} \to X$ be a covering map. Then the **monodromy functor** of p is the functor $F_p : \pi(X) \to \mathsf{Set}$ defined as

- $F_p(x) := p^{-1}(\{x\})$;
- $F_p(u : x \to y) := (\tilde{x} \mapsto (\tilde{x} \cdot u)(1))$.

$\circ$

Proposition 2.3. *The monodromy functor F_p is a functor for every covering map $p : \tilde{X} \to X$.*

Proof. Follows from the uniqueness of the lifted paths. ■

Definition 2.4. For X a topological space, define $\Phi_X : \mathsf{Cov}(X) \to \mathsf{Set}^{\pi(X)}$ as

- $\Phi_X(p : \tilde{X} \to X) := F_p$;
- $\Phi_X(h : p \to p') := (\tau_x : \tilde{x} \mapsto h\tilde{x})$.

$\circ$

Proposition 2.5. *Φ_X is a well-defined functor for every topological space X.*

Proof. To show that Φ_X is well-defined, we only need to show that $\tau := \Phi_X(h : p_1 \to p_2)$ *is* in fact a natural transformation $\tau : F_{p_1} \Rightarrow F_{p_2}$. But we have the commutative diagram

$$\begin{tikzcd}
p_1^{-1}(\{x\}) \arrow[rrr, "\tau_x"] \arrow[ddd, "F_{p_1}(u)"'] & & & p_2^{-1}(\{x\}) \arrow[ddd, "F_{p_2}(u)"] \\
& \tilde{x} \arrow[r, mapsto] \arrow[d, mapsto] & h\tilde{x} \arrow[d, mapsto] & \\
& (\tilde{y} \cdot u)(1) \arrow[r, mapsto] & h((\tilde{y} \cdot u)(1)) = (h\tilde{y} \cdot u)(1) & \\
p_1^{-1}(\{y\}) \arrow[rrr, "\tau_y"'] & & & p_2^{-1}(\{y\})
\end{tikzcd}$$

The equality is due to uniqueness of path lifting. Hence Φ_X is well-defined, and it's clear that it's a functor. ■

Definition 2.6. A topological space X is **semi-locally simply connected** (abbreviated slsc) if any neighborhood around any point $x \in X$ contains a neighborhood U of x such that any loop at x in U is contractible in X. Equivalently, $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is the zero map $\circ$

Definition 2.7. Given a functor $F : \pi(X) \to \mathsf{Set}$, define

- $\tilde{X}_F := \coprod_{x \in X} F(x)$;
- $p_F : \tilde{X}_F \to X$, where p_F" $F(x) := \{x\}$.

$\circ$

Lemma 2.8. *Let X be a topological space and $F : \pi(X) \to \mathsf{Set}$ a functor. If X is slsc and lpc then there is a topology on $\tilde{X}_F$ such that p_F is a covering map.*

Proof. Let $x \in X$ and $U \subseteq X$ an open pc neighborhood of x such that any loop in U on x is nullhomotopic in X. This implies that there's a unique path homotopy class u_y from x to any $y \in U$. Define $f : U \times F(x) \to p_F^{-1}(U)$ as $f(y, z) := F(u_y)(z)$. Then f is bijective as $f \restriction (\{y\} \times F(x))$ for every $y \in U$ and $f = \coprod_{y \in U} f \restriction (\{y\} \times F(x))$.

For each $z \in F(x)$ set $(U, z) := f$" $(U \times \{z\})$. By assumption, X has a basis of such U as described above. Now the sets (U, z) form a basis for a topology on $\tilde{X}$. Indeed, assume $w \in (U, z) \cap (V, z')$. Then there is some $y \in U$ and $y' \in V$ such that $F(u_y)(z) = w = F(u_{y'})(z')$. But then $w \in F(y) \cap F(y')$, meaning $y = y' \in U \cap V$ as the fibers under F are disjoint. Now $F(u_y)(z) = F(u_y)(z')$, so $z = z'$ as $F(u_y)$ is a bijection. Thus $w \in (U \cap V, z) \subseteq (U, z) \cap (V, z')$.

Clearly $p_F^{-1}(U) = \coprod_{y \in U} \tilde{U}_y$ with $p_F \restriction \tilde{U}_y$ a homeomorphism, so p_F is also continuous and hence a covering map. ■

Definition 2.9. Let X be a topological space which is both slsc and lpc. Then define $\Psi_X : \mathsf{Set}^{\pi(X)} \to \mathsf{Cov}(X)$ given by

- $\Psi_X(F) := p_F$;
- $\Psi_X(\tau : F \Rightarrow G) := (\coprod_{x \in X} \tau_x : \coprod_{x \in X} F(x) \to \coprod_{x \in X} G(x))$.

$\circ$

Theorem 2.10. *Assume X is a topological space which is both slsc and lpc. Then*

$$\Phi_X : \mathit{Cov}(X) \cong \mathit{Set}^{\pi(X)}.$$

Proof. We have that

- $\Phi_X\Psi_X(F) = \Phi_X(p_F) = F_{p_F} = F$;
- $\Phi_X\Psi_X(\tau : F \Rightarrow G) = \Phi_X(\coprod_{x\in X}\tau_x) = (\tau'_x : \tilde{x} \mapsto \coprod_{x\in X}\tau_x(\tilde{x})) = \tau$;
- $\Psi_X\Phi_X(p) = \Psi_X(F_p) = p_{F_p} = p$;
- $\Psi_X\Phi_X(h : p \to p') = \Psi_X(\tau_x : \tilde{x} \mapsto h\tilde{x}) = \coprod_{x\in X}\tau_x = h$.

Hence it only remains to check that the topologies on $\tilde{X}$ and $\coprod_{x\in X} F_p(x) = \coprod_{x\in X} p^{-1}(\{x\})$ are the same. **missing!** ■

Definition 2.11. Let G be a group. Then $G\mathsf{Set}$ is the category of G-sets and G-maps between them. ○

Proposition 2.12. *Let X be a topological space which is pc, lpc and slsc. Then*

$$\mathit{Cov}(X) \simeq \pi_1(X, x_0)\mathit{Set}$$
$$(p : \tilde{x} \to x) \mapsto p^{-1}(\{x_0\})$$

is an equivalence of categories.

Proof. The inclusion $\pi_1(X, x_0) \to \pi(X)$ is an equivalence since it's essentially surjective as X is pc and clearly fully faithful. Then the induced functor

$$\mathsf{Set}^{\pi(X)} \to \mathsf{Set}^{\pi_1(X,x_0)}$$

is also an equivalence as functors preserve equivalences. But notice that

$$\mathsf{Set}^{\pi_1(X,x_0)} \cong \pi_1(X, x_0)\mathsf{Set}$$

via the functors $G : \mathsf{Set}^{\pi_1(X,x_0)} \to \pi_1(X,x_0)\mathsf{Set} : H$ given by

- $G(F) := F(x_0)$;
- $G(\tau : F \Rightarrow F') := (\tau_{x_0} : F(x_0) \to F'(x_0))$;
- $H(A)(x_0) := A$ and $H(A)(g : x_0 \to x_0) := (a \mapsto a.g)$;
- $H(f : A \to B) := (\tau_{x_0} := f)$.

missing details!

Hence we now have that

$$\mathsf{Cov}(X) \cong \mathsf{Set}^{\pi(X)} \simeq \mathsf{Set}^{\pi_1(X,x_0)} \cong \pi_1(X,x_0)\mathsf{Set}.$$

■

Definition 2.13. Let G be a group. Let $\mathsf{Conj(G)}$ be the category consisting of subgroups of G and arrows $\mathrm{Inn}(H_2 n) : H_1 \to H_2$, where $nH_1n^{-1} \subseteq H_2$ and $\mathrm{Inn}(H_2 n)$ is conjugation with the unique element in $\{x \in H_2 n \mid \forall h \in H_2 - \{1\} : h \nmid x\}$. ○

Definition 2.14. Let $\mathsf{Cov}_0(X)$ be the full subcategory of $\mathsf{Cov}(X)$ generated by all the pc covering spaces over X. ○

We aim to prove the following:

Theorem 2.15. *Let X be a topological space which is pc, lpc and slsc. Then*

$$\mathit{Cov}_0(X) \simeq \mathit{Conj}(\pi_1(X,x_0))$$

is an equivalence of categories.

To get there, however, requires working through some more categories.

Definition 2.16. The **orbit category** $\mathcal{O}_G$ of a group G is the full subcategory of $G\mathsf{Set}$, generated by the transistive G-sets, i.e. the ones with a single orbit. ○

Lemma 2.17. *Let X be a topological space which is pc, lpc and slsc. Then*

$$\begin{aligned} \mathsf{Cov}_0(X) &\simeq \mathcal{O}_{\pi_1(X,x_0)} \\ (p : \tilde{X} \to X) &\mapsto p^{-1}(\{x_0\}) \end{aligned}$$

is an equivalence of categories.

Proof. We show that $\tilde{X}$ is pc iff $p^{-1}(\{x_0\})$ is a transitive $\pi_1(X, x_0)$-set with action $\tilde{x}.u := (\tilde{x} \cdot u)(1)$, because then we can just restrict the equivalence from Proposition 2.12 to $\mathsf{Cov}_0(X)$ and we're done. ∎