Přikrý forcing theory II

Definition. Let κ be a singular cardinal and $\langle \kappa_i \mid i < \operatorname{cof}\kappa \rangle$ an increasing sequence of regular cardinals cofinal in κ. [Often we'll assume $\operatorname{cof}\kappa < \kappa_0$.] Then $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a scale in $\prod_i \kappa_i$ if

- $\forall \alpha < \kappa^+ : f_\alpha \in \prod_i \kappa_i$;
- $\forall \alpha < \beta < \kappa^+ : f_\alpha <^* f_\beta$;
- $\forall f \in \prod_i \kappa_i \, \exists \alpha \in \kappa^+ : f <^* f_\alpha$. $\dashv$

Fact (Shelah). If κ is singular then for any increasing cofinal in κ sequence of regular cardinals $\langle \kappa_i \mid i < \operatorname{cof}\kappa \rangle$ with $\operatorname{cof}\kappa < \kappa_0$, there's a scale in $\prod_i \kappa_i$. $\dashv$

This is a non-trivial ZFC theorem — see Eisworth's handbook chapter, or Burke-Magidor (~'92) in APAL.

Definition. $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a very good scale in $\prod_i \kappa_i$ if it's a scale and for every $\alpha < \kappa^+$, if $\operatorname{cof} \alpha \in (\operatorname{cof}\kappa, \kappa)$ then there's a club $C \subseteq \kappa$ and $i < \operatorname{cof}\kappa$ such that for every $\beta, \gamma \in C$ and $j \in (i, \operatorname{cof}\kappa)$,

$$\beta < \gamma \Rightarrow f_\beta(j) < f_\gamma(j). \qquad \dashv$$

Definition. Let λ, κ be cardinals, $\lambda \leq \kappa$. Then $\square_{\kappa,\lambda}$ holds if there's a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \text{ limit} \rangle$ such that for each α,

1) $\emptyset \neq \mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{C}_\alpha| \leq \lambda$ and $C \in \mathcal{C}_\alpha$ implies C club in α;
2) $\operatorname{cof} \alpha < \kappa \Rightarrow \forall C \in \mathcal{C}_\alpha : \operatorname{ot}(C) < \kappa$;
3) $\forall C \in \mathcal{C}_\alpha \, \forall \beta \in \lim(C) : C \cap \beta \in \mathcal{C}_\beta$ $\dashv$

Theorem (Cummings-Foreman-Magidor). Let κ be singular, $\lambda < \kappa$ and assume $\square_{\kappa,\lambda}$ holds. Then there's a very good scale on κ — i.e. on some cofinal sequence $\langle \kappa_i \mid i < \operatorname{cof} \kappa \rangle$.

Proof. Pick an increasing cofinal $\vec{\kappa_i}$ in κ with $\kappa_0 > \operatorname{cof}(\kappa) + \lambda$. Let $\vec{\mathcal{C}_\alpha}$ witness $\square_{\kappa,\lambda}$. Build $\langle g_\alpha \mid \alpha < \kappa^+ \rangle$ recursively, where we inductively make sure that $\forall \alpha < \kappa^+ : f_\alpha < g_\alpha$, with $\vec{f_\alpha}$ being the scale in $\prod_i \kappa_i$.

For $\alpha = 0$ simply let $g_0 > f_0$. For successors we again simply choose $g_{\alpha+1} > f_{\alpha+1}, g_\alpha$. Assume lastly that α is a limit. We need to ensure that

a) $f_\alpha < g_\alpha$
b) $\forall \beta < \alpha : g_\beta <^* g_\alpha$
c) $\forall i < \operatorname{cof} \kappa : \sup\{\sup\{g_\beta(i) \mid \beta \in C\} \mid C \in \mathcal{C}_\alpha \wedge |C| < \kappa_i\} < g_\alpha(i)$

Points (a)+(b) are simple to achieve, and (c) is where we use regularity of κ_i. We now check that this works. So let $\alpha < \kappa^+$ satisfy $\operatorname{cof} \alpha \in (\operatorname{cof} \kappa, \kappa)$, and pick $C \in \mathcal{C}_\alpha$. Let $i < \operatorname{cof} \kappa$ be such that $|C| < \kappa_i$, and let $\beta, \gamma \in$ $\lim C =: C^*$

and $j\in(i,\mathrm{cof}\,\kappa)$ be given. Assume that $\beta<\gamma$. Then by (3) of the definition of $\square_{\kappa,\lambda}$, $C\cap\beta\in\mathcal{C}_\beta$ and $C\cap\gamma\in\mathcal{C}_\gamma$. Also, $\beta\in C\cap\gamma$ and $|C\cap\gamma|<\kappa_j$. We chose $g_\beta(j)<g_\gamma(j)$ in (c). $\square$

Remark. The proofs show that we (probably) need the assumption that $\square_{\kappa,\lambda}$ holds for some $\lambda<\kappa$ — $\square_{\kappa,<\kappa}$ probably won't be good enough. $\dashv$

Now assume $V\models\ulcorner\kappa$ is measurable$\urcorner$, let $\mathcal{U}$ be a measure on κ and $g\subseteq\mathbb{P}_{\mathcal{U}}$ is V-generic with Příkrý sequence $\langle\kappa_i\mid i<\omega\rangle$.

Theorem (Jech-Cummings-Foreman-Magidor). There's a very good scale in $\prod_{i<\omega}\kappa_i^+$.

Proof. Pick representatives in $\mathrm{Ult}(V,\mathcal{U})$ for the ordinals less than κ^+, i.e. $\langle f_\alpha\mid\alpha<\kappa^+\rangle$ such that $\mathrm{Ult}(V,\mathcal{U})\models[f_\alpha]_{\mathcal{U}}=\alpha$.

Define $g_\alpha(n):=f_\alpha(\kappa_n)$ for all $n<\omega$. We'll postpone showing that $\vec{g_\alpha}$ is a scale in $\prod_{i<\omega}\kappa_i^+$ and instead focus on showing that it's very good.

Fix $\alpha<\kappa^+$ such that $\omega<\mathrm{cof}(\alpha)\overset{V[g]}{=:}\lambda$, so that $\lambda<\kappa$ by singularity of κ. We saw last time that $V\models\mathrm{cof}(\alpha)=\lambda$. Choose $D\in V$ such that D is club in α with $\mathrm{ot}(D)=\lambda$. Let $A:=\{\beta<\kappa\mid\langle f_\gamma(\beta)\mid\gamma\in D\rangle$ is strictly increasing$\}$.

As $j(D)=j``D$ (since $\lambda<\kappa$), we have

$$\langle j(f)_\gamma(\kappa) \mid \gamma \in j(D)\rangle = \langle j(f_\gamma)(\kappa) \mid \gamma \in D\rangle$$
$$= \langle \gamma \mid \gamma \in D\rangle,$$

which is strictly increasing, so that $\kappa \in j(A)$ and therefore $A \in \mathcal{U}$. Then $\exists m \in \omega \forall n \geq m (\kappa_n \in A)$, so for $n \geq m$ we get $\langle f_\gamma(\kappa_n) \mid \gamma \in D\rangle = \langle g_\gamma(n) \mid \gamma \in D\rangle$ is strictly increasing, making $\vec{g_\alpha}$ a very good scale. $\square$