Remarkables are strongly unfoldable

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Theorem 0.1. *Every remarkable cardinal is strongly unfoldable.*

Proof. Let κ be remarkable, $\theta \gg \kappa$ and let $j : M \to N$ and $\pi : M \to H_\theta$ witness the remarkability of $\kappa \in \operatorname{ran} \pi$. It suffices to show that $\bar{\kappa} := \pi^{-1}(\kappa)$ is strongly unfoldable in M. Let $A \subseteq \bar{\kappa}$ be fixed, where $A \in M$. Let $\alpha \in (\bar{\kappa}, \bar{\kappa}^{+M})$ be such that $M|\alpha \models \mathsf{ZFC}^-$, $A \in M|\alpha$ and

$$M \models (M|\alpha)^{<\kappa} \subseteq M|\alpha.$$

Note that we can pick θ such that $\bar{\kappa}$ is *not* the largest cardinal in M. Let $\lambda < o(M)$ be given and let E be the $(\bar{\kappa}, \lambda)$-extender derived from $j \restriction (M|\alpha)$ and $k : \mathrm{Ult}(M|\alpha, E) \to N$ be the factor map. Then, in M, i_E and $M|\alpha$ witness unfoldability of $\bar{\kappa}$, and as $k \restriction \lambda$

$$V_\lambda^M = V_\lambda^N \subseteq \mathrm{Ult}(M|\alpha, E),$$

showing that $\bar{\kappa}$ is strongly unfoldable in M, which is what we wanted to show. ∎