Taking the blue pill: Virtual Set Theory

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Abstract

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Introduction

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Notation

We will denote the class of ordinals by On. For X, Y sets we denote by ${}^X Y$ the set of all functions from X to Y. For an infinite cardinal κ, we let H_κ be the set of sets X such that the cardinality of the transitive closure of X is $<\kappa$. ZF^- will denote ZF with the Collection scheme but without the Power Set axiom, following the results of [Gitman et al., 2015]. The symbol $\lightning$ will denote a contradiction and $\mathscr{P}(X)$ denotes the power set of X. We will denote elementary embeddings $\pi\colon (\mathcal{M}, \in) \to (\mathcal{N}, \in)$ by simply $\pi\colon\ \mathcal{M} \to \mathcal{N}$.

Part I

Fantastic Cardinals and Where to Find Them

1 | Part I Introduction

1.1 Filters on small structures

Definition 1.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa+1 \subseteq \mathcal{M}$ and $(\mathcal{M}, \in) \models \mathsf{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$, $\mathcal{M}$ is a **κ-model**.[1] $\circ$

Recall that μ is an **$\mathcal{M}$-measure** if $(\mathcal{M}, \in, \mu) \models \ulcorner\mu$ is a κ-complete ultrafilter on $\kappa\urcorner$.

Definition 1.2. Let $\mathcal{M}$ be a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

- **weakly amenable** if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with $\mathcal{M}$-cardinality κ;
- **countably complete** if $\bigcap \vec{X} \neq \emptyset$ for every ω-sequence $\vec{X} \in {}^\omega\mu$.

$\circ$

Proposition 1.3 (Folklore)**.** *Let $\mathcal{M}$ be a weak κ-model, μ an $\mathcal{M}$-measure and $j : \mathcal{M} \to \mathcal{N}$ the associated ultrapower embedding. Then μ is weakly amenable if and only if j is **κ-powerset preserving**, meaning that $\mathcal{M} \cap \mathscr{P}(\kappa) = \mathcal{N} \cap \mathscr{P}(\kappa)$.* ■

1.2 Embeddings between small structures

A key folklore lemma which we will frequently need when dealing with elementary embeddings existing in generic extensions is the following.

Lemma 1.4 (Countable Embedding Absoluteness)**.** *Let $\mathcal{M}, \mathcal{N}$ be sets, $\mathcal{P}$ a transitive class with $\mathcal{M}, \mathcal{N} \in \mathcal{P}$, and let $\pi\colon \mathcal{M} \to \mathcal{N}$ be an elementary*

[1] Note that our (weak) κ-models do not have to be transitive, in contrast to the models considered in [Gitman, 2011] and [Gitman and Welch, 2011]. Not requiring the models to be transitive was introduced in [Holy and Schlicht, 2018].

embedding. Assume that

$$\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC} + \ulcorner \mathcal{M} \text{ is countable} \urcorner$$

and fix any finite $X \subseteq \mathcal{M}$. *Then* $\mathcal{P}$ *contains an elementary embedding* $\pi^*\colon \mathcal{M} \to \mathcal{N}$ *which agrees with* π *on* X. *If* π *has a critical point and if* $\mathcal{M}$ *is transitive then we can also assume that* $\operatorname{crit}\pi = \operatorname{crit}\pi^*$.[2]

Proof. Let $\{a_i \mid i < \omega\} \in \mathcal{P}$ be an enumeration of $\mathcal{M}$ and set $\mathcal{M} \upharpoonright n := \{a_i \mid i < n\}$. Then, in $\mathcal{P}$, build the tree $\mathcal{T}$ of all partial isomorphisms between $\mathcal{M} \upharpoonright n$ and $\mathcal{N}$ for $n < \omega$, ordered by extension. Then $\mathcal{T}$ is illfounded in V by assumption, so it's also illfounded in $\mathcal{P}$ since $\mathcal{P}$ is transitive and $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC}$. The branch then gives us the embedding π^*, and if $\operatorname{crit}\pi$ exists then we can ensure that it agrees with π on the critical point and finitely many values by adding these conditions to $\mathcal{T}$. ■

We'll need the following well-known lemma.

Lemma 1.5 (Ancient Kunen Lemma)**.** *Let* κ *be regular,* $\mathcal{M}, \mathcal{N}$ *weak* κ*-models,* $\theta \in (\kappa, o(\mathcal{M}))$ *a regular* $\mathcal{M}$*-cardinal, and* $\pi\colon \mathcal{M} \to \mathcal{N}$ *an elementary embedding with* $\operatorname{crit}\pi = \kappa$ *and* $H_\theta^{\mathcal{M}} \subseteq \mathcal{N}$. *Then for every* $X \in H_\theta^{\mathcal{M}}$ *with* $\operatorname{card}^{\mathcal{M}}(X) = \kappa$ *it holds that* $\pi \upharpoonright X \in \mathcal{N}$.

Proof. Let $f\colon \kappa \to X$, $f \in \mathcal{M}$, be a bijection and note that $\pi(x) = \pi(f)(f^{-1}(x))$ for all $x \in X$, so it suffices that $f, \pi(f) \in \mathcal{N}$, which is true since $f \in H_\theta^{\mathcal{M}} \subseteq \mathcal{N}$. ■

[2] We are using transitivity of $\mathcal{M}$ to ensure that the *ordinal* $\operatorname{crit}\pi$ exists.

2 | Virtual large cardinals

In this chapter we investigate the properties of virtual versions of well-known large cardinals, including measurables, strongs, supercompacts, woodins and vopěnkas. This entails firstly analysing the relationships between them, and secondly looking at more general properties in terms of their behaviour in canonical inner models as well as their indestructibility. This virtual perspective also allows us to analyse virtualised versions of large cardinals that are otherwise inconsistent with ZFC, such as the berkeley cardinals.

2.1 Strongs & supercompacts

We start out with measurables, strongs and supercompacts. Their (non-virtual) definitions can be found in Appendix A.

Definition 2.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is...

- **faintly θ-measurable** if, in a forcing extension, there is a transitive class $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$;
- **faintly θ-strong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$;
- **faintly θ-supercompact** if it's faintly θ-measurable, ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

We further replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we attach a **"pre"** if we don't assume that $\pi(\kappa) > \theta$, and we will leave out θ when it holds for all regular $\theta > \kappa$. ○

As a quick example of this terminology, a *faintly prestrong cardinal* is a cardinal κ such that for all regular $\theta > \kappa$, κ is faintly θ-measurable with $H_\theta^V \subseteq \mathcal{N}$.

We note that even small cardinals can be faintly measurable: we may for instance have a precipitous ideal on ω_1; see [Jech, 2006, Theorem 22.33].

The "virtually" adverb implies that the cardinals are in fact large cardinals in the usual sense, as Proposition 2.2 below shows.

Proposition 2.2 (Virtualised folklore)**.** *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is 1-iterable.*

PROOF. Let κ be virtually θ-measurable, witnessed by a forcing $\mathbb{P}$, a transitive $\mathcal{N} \subseteq V$ and an elementary $\pi\colon H_\theta^V \to \mathcal{N}$ with $\pi \in V^{\mathbb{P}}$. If κ isn't a strong limit then we have a surjection $\pi(f)\colon \mathscr{P}(\alpha) \to \pi(\kappa)$ with $\operatorname{ran}\pi(f) = \operatorname{ran} f \subseteq \kappa$ for some $\alpha < \kappa$, $\lightning$. Note that we used $\mathcal{N} \subseteq V$ to ensure that $\mathscr{P}(\alpha)^V = \mathscr{P}(\alpha)^{\mathcal{N}}$. The same argument shows that κ is regular. By restricting the generic embedding and using that $\mathscr{P}(\kappa)^V = \mathscr{P}(\kappa)^N$ as $\mathcal{N} \subseteq V$ and $\mathscr{P}(\kappa)^V \subseteq \mathcal{N}$, we get that κ is 1-iterable. ■

Along with the above definition of faintly supercompactness we can also virtualise Magidor's characterisation of supercompact cardinals[1], which was one of the original characterisations of the remarkable cardinals in [Schindler, 2000a].

Definition 2.3. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-Magidor-supercompact** if there are $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ such that $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. ○

In the virtual world these two versions of supercompacts remain equivalent, but they also turn out to be equivalent to the virtually strongs:

Theorem 2.4 (Gitman-Schindler)**.** *For an uncountable cardinal κ, the following are equivalent.*[2]

(i) *κ is virtually strong;*
(ii) *κ is virtually supercompact;*
(iii) *κ is virtually Magidor-supercompact.*

PROOF. *(ii)* $\Rightarrow$ *(i)* is simply by definition.

[1] See Appendix A for the non-virtual version of this characterisation.
[2] A cardinal satisfying any/all of these conditions is usually called **remarkable**.

$(i) \Rightarrow (iii)$: Fix $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi\colon H^V_{(2^{<\theta})^+} \to \mathcal{M}$ with[3] $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H^V_{(2^{<\theta})^+} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. Since $H^V_\theta, H^{\mathcal{M}}_{\pi(\theta)} \in \mathcal{M}$, Countable Embedding Absoluteness 1.4 implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H^V_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ with $\operatorname{crit}\pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H^V_\theta = H^{\mathcal{M}}_\theta$ as $\mathcal{M} \subseteq V$ and $H^V_\theta \subseteq \mathcal{M}$, elementarity of π now implies that $H^V_{(2^{<\theta})^+}$ has ordinals $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\sigma\colon H^V_{\bar\theta} \to H^V_\theta$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa$. This shows (iii).

$(iii) \Rightarrow (ii)$: Fix $\theta > \kappa$ and $\delta := (2^{<\theta})^+$. By (iii) there exist ordinals $\bar\kappa < \bar\delta < \kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\delta} \to H^V_\delta$ with $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. We will argue that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, so that by elementarity κ is virtually θ-supercompact in H^V_δ and hence also in V by the choice of δ. Consider the restriction

$$\sigma := \pi \restriction H^V_{\bar\theta} : H^V_{\bar\theta} \to H^V_\theta.$$

Note that H^V_θ is closed under $<\bar\theta$-sequences (and more) in V. Now define

$$X := \bar\theta{+}1 \cup \{x \in H^V_\theta \mid \exists y \in H^V_{\bar\theta} \exists p \in \operatorname{Col}(\omega, H^V_{\bar\theta})\colon p \Vdash \dot\sigma(\check y) = \check x\} \in V.$$

Note that $|X| = \left|H^V_{\bar\theta}\right| = 2^{<\bar\theta}$ and that $\operatorname{ran}\sigma \subseteq X$. Now let $\overline{\mathcal{M}} \prec H^V_\theta$ be such that $X \subseteq \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{\mathcal{M}}$ of size $(2^{<\bar\theta})^{<\bar\theta} = 2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{\mathcal{M}}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}| = 2^{<\bar\theta} < \bar\delta$, making $\mathcal{M} \in H^V_{\bar\delta}$.

Countable Embedding Absoluteness 1.4 then implies that $H^V_{\bar\delta}$ has a generic elementary embedding $\sigma^*\colon H^V_{\bar\theta} \to \mathcal{M}$ with $\operatorname{crit}\sigma^* = \bar\kappa$, showing that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, which is what we wanted to show. ∎

Remark 2.5. The above proof in fact shows something stronger: if κ is virtually $(2^{<\theta})^+$-strong then it is virtually θ-supercompact, and if it's virtually

[3] The domain of π is $H^V_{(2^{<\theta})^+}$ to ensure that $H^V_\theta \in \operatorname{dom}\pi$.

$(2^{<\theta})^+$-Magidor-supercompact then it's virtually θ-supercompact. It's open whether they are equivalent level-by-level (see Question 4.1).

A key difference between the normal large cardinals and the virtual kind is that we don't have a virtual version of the Kunen inconsistency[4]: it's perfectly valid to have a generic elementary embedding $H_\theta^V \to H_\theta^V$ with θ much larger than the critical point.

Proposition 2.6 (Folklore). *If $0^\sharp$ exists then there are inaccessible cardinals $\kappa < \theta$ such that, in a generic extension of L, there is an elementary embedding $\pi\colon L_\theta \to L_\theta$. In other words, π witnesses a strong failure of the virtualised Kunen inconsistency.*

PROOF. From $0^\sharp$ we get an elementary embedding $j\colon L \to L$. Let $C \subseteq \mathsf{On}$ be the proper class club of limit points of j above $\operatorname{crit} j$, which then contains an inaccessible cardinal θ as there are stationarily many such. Restrict j to $\pi := j \restriction L_\theta \to \mathcal{N}$ and note that $\mathcal{N} = L_\theta$ by condensation and because θ is a limit point of j. Let $\kappa := \operatorname{crit} \pi$. Now an application of Countable Embedding Absoluteness 1.4 shows that a generic extension of L contains an elementary embedding $\tilde{\pi} : L_\theta \to L_\theta$ with $\operatorname{crit} \tilde{\pi} = \kappa$. ■

This becomes important when dealing with the "pre"-versions of the large cardinals. We next move to a virtualisation of the α-superstrong cardinals.

Definition 2.7 (N.). Let θ be a regular uncountable cardinal and and α an ordinal. Then a cardinal $\kappa < \theta$ is **faintly (θ, α)-superstrong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$[5]. We replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we say that κ is **faintly α-superstrong** if it's faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong.[6] ○

[4]See Appendix A for the Kunen inconsistency.
[5]Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.
[6]Note that the conventions stated here are different from the ones in Definition 2.1.

As in the non-virtual case, the virtually superstrongs supercede the virtually strongs in consistency strength. Note that this then also implies that the superstrongs are stronger than the virtually supercompacts, which is *not* the case outside the virtual world.

Proposition 2.8 (N.). *If κ is faintly superstrong then H_κ has a proper class of virtually strong cardinals.*

PROOF. Fix a regular $\theta > \kappa$ and a generic embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) < \theta$. Then $\pi(\kappa)$ is a V-cardinal, so that $H_{\pi(\kappa)}^V$ thinks that κ is virtually strong. This implies that H_κ^V thinks there is a proper class of virtually strong cardinals, using that $H_\kappa^V \prec H_{\pi(\kappa)}^V$. ■

The following theorem then shows that the only thing stopping prestrongness from being equivalent to strongness is the existence of "Kunen inconsistencies".

Theorem 2.9 (N.). *Let θ be an uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-prestrong iff either*

(i) *κ is virtually θ-strong; or*

(ii) *κ is virtually (θ, ω)-superstrong.*

PROOF. ($\Leftarrow$) is trivial, so we show ($\Rightarrow$). Let κ be virtually θ-prestrong. Assume (i) fails, meaning that there's a generic extension $V^{\mathbb{P}}$ and an elementary embedding $\pi \in V^{\mathbb{P}}$ such that $\pi\colon H_\theta^V \to \mathcal{N}$ for some transitive $\mathcal{N}$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) \leq \theta$. Assume $\pi^n(\kappa)$ is defined for all $n < \omega$ and define $\lambda := \sup_{n<\omega} \pi^n(\kappa)$. If $\lambda \leq \theta$ then κ is virtually (θ, ω)-superstrong by definition, so assume that there's some least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$.

This means that κ is virtually ν-strong for every regular $\nu \in (\kappa, \pi^n(\kappa))$, which is a Δ_0-statement in $\{H_{\nu^+}^V\}$ and hence downwards absolute to $H_{\pi^n(\kappa)}^V$. This means that κ is virtually strong in $H_{\pi^n(\kappa)}^V$ and also that $\pi^n(\kappa)$ is virtually strong in $H_{\pi^{n+1}(\kappa)}^{\mathcal{N}}$ by elementarity, and so in particular virtually θ-

strong in $\mathcal{N}$. This means that there's some generic elementary embedding

$$\sigma\colon H_\theta^{\mathcal{N}} \to \mathcal{M}$$

with $H_\theta^{\mathcal{N}} \subseteq \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{N}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. We can now restrict σ to its critical point $\pi^n(\kappa)$ to get that

$$H^V_{\pi^n(\kappa)} = H^{\mathcal{N}}_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))},$$

using that $H_\theta^V = H_\theta^{\mathcal{N}}$ holds as π is a virtual embedding. Since κ is virtually strong in $H^V_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$. In particular, κ is virtually θ-strong in $\mathcal{M}$, and as $H_\theta^{\mathcal{M}} = H_\theta^{\mathcal{N}} = H_\theta^V$, this means that κ is virtually θ-strong in V, contradicting (i). ■

We then get the following consistency result.

Corollary 2.10 (N.). *For any uncountable regular θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

Proof. The above Proposition 2.8 and Theorem 2.9 show that virtually θ-prestrongs are equiconsistent with virtually θ-strongs. Now note that Countable Embedding Absoluteness 1.4 and condensation in L imply that every faintly θ-measurable cardinal is virtually θ-prestrong in L. ■

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_\theta^V \to H_\theta^V$ with $\operatorname{crit}\pi = \kappa$. We firstly note that the virtually ω-superstrongs coincide with the virtually rank-into-ranks.

Proposition 2.11 (N.). *A regular uncountable cardinal κ is virtually ω-superstrong iff it is virtually rank-into-rank.*

Proof. If κ is virtually ω-superstrong, witnessed by a generic embedding $\pi\colon H_\theta^V \to \mathcal{N}$, then $\lambda := \sup_{n<\omega} \pi^n(\kappa)$ is well-defined. By restricting π to $\pi \restriction H_\lambda^V \colon H_\lambda^V \to H_\lambda^V$ we get a witness to κ being virtually λ-rank-into-rank.

Conversely, if κ is θ-rank-into-rank, witnessed by a generic embedding $\pi\colon H_\theta^V \to H_\theta^V$, then one readily checks that π also witnesses that κ is virtually ω-superstrong. ■

2.2 Woodins & Vopěnkas

In this section we will analyse the virtualisations of the woodin and vopěnka cardinals, which can be seen as "boldface" variants of strongs and supercompacts.

Definition 2.12. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, A)**-strong** for a set $A \subseteq H_\theta^V$ if there exists a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, A) \to (\mathcal{M}, \in, B)$$

with $\mathcal{M}$ transitive, such that $\operatorname{crit} \pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta^V \subseteq \mathcal{M}$ and $B \cap H_\theta^V = A$. We say that κ is **faintly** (θ, A)**-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$ and say that κ is **faintly** (θ, A)**-extendible** if $\mathcal{M} = H_\mu^V$ for some V-cardinal μ. We will leave out θ if it holds for all regular $\theta > \kappa$. ◦

Definition 2.13. A cardinal δ is **faintly woodin** if given any $A \subseteq H_\delta^V$ there exists a faintly $({<}\delta, A)$-strong cardinal $\kappa < \delta$. ◦

As with the previous definitions, for both of the above two definitions we substitute "faintly" for **virtually** when $\mathcal{M} \subseteq V$, and substitute "strong", "supercompact" and "woodin" for **prestrong**, **presupercompact** and **prewoodin** when we don't require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with the real woodin cardinals, virtually woodin cardinals are mahlo. This contrasts the virtually prewoodins since [Wilson, 2019a], together with Theorem 2.20 below, shows that they can be singular.

Proposition 2.14 (Virtualised folklore)**.** *Virtually woodin cardinals are mahlo.*

Proof. Let δ be virtually woodin. Note that δ is a limit of weakly compact cardinals by Proposition 2.2, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f\colon\alpha\to\delta$ with $f(0)>\alpha$ and $\alpha<\delta$, and note that f cannot have any closure points. Fix a virtually $({<}\delta,f)$-strong cardinal $\kappa<\delta$; we claim that κ is a closure point for f, which will yield our desired contradiction.

Let $\gamma<\kappa$ and choose a regular $\theta\in(f(\gamma),\delta)$. We then have a generic embedding $\pi\colon(H_\theta^V,\in,f\cap H_\theta^V)\to(\mathcal{N},\in,f^+)$ with $H_\theta^V\subseteq\mathcal{N}$, $\mathcal{N}\subseteq V$, $\operatorname{crit}\pi=\kappa$, $\pi(\kappa)>\theta$ and f^+ is a function such that $f^+\cap H_\theta^V=f\cap H_\theta^V$. But then $f^+(\gamma)=f(\gamma)<\pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma)<\kappa$, making κ a closure point for f, $\lightning$. This shows that δ is inaccessible.

As for mahloness, let $C\subseteq\delta$ be a club and $\kappa<\delta$ a virtually $({<}\delta,C)$-strong cardinal. Let $\theta\in(\min C,\delta)$ and let $\pi\colon H_\theta^V\to\mathcal{N}$ be the associated generic elementary embedding. Then for every $\gamma<\kappa$ there exists an element of C below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an element of C. As κ is regular, this shows that δ is mahlo. ■

The well-known equivalence of the "function definition" and "A-strong" definition of woodin cardinals[7] holds if we restrict ourselves to *virtually* woodins, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Proposition 2.15 (Dimopoulos-Gitman-N.)**.** *For an uncountable cardinal δ, the following are equivalent.*

(i) *δ is virtually woodin.*
(ii) *for every $A\subseteq H_\delta^V$ there exists a virtually $({<}\delta,A)$-supercompact $\kappa<\delta$.*
(iii) *for every $A\subseteq H_\delta^V$ there exists a virtually $({<}\delta,A)$-extendible $\kappa<\delta$.*

[7] See Appendix A for this characterisation of (non-virtual) woodin cardinals.

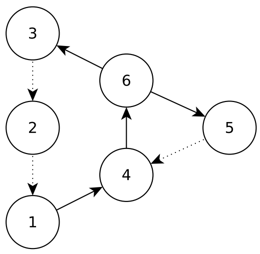

Figure 2.1: Proof strategy of Proposition 2.15, dotted lines are trivial implications.

(iv) *for every function* $f\colon\delta\to\delta$ *there are regular cardinals* $\kappa<\theta<\delta$, *where* κ *is a closure point for* f, *and a generic elementary* $\pi\colon H_\theta^V\to\mathcal{M}$ *such that* $\operatorname{crit}\pi=\kappa$, $H_\theta^V\subseteq\mathcal{M}$, $\mathcal{M}\subseteq V$ *and* $\theta=\pi(f\restriction\kappa)(\kappa)$.

(v) *for every function* $f\colon\delta\to\delta$ *there are regular cardinals* $\kappa<\theta<\delta$, *where* κ *is a closure point for* f, *and a generic elementary* $\pi\colon H_\theta^V\to\mathcal{M}$ *such that* $\operatorname{crit}\pi=\kappa$, ${}^{<\pi(f)(\kappa)}\mathcal{M}\subseteq\mathcal{M}$, $\mathcal{M}\subseteq V$ *and* $\theta=\pi(f\restriction\kappa)(\kappa)$.

(vi) *for every function* $f\colon\delta\to\delta$ *there are regular cardinals* $\bar\theta<\kappa<\theta<\delta$, *where* κ *is a closure point for* f, *and a generic elementary embedding* $\pi\colon H_{\bar\theta}^V\to H_\theta^V$ *with* $\pi(\operatorname{crit}\pi)=\kappa$, $f(\operatorname{crit}\pi)=\bar\theta$ *and* $f\restriction\kappa\in\operatorname{ran}\pi$.

Proof. Firstly note that *(iii)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i)* and *(v)* $\Rightarrow$ *(iv)* are simply by definition.

$\boxed{(i)\Rightarrow(iv)}$ Assume δ is virtually woodin, and fix a function $f\colon\delta\to\delta$. Let $\kappa<\delta$ be virtually $({<}\delta,f)$-strong and let $\theta:=\sup_{\alpha\leq\kappa}f(\alpha)+1$. Then there's a generic elementary embedding $\pi\colon(H_\theta^V,\in,f\cap H_\theta^V)\to(\mathcal{M},\in,f^+)$ where $f^+\restriction\kappa=f\restriction\kappa$, $\mathcal{M}\subseteq V$ and $\pi(\kappa)>\theta$. We firstly want to show that κ is a closure point for f, so let $\alpha<\kappa$. Then

$$f(\alpha)=f^+(\alpha)=\pi(f)(\alpha)=\pi(f)(\pi(\alpha))=\pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha<\kappa$. Now, if κ wasn't a closure point for f then, letting $\alpha<\kappa$ be the least such that $f(\alpha)\geq\kappa$,

$$\theta>f(\alpha)=\pi(f(\alpha))>\theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument wouldn't work if we had only assumed δ to be virtually prewoodin. Lastly, θ-strongness implies that $H_\theta^V \subseteq \mathcal{M}$, and $\mathcal{M} \subseteq V$ holds by assumption.

$\boxed{(iv) \Rightarrow (vi)}$ Assume (iv) holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$. By (iv) there's a $\kappa < \delta$ which is a closure point of g and there's a regular $\theta \in (\kappa, \delta)$ and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$. We want to find a regular $\bar\theta < \kappa$ and another elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f(\operatorname{crit}\sigma) = \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\sigma$.

Note that $\mathcal{M} \subseteq V$ and $H_\theta^V \subseteq \mathcal{M}$ implies that $H_\theta^V = H_\theta^{\mathcal{M}}$, so that both H_θ^V and $H_{\pi(\theta)}^{\mathcal{M}}$ are elements of $\mathcal{M}$ (we introduced g to ensure that $\pi(\theta)$ makes sense). An application of Countable Embedding Absoluteness 1.4 then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H_\theta^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$ and $\pi(f \restriction \kappa) \in \operatorname{ran}\pi^*$.

By elementarity of π, H_θ^V has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f \restriction \kappa \in \operatorname{ran}\sigma$ and $\bar\theta = f(\operatorname{crit}\sigma)$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta \to \delta$ be given. Define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$, so that by (vi) there exist regular $\bar\kappa < \bar\theta < \kappa < \theta$ such that κ is a closure point for g and there exists a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) = \bar\theta$ and $g \restriction \kappa \in \operatorname{ran}\pi$.

Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 2.4 we get a transitive $\mathcal{M} \in H_{g(\bar\kappa)}^V$ closed under $<f(\bar\kappa)$-sequences, and $H_{g(\bar\kappa)}^V$ has a generic elementary embedding $\sigma\colon H_{f(\bar\kappa)}^V \to \mathcal{M}$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa > f(\bar\kappa)$. In other words, $\bar\kappa$ is virtually $f(\bar\kappa)$-supercompact in $H_{\bar\theta}^V$. Elementarity of π then implies that κ is virtually $\pi(f)(\kappa)$-supercompact in H_θ^V, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all α such that $(H_\alpha^V, \in, A \cap H_\alpha^V) \prec (H_\delta^V, \in, A)$. Let $f\colon \delta \to \delta$ be given as $f(\alpha) = \langle \alpha_0, \alpha_1 \rangle$ with $\langle -, - \rangle$ being the Gödel pairing function, where α_0 is the first limit of elements of C above α and the α_1's are chosen such that $\{\alpha_1 \mid \alpha < \beta\}$ encodes $A \cap \beta$. This definition makes sense since δ is inaccessible by Proposition 2.2.

Let $\kappa < \delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa$, $\theta > \kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$, and $f \upharpoonright \kappa \in \operatorname{ran}\pi$. We claim that $\bar\kappa := \operatorname{crit}\pi$ is virtually $({<}\delta, A)$-extendible. To see this, it suffices by the definition of C to show that

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \models \ulcorner\bar\kappa \text{ is virtually } (A \cap H_\kappa)\text{-extendible}\urcorner, \tag{1}$$

since $\kappa \in C$ because it is a closure point of f. Let $\beta := \min(C - \bar\kappa) < \bar\theta$ and note that β exists as $f(\bar\kappa) = \bar\theta$ so the definition of f says that $\bar\theta$ is a limit of elements of C above $\bar\kappa$. It then holds that $(H^V_{\bar\kappa}, \in, A \cap H^V_{\bar\kappa}) \prec (H^V_\beta, \in, A \cap H^V_\beta)$ as both $\bar\kappa$ and β are elements of C. Since f encodes A in the manner previously described and $\pi^{-1}(f) \upharpoonright \bar\kappa = f \upharpoonright \bar\kappa$, we get that $\pi(A \cap H^V_{\bar\kappa}) = A \cap H^V_\kappa$ and thus

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \prec (H^V_{\pi(\beta)}, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H^V_\beta)$. Now, as $(H^V_\gamma, \in, A \cap H^V_\gamma)$ and $(H^V_{\pi(\gamma)}, \in, A^* \cap H^V_{\pi(\gamma)})$ are elements of $H^V_{\pi(\beta)}$ for every $\gamma < \kappa$, Countable Embedding Absoluteness 1.4 implies that $H^V_{\pi(\beta)}$ sees that $\bar\kappa$ is virtually $({<}\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ∎

Remark 2.16. The above proof shows that the $\mathcal{M} \subseteq V$ assumptions can be replaced by "sufficient" agreement between $\mathcal{M}$ and V: for (i)-(iii) this means that $H^{\mathcal{M}}_\theta = H^V_\theta$ whenever $\mathcal{M}$ is the codomain of a virtual (θ, A)-strong/supercompact/extendible embedding, and in (iv)-(v) this means that $H^{\mathcal{M}}_{\pi(f)(\kappa)} = H^V_{\pi(f)(\kappa)}$. The same thing holds in the "lightface" setting of Theorem 2.4.

We will now step away from the woodins for a little bit, and introduce the vopěnkas. In anticipation of the next section we will work with the class-sized version here, but all the following results work equally well for inaccessible virtually vopěnka cardinals[8].

Definition 2.17 (GBC)**.** The **Generic Vopěnka Principle** (gVP) states that for any class C consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$. ○

We will be using a standard variation of gVP involving the following *natural sequences*.

Definition 2.18 (GBC)**.** Say that a class function $f\colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. ○

Definition 2.19 (GBC)**.** Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ is **natural** if there exists an indexing function $f\colon \mathsf{On} \to \mathsf{On}$ and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$ for every α. Denote this indexing function by $f^{\vec{\mathcal{M}}}$ and the unary relations as $R^{\vec{\mathcal{M}}}_\alpha$. ○

The following Theorem 2.20 is then the main theorem of this section. Firstly it shows that inaccessible cardinals are virtually vopěnka iff they are virtually prewoodin, but also that adding the "virtually" adverb doesn't do anything in this context, in contrast to Theorem 2.47.

Theorem 2.20 (GBC, Dimopoulos-Gitman-N.)**.** *The following are equivalent:*

(i) *gVP holds;*
(ii) *For any natural On-sequence $\vec{\mathcal{M}}$ there exists a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for some $\alpha < \beta$;*
(iii) *On is virtually prewoodin;*
(iv) *On is faintly prewoodin.*

[8] Note however that we have to require inaccessibility here: see [Wilson, 2019a] for an analysis of the singular virtually vopěnka cardinals.

Redundant; mention this properly above

PROOF. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume On is faintly prewoodin and fix some On-sequence $\vec{\mathcal{M}} := \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ of structures in a common language. Let κ be $({<}\mathsf{On}, \vec{\mathcal{M}})$-prestrong and fix some regular $\theta > \kappa$ satisfying that $\mathcal{M}_\alpha \in H_\theta^V$ for every $\alpha < \theta$, and fix a generic elementary embedding

$$\pi \colon (H_\theta^V, \in, \vec{\mathcal{M}}) \to (\mathcal{N}, \in, \mathcal{M}^*)$$

with $H_\theta^V \subseteq \mathcal{N}$ and $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$. Set $\kappa := \operatorname{crit} \pi$.

We have that $\pi \restriction \mathcal{M}_\kappa \colon \mathcal{M}_\kappa \to \mathcal{M}^*_{\pi(\kappa)}$, but we need to reflect this embedding down below θ as we don't know whether $\mathcal{M}^*_{\pi(\kappa)}$ is on the $\vec{\mathcal{M}}$ sequence. Working in the generic extension, we have

$$\mathcal{N} \models \exists \bar\kappa < \pi(\kappa) \exists \dot\sigma \in V^{\mathrm{Col}(\omega, \mathcal{M}^*_{\bar\kappa})} \colon \ulcorner \dot\sigma \colon \mathcal{M}^*_{\bar\kappa} \to \mathcal{M}^*_{\pi(\kappa)} \text{ is elementary} \urcorner.$$

Here κ realises $\bar\kappa$ and $\pi \restriction \mathcal{M}_\kappa$ realises σ. Note that $\mathcal{M}^*_\kappa = \mathcal{M}_\kappa$ since we ensured that $\mathcal{M}_\kappa \in H_\theta^V$ and we are assuming that $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$, so the domain of σ $(= \pi \restriction \mathcal{M}_\kappa)$ *is* $\mathcal{M}^*_\kappa$ — also note that σ exists in a $\mathrm{Col}(\omega, \mathcal{M}_\kappa)$ extension of $\mathcal{N}$ by an application of Countable Embedding Absoluteness 1.4. Now elementarity of π implies that

$$H_\theta^V \models \exists \bar\kappa < \kappa \exists \dot\sigma \in V^{\mathrm{Col}(\omega, \mathcal{M}_{\bar\kappa})} \colon \ulcorner \dot\sigma \colon \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa \text{ is elementary} \urcorner,$$

which is upwards absolute to V, from which we can conclude that $\sigma \colon \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa$ witnesses that gVP holds.

$(ii) \Rightarrow (iii)$: Assume (ii) holds and assume that On is not virtually prewoodin, which means that there exists some class A such that there are no virtually A-prestrong cardinals. This allows us to define a function $f \colon \mathsf{On} \to \mathsf{On}$ as $f(\alpha)$ being the least regular $\eta > \alpha$ such that α is not virtually (η, A)-prestrong.

We also define $g \colon \mathsf{On} \to \mathsf{On}$ as taking α to the least strong limit cardinal above α which is a closure point for f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A \cap H^V_{g(\alpha)}$.

(ii) supplies us with $\alpha < \beta$ and a generic elementary embedding[9]

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}).$$

Since $g(\alpha)$ is a closure point for f it holds that $f(\operatorname{crit}\pi) < g(\alpha)$, so fixing a regular $\theta \in (f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-prestrong, contradicting the definition of f. Hence On is virtually prewoodin. ■

2.2.1 Weak Vopěnka

Scrap this section?

We now move to a *weak* variant of gVP, introduced in a category-theoretic context in [Adámek and Rosický, 1994]. It starts with the following equivalent characterisation of gVP, which is the virtual analogue of the characterisation shown in [Adámek and Rosický, 1994].

Lemma 2.21 (GBC, Virtualised Adámek-Rosický)**.** *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that*[10]

(i) gVP

(ii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that
- *there is a generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

(iii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that
- *there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

Proof. Note that the only difference between (ii) and (iii) is that the homomorphism exists in V, making $(ii) \Rightarrow (iii)$ trivial.

[9]Note that $V_{g(\alpha)} = H^V_{g(\alpha)}$ since $g(\alpha)$ is a strong limit cardinal.
[10]This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [Adámek and Rosický, 1994].

(iii) $\Rightarrow$ *(i)*: Assume that gVP fails, meaning by Theorem 2.20 that we have a natural On-sequence $\vec{\mathcal{M}}_\alpha$ such that, in every generic extension, there's no homomorphism between any two disctinct $\mathcal{M}_\alpha$'s. Define an On-sequence $\langle \mathcal{N}_\kappa \mid \kappa \in \text{Card}\rangle$ as

$$\mathcal{N}_\kappa := \coprod_{\xi \leq \kappa} \mathcal{M}_\xi = \{(x, \xi) \mid \xi \leq \kappa \wedge \xi \in \text{Card} \wedge x \in \mathcal{M}_\xi\},^{11}$$

with a unary relation R^* given as $R^*(x, \xi)$ iff $\mathcal{M}_\xi \models R(x)$ and a binary relation $\sim^*$ given as $(x, \xi) \sim^* (x', \xi')$ iff $\xi = \xi'$. Whenever we have a homomorphism $f\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ we then get an induced homomorphism $\tilde{f}\colon \mathcal{M}_0 \to \mathcal{M}_\xi$, given as $\tilde{f}(x) := f(x, 0)$, where $\xi \leq \kappa$ is given by preservation of $\sim^*$.

For any two cardinals $\kappa < \lambda$ we have a homomorphism $j_{\kappa\lambda}\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ in V, given as $j_{\kappa\lambda}(x, \xi) := (x, \xi)$. This embedding must also be the *unique* such embedding in all generic extensions, as otherwise we get a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s. Furthermore, there can't be any homomorphism $\mathcal{N}_\lambda \to \mathcal{N}_\kappa$ as that would also imply the existence of a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s.

(i) $\Rightarrow$ *(ii)*: Assume that we have an On-sequence $\vec{\mathcal{M}}_\alpha$ as in the theorem, with generic homomorphisms $j_{\alpha\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ that are unique in all generic extensions for every $\alpha \leq \beta$, with no generic homomorphisms going the other way.

We first note that we can for every $\alpha \leq \beta$ choose the $j_{\alpha\beta}$ in a $\text{Col}(\omega, \mathcal{M}_\alpha)$-extension, by a proof similar to the proof of Lemma 1.4 and using the uniqueness of $j_{\alpha\beta}$. Next, fix a proper class $C \subseteq \mathsf{On}$ such that $\alpha \in C$ implies that

$$\sup_{\xi \in C \cap \alpha} |\mathcal{M}_\xi|^V < |\mathcal{M}_\alpha|^V .$$

and note that this implies that $V[g] \models |\mathcal{M}_\xi| < |\mathcal{M}_\alpha|$ for every V-generic $g \subseteq \text{Col}(\omega, \mathcal{M}_\xi)$. This means that for every $\alpha \in C$ we may choose some $\eta_\alpha \in \mathcal{M}_\alpha$ which is *not* in the range of any $j_{\xi\alpha}$ for $\xi < \alpha$. But now define first-order structures $\langle \mathcal{N}_\alpha \mid \alpha \in C\rangle$ as $\mathcal{N}_\alpha := (\mathcal{M}_\alpha, \eta_\alpha)$. Then, by our assumption on the $\mathcal{M}_\alpha$'s and construction of the $\mathcal{N}_\alpha$'s, there can be no generic homomorphism between any two distinct $\mathcal{N}_\alpha$, showing that gVP fails. ■

Note that the proof of the above lemma shows that we without loss of generality may assume that the generic homomorphism in (i) exists in V, which we record here:

Lemma 2.22 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ satisfying that*[12]

(i) there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;

(ii) there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.

■

The *weak* version of gVP is then simply "flipping the arrows around" in the above characterisation of gVP.

Definition 2.23 (GBC). **Generic Weak Vopěnka's Principle** (gWVP) states that there does *not* exist an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ such that

- there is a generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\alpha \leq \beta$, which is unique in all generic extensions;
- there is *no* generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for any $\alpha < \beta$.

○

Denoting the corresponding non-generic principle by WVP [Wilson, 2019b] showed the following.

Theorem 2.24 (Wilson). *WVP is equivalent to On being a Woodin cardinal.*

Given our 2.20 we may then suspect that in the virtual world these two are equivalent, which indeed turns out to be the case. We will be roughly following the argument in [Wilson, 2019b], but we have to diverge from it at several points in which they're using the fact that they're working with class-sized elementary embeddings.

[12] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [Adámek and Rosický, 1994].

Indeed, in that paper they establish a correspondence between elementary embeddings and certain homomorphisms, a correspondence we won't achieve here. Proving that the elementary embeddings we *do* get are non-trivial seems to furthermore require extra assumptions on our structures. Let's begin.

Define for every strong limit cardinal λ and Σ_1-formula φ the relations

$$R^\varphi := \{x \in V \mid (V, \in) \models \varphi[x]\}$$
$$R^\varphi_\lambda := \{x \subseteq H^V_\lambda \mid \exists y \in R^\varphi : y \cap H^V_\lambda = x\}$$

and given any class A define the structure

$$\mathscr{P}_{\lambda,A} := (H^V_{\lambda^+}, R^\varphi_\lambda, \{\lambda\}, A \cap H^V_\lambda)_{\varphi\in\Sigma_1}.$$

Say that a homomorphism $h \colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ is **trivial** if $h(x) \cap H^V_\eta = x \cap H^V_\eta$ for every $x \in H^V_{\lambda^+}$. Note that h can only be trivial if $\eta \leq \lambda$ since $h(\lambda) = \eta$.

Lemma 2.25 (GBC, Gitman-N.)**.** *Let λ be a singular strong limit cardinal, η a strong limit cardinal and $A \subseteq V$ a class. If there exists a non-trivial generic homomorphism $h \colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ then there's a non-trivial generic elementary embedding*

$$\pi \colon (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that, letting $\nu := \min\{\lambda, \eta\}$, it holds that $H^V_\nu \subseteq \mathcal{M}$, $A \cap H^V_\nu = B \cap H^V_\nu$ and $\operatorname{crit}\pi < \nu$.

Proof. Assume that we have a non-trivial homomorphism $h \colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ in a forcing extension $V[g]$, define in $V[g]$ the set

$$\mathcal{M}^* := \{\langle b, f\rangle \mid b \in [H_\nu]^{<\omega} \wedge f \in H^V_{\lambda^+} \wedge f \colon H^V_\lambda \to H^V_\lambda\},$$

and define the relation $\in^*$ on $\mathcal{M}^*$ as

$$\langle b_0, f_0\rangle \in^* \langle b_1, f_1\rangle \quad \text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) \in f_1(y)\}).$$

Claim 2.26. $\in^*$ is wellfounded.

PROOF OF CLAIM. Assume not and let $\cdots \in^* \langle b_1, f_1\rangle \in^* \langle b_0, f_0\rangle$ be an $\in^*$-decreasing chain, which by definition means that, for every $n < \omega$,

$$b_{n+1}b_n \in h(\{xy \in [H_\lambda^V]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}). \tag{1}$$

Define the relation $R(v_0, v_1, v_2)$ on H_λ^V as

$$R(X, f, g) \text{ iff } X = \{xy \in [H_\lambda^V]^{<\omega} \mid f(x) \in g(y)\}.$$

This relation is equal to R_λ^φ for some φ, so h moves $\langle X, f, g\rangle \in R_\lambda^\varphi$ to

$$\langle h(X), h(f), h(g)\rangle \in R_\eta^\varphi,$$

meaning that

$$h(\{xy \in [H_\lambda^V]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}) = \{xy \in [H_\eta^V]^{<\omega} \mid f_{n+1}^*(x) \in f_n^*(y)\}$$

for some f_n^* such that $f_n^* \cap H_\eta^V = h(f_n)$ for all $n < \omega$. But now (1) implies that

$$b_{n+1}b_n \in \{xy \in [H_\eta^V]^{<\omega} \mid f_{n+1}^*(x) \in f_n^*(y)\}$$

and so $h(f_{n+1})(x) = f_{n+1}^*(x) \in f_n^*(y) = h(f_n)(y)$, giving an $\in$-decreasing sequence in $V[g]$ using transitivity of H_η^V, a contradiction!

Hence $\in^*$ is wellfounded. $\dashv$

$\mathcal{M}^*$ is a set, so $\in^*$ is trivially set-like. This means that we can take the transitive collapse $(\mathcal{M}, \in) \cong (\mathcal{M}^*, \in^*)$, and we note that $\mathcal{M} = \{[b, f] \mid \langle b, f\rangle \in \mathcal{M}^*\}$, where $[b, f] := \{[\bar{b}, \bar{f}] \mid \langle \bar{b}, \bar{f}\rangle \in^* \langle b, f\rangle\}$.

We now get a version of Łoś' Theorem whose proof is straight-forward, using that h preserves all Σ_1-relations and that $H_\lambda^V \models \mathsf{ZFC}^-$.

Claim 2.27. For every formula $\varphi(v_1, \ldots, v_n)$ and every $[b_1, f_1], \ldots, [b_n, f_n] \in \mathcal{M}$ the following are equivalent:

(i) $(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]$;

(ii) $b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\})$.

Proof of claim. The proof is straightforward, using that h preserves Σ_1-relations. We prove this by induction on φ. If φ is $v_i \in v_j$ then we have that

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &[b_i, f_i] \in [b_j, f_j] \\
\Leftrightarrow &\langle b_i, f_i \rangle \in^* \langle b_j, f_j \rangle \\
\Leftrightarrow &b_i b_j \in h(\{a_i a_j \in [H_\lambda^V]^{<\omega} \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\psi \wedge \chi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \models \psi[[b_1, f_1], \dots, [b_n, f_n]] \wedge \chi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \dots, f_n(a_n)]\}) \cap \\
&\qquad\qquad h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \chi[f_1(a_1), \dots, f_n(a_n)]\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\neg\psi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \models \neg\psi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \not\models \psi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &b_1 \cdots b_n \notin h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \dots, f_n(a_n)]\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).
\end{aligned}$$

Finally, if φ is $\exists x\psi$ then

$$\begin{aligned}
&(\mathcal{M},\in) \models \varphi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&(\mathcal{M},\in) \models \exists x\psi[x,[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle \in \mathcal{M}^*\colon (\mathcal{M},\in) \models \psi[[b,f],[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle \in \mathcal{M}^*\colon bb_1\cdots b_n \in h(\{aa_1\cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f(a),f_1(a_1),\dots,f_n(a_n)]\})\\
\Leftrightarrow&b_1\cdots b_n \in h(\{a_1\cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1),\dots,f_n(a_n)]\}).
\end{aligned}$$

This finishes the proof. $\dashv$

Next up, we have the following standard lemma, which implies that $H^V_\eta \subseteq \mathcal{M}$:

Claim 2.28. For all $y \in H^V_\eta$ we have $y = [\langle y\rangle, \mathrm{pr}]$, where $\mathrm{pr}(\langle x\rangle) := x$.

Proof of claim. We prove this by $\in$-induction on $y \in H^V_\eta$, so suppose that $y' = [\langle y'\rangle, \mathrm{pr}]$ for every $y' \in y$, which implies that $y \subseteq \mathcal{M}$ by transitivity of $\mathcal{M}$. We then get that, for every $[b,f] \in \mathcal{M}$,

$$\begin{aligned}
[b,f] \in [\langle y\rangle, \mathrm{pr}] &\Leftrightarrow b\langle y\rangle \in h(\{a\langle x\rangle \mid f(a) \in \mathrm{pr}(\langle x\rangle)\})\\
&\Leftrightarrow \exists y' \in y\colon b\langle y'\rangle \in h(\{a\langle x\rangle \mid f(a) = x\})\\
&\Leftrightarrow \exists y' \in y\colon [b,f] = [\langle y'\rangle, \mathrm{pr}] = y'\\
&\Leftrightarrow [b,f] \in y,
\end{aligned}$$

showing that $y = [\langle y\rangle, \mathrm{pr}]$. $\dashv$

Now define

$$B := \{[b,f] \in \mathcal{M} \mid b \in h(\{x \in H^V_\lambda \mid f(x) \in A\})\}.$$

and, in $V[g]$, let $\pi\colon (H^V_\lambda, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$ be given as $\pi(x) := [\langle\rangle, c_x]$.

Claim 2.29. π is elementary.

Proof of claim. For $x_1, \dots, x_n \in H_\lambda^V$ it holds that

$$\begin{aligned} &(\mathcal{M}, \in, B) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\ \Leftrightarrow\ &(\mathcal{M}, \in) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\ \Leftrightarrow\ &\langle\rangle \in h(\{\langle\rangle \mid \mathscr{P}_{\lambda,A} \models \varphi[x_1, \dots, x_n]\}) \\ \Leftrightarrow\ &(H_{\lambda^+}^V, \in, A \cap H_\lambda^V) \models \varphi[x_1, \dots, x_n] \end{aligned}$$

and we also get that, for every $x \in H_\lambda^V$,

$$x \in A \Leftrightarrow \langle\rangle \in h(\{a \in H_\lambda^V \mid x \in A\}) \Leftrightarrow \pi(x) \in B,$$

which shows elementarity. $\dashv$

We next need to show that $B \cap H_\nu^V = A \cap H_\nu^V$, so let $x \in H_\nu^V$. Note that $x = [\langle x\rangle, \mathrm{pr}]$ by Claim 2.28, which means that

$$x \in B \Leftrightarrow \langle x\rangle \in h(\{\langle y\rangle \in H_\lambda^V \mid y \in A\}) \Leftrightarrow x \in A.$$

The last thing we need to show is that $\operatorname{crit}\pi < \nu$. We start with an analogous result about h.

Claim 2.30. There exists some $b \in H_\nu^V$ such that $h(b) \neq b$.

Proof of claim. Assume the claim fails. We now have two cases.

$\boxed{\textbf{Case 1: } \lambda \geq \eta}$

By non-triviality of h there's an $x \in H_{\lambda^+}^V$ such that $h(x) \cap H_\eta^V \neq x \cap H_\eta^V$, which means that there exists an $a \in H_\eta^V$ such that $a \in h(x) \Leftrightarrow a \notin x$.

If $a \in x$ then $\{a\} = h(\{a\}) \subseteq h(x)$,[13] making $a \in h(x)$, $\lightning$, so assume instead that $a \in h(x)$. Since η is a strong limit cardinal we may fix a

[13]Note that as h preserves Σ_1 formulas it also preserves singletons and boolean operations.

cardinal $\theta < \eta$ such that $a \in H_\theta^V$ and $H_\theta^V \in H_\eta^V$. We then have that[14]

$$\{a\} \subseteq h(x) \cap H_\theta^V = h(x) \cap h(H_\theta^V) = h(x \cap H_\theta^V) = x \cap H_\theta^V,$$

so that $a \in x$, $\lightning$.

$\boxed{\textbf{Case 2: } \lambda < \eta}$

In this case we are assuming that $h \restriction H_\lambda^V = \mathrm{id}$, but $h(\lambda) = \eta > \lambda$. Since λ is singular we can fix some $\gamma < \lambda$ and a cofinal function $f\colon \gamma \to \lambda$. Define the relation

$$R = \{(\alpha, \beta, \bar\alpha, \bar\beta, g) \mid \ulcorner g \text{ is a cofinal function } g\colon \alpha \to \beta \urcorner \wedge g(\bar\alpha) = \bar\beta\}.$$

Then $R(\gamma, \lambda, \alpha, f(\alpha), f)$ holds by assumption for every $\alpha < \gamma$, so that R holds for some $(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*)$ such that

$$\begin{aligned}(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*) \cap H_\eta^V &= (h(\gamma), h(\lambda), h(\alpha), h(f(\alpha)), h(f)) \\ &= (\gamma, \eta, \alpha, f(\alpha), h(f)),\end{aligned}$$

using our assumption that h fixes every $b \in H_\lambda^V$. Since γ, α and $f(\alpha)$ are transitive and bounded in H_λ^V it holds that $h(\gamma) = \gamma^*$, $h(\alpha) = \alpha^*$ and $h(f(\alpha)) = f(\alpha)^*$. Also, since $\mathrm{dom}(f^*) = \gamma = \mathrm{dom}(f)$ we must in fact have that $f^* = h(f)$. But this means that $h(f)\colon \gamma \to \eta$ is cofinal and $\mathrm{ran}(h(f)) \subseteq \lambda$, a contradiction! $\dashv$

To use the above Claim 2.30 to conclude anything about π we'll make use of the following standard lemma.

Claim 2.31. For any $x \in H_\lambda^V$ it holds that $h(x) \cap H_\eta^V = \pi(x) \cap H_\eta^V$.

[14] Note that we're using $\lambda \geq \eta$ here to ensure that $H_\theta^V \in \mathrm{dom}\, h$.

Proof of claim. For any $n < \omega$ and $\langle a_1, \dots, a_n\rangle \in [H_\eta^V]^n$ we have that

$$\begin{aligned}
&\langle a_1, \dots, a_n\rangle \in \pi(x) \\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle a_1, \dots, a_n\rangle \in \pi(x) \\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle [\langle a_1\rangle, \mathrm{pr}], \dots, [\langle a_n\rangle, \mathrm{pr}]\rangle \in [\langle\rangle, c_x] \\
\Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(\{\langle x_1, \dots, x_n\rangle \mid \mathscr{P}_{\lambda,A} \models \langle x_1, \dots, x_n\rangle \in x\}) \\
\Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(x),
\end{aligned}$$

showing that $h(x) \cap H_\eta^V = \pi(x) \cap H_\eta^V$. ⊣

Now use Claim 2.30 to fix a $b \in H_\nu^V$ which is moved by h. Claim 2.31 then implies that

$$\pi(b) \cap H_\eta^V = h(b) \cap H_\eta^V = h(b) \neq b = b \cap H_\eta^V,$$

showing that $\pi(b) \neq b$ and hence $\operatorname{crit}\pi < \nu$. This finishes the proof of the lemma. ■

Theorem 2.32 (GBC, Gitman-N.)**.** *gVP is equivalent to gWVP.*

Proof. ($\Rightarrow$): Assume gVP holds and gWVP fails, and let $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ be an On-sequence of first-order structures such that for every $\alpha \leq \beta$ there exists a generic homomorphism

$$j_{\beta\alpha}\colon\ \mathcal{M}_\beta \to \mathcal{M}_\alpha$$

in some $V[g]$ which is unique in all generic extensions, with no generic homomorphisms going the other way. Here we may assume, as in the proof of Lemma 2.21, that $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\beta)$. We can then find a proper class $C \subseteq \mathsf{On}$ such that $|\mathcal{M}_\alpha|^V < |\mathcal{M}_\beta|^V$ for every $\alpha < \beta$ in C. By gVP there are then

$\alpha < \beta$ in C and a generic homomorphism

$$\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta\,.$$

in some $V[h]$, where again we may assume that $h \subseteq \mathrm{Col}(\omega, \mathcal{M}_\alpha)$. But then $\pi \circ j_{\beta\alpha} = \mathrm{id}$ by uniqueness of $j_{\beta\beta} = \mathrm{id}$, which means that $j_{\beta\alpha}$ is injective in $V[g\times h]$ and hence also in $V[g]$. But then $|\mathcal{M}_\beta|^{V[g]} \leq |\mathcal{M}_\alpha|^{V[g]}$, which implies that $|\mathcal{M}_\beta|^V \leq |\mathcal{M}_\alpha|^V$ by the $|\mathcal{M}_\beta|^{+V}$-cc of $\mathrm{Col}(\omega, \mathcal{M}_\beta)$, contradicting the definition of C.

($\Leftarrow$): Assume that gVP fails, which by Theorem 2.20 is equivalent to On not being faintly prewoodin. This means that there exists a class A such that there are no faintly A-prestrong cardinals. We can therefore assign to any cardinal κ the least cardinal $f(\kappa) > \kappa$ such that κ is not faintly $(f(\kappa), A)$-prestrong.

Also define a function $g\colon \mathsf{On} \to \mathrm{Card}$ as taking an ordinal α to the least singular strong limit cardinal above α closed under f. Then we're assuming that there's no non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

with $H^V_{g(\alpha)} \subseteq \mathcal{M}$ and $B \cap H^V_{g(\alpha)} = A \cap H^V_{g(\alpha)}$. Assume towards a contradiction that for some α, β there is a non-trivial generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$. Lemma 2.25 then gives us a non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that $H^V_\nu \subseteq \mathcal{M}$ with $\nu := \min\{g(\alpha), g(\beta)\}$ and $A \cap H^V_\nu = B \cap H^V_\nu$, a contradiction! Therefore every generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$ is trivial. Since there is a unique trivial homomorphism when $\alpha \geq \beta$ and no trivial homomorphism when $\alpha < \beta$ since $g(\alpha)$ is sent to $g(\beta)$, the sequence of structures

$$\langle \mathscr{P}_{g(\alpha),A} \mid \alpha \in \mathsf{On}\rangle$$

is a counterexample to gWVP, which is what we wanted to show. ■

2.3 Berkeleys

Berkeley cardinals was introduced by Woodin at University of California, Berkeley around 1992, and was introduced as a large cardinal candidate that would be inconsistent with ZF. They trivially imply the Kunen inconsistency and are therefore at least inconsistent with ZFC, but that's as far as it currently goes. In the virtual setting the virtually berkeley cardinals, like all the other virtual large cardinals, are simply downwards absolute to L. See Appendix A for a quick overview of what is currently known about the non-virtual Berkeley cardinals.

It turns out that virtually berkeley cardinals are natural objects, as the main theorem of this section shows that these large cardinals are precisely what separates virtually prewoodins from the virtually woodins, as well as separating virtually vopěnka cardinals from mahlo cardinals.

Definition 2.33. Say that a cardinal δ is **virtually proto-berkeley** if for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi < \delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ then δ is **virtually berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C \subseteq \delta$ we say δ is **virtually club berkeley**. ○

Virtually (proto-)berkeley cardinals turn out to be equivalent to their "boldface" versions, the proof of which is a straight-forward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [Cutolo, 2017].

Proposition 2.34 (Virtualised Cutolo)**.** *If δ is virtually proto-berkeley then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with* $\operatorname{crit}\pi < \delta$*. If δ is virtually berkeley then we can furthermore ensure that* $\operatorname{crit}\pi$ *is arbitrarily large below δ.*

Proof. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{\{\langle A, x\rangle \mid x \in \mathcal{M}\}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: a is the first element in the pairs belonging to the set of highest rank, and $\mathcal{M}$ is what remains if we remove the set with the highest rank. But this means that a generic elementary embedding $\pi\colon \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and a, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the wanted conclusion. ■

The following is a straight-forward virtualisation of the usual definition of the vopěnka filter (see e.g. [Kanamori, 2008]).

Definition 2.35 (GBC)**.** Define the **virtually vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$. ○

Theorem 2.20 shows that $\emptyset$ is in the virtually vopěnka filter iff gVP fails, in analogy with the non-virtual case. Normality also holds in the virtual context, as the following proof shows.

Lemma 2.36 (GBC, Virtualised folklore)**.** *The virtually vopěnka filter is a normal filter.*

Proof. Let F be the virtually vopěnka filter. We first show that F is actually a filter. If $X \in F$ and $Y \supseteq X$ then $Y \in F$ simply by definition of F. If $X, Y \in F$, witnessed by natural sequences $\vec{\mathcal{M}}$ and $\vec{\mathcal{N}}$, then $X \cap Y \in F$ as well, witnessed by the natural sequence $\vec{\mathcal{P}}$ induced by the indexing function $f^{\vec{\mathcal{P}}} := \max(f^{\vec{\mathcal{M}}}, f^{\vec{\mathcal{N}}})$ and unary relations $R^{\vec{\mathcal{P}}}_\alpha := \operatorname{Code}(\langle R^{\vec{\mathcal{M}}}_\alpha, R^{\vec{\mathcal{N}}}_\alpha\rangle)$. Indeed, if $\pi\colon \mathcal{P}_\alpha \to \mathcal{P}_\beta$ is a generic elementary embedding with critical point μ then μ is also the critical point of both $\pi \restriction \mathcal{M}_\alpha\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ and $\pi \restriction \mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$.

For normality, let $X \in F^+$ be F-positive, where we recall that this means that $X \cap C \neq \emptyset$ for every $C \in F$, and let $f\colon X \to \mathsf{On}$ be regressive. We want to show that f is constant on an F-positive set.

Assume this fails, meaning that there are natural sequences $\vec{\mathcal{M}}^\gamma$ for γ such that for any generic elementary $\pi\colon\ \mathcal{M}^\gamma_\alpha \to \mathcal{M}^\gamma_\beta$ satisfies that $f(\operatorname{crit}\pi) \neq \gamma$. Define a new natural sequence $\vec{\mathcal{N}}$ as induced by the indexing function $g\colon \mathsf{On} \to \mathsf{On}$ given as $g(\alpha) := \sup_{\gamma<\alpha} \operatorname{rk} \mathcal{M}^\gamma_\alpha + \omega$ and unary relations $R^{\vec{\mathcal{N}}}_\alpha$ given as

$$R^{\vec{\mathcal{N}}}_\alpha := \operatorname{Code}(\langle\langle \mathcal{M}^\gamma_\alpha \mid \gamma < \alpha\rangle, f \restriction \alpha\rangle).$$

Now since X is F-positive there exists a generic elementary embedding $\pi\colon\ \mathcal{N}_\alpha \to \mathcal{N}_\beta$ with $\operatorname{crit}\pi \in X$. As $f(\operatorname{crit}\pi) < \operatorname{crit}\pi$ we get that $\pi(f(\operatorname{crit}\pi)) = f(\operatorname{crit}\pi)$, so that we have a generic elementary embedding

$$\pi \restriction \mathcal{M}^{f(\operatorname{crit}\pi)}_\alpha\colon\ \mathcal{M}^{f(\operatorname{crit}\pi)}_\alpha \to \mathcal{M}^{f(\operatorname{crit}\pi)}_\beta,$$

but this contradicts the definition of $\vec{\mathcal{M}}^{f(\operatorname{crit}\pi)}$! Thus F is normal. ■

The reason why we are being careful in showing all these analogous properties for the virtual vopěnka filter is that not all properties carry over. Indeed, note that uniformity of filters is non-trivial as we're working with proper classes[15], and we will see in Theorem 2.40 shows that uniformity of this filter is equivalent to there being no virtually berkeley cardinals — the following lemma is the first implication.

Lemma 2.37 (GBC, N.)**.** *Assume gVP and that there are no virtually berkeley cardinals. Then the virtually vopěnka filter F on On contains every class club C.*

Proof. The crucial extra property we get by assuming that there aren't any virtually berkeleys is that F becomes uniform, i.e. contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin$

[15]This boils down to the fact that the class club filter is not provably normal in GBC, see [Gitman et al., 2019]

F. Let M be a transitive set with $\delta \subseteq M$ and $\gamma < \delta$ a cardinal. As $(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function $f\colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\alpha + 1, \delta + 1)$$

and unary relations $R_\alpha := \langle M, \mathcal{N}_\alpha \rangle$. If $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit}\pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$, then $\pi(R_\alpha) = R_\beta$ implies that $\pi \restriction \mathcal{M}\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi \leq \delta$. We also get that $\operatorname{crit}\pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit}\pi}\colon \mathcal{N}_{\operatorname{crit}\pi} \to \mathcal{N}_{\pi(\operatorname{crit}\pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and hence $\operatorname{crit}\pi > \gamma$ as $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually berkeley, a contradiction. Thus $\operatorname{crit}\pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$.

Note that the class $C_0 \subseteq \mathsf{On}$ of limit ordinals is in F, since it's the diagonal intersection of the tails $(\alpha + 1, \mathsf{On})$. Now let $C \subseteq \mathsf{On}$ be a class club, and let $C = \{a_\alpha \mid \alpha < \mathsf{On}\}$ be its increasing enumeration. Then $C \supseteq C_0 \cap \triangle_{\alpha < \mathsf{On}}(a_\alpha, \mathsf{On})$, implying that $C \in F$. ■

Theorem 2.38 (GBC, N.). *If there are no virtually berkeley cardinals then On is virtually prewoodin iff On is virtually woodin.*

Proof. Assume On is virtually prewoodin, so gVP holds by Theorem 2.20 and we can let F be the virtually vopěnka filter. The assumption that there aren't any virtually berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails — we will follow the proof of Theorem 2.20.

Failure means that there is some class A and some class club C such that there are no virtually A-prestrong cardinals in C. Since there are no virtually berkeley cardinals, Lemma 2.37 imples that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as sending α to the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-prestrong if $\alpha \in C$, and set $f(\alpha) := \alpha$ if $\alpha \notin C$. Also define $g\colon \mathsf{On} \to \mathsf{On}$ as $g(\alpha)$ being the least strong limit cardinal in C above α which is a closure point for f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := \mathrm{Code}(\langle A \cap H^V_{g(\alpha)}, \mathcal{N}_\alpha\rangle)$ and apply gVP to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi \restriction (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)})\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}),$$

making $\operatorname{crit}\pi$ virtually $(g(\alpha), A)$-prestrong and thus $\operatorname{crit}\pi \notin C$. But as we also get the embedding $\pi \restriction \mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\operatorname{crit}\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \mathrm{Card} \mid (H^V_\kappa, \in, A \cap H^V_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-prestrong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it isn't virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, A \cap H^V_\theta) \to (M, \in, B)$$

with $\operatorname{crit}\pi = \kappa$, $H^V_\theta \subseteq M$, $M \subseteq V$, $A \cap H^V_\theta = B \cap H^V_\theta$ and $\pi(\kappa) < \theta$.

Now $\pi(\kappa)$ is inaccessible, and $(H^V_{\pi(\kappa)}, \in, A \cap H^V_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$ believes that κ is virtually $(A \cap H^V_{\pi(\kappa)})$-strong as in the proof of Theorem 2.9, meaning that $(H^V_\kappa, \in, A \cap H^V_\kappa)$ believes that there is a proper class of virtually $(A \cap H^V_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually woodin. ■

Theorem 2.39 (GBC, N.). *If there exists a virtually berkeley cardinal δ then gVP holds and On is not mahlo.*

Proof. If On was Mahlo then there would in particular exist an inaccessible cardinal $\kappa > \delta$, but then $H_\kappa^V \models \ulcorner$there exists a virtually berkeley cardinal$\urcorner$, contradicting the incompleteness theorem.

To show gVP we show that On is virtually prewoodin, which is equivalent by Theorem 2.20. Fix therefore a class A — we have to show that there exists a virtually A-prestrong cardinal. For every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta\colon (H_\theta^V, \in, A \cap H_\theta^V) \to (H_\theta^V, \in, A \cap H_\theta^V)$$

with $\operatorname{crit}\pi < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-prestrong, making On virtually prewoodin. ∎

Theorem 2.40 (GBC, N.). *The following are equivalent:*

(i) *gVP implies that On is mahlo;*

(ii) *On is virtually prewoodin iff On is virtually woodin;*

(iii) *There are no virtually berkeley cardinals.*

Proof. $(iii) \Rightarrow (ii)$ is Theorem 2.38, and the contraposed version of $(i) \Rightarrow (iii)$ is Theorem 2.39. For $(ii) \Rightarrow (i)$ note that gVP implies that On is virtually prewoodin by Theorem 2.20, which by (ii) means that it's virtually woodin and the usual proof shows that virtually woodins are mahlo[16], showing (i). ∎

This also immediately implies the following equiconsistency, as virtually berkeley cardinals have strictly larger consistency strength than virtually woodin cardinals.

[16]See e.g. Exercise 26.10 in [Kanamori, 2008].

Corollary 2.41 (N.). *The existence of an inaccessible virtually prewoodin cardinal is equiconsistent with the existence of an inaccessible virtually woodin cardinal.* ■

2.4 Behaviour in canonical inner models

Most of the cardinals turn out to downwards absolute to most inner models, including L:

Proposition 2.42. *For any regular uncountable cardinal θ, faintly θ-measurable cardinals are downwards absolute to any transitive class $\mathcal{U} \subseteq V$ satisfying $\mathsf{ZF}^- + \mathsf{DC}$.*

Proof. Let κ be faintly θ-measurable, witnessed by a forcing poset $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there's a transitive $\mathcal{M}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a transitive class $\mathcal{U} \subseteq V$ which satisfies $\mathsf{ZF}^- + \mathsf{DC}$. Restricting the embedding to $\pi \restriction H_\theta^{\mathcal{U}} \colon H_\theta^{\mathcal{U}} \to \mathcal{N}$ we can now apply the Countable Absoluteness Lemma 1.4 to $\pi \restriction H_\theta^{\mathcal{U}}$ to get that there exists an embedding $\pi^*\colon H_\theta^{\mathcal{U}} \to \mathcal{N}^*$ in a generic extension of U, making κ faintly θ-measurable in $\mathcal{U}$. ■

Proposition 2.43 (N.). *Let θ be a regular uncountable cardinal.*

(i) $L \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-prestrongs*$\urcorner$.

(ii) *Assume that $L[\mu]$ exists. It then holds that*
$L[\mu] \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-measurables*$\urcorner$.

(iii) *Assume there is no inner model with a woodin. It then holds that*
$K \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-measurables*$\urcorner$.

Proof. For (i) simply note that if $\pi\colon L_\theta \to \mathcal{N}$ is a generic elementary embedding with $\mathcal{N}$ transitive, then by condensation we have that $\mathcal{N} = L_\gamma$ for some $\gamma \geq \theta$, so that π also witnesses the virtual θ-prestrongness of $\operatorname{crit}\pi$.

(ii): Assume that $V = L[\mu]$ for notational simplicity and let κ be faintly θ-measurable, witnessed by a generic elementary embedding $\pi\colon L_\theta[\mu] \to \mathcal{N}$ existing in some generic extension $V[g]$. By condensation we get that $\mathcal{N} =$

$L_\gamma[\overline{\mu}]$ for some $\gamma \geq \theta$ and $\overline{\mu} \in V[g]$, but we're not guaranteed that $\overline{\mu} \in V$ here. Let λ be the unique measurable cardinal of $V = L[\mu]$.

Note that $\bar{\mu}$ is a measure on $\pi(\lambda) \geq \lambda$. If $\pi(\lambda) = \lambda$ then $L[\mu] = L[\bar{\mu}]$ by [Kanamori, 2008, Theorem 20.10] and we trivially get that $\mathcal{N} \subseteq V$. Assume thus that $\pi(\lambda) > \lambda$, which implies that $L[\bar{\mu}]$ is an internal iterate of $L[\mu]$ by [Kanamori, 2008, Theorem 20.12]. In particular it then holds that $L[\bar{\mu}] \subseteq L[\mu]$, so again we get that $\mathcal{N} \subseteq V$.

(*iii*): Assume that $V = K = L[\mathcal{E}]$ and fix a faintly θ-measurable cardinal κ, witnessed by a generic embedding $\pi\colon L_\theta[\mathcal{E}] \to \mathcal{N} = L_\gamma[\overline{\mathcal{E}}]$ in some generic extension $V[g]$. Now coiterate $L[\mathcal{E}]$ with $L[\overline{\mathcal{E}}]$, and denote the last models by $\mathcal{P}$ and $\mathcal{Q}$. Since $K = K^{V[g]}$ and as K is universal we get that $\mathcal{Q} \trianglelefteq \mathcal{P}$. Then the $L[\overline{\mathcal{E}}]$-to-$\mathcal{Q}$ branch did not drop, giving us an iteration embedding $i\colon L[\overline{\mathcal{E}}] \to \mathcal{Q}$.

Note that $\operatorname{crit} i \geq \kappa$ as $\overline{\mathcal{E}}$ is simply the pointwise image of $\mathcal{E}$ under π, so nothing below κ is touched and is therefore not used in the comparison either. This means that $\operatorname{crit}(i \circ \pi) = \kappa$, so that $(i \circ \pi)\colon L_\theta[\mathcal{E}] \to \mathcal{Q}$ witnesses that κ is virtually θ-measurable, since $\mathcal{Q} \trianglelefteq \mathcal{P}$ implies that $\mathcal{Q} \subseteq K$. ■

Note that the proofs of (*ii*) and (*iii*) above do not show that κ is virtually θ-prestrong, as it might still be the case that $\bar{\mu} \neq \mu$ or $\bar{\mathcal{E}} \neq \mathcal{E}$, so we cannot conclude that $L_\theta[\mu] \subseteq L_\theta[\bar{\mu}]$ or $L_\theta[\mathcal{E}] \subseteq L_\theta[\overline{\mathcal{E}}]$. It might still hold however; see Question 4.3.

2.5 Separation results

Having proving many positive results about the relations between the virtual large cardinals in the previous sections, this section is dedicated to the negatives. More precisely, we will aim to *separate* many of the defined notions (potentially under suitable large cardinal assumptions).

Our first separation result is that the virtuals form a level-by-level hierarchy.

Theorem 2.44 (N.). *Let $\alpha < \kappa$ and assume that κ is faintly $\kappa^{+\alpha+2}$-measurable. Then*

$$L_\kappa \models \ulcorner \textit{There's a proper class of } \lambda \textit{ which are virtually } \lambda^{+\alpha+1}\textit{-strong}\urcorner.$$

PROOF. Write $\theta := \kappa^{+\alpha+1}$. Then by Theorem 2.9 we get that either κ is faintly θ^+-strong in L or otherwise, in particular, L_κ thinks that there's a proper class of remarkables. In the second case we also get that L_κ thinks that there's a proper class of λ such that λ is virtually $\lambda^{+\alpha+1}$-strong and we'd be done, so assume the first case. Then $L_\kappa \prec_2 L_{\theta^+}$, so define for each $\xi < \kappa$ the sentence ψ_ξ as

$$\psi_\xi :\equiv \exists \lambda < \xi \colon \ulcorner \lambda \text{ is virtually } \lambda^{+\alpha+1}\text{-strong}\urcorner.$$

Then ψ_ξ is $\Sigma_2(\{\alpha, \xi\})$ since being virtually β-strong is a $\Delta_2(\{\beta\})$-statement. As $L_{\theta^+} \models \psi_\xi$ for all $\xi < \kappa$ we also get that $L_\kappa \models \psi_\xi$ for all $\xi < \kappa$, which is what we wanted to show. ■

As we are only assuming κ to be *faintly* measurable in the above, this also shows that the faintly $\kappa^{+\alpha+1}$-measurable cardinals κ form a strict hierarchy whenever $\alpha < \kappa$.

A separation result in a similar vein is the following, showing that it is consistent to have an inaccessible faintly measurable cardinal which is not weakly compact.

Proposition 2.45 (N.). *Assuming κ is measurable, there's a generic extension of V in which κ is inaccessible and faintly measurable, but not weakly compact.*

PROOF. Let $\mathbb{P}$ be the forcing notion that adds a κ-Suslin tree $\mathcal{T}$. By [Kunen, 1978] it then holds that $\mathbb{P} * \mathcal{T} \cong \mathrm{Add}(\kappa^+, 1)$, a $<\kappa^+$-closed forcing, which preserves the measurability of κ. Further, the $\mathbb{P}$ forcing is shown to preserve the inaccesibility of κ, making κ inaccesible and faintly measurable in $V[g]$. Lastly, it cannot be weakly compact in $V[g]$ because $\mathcal{T}$ is a κ-tree without a branch, by definition. ■

Next, we show that the "virtually" adverb *does* yield cardinals different from the faintly ones. This is trivial in general as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case. For a slightly more fine-grained distinction let's define an intermediate large cardinal between the faintly and virtual.

Definition 2.46. Let $\kappa < \theta$ be infinite regular cardinals. Say that κ is **faintly θ-power-Φ** for $\Phi \in \{\text{measurable}, \text{prestrong}, \text{strong}\}$ if it is faintly θ-Φ, witnessed by an embedding $\pi\colon H_\theta^V \to \mathcal{N}$, and $\mathscr{P}^V(\kappa) = \mathscr{P}^{\mathcal{N}}(\kappa)$. $\circ$

Note that the proof of Lemma 2.2 shows that faintly power-measurables are also 1-iterable and so in particular weakly compact. Our separation result is then the following.

Theorem 2.47 (Gitman-N.)**.** *For $\Phi \in \{\mathit{measurable}, \mathit{prestrong}, \mathit{strong}\}$, if κ is virtually Φ then there exist forcing extensions $V[g]$ and $V[h]$ such that*

(i) In $V[g]$, κ is inaccessible and faintly Φ but not faintly power-Φ; and

(ii) In $V[h]$, κ is faintly power-Φ but not virtually Φ.

Proof. We start with *(i)*. Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\lambda < \kappa$, and let $g \subseteq \mathbb{P}_\kappa$ be V-generic. Note that κ remains inaccessible in $V[g]$. Fix a regular $\theta > \kappa$ and let $\mathbb{Q}_\theta$ be a forcing witnessing that κ is virtually θ-measurable.

Since κ is *virtually* measurable we may without loss of generality assume that $\mathbb{Q}_\theta = \mathrm{Col}(\omega, \theta)$ by applying Countable Embedding Absoluteness 1.4. Fixing a $V[g]$-generic $h \subseteq \mathbb{Q}_\theta$ we get a transitive $\mathcal{N} \subseteq V$ and in $V[h]$ an elementary embedding

$$\pi\colon H_\theta^V \to \mathcal{N}$$

with $\operatorname{crit}\pi = \kappa$. Let's now work in $V[g][h] = V[h][g] = V[g \times h]$, in which we still have access to π. The lifting criterion[17] is trivial for $\mathbb{P}_\kappa$, so we get an $\mathcal{N}$-generic $\tilde{g} \subseteq \pi(\mathbb{P}_\kappa)$ and an elementary

$$\pi^+ \colon H_\theta^{V[g]} \to \mathcal{N}[\tilde{g}]$$

with $\pi \subseteq \pi^+$. Note here that without loss of generality $\pi(\kappa)$ is countable as otherwise we replace $\mathcal{N}$ by a countable hull, so we can indeed construct such a $\tilde{g}$. By elementarity of π it holds that

$$\pi(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \prod_{\lambda \in [\kappa, \pi(\kappa))} \operatorname{Add}(\lambda, 1), \tag{1}$$

so that $\mathcal{N}[\tilde{g}] \not\subseteq V[g]$ as it in particular contains a new subset of κ. If $\Phi =$ measurable then we're done at this point. For $\Phi =$ prestrong we simply note that $g \in \mathcal{N}[\tilde{g}]$ by (1) so that $H_\theta^{V[g]} \subseteq \mathcal{N}[\tilde{g}]$ as well, and since π^+ lifts π it holds that $\pi^+(\kappa) = \pi(\kappa) > \theta$ in the $\Phi =$ strong case.

As for *(ii)*, we simply change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$, which means that $\pi(\mathbb{P}_\kappa)$ doesn't add any subsets of κ and κ thus remains faintly power-Φ. By choosing $\theta > \kappa^+$ it *does* add a subset to κ^+ however, showing that κ is not virtually Φ. ∎

Note however, that in contrast to the above separation result, Theorem 2.20 showed that the faintly-virtually distinction vanishes when we're dealing with woodin cardinals.

Our next separation result is concerning the virtually prestrong and virtually strong cardinals.

Corollary 2.48 (N.). *There exists a virtually rank-into-rank cardinal iff there is an uncountable cardinal θ and a virtually θ-prestrong cardinal which is not virtually θ-strong.*

Proof. ($\Leftarrow$) is directly from the above Proposition 2.11 and Theorem 2.9.

[17] See Appendix B for the definition and characterisations of this criterion.

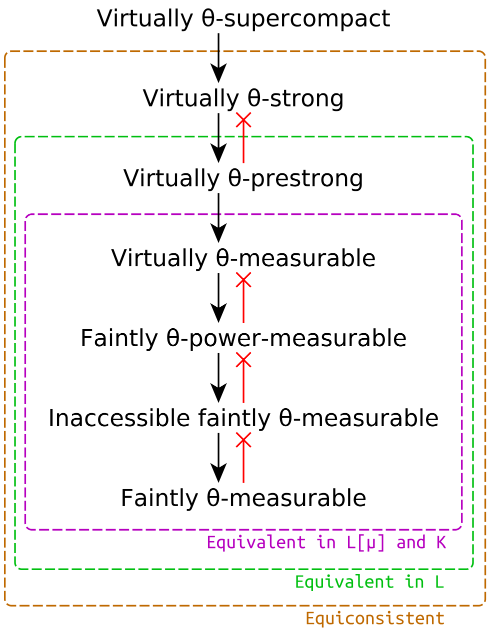

Figure 2.2: Direct implications between virtuals, where the red lines with crosses indicate that ZFC doesn't prove the reverse implication.

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let (κ, θ) be the lexicographically least pair such that κ is virtually θ-rank-into-rank, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong then it would be Σ_2-reflecting, so that the statement that there exists a virtually rank-into-rank cardinal would reflect down to H_κ^V, contradicting the minimality of κ. ■

Figure 2.2 summarises the separation results along with the results from Section 2.4. Note that it *might* be the case that virtually θ-measurables are always virtually θ-prestrong (and hence also equivalent in $L[\mu]$ and K below a woodin cardinal); see Question 4.3.

2.6 Indestructibility

It is well-known that supercompact cardinals κ can be made indestructible by all $<\kappa$-directed closed forcings by a suitable *Laver preparatory forcing*, which is the main theorem in the seminal paper [Laver, 1978]. A natural question, then, is whether similar results hold for the faintly and virtual versions. We noted in Proposition 2.2 that the virtuals are weakly compact, so the following theorem from [Jensen et al., 2009] shows that the consistency strength of indestructible virtual supercompacts is very large, potentially even in the realm of supercompacts themselves.

Theorem 2.49 (Schindler)**.** *The consistency strength of a weakly compact cardinal κ which is indestructible by $<\kappa$-directed closed forcing is larger than the consistency strength of a proper class of strong cardinals and a proper class of woodin cardinals.*

This gets close to resolving the question about the indestructible virtuals, so what about the faintly supercompact cardinals? To make things a bit easier for ourselves, let us make the notion a bit stronger.

Definition 2.50. Fix uncountable cardinals $\kappa < \theta$. Then κ is **generically setwise θ-supercompact** if there exists a generic extension $V[g]$, a transitive $\mathcal{N} \in V[g]$ and a generic elementary embedding $\pi \colon H_\theta^V \to \mathcal{N}$, $\pi \in V[g]$, with $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and $V[g] \models {}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$. If it holds for all $\theta > \kappa$ then we say that κ is **generically setwise supercompact**. ◦

Note that the only difference between a generically setwise θ-supercompact cardinal and a virtually θ-supercompact cardinal is that the former is closed under sequences *in the generic extension*, where the latter is only closed under sequences in V; i.e., that $V \cap {}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$.

Ostensibly this seems to be an incredibly strong notion, as the target model now inherits a lot of structure from the generic extension. A first stab at an upper consistency bound could be to note that if there exists a proper class of woodin cardinals then ω_1 is generically setwise supercompact. This can be shown using the countable stationary tower, see [Larson, 2004].

But, surprisingly, the following result from [Usuba, TBA] shows that they can exist in L.

Theorem 2.51 (Usuba). *If κ is virtually extendible then* $\mathrm{Col}(\omega, {<}\kappa)$ *forces that ω_1 is generically setwise supercompact.* ■

It turns out that this slightly stronger notion *does* have indestructiblity properties. We warm up by firstly showing that they are indestructible by small forcing.

Proposition 2.52 (N.-Schlicht). *Generic setwise supercompactness of κ is indestructible for forcing notions of size $< \kappa$.*

Proof. Fix a forcing $\mathbb{P}$ of size ${<}\kappa$ and assume without loss of generality that $\mathbb{P} \in H^V_\kappa$, and fix also a cardinal $\theta > \kappa$. Using the setwise supercompactness of κ we may fix a forcing $\mathbb{Q}$ and a V-generic $h \subseteq \mathbb{Q}$ such that in $V[h]$ we have an elementary $\pi\colon M := H^V_\theta \to \mathcal{N}$ in $V[h]$ with $\mathcal{N}$ ${<}\theta$-closed.

Let $g \subseteq \mathbb{P}$ be $V[h]$-generic and work in $V[g \times h]$. By the lifting criterion we get a lift $\tilde{\pi}\colon M[g] \to \mathcal{N}[g]$ of π. If κ is a limit cardinal, then we may choose a cardinal $\lambda < \kappa$ such that $\mathbb{P} \in H^V_\lambda$. Since $\mathbb{P}$ has the λ^+-cc in V we get that $\pi(\mathbb{P}) = \mathbb{P}$ has the λ^+-cc in $\mathcal{N}$ and hence in $V[h]$ as well, making $\mathcal{N}[g]$ ${<}\theta$-closed by Lemma B.1 and we're done.

If $\kappa = \nu^+$, then there are no cardinals between ν and $\pi(\kappa)$ in $\mathcal{N}$ and hence $|\theta| \leq \nu$. Thus it suffices to show that $\mathcal{N}[g]$ is ν-closed. Since $\pi(\mathbb{P}) = \mathbb{P}$ has size $\leq\nu$ in V, it has the $\nu^{+V[h]}$-cc in $V[h]$. Therefore $\mathcal{N}[g]$ is again ${<}\theta$-closed by Lemma B.1. ■

Next, we show that these generic setwise supercompact cardinals κ *are* in fact indestructible for ${<}\kappa$-directed closed forcings, without having to do any preparation forcing at all.

Theorem 2.53 (N.-Schlicht). *Generic setwise supercompactness of κ is indestructible for ${<}\kappa$-directed closed forcings.*

Proof. Suppose that κ is generically setwise supercompact, $\mathbb{P}$ is a ${<}\kappa$-directed closed forcing and g is $\mathbb{P}$-generic over V. We'll show that κ is generically setwise supercompact in $V[g]$.

In V fix a regular $\theta > \kappa$ such that $\mathbb{P} \in H^V_\theta$, and let $\mathbb{Q}$ be the forcing given by the definition of setwise supercompactness. Let h be $\mathbb{Q}$-generic over $V[g]$. Let $\pi\colon H^V_\theta \to \mathcal{N}$ be as in the definition of generically setwise supercompactness, so that $\pi \in V[h] \subseteq V[g \times h]$. Work in $V[g \times h]$.

We may assume that $\theta = \theta^{<\theta}$ holds, as otherwise we just replace $\mathbb{Q}$ with $\mathbb{Q} * \mathrm{Col}(\theta, \theta^{<\theta})$ — we retain the $<\theta$-closure of $\mathcal{N}$ because $\mathrm{Col}(\theta, \theta^{<\theta})$ is $<\theta$-closed. We can further assume that $|\mathcal{N}| = \theta^{<\theta} = \theta$, as otherwise we can take a hull of $\mathcal{N}$ containing $\mathrm{ran}(\pi)$ and recursively close under $<\theta$-sequences, ending up with a $<\theta$-closed elementary substructure $\mathcal{H} \prec \mathcal{N}$ containing $\mathrm{ran}(\pi)$ – now replace $\mathcal{N}$ by the transitive collapse of $\mathcal{H}$.

Claim 2.54. There's a $\pi(\mathbb{P})$-generic filter $\tilde{g}$ over $\mathcal{N}$ that extends $\pi[g]$.

Proof of claim. Since $\mathcal{N}$ is (in particular) $|\mathbb{P}|$-closed in $V[h]$ and $\mathbb{P}$ is trivially $|\mathbb{P}|^+$-cc, Lemma B.1 implies that $\mathcal{N}$ is still $|\mathbb{P}|$-closed in $V[g \times h]$. As below, we thus still get that $\pi[g] \in \mathcal{N}$. Now work in $V[h]$, where we have full $<\theta$-closure of $\mathcal{N}$.

Since $\pi(\mathbb{P})$ is directed, there's a condition $q \leq \pi[g]$ in $\pi(\mathbb{P})$. Using the fact that $|\mathcal{N}| = \theta$ and $\pi(\mathbb{P})$ is $<\theta$-closed, we can construct a $\pi(\mathbb{P})$-generic filter $\tilde{g}$ over $\mathcal{N}$ with $q \in g$.[18] Then $\tilde{g}$ is as required. $\dashv$

Since we now have that $\pi[g] \subseteq \tilde{g}$ by the claim, the lifting criterion implies that we can lift π to $\tilde{\pi}\colon H^V_\theta[g] \to \mathcal{N}[\tilde{g}]$.

It thus remains to see that $\mathcal{N}[\tilde{g}]$ is $<\theta$-closed. To see this, take a sequence $\vec{x} = \langle x_i \mid i < \gamma\rangle$ with $\gamma < \theta$ and $x_i \in \mathcal{N}[\tilde{g}]$ and find names $\sigma_i \in \mathcal{N}$ with $\sigma_i^{\tilde{g}} = x_i$ for all $i < \gamma$. Since ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ we have that $\vec{\sigma} = \langle \sigma_i \mid i < \gamma\rangle \in \mathcal{N}$, and from $\vec{\sigma}$ we obtain a canonical name $\vec{\sigma}^\bullet \in \mathcal{N}$ with $\vec{\sigma}^{\bullet\tilde{g}} = \vec{x} \in \mathcal{N}[\tilde{g}]$. ∎

Investigating further, we also show indestructibility for some forcings that do not fall into the above-mentioned categories.

[18] Namely, enumerate the dense subsets of $\pi(\mathbb{P})$ that are elements of $\mathcal{N}$ in order type θ and use the fact that the initial segments of the sequence, and of the corresponding sequence of conditions that we construct, are in $\mathcal{N}$.

Proposition 2.55 (N.-Schlicht)**.** *Generic setwise supercompactness of a regular cardinal κ is indestructible for* $\mathrm{Add}(\omega,\kappa)$*. If κ is a successor cardinal then it is also indestructible for* $\mathrm{Col}(\omega,{<}\kappa)$*.*

PROOF. Let g be $\mathrm{Add}(\omega,\kappa)$-generic over V. In V fix a regular $\theta>\kappa$ and let $\mathbb{Q}$ be the forcing given by the definition of generic setwise supercompactness. Let h be $\mathbb{Q}$-generic over $V[g]$ and work in $V[g\times h]$.

Let $\pi\colon H^V_\theta\to\mathcal{N}$ be as in the definition of generically setwise supercompactness. Moreover, let $\tilde g$ be $\mathrm{Add}(\omega,\pi(\kappa))$-generic over $V[g\times h]$. Since $\pi[g]=g$, the lifting criterion allows us to extend π to some $\tilde\pi\colon H^V_\theta[g]\to\mathcal{N}[g\times\tilde g]$. To show that $\mathcal{N}[g\times\tilde g]$ is ${<}\theta$-closed in $V[g\times h\times\tilde g]$, it suffices that $\mathrm{Add}(\omega,\pi(\kappa))$ has the ccc by Lemma B.1.

For $\mathrm{Col}(\omega,{<}\kappa)$, we proceed similarly. Assume that $\kappa=\nu^+$. Take $\mathrm{Col}(\omega,{<}\kappa)$-, $\mathbb{Q}$- and $\mathrm{Col}(\omega,{<}\pi(\kappa))$-generic filters g, h and $\tilde g$. π and $\mathcal{N}$ are as above. Since $\nu<\kappa<\theta<\pi(\kappa)$ and there are no cardinals between ν and $\pi(\kappa)$ (in $\mathcal{N}$ and thus also in $V[h]$), ${<}\theta$-closure means ν-closure (in any model containing $V[h]$). By Lemma B.1, it's thus sufficient to know that $\mathrm{Col}(\omega,{<}\pi(\kappa))$ has the ν^+-cc in $V[g\times h]$. This is because $\pi(\kappa)=\nu^{+\mathcal{N}}\leq\nu^{+V[g\times h]}$. ■

Usuba's Theorem 2.51 shows that the *consistency strength* of these generically setwise supercompact cardinals is small, but do they appear naturally anywhere? The following result shows that we cannot find any in neither L nor $L[\mu]$.

Proposition 2.56 (N.-Schlicht)**.** *No cardinal κ is generically setwise supercompact in neither L nor $L[\mu]$ with μ being a normal ultrafilter.*

PROOF. Assume first that $V=L$ and that κ is generically setwise supercompact. Let g be a generic filter and $\pi\colon L_\theta\to\mathcal{N}$ an embedding in $V[g]$ with $\pi\restriction L_{\kappa^{+L}}\in\mathcal{N}$. Then $\mathcal{N}=L_\alpha$ for some α by condensation and thus $\pi\restriction H_{\kappa^{+L}}\in L$. But this would induce a ${<}\kappa$-complete ultrafilter on κ, contradicting $V=L$.

Assume now that $V = L[\mu]$ and that κ is generically setwise supercompact, witnessed by a generic embedding $\pi\colon L_\theta[\mu] \to L_\alpha[\bar{\mu}]$. In particular this means that $\pi \restriction L_{\kappa^{+L[\mu]}}[\mu] \in L_\alpha[\bar{\mu}]$. If $\operatorname{crit}\mu < \kappa$ then $\mu = \bar{\mu}$ and $\mathscr{P}^{L[\mu]}(\kappa) = \mathscr{P}^{L[\bar{\mu}]}$, so that both $\pi(\kappa)$ and κ are now measurable cardinals in $L[\bar{\mu}]$, contradicting [Kanamori, 2008, Lemma 20.2]. So $\operatorname{crit}\mu \geq \kappa$.

If $\pi(\operatorname{crit}\mu) > \operatorname{crit}\mu$ then by [Kanamori, 2008, Theorem 20.12] we get that $L[\bar{\mu}]$ is an iterate of $L[\mu]$. But iteration embeddings preserve the subsets of their critical point, so again we have that $\mathscr{P}^{L[\mu]}(\kappa) = \mathscr{P}^{L[\bar{\mu}]}$ and we get the same contradiction as before.

Lastly, if $\operatorname{crit}\mu > \kappa$ and $\pi(\operatorname{crit}\mu) = \operatorname{crit}\mu$ then $\mu = \bar{\mu}$ by [Kanamori, 2008, Theorem 20.10], so we get a contradiction as in the $\operatorname{crit}\mu < \kappa$ ■

3 | Set-theoretic connections

Moving away from the pure theory of the virtual large cardinals from Chapter 2, we now move to connections between these large cardinals and common set-theoretic objects of study: filters, games and ideals.

3.1 Filters & Games

This section covers the content in the paper [Nielsen and Welch, 2019], which started out as a further analysis of the results in [Holy and Schlicht, 2018] and somewhat surprisingly we ended up in the realm of virtual large cardinals. As is custom in Mathematics, we will pretend that this was the goal all along.

We will in this section be dealing with certain small structures dubbed *weak κ-models* and filters on them of various types. We therefore start with a couple of definitions.

Definition 3.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa+1\subseteq\mathcal{M}$ and $(\mathcal{M},\in)\models\mathsf{ZFC}^-$. If furthermore ${}^{<\kappa}\mathcal{M}\subseteq\mathcal{M}$ then $\mathcal{M}$ is a **κ-model**. ○

Definition 3.2. Let κ be a cardinal, $\mathcal{M}$ a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

- **$\mathcal{M}$-normal** if $(\mathcal{M},\in,\mu)\models\forall\vec{X}\in{}^\kappa\mu:\triangle\vec{X}\in\mu$;
- **genuine** if $|\triangle\vec{X}|=\kappa$ for every κ-sequence $\vec{X}\in{}^\kappa\mu$;
- **normal** if $\triangle\vec{X}$ is stationary in κ for every κ-sequence $\vec{X}\in{}^\kappa\mu$;
- 0-**good**, or simply **good**, if it has a well-founded ultrapower;
- α-**good** for $\alpha>0$ if it is weakly amenable and has α-many well-founded iterates.

○

Note that a genuine $\mathcal{M}$-measure is $\mathcal{M}$-normal and countably complete, and a countably complete weakly amenable $\mathcal{M}$-measure is α-good for all ordinals α. In [Holy and Schlicht, 2018] they provide the following characterisation of the normal measures.

Lemma 3.3 (Holy-Schlicht)**.** *Let $\mathcal{M}$ be a weak κ-model and μ and $\mathcal{M}$-measure. Then μ is normal iff $\triangle\vec{X}$ is stationary for some enumeration $\vec{X}$ of μ.*

Proof. ($\Rightarrow$) is trivial, so assume that $\vec{X}$ is an enumeration of μ such that $\triangle\vec{X}$ is stationary. Let $\vec{Y} \in {}^\kappa\mu$ be a κ sequence and define $g\colon \kappa \to \kappa$ such that $Y_\alpha = X_{g(\alpha)}$ for $\alpha < \kappa$. Letting $C_g \subseteq \kappa$ be the club of closure points of g we get that $\triangle\vec{X} \cap C_g \subseteq \triangle\vec{Y} \cap C_g$, making $\triangle\vec{Y}$ stationary. ■

The α-Ramsey cardinals in [Holy and Schlicht, 2018] are based upon the following game[1].

Definition 3.4 (Holy-Schlicht)**.** For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, a limit ordinal $\gamma \leq \kappa$ and a regular cardinal $\theta > \kappa$ define the game $wfG^\theta_\gamma(\kappa)$ of length γ as follows.

$$\begin{array}{lccccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \mathrm{II} & & \mu_0 & & \mu_1 & & \mu_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a κ-model and μ_α is a filter for all $\alpha < \gamma$, such that μ_α is an $\mathcal{M}_\alpha$-measure, the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing and $\langle \mathcal{M}_\xi \mid \xi < \alpha\rangle, \langle \mu_\xi \mid \xi < \alpha\rangle \in \mathcal{M}_\alpha$ for every $\alpha < \gamma$. Letting $\mu := \bigcup_{\alpha<\gamma}\mu_\alpha$ and $\mathcal{M} := \bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$, player II wins iff μ is an $\mathcal{M}$-normal good $\mathcal{M}$-measure. ∘

We will also be using the following fact from [Holy and Schlicht, 2018, Lemma 3.3], that the games $wfG^\theta_\gamma(\kappa)$ do not depend upon the values of θ:

Lemma 3.5 (Holy-Schlicht)**.** *Let γ be a limit ordinal with $\operatorname{cof}\gamma \neq \omega$. Then $wfG^{\theta_0}_\gamma(\kappa)$ and $wfG^{\theta_1}_\gamma(\kappa)$ are equivalent for any regular $\theta_0, \theta_1 > \kappa$.* ■

[1]See Appendix **??** for a refresher on infinite game theory.

We will be working with a variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I but more of player II. It will turn out that this change of game is innocuous, as Proposition 3.9 will show that they are equivalent.

Definition 3.6 (Holy-N.-Schlicht)**.** Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following game $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ with $(\gamma{+}1)$-many rounds:

$$\begin{array}{lccccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & & \mathcal{M}_\gamma & \\ \text{II} & & \mu_0 & & \mu_1 & & \cdots & & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leq \gamma$, μ_α is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$, μ_γ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure and the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore require that $\mathcal{M}_\alpha = \bigcup_{\xi<\alpha} \mathcal{M}_\xi$, $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$ and that μ_α is ζ-good. Player II wins iff they could continue to play throughout all $(\gamma{+}1)$-many rounds. ∘

For convenience we will write $\mathcal{G}^\theta_\gamma(\kappa)$ for the game $\mathcal{G}^\theta_\gamma(\kappa, 0)$, and $\mathcal{G}_\gamma(\kappa)$ for $\mathcal{G}^\theta_\gamma(\kappa)$ whenever $\operatorname{cof}\gamma \neq \omega$, as again the existence of winning strategies in these games doesn't depend upon a specific θ. Note that we assume that $\kappa = \kappa^{<\kappa}$ is uncountable in the definition of the games that we're considering, so this is a standing assumption throughout this section, whenever any one of the above two games are considered.

Definition 3.7. Define the **Cohen game** $\mathcal{C}^\theta_\gamma(\kappa)$ as $\mathcal{G}^\theta_\gamma(\kappa)$ but where we require that $|\mathcal{M}_\alpha - H_\kappa| < \gamma$ for every $\alpha < \gamma$, i.e. that we only allow player I to add $<\gamma$ new elements to the models in each round, and where we only require $\mathcal{M}_\alpha \models \mathsf{ZFC}^-$ and $\mathcal{M}_\alpha \prec H_\theta$ for $\alpha \leq \gamma$ limit.[2]

Also define the **weak Cohen game** $\mathcal{C}^-_\gamma(\kappa)$ in analogy with $\mathcal{G}^-_\gamma(\kappa)$. ∘

Proposition 3.8 (N.)**.** *Assume $\gamma^{\aleph_0} = \gamma$ and let κ be regular. Then $\mathcal{C}^-_\gamma(\kappa)$ is equivalent to $\mathcal{C}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. In particular, if CH holds then $\mathcal{C}^-_{\omega_1}(\kappa)$ is equivalent to $\mathcal{C}^\theta_{\omega_1}(\kappa)$ for all regular $\theta > \kappa$.*

[2] $\mathcal{C}^\theta_\omega(\kappa)$ is similar to the $H(F, \lambda)$-games in [Donder and Levinski, 1989].

PROOF. The assumption that $\gamma^{\aleph_0} = \gamma$ allows us to ensure that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ for all $\alpha < \gamma$. If player I has a winning strategy in $\mathcal{C}^\theta_\gamma(\kappa)$ for some regular $\theta > \kappa$ then they still win if we require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ (since they're only enlargening their models, making it even harder for player II to win), in which case the final measure μ_γ is countably complete and hence automatically has a wellfounded ultrapower.

If player II has a winning strategy in $\mathcal{C}^-_\gamma(\kappa)$ then they still win if player I plays $\mathcal{M}_\alpha$ such that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$, again ensuring that μ_γ has a wellfounded ultrapower. ■

Proposition 3.9 (Holy-N.-Schlicht)**.** *$\mathcal{G}^\theta_\gamma(\kappa)$, $\mathcal{G}^\theta_\gamma(\kappa, 1)$ and $wfG^\theta_\gamma(\kappa)$ are all equivalent for all limit ordinals $\gamma \leq \kappa$, and $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$ whenever* $\operatorname{cof}\gamma > \omega$ *and $\zeta \in \mathit{On}$.*

PROOF. We start by showing the latter statement, so assume that $\operatorname{cof}\gamma > \omega$. Consider now the auxilliary game, call it $\mathcal{G}$, which is exactly like $\mathcal{G}^\theta_\gamma(\kappa, 0)$, but where we also require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ and $\langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle, \langle \mu_\xi \mid \xi \leq \alpha\rangle \in \mathcal{M}_{\alpha+1}$ for every $\alpha < \gamma$.

Claim 3.10. $\mathcal{G}$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$.

PROOF OF CLAIM. If player I has a winning strategy in $\mathcal{G}$ then they also have one in $\mathcal{G}^\theta_\gamma(\kappa)$, by doing exactly the same. Analogously, if player II has a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have one in $\mathcal{G}$. If player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ then we can construct a winning strategy σ' in $\mathcal{G}$, which is defined as follows. Fix some $\alpha \leq \gamma$ and, writing $\vec{\mathcal{M}}_\xi := \langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle$ and $\vec{\mu}_\xi := \langle \mu_\xi \mid \xi \leq \alpha\rangle$, we set

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) := \operatorname{Hull}^{H_\theta}(\sigma(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) \cup {}^\omega\mathcal{M}_\alpha \cup \{\vec{\mathcal{M}}_\xi, \vec{\mu}_\xi\}),$$

i.e. that we're simply throwing in the sequences into our models and making sure that we're still an elementary substructure of H_θ. This new strategy σ' is clearly winning. Assuming now that τ is a winning strategy for player II in $\mathcal{G}$, we define a winning strategy τ' for player II in $\mathcal{G}^\theta_\gamma(\kappa)$

by letting $\tau'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle)$ be the result of throwing in the appropriate sequences into the models $\mathcal{M}_\xi$, applying τ to get a measure, and intersecting that measure with $\mathcal{M}_\alpha$ to get an $\mathcal{M}_\alpha$-measure. $\dashv$

Now, letting $\mathcal{M}_\gamma$ be the final model of a play of $\mathcal{G}$, $\operatorname{cof}\gamma > \omega$ implies that any ω-sequence $\vec{X} \in \mathcal{M}_\gamma$ really is a sequence of elements from some $\mathcal{M}_\xi$ for $\xi < \gamma$, so that $\vec{X} \in \mathcal{M}_{\xi+1}$ by definition of $\mathcal{G}$, making $\mathcal{M}_\gamma$ closed under ω-sequences and thus also μ_γ countably complete. Since γ is a limit ordinal and the models contain the previous measures and models as elements, the proof of e.g. Theorem 5.6 in [Holy and Schlicht, 2018] shows that μ_γ is also weakly amenable, making it ζ-good for all ordinals ζ.

Now we deal with the first statement, so fix a limit ordinal γ. Firstly $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa, 1)$ as above, since both are equivalent to the auxilliary game $\mathcal{G}$ when γ is a limit ordinal. So it remains to show that $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $wfG^\theta_\gamma(\kappa)$. If player I has a winning strategy σ in $wfG^\theta_\gamma(\kappa)$ then define a winning strategy σ' for player I in $\mathcal{G}^\theta_\gamma(\kappa)$ as

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle) := \sigma(\langle \mathcal{M}_0, \mu_0 \rangle^\frown \langle \mathcal{M}_{\xi+1}, \mu_{\xi+1} \mid \xi + 1 \leq \alpha \rangle)$$

and for limit ordinals $\alpha \leq \gamma$ set $\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi < \alpha \rangle) := \bigcup_{\xi<\alpha} \mathcal{M}_\xi$; i.e. they simply follow the same strategy as in $wfG^\theta_\gamma(\kappa)$ but plugs in unions at limit stages. Likewise, if player II had a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have a winning strategy in $wfG^\theta_\gamma(\kappa)$, this time just by skipping the limit steps in $\mathcal{G}^\theta_\gamma(\kappa)$.

Now assume that player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ and that player I *doesn't* have a winning strategy in $wfG^\theta_\gamma(\kappa)$. Then define a strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$ as follows. Let $s = \langle \mathcal{M}_\alpha, \mu_\alpha \mid \alpha \leq \eta \rangle$ be a partial play of $wfG^\theta_\gamma(\kappa)$ and let s' be the modified version of s in which we have 'inserted' unions at limit steps, just as in the above paragraph. We can assume that every μ_α in s' is good and $\mathcal{M}_\alpha$-normal as otherwise player II has already lost and player I can play anything. Now, we want to show that s' is a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$. All the models in s are κ-models, so in particular weak κ-models.

Claim 3.11. Every μ_α in s' is normal.

Proof of claim. Assume without loss of generality that $\alpha = \eta$. Let player I play any legal response $\mathcal{M}$ to s in $wfG^\theta_\gamma(\kappa)$ (such a response always exists). If player II can't respond then player I has a winning strategy by simply following $s^\frown\langle\mathcal{M}\rangle$, $\lightning$, so player II *does* have a response μ to $s^\frown\mathcal{M}$. But now the rules of $wfG^\theta_\gamma(\kappa)$ ensures that $\mu_\eta \in \mathcal{M}$, so since

$$(\mathcal{M}, \in, \mu) \models \forall\vec{X} \in {}^\kappa\mu : \ulcorner \triangle\vec{X} \text{ is stationary in } \kappa\urcorner,$$

we then also get that $\mathcal{M} \models \ulcorner\triangle\mu_\eta$ is stationary in $\kappa\urcorner$ since $\mu_\eta \subseteq \mu$, so elementarity of $\mathcal{M}$ in H_θ implies that $\triangle\mu_\eta$ really *is* stationary in κ, making μ_η normal. $\dashv$

This makes s' a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$, so we may form the weak κ-model $\tilde{\mathcal{M}}_\eta := \sigma(s')$. Now let $\mathcal{M}_\eta \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_\eta \subseteq \mathcal{M}_\eta$ and $s \in \mathcal{M}_\eta$ and set $\sigma'(s) := \mathcal{M}_\eta$. This defines the strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$, which is winning since the winning condition for the two games is the same for γ a limit.[3]

Next, assume that player II has a winning strategy τ in $wfG^\theta_\gamma(\kappa)$. We recursively define a strategy $\tilde{\tau}$ for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ as follows. If $\tilde{\mathcal{M}}_0$ is the first move by player I in $\mathcal{G}^\theta_\gamma(\kappa)$, let $\mathcal{M}_0 \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_0 \subseteq \mathcal{M}_0$, making $\mathcal{M}_0$ a valid move for player I in $wfG^\theta_\gamma(\kappa)$. Write $\mu_0 := \tau(\langle\mathcal{M}_0\rangle)$ and then set $\tilde{\tau}(\langle\tilde{\mathcal{M}}_0\rangle)$ to be $\tilde{\mu}_0 := \mu_0 \cap \tilde{\mathcal{M}}_0$, which again is normal by the same trick as above, making $\tilde{\mu}_0$ a legal move for player II in $\mathcal{G}^\theta_\gamma(\kappa)$. Successor stages $\alpha+1$ in the construction are analogous, but we also make sure that $\langle\mathcal{M}_\xi \mid \xi < \alpha+1\rangle, \langle\mu_\xi \mid \xi < \alpha+1\rangle \in \mathcal{M}_{\alpha+1}$. At limit stages τ outputs unions, as is required by the rules of $\mathcal{G}^\theta_\gamma(\kappa)$. Since the union of all the μ_α's is good as τ is winning, $\tilde{\mu}_\gamma := \bigcup_{\alpha<\gamma}\tilde{\mu}_\alpha$ is good as well, making $\tilde{\tau}$ winning

[3] More precisely, that σ is winning in $\mathcal{G}^\theta_\gamma(\kappa)$ means that there's a sequence $\langle f_n : \kappa \to \kappa \mid n < \omega\rangle$ with the f_n's all being elements of the last model $\tilde{\mathcal{M}}_\gamma$, witnessing the illfoundedness of the ultrapower. But then all these functions will also be elements of the union of the $\mathcal{M}_\alpha$'s, since we ensured that $\mathcal{M}_\alpha \supseteq \tilde{\mathcal{M}}_\alpha$ in the construction above, making the ultrapower of $\bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$ by $\bigcup_{\alpha<\gamma}\mu_\alpha$ illfounded as well.

and we are done. ■

We now arrive at the definitions of the cardinals we will be considering. They were in [Holy and Schlicht, 2018] only defined for γ being a cardinal, but given the above result we generalise it to all ordinals γ.

Definition 3.12. Let κ be a cardinal and $\gamma \leq \kappa$ an ordinal. Then κ is **γ-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. We furthermore say that κ is **strategic γ-Ramsey** if player II *does* have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$.

Define **(strategic) genuine γ-Ramseys** and **(strategic) normal γ-Ramseys** analogously, but where we require the last measure μ_γ to be genuine and normal, respectively. ○

Definition 3.13 (N.)**.** A cardinal κ is **$<\gamma$-Ramsey** if it is α-Ramsey for every $\alpha < \gamma$, **almost fully Ramsey** if it is $<\kappa$-Ramsey and **fully Ramsey** if it is κ-Ramsey.

Further, say that κ is **coherent $<\gamma$-Ramsey** if it's strategic α-Ramsey for every $\alpha < \gamma$ and that there exists a choice of winning strategies τ_α in $\mathcal{G}_\alpha(\kappa)$ for player II satisfying that $\tau_\alpha \subseteq \tau_\beta$ whenever $\alpha < \beta$. In other words, there is a single strategy τ for player II in $\mathcal{G}_\gamma(\kappa)$ such that τ is a winning strategy for player II in $\mathcal{G}_\alpha(\kappa)$ for every $\alpha < \gamma$.[4] ○

This is not the original definition of (strategic) γ-Ramsey cardinals however, as this involved elementary embeddings between weak κ-models – but as the following theorem of [Holy and Schlicht, 2018] shows, the two definitions coincide whenever γ is a regular cardinal.

Theorem 3.14 (Holy-Schlicht)**.** *For regular cardinals λ, a cardinal κ is λ-Ramsey iff for arbitrarily large $\theta > \kappa$ and every $A \subseteq \kappa$ there is a weak κ-model $\mathcal{M} \prec H_\theta$ with $\mathcal{M}^{<\lambda} \subseteq \mathcal{M}$ and $A \in \mathcal{M}$ with an $\mathcal{M}$-normal 1-good $\mathcal{M}$-measure μ on κ.* ■

[4] Note that, with this terminology, "coherent" is a stronger notion than "strategic". We could've called the cardinals *coherent strategic $<\gamma$-Ramseys*, but we opted for brevity instead.

3.1.1 The finite case

In this section we are going to consider properties of the n-Ramsey cardinals for finite n. Note in particular that the $\mathcal{G}_n^\theta(\kappa)$ games are determined, making the "strategic" adjective superfluous in this case. We further note that the θ's are also dispensible in this finite case:

Proposition 3.15 (N.)**.** *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n \prec H_\theta$.*

Proof. $\Leftarrow$ is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays $\mathcal{M}_k$ in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \mathrm{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of $\mathcal{M}_k$, and play $\mathcal{M}_k^*$ in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ-responses to the $\mathcal{M}_k^*$'s and let player II play the μ_k's in $\mathcal{G}_n(\kappa)$ as well.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\mathrm{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω-sequence $\vec{f} := \langle f_k \mid k < \omega\rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \mathrm{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\mathrm{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \to H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \operatorname{ran}\pi$.

Now $\bar{A} = A$ for every $A \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the $\bar{f}_k$'s witness that $\mathrm{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\mathrm{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \mathrm{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ's anymore we note that n-Ramseyness can now be described using a Π^1_{2n+2}-formula and normal n-Ramseyness using a Π^1_{2n+3}-formula.

We have the following characterisations, as proven in [Abramson et al., 1977].

Theorem 3.16 (Abramson et al.)**.** *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then* *(i)* κ *is weakly compact if and only if it is* 0*-Ramsey;*

(ii) κ *is weakly ineffable if and only if it is genuine* 0*-Ramsey;*

(iii) κ *is ineffable if and only if it is normal* 0*-Ramsey.*

PROOF. This is mostly just changing the terminology in [Abramson et al., 1977] to the current game-theoretic one, so we only show (i).

Theorem 1.1.3 in [Abramson et al., 1977] shows that κ is weakly compact if and only if every κ-sized collection of subsets of κ is measured by a $<\kappa$-complete measure, in the sense that every $<\kappa$-sequence (in V) of measure one sets has non-empty intersection.

For the $\Rightarrow$ direction we can let player II respond to any $\mathcal{M}_0$ by first getting the $<\kappa$-complete $\mathcal{M}_0$-measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \to \mathrm{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π, which is $\mathcal{M}_0$-normal and good. For $\Leftarrow$, if $X \subseteq \mathscr{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ-model $\mathcal{M}_0 \prec H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play $\mathcal{M}_0$ in $\mathcal{G}_0(\kappa)$ we get some $\mathcal{M}_0$-normal good $\mathcal{M}_0$-measure μ_0 on κ. Since $\mathcal{M}_0$ is closed under $<\kappa$-sequences we get that μ_0 is $<\kappa$-complete. ■

Indescribability

In this section we aim to prove that n-Ramseys are Π^1_{2n+1}-indescribable and that normal n-Ramseys are Π^1_{2n+2}-indescribable, which will also establish that the hierarchy of alternating n-Ramseys and normal n-Ramseys forms a strict hierarchy. Recall the following definition.

Definition 3.17. A cardinal κ is Π^1_n**-indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ◦

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π^1_1-indescribable — see [Abramson et al., 1977].

Theorem 3.18 (N.)**.** *Every* n*-Ramsey* κ *is* Π^1_{2n+1}*-indescribable for* $n < \omega$*.*

Proof. Let κ be n-Ramsey and assume that it is not Π^1_{2n+1}-indescribable, witnessed by a Π_{2n+1}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$ and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$-many quantifiers occuring in φ in $(n+1)$-many steps. We will here describe the first two steps with the remaining steps following the same pattern.

First step. Write $\varphi(v) \equiv \forall v_1 \psi(v, v_1)$ for a Σ_{2n}-formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A^{(0)}_\alpha \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let $\mathcal{M}_0$ be a weak κ-model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good $\mathcal{M}_0$-normal $\mathcal{M}_0$-measure μ_0 on κ, using the 0-Ramseyness of κ. Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \mathrm{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. $\mathcal{M}_0$-normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, $\mathcal{M}_0$-normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}]\urcorner. \tag{1}$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0-formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

Second step. Write $\psi(v, v_1) \equiv \exists v_2 \forall v_3 \chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$-formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3 \chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \tag{2}$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, v_3]. \tag{3}$$

Fix witnesses $A^{(1)}_\alpha \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S^{(0)}_\alpha := \{\xi < \kappa \mid A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S^{(0)}_\alpha \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and $\mathcal{M}_0$-normality of μ_0 then implies that $S^{(0)}_\alpha \in \mu_0$ is equivalent to

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ-model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an $\mathcal{M}_1$-normal $\mathcal{M}_1$-measure on κ, using the 1-Ramseyness of κ, so that $\mathcal{M}_1$-normality of μ_1 yields that $\triangle\vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle\vec{S}^{(0)}$ if and only if $A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A^{(0)}_\xi = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \mathrm{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A^{(1)}_\alpha$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha].$$

Now this, paired with the above observation regarding $\triangle\vec{S}^{(0)}$, means that for every $\alpha \in \triangle\vec{S}^{(0)} \cap \mathrm{Lim}$ we have that

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha],$$

so that $\mathcal{M}_1$-normality of μ_1 and Łoś' lemma implies that

$$\mathrm{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg\chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]\urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$-many steps, ending with a Δ_0-formula $\phi(v, v_1, \ldots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\mathrm{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n)}]\urcorner$. But now absoluteness of $\neg\phi$ means that $V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n)}]$, contradicting (4). ■

Note that this is optimal, as n-Ramseyness can be described by a Π^1_{2n+2}-formula. As a corollary we then immediately get the following.

Corollary 3.19 (N.). *Every $<\omega$-Ramsey cardinal is Δ^2_0-indescribable.* ■

The second indescribability result concerns the normal n-Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π^1_2-indescribable — see [Abramson et al., 1977].

Theorem 3.20 (N.). *Every normal n-Ramsey κ is Π^1_{2n+2}-indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 3.18, as we would end up with a Π^1_1 statement in an ultrapower, and as Π^1_1 statements are not upwards absolute in general we would not be able to get our contradiction.

Proof. Let κ be normal n-Ramsey and assume that it is not Π^1_{2n+2}-indescribable, witnessed by a Π_{2n+2}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n-Ramsey to perform the same $n+1$ steps as in the proof of Theorem 3.18. This gives us a Σ_1-formula $\phi(v, v_1, \ldots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \cdots, \mathcal{A}^{(n)} \rangle$, $\langle B^{(0)}, \ldots, B^{(n-1)} \rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n \rangle$ of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{1}$$

and, for μ_n-many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \ldots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, A^{(n)}_\alpha].$$

Now form $S^{(n)}_\alpha \in \mu_n$ as in the proof of Theorem 3.18. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 3.18 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for $k < n$), but we can now use normality[5] of μ_n to ensure that we *do* have that $\triangle\vec{S}^{(n)}$ is stationary in κ. This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \ldots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \quad (2)$$

Now note that since κ is inaccessible it is Σ^1_1-indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of [Abramson et al., 1977] shows that the set of reflection points of Σ^1_1-formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \ldots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n-Ramseyness can be described by a Π^1_{2n+3}-formula. In particular this then means that every $(n+1)$-Ramsey is a normal n-Ramsey stationary limit of normal n-Ramseys, and every normal n-Ramsey is an n-Ramsey stationary limit of n-Ramseys, making the hierarchy of alternating n-Ramseys and normal n-Ramseys a strict hierarchy.

Downwards absoluteness to L

Our absoluteness result below, Theorem 3.22, is inspired by arguments in [Abramson et al., 1977], and uses the following lemma from that paper.

[5] Recall that this is stronger than just requiring it to be $\mathcal{M}_n$-normal — we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.

Lemma 3.21 (Abramson et al)**.** *There is a Π^1_1 formula $\varphi(A)$ such that, for any ordinal α, $(V_\alpha, V_{\alpha+1}) \models \varphi[A]$ iff α is a regular cardinal and A is a non-constructible subset of α.*[6] ■

Theorem 3.22 (N.)**.** *Genuine- and normal n-Ramseys are downwards absolute to L, for every $n < \omega$.*

Proof. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ-model — we want to find a genuine $\mathcal{M}$-measure inside L. By assumption we *can* find such a measure μ in V; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa\rangle \in L$ of $\mathscr{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

Claim 3.23. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

Proof of claim. Let $\vec{B}$ be the **μ-positive part** of $\vec{A}$, meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle\vec{B}$ has size κ, so we can pick $\delta \in \triangle\vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L. ⊣

Now let φ be the Π^1_1 formula given by Lemma 3.21. If we therefore assume that $T \notin L$ then $(V_\kappa, V_{\kappa+1}) \models \varphi[T]$, which by Π^1_1-indescribability of κ means that there exists some $\alpha < \kappa$ such that $(V_\alpha, V_{\alpha+1}) \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle\mu)^L = \triangle\mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V. The cases where κ is a genuine- or normal n-Ramsey cardinal is analogous. ■

Since $(n+1)$-Ramseys are normal n-Ramseys we then immediately get the following.

Corollary 3.24 (N.)**.** *Every $(n+1)$-Ramsey is normal n-Ramsey in L, for every $n < \omega$. In particular, $<\omega$-Ramseys are downwards absolute to L.* ■

[6] This appears as Lemma 4.1.2 in [Abramson et al., 1977].

Complete ineffability

In this section we provide a characterisation of the *completely ineffable* cardinals[7] in terms of the α-Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$-Ramsey cardinals, namely the *coherent $<\omega$-Ramseys* as defined in 3.13. Note that a coherent $<\omega$-Ramsey is precisely a cardinal satisfying the ω-filter property, as defined in [Holy and Schlicht, 2018].

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is closely related to the proof of Theorem 3.20 (the indescribability of normal n-Ramseys), but the main difference is that we want everything to occur locally inside our weak κ-models. We'll need another lemma from [Abramson et al., 1977].

Lemma 3.25 (Abramson et al). *Let κ be inaccessible, $X \subseteq \kappa$ and φ a Σ^1_1-formula such that $(V_\kappa, \in, X) \models \varphi[X]$. Then*

$$\{\alpha < \kappa \mid (V_\alpha, \in, X \cap V_\alpha) \models \varphi[X \cap V_\alpha]\}$$

is a club. ■

Theorem 3.26 (N.). *Every coherent $<\omega$-Ramsey is a stationary limit of $<\omega$-Ramseys.*

Proof. Let κ be coherent $<\omega$-Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 \prec H_\theta$ be a weak κ-model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ-model $\mathcal{M} \prec H_\theta$ with an $\mathcal{M}$-normal $\mathcal{M}$-measure $\mu := \bigcup_{n<\omega} \mu_n$ on κ.

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } <\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by $\mathcal{M}$-normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k+1$ steps as in the proof of Theorem 3.20 with $\varphi(\xi)$ being $\ulcorner \xi \text{ is } k\text{-Ramsey} \urcorner$, so

[7] See Appendix A for a definition of the completely ineffable cardinals.

that we get a weak κ-model $\mathcal{M}_{k+1} \prec H_\theta$, an $\mathcal{M}_{k+1}$-normal $\mathcal{M}_{k+1}$-measure $\tilde{\mu}_{k+1}$ on κ, a Σ_1-formula $\varphi(v, v_1, v_2, \dots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \dots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \dots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \tag{2}$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg\varphi[\xi, A^{(0)}_\xi, B^{(0)} \cap V_\xi, A^{(1)}_\xi, B^{(1)} \cap V_\xi, \dots, A^{(k-1)}_\xi, B^{(k-1)} \cap V_\xi, A^{(k)}_\xi], \tag{3}$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \mathrm{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 3.18.

Since κ in particular is Σ^1_1-indescribable, Lemma 3.25 implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ-model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n+1)$-steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of $\mathcal{M}_{k+1}$. In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ, $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A^{(i)}_\xi \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa \rangle$ and note that since $\vec{S}$ is definable it is an element of $\mathcal{M}_{k+2}$ as well. Then $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $\triangle\vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \triangle\vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \triangle\vec{S}$ we see, as in the proof of Theorem 3.18, that

$$V_{\zeta+1} \models \varphi[\zeta, A^{(0)}_\zeta, B^{(0)} \cap V_\zeta, A^{(1)}_\zeta, B^{(1)} \cap V_\zeta, \dots, A^{(k)}_\zeta]$$

since $\triangle\vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} \prec H_\theta$ we have that $\mathcal{M}$ is correct about stationary subsets of κ, meaning that κ is a stationary limit of $<\omega$-Ramseys. ■

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

Definition 3.27. A collection $R \subseteq \mathscr{P}(\kappa)$ is a **stationary class** if

(i) $R \neq \emptyset$;

(ii) every $A \in R$ is stationary in κ;

(iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

$\circ$

Definition 3.28. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$ there is an $H \in R$ homogeneous for f. $\circ$

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in [Abramson et al., 1977].

Theorem 3.29 (N.). *A cardinal κ is completely ineffable if and only if it is coherent $<\omega$-Ramsey.*

Proof. ($\Leftarrow$): Assume κ is coherent $<\omega$-Ramsey, witnessed by strategies $\langle \tau_n \mid n < \omega \rangle$. Let $f : [\kappa]^2 \to 2$ be arbitrary and form the sequence $\langle A^f_\alpha \mid \alpha < \kappa \rangle$ as

$$A^f_\alpha := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let $\mathcal{M}_f$ be a transitive weak κ-model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated $\mathcal{M}_f$-measure on κ given by τ_0.[8] 1-Ramseyness of κ ensures that μ_f is normal, meaning $\triangle \mu_f$ is stationary in κ. Define a new sequence $\vec{B}^f$ as the μ_f-positive part of $\vec{A}^f$.[9] Then $B^f_\alpha \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\triangle \vec{B}^f$ is stationary.

Let now $\mathcal{M}'_f$ be a new transitive weak κ-model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an $\mathcal{M}'_f$-measure $\mu'_f \supseteq \mu_f$ on κ. Then $\triangle \vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \in \mu_f\}$ and $\triangle \vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \notin \mu_f\}$ are both elements of $\mathcal{M}'_f$, so one of them is in μ'_f; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f.

[8]Technically we would have to require that $\mathcal{M}_f \prec H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0, but note that we could simply get a measure on $\mathrm{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to $\mathcal{M}_f$. We will use this throughout the proof.

[9]The *μ-positive part* was defined in Claim 3.23.

Now let $g : [H_f]^2 \to 2$ be arbitrary and again form

$$A_\alpha^g := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ-model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$-measure $\mu_{f,g} \supseteq \mu'_f$ on κ. As before we then get a stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g. We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists \vec{f} : H_{\vec{f}} \subseteq A\},$$

where the $\vec{f}$'s range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \to 2$ and $f_{k+1} : [H_{f_k}] \to 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \to 2$, we can find $H \in R$ which is homogeneous for f. Indeed, if we let $\vec{f}$ be such that $H_{\vec{f}} \subseteq A$, which exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

($\Rightarrow$): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n-Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play $\mathcal{M}_0$ and enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A_\alpha^0 \mid \alpha < \kappa \rangle$ such that $A_\xi^0 \subseteq A_\zeta^0$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A_\xi^0$. Let $<_{\text{lex}}^\alpha$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <_{\text{lex}}^{\min(\alpha,\beta)} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f, using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \restriction \alpha \mid \xi \in H_0 \wedge \xi > \alpha \rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least such η, which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C.

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \restriction \zeta = r_\gamma \restriction \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \restriction \zeta$ holds for every $\gamma \in H_0$, $\gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^0_\xi$ iff $\gamma \in A^0_\xi$. Define thus the $\mathcal{M}_0$-measure μ_0 on κ as

$$\begin{aligned} \mu_0(A^0_\xi) = 1 \quad &\text{iff} \quad (\forall \beta \in H_0 \cap C)(\beta > \xi \to \beta \in A^0_\xi) \\ &\text{iff} \quad (\exists \beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A^0_\xi), \end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa \rangle$ be the μ_0-positive part of $\vec{A}^0$, it is also simple to check that $H_0 \cap C \subseteq \triangle \vec{B}$, making μ_0 normal and hence also both $\mathcal{M}_0$-normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n-Ramsey and let $\langle \mathcal{M}_0, \mu_0, \dots, \mathcal{M}_n, \mu_n, \mathcal{M}_{n+1} \rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}$ as $\vec{A}^{n+1} = \langle A^{n+1}_\xi \mid \xi < \kappa \rangle$, again satisfying that $\xi \leq \zeta$ whenever $A^{n+1}_\xi \subseteq A^{n+1}_\zeta$, but also such that given any $\xi < \kappa$ there are $\zeta, \zeta' \in (\xi, \kappa)$ satisfying that $A^{n+1}_\zeta \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and $A^{n+1}_{\zeta'} \in (\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}) - \mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n \in R$ and C be club in κ such that $H_n \cap C \subseteq \triangle \mu_n$, which exist by our inductive assumption. For $\alpha < \kappa$ define $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^{n+1}_\xi$, and define a colouring $f : [H_n]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

As $H_n \in R$ there is an $H_{n+1} \in R$ homogeneous for f. Just as before, define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least $\eta \in [\alpha, \kappa)$ such that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_{n+1}$ with $\eta \leq \beta < \gamma$, and let D be the club of fixed points of g. As above we get that given any $\alpha, \gamma \in H_{n+1} \cap D$ with $\alpha \leq \gamma$ and $\xi < \alpha$,

$\alpha \in A_\xi^{n+1}$ iff $\gamma \in A_\xi^{n+1}$. Define then the $\mathcal{M}_{n+1}$-measure μ_{n+1} on κ as

$$\begin{aligned}\mu_{n+1}(A_\xi^{n+1}) = 1 \quad &\text{iff} \quad (\forall \beta \in H_{n+1} \cap D \cap C)(\beta > \xi \to \beta \in A_\xi^{n+1}) \\ &\text{iff} \quad (\exists \beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A_\xi^{n+1}).\end{aligned}$$

Then $H_{n+1} \cap D \cap C \subseteq \triangle \mu_{n+1}$, making μ_{n+1} normal, $\mathcal{M}_{n+1}$-normal and good, just as before. It remains to show that $\mu_n \subseteq \mu_{n+1}$. Let thus $A \in \mu_n$ be given, and say $A = A_\xi^{n+1} = A_\eta^n$, where $\vec{A}^n$ was the enumeration of $\mathscr{P}(\kappa) \cap \mathcal{M}_n$ used at the n'th stage. Then by definition of μ_n we get that for every $\beta \in H_n \cap C$ with $\beta > \eta$, $\beta \in A_\eta^n$. We need to show that

$$(\exists \beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A_\xi^{n+1})$$

holds. But here we can simply pick a $\beta > \max(\xi, \eta)$ with $\beta \in H_{n+1} \cap D \cap C \subseteq H_n \cap C$. This shows that $\mu_n \subseteq \mu_{n+1}$, making κ $(n+1)$-Ramsey and thus inductively also coherent $<\omega$-Ramsey. ■

3.1.2 The countable case

This section covers the (strategic) γ-Ramsey cardinals whenever γ has countable cofinality. This case is special because, as we cannot ensure that the final measure in $\mathcal{G}_\gamma^\theta(\kappa)$ is countably complete and so the existence of winning strategies *might* depend on θ, in contrast with the uncountable cofinality case.

[Strategic] ω-Ramsey cardinals

We now move to the strategic ω-Ramsey cardinals and their relationship to the (non-strategic) ω-Ramseys.

Theorem 3.30 (Schindler-N.)**.** *Let $\kappa < \theta$ be regular cardinals. Then κ is faintly θ-measurable iff player II has a winning strategy in $\mathcal{C}_\omega^\theta(\kappa)$.*

Proof. $(\Leftarrow)$: Fix a winning strategy σ for player II in $\mathcal{C}_\omega^\theta(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, H_\theta^V)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega \rangle$ of weak κ-models $\mathcal{M}_n \prec H_\theta^V$ such that $H_\theta^V \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H_θ^V-normal H_θ^V-measure μ on κ.

We claim that $\mathrm{Ult}(H_\theta^V, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n\colon \kappa \to \theta$ such that $g_n \in H_\theta^V$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f\colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{C}_\omega^\theta(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $C_\omega^\theta(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H_\theta^V, \mu)$ is wellfounded, so that the ultrapower embedding $\pi\colon H_\theta^V \to \mathrm{Ult}(H_\theta^V, \mu)$ witnesses that κ is faintly θ-measurable.

$(\Rightarrow)$: Assume that κ is faintly θ-measurable. Let $\mathbb{P}$ be a forcing $\dot\mu$ a $\mathbb{P}$-name for an H_θ^V-normal H_θ^V-measure on κ and $\dot\pi$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}_\omega^\theta(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f_i^n \mid i < k\rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \check\mu \cap \mathcal{M}_n = \check\mu_n \cap \forall i < \check k \colon \dot\pi(\check f_i^n)(\check\kappa) = \check\alpha_i^n \urcorner,$$

with $\mu_n, \alpha_i^n \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f_{i_n}^{k_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check\alpha_{i_{n+1}}^{k_{n+1}} < \check\alpha_{i_n}^{k_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

We note that the above Theorem along with our results from Chapter 2 shows that winning the Cohen games doesn't guarantee weak compactness.

Corollary 3.31 (N.). *Let κ be inaccessible.*

(i) If player II wins $\mathcal{C}^\theta_\omega(\kappa)$ for all regular $\theta > \kappa$ then κ is not necessarily weakly compact;

(ii) If player II wins $\mathcal{C}_\kappa(\kappa)$ then κ is weakly compact.

PROOF. The first claim is directly by Proposition 2.45 and Theorem 3.30, and the second claim is because the hypothesis implies that player II wins $\mathcal{G}_0(\kappa)$ so that inaccesibility of κ makes κ weakly compact — see e.g. [Gitman, 2011] for this characterisation of weak compactness. ■

Here's a near-analogous result of Theorem 3.30 for the $\mathcal{G}^\theta_\omega(\kappa)$ game.

Theorem 3.32 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$, and if player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$ then κ is faintly θ-power-measurable. In particular, $\mathcal{G}^\theta_\omega(\kappa)^L \sim \mathcal{C}^\theta_\omega(\kappa)^L$.*

PROOF. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi\colon H^V_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a name $\dot\mu$ and a $\mathbb{P}$-condition p such that

$$p \Vdash \ulcorner \dot\mu \text{ is a weakly amenable } \check H_\theta\text{-normal } \check H_\theta\text{-measure with a wellfounded ultrapower}\urcorner.$$

We now define a strategy σ for player II in $\mathcal{G}^\theta_\omega(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H^V_\theta$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n\colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \le p$, $p_n \le p_k$ for every $k \le n$ and that

$$p_n \Vdash \ulcorner \dot\mu \cap \check{\mathcal{M}}_n = \check\mu_n \cap \check\mu_n = \dot\mu \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma 1.5 we get that $\pi \upharpoonright \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega} \mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot{\pi}(\check{g}_{n+1})(\check{\kappa}) < \dot{\pi}(\check{g}_n)(\check{\kappa}) \urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check{\pi}_{n+1}(\check{g}_{n+1})(\check{\kappa}) < \check{\pi}_n(\check{g}_n)(\check{\kappa}) \urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy. ■

We get the following immediate corollary.

Corollary 3.33 (N.-Schindler)**.** *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ■

Note also that the proof of Theorem 3.32 shows that whenever κ is strategic ω-Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H_ν^V-normal H_ν-measure on κ.

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω-Ramsey cardinals and ω-Ramsey cardinals lie, namely that strategic ω-Ramseys are equiconsistent with *remarkables* and ω-Ramseys are strictly below. Theorem 4.8 of [Gitman and Welch, 2011] showed that 2-iterables are limits of remarkables, and our Propositions 3.9 and 3.41 shows that ω-Ramseys are limits of 1-iterables, so that the strategic ω-Ramseys and the ω-Ramseys both lie strictly between the 2-iterables and 1-iterables. It was shown in [Holy and Schlicht, 2018] that ω-Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by [Schindler, 2000b], and [Gitman and Schindler, 2018] showed the following two equivalent formulations.

Definition 3.34. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

(i) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H^V_\lambda \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H^V_\nu \to M$ with critical point κ and $\pi(\kappa) > \lambda$;

(ii) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^\lambda M \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H^V_\nu \to M$ with critical point κ and $\pi(\kappa) > \lambda$.

$\circ$

Theorem 3.35 (N.)**.** *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models \ulcorner$there is a proper class of virtually measurables$\urcorner$. In particular, the two notions are equiconsistent.*

Proof. Virtually measurables are downwards absolute to L by Lemma 2.42, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H^V_\lambda \subseteq M$ and forcing poset $\mathbb{P}$ it holds that, in $V^{\mathbb{P}}$, there's no elementary embedding $\pi : H^V_\nu \to M$ with $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν-measurable to fix a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's an elementary $\pi : H^V_\nu \to M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H^V_\nu = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we're assuming that κ isn't remarkable. Then by restricting the generic embedding to H^V_κ we get that $H^V_\kappa \prec H^M_{\pi(\kappa)} = H^V_{\pi(\kappa)}$, using that $\pi(\kappa) < \nu$ and $H^V_\nu = H^M_\nu$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H^V_ν since $\pi(\kappa) < \nu$, and as $H^V_\nu \prec_1 V$ we get that $\pi(\kappa)$ *is* a cardinal. But then, again using that $H_{\pi(\kappa)} \prec_1 V$, κ is virtually measurable in $H^V_{\pi(\kappa)}$ since being virtually measurable is Π_2. This means that for every $\xi < \kappa$ it holds that

$$H^V_{\pi(\kappa)} \models \exists\alpha > \xi : \ulcorner\alpha \text{ is virtually measurable}\urcorner,$$

implying that $H^V_\kappa \models \ulcorner$There is a proper class of virtually measurables$\urcorner$. ■

Now Theorem 3.35 and Corollary 3.33 yield the following immediate corollary.

Corollary 3.36 (N.-Schindler)**.** *Let κ be strategic ω-Ramsey. Then either κ is remarkable in L or otherwise*

$$L_\kappa \models \ulcorner\textit{there is a proper class of strategic } \omega\textit{-Ramseys}\urcorner.$$

In particular, the two notions are equiconsistent. ■

Now, using these results we show that the strategic ω-Ramseys have strictly stronger consistency strength than the ω-Ramseys.

Theorem 3.37 (N.)**.** *Remarkable cardinals are strategic ω-Ramsey limits of ω-Ramsey cardinals.*

Proof. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ-sequences and a generic elementary embedding $\pi : H^V_\nu \to M$ for some $\nu > 2^\kappa$. We will show that κ is ω-Ramsey in M. Note that remarkables are clearly virtually measurable, and thus by Theorem 3.32 also strategic ω-Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}^\theta_\omega(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}^\theta_\omega(\kappa)^M$. Since M is closed under 2^κ-sequences it means that $\mathscr{P}(\mathscr{P}(\kappa)) \subseteq M$ and thus that M contains all possible filters on κ. We let player II follow τ, which produces a play $\sigma * \tau$ in which player II wins. But all player II's moves are in $\mathscr{P}(\mathscr{P}(\kappa))$ and hence in M, and as M is furthermore closed under ω-

sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω-Ramsey in M.

This also implies that κ is a limit of ω-Ramseys in H_ν. But as κ is remarkable it holds that $H_\kappa \prec_2 V$, in analogy with the same property for strongs and supercompacts, and as being ω-Ramsey is a Π_2-notion this means that κ *is* a limit of ω-Ramseys. ■

This immediately yields the following corollary.

Corollary 3.38 (N.-Schindler)**.** *If κ is a strategic ω-Ramsey cardinal then*

$$L_\kappa \models \ulcorner \textit{there is a proper class of } \omega\textit{-Ramseys} \urcorner. \qquad \dashv$$

(ω, α)-Ramsey cardinals

A natural generalisation of the γ-Ramsey definition is to require more iterability of the last measure. Of course, by Proposition 3.9 we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when $\operatorname{cof}\gamma > \omega$ so the next definition is only interesting whenever $\operatorname{cof}\gamma = \omega$.

Definition 3.39 (N.)**.** Let α, β be ordinals. Then a cardinal κ is **(α, β)-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ for all regular $\theta > \kappa$.[10] ○

Definition 3.40 (Gitman)**.** A cardinal κ is **α-iterable** if for every $A \subseteq \kappa$ there exists a *transitive* weak κ-model $\mathcal{M}$ with $A \in \mathcal{M}$ and an α-good $\mathcal{M}$-measure μ on $\mathcal{M}$. ○

Proposition 3.41. *If $\beta > 0$ then every (α, β)-Ramsey is a β-iterable stationary limit of β-iterables.*

Proof. Let $(\mathcal{M}, \in, \mu)$ be a result of a play of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$ in which player II won. Then the transitive collapse of $(\mathcal{M}, \in, \mu)$ witnesses that κ is β-iterable, since μ is β-good by definition of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$.

[10] Note that an α-Ramsey cardinal is the same as an $(\alpha, 0)$-Ramsey cardinal.

That κ is β-iterable is reflected to some H_θ, so let now $(\mathcal{N}, \in, \nu)$ be a result of a play of $\mathcal{G}_\alpha^\theta(\kappa, \beta)$ in which player II won. Then $\mathcal{N} \prec H_\theta$, so that κ is also β-iterable in $\mathcal{N}$. Since being β-iterable is witnessed by a subset of κ and $\beta > 0$ implies[11] that we get a κ-powerset preserving $j : \mathcal{N} \to \mathcal{P}$, $\mathcal{P}$ also thinks that κ is β-iterable, making κ a stationary limit of β-iterables by elementarity. ■

We now move towards Theorem 3.45 which gives an upper consistency bound for the (ω, α)-Ramseys. We first recall a few definitions and a folklore lemma.

Definition 3.42. For an infinite ordinal α, a cardinal κ is **α-Erdős** for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \to \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c; i.e. that $|c``[H]^n| \leq 1$ holds for every $n < \omega$. ○

Definition 3.43. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$.[12] ○

Lemma 3.44 (Folklore). *Let κ be α-Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure $\mathcal{M}$ in a countable language $\mathcal{L}$ with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

Proof. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all $\mathcal{L}$-formulas and define $c : [C]^{<\omega} \to \kappa$ as follows. For an increasing sequence $\alpha_1 < \cdots < \alpha_{2n} \in C$ let

$$\begin{aligned} c(\{\alpha_1, \dots, \alpha_{2n}\}) := \text{ the least } \lambda < \alpha_1 \text{ such that} & \\ & \exists \delta_1 < \cdots \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \dots, \delta_k \rangle \wedge \\ & \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}] \end{aligned}$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α-Erdős we get a homogeneous $I \in [C]^\alpha$ for c; i.e. that $|c``[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \dots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \dots, \alpha_{2n} \in I$, as otherwise

[11] Recall that β-good for $\beta > 0$ in particular implies weak amenability.
[12] Note that this terminology is not at all related to remarkable *cardinals*.

there exists an $m < \omega$ and $\delta_1 < \cdots \delta_k$ such that for any $\alpha_1 < \ldots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}]. \qquad (\dagger)$$

But then simply pick $\alpha_1 < \ldots \alpha_{2n} < \alpha'_1 < \cdots < \alpha'_{2n}$ so that both $\{\alpha_1, \ldots, \alpha_{2n}\}$ and $\{\alpha'_1, \ldots, \alpha'_{2n}\}$ witnesses (†); then either $\{\alpha_1, \ldots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \ldots, \alpha_n, \alpha'_{n+1}, \ldots, \alpha'_{2n}\}$ also witnesses that (†) fails, $\lightning$. ■

Theorem 3.45 (N.). *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α-Erdős cardinal is a limit of (ω, α)-Ramsey cardinals.*

Proof. Let κ be α-Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 3.44 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I. We will show that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in [Holy and Schlicht, 2018] it suffices to find a weak ι-model $\mathcal{M} \prec H_\theta$ and an α-good $\mathcal{M}$-measure on ι. Define

$$\mathcal{M} := \operatorname{Hull}^{H_\theta}(\iota \cup I) \prec H_\theta$$

and let $\pi : I \to I$ be the right-shift map. Since I is remarkable, I $(= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \ldots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι-powerset preserving we get that $\mathcal{M} \prec H_\theta$ is a weak ι-model satisfying ZFC^- with a 1-good $\mathcal{M}$-measure μ_j on ι. Furthermore, as we can linearly iterate $\mathcal{M}$ simply by applying j we get an α-iteration of $\mathcal{M}$ since there are α-many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \ldots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}_\omega^\theta(\eta,\alpha)$. If we repeat this procedure for all regular $\theta>\kappa$ we get by the pidgeon hole principle that κ is a limit of (ω,α)-Ramsey cardinals. ■

As Theorem 4.5 in [Gitman and Schindler, 2018] shows that $(\alpha+1)$-iterable cardinals have α-Erdős cardinals below them for $\alpha\geq\omega$ additively closed, this shows that the (ω,α)-Ramseys form a strict hierarchy. Further, as α-Erdős cardinals are consistent with $V=L$ when $\alpha<\omega_1^L$ and ω_1-iterable cardinals aren't consistent with $V=L$, we also get that (ω,α)-Ramsey cardinals are consistent with $V=L$ if $\alpha<\omega_1^L$ and that they aren't if $\alpha=\omega_1$.

[Strategic] $(\omega{+}1)$-Ramsey cardinals

The next step is then to consider $(\omega{+}1)$-Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in [Mitchell, 1979] and in the proof of Lemma 1.3 in [Donder et al., 1981] — see also [Dodd, 1982] and [Gitman, 2011].

Theorem 3.46 (Dodd, Mitchell). *A cardinal κ is Ramsey if and only if every $A\subseteq\kappa$ is an element of a weak κ-model $\mathcal{M}$ such that there exists a weakly amenable countably complete $\mathcal{M}$-measure on κ.* ■

The following theorem then supplies us with a lower bound for the strength of the $(\omega{+}1)$-Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 3.57, but we include this Ramsey lower bound as well for completeness.

Theorem 3.47 (N.). *Every $(\omega{+}1)$-Ramsey cardinal is a Ramsey limit of Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey and $A\subseteq\kappa$. Let σ be a strategy for player I in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A\in\mathcal{M}_0$ and $\mu_\alpha\in\mathcal{M}_{\alpha+1}$ for all $\alpha\leq\omega$. Then σ isn't winning as κ is $(\omega{+}1)$-Ramsey, so we may fix a play $\sigma*\vec{\mu}_\alpha$ of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable

$\mathcal{M}_\omega$-measure on κ, and by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 3.46.

Since κ is Ramsey, $\mathcal{M}_\omega \models \ulcorner\kappa$ is Ramsey$\urcorner$ as well. Letting $j : \mathcal{M}_\omega \to \mathcal{N}$ be the κ-powerset preservering embedding induced by μ_ω, we also get that $\mathcal{N} \models \ulcorner\kappa$ is Ramsey$\urcorner$ by κ-powerset preservation. This then implies that κ *is* a stationary limit of Ramsey cardinals inside $\mathcal{M}_\omega$, and thus also in V by elementarity. ■

As for the *consistency* strength of the strategic (ω+1)-Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal — see e.g. p. 249 in [Kanamori and Magidor, 1978].

Theorem 3.48 (N.). *If κ is a strategic $(\omega{+}1)$-Ramsey cardinal then, in $V^{\mathrm{Col}(\omega,2^\kappa)}$, there's a transitive class N and an elementary embedding $j : V \to N$ with $\operatorname{crit} j = \kappa$. In particular, the existence of a strategic $(\omega{+}1)$-Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

Proof. Set $\mathbb{P} := \mathrm{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$. Let $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name of an ω-sequence $\langle\mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n \prec H^V_{\kappa^+}$ and $\mathscr{P}(\kappa)^V \subseteq \bigcup_{n<\omega}\mathcal{M}_n$, and let $\dot{\mu}$ be a $\mathbb{P}$-name for the ω-sequence of σ-responses to the $\mathcal{M}_n$'s in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)^V$.

Assume that there's a $\mathbb{P}$-condition p which forces the generic ultrapower $\mathrm{Ult}(V, \bigcup_n \dot{\mu}_n)$ to be illfounded, meaning that we can fix a $\mathbb{P}$-name $\dot{f}$ for an ω-sequence $\langle f_n \mid n < \omega\rangle$ such that

$$p \Vdash \dot{X}_n := \{\alpha < \kappa \mid \dot{f}_{n+1}(\alpha) < \dot{f}_n(\alpha)\} \in \bigcup_{n<\omega} \dot{\mu}_n.$$

Now, in V, we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} \prec H_\theta$ such that $\dot{\mathcal{M}}, \dot{\mu}, \dot{f}, H^V_{\kappa^+}, \sigma, p \in \mathcal{N}$. We can find an $\mathcal{N}$-generic $g \subseteq \mathbb{P}^{\mathcal{N}}$ in V with $p \in g$ since $\mathcal{N}$ is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}^g_n * \dot{\mu}^g_n$ is a play of $\mathcal{G}^{\kappa^+}_\omega(\kappa)^V$ which is according to σ, meaning that $\bigcup_{n<\omega}\dot{\mu}^g_n$ is normal and in particular countably complete (in V). Then $\bigcap_{n<\omega} \dot{X}_n^g \neq \emptyset$, but if $\alpha \in \bigcap_{n<\omega} \dot{X}_n^g$ then $\langle \dot{f}_n^g(\alpha) \mid n < \omega \rangle$ is a strictly decreasing ω-sequence of ordinals, $\lightning$. This means that $\mathrm{Ult}(V, \bigcup_n \mu_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in [Kellner and Shelah, 2011]. ■

The above Theorem 3.48 then answers Question 9.2 in [Holy and Schlicht, 2018] in the negative, asking if λ-Ramseys are strategic λ-Ramseys for uncountable cardinals λ, as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.

3.1.3 The general case

Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from [Gitman, 2011] and investigate further connections between these and the α-Ramsey cardinals. First, a definition.

Definition 3.49 (Gitman). A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ. If furthermore $\mathcal{M} \prec H_{\kappa^+}$ then we say that κ is **super Ramsey**. ○

Note that since the model $\mathcal{M}$ in question is a *κ-model* it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$-sequences. [Gitman, 2011] shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M} \models \mathsf{ZFC}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ.

Now, a first connection between the α-Ramseys and the strongly- and super Ramseys is the result in [Holy and Schlicht, 2018] that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

Theorem 3.50 (N.-Welch)**.** *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

Proof. Let κ be strongly Ramsey and let $\mathcal{M} \models \mathsf{ZFC}$ be a transitive κ-model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^{\mathcal{M}}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of $\mathcal{M}$ by weak amenability of μ. As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside $\mathcal{M}$, so $\mathcal{M}$ sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \to \mathcal{N}$ be the induced embedding, κ-powerset preservation of j_μ implies that μ is also a weakly amenable $\mathcal{N}$-normal $\mathcal{N}$-measure on κ. This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in $\mathcal{N}$, entailing that it is a stationary limit of almost fully Ramseys in $\mathcal{M}$. But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in [Holy and Schlicht, 2018], so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and $\mathcal{M}$ being transitive implies that κ really *is* a stationary limit of almost fully Ramseys. ■

Downwards absoluteness to K

Lastly, we consider the question of whether the α-Ramseys are downwards absolute to K, which turns out to at least be true in many cases. The below Theorem 3.52 then also answers Question 9.4 from [Holy and Schlicht, 2018] in the positive, asking whether α-Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of $0^\P$.

Definition 3.51. $0^\P$ is "the sharp for a strong cardinal", meaning the minimal sound active mouse $\mathcal{M}$ with $\mathcal{M} \mid \mathrm{crit}(\dot{F}^{\mathcal{M}}) \models \ulcorner$There exists a strong cardinal$\urcorner$, with $\dot{F}^{\mathcal{M}}$ being the top extender of $\mathcal{M}$. ○

Theorem 3.52 (N.-Welch). *Assume $0^\P$ does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ-Ramsey. Then $K \models$ $\ulcorner\kappa$ is a λ-Ramsey cardinal$\urcorner$.*

Proof. Note first that $\kappa^{+K} = \kappa^+$ by [Schindler, 1997], since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ-models $\mathcal{M} \prec K|\kappa^+$. We can then define a strategy $\tilde\sigma$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde\sigma(\emptyset) := \text{Hull}^{H_{\kappa^+}}(K|\kappa \cup \sigma(\emptyset))$. Assuming now that $\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$, we have two cases. If $\tilde\mu_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle\mathcal{M}_\alpha \mid \alpha < \gamma\rangle$ be the corresponding models played in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde\sigma(\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle) := \text{Hull}^{H_{\kappa^+}}(K|\kappa \cup \sigma(\langle\mathcal{M}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle)),$$

and otherwise let $\tilde\sigma$ play arbitrarily. As κ is λ-Ramsey (in V) there exists a play $\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha \leq \lambda\rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ \prec K|\kappa^+$ so let $\mathcal{N}$ be the transitive collapse of $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+$. But if $j : \mathcal{N} \to K|\kappa^+$ is the uncollapse then $\text{crit}\, j$ is both an $\mathcal{N}$-cardinal and also $> \kappa$ because we ensured that $K|\kappa \subseteq \mathcal{N}$. This means that $j = \text{id}$ because κ is the largest $\mathcal{N}$-cardinal by elementarity in $K|\kappa^+$, so that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ = \mathcal{N}$ is a transitive elementary substructure of $K|\kappa^+$, making it an initial segment of K.

Now, since $\mu := \tilde\mu_\lambda$ is a countably complete weakly amenable $K|o(\mathcal{N})$-measure[13], the "beaver argument"[14] shows that $\mu \in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu \cap \mathcal{N} \in K$ whenever player I plays $\mathcal{N}$. Since $\mu = \tilde\mu_\lambda$ we also have that $\mu \cap \mathcal{M}_\alpha = \tilde\mu_\alpha \cap \mathcal{M}_\alpha$, so that σ will eventually play $\mathcal{N}$, making τ win against σ.[15] ■

Note that the only thing we used $\text{cof}\,\lambda > \omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine-

[13] Here we use that $\mathcal{N} \lhd K$.

[14] See Appendix C for details regarding the beaver argument.

[15] Note that τ is not necessarily a winning strategy — all we know is that it is winning against this particular strategy σ.

or normal α-Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

Corollary 3.53 (N.-Welch)**.** *Assume $0^\P$ does not exist and let α be any limit ordinal. Then every genuine- and every normal α-Ramsey cardinal is downwards absolute to K. In particular, if α is a limit of limit ordinals then every $<\alpha$-Ramsey cardinal is downwards absolute to K as well.* ■

Indiscernible games

We now move to the strategic versions of the α-Ramsey hierarchy. The first thing we want to do is define *α-very Ramsey cardinals*, introduced in [Sharpe and Welch, 2011], and show the tight connection between these and the strategic α-Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 3.43.

Definition 3.54. A **good set of indiscernibles** for a structure $\mathcal{M}$ is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for $\mathcal{M}$ such that $\mathcal{M}|\iota \prec \mathcal{M}$ for any $\iota \in I$. ○

Definition 3.55 (Sharpe-Welch)**.** Define the **indiscernible game** $G^I_\gamma(\kappa)$ in γ many rounds as follows

I	$\mathcal{M}_0$		$\mathcal{M}_1$		$\mathcal{M}_2$		$\cdots$
II		I_0		I_1		I_2	$\cdots$

Here $\mathcal{M}_\alpha$ is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for $\mathcal{M}_\alpha$ and the I_α's are $\subseteq$-decreasing. Player II wins iff they can continue playing through all the rounds. ○

Definition 3.56 (Sharpe-Welch)**.** A cardinal κ is **γ-very Ramsey** if player II has a winning strategy in the game $G^I_\gamma(\kappa)$. ○

The next couple of results concerns the connection between the strategic α-Ramseys and the α-very Ramseys. We start with the following.

Theorem 3.57 (N.). *Every $(\omega+1)$-Ramsey is an ω-very Ramsey stationary limit of ω-very Ramseys.*

PROOF. Let κ be $(\omega+1)$-Ramsey. We will describe a winning strategy for player II in the indiscernible game $G^I_\omega(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G^I_\omega(\kappa)$ then let player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ play

$$\mathcal{H}_0 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) \prec H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ which starts off with $\mathcal{H}_0$ and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}^{(\alpha)}_0 := \mathcal{M}_\alpha$ and $\mu^{(\alpha)}_0 := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{lccccccc} \mathrm{I} & \mathcal{H}^{(0)}_0 & & \cdots & & \mathcal{H}^{(\omega)}_0 & & \mathcal{H}^{(\omega+1)}_0 & \\ \mathrm{II} & & \mu^{(0)}_0 & & \cdots & & \mu^{(\omega)}_0 & & \mu^{(\omega+1)}_0 \end{array}$$

By the choice of player I's strategy we get that $\mu^{(\omega)}_0$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ (it's even normal). Now Lemma 2.9 of [Sharpe and Welch, 2011] gives us a set of good indiscernibles $I_0 \in \mu^{(\omega)}_0$ for $\mathcal{M}_0$, as $\mathcal{M}_0 \in \mathcal{H}^{(\omega)}_0$ and $\mu^{(\omega)}_0$ is a countably complete weakly amenable $\mathcal{H}^{(\omega)}_0$-normal $\mathcal{H}^{(\omega)}_0$-measure on κ. Let player II play I_0 in $G^I_\omega(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G^I_\omega(\kappa)$.

$$\begin{array}{llll} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 \\ \mathrm{II} & & I_0 & \end{array}$$

Since $\mu^{(\omega)}_0 = \bigcup_n \mu^{(n)}_0$ we must have that $I_0 \in \mu^{(n_0)}_0$ for some $n_0 < \omega$. In the (n_0+1)'st round of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}^{(k)}_0, \mu^{(k)}_0 \mid k \leq n_0\rangle\}) \prec H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ thus looks like the following so far.

$$\begin{array}{lcccccc} \text{I} & \mathcal{H}_0^{(0)} & & \cdots & \mathcal{H}_0^{(n_0)} & & \mathcal{H}_1 \\ \text{II} & & \mu_0^{(0)} & \cdots & & \mu_0^{(n_0)} & \end{array}$$

Now player II in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ is not losing at round n_0, so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I_1' \in \mu_1^{(n_1)}$ of good indiscernibles for $\mathcal{M}_1$, where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G^I_\omega(\kappa)$ play $I_1 := I_0 \cap I_1' \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G^I_\omega(\kappa)$, making κ ω-very Ramsey.

As for showing that κ is a stationary limit of ω-very Ramseys, let $\mathcal{M} \prec H_{\kappa^+}$ be a weak κ-model with a weakly amenable countably complete $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ, which exists by Theorem 3.47 as κ is $(\omega+1)$-Ramsey. Then by elementarity $\mathcal{M} \models \ulcorner\kappa$ is ω-very Ramsey$\urcorner$ and since κ being ω-very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ-ultrapower, meaning that κ is a stationary limit of ω-very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

Theorem 3.58 (N.)**.** *For limit ordinals α, every coherent $<\omega\alpha$-Ramsey is $\omega\alpha$-very Ramsey.*

Proof. This is basically the same proof as the proof of Theorem 3.57. We do the "going-back" trick in ω-chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}^{\kappa^+}_{\omega\alpha}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$-Ramseyness. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω-chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 3.57. ■

As for going from the α-very Ramseys to the strategic α-Ramseys we got the following.

Theorem 3.59 (N.)**.** *For γ any ordinal, every coherent $<\gamma$-very Ramsey*[16] *is coherent $<\gamma$-Ramsey.*[17]

Proof. The reason why we work with $<\gamma$-Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ-very Ramsey and let τ be a winning strategy for player II in $G^I_\zeta(\kappa)$. Let $\mathcal{M}_\alpha \prec H_\theta$ be any move by player I in the α'th round of $\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in $\mathcal{M}_\alpha$ and form now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G^I_\zeta(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for $\mathcal{N}_\alpha$ such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in [Sharpe and Welch, 2011] we get a structure $\mathcal{P}_\alpha$ with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a $\mathcal{P}_\alpha$-measure $\tilde{\mu}_\alpha$ on κ, generated by I_α.[18] Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α's are generated by the I_α's, the μ_α's are $\subseteq$-increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$-Ramsey. ∎

The following result is then a direct corollary of Theorems 3.58 and 3.59.

[16] Here the coherency again just means that the winning strategies σ_α for player II in $G^I_\alpha(\kappa)$ are $\subseteq$-increasing.

[17] Here a "coherent $<\gamma$-very Ramsey cardinal" is defined from γ-very Ramseys in the same way as coherent $<\gamma$-Ramsey cardinals is defined from γ-Ramseys. When γ is a limit ordinal then coherent $<\gamma$-very Ramseys are precisely the same as γ-very Ramseys, so this is solely to "subtract one" when γ is a successor ordinal — i.e. a coherent $<(\gamma+1)$-very Ramsey cardinal is the same thing as a γ-very Ramsey cardinal.

[18] By *generated* here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α.

Corollary 3.60 (N.)**.** *For limit ordinals α, κ is $\omega\alpha$-very Ramsey iff it is coherent $<\omega\alpha$-Ramsey. In particular, κ is λ-very Ramsey iff it is strategic λ-Ramsey for any λ with uncountable cofinality.* ■

We can now use this equivalence to transfer results from the α-very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in [Feng, 1990]. A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in [Sharpe and Welch, 2011]. We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in [Sharpe and Welch, 2011].

Theorem 3.61 (Sharpe-Welch)**.** *A cardinal is completely Ramsey if and only if it is ω-very Ramsey.* ■

This, together with Theorem 3.57, immediately yields the following strengthening of Theorem 3.47.

Corollary 3.62 (N.)**.** *Every $(\omega+1)$-Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* ■

The above Theorem 3.59 also yields the following consequence.

Corollary 3.63 (N.)**.** *Every completely Ramsey cardinal is completely ineffable.*

Proof. From Theorem 3.61 we have that being completely Ramsey is equivalent to being ω-very Ramsey, so the above Theorem 3.59 then yields that a completely Ramsey cardinal is coherent $<\omega$-Ramsey, which we saw in Theorem 3.29 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 3.60 yields that strategic ω_1-Ramsey cardinals are ω_1-very Ramsey, and Theorem 3.50 in [Sharpe and Welch, 2011] states that ω_1-very Ramseys are measurable in the core model K, assuming

$0^\P$ doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

Theorem 3.64 (Welch)**.** *Assuming $0^\P$ doesn't exist, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Let κ be strategic ω_1-Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \text{Col}(\omega_1, \kappa^+)$ is V-generic. Since $\text{Col}(\omega_1, \kappa^+)$ is ω-closed, V and $V[g]$ have the same countable sequences of V, so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V.

Now let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of regular K-cardinals cofinal in κ^+, let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \text{Hull}^{H_\theta}(K|\kappa_\alpha) \prec H_\theta$ and player II follow τ. This results in a countably complete weakly amenable K-measure μ_{ω_1}, which the "beaver argument"[19] then shows is actually an element of K, making κ measurable in K. ■

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1-Ramsey cardinal is also measurable in Steel's core model below a Woodin[20], a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from [Schindler, 2006a].[21]

Theorem 3.65 (Schindler)**.** *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ, and let $\pi : V \to \text{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.* ■

[19]See Appendix C for details regarding the beaver argument.
[20]See Appendix C.
[21]That paper assumes the existence of a measurable as well, but by [Jensen and Steel, 2013a] we can omit that here.

Theorem 3.66 (Schindler). *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of weak κ-models $M_\alpha \prec H_\theta^V$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H_{\kappa^+}^V \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\mathrm{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Let $j : M_{\omega_1} \to \mathrm{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H_{\kappa^+}^V \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ with N transitive.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 3.65 applies, meaning that $\pi \restriction K : K \to K^N$ is an iteration map with critical point κ, making κ measurable in K. ■

3.2 Ideals

Historically, the idea of considering elementary embeddings existing only in generic extensions has been around for a while, but it all started as an analysis of *ideals*. *Precipitous ideals* were introduced in [Galvin et al., 1978] and further analysed in [Jech et al., 1980], being ideals that give rise to wellfounded generic ultrapowers[22].

In this section we will introduce the *ideally measurable cardinals*, essentially just switching perspective from the ideals themselves to the cardinals they are on. We then proceed to show how these cardinals relate to "pure" generic cardinals, being proper class versions of the faintly measurable cardinals that we have considered throughout Chapter 2. We start with a definition of the latter.

[22]See Appendix E for some preliminaries on these concepts.

Definition 3.67 (GBC)**.** A cardinal κ is **generically measurable** if there is a generic extension $V[g]$, a transitive class $\mathcal{N} \subseteq V[g]$ and a generic elementary embedding $\pi\colon V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$. ∘

Note that, trivially, every generically measurable cardinal is faintly measurable. The corresponding ideal version of this is then the following.

Definition 3.68. A cardinal κ is **ideally measurable** if there exists an ideal $\mathcal{I}$ on θ such that the generic ultrapower $\mathrm{Ult}(V, \mathcal{I})$ is wellfounded in $V^{\mathbb{P}}$ for $\mathbb{P} := \mathscr{P}^V(\kappa)/\mathcal{I}$. ∘

It should also be noted that [Claverie and Schindler, 2016] generalised the concept of ideally measurables to *ideally strong cardinals* by introducing the concept of *ideal extenders* to capture the strongness properties.

Throughout this section we will be interested in how properties of the *forcings* affect the large cardinal structure of a critical point of a generic embedding. We thus define the following.

Definition 3.69. Let θ be a regular uncountable cardinal, $\kappa < \theta$ a cardinal and $\Phi(\kappa)$ a poset property[23]. Then κ is $\Phi(\kappa)$ **faintly θ-measurable** if it is faintly θ-measurable, witnessed by a forcing poset satisfying $\Phi(\kappa)$. Similarly, κ is $\Phi(\kappa)$ **generically measurable** if it is generically measurable with the associated forcing satisfying $\Phi(\kappa)$. ∘

Note that ω-distributive faintly θ-measurable cardinals are equivalent to ω-distributive generically measurable cardinals for all regular θ since wellfoundedness becomes automatic.

Definition 3.70. A poset property $\Phi(\kappa)$ is **ideal-absolute** if whenever κ satisfies that there's a $\Phi(\kappa)$ forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure μ on κ, then there's an ideal I on κ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(v)$. ∘

[23] Examples of these are having the κ-chain condition, being κ-closed, κ-distributive, κ-Knaster, κ-sized and so on. Formally speaking, $\Phi(\kappa)$ is a first-order formula $\varphi(\kappa, \mathbb{P})$ which is true iff $\mathbb{P}$ is a poset, κ is a cardinal and some first-order formula $\psi(\kappa, \mathbb{P})$ is true.

Note that this is *almost* saying that $\Phi(\kappa)$ ideally measurables are equivalent to $\Phi(\kappa)$ generically measurables, but the only difference is that these definitions require well-foundedness of the target model.

A typical ideal that we will be utilising is the following.

Definition 3.71. Let κ be a regular cardinal, $\mathbb{P}$ a poset and $\dot{\mu}$ a $\mathbb{P}$-name for a V-normal V-measure on κ. Then the **induced ideal** is

$$\mathcal{I}(\mathbb{P}, \dot{\mu}) := \{X \subseteq \kappa \mid \big|\big|\check{X} \in \dot{\mu}\big|\big|_{\mathcal{B}(\mathbb{P})} = 0\},$$

where $\mathcal{B}(\mathbb{P})$ is the boolean completion of $\mathbb{P}$. ∘

Note that if the generic measure μ is V-normal then $\mathcal{I}(\mathbb{P}, \dot{\mu})$ is also normal. This ideal will witness our first ideal-absoluteness result, which is a simple rephrasing of a folklore result.

Theorem 3.72 (Folklore)**.** *"The κ^+-chain condition" is ideal-absolute.*

Proof. Assume $\mathbb{P}$ has the κ^+-chain condition such that there's a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot{\mu})$ — we will show that $\mathscr{P}(\kappa)/I$ has the κ^+-chain condition. Assume not and let $\langle X_\alpha \mid \alpha < \kappa^+\rangle$ be an antichain of $\mathscr{P}(\kappa)/I$, which by normality of I we may assume is pairwise almost disjoint. But this then makes $\langle \big|\big|\check{X}_\alpha \in \dot{\mu}\big|\big|_{\mathcal{B}(\mathbb{P})} \mid \alpha < \kappa^+\rangle$ an antichain of $\mathbb{P}$ of size κ^+, $\lightning$. ■

We next move to distributivity. This property is especially interesting in the context of our generic large cardinals, as an ideal I on some cardinal κ is ω-distributive precisely if it's precipitous[24], so that carrying an ω-distributive ideal coincides with our definition of *ideally measurable.*

Theorem 3.73 (N.)**.** *"$<\lambda$-distributivity" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*

[24]See [Jech et al., 1980] and [Foreman, 1983].

Proof. Assume that $\mathbb{P}$ is a $<\lambda$-distributive forcing such that there exists a $\mathbb{P}$-name $\dot\mu$ for a V-normal V-measure on κ. Let $\mathcal{I} := \mathcal{I}(\mathbb{P}, \dot\mu)$ — we'll show that $\mathscr{P}(\kappa)/\mathcal{I}$ is $<\lambda$-distributive.

Let $\gamma < \lambda$ and let $\vec{\mathcal{A}}$ be a γ-sequence of maximum antichains $\mathcal{A}_\alpha \subseteq \mathscr{P}(\kappa)/\mathcal{I}$ such that $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ for $\alpha \leq \beta$. We have to show that there's a maximal antichain $\mathcal{A}$ which refines all the antichains in $\vec{\mathcal{A}}$.

Now define for every $\alpha < \gamma$ the sets

$$\mathcal{A}^*_\alpha := \{||\check{X} \in \dot\mu||_{\mathcal{B}(\mathbb{P})} \mid X \in \mathcal{A}_\alpha\}.$$

Note that $\mathcal{A}^*_\alpha$ is an antichain in $\mathbb{P}$. They're also maximal, because if $p \in \mathbb{P}$ was incompatible with every condition in $\mathcal{A}^*_\alpha$ then, letting $X := \bigcap \mathcal{A}_\alpha$, we have that p is compatible with $||\check{X} \in \dot\mu||_{\mathcal{B}(\mathbb{P})}$, so that $X \in \mathcal{I}^+$. But X is incompatible with everything in $\mathcal{A}_\alpha$, contradicting that $\mathcal{A}_\alpha$ is maximal.

By $<\lambda$-distributivity of $\mathbb{P}$ we get an antichain $\mathcal{A}^*$ which refines all the antichains in $\vec{\mathcal{A}}^*$. But note that for every $p \in \mathcal{A}^*$, if we define $s_p(\alpha)$ to be the unique $a \in \mathcal{A}_\alpha$ such that $p \leq a$, then it holds that $p \leq ||\Delta s_p \in \dot\mu||_{\mathcal{B}(\mathbb{P})}$,[25] so that $\Delta b_p \in \mathcal{I}^+$. Now $\mathcal{A} := \{\Delta b_p \mid p \in \mathcal{A}^*\}$ gives us a maximal antichain consisting of limit points of branches of $\mathcal{T}$. ■

The main technical result of this section is then the following. In an unpublished paper, Foreman proved the following.

Theorem 3.74 (Foreman)**.** *Let κ be a regular cardinal such that $2^\kappa = \kappa^+$, and let $\lambda \leq \kappa^+$ be an infinite successor cardinal. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal precipitous ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ has a dense $<\lambda$-closed subset of size κ^+.*

Here we improve that result by not relying on the CH-assumption, reaching the conclusion for all regular infinite λ and also showing (κ, κ)-distributivity of the ideal forcing. The argument follows the same overall structure as the original, with more technicalities to achieve the stronger result.

[25] Here we're using that $\lambda \leq \kappa^+$ to ensure that the diagonal intersection is in the measure.

Theorem 3.75 (Foreman-N.). *Let κ be a regular cardinal and $\lambda \leq \kappa^+$ be regular infinite. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is (κ,κ)-distributive and has a dense $<\lambda$-closed subset of size κ^+.*

Proof. Set $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$ if $2^\kappa > \kappa^+$ and $\mathbb{P} := \{\emptyset\}$ otherwise. If κ is measurable then the dual ideal to the measure on κ satisfies all of the wanted properties, so assume that κ is not measurable. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \kappa^+ \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \kappa^+$;
- $\mathcal{N}_{\gamma+1} \prec H^V_{\kappa^+}$ is a κ-model for every $\gamma < \kappa^+$;
- $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$ for limit ordinals $\delta < \kappa^+$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for $\gamma < \beta < \kappa^+$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\kappa^+} \mathcal{N}_\gamma$.

Define now the auxilliary game $\mathcal{G}(\kappa)$ of length λ as follows.

$$\begin{array}{lccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \cdots \\ \mathrm{II} & & p_0, \mathcal{M}_0, \mu_0, Y_0 & & p_1, \mathcal{M}_1, \mu_1, Y_1 & \cdots \end{array}$$

Here $\langle \alpha_\gamma \mid \gamma < \lambda \rangle$ is an increasing continuous sequence of ordinals bounded in κ^+, $\vec{p}_\gamma$ is a decreasing sequence of $\mathbb{P}$-conditions satisfying that

$$p_\gamma \Vdash \ulcorner \check{\mathcal{M}}_\gamma = \pi(\check{\alpha}_\gamma) \wedge \check{\mu}_\gamma \text{ is a } \check{\mathcal{M}}_\gamma\text{-normal } \check{\mathcal{M}}_\gamma\text{-measure on } \check{\kappa} \urcorner$$

such that $Y_\gamma = \Delta_{\xi<\kappa} X^{\mu_\gamma}_\xi$, where $\vec{X}^{\mu_\gamma}_\xi \in H^V_{\kappa^+}$ is the $<_{\kappa^+}$-least enumeration of μ_γ.[26] We require that the μ_γ's are $\subseteq$-increasing, and player II wins iff she can continue playing throughout all λ rounds. Let $\mu_\lambda := \bigcup_{\xi<\lambda} \mu_\xi$ be the **final measure** of the play.

To every limit ordinal $\eta < \kappa^+$ define the **restricted auxilliary game** $\mathcal{G}(\kappa) \restriction \eta$ in which player I is only allowed to play ordinals $<\eta$. Note that a strategy τ for player II is winning in $\mathcal{G}(\kappa)$ if and only if it's winning in $\mathcal{G}(\kappa) \restriction \eta$ for all $\eta < \kappa^+$, simply because all sequences of ordinals played by player I are bounded in κ^+.

[26]We use that $\mathbb{P}$ is κ-closed to get the p_γ's as well as to ensure that $\mathcal{M}_\gamma, \mu_\gamma \in V$.

Note that μ_λ is precisely the tail measure on κ defined by the Y_γ's; i.e. that $X \in \mu_\lambda$ iff there exists a $\delta < \lambda$ such that $|Y_\delta - X| < \kappa$. From this it's simple to see that $\mathcal{G}(\kappa)$ is equivalent to $\mathcal{G}^-_\lambda(\kappa)$, so player II has a winning strategy τ_0 in $\mathcal{G}(\kappa)$.

For any winning strategy τ in $\mathcal{G}(\kappa) \upharpoonright \eta$ and to every partial play p of $\mathcal{G}(\kappa) \upharpoonright \eta$ consistent with τ, define the associated **hopeless ideal**[27]

$$I^\tau_p \upharpoonright \eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa) \upharpoonright \eta, \\ X \text{ is } \textit{not} \text{ in the final measure}\}$$

Claim 3.76. Every hopeless ideal $I^\tau_p \upharpoonright \eta$ is normal and (κ, κ)-distributive.

Proof of claim. For normality, if $\langle Z_\gamma \mid \gamma < \kappa \rangle$ is a sequence of elements of I^τ_p such that $Z := \nabla_\gamma Z_\gamma$ is I^τ_p-positive, then there exists a play of $\mathcal{G}(\kappa) \upharpoonright \eta$ in which player II follows τ such that Z lies in the final measure. If we let player I play sufficiently large ordinals in $\mathcal{G}(\kappa) \upharpoonright \eta$ we may assume that $\langle Z_\gamma \mid \gamma < \kappa \rangle$ is a subset and an element of the final model as well, meaning that one of the Z_γ's also lies in the final measure, $\lightning$.

We now show (κ, κ)-distributivity. Let $\mathcal{U} \subseteq \mathscr{P}(\kappa)/I^\tau_p$ be an unrooted tree of height κ such that every level $\mathcal{U}_\alpha$ is a maximal antichain of size $\leq \kappa$. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of branches of $\mathcal{U}$. Pick $X \in \mathcal{U}$ and let p be a play of $\mathcal{G}(\kappa) \upharpoonright \eta$ consistent with τ with limit model $\mathcal{M}$ and limit measure μ, such that $X \in \mu$.

By letting player I in p play sufficiently large ordinals, we may assume that $\mathcal{U} \subseteq \mathcal{M}$, using that $|\mathcal{U}| \leq \kappa$, and also that $b_X := \mathcal{U} \cap \mu \in \mathcal{M}$. This means that $d_X := \Delta b_X \in \mathscr{P}(\kappa)/I^\tau_p$ is a limit point of the branch b_X through $\mathcal{U}$, so that $\mathcal{A} := \{d_X \mid X \in \mathcal{U}\}$ is a maximal antichain of limit points of branches of $\mathcal{U}$, making $\mathscr{P}(\kappa)/I^\tau_p$ (κ, κ)-distributive. $\dashv$

Fix some limit ordinal $\eta < \kappa^+$. We will recursively construct a tree $\mathcal{T}^\eta$ of height λ which consists of subsets $X \subseteq \kappa$, ordered by reverse inclusion. Dur-

[27] This terminology is due to Matt Foreman.

ing the construction of the tree we will inductively maintain the following properties of $\mathcal{T}^\eta \restriction \alpha$ for $\alpha \leq \lambda$:

- Tree strategy: For every $\gamma < \alpha$ there is a winning strategy τ^η_γ for player II in $\mathcal{G}(\kappa) \restriction \eta$ such that for every $\beta < \gamma$, the β'th move by τ^η_γ is an element of $\mathcal{T}^\eta_\beta$ and τ^η_γ is consistent with τ^η_β for the first β-many rounds.
- Unique pre-history: Given any $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$ there's a unique partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ^η_β ending with Y — we define $I^\tau_Y := I^\tau_p$ for τ being any winning strategy for player II in $\mathcal{G}(\kappa) \restriction \eta$ satisfying that p is consistent with τ^η_β.
- Cofinally many responds: Let $\beta + 1 < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$, and set p to be the unique partial play of $\mathcal{G}(\kappa) \restriction \eta$ given by the unique pre-history of Y. Then the $\mathcal{T}^\eta$-successors of Y consists of player II's τ^η_β-responds to τ^η_β-partial plays extending p such that player I's last move in these partial plays are cofinal in η.[28]
- Positivity: If $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$ then Y is $I^{\tau^\eta_\gamma}_X$-positive for every $\gamma < \beta$ and every $X \in \mathcal{T}^\eta \restriction \gamma + 1$ with $X \leq_{\mathcal{T}^\eta} Y$.[29]
- Almost disjointness property: Every level $\mathcal{T}^\eta_\beta$ consists of pairwise almost disjoint sets.[30]
- Hopeless ideal coherence: $I^{\tau^\eta_\beta}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_\beta}_Y \cap \mathscr{P}(Y)$ for every $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$.

Note that what we're really aiming for is achieving the hopeless ideal coherence, since that enables us to ensure that if $X, Y \in \mathcal{T}^\eta$ and $X \subseteq Y$ then really $X \geq_{\mathcal{T}^\eta} Y$ — i.e. that we "catch" both X and Y in the same play of $\mathcal{G}(\kappa) \restriction \eta$. The rest of the properties are inductive properties we need to ensure this.

[28]The reason why we're dealing with the *restricted* auxilliary games is to achieve this property.

[29]This actually follows from the cofinally many responds, but we include it here for transparency.

[30]Two subsets $X, Y \subseteq \kappa$ are *almost disjoint* if $|X \cap Y| < \kappa$.

Set $\mathcal{T}^\eta_0 := \{\kappa\}$. Assume that we've built $\mathcal{T}^\eta \restriction \alpha + 1$ satisfying the inductive assumptions[31] and let $Y \in \mathcal{T}^\eta_\alpha$ — we need to specify what the $\mathcal{T}^\eta$-successors of Y are. Since κ is weakly compact and not measurable it holds by Proposition 6.4 in [Kanamori, 2008] that $\mathrm{sat}(I_Y^{\tau^\eta_\alpha}) \geq \kappa^+$, so we can fix a maximal antichain $\langle X^Y_\gamma \mid \gamma < \eta\rangle$ of $I_Y^{\tau^\eta_\alpha}$-positive sets. By κ-completeness of $I_Y^{\tau^\eta_\alpha}$ we can by Exercise 22.1 in [Jech, 2006] even ensure that all of the X^Y_γ's are pairwise disjoint.

To every $\gamma < \eta$ we fix a partial play p of even length of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ^η_α such that the last ordinal β^Y_γ in p played by player I is greater than or equal to γ and X^Y_γ has measure one with respect to the last measure in p. We then define the $\mathcal{T}^\eta$-successors of Y to be player II's τ^η_α-responses to the β_γ's (which are subsets of the X^Y_γ's modulo a bounded set and are therefore pairwise almost disjoint).

For limit stages $\delta < \lambda$ we apply τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ to get $\mathcal{T}^\eta_\delta$.

We now have to check that the inductive assumptions still hold; let's start with the tree strategy. Assume that we have a partial play p of length $2 \cdot \alpha + 1$ of $\mathcal{G}(\kappa) \restriction \eta$, i.e. the last move in p is by player II, consistent with τ^η_α; write ξ_p for player I's last move in p and Y_p for player II's response to ξ_p, which is also the last move in p. We can then pick a $\zeta < \eta$ such that $\beta^{Y_p}_\zeta > \xi_p$ by the cofinally many responds property and let $\tau^\eta_{\alpha+1}(p)$ be player II's τ^η_α-response to the partial play leading up to $\beta^{Y_p}_\zeta$. After this $(\alpha+1)$'th round we just set $\tau^\eta_{\alpha+1}$ to follow τ_0. It's clear that $\tau^\eta_{\alpha+1}$ satisfies the required properties.

Before we move on to checking the remaining inductive assumptions, let's pause to get some intuition about the tree strategies. In the definition of $\tau^\eta_{\alpha+1}$ above, we took a partial play consistent with τ^η_α, applied τ_0 for a while, took note of player II's last τ_0-response and then included *only that* response in our new $\tau^\eta_{\alpha+1}$ partial play. This means that to every τ^η_α-partial play there's an ostensibly much longer τ_0-partial play into which τ^η_α embeds; so we can look at the τ^η_α-partial plays as being "collapsed" τ_0-partial plays.

Given the above tree strategy, $\mathcal{T}^\eta_{\alpha+1}$ clearly satisfies the cofinally many responds property and the positivity property, simply by construction. For the unique pre-history, let $Y \in \mathcal{T}^\eta_{\alpha+1}$ and assume it has two distinct im-

[31] In particular, we assume that τ^η_α is defined.

mediate $\mathcal{T}^\eta$-predecessors $Z_0, Z_1 \in \mathcal{T}^\eta_\alpha$. But then $Y \subseteq Z_0 \cap Z_1$ and Y is $I^{\tau^\eta_\alpha}_{Z_0}$-positive by the positivity assumption, contradicting that Z_0 and Z_1 are almost disjoint by the almost disjointness property. Given the unique pre-history we then also get the almost disjointness property.

Claim 3.77. $\mathcal{T}^\eta \restriction \alpha + 2$ satisfies the hopeless ideal coherence property.

Proof of claim. Let $Y \in \mathcal{T}^\eta_{\alpha+1}$ — we have to show that

$$I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_{\alpha+1}}_Y \cap \mathscr{P}(Y). \tag{1}$$

It's clear that $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \subseteq I^{\tau^\eta_{\alpha+1}}_Y$, so let $Z \in I^{\tau^\eta_{\alpha+1}}_Y \cap \mathscr{P}(Y)$ and assume for a contradiction that Z is $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle}$-positive. Letting $\vec\alpha_\xi * \vec Y_\xi$ be a play of $\mathcal{G}(\kappa) \restriction \eta$ consistent with $\tau^\eta_{\alpha+1}$ such that Z is in the final measure, the definition of $\tau^\eta_{\alpha+1}$ yields that $Y_\alpha \in \mathcal{T}^\eta_{\alpha+1}$. As $Z \in I^{\tau^\eta_{\alpha+1}}_Y$ we have to assume that $Y \neq Y_\alpha$, so that the almost disjointness property implies that

$$|Y \cap Y_\alpha| < \kappa, \tag{2}$$

By the choice of $\vec\alpha_\xi * \vec Y_\xi$ there's some $\delta \in (\alpha, \lambda)$ such that $|Y_\delta - Z| < \kappa$, i.e. that Y_δ is a subset of Z modulo a bounded set, since the Y_α's generate the final measure of the play. But then $Y_\delta \subseteq Y_\alpha$ by the rules of $\mathcal{G}(\kappa) \restriction \eta$, and also that $|Y_\delta - Y| < \kappa$ since $Z \subseteq Y$. But this means that $Y \cap Y_\alpha$ is $I^{\tau^\eta_{\alpha+1}}_Y$-positive since Y_δ is, contradicting (2). This shows (1). $\dashv$

This finishes the construction of $\mathcal{T}^\eta_{\alpha+1}$. For limit levels $\delta < \lambda$ we define τ^η_δ as simply applying τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ — showing that the inductive assumptions hold at $\mathcal{T}^\eta_\delta$ is analogous to the above arguments, so we're now done with the construction of $\mathcal{T}^\eta$. Let $\tau^\eta := \bigcup_{\alpha<\lambda} \tau^\eta_\alpha \restriction^{<\alpha} H_{\kappa^+}$ and define[32] $\mathcal{I}^\eta := I^{\tau^\eta}_{\langle\rangle}$.

Now note that $\mathcal{I}^{\eta+1} \subseteq \mathcal{I}^\eta$ and $\mathcal{T}^\eta \subseteq \mathcal{T}^{\eta+1}$ for every $\eta < \kappa^+$ — set $\mathcal{I} := \bigcap_{\eta<\kappa^+} \mathcal{I}^\eta$ and $\mathcal{T} := \bigcup_{\eta<\kappa^+} \mathcal{T}^\eta$. We showed that all hopeless ideals are

[32] Note that the tree strategy property above ensures that the strategies *do* line up, so that τ^η is a well-defined strategy as well.

κ-complete, normal and (κ,κ)-distributive, so this holds in particular for the $\mathcal{I}^\eta$'s and thus also for $\mathcal{I}$.

We claim that $\mathcal{T}$ is dense in $\mathscr{P}(\kappa)/\mathcal{I}$.[33] Let X be an $\mathcal{I}$-positive set, making it $\mathcal{I}^\eta$-positive for some $\eta<\kappa^+$, meaning that there's a play $\vec{\alpha}_\gamma * \tau^\eta$ of $\mathcal{G}(\kappa)\restriction\eta$ such that X is in the final measure, which means that $|Y_\delta - X|<\kappa$ for some large $\delta<\lambda$ and in particular that $Y_\delta - X\in\mathcal{I}$. But $Y_\delta\in\mathcal{T}^\eta\subseteq\mathcal{T}$ by definition of τ^η, which shows that $\mathcal{T}$ is dense.

It remains to show that $\mathcal{T}$ is $<\lambda$-closed. If $\lambda=\omega$ then this is trivial, so assume that $\lambda\geq\omega_1$. Let $\beta<\lambda$ and let $\langle Z_\alpha \mid \alpha<\beta\rangle$ be a $\subseteq$-decreasing sequence of elements $Z_\alpha\in\mathcal{T}$. We can fix some $\eta<\kappa^+$ such that $Z_\alpha\in\mathcal{T}^\eta$ for every $\alpha<\beta$ by regularity of κ^+, and since the Z_α's are $\subseteq$-decreasing they must also be $\leq_{\mathcal{T}^\eta}$-increasing by the hopeless ideal coherence for $\mathcal{T}^\eta$[34].

Let $\tilde{Z}\in\mathcal{T}^\eta$ be player II's τ^η-response to the unique partial play of $\mathcal{G}(\kappa)\restriction\eta$ corresponding to the branch containing the Z_α's, and pick $Z\in\mathcal{T}^\eta$ such that $\left|Z-\tilde{Z}\right|<\kappa$ and $Z\geq_{\mathcal{T}^\eta} Z_\alpha$ for all $\alpha<\beta$, again by the density claim and the hopeless ideal coherence. Then Z witnesses $<\lambda$-closure of $\mathcal{T}$.[35] ■

With a bit more work we can from this result then derive the following equivalences.

Corollary 3.78 (N.). *Let κ be a regular cardinal and $\lambda\in[\omega_1,\kappa^+]$ be regular. Then the following are equivalent:*

(i) *κ is $<\lambda$-closed faintly power-measurable;*
(ii) *κ is $<\lambda$-closed ideally power-measurable;*
(iii) *κ is (κ,κ)-distributive $<\lambda$-closed faintly measurable;*
(iv) *κ is (κ,κ)-distributive $<\lambda$-closed ideally measurable;*
(v) *Player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$.*

[33] This means that given any $\mathcal{I}$-positive set X there's a $Y\in\mathcal{T}$ such that $Y-X\in\mathcal{I}$.
[34] This is the only place in which we're using hopeless ideal coherence.
[35] We're using that λ is regular to get Z.

Proof. $(v) \Rightarrow (iv)$ is Theorem 3.75 above[36] and $(iv) \Rightarrow (iii) + (ii)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial, so we show $(i) \Rightarrow (v)$.

Assume κ is $<\lambda$-closed faintly power-measurable, so there's a $<\lambda$-closed forcing $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there exists a transitive class N and a κ-powerset preserving elementary embedding $\pi\colon V \to N$. Write μ for the induced weakly amenable V-normal V-measure on κ. Now, back in V, define a strategy σ for player II in $G_\lambda(\kappa)$ as follows.

Whenever player I plays some model M_α then we let player II respond with a filter μ_α such that, for some $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash \ulcorner \check{\mu}_\alpha = \dot{\mu} \cap \check{M}_\alpha \urcorner$ — such a filter exists because μ is weakly amenable. We require the p_α's to be decreasing, which is possible by $<\lambda$-closure. Now, all the μ_α's are clearly M_α-normal M_α-measures on κ, which makes σ a winning strategy. ∎

Note that the above results all relied on λ being uncountable to achieve wellfoundedness of the generic ultrapower. If we simply ignore this wellfoundedness aspect then we get the following similar equivalence in the $\lambda = \omega$ case, which then also includes completely ineffable cardinals.

Corollary 3.79 (N.). *Let κ be a regular cardinal. Then the following are equivalent:*[37]

- *(i) There exists a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a weakly amenable V-normal V-measure on κ;*
- *(ii) There exists a (κ, κ)-distributive forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure on κ;*
- *(iii) κ carries a normal (κ, κ)-distributive ideal;*
- *(iv) Player II has a winning strategy in $\mathcal{G}^-_\omega(\kappa)$;*
- *(v) κ is completely ineffable.*

Proof. $(iv) \Leftrightarrow (v)$ was shown in Theorem 3.29, and $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. $(i) \Rightarrow (iv)$ is as $(i) \Rightarrow (v)$ in Corollary 3.78, and

[36] Here wellfoundedness of the generic ultrapower is automatic since λ has uncountable cofinality.

[37] Points (i) and (ii) look a lot like the definition of faintly power-measurable and (κ, κ)-distributive ideally measurable, but here we're not requiring the ultrapowers to be wellfounded, so that would be stretching the definition of being measurable.

$(iv) \Rightarrow (iii)$ is Theorem 3.75. ■

As an immediate consequence we then get another ideal-absoluteness result.

Corollary 3.80. *"(κ,κ)-distributive $<\lambda$-closed" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■

We get the following similar results for the $\mathcal{C}$-games[38].

Theorem 3.81 (N.). *Let κ and $\lambda \leq \kappa^+$ be regular infinite cardinals such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. If player II has a winning strategy in $\mathcal{C}^-_\lambda(\kappa)$ then κ carries a λ-complete ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $Add(\lambda, 1)$.*

Proof. If $\lambda = \kappa^+$ then we're done by Theorem 3.75, since $\mathcal{G}_{\kappa^+}(\kappa)$ is equivalent to $\mathcal{C}_{\kappa^+}(\kappa)$, so assume that $\lambda \leq \kappa$. We follow the proof of Theorem 3.75 closely. Set $\mathbb{P} := \mathrm{Col}(\lambda, 2^\kappa)$. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \lambda\rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \lambda$;
- $\kappa+1 \subseteq \mathcal{N}_\gamma$ and $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for every $\gamma < \lambda$;
- If $\delta < \lambda$ is a limit ordinal then $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$, $\mathcal{N}_\delta \prec H_{\kappa^+}$ and $\mathcal{N}_\delta \models \mathsf{ZFC}^-$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for all $\gamma < \beta < \lambda$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\lambda} \mathcal{N}_\gamma$.

Define the auxilliary game $\mathcal{G}(\kappa)$ as in the proof of Theorem 3.75 but where player I plays ordinals $\alpha_\eta < \lambda$ and where we use the above $\mathcal{N}_\gamma$'s. Here we only need $<\lambda$-closure of $\mathbb{P}$ to get an equivalence between $\mathcal{G}(\kappa)$ and $\mathcal{C}^-_\lambda(\kappa)$, since $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for all $\gamma < \lambda$.

To every limit ordinal $\eta < \lambda$ we define the restricted auxilliary game $\mathcal{G}(\kappa) \restriction \eta$ as in the proof of Theorem 3.75, and to every winning strategy τ in $\mathcal{G}(\kappa) \restriction \eta$ and partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ define the associated

[38]Theorem 3.81 is the reason for naming the $\mathcal{C}$-games "Cohen games".

hopeless ideal[39]

$$I_p^\tau \restriction \eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction\eta,$$
$$X \text{ is } \textit{not} \text{ in the final measure}\}.$$

As in the proof of Claim 3.76 we get that every hopeless ideal is λ-complete.

Now, if κ is measurable then we trivially get the conclusion,[40] so assume κ isn't measurable. Then $\mathrm{sat}(\kappa) \geq \lambda$ since $2^{<\theta} < \kappa$ for every $\theta < \lambda$,[41] so that we can continue exactly as in the proof of Theorem 3.75 to construct (λ-sized) trees $\mathcal{T}^\eta$ and winning strategies τ^η for all limit ordinals $\eta < \lambda$ such that, setting $\mathcal{I} := \bigcap_{\eta<\lambda} I_{\langle\rangle}^{\tau^\eta}$ and $\mathcal{T} := \bigcup_{\eta<\lambda} \mathcal{T}^\eta$, $\mathcal{T}$ is a dense $<\lambda$-closed subset of $\mathscr{P}(\kappa)/\mathcal{I}$ of size λ, so that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $\mathrm{Add}(\lambda, 1)$. ■

Corollary 3.82 (N.). *Let κ and $\lambda \in [\omega_1, \kappa^+]$ be regular such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. Then the following are equivalent:*

(i) κ is $<\lambda$-closed faintly measurable;
(ii) κ is $<\lambda$-closed ideally measurable;
(iii) κ is $<\lambda$-closed λ-sized faintly measurable;
(iv) κ is $<\lambda$-closed λ-sized ideally measurable;
(v) Player II has a winning strategy in $\mathcal{C}_\lambda(\kappa)$.

PROOF. *(iv)* $\Rightarrow$ *(iii)* + *(ii)*, *(ii)* $\Rightarrow$ *(i)* and *(iii)* $\Rightarrow$ *(i)* all trivial, and *(i)* $\Rightarrow$ *(v)* is like *(i)* $\Rightarrow$ *(v)* in Corollary 3.78, and *(v)* $\Rightarrow$ *(iv)* is Theorem 3.81. ■

Again, if we ignore wellfoundedness then we get the same equivalence in the $\lambda = \omega$ case:

Corollary 3.83 (N.). *Let κ be regular infinite. Then:*

(i) Player II has a winning strategy in $\mathcal{C}_\omega^-(\kappa)$; and

[39] This terminology is due to Matt Foreman.
[40] Take $\mathcal{I}(\mathrm{Add}(\lambda,1), \check{\mu})$ for μ the measure on κ.
[41] See Proposition 16.4 in [Kanamori, 2008].

(ii) κ *carries an ideal* I *such that* $\mathscr{P}(\kappa)/I$ *is forcing equivalent to* $Add(\omega, 1)$*.*

PROOF. Player II has a winning strategy in $\mathcal{C}^-_\omega(\kappa)$ as we're simply measuring finitely many sets without any demand for wellfoundedness, showing (i). Since $2^{<n} < \kappa$ for all $n < \omega$ as κ is infinite, Theorem 3.81 then implies (ii). ■

Corollary 3.84. *"<λ-closed λ-sized" is ideal-absolute for all regular* $\lambda \in [\omega, \kappa^+]$*.* ■

4 | Further questions

4.1 Virtually strongs & supercompacts

Question 4.1. Are virtually θ-strong cardinals, virtually θ-supercompacts and virtually θ-supercompacts ala Magidor all equivalent, for any uncountable regular cardinal θ?

4.2 Behaviour in core models

Question 4.2. What happens in larger core models? It seems that in both $L[\mu]$ and K below $0^{\P}$ we get that generically θ-measurables are equivalent to virtually θ-measurables, but the measurable in $L[\mu]$ is virtually measurable and not virtually κ^{++}-strong. What happens to winning strategies in $\mathcal{G}^\theta_\omega(\kappa)$ then?

4.3 Separation results

Question 4.3. Let θ be an uncountable cardinal. Is every virtually θ-measurable cardinal also virtually θ-prestrong? How about if we assume that $V = L[\mu]$ or $V = K$ with K being the core model below a woodin cardinal?

Question 4.4. Can we find a virtually ∞-measurable which isn't measurable?

4.4 Berkeleys

Question 1.7 in [Wilson, 2018] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually prestrong cardinal, and Lemmata 2.5 and 2.8 in the same paper also shows that being ω-Erdős is equivalent to being virtually club berkeley and that the least such is also the least virtually berkeley.[1]

Furthermore, they also showed that a non-Σ_2-reflecting virtually prestrong cardinal is equivalent to a virtually prestrong cardinal which isn't virtually strong. We can therefore reformulate their question to the following equivalent question.

Question 4.5 (Wilson)**.** If there exists a virtually prestrong cardinal which is not virtually strong, is there then a virtually berkeley cardinal?

[Wilson, 2018] showed that their question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem 2.9 we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 4.6. If there exists a cardinal κ which is virtually (θ, ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually berkeley cardinal?

Our results above at least gives a partially positive result:

Corollary 4.7 (N.)**.** *If there exists a virtually A-prestrong cardinal for every class A and there are no virtually strong cardinals, then there exists a virtually berkeley cardinal.*

Proof. The assumption implies by definition that On is virtually prewoodin but not virtually woodin, so Theorem 2.40 supplies us with the desired. ■

The assumption that there is a virtually A-prestrong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 2.40 shows that this is necessary, which might lead one to think that the question could have a negative answer.

[1] Note that this also shows that virtually club berkeley cardinals and virtually berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

4.5 Games

Question 4.8. If κ is generically θ-power-measurable, does player II then have a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$?

4.6 Ideals

Question 4.9. Is "ω-distributive (κ, κ)-distributive" ideal-absolute? Does it correspond to generically power-measurables?

Part II

A Virtual Equiconsistency

5 | Part II Introduction

5.1 A virtual hypothesis

Write introduction, history of the result etc

Rephrase this in terms of generic embeddings?

Denote by DI the theory

$$\mathsf{ZFC} + \mathsf{CH} + \text{there is an } \omega_1\text{-dense ideal on } \omega_1$$

and by DI^+ the theory

$\mathsf{ZFC} + \mathsf{CH} +$ there is an ω_1-dense ideal on ω_1 such that the induced generic embedding restricted to the ordinals is independent of the generic object.

In this paper, we will give a full proof of the following result,

Theorem 5.1. *The following theories are equiconsistent:*

(i) $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular,*

(ii) DI^+

Most of our results don't require the full strength of DI^+ and we've stated the needed assumptions in these cases, but the reader may simply assume DI^+ for the remainder of this paper if they wish to.

5.2 Core models

As we will be utilising the *core model induction* to prove the lower consistency bounds of our hypothesis DI^+, we give here an idea of what we mean by the *core model*. A convenient feature of core model theory is that most of the technical details regarding the construction is not needed for applications; it simply suffices to know only its abstract properties. We will provide a glimpse of the construction at the end of this section, as this will be useful when we create variants of the core model in Chapter 6. To see the full construction we refer the interested reader to [Nielsen, 2016], [Zeman, 2001] and [Jensen and Steel, 2013b].

The **core model**[1] K of a universe is the roughly speaking the subuniverse that strikes a balance between retaining the complexity of the universe while being as simple as possible. The problem is then making all of this precise. Some aspects of the definition is agreed upon by most researchers:

(i) We choose to define the *complexity* of a universe by its large cardinal structure. This is based on the empirical fact that large cardinals seem to capture the strength of every "naturally defined" hypothesis, and gives us a convenient yard stick. For instance, a universe containing a measurable cardinal is more complex than L, as Scott's Theorem, see [Kanamori, 2008, Corollary 5.5], shows that L cannot contain any measurable cardinals (or any large cardinals stronger than measurables);

(ii) We further postulate that L is the simplest universe there is, and the simplicity of a universe should therefore be measured in terms of how much it resembles L. We will be more precise about what it means to "resemble L" below, but with this intuitive notion is should at least be clear that, say, L is simpler than $L[\mu]$.

Even though (i) captures what we mean by complexity, it leaves much to be desired. For instance, as the structure of the large cardinal hierarchy can only be verified empirically, we might end up in an unfortunate situation where we simply do not know whether a given universe is more complex than another one[2]. The famous example of this is the current situation with the superstrong- and strongly compact cardinals, that we simply do not know which one is stronger[3]. Thus, given a universe whose strength corresponds to that of a strongly compact and another one at the level of superstrongs, we would not be able to say which one is more complex.

[1] K is short for *Kern*, meaning *core* in German.

[2] It might also be the case that the large cardinal hierarchy is not linear at all.

[3] Although the general consensus is that the strongly compact cardinals should be equiconsistent with the supercompacts, making them stronger than the superstrongs.

To remedy this unfortunate situation, we choose instead to define the complexity of a universe in terms of an intermediate property. A universe satisfying this property should then entail that it inherits the large cardinal structure of its surrounding universe. All the intermediate properties currently being used are all instances of a general phenomenon called *covering*. The intuitive idea is that every set in the universe can be "approximated" by a set in the subuniverse, and arose from a seminal theorem of Jensen, see [Schindler, 2014, Theorem 11.56], stating that $0^\sharp$ exists if and only if *strong covering* fails for L, defined as follows.

Definition 5.2 (Jensen)**.** We say that **strong covering** holds for universes $\mathcal{U} \subseteq \mathcal{V}$ if to every $\alpha < o(\mathcal{V})$ and $X \in \mathscr{P}^{\mathcal{V}}(\alpha)$ there exists $A \in \mathcal{U}$ such that $X \subseteq A$ and $\mathrm{Card}^{\mathcal{V}}(X) = \mathrm{Card}^{\mathcal{V}}(A)$. ◦

We can then interpret Jensen's result as saying that, if the complexity of the surrounding universe $\mathcal{V}$ is below the strength of $0^\sharp$ then L is a good candidate for K. In a complex universe we would therefore be looking for the core model among subuniverses more complex than L, and it turns out that also requiring strong covering to hold in such models is too much to ask; the current definition of covering has thus been weakened to the following.

Definition 5.3. We say that **(weak) covering** holds for universes $\mathcal{U} \subseteq \mathcal{V}$ if $\mathrm{cof}^{\mathcal{V}}(\alpha^{+\mathcal{U}}) = \mathrm{Card}^{\mathcal{V}}(\alpha^{+\mathcal{U}})$ holds for any ordinal α with $\alpha^{+\mathcal{U}} \geq \aleph_2^{\mathcal{V}}$. ◦

This statement might seem very distant from the strong version, but one can think of weak covering as saying that $\mathcal{U}$ "knows" the true cofinality of its successor cardinals $\kappa \geq \aleph_2^{\mathcal{V}}$ within the error margin $\varepsilon := \kappa^{+\mathcal{U}} - \mathrm{Card}^{\mathcal{V}}(\kappa)$. More concretely, we could equivalently define weak covering as $\mathcal{U}$ containing all cofinal maps $f\colon \gamma \to \kappa$ in $\mathcal{V}$ for every $\gamma \in \mathrm{Card}^{\mathcal{V}}(\kappa)$, making it closer in spirit to the strong covering property.

When it comes to *(ii)* we have to define what we mean by "resembling L". Ultimately this boils down to the current working definition of a *mouse* and is still work in progress. If our universe is no more complex than the strength of a Woodin cardinal however, then we know what the correct

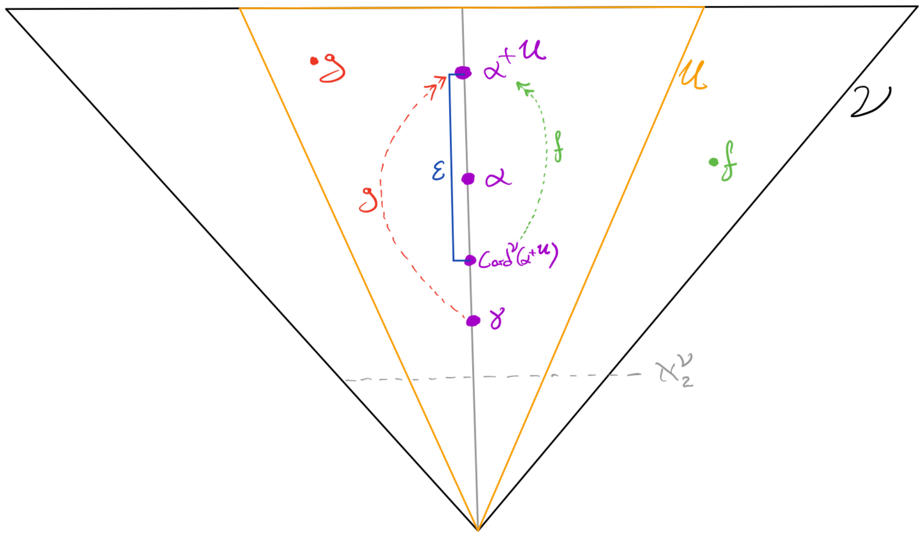

Figure 5.1: Weak covering property

definition of a mouse is, and hence also what "resembling L" would mean in this context. The definition of mice along with the assumption of covering then turns out to imply that the core model will indeed inherit the large cardinal strength of the universe[4].

To construct the core model one could then take a bottom-up approach, starting with L and then carefully include the complexity of the universe while remaining similar to L[5]. Alternatively, a top-down approach would be to define a structure which has *all* the complexity of the universe, and then showing that this structure indeed exhibits these L-like properties[6].

The construction of K takes the bottom-up approach. The first step towards this is the construction of K^c[7], which we build by recursion on the ordinals. We start with $K_0^c := \emptyset$ and at every successor ordinal α we do one of two things:

[4]To show this one first uses covering to show that K is *universal*, i.e. that it wins every coiteration. With universality at hand, a comparison argument with any $L[\vec{E}]$-model containing a large cardinal will then show that K will have an inner model with the large cardinal in question.

[5]This is the strategy undertaken by Steel and Sargsyan.

[6]Woodin is pursuing this path.

[7]The "c" stands for *certified*, as the extenders we put on the sequence was historically called *certified extenders*.

(i) If there exists a "nice" extender indexed at α then we put it onto the extender sequence of $\mathfrak{C}(K^c_\alpha)$, where $\mathfrak{C}(X)$ is the transitive collapse of a certain hull of X[8];

(ii) Otherwise we let $K^c_\alpha := \mathcal{J}(\mathfrak{C}(K^c_{\alpha-1}))$, with $\mathcal{J}(x) := \text{rud}(\text{trcl}(x \cup \{x\}))$ being the usual operator we use to build L with Jensen's hierarchy.

In other words, we are essentially building L with extenders attached onto it in a canonical fashion. Taking cores at every step will ensure that the initial segments will be *sound*, which ultimately is what guarantees iterability of K^c. The fact that we put on all the relevant extenders from V is what will ensure the covering property of the model. It turns out that K^c isn't exactly what we want however, as it relies *too much* on the surrounding universe, in contrast with L whose construction procedure builds the exact same model in every universe. To attain this *canonicity* we are again taking certain "thick" hulls of K^c (again, think of it as removing the noise). The resulting construction *almost* gives us what we want and is dubbed *pseudo-K*. The problem with this is that the technicalities of the construction uses certain properties of a fixed cardinal Ω, so to build the true core model we "glue" these pseudo-K's together.

The takeaway here is that whenever we're working with an initial segment of K then that segment will be build using the recursive steps (i) and (ii) above, carefully including extenders from V.

5.3 Mice and games

Define mice, $M_n^\sharp(x)$, iteration game, exit extender, cutpoint, condenses well, relativises well

5.4 Core model induction

Before we start with the actual induction, this section will attempt to give the reader an overview of what's going to happen, which will hopefully make it easier to understand the lemmata along the way to the finish line.

[8] Think of $\mathfrak{C}(X)$ as "removing the noise of X".

A core model induction is a way of producing determinacy models from strong hypotheses. What we're trying to do is to show that many subsets of the reals are determined, so one place to start could be the projective hierarchy, as Martin has shown that ZFC alone proves that all the Borel sets are determined.

To show that the projective sets are determined, we use the Müller-Neeman-Woodin result that $\boldsymbol{\Sigma^1_{n+1}}$-determinacy is equivalent to the existence and iterability of $M_n^\sharp(x)$ for every real x. So from our given hypothesis we then, somehow, manage to show that all these mice exist, giving us projective determinacy.

A next step could then be to notice that the projective sets of reals are precisely those reals belonging to $J_1(\mathbb{R})$, so we would then want to show that *all* the sets of reals in $L(\mathbb{R})$ are determined, by an induction on the levels. Kechris-Woodin-Solovay shows that we only need to check that the sets of reals in $J_{\alpha+1}(\mathbb{R})$ for so-called *critical* ordinals α are determined.

Check authors.

This is convenient, since Steel (see [Steel, 1983]) has characterised these critical ordinals and showed that they fall into a handful of cases, the notable ones being the *inadmissible case* and the *end of gap case*. Long story short, the so-called *Witness Equivalence* shows that to prove $J_{\alpha+1}(\mathbb{R}) \models \mathsf{AD}$ for a critical ordinal α, it suffices to show that $M_n^F(x)$ exists and is iterable for a certain operator F, in analogy with what happened with the projective sets.

This part of the induction, showing $\mathsf{AD}^{L(\mathbb{R})}$, is an instance of an *internal* core model induction: we're showing that all the sets of reals in some fixed inner model are determined. Crucially, for these internal core model inductions to work, we need a *scale analysis* of the model at hand. In this paper we will be working with the *lower part model* $\mathrm{Lp}(\mathbb{R})$, which contains all the sets of reals of $L(\mathbb{R})$ and more, and generalisations of such lower part models. The scale analysis of $\mathrm{Lp}(\mathbb{R})$ is shown in Steel , and the scale analysis for the generalised versions is shown in Trang and Schlutzenberg .

Reference Scales in $K(\mathbb{R})$ and the companion paper.

Reference their scale paper.

Our first internal step will thus show that every set of reals in $\mathrm{Lp}(\mathbb{R})$ is determined, which consistency-wise is a tad stronger than having a limit of Woodin cardinals. This first step can be seen as showing that all sets of reals in the pointclass $(\Sigma^2_1)^{\mathrm{Lp}(\mathbb{R})}$ are determined, since being in this pointclass precisely means that you belong to an initial segment of $\mathrm{Lp}(\mathbb{R})$.

Boldface?

The *external* core model induction takes this further. If we define Γ_∞ to be the set of all determined sets of reals, we want to see how big this pointclass is. We organise this by looking at the so-called *Solovay sequence* $\langle\theta_\alpha \mid \alpha \leq \Omega\rangle$ of $L(\Gamma_\infty, \mathbb{R})$, whose length can be seen as a measure of "how many determined sets of reals there are" in a context without the axiom of choice.

Even equal I think

If $\Omega = 0$ then it can be shown that $L(\Gamma_\infty, \mathbb{R})$ and $\mathrm{Lp}(\mathbb{R})$ have the same sets of reals, so if we want to show that $\Omega > 0$ then it suffices to find some determined set of reals which is not in $\mathrm{Lp}(\mathbb{R})$. This is done by producing a so-called $(\Sigma^2_1)^{L(\Gamma_\infty,\mathbb{R})}$-fullness preserving hod pair $(\mathcal{P}_0, \Lambda_0)$, which will have the property that $\Lambda_0 \notin \mathrm{Lp}(\mathbb{R})$ and that Λ_0 is determined when viewed as a set of reals. This yields a contradiction to $\Omega = 0$, so we must have that $\Omega > 0$.

Next, if we assume that $\Omega = 1$ then we show that $L(\Gamma_\infty, \mathbb{R})$ and $\mathrm{Lp}^{\Lambda_0}(\mathbb{R})$ have the same sets of reals, where $\mathrm{Lp}^{\Lambda_0}(\mathbb{R})$ is one of the generalised versions of $\mathrm{Lp}(\mathbb{R})$ we mentioned above. We do another internal induction to show that every set of reals in $\mathrm{Lp}^{\Lambda_0}(\mathbb{R})$ is determined, and then proceed to construct a $\Sigma^2_1(\Lambda_0)^{L(\Gamma_\infty,\mathbb{R})}$-fullness preserving hod pair $(\mathcal{P}_1, \Lambda_1)$, which again has the property that $\Lambda_1 \notin \mathrm{Lp}^{\Lambda_0}(\mathbb{R})$, so that we must have $\Omega > 1$.

Now assume that $\Omega = 2$ — this step will look like the general *successor case*. This time we're working with $\mathrm{Lp}^{\Gamma_0,\Lambda_1}(\mathbb{R})$, where $\Gamma_0 := \Gamma(\mathcal{P}_0, \Lambda_0)$ is a pointclass generated by $(\mathcal{P}_0, \Lambda_0)$. We again produce a $\Sigma^2_1(\Lambda_1)^{\mathrm{Lp}^{\Gamma_0,\Lambda_1}(\mathbb{R})}$-fullness preserving hod pair $(\mathcal{P}_2, \Lambda_2)$ with $\Lambda_2 \notin \mathrm{Lp}^{\Gamma_0,\Lambda_1}(\mathbb{R})$, showing that $\Omega > 2$.

As for the limit case, if we assume that $\Omega = \gamma$, we let $\Gamma_\gamma := \bigcup_{\alpha<\gamma} \Gamma_\alpha$ and coiterate all the previous mice to get some $\mathcal{P}_\gamma$, which we then have to show has a $\Sigma^2_1(\Lambda_\gamma)^{\mathrm{Lp}^{\Gamma_\gamma,\oplus_{\alpha<\gamma}\Lambda_\alpha}}$-fullness preserving iteration strategy Λ_γ. As before, this strategy will be determined as a set of reals and won't be in $L(\Gamma_\infty, \mathbb{R})$, a contradiction, which shows that $\Omega > \gamma$.

In this paper, we will be able to do the tame case, the successor case, and the limit case when Ω is singular, which shows that we end up getting that Ω is regular.

6 | Internal core model induction

6.1 Operators and hybrid mice

We'll need a generalisation of the concept of mice as we move up to the higher reaches of the core model induction, a generalisation usually known as either *hybrid mice* or *operator mice*. The basic concept is simple. When we're constructing "pure" mice we're traversing the $\mathcal{J}$-hierarchy, applying the $\mathcal{J}(x) := \text{rud}(x \cup \{x\})$ operator at every step, and taking unions at limit stages. In the hybrid case we're simply replacing $\mathcal{J}$ with another *operator* $\mathcal{F}$, again applying it at every successor stage and taking unions at limits.

Figuring out what operators we're allowed to pick is the hard part, as we want to maintain all the fine structure that we get in the "pure" case. This has been done in great detail in [Schlutzenberg and Trang, 2016], and we'll introduce a particularly simple case of their general definition here.

Definition 6.1 (Schlutzenberg-Trang)**.** For a set x write $\rho_x : x \to \text{rk}\, x$ for the rank function of x and define the **rank closure** $\hat{x} := \text{trcl}(\{x, \rho_x\})$ of x and the **cone** $C_x := \{\hat{y} \in H_\kappa \mid x \in \mathcal{J}_1(\hat{y})\}$ over x. ◦

Definition 6.2 (Schlutzenberg-Trang)**.** Let κ be an infinite cardinal, D a set of self-wellordered[1] sets and fix $b \in H_\kappa$. An **operator on H_κ over b with support** D is a partial function

$$\mathcal{F} \colon H_\kappa \dashrightarrow H_\kappa$$

such that $D \cap C_b \subseteq \text{dom}\, \mathcal{F}$ and $\text{dom}\, \mathcal{F}$ is closed under both unions and applications of $\mathcal{F}$. We also call C_b the **cone over** b and b the **base of** C_b. ◦

[1] A set x is *self-wellordered* if there's a wellorder of x in $\mathcal{J}(x)$.

Before we move on, let's take a step back and have a look at a few examples of operators. As this is supposed to be a generalisation of the $\mathcal{J}$-function we'd want that to also be an operator, of course. Indeed, if $V = L$ then note that $\hat{x} = x$ for all x, so if we take $\emptyset$ to be our base then $C_a = J_\kappa$, making $\operatorname{dom}\mathcal{J} = J_\kappa$ trivially closed under both unions and applications of $\mathcal{J}$.

We could also let x be any set, assume that $V = L(\hat{x})$ and consider the operator $\mathcal{J}_x$, which is simply applying $\mathcal{J}$ but with base $\hat{x}$ instead of $\emptyset$. A similar argument as above would show that this is an operator on $J_\kappa(\hat{x})$ $(= H_\kappa^{L(\hat{x})})$ as well.

A slightly more sophisticated example would be $\mathcal{F} := (-)^\sharp$, where $x^\sharp$ is the smallest initial segment of $\mathrm{Lp}(x)$ with a measure. We can consider $\mathcal{F}$ as an operator on HC over $\emptyset$, where $\mathcal{F}(\emptyset) = 0^\sharp$ and $\mathcal{F}^\omega(\emptyset) = \bigcup_{n<\omega}\mathcal{F}^n(\emptyset)$ is the smallest $\sharp$-closed structure.

Elaborate

Check the last example

We now, somewhat informally[2], define what hybrid mice are. Namely, an **$\mathcal{F}$-mouse on** $\hat{x}$ is a structure $\mathcal{M}$ built by successively applying $\mathcal{F}$ to $\hat{x}$ and taking unions at limits. We can stop the construction at any point and denote the amount of $\mathcal{F}$-applications by $l(\mathcal{M})$, the **length** of $\mathcal{M}$. The **initial segments** of $\mathcal{M}$ are simply the intermediate models up to $\mathcal{M}$, being of the form $\mathcal{F}^\alpha(\hat{x})$ for some $\alpha < l(\mathcal{M})$.

As with regular mice, we require that they are *amenable*, *acceptable* and that proper initial segments $\mathcal{F}$-mice are *sound*. An extra property that we require in the hybrid context is that every proper initial segment $\mathcal{P} \lhd \mathcal{M}$ is *$<\omega$-condensing*, roughly stating that if $\pi\colon \mathcal{N} \to \mathcal{P}$ is a sufficiently elementary embedding from a "nice" $\mathcal{F}$-structure then either $\mathcal{N}$ is an initial segment of $\mathcal{P}$ or an element of an ultrapower of $\mathcal{P}$[3].

Definition 6.3. Let κ be an infinite cardinal and $\mathcal{F}$ an operator on H_κ over $b \in H_\kappa$. We then define the **$\mathcal{F}$-lower part model on** b as

$$\mathrm{Lp}^{\mathcal{F}}(b) := \{\mathcal{M} \mid \mathcal{M} \text{ is a sound } \mathcal{F}\text{-mouse projecting to } b\}$$ ∘

[2] For a comprehensive definition, see [Schlutzenberg and Trang, 2016, Section 2].

[3] For a proper definition of these concepts in the hybrid context, see [Schlutzenberg and Trang, 2016, Section 2].

Proposition 6.4. *Let κ be an infinite cardinal and $\mathcal{F}$ an operator on H_κ over $b \in H_\kappa$. Assuming $\mathsf{DC}_{\hat{b}}$ holds, $\mathrm{Lp}^{\mathcal{F}}(b)$ is itself an $\mathcal{F}$-premouse.*

Proof. If $\mathcal{M} \lhd \mathrm{Lp}^{\mathcal{F}}(b)$ then $\mathcal{M} = \mathrm{cHull}^{H_\nu}(\hat{b} \cup o(\mathcal{M}))$, so that there's an isomorphism from $\hat{b}^{<\omega}$ onto $\mathcal{M}$. Now, using $\mathsf{DC}_{\hat{b}}$, we can take a countable hull containing $\hat{b}$, meaning that without loss of generality we may assume that $\hat{b}$, and hence also $\mathcal{M}$, is countable.

This means that whenever we have two $\mathcal{M}, \mathcal{N} \lhd \mathrm{Lp}^{\mathcal{F}}(b)$ we may assume that they're both countable, which means that a comparison argument shows that one of them is an initial segment of the other. This means that all the mice in $\mathrm{Lp}^{\mathcal{F}}(b)$ line up, which implies that all the axioms for being an $\mathcal{F}$-premouse trivially hold for $\mathrm{Lp}^{\mathcal{F}}(b)$. ■

Further, $\mathrm{Lp}^{\mathcal{F}}$ is itself an operator on H_κ over b and we write $\mathrm{Lp}^{\mathcal{F}}_\alpha(b) := (\mathrm{Lp}^{\mathcal{F}})^\alpha(b)$.

Check that the definition of $\mathrm{Lp}^{\mathcal{F}}$ is correct and that it *does* in fact have the two properties above.

Definition 6.5 (Schindler-Steel)**.** Let κ be an infinite cardinal and $\mathcal{F}$ an operator on H_κ over $b \in H_\kappa$. Then $\mathcal{F}$ **condenses well** if whenever $g \subseteq \mathrm{Col}(\omega, \kappa)$ is V-generic and that there are models $\overline{\mathcal{M}}, \mathcal{M} \in H_\kappa$ and $\overline{\mathcal{M}}^+ \in V[g]$, all on b, with

(i) $|\overline{\mathcal{M}}| = |b| \cdot \aleph_0$;
(ii) $\overline{\mathcal{M}} \in \overline{\mathcal{M}}^+$;
(iii) $\overline{\mathcal{M}}^+ = \mathrm{Hull}_1^{\overline{\mathcal{M}}^+}(\overline{\mathcal{M}})$;
(iv) Either
   (a) There's a map $\pi \colon \overline{\mathcal{M}}^+ \to \mathcal{F}(\mathcal{M})$ in $V[g]$ with $\pi(\overline{\mathcal{M}}) = \mathcal{M}$ and $\pi \restriction (b \cup \{b\}) = \mathrm{id}$ which is Σ_0-cofinal or Σ_2-elementary; or
   (b) There's a model $\mathcal{P} \in \operatorname{dom} \mathcal{F}$ on b with $\mathcal{F}(\mathcal{P}) \in H_\kappa$ and maps $i \colon \mathcal{F}(\mathcal{P}) \to \overline{\mathcal{M}}^+$ and $\pi \colon \overline{\mathcal{M}}^+ \to \mathcal{F}(\mathcal{M})$, both in $V[g]$ but with their composition in V, with $i(\mathcal{P}) = \overline{\mathcal{M}}$, $\pi(\overline{\mathcal{M}}) = \mathcal{M}$,

$$i \restriction (b \cup \{b\}) = \pi \restriction (b \cup \{b\}) = \mathrm{id},$$

i is Σ_0-cofinal or Σ_2 elementary, and π is a weak Σ_1-embedding. Then $\overline{\mathcal{M}}^+ = \mathcal{F}(\overline{\mathcal{M}}) \in V$. ∘

It turns out that *condenses well* is a bit too strong to do proper core model theory, as shown in [Schlutzenberg and Trang, 2016], and in that paper they propose a technical weakening of this concept which they call *condenses finely*, whose definition is of a similar spirit as the above. We therefore formally require that our desired operators only condense finely, but as all the operators that we will encounter "in the wild" in this thesis condense well, we will omit the definition of fine condensation here.

Definition 6.6. An operator $\mathcal{F}$ **determines itself on generic extensions** if there exists a formula $\varphi(v_0, v_1)$ such that whenever $\mathcal{M}$ is an $\mathcal{F}$-premouse with

$$\mathcal{M} \models \mathsf{KP} + \ulcorner\text{there are arbitrarily large cardinals}\urcorner,$$

κ is an $\mathcal{M}$-cardinal and $g \subseteq \mathrm{Col}(\omega, \kappa)$ is $\mathcal{M}$-generic, then $\mathsf{HC}^{\mathcal{M}[g]}$ is closed under $\mathcal{F}$ and $\mathcal{F} \restriction \mathsf{HC}^{\mathcal{M}[g]} = (\tau_\kappa^{\mathcal{M}})^g$, with $\tau_\kappa^{\mathcal{M}}$ being the unique τ such that $\mathcal{M} \models \varphi[\kappa, \tau]$. ∘

Definition 6.7. Let κ be an infinite cardinal and $\mathcal{F}$ an operator on H_κ over $b \in H_\kappa$. We then say that $\mathcal{F}$ is **radiant**[4] if $\mathcal{F}$ condenses finely and determines itself on generic extensions. ∘

6.2 Core model dichotomy

Lemma 6.8 (Mesken-N.)**.** *Let θ be a regular uncountable cardinal or $\theta = \infty$ and let $\mathcal{N}$ be a tame hybrid mouse operator on H_θ which relativises well. Then $\mathcal{N}$ is countably iterable iff it's (θ, θ)-iterable, guided by $\mathcal{N}$. Furthermore, for every $x \in H_\theta$, if $M_1^{\mathcal{N}}(x)$ exists and is countably iterable, then it's also (θ, θ)-iterable, guided by $\mathcal{N}$.*

[4]The terminology is meant to suggest that the operator is preserved when moving in "any direction": down to smaller models or up to larger forcing extensions.

Change this to model operators; perhaps change parts of the proof and/or assumptions needed.

PROOF. Fix $x \in H_\theta$ and assume that $\mathcal{N}(x)$ is countable iterable. We first show that $\mathcal{N}(x)$ is (θ, θ)-iterable. Let $\mathcal{T} \in H_\theta$ be a normal tree of limit length on $\mathcal{N}(x)$. Let $\eta \gg \mathrm{rk}(\mathcal{T})$ and let

$$\mathcal{H} := \mathrm{cHull}^{H_\eta}(\{x, \mathcal{N}(x), \mathcal{T}\})$$

with uncollapse $\pi \colon \mathcal{H} \to H_\eta$. Set $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran} \pi$. Note that $\overline{\mathcal{N}(x)} = \mathcal{N}(\overline{x})$ since $\mathcal{N}$ relativises well. Now $\overline{\mathcal{T}}$ is a normal, countable iteration tree on $\mathcal{N}(\overline{x})$ and hence our iteration strategy yields a wellfounded cofinal branch $\bar{b} \in V$ for $\overline{\mathcal{T}}$. Note that $\overline{\mathcal{Q}} := \mathcal{Q}(\bar{b}, \overline{\mathcal{T}})$ exists, since if $\bar{b}$ drops then there's nothing to do, and otherwise we have that

$$\rho_1(\mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}}) = \rho_1(\mathcal{N}(\overline{x})) = \mathrm{rk}\, \overline{x} < \delta(\overline{\mathcal{T}}),$$

so $\delta(\overline{\mathcal{T}})$ is not definably Woodin over $\mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}}$, as there is a definable surjection from $\rho_1(\mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}})$ onto $\delta(\overline{\mathcal{T}})$.

Claim 6.9. $\overline{\mathcal{Q}} \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$

PROOF OF CLAIM. If $\overline{\mathcal{Q}} = \mathcal{M}(\overline{\mathcal{T}})$ then the claim is trivial, so assume that $\mathcal{M}(\overline{\mathcal{T}}) \triangleleft \overline{\mathcal{Q}}$. Note that $\overline{\mathcal{Q}} \trianglelefteq M_{\bar{b}}^{\overline{\mathcal{T}}}$ by definition of $\mathcal{Q}$-structures, and that $M_{\bar{b}}^{\overline{\mathcal{T}}}$ satisfies (2) of the definition of relativises well, meaning that

$$M_{\bar{b}}^{\overline{\mathcal{T}}} \models \ulcorner \forall \eta \forall \zeta > \eta : \text{if } \eta \text{ is a cutpoint then } M_{\bar{b}}^{\overline{\mathcal{T}}} | \zeta \not\models \varphi_{\mathcal{N}}[\bar{x}, p_{\mathcal{N}}] \urcorner. \tag{1}$$

This statement is Π^1_2 and $\overline{\mathcal{Q}}$ is Π^1_2-correct since it contains a Woodin cardinal, so that $\mathcal{Q}$ satisfies the statement as well. Since $\mathcal{N}$ is tame we get that $\delta(\overline{\mathcal{T}})$ is a cutpoint of $\overline{\mathcal{Q}}$, so that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}} | \delta(\overline{\mathcal{T}}))$ is *not* a proper initial segment of $\overline{\mathcal{Q}}$. Further, as we're assuming that both $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}}$ are $(\omega_1{+}1)$-iterable above $\delta(\overline{\mathcal{T}})$ the same thing holds for $\overline{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}}$, so that we can compare $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ with $\overline{\mathcal{Q}}$ (in V). Let

$$(\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})), \overline{\mathcal{Q}}) \rightsquigarrow (\mathcal{P}, \mathcal{R})$$

be the result of the coiteration. We claim that $\mathcal{R} \trianglelefteq \mathcal{P}$. Suppose $\mathcal{P} \triangleleft \mathcal{R}$. Then there is no drop in $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ and in fact $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P}$ since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ projects to $\delta(\overline{\mathcal{T}})$. Furthermore, as we established that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ isn't a proper initial segment of $\overline{\mathcal{Q}}$ it can't be a proper initial segment of $\mathcal{R}$ either, as the coiteration is above $\delta(\overline{\mathcal{T}})$. But we're assuming that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P} \triangleleft \mathcal{R}$, a contradiction. So $\mathcal{R} \trianglelefteq \mathcal{P}$.

Since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\overline{\mathcal{Q}}$ agree up to $\delta(\overline{\mathcal{T}})$ and there is no drop $\overline{\mathcal{Q}} \rightsquigarrow \mathcal{R}$ we have that $\overline{\mathcal{Q}} = \mathcal{R}$. If $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ doesn't move either we're done, so assume not. Let F be the first exit extender of $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ in the coiteration. We have $\mathrm{lh}(F) \leq o(\overline{\mathcal{Q}})$, $\overline{\mathcal{Q}} \trianglelefteq \mathcal{P}$ and $\mathrm{lh}(F)$ is a cardinal in $\mathcal{P}$.

As $\overline{\mathcal{Q}}$ is $\delta(\overline{\mathcal{T}})$-sound and projects to $\delta(\overline{\mathcal{T}})$ it follows that $J(\overline{\mathcal{Q}}|\,\mathrm{lh}(F))$ collapses $\mathrm{lh}(F)$, so it has to be the case that $\overline{\mathcal{Q}}|\,\mathrm{lh}(F) = \mathcal{P}$ and thus $o(\mathcal{P}) = \mathrm{lh}(F)$. But this means that $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ even though we assumed that $\mathcal{N}(\mathcal{M}(\mathcal{T})) \rightsquigarrow \mathcal{P}$ moved, a contradiction. $\dashv$

Now, in a sufficiently large collapsing extension extension of $\mathcal{H}$, $\bar{b}$ is the unique cofinal, wellfounded branch of $\overline{\mathcal{T}}$ such that $\mathcal{Q}(\bar{b}, \overline{\mathcal{T}}) \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ exists. Hence, by the homogeneity of $\mathrm{Col}(\omega, \theta)$, $\bar{b} \in H$. By elementarity there is a unique cofinal, wellfounded branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$. This proves that M is (uniquely) On-iterable and virtually the same argument yields the iterability of M via successor-many stacks of normal trees.

To show that M is fully iterable, it remains to be seen that the unique iteration strategy (guided by $\mathcal{N}$) of M outlined above leads to wellfounded direct limits for stacks of normal trees on M of limit length. Let λ be a limit ordinal and $\vec{\mathcal{T}} = (\mathcal{T}_i \mid i < \lambda)$ a stack according to our iteration strategy. Suppose $\lim_{i<\lambda} \mathcal{M}_\infty^{\mathcal{T}_i}$ is illfounded.

Redefine $\eta \gg \mathrm{rk}(\vec{\mathcal{T}})$, $\mathcal{H} := \mathrm{cHull}^{H_\eta}(\{x, M, \vec{\mathcal{T}}\})$ and $\pi : \mathcal{H} \to H_\eta$ the uncollapse, again with $\bar{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran} \pi$. By elementarity we get that $\mathcal{H} \models \ulcorner \lim_{i<\bar{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i}$ is illfounded$\urcorner$. But $\overline{\vec{\mathcal{T}}}$ is countable and according to the iteration strategy guided by $\mathcal{N}$, so that

$$V \models \ulcorner \lim_{i<\bar{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i} \text{ is wellfounded.}\urcorner$$

Now note that $(\lim_{i<\bar{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^{\mathcal{H}} = (\lim_{i<\bar{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^V$ and wellfoundedness is absolute between $\mathcal{H}$ and V, a contradiction.

Now assume that $M_1^{\mathcal{N}}(x)$ exists for some $x \in H_\theta$, and that it's countably iterable. We then do exactly the same thing as with $\mathcal{N}(x)$ *except* that in the claim we replace (1) with

$$\overline{\mathcal{Q}} \models \forall\eta(\overline{\mathcal{Q}}|\eta \not\models \ulcorner\delta(\overline{\mathcal{T}}) \text{ is not Woodin}\urcorner),$$

so that if $\mathcal{P} \lhd \mathcal{R}$ then $\delta(\overline{\mathcal{T}})$ is still Woodin in $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$, contradicting the defining property of $M_1^{\mathcal{N}}(x)$ (and thus also of $\mathcal{R}$). The rest of the proof is a copy of the above. ■

Theorem 6.10 (Hybrid core model dichotomy). *Let θ be a $\beth$-fixed point or $\theta = \infty$, and let $\mathcal{F}$ be a model operator on H_θ that condenses well. Let $x \in H_\theta$. Then either:*

(i) The core model $K^{\mathcal{F}}(x)|\theta$ exists and is (θ, θ)-iterable; or

(ii) $M_1^{\mathcal{F}}(x)$ exists and is (θ, θ)-iterable.

Proof. Assume first that $K^{c,\mathcal{F}}(x)|\theta$ reaches a premouse which isn't $\mathcal{F}$-small; let $\mathcal{N}_\xi$ be the first part of the construction witnessing this. Then $\mathfrak{C}(\mathcal{N}_\xi) = M_1^{\mathcal{F}}(x)$, and by Lemma 6.8 it suffices to show that $M_1^{\mathcal{F}}(x)$ is countably iterable. Insert argument?

Show that $M_1^{\mathcal{F}}(x)$ is countably iterable.

We can thus assume that $K^{c,\mathcal{F}}(x)|\theta$ is $\mathcal{F}$-small. Note that if $K^{c,\mathcal{F}}(x)|\theta$ has a Woodin cardinal then because the model is $\mathcal{F}$-closed we contradict $\mathcal{F}$-smallness, so the model has no Woodin cardinals either, making it (θ, θ)-iterable.

Let $\kappa < \theta$ be any uncountable cardinal and let $\Omega := \beth_\kappa(\kappa)^+$. Note that $\Omega < \theta$ since we assumed that θ is a $\beth$-fixed point and $\kappa < \theta$. If Ω is a limit cardinal in $K^{c,\mathcal{F}}(x)|\theta$ then let $\mathcal{S} := \mathrm{Lp}(K^{c,\mathcal{F}}(x)|\Omega)$ and otherwise let $\mathcal{S} := K^{c,\mathcal{F}}(x)|\Omega$. Then by Lemma 3.3 of [Jensen et al., 2009] we get that $\mathcal{S}$ is countably iterable, with largest cardinal Ω in the "limit cardinal case".

This also means that Ω isn't Woodin in $L[\mathcal{S}]$, as it's trivial in the case where Ω is a successor cardinal of $K^{c,\mathcal{F}}(x)|\theta$ by our case assumption, and in the "limit cardinal case" it also holds since

$$K^{c,\mathcal{F}}(x)|\Omega^{+K^{c,\mathcal{F}}(x)|\theta} \subseteq \mathcal{S}.$$

By [Fernandes, 2018] and [Jensen and Steel, 2013a] this means that we can build $K^{\mathcal{F}}(x)|\kappa$, as the only places they use that there's no inner model with a Woodin are to guarantee that $K^{c,\mathcal{F}}(x)|\Omega$ exists and has no Woodin cardinals, and in Lemma 4.27 of [Jensen and Steel, 2013a] in which they only require that Ω isn't Woodin in $L[\mathcal{S}]$.

As $\kappa < \theta$ was arbitrary we then get that $K^{\mathcal{F}}(x)|\theta$ exists. Note that $K^{\mathcal{F}}(x)|\theta$ has no Woodin cardinals either and is $\mathcal{F}$-small, so that $\mathcal{Q}$-structures trivially exist, making it (θ,θ)-iterable. ■

6.3 Mouse witness equivalence

Definition 6.11. Define coarse (k,U,x)-Woodin pairs

∘

Definition 6.12. Let $\mathcal{F}$ be a total condensing operator and let α be an ordinal. Then the **coarse mouse witness condition at α with $\mathcal{F}$**, written $W^*_\alpha(\mathcal{F})$, states that given any scaled-co-scaled $U \subseteq \mathbb{R}$ whose associated sequences of prewellorderings are elements of $\mathrm{Lp}^{\mathcal{F}}_\alpha(\mathbb{R})$, we have for every $k < \omega$ and $x \in \mathbb{R}$ a coarse (k,U,x)-Woodin pair (N,Σ) with $\Sigma\restriction\mathsf{HC} \in \mathrm{Lp}^{\mathcal{F}}_\alpha(\mathbb{R})$. ∘

Check if this is a reasonable definition.

Theorem 6.13 (Hybrid witness equivalence)**.** *Let $\theta > 0$ be a cardinal, $g \subseteq \mathrm{Col}(\omega,{<}\theta)$ V-generic, $\mathbb{R}^g := \bigcup_{\alpha<\theta}\mathbb{R}^{V[g\restriction\alpha]}$, $\mathcal{F}$ a total radiant operator* *and* α *a critical ordinal of* $\mathrm{Lp}^{\mathcal{F}}(\mathbb{R}^g)$*. Assume that*

$$\mathrm{Lp}^{\mathcal{F}}(\mathbb{R}^g) \models \mathsf{DC} + \ulcorner W^*_\beta(\mathcal{F}) \textit{ holds for all } \beta \leq \alpha \urcorner.$$

Then there is a hybrid mouse operator $\mathcal{N} \in V$ *on* $H_{\aleph_1^{V[g]}}$ *such that*

$$\mathrm{Lp}^{\mathcal{F}}(\mathbb{R}^g) \models W^*_{\alpha+1}(\mathcal{F}) \quad \textit{iff} \quad V \models \ulcorner M_n^{\mathcal{N}} \textit{ is total on } H_{\aleph_1^{V[g]}} \textit{ for all } n < \omega \urcorner$$

Furthermore, if $\theta < \aleph_1^V$ *then we only need to assume that* $\mathcal{F}$ *is total and condensing.*

Be more explicit about what the given operator $\mathcal{N}$ looks like.

7 External core model induction

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7.1 HOD mice

Provide overview of this section.

7.1.1 Iteration strategies

At some point we should mention that we adopt John‘s convention of hiding the degree of iteration trees, always taking the maximal possible degree. And that all of our trees are (stacks of) normal trees.

Definition 7.1. Let $\vec{\mathcal{T}}$ be a stack of normal trees. We write $\mathrm{lh}(\vec{\mathcal{T}})$ for the length of $\vec{\mathcal{T}}$ and $\mathcal{T}_\alpha$ for the α'th tree in $\vec{\mathcal{T}}$, so that

$$\vec{\mathcal{T}} = (\mathcal{T}_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}})).$$

For $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$, $\gamma < \mathrm{lh}(\mathcal{T}_\alpha)$, $\eta < \mathrm{lh}(\mathcal{T}_\beta)$ we let $\mathcal{M}_\gamma^{\mathcal{T}_\alpha}$ be the model with index γ in the tree $\mathcal{T}_\alpha$ and write

$$\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} \colon \mathcal{M}_\gamma^{\mathcal{T}_\alpha} \to \mathcal{M}_\eta^{\mathcal{T}_\beta}$$

for the corresponding embedding, provided it exists. We also write

$$\pi^{\vec{\mathcal{T}}}_{\alpha,\beta} \colon \mathcal{M}_0^{\mathcal{T}_\alpha} \to \mathcal{M}_0^{\mathcal{T}_\beta}.$$

If $\vec{\mathcal{T}}$ has a last model, i.e. if $\mathrm{lh}(\vec{\mathcal{T}}) = \xi + 1$ and $\mathcal{M}_\infty^{\mathcal{T}_\xi}$ exists, we let $\mathcal{M}_\infty^{\vec{\mathcal{T}}} := \mathcal{M}_\infty^{\mathcal{T}_\xi}$ and $\pi^{\vec{\mathcal{T}}} \colon \mathcal{M}_0^{\mathcal{T}_0} \to \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ be the associated embedding. ◦

Definition 7.2. Let Σ be an iteration strategy and $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$. We write $\Sigma_{\vec{\mathcal{T}},N}$ for the iteration strategy on N given by

$$\Sigma_{\vec{\mathcal{T}},N}(\vec{\mathcal{U}}) := \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}).$$

We call $\Sigma_{\vec{\mathcal{T}},N}$ the $(\vec{\mathcal{T}}, N)$**-tail strategy**of Σ. ◦

Definition 7.3. at the very end we should remove those definitions that we didn't need

Let Σ be an iteration strategy.

(i) Σ has the **Dodd-Jensen property** if for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$ and all $\pi \colon \mathcal{M}_\Sigma \to_{\Sigma_1} N$ we have $\pi^{\vec{\mathcal{T}}}(\alpha) \le \pi(\alpha)$ for all $\alpha \in o(\mathcal{M}_\Sigma)$.

(ii) Σ has the **positional Dodd-Jensen property** if $\Sigma_{\vec{\mathcal{T}},N}$ has the Dodd-Jensen property for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iii) Σ is **weakly positional** if $\Sigma_{\vec{\mathcal{T}},N} = \Sigma_{\vec{\mathcal{U}},N}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iv) Σ is **positional** if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly positional for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(v) Σ is **weakly commuting** if $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{U}}}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vi) Σ is **commuting** if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly commuting for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vii) Σ is **weakly pullback consistent** if $\Sigma^{\vec{\mathcal{T}}} = \Sigma$ for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(viii) Σ is **pullback consistent** if $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly pullback consistent for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

◦

$$\begin{tikzcd}
 & N \\
\mathcal{M}_\Sigma \arrow[ur, "\pi^{\vec{\mathcal{T}}}"] \arrow[dr, "\pi_c^{\vec{\mathcal{U}}}"'] & \\
 & \mathcal{M}_c^{\vec{\mathcal{U}}} \arrow[uu, "\pi"']
\end{tikzcd}$$

Figure 7.1: Branch condensation

Definition 7.4. If Σ is positional, $\Sigma_{\vec{\mathcal{T}},N}$ doesn't depend on $\vec{\mathcal{T}}$ and hence we simply write Σ_N for this tail strategy. $\circ$

Definition 7.5. An iteration strategy Σ has **branch condensation** (see Figure 7.1) if for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}_\Sigma$ such that

(i) $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ are according to Σ,

(ii) $\vec{\mathcal{U}}$ is a stack of successor length $\gamma+1$ and $\vec{\mathcal{U}}$'s last component $\mathcal{U}_\gamma$ is of limit length,

(iii) $\vec{\mathcal{T}}$ has a last model N such that $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$,

(iv) there is some branch c such that $\pi_c^{\vec{\mathcal{U}}}$ exists and for some $\pi\colon \mathcal{M}_c^{\vec{\mathcal{U}}} \to_{\Sigma_1} N$ we have $\pi^{\vec{\mathcal{T}}} = \pi \circ \pi_c^{\vec{\mathcal{U}}}$.

Then $c = \Sigma(\vec{\mathcal{U}})$. $\circ$

Definition 7.6. $(\mathcal{M}, \mathcal{T})$ **is a hull of** $(\mathcal{N}$,$\mathcal{U})$ if there are

(i) an embedding, $\pi\colon \mathcal{M} \to_{\Sigma_1} \mathcal{N}$ and

(ii) an order-preserving map $\sigma\colon \operatorname{lh}(\mathcal{T}) \to \operatorname{lh}(\mathcal{U})$

such that

(i) $\alpha \leq_{\mathcal{T}} \beta \iff \sigma(\alpha) \leq_{\mathcal{U}} \sigma(\beta)$

(ii) $[\alpha, \beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}} = \emptyset \iff [\sigma(\alpha), \sigma(\beta)]_{\mathcal{U}} \cap \mathcal{D}^{\mathcal{U}} = \emptyset$,

(iii) $\pi_\alpha\colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_{\sigma(\alpha)}^{\mathcal{U}}$ and $\pi_\alpha(E_\alpha^{\mathcal{T}}) = E_{\sigma(\alpha)}^{\mathcal{U}}$,

(iv) for $\beta < \alpha$ we have $\pi_\alpha \restriction \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1 = \pi_\beta \restriction \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1$,

(v) for $\alpha \leq_{\mathcal{T}} \beta$ with $[\alpha, \beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}}$ we have $\pi_\beta \circ \pi_{\alpha,\beta}^{\mathcal{T}} = \pi_{\sigma(\alpha),\sigma(\beta)}^{\mathcal{U}} \circ \pi_\alpha$,

(vi) if $\beta = \mathrm{pred}_{\mathcal{T}}(\alpha+1)$, then $\sigma(\beta) = \mathrm{pred}_{\mathcal{U}}(\sigma(\alpha+1))$ and $\pi_{\alpha+1}([a,f]_{E^{\mathcal{T}}_{\alpha}}) = [\pi_\alpha(a), \pi_\beta(f)]_{E^{\mathcal{T}}_{\sigma(\alpha)}}$ and

(vii) $0 \leq_{\mathcal{U}} \sigma(0)$, $[0,\sigma(0)] \cap \mathcal{D}^{\mathcal{U}} = \emptyset$ and $\pi_0 = \pi^{\mathcal{U}}_{0,\sigma(0)} \circ \pi$,

(See Figure 7.2) ○

Definition 7.7. Let $\mathcal{M},\mathcal{N}$ be layered hybrid premice and $\vec{\mathcal{T}},\vec{\mathcal{U}}$ be stacks of normal trees on $\mathcal{M},\mathcal{N}$ respectively. $(\mathcal{M},\vec{\mathcal{T}})$ **is a hull of** $(\mathcal{N},\vec{\mathcal{U}})$ if there are

(i) an order presercing map $\sigma\colon \mathrm{lh}(\vec{\mathcal{T}}) \to \mathrm{lh}(\vec{\mathcal{U}})$,

(ii) a sequence $(\sigma_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}))$ of order preserving maps $\sigma_\alpha\colon \mathrm{lh}(\mathcal{T}_\alpha) \to \mathrm{lh}(\mathcal{U}_{\sigma(\alpha)})$,

(iii) $(\pi_{\alpha,\beta} \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}) \wedge \beta < \mathrm{lh}(\mathcal{T}_\alpha))$ such that

  (a) $\pi_{0,0} = \pi^{\vec{\mathcal{U}}}_{0,\sigma(0)}$ (so that $\pi_{0,0} = \mathrm{id}$ if $\sigma(0) = 0$),

  (b) for $\alpha < \mathrm{lh}(\vec{\mathcal{T}})$

$$\pi_{\alpha,0}\colon \mathcal{M}^{\vec{\mathcal{T}}}_{\alpha} \to_{\Sigma_1} \mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}$$

  and $(\mathcal{M}^{\vec{\mathcal{T}}}_{\alpha}, \mathcal{T}_\alpha)$ is a $(\pi_{\alpha,0},\sigma_0)$-hull of $(\mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}, \mathcal{U}_{\sigma(\alpha)})$,

  (c) $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$ and $\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)}$ exists, then $\pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))}$ exists and

$$\pi_{\beta,\eta} \circ \pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} = \pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))} \circ \pi_{\alpha,\gamma}.$$

(See Figure 7.3) ○

Definition 7.8. Let $\mathcal{M}$ be a layered hybrid premouse and Σ be a (partial) iteration strategy for $\mathcal{M}$. Σ has **hull condensation** if the following holds true for any two stacks $\vec{\mathcal{T}},\vec{\mathcal{U}}$ on $\mathcal{M}$. If $\vec{\mathcal{U}}$ is according to Σ and $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$, then $\vec{\mathcal{T}}$ is according to Σ. ○

Lemma 7.9. *Let Σ be an iteration strategy. Then the following hold true.*

(i) If Σ has hull condensation then it is pullback consistent.

(ii) If Σ is positional and pullback consistent then it is commuting.

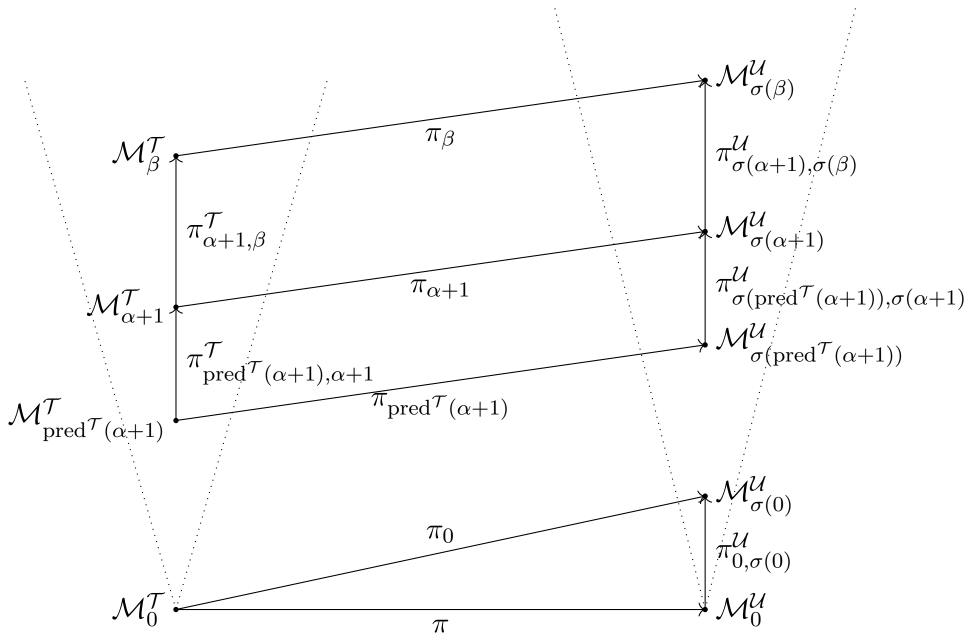

Figure 7.2: $\mathcal{T}$ is a hull of $\mathcal{U}$

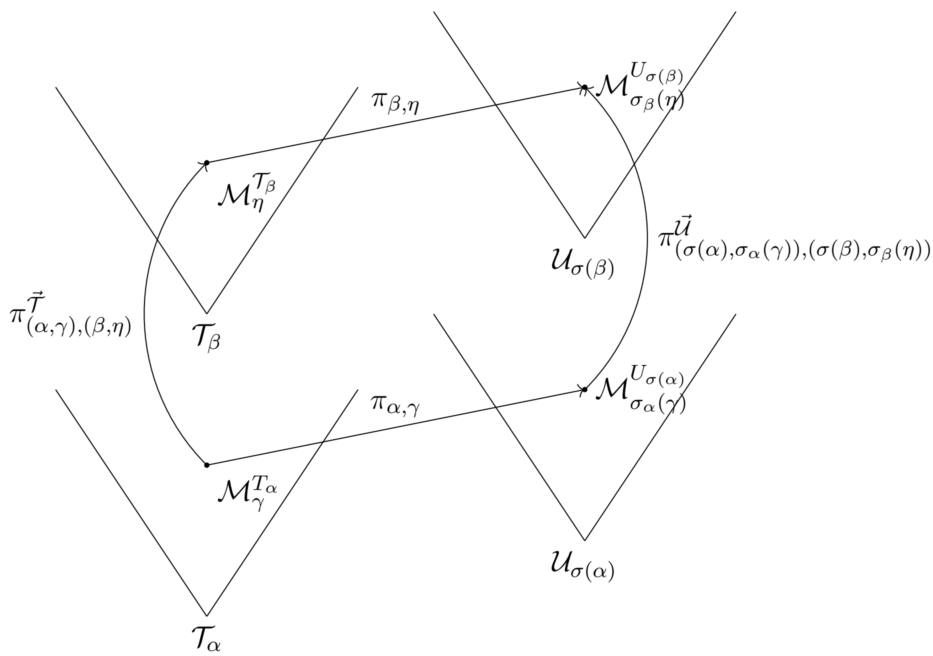

Figure 7.3: $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$

PROOF. See [Sargsyan, 2015b, Proposition 2.36]. ■

7.1.2 Layered hybrid mice

Define strategy mice as a particular kind of hybrid mice, hod mice/pairs and put in positional and commuting in the definition, state comparison. Introduce derived models of hod mice and how they relate to the Solovay hierarchy. Define Σ-mouse

Definition 7.10. Let $\mathcal{M}$ be a transitive set (or structure). We let $o(\mathcal{M}) := \mathcal{M} \cap \mathsf{On}$ be the ordinal height of $\mathcal{M}$. ∘

Definition 7.11. Let $\mathcal{M}$ be a (hybrid) premouse and $\alpha \leq o(\mathcal{M})$. We let

(i) $\mathcal{M}||\alpha$ be the initial segment of $\mathcal{M}$ of height α including its top extender and

(ii) $\mathcal{M}|\alpha$ be the passive initial segment of $\mathcal{M}$ of height α, i.e. $\mathcal{M}||\alpha$ but without the top extender.

∘

Definition 7.12. Let $\mathcal{M}$ be a $\mathcal{J}$-structure [1] and $\alpha \leq o(\mathcal{M})$. We write $\mathcal{J}_\alpha^{\mathcal{M}}$ for the αth level of $\mathcal{M}$'s construction. ∘

Definition 7.13. A **potential layered hybrid premouse** (over X) is an acceptable $\mathcal{J}$-structure of the form $\mathcal{M} = (J_\alpha^{\vec{E},f}(X); \in, \vec{E}, B, f)$ X such that

(i) $\vec{E}$ is a fine extender sequence (over X),

(ii) f is a function with domain $Y \subseteq \alpha$ such that $f(\gamma)$, for each $\gamma \in Y$, is a shift of an amenable function that typically codes part of an iteration strategy for $\mathcal{M}$,

We will often write $\vec{E}^{\mathcal{M}}, f^{\mathcal{M}}, Y^{\mathcal{M}}$ for $\vec{E}, f, Y$ as above. If all proper initial segments of $\mathcal{M}$ are sound, we say that $\mathcal{M}$ is a **layered hybrid premouse**. ∘

[1] See [Zeman, 2011] for the basics on $\mathcal{J}$-structures, premice and their fine structure.

In our case, assuming X is a self-well-ordered set, $Y^{\mathcal{M}}$ is determined by the **standard indexing scheme** (see [Sargsyan, 2015b, Definition 1.18]).

Definition 7.14. Let Σ be a strategy for a layered hybrid premouse $\mathcal{M}$. For $\alpha \leq o(M)$ we let $\Sigma_{\mathcal{M}|\alpha}$ be the id-pullback iteration strategy on $\mathcal{M}|\alpha$ induced by Σ, i.e. a stack $\vec{\mathcal{T}}$ on $\mathcal{M}|\alpha$ is according to $\Sigma_{\mathcal{M}|\alpha}$ iff $\mathrm{id}\,\vec{\mathcal{T}}$ on $\mathcal{M}$, given by the copy construction via id (see [Steel, 2010, 4.1]), is according to Σ. $\circ$

Definition 7.15. A **layered strategy premouse** $\mathcal{M}$ is a layered hybrid premouse such that

(i) $f^{\mathcal{M}}(\gamma)$ codes a partial iteration strategy $\Sigma^{\mathcal{M}}_\gamma$ for $\mathcal{M}|\gamma$ and

(ii) For $\gamma_0, \gamma_1 \in Y^{\mathcal{M}}$, if $\gamma_0 < \gamma_1$ then $(\Sigma^{\mathcal{M}}_{\gamma_1})_{\mathcal{M}|\gamma_0} \subseteq \Sigma^{\mathcal{M}}_{\gamma_0}$.

We also write $\Sigma^{\mathcal{M}}$ for the the strategy coded by $f^{\mathcal{M}}$. $\circ$

Definition 7.16. Let $\mathcal{M}$ be a layered strategy premouse and Σ be an iteration strategy for $\mathcal{M}$. $\mathcal{M}$ is a **Σ-premouse** if $\Sigma^{\mathcal{M}} \subseteq \Sigma$. $\circ$

Definition 7.17. Let Σ be an iteration strategy. We write $\mathcal{M}_\Sigma$ for the (layered hybrid) premouse $\mathcal{N}$ such that Σ is an iteration strategy for $\mathcal{N}$. We also let $I(\mathcal{M}_\Sigma, \Sigma)$ be the set of pairs $(\vec{\mathcal{T}}, N)$ such that $\vec{\mathcal{T}}$ is a stack of normal trees on $\mathcal{M}_\Sigma$ according to Σ, $\pi^{\vec{\mathcal{T}}}$ exists and N is the last model of $\vec{\mathcal{T}}$. We also let

$$pI(\mathcal{M}_\Sigma, \Sigma) := \{N \mid \exists \vec{\mathcal{T}} \colon (\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)\}.$$

$\circ$

Definition 7.18. A Σ-premouse $\mathcal{M}$ is a **Σ-mouse** if there is a $\omega_1 + 1$-iteration strategy Λ such that all $\mathcal{N} \in pI(\mathcal{M}, \Lambda)$, $\mathcal{N}$ are themselves Σ-premice. $\circ$

Definition 7.19. Let a be a transitive self-well-ordered set and let Σ be an iteration strategy with hull-condensation such that $\mathcal{M}_\Sigma \in a$ and let Γ be a pointclass which is closed under Boolean operations and continuous images and preimages. Define the (Γ, Σ)-Lp stack over a recursively as follows:

(i) $\mathrm{Lp}_0^{\Gamma,\Sigma}(a) := a \cup \{a\}$,

(ii) $\mathrm{Lp}_{\alpha+1}^{\Gamma,\Sigma}(a)$ is the union of all sound Σ-mice over $\mathrm{Lp}_\alpha^{\Gamma,\Sigma}(a)$ that projects to $o(\mathrm{Lp}_\alpha^{\Gamma,\Sigma}(a))$ and which has an iteration strategy in Γ,

(iii) $\mathrm{Lp}_\lambda^{\Gamma,\Sigma}(a) := \bigcup_{\alpha<\lambda} \mathrm{Lp}^{\Gamma,\Sigma}(a)$ for limit λ.

We also let $\mathrm{Lp}^{\Gamma,\Sigma}(a) := \mathrm{Lp}_1^{\Gamma,\Sigma}(a)$. ○

7.1.3 HOD mice

Definition 7.20. Suppose $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ is a layered strategic premouse. $\mathcal{P}$ is a HOD-**premouse**[2] provided the following hold: Let $\lambda = \mathrm{otp}(Y^\mathcal{P})$, $(\gamma_\beta \mid \beta < \lambda)$ be the strictly increasing enumeration of $Y^\mathcal{P}$ and let, for $\beta < \lambda$, $\mathcal{P}(\beta) := \mathcal{P}||\gamma_\beta$ and moreover $\mathcal{P}(\lambda) := \mathcal{P}$. Then there is a continuous, strictly increasing sequence $(\delta_\beta \mid \beta \leq \lambda)$ of $\mathcal{P}$-cardinals such that

(i) $B = \emptyset$,

(ii) $Y^\mathcal{P} \subseteq \delta_\lambda$,

(iii) $(\delta_\beta \mid \beta \leq \lambda)$ is sequence of Woodin cardinals and their limits in $\mathcal{P}$ and

(iv) for all $\beta \leq \lambda$

 (a) δ_β is a strong cutpoint of $\mathcal{P}$,

 define strong cut-point

 (b) $\mathcal{P}(\beta) \models \ulcorner\mathsf{ZFC}\text{-Replacement}\urcorner$,

 (c) $\mathcal{P}(\beta) = \mathcal{O}_{\delta_\beta}^{\mathcal{P},\omega}$ [3],

 (d) if β is a limit then $\delta_\beta^{+\mathcal{P}} = \delta_\beta^{+\mathcal{P}(\beta)}$,

 (e) if $\beta < \lambda$ then $f(\gamma_\beta)$ codes a $(o(\mathcal{P}), o(\mathcal{P}))$-strategy, call it $\Sigma_\beta^\mathcal{P}$, for $\mathcal{P}(\beta)$ with hull condensation [4],

 (f) if $\alpha < \beta < \lambda$, then $(\Sigma_\beta^\mathcal{P})_{\mathcal{P}(\alpha)} = \Sigma_\alpha^\mathcal{P}$,

 confirm with Grigor that this is what he had in mind

 (g) if $\beta < \lambda$ and $\eta \in (\delta_\beta, \delta_{\beta+1})$ is a $\mathcal{P}$-successor cardinal, then $\mathcal{P}|\eta$ is a $\Sigma_{\gamma_\beta}^\mathcal{P}$-premouse over $\mathcal{P}(\beta)$ which is $(o(\mathcal{P}), o(\mathcal{P}))$-iterable for stacks above δ_β.

(v) $\forall n < \omega \colon \mathcal{P} \models \delta_\lambda^{+n}$ exists and $o(\mathcal{P}) = \sup_{n<\omega}(\delta_\lambda^{+n})^\mathcal{P}$.

[2] These are in fact HOD-premice below $\ulcorner\mathsf{AD}_\mathbb{R} + \Theta$ is measurable$\urcorner$ in [Sargsyan, 2015b]. However, since all of our HOD-mice are of this form, we omit this.

[3] see [Sargsyan, 2015b, Definition 1.26]

[4] note that $\Sigma_\beta^\mathcal{P} \subseteq \mathcal{P}$ is an internal strategy, i.e. only defined on trees that are elements of $\mathcal{P}$

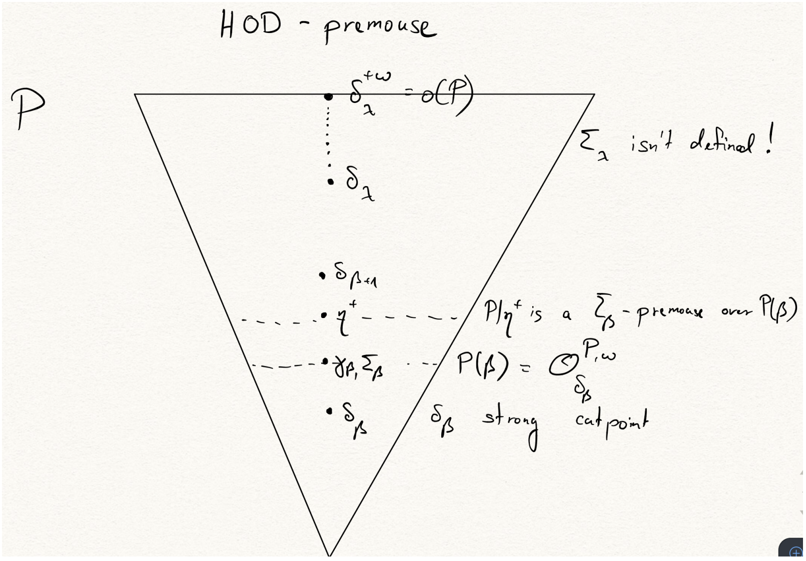

Figure 7.4: HOD premouse

See Figure 7.4. We will often write $\delta_\beta^{\mathcal{P}}, \gamma_\beta^{\mathcal{P}}, \lambda^{\mathcal{P}}$ for $\delta_\beta, \gamma_\beta, \lambda$ as above and moreover let $\delta^{\mathcal{P}} := \delta_\lambda$. ○

include an intuitive description of HOD-mice

Definition 7.21. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse. We let

$$\mathcal{P}^- = \begin{cases} P|\gamma_{\lambda^{\mathcal{P}}-1} & \text{, if } \lambda^{\mathcal{P}} \text{ is a successor ordinal,} \\ \mathcal{P}|\delta^{\mathcal{P}} & \text{, otherwise.} \end{cases}$$

See Figure 7.5

add picture and figure out why we don't just let $\mathcal{P}^- = \mathcal{P}(\gamma^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1})$ in the successor case.

○

Definition 7.22. Let $\mathcal{P}, \mathcal{Q}$ be HOD-premice. We write $\mathcal{P} \trianglelefteq_{\mathrm{HOD}} \mathcal{Q}$ if there is some $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We also write $\mathcal{P} \triangleleft_{\mathrm{HOD}} \mathcal{Q}$ if $\mathcal{P} \trianglelefteq_{\mathrm{HOD}} \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$. In this case we say that $\mathcal{P}$ is a (proper) HOD-**initial segment** of $\mathcal{Q}$. ○

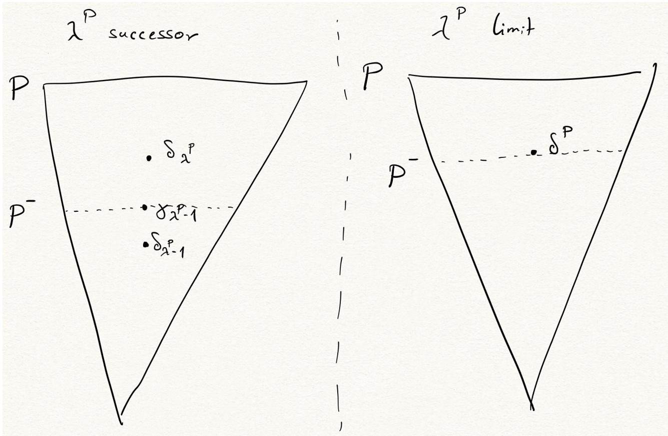

Figure 7.5: $\mathcal{P}^-$

Definition 7.23. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse and $\alpha \leq \lambda^{\mathcal{P}}$.

(i) If $\alpha < \lambda^{\mathcal{P}}$, we let $\Sigma^{\mathcal{P}}_\alpha$ be the internal iteration strategy of $\mathcal{P}(\alpha)$ coded by $f(\alpha)$ and

(ii) $\Sigma^{\mathcal{P}}_{<\alpha} := \bigoplus_{\beta<\alpha} \Sigma^{\mathcal{P}}_\beta$.

We also let $\Sigma^{\mathcal{P}} := \Sigma^{\mathcal{P}}_{<\lambda^{\mathcal{P}}}$. ◦

Remark 7.24. By the agreement of the internal iteration strategies of HOD-premice (item 4f in Theorem 7.20), $\Sigma^{\mathcal{P}}_\alpha$ already includes all of the information of $\Sigma^{\mathcal{P}}_{<\alpha}$ and can be identified with $\Sigma^{\mathcal{P}}_{<\alpha+1}$. reference broken

Definition 7.25. Let $\mathcal{P}$ be a HOD-premouse. Σ is a (κ, λ)**-iteration strategy** for $\mathcal{P}$ if it is a winning strategy for player II in the iteration game $\mathcal{G}(\mathcal{P}, \kappa, \lambda)$ and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, then $\mathcal{Q}$ is a HOD-premouse such that $\Sigma^{\mathcal{Q}} = \Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \cap \mathcal{Q}$. ◦ add reference

Remark 7.26. In particular, $\Sigma^{\mathcal{P}} = \Sigma \cap \mathcal{P}$, i.e. Σ extends the internal iteration strategy of $\mathcal{P}$.

Definition 7.27. $(\mathcal{P}, \Sigma)$ is a HOD-**pair** if

(i) $\mathcal{P}$ is a HOD-premouse and

(ii) Σ is a $(\omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{P}$ with hull condensation.

the definition of hod pair is different in both versions of Grigor‘s thesis. Verify that this is the intended one.

$\circ$

7.1.4 HOD analysis

gather all the information we need on HOD – this can be found in Grigor‘s thesis

Definition 7.28. Let $(P, \Sigma), (Q, \Lambda)$ be HOD-pairs. We let $(\mathcal{P}, \Sigma) \leq_{\text{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$, i.e. there is a $(\mathcal{P}, \Sigma)$-iterate $(\mathcal{T}, R)$ and a $(\mathcal{Q}, \Lambda)$-iterate $(\mathcal{U}, S)$ such that

$$\mathcal{R} \trianglelefteq_{\text{HOD}} \mathcal{S} \text{ and } \Sigma_{\mathcal{R},\mathcal{T}} = \Lambda_{\mathcal{R},\mathcal{U}}.$$

We also let $(\mathcal{P}, \Sigma) <_{\text{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma) \leq_{\text{DJ}} (\mathcal{Q}, \Lambda)$ and $(\mathcal{Q}, \Lambda) \not\leq_{\text{DJ}} (\mathcal{P}, \Sigma)$. $\circ$

Definition 7.29. Let $(\mathcal{P}, \Sigma)$ be a HOD-pair such that Σ has branch condensation and is fullness preserving. Then we define $\alpha(\mathcal{P}, \Sigma)$ to be the order type of $(\mathcal{P}, \Sigma)$ with respect to $\leq_{\text{DJ}}$. $\circ$

Remark 7.30. As in the case of ordinary premice, $\leq_{\text{DJ}}$ (or rather $<_{\text{DJ}}$) is a wellfounded relation. The interesting question is whether it‘s total.

Theorem 7.31 (Sargsyan). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are fullness preserving. Then $(\mathcal{P}, \Sigma) \leq_{\text{DJ}} (\mathcal{Q}, \Lambda)$ or $(\mathcal{Q}, \Lambda) \leq_{\text{DJ}} (\mathcal{P}, \Sigma)$.*

Proof. [Sargsyan, 2015b, Theorem 5.10]. ■

Theorem 7.32 (Sargsyan). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are Γ-fullness preserving for some pointclass Γ which is closed under continuous images and preimages. Suppose further that there is a good pointclass*

Γ^* *such that* $\Gamma \cup \{\mathrm{Code}(\Sigma), \mathrm{Code}(\Lambda)\} \subseteq \Delta_{\underset{\sim}{\Gamma^*}}$. *Then* $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ *or* $(\mathcal{Q}, \Lambda) \leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$. define Code Σ

Proof. [Sargsyan, 2015b, Theorem 2.33]. ■

Definition 7.33. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. We let

(i) $\mathcal{F}(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \mid \mathcal{Q} \in pB(\mathcal{P}, \Sigma)\}$ and

(ii) $\mathcal{F}^+(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \mid \mathcal{Q} \in pI(\mathcal{P}, \Sigma)\}$.

∘

Remark 7.34. By [Sargsyan, 2015b, Corollary 2.44] Σ is commuting, so that $\Sigma_{\mathcal{Q}}$ is indeed well-defined.

Definition 7.35. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$. We let $\mathcal{Q} \leq^{\mathcal{P},\Sigma} \mathcal{R}$ if

(i) $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and $R \in pI(\mathcal{Q}, \Sigma_Q)$ or

(ii) $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \leq_{\mathrm{DJ}} (\mathcal{R}, \Sigma_{\mathcal{R}})$.

∘

Lemma 7.36 (Sargsyan)**.** $\leq^{\mathcal{P},\Sigma}$ *is directed.*

Proof. [Sargsyan, 2015b, Lemma 4.17]. ■

Definition 7.37. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q}, \mathcal{R} \in \mathrm{pI}(\mathcal{P}, \Sigma) \cup \mathrm{pB}(\mathcal{P}, \Sigma)$ be such that, for some $\alpha \leq^{\mathcal{R}}$, $\mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. We let

$$\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q} \to \mathcal{R}(\alpha)$$

be the iteration map given by $\Sigma_{\mathcal{Q}}$. We let

(i) $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ be the direct limit of $\mathcal{F}(\mathcal{P}, \Sigma)$ with respect to the embeddings $\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}}$ for $\mathcal{Q}, \mathcal{R} \in pB(\mathcal{P}, \Sigma)$ such that there is an $\alpha \leq \lambda^{\mathcal{R}}\mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$); and let

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ be the direct limit of $\mathcal{F}(\mathcal{P}, \Sigma)$ with respect to the embeddings $\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}}$ for $\mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma)$ such that $\mathcal{Q} \leq^{\Sigma}_{\mathcal{Q},\mathcal{R}} \mathcal{R}$).

For $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ we let

(i) $\pi^{\Sigma}_{\mathcal{Q},\infty} \colon \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$

(ii) $\sigma^{\Sigma}_{\mathcal{R},\infty} \colon \mathcal{R} \to \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$

be the direct limit maps. ◦

Definition 7.38. Let $(\mathcal{P}, \Sigma)$ be as above. We let

(i) $\delta_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$,

(ii) $\delta^+_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ and

(iii) $\lambda_\infty(\mathcal{P}, \Sigma) := \lambda^{\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)}$.

◦

Lemma 7.39 (Sargsyan)**.** *Let Γ be a pointclass closed under Wadge reducibility. Suppose $(\mathcal{P}, \Sigma)$ is a* HOD*-pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal and Σ has branch condensation and is Γ-fullness preserving. Then*

(i) $\delta_\infty(\mathcal{P}, \Sigma) = \delta^+_\infty(\mathcal{P}, \Sigma)$ *and*

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)|\delta^+_\infty = \mathcal{M}_\infty(\mathcal{P}, \Sigma)$.

Proof. [Sargsyan, 2015b, Lemma 4.18]. ■

We will likely not need the entire theorem and should reduce it to the part that we need once we are done.

Theorem 7.40 (Sargsyan)**.** *Assume AD^+, let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $\mathcal{H} =$* HOD$^{L(\Gamma,\mathbb{R})}$*. Then the following holds:*

define ϕ

define Ω^Γ

define θ^Γ_α

(i) If $L(\Gamma, \mathbb{R}) \models \phi$ then for all $(\mathcal{P}, \Sigma) \in \Gamma$ such that $\alpha(\mathcal{P}, \Sigma) < \Omega^\Gamma$ we have, for all $\alpha \leq \alpha(\mathcal{P}, \Sigma)$,

(a) $\delta^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}_\alpha = \theta^\Gamma_\alpha$ and

(b) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}; \in, \vec{E}^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} \restriction \theta^\Gamma_\alpha, \Lambda \restriction \theta^\Gamma_\alpha)$,

where Λ *is the iteration strategy coded by* $f^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}$.

(ii) *If* $L(\Gamma,\mathbb{R}) \models \psi$ *then for all* $\alpha \leq \Omega^\Gamma$ define ψ

(a) $\delta_\alpha^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} = \theta^\Gamma_\alpha$ *and*

(b) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}; \in, \vec{E}^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} \restriction \theta^\Gamma_\alpha, \Lambda \restriction \theta^\Gamma_\alpha)$.

(iii) *Suppose* $\Gamma^* \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \models \mathsf{AD}^+$ *and there is a* HOD*-pair* $(\mathcal{P},\Sigma) \in \Gamma^*$ *such that*

(a) Σ *has branch condensation and is* Γ*-fullness preserving,*

(b) $\lambda^{\mathcal{P}}$ *is a successor ordinal,* $\mathrm{Code}(\Sigma_{\mathcal{P}^-}) \in \Gamma$ *and* $L(\Gamma,\mathbb{R})$ *models that* $(\mathcal{P},\Sigma_{\mathcal{P}^-})$ *is a suitable pair such that* $\alpha(\mathcal{P}^-,\Sigma_{\mathcal{P}^-}) = \alpha$, define suitable pair

(c) *there is a sequence* $(B_i \mid i < \omega) \subseteq \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *which guides* Σ *and* define $\mathbb{B}(..)$ and what it means to be guided

(d) *for any* $B \in \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *there is some* $\mathcal{R} \in \mathrm{pI}(\mathcal{P},\Sigma)$ *such that* $\Sigma_{\mathcal{R}}$ *respects* B. define respects B

Then $L(\Gamma,\mathbb{R}) \models \psi$ *and* $\mathcal{M}_\infty(\mathcal{P},\Sigma) = \mathcal{M}^+_\infty(\mathcal{P},\Sigma)$.

Proof. [Sargsyan, 2015b, Theorem 4.24]. ■

7.2 The induction start

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7.3 The successor case

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7.4 The countable cofinality case

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7.5 The singular case

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8 | Inner model direction

8.1 Determinacy in mice from DI

Proposition 8.1 (Folklore?). *If ω_1 carries a saturated ideal then mouse reflection holds at ω_1.*

Proof. Let $\mathcal{N}$ be a mouse operator defined on HC and fix some $x \in H_{\omega_2}$; we want to show that $\mathcal{N}(x)$ is defined. Let $j : V \to M$ be the generic ultrapower with $\operatorname{crit} j = \omega_1^V$ and note that $j(\omega_1^V) = \omega_1^M = \omega_1^{V[g]} = \omega_2^V$ by saturation of the ideal. This means in particular that $\mathsf{HC} \prec H_{\omega_2}^M$. Since

$$\mathsf{HC} \models \ulcorner \mathcal{N}(y) \text{ exists for all sets } y \urcorner$$

we get that $H_{\omega_2}^M$ believes the same is true. But $H_{\omega_2}^V \subseteq H_{\omega_2}^M$ since $\operatorname{crit} j = \omega_1^V$, so that in particular $H_{\omega_2}^M$ believes that $x^\sharp$ exists. Since M is closed under ω-sequences in $V[g]$ by Proposition E.5, we get that $x^\sharp$ exists in $V[g]$ and hence also in V as set forcing can't add sharps. ■

Prove this or give a reference.

Proposition 8.2 (Folklore?). *If ω_1 carries a precipitous ideal then HC is closed under sharps. If the ideal is furthermore saturated then H_{ω_2} is closed under sharps.*

Proof. Proposition 8.1 gives the latter statement if we show the former, so fix an $x \in \mathsf{HC}$ and let $j : V \to M$ be the generic ultrapower from a precipitous ideal on ω_1^V. Since $j(x) = x$ we get that $j : L[x] \to L[x]$ with $\operatorname{crit} j > \operatorname{rk} x$, implying that $x^\sharp$ exists in the generic extension. But set forcing can't add sharps so $x^\sharp$ exists in V as well. ■

Add argument or reference.

Definition 8.3. Let $j : V \to M$ be an elementary embedding in some $V[g]$ and let F be a model operator. Then F is **j-radiant** if it condenses well, determines itself on generic extensions and satisfies the **extension property**, which says that $F \subseteq j(F)$ and $j(F) \upharpoonright \mathsf{HC}^{V[g]}$ is definable in $V[g]$. $\circ$

Lemma 8.4 (DI)**.** *M_1^F is total on H_{ω_2} for any j-radiant model operator F on H_{ω_2}.*

Proof. We want to use the hybrid core model dichotomy 6.10, but the problem is that F is not total. We solve this by going to a smaller model; the model $W := L^F_{\omega_2^V}(\mathbb{R})$ will be a first attempt (note that $\mathbb{R} \in \operatorname{dom} F$ as we're assuming CH). To be able to apply the dichotomy in a model we need it to satisfy ZFC. The following claim is the first step towards this.

Claim 8.5. Given any real x, $L^F_{\omega_2}(x) \models \ulcorner\omega_1^V$ is inaccessible$\urcorner$.

Proof of claim. Letting $j : V \to M$ be the generic elementary embedding, note that j doesn't move x, so that

$$j \upharpoonright L^F_{\omega_2^V}(x) : L^F_{\omega_2^V}(x) \to L^{j(F)}_{\omega_2^M}(x).$$

Since F has the extension property, $L^{j(F)}_{\omega_2^M}(x)$ is just an end-extension of $L^F_{\omega_2^V}(x)$. In particular ω_1^V is still a cardinal in there, meaning that, for every $\alpha < \omega_1^V$,

$$L^{j(F)}_{\omega_1^M}(x) \models \ulcorner\text{there's a cardinal} > \alpha\urcorner.$$

By elementarity this makes ω_1^V a limit cardinal in $L^F_{\omega_2^V}(x)$ and by GCH in $L^F_{\omega_2^V}(x)$ it's inaccessible. $\dashv$

This claim is now transferred to M, and as $\mathbb{R}^V$ is a real from the point of view of M, we get that

$$L^{j(F)}_{\omega_2^M}(\mathbb{R}^V) \models \ulcorner \omega_1^M \text{ is inaccessible} \urcorner.$$

Noting that $\omega_1^M = \omega_2^V$ and again using the extension property of F, we get that $W \models \mathsf{ZF}$. We don't get choice in W as it doesn't contain a wellorder of the reals, so we we'll work with $W[h]$ instead, where $h \subseteq \mathrm{Col}(\omega_1, \mathbb{R})^W$ is W-generic. Since we're assuming CH we get that $g \in V$, making $W[h] \in V$ as well, $W[h]$ is still closed under F since F determines itself on generic extensions, and $W[h] \models \mathsf{ZFC}$.

We can now apply the hybrid core model dichotomy 6.10 inside $W[h]$ to conclude that, for every real x, either $K^F(x)^{W[h]}$ exists or $M_1^F(x)$ exists (note that (ω_1, ω_1)-iterability is absolute between $W[h]$ and V since $W[h]$ contains all the reals). Since mouse reflection holds at ω_1 by Proposition 8.1 if the latter conclusion held at all reals x then we would also get that M_1^F is total on H_{ω_2} and we'd be done. So assume $K := K^F(x)^{W[h]}$ exists.

Claim 8.6. $j(K) \in V$.

Proof of claim. This is where we'll be using homogeneity of our ideal. Firstly K is definable in $W[h]$ and thus also in W by homogeneity of $\mathrm{Col}(\omega_1, \mathbb{R})$, so that $j(K)$ is definable in $j(W)$. But $j(W)$ is definable in $V[g]$ as the unique $j(F)$-premouse over $\mathbb{R}$ of height ω_1, making $j(K)$ definable in $V[g]$ with $j(F) \restriction \mathsf{HC}$ as a parameter. But $j(F) \restriction \mathsf{HC}$ is definable in $V[g]$ since F satisfies the extension property, so homogeneity of our ideal implies that $j(F) \in V$ and hence $j(K) \in V$ as well. $\dashv$

This claim also implies that ω_1^V is inaccessible in K, as if it wasn't, say $\omega_1^V = \lambda^{+K}$, then $\omega_2^V = j(\omega_1^V) = j(\lambda)^{+j(K)} = \lambda^{+j(K)}$, so that ω_2^V isn't a cardinal in V, $\lightning$.

We then also get that $(\omega_1^V)^{+j(K)} < \omega_2^V$, since if they were equal then elementarity would imply that ω_1^V was a successor in K, $\lightning$.

Since $K|\omega_1^V = j(K)|\omega_1^V$, elementarity and the above implies that

$$j^2(K)|(\omega_1^V)^{+j^2(K)} = j(K)|(\omega_1^V)^{+j(K)},$$

which makes sense as $j(K) \in V$.

Let now E be the (ω_1^V, ω_2^V)-extender derived from $j \restriction j(K)$, and note that $E \restriction \alpha \in M$ for every $\alpha < \omega_2^V = \omega_1^M$ as M is closed under countable sequences in $V[g]$.

Claim 8.7. $E \restriction \alpha$ is on the $j(K)$-sequence for every $\alpha < \omega_2^V$.

Proof of claim. We need to show that

Why is this sufficient?

$$j(W) \models \ulcorner \langle \langle j(K), \mathrm{Ult}(j(K), E \restriction \alpha) \rangle, \alpha \rangle \text{ is } \mathsf{On}\text{-iterable} \urcorner.$$

Assume not. Then by reflection we get, in $j(W)$, a countable $\overline{K}$ and an elementary $\sigma : \overline{K} \to \mathrm{Ult}(j(K), E \restriction \alpha)$ with $\sigma \restriction \alpha = \mathrm{id}$ and $\langle \langle j(K), \overline{K} \rangle, \alpha \rangle$ isn't ω_1-iterable.

What kind of reflection?

Let $k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$ be the factor map with $k \restriction \alpha = \mathrm{id}$ and define $\psi := k \circ \sigma : \overline{K} \to j^2(K)$, so that $(k \circ \sigma) \restriction \alpha = \mathrm{id}$. We have both ψ and $\overline{K}$ in M, which is the generic ultrapower $\mathrm{Ult}(V, g)$, so we also get that $\psi = [\vec{\psi}_\xi]_g$, $\overline{K} = [\vec{K}_\xi]_g$ and $\alpha = [\vec{\alpha}_\xi]_g$. We need to show that

For g-almost every $\xi < \omega_1^V$ it holds that $W \models \ulcorner \langle \langle K, K_\xi \rangle, \alpha_\xi \rangle$ is ω_1-iterable$\urcorner$

By Łoś' Lemma we have that, in V and hence also in $V[g]$, there are embeddings $\psi_\xi : K_\xi \to j(K)$ with $\psi_\xi \restriction \alpha_\xi = \mathrm{id}$ for g-almost every $\xi < \omega_1^V$. As $j(W)$ is closed under countable sequences in $V[g]$ it sees that the K_ξ's are countable, so that an application of absoluteness of wellfoundedness shows that $j(W)$ also has elementary embeddings $\psi_\xi^* : K_\xi \to j(K)$ with $\psi_\xi^* \restriction \alpha_\xi$.

Include this argument perhaps.

But $j(K) = K^{j(F)}(x)^{j(W[h])}$, so $j(W[h])$ sees that $\langle \langle K, K_\xi \rangle, \alpha_\xi \rangle$ is ω_1-iterable, which is therefore also true in W since $W \cap \mathbb{R} \subseteq \mathbb{R}^{V[g]} = j(W[h]) \cap \mathbb{R}$. ⊣

Our desired contradiction is then showing that K has a Shelah cardinal,

Insert argument?

which is impossible. Let $f : \omega_1^V \to \omega_1^V$ be a function in $j(K)$ and pick some $\alpha \in (j(f)(\kappa), \omega_2^V)$. Letting

$$k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$$

be the factor map, we get that $\mathrm{crit}\, k \geq \alpha$ by coherence of extenders on the K-sequence and hence that $i_{E \restriction \alpha}(f)(\omega_1^V) < \alpha$ as well. This shows that ω_1^V is Shelah in $j(K)$ and hence K has a Shelah cardinal by elementarity, $\lightning$. ■

Specify niceness.

Theorem 8.8 (DI). $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R}) \models$ *AD for all "nice" Γ and Σ.*

Proof.

Show that all the operators occuring in the $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R})$ induction are j-radiant.

■

8.2 Ω is not zero

Collapse all these Ω sections into one, where the abstract results are moved into the internal/external CMI chapters

Define

$$\Gamma_0 := \{A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \models \mathsf{AD} + \Omega = 0\}.$$

Lemma 8.9 (DI). $\Gamma_0 = \mathrm{Lp}(\mathbb{R}) \cap \mathscr{P}(\mathbb{R})$.

Proof. ($\supseteq$) Let $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$ and let $A \subseteq \mathbb{R}$ be an element of $\mathcal{M}$. Since $\mathcal{M}$ projects to $\mathbb{R}$ and is sound, we get that A is OD_x for a real x, so that everything in $L(A, \mathbb{R})$ is also ordinal definable in a real as well. Since $\mathrm{Lp}(\mathbb{R}) \models \mathsf{AD}$

Check this proof.

we then get that $\mathsf{AD} + \Omega = 0$ holds in $L(A, \mathbb{R})$, making $A \in \Gamma_0$.

($\subseteq$) Let $A \in \Gamma_0$. Since we're assuming CH we get that $V[g] \models |\mathbb{R}| = \aleph_1^V = \aleph_0$, so fix a generic bijection $b : \omega \to \mathbb{R}^V$ in $V[g]$. Define $a_b \in \mathbb{R}$ as

$n \in a_b$ iff $b(n) \in A$. As $L(A, \mathbb{R}) \models \mathsf{AD} + \theta_0 = \Theta$ it holds that A is $\mathrm{OD}_z^{L(A,\mathbb{R})}$ for $z \in \mathbb{R}$, so that

$$A = j(A) \cap \mathbb{R}^V \in \mathrm{OD}_{z,\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}.$$

In particular, as A and $\mathbb{R}^V$ are definable from b and a_b is definable from b, we get that $a_b \in \mathrm{OD}_b^{L(j(A),\mathbb{R}^{V[g]})}$. By MC we then get that there's some b-premouse $\mathcal{M} \in L(j(A), \mathbb{R}^{V[g]})$ projecting to b with $a_b \in \mathcal{M}$ and a Σ such that

$$L(j(A), \mathbb{R}^{V[g]}) \models \ulcorner\Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{M}\urcorner.$$

Why is it that we have to go through b in this fashion? Can't we just use MC and get $\mathcal{N}$ without going through $\mathcal{M}$? Is it because $L(j(A), \mathbb{R}^{V[g]})$ doesn't know that $\mathbb{R}^V$ is countable?

From this $\mathcal{M}$ we can then get an $\mathbb{R}^V$-premouse $\mathcal{N} \in L(j(A), \mathbb{R}^{V[g]})$ projecting to $\mathbb{R}^V$ with $A \in \mathcal{N}$ and

$$L(j(A), \mathbb{R}^{V[g]}) \models \ulcorner\Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{N}\urcorner.$$

Now $\mathcal{N}$ is $\mathrm{OD}_{\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}$, and since we don't have divergent models of AD^+ it holds that, letting $\Theta^{j(A)} := \Theta^{L(j(A),\mathbb{R}^{V[g]})}$,

$$V[g] \models L(j(A), \mathbb{R}) = L(P_{\Theta^{j(A)}}(\mathbb{R})).$$

This means that $\mathcal{N} \in \mathrm{OD}_{\mathbb{R}^V}^{V[g]}$, so that homogeneity of I we get that $\mathcal{N} \in V$. It remains to show that $\mathcal{N} \trianglelefteq \mathrm{Lp}(\mathbb{R}^V)$, meaning that we need to show that $\mathcal{N}$ is countably $(\omega_1 + 1)$-iterable in V. But letting $\overline{\mathcal{N}} \to \mathcal{N}$ be a countable hull in V we get that $j(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$, so that elementarity of j implies that $\Sigma \restriction V \in V$ is an $\omega_1^{V[g]} = \omega_2^V$-iteration strategy for $\overline{\mathcal{N}}$ and we're done. ■

Why's this?

Is this really this iterable?

Proposition 8.10 (DI). $\mathrm{cof}^V(\Theta^{\mathrm{Lp}(\mathbb{R})}) = \omega$.

Proof.

See Ketchersid's Thesis 3.17 or 7.4.2 in the CMI book. Perhaps we don't need it though, following Wilson's thesis.

■

Theorem 8.11. *Let Γ be an inductive-like pointclass. If $\mathcal{M}$ is a suitable quasi-iterable premouse, $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ is closed under recursive join and the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is both total on $\mathcal{M}$ and has the full factors property, then there's a unique Γ-fullness preserving (ω_1, ω_1)-strategy Φ for $\mathcal{M}$ such that, for every quasi-iterate $\mathcal{P}$ of $\mathcal{M}$,*

- *$\mathcal{P}$ is a non-dropping Φ-iterate of $\mathcal{M}$; and*
- *the Φ-iteration map $i : \mathcal{M} \to \mathcal{P}$ equals the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\mathcal{P}}$.*

Let $\Phi_{\mathcal{M}}$ be the unique strategy for $\mathcal{M}$ as in the above theorem. We now improve this to include branch condensation.

The 3d argument is quite similar to the proof of Theorem 7.19 in the outline.

Theorem 8.12. *Let Γ be an inductive-like pointclass and assume that $\boldsymbol{\Delta}_\Gamma$ is determined and that Γ-MC holds. Let $\mathcal{M}$ be an ω-suitable quasi-iterable premouse such that $\mathcal{D}(\mathcal{M}) \equiv \mathcal{M}_\Gamma$, let $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ be closed under recursive join, assume $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and that it has the full factors property. Let $\Phi := \Phi_{\mathcal{M}}$. Then there's a $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ has $\mathcal{A}$-condensation, and hence also branch condensation, for every $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$.*

This is the companion of Γ, see Trevor's thesis. I'm not sure if we can find $\mathcal{M}$ like this, however.

Proof. Assume not and fix $A \in \mathrm{Env}(\Gamma)$ such that given any $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ there's a $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ doesn't have A-condensation. Applying this inductively, we get a sequences $\langle \mathcal{Q}^0_n, \mathcal{R}^0_n, \mathcal{T}^0_n, \pi^0_n, \sigma^0_n, j^0_n \mid n < \omega \rangle$ such that

(i) $\mathcal{Q}^0_0 := \mathcal{M}$;

(ii) $\pi^0_n : \mathcal{Q}^0_n \to \mathcal{Q}^0_{n+1}$ is the iteration map through a tree of successor length, according to Φ;

$$
\begin{tikzcd}
\mathcal{Q}_0^0 \arrow[rr, "\pi_0^0"] \arrow[dr, "\sigma_0^0"'] \arrow[dd, "\tau_0^0"'] & & \mathcal{Q}_1^0 \arrow[rr, "\pi_1^0"] \arrow[dr, "\sigma_1^0"'] \arrow[dd, "\tau_1^0"'] & & \mathcal{Q}_2^0 \arrow[r, "\pi_2^0"] \arrow[dd, "\tau_2^0"'] & \cdots & \mathcal{Q}_\omega^0 \arrow[dd, "\tau_\omega^0"'] \\
& \mathcal{R}_0^0 \arrow[ur, "j_0^0"'] \arrow[dd, "k_0^0"] & & \mathcal{R}_1^0 \arrow[ur, "j_1^0"'] \arrow[dd, "k_1^0"] & & & \\
\mathcal{Q}_0^1 \arrow[rr, "\pi_0^1"] \arrow[dr, "\sigma_0^1"'] \arrow[dd, "\tau_0^1"'] & & \mathcal{Q}_1^1 \arrow[rr, "\pi_1^1"] \arrow[dr, "\sigma_1^1"'] \arrow[dd, "\tau_1^1"'] & & \mathcal{Q}_2^1 \arrow[r, "\pi_2^1"] \arrow[dd, "\tau_2^1"'] & \cdots & \mathcal{Q}_\omega^1 \arrow[dd, "\tau_\omega^1"'] \\
& \mathcal{R}_0^1 \arrow[ur, "j_0^1"'] \arrow[dd, "k_0^1"] & & \mathcal{R}_1^1 \arrow[ur, "j_1^1"'] \arrow[dd, "k_1^1"] & & & \\
\mathcal{Q}_0^2 \arrow[rr, "\pi_0^2"] \arrow[dr, "\sigma_0^2"'] \arrow[d, "\tau_0^2"'] & & \mathcal{Q}_1^2 \arrow[rr, "\pi_1^2"] \arrow[dr, "\sigma_1^2"'] \arrow[d, "\tau_1^2"'] & & \mathcal{Q}_2^2 \arrow[r, "\pi_2^2"] \arrow[d, "\tau_2^2"'] & \cdots & \mathcal{Q}_\omega^2 \arrow[d, "\tau_\omega^2"'] \\
\cdot & \mathcal{R}_0^2 \arrow[ur, "j_0^2"'] \arrow[dd, "k_0^2"] & \cdot & \mathcal{R}_1^2 \arrow[ur, "j_1^2"'] \arrow[dd, "k_1^2"] & \cdot & & \cdot \\
& & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
& \cdot & & \cdot & & & \\
\mathcal{Q}_0^\omega \arrow[rr, "\pi_0^\omega"] \arrow[dr, "\sigma_0^\omega"'] & & \mathcal{Q}_1^\omega \arrow[rr, "\pi_1^\omega"] \arrow[dr, "\sigma_1^\omega"'] & & \mathcal{Q}_2^\omega \arrow[r, "\pi_2^\omega"] & \cdots & \mathcal{Q}_\omega^\omega \\
& \mathcal{R}_0^\omega \arrow[ur, "j_0^\omega"'] & & \mathcal{R}_1^\omega \arrow[ur, "j_1^\omega"'] & & &
\end{tikzcd}
$$

Figure 8.1: The three-dimensional argument in Theorem 8.12

(iii) $\sigma_n^0 : \mathcal{Q}_n^0 \to \mathcal{R}_n^0$ an iteration map through a tree of limit length, according to Φ;

(iv) $j_n^0 : \mathcal{R}_n^0 \to \mathcal{Q}_{n+1}^0$ is elementary such that $\pi_n^0 = j_n^0 \circ \sigma_n^0$;

(v) $(j_n^0)^{-1}(\tau_{A,j_n^0(\kappa)}^{\mathcal{Q}_{n+1}^0}) \neq \tau_{A,\kappa}^{\mathcal{R}_n^0}$ for every $\mathcal{R}_n^0$-cardinal $\kappa \geq \delta_0^{\mathcal{R}_n^0}$.

Let $\mathcal{Q}_\omega^0$ be the direct limit of the $\mathcal{Q}_n^0$'s under the π_n^0 maps. Also let $\langle x_n \mid n < \omega \rangle$ enumerate the reals of $\mathcal{M}_\Gamma$ and pick $s \in [\mathsf{On}]^{<\omega}$ and a formula φ such that

$$\forall x \in \mathbb{R}(x \in A \Leftrightarrow \mathcal{M}_\Gamma \models \varphi[x, s]).$$

Our strategy now is now firstly to capture all the x_n's so that the derived models of the resulting structures become equal to $\mathcal{M}_\Gamma$. See Figure 8.1.

Perform a genericity iteration of $\mathcal{Q}_0^0$ above $\delta_0^{\mathcal{Q}_0^0}$ to $\mathcal{Q}_0^1$ to make x_0 generic over $\mathcal{Q}_0^1$ at $\delta_1^{\mathcal{Q}_0^1}$, while lifting the genericity iteration tree via the copy construction to the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and picking branches on the genericity iteration tree on $\mathcal{Q}_0^0$ by using $\Phi_{\mathcal{Q}_\omega^0}$ on the lifted tree on $\mathcal{Q}_\omega^0$. Let $\tau_0^0 : \mathcal{Q}_0^0 \to \mathcal{Q}_0^1$ be the genericity iteration map and $\mathcal{W}_0$ the last model of the lifted tree on $\mathcal{Q}_\omega^0$.

Now perform another genericity iteration of the last model of the lifted iteration tree on $\mathcal{R}_0^0$ above its δ_0 to $\mathcal{R}_0^1$ to make x_0 generic over $\mathcal{R}_0^1$ at $\delta_1^{\mathcal{R}_0^1}$, with branches being picked by lifting the iteration tree to $\mathcal{W}_0$ and using the branches according to $\Phi_{\mathcal{W}_0}$. Let $k_0^0 : \mathcal{R}_0^0 \to \mathcal{R}_0^1$ be the iteration embedding, $\sigma_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_0^1$ be the shift of σ_0^0 followed by latter genericity iteration, and $\mathcal{W}_1$ the last model of the lifted tree on $\mathcal{W}_0$.

Do a third genericity iteration of the last model of the lifted stack on $\mathcal{Q}_1^0$ above its δ_0 to $\mathcal{Q}_1^1$ to make x_0 generic at $\delta_1^{\mathcal{Q}_1^1}$, with branches being picked by lifting the tree to $\mathcal{W}_1$ and using branches picked by $\Phi_{\mathcal{W}_1}$. Let $\tau_1^0 : \mathcal{Q}_1^0 \to \mathcal{Q}_1^1$ be the iteration embedding, $j_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_1^1$ be the natural map, and $\pi_0^1 := j_0^1 \circ \sigma_0^1$.

Now continue this process to make x_0 generic over the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and let $\mathcal{Q}_\omega^1$ be the direct limit of the $\mathcal{Q}_n^1$'s under the π_n^1 maps. Then start at $\mathcal{Q}_0^1$ and repeat the same thing to make x_1 generic at the respective δ_2's and so on. Let $\mathcal{Q}_i^\omega$ be the direct limit of the $\mathcal{Q}_i^n$'s under the τ_i^n maps, $\mathcal{R}_i^\omega$

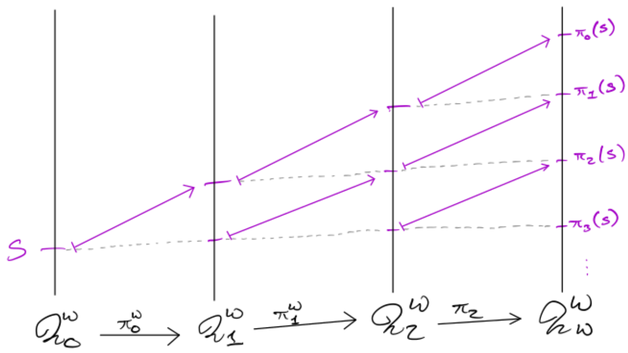

Figure 8.2: The argument in Claim 8.13.

the direct limit of the $\mathcal{R}^n_i$'s under the k^n_i maps and $\mathcal{Q}^n_\omega$ the direct limit of the $\mathcal{Q}^n_i$'s under the π^n_i maps.

By construction we get that the π^0_n's and τ^n_ω's are all by Φ and its tails, and that $\mathcal{Q}^\omega_\omega$ is wellfounded and Lp^Γ-full, so that the $\mathcal{Q}^\omega_n$'s and the $\mathcal{R}^\omega_n$'s are also wellfounded and Lp^Γ-full.

Claim 8.13. There exists some $k<\omega$ such that π^ω_n fixes s for every $n\geq k$.

Proof of claim. It suffices to show that $(\pi^\omega_n(\xi)\mid n<\omega)$ is eventually constant for all $\xi\in s$. Suppose this isn't the case. Fix $\xi\in s$ and a strictly increasing sequence $(i_n\mid n<\omega)$ such that $\pi^\omega_{i_n}(\xi)>\xi$ for all $n<\omega$. For $m<n<\omega$ we then have

$$\pi^\omega_{i_m,\infty}(\xi)=\pi^\omega_{i_n,\infty}\circ\pi^\omega_{i_m,i_n}(\xi)\geq\pi^\omega_{i_n,\infty}\circ\pi^\omega_{i_m}(\xi)>\pi^\omega_{i_n,\infty}(\xi),$$

so that $(\pi^\omega_{i_n}(\xi)\mid n<\omega)$ is a strictly decreasing sequence of ordinals in $\mathcal{Q}^\omega_\omega$ – contradicting its wellfoundedness. See Figure 8.2. ⊣

Let $k<\omega$ be as in the claim, and note that the j^ω_n's also fix s for $n\geq k$. Since $\mathcal{D}(\mathcal{R}^\omega_n)=\mathcal{M}_\Gamma$ for every $n<\omega$, the $\mathcal{Q}^\omega_n$'s and the $\mathcal{R}^\omega_n$'s have uniform definitions for the term relations for A when $n\geq k$, yielding that j^ω_n pulls back the term relation correctly whenever $n\geq k$, $\lightning$. ■

Theorem 8.14 (DI^+)**.** $\mathrm{Lp}(\mathbb{R}) \models \ulcorner$*there's a fullness preserving hod pair below* $\omega_1\urcorner$.

Proof.

Show the above requirements in Wilson's theorem is satisfied? Double check the statement.

■

Theorem 8.15 (DI^+)**.** *There is a model M containing all the reals such that* $M \models \mathsf{AD}^+ + \theta_0 < \Theta$.

Proof.

Let $(\mathcal{M}, \Sigma)$ be a fullness preserving hod pair in $\mathrm{Lp}(\mathbb{R})$ given by the above theorem. Then $\Sigma \notin \mathrm{Lp}(\mathbb{R})$ by the proof of 7.4.3 in the CMI book, and in particular $\Sigma \notin \Gamma_0$. Then $M := L(\Sigma, \mathbb{R})$ is the wanted model.

■

8.3 Ω is not a successor

Definition 8.16. Let $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ be hod pairs below ω_1. We then say that $(\mathcal{Q}, \Lambda)$ **extends** $(\mathcal{P}, \Sigma)$, or is an **extension** of $(\mathcal{P}, \Sigma)$, if there exists some $\alpha < \lambda^{\mathcal{Q}}$ such that

(i) $\mathcal{Q}(\alpha) \in pI(\mathcal{P}, \Sigma)$; and

(ii) $\Sigma_{\mathcal{Q}(\alpha)} = \Lambda_{\mathcal{Q}(\alpha)}$.

We say that $(\mathcal{P}, \Sigma)$ **can be extended** if there exists an extension of $(\mathcal{P}, \Sigma)$.

∘

Theorem 8.17 (DI^+)**.** *Every hod pair below ω_1 can be extended.*

Rough steps in the proof:

(i) Show that $M_1^{\sharp,\Sigma}$ exists

(ii) $\mathrm{Lp}^{\Sigma}(\mathbb{R}) \models \mathsf{AD}^+$ for some appropriate definition of $\mathrm{Lp}^{\Sigma}(\mathbb{R})$

(iii) The $\Omega > 0$ argument should show that there's an $A \notin \mathrm{Lp}^{\Sigma}(\mathbb{R})$ such that $L(A, \mathbb{R}) \models \mathsf{AD}^+$ and $\Sigma <_W A$

(iv) Show $L(A, \mathbb{R})$ then has the desired $(\mathcal{Q}, \Lambda)$ (this step has already been done and can be black boxed)

8.4 Ω does not have countable cofinality

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

8.5 Ω is not singular

Theorem 8.18 (DI^+)**.** *Assume there exists a sequence of hod pairs $(\mathcal{P}_\alpha, \Sigma_\alpha)$ below ω_1 with $(\mathcal{P}_{\alpha+1}, \Sigma_{\alpha+1})$ extending $(\mathcal{P}_\alpha, \Sigma_\alpha)$ for every α. Then either*

(i) There exists a hod pair $(\mathcal{H}, \Lambda)$ below ω_1 such that $\lambda^{\mathcal{H}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$; or

(ii) There exists an $\mathcal{M}$ containing all the reals such that $\mathcal{M} \models \mathsf{AD}_\mathbb{R} +$ Θ is regular.

Rough steps in the proof:

(i) Do the easier countable cofinality case

(ii) Coiterate all the hod pairs to some $(\mathcal{P}, \Sigma)$, which has $\lambda := \lambda^{\mathcal{P}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$

(iii) If λ has non-measurable cofinality then $(\mathcal{P}, \Sigma)$ is the hod pair that we're looking for, so assume this is not the case

(iv) Take the derived model $\mathcal{D}(\mathcal{P}, \lambda)$, which then satisfies $\mathsf{AD}_\mathbb{R}+\mathsf{DC}+\Omega = \lambda$, where DC is because λ has uncountable cofinality

This is wrong, as we can't take this derived model. Instead we should form a directed system of all "nice" hod pairs having λ's below $\lambda^{\mathcal{P}}$ and take the Lp-closure of that, which should then be an initial segment of hod; call it $\mathcal{H}$.

(v) Show that $\mathcal{H}\,|\delta^{\mathcal{H}}$ is the union of M^{α}_{∞} for $\alpha < \lambda$, where M^{α}_{∞} is the hod limit of

$$\mathcal{F}_{\alpha} := \{(\mathcal{Q}, \Psi) \mid \mathrm{Ult}(V, g) \models \ulcorner(\mathcal{Q}, \Psi) \text{ is a hod pair and } \lambda^{\mathcal{Q}} = \alpha\urcorner\}.$$ [1]

Let Φ be the join of the strategies of the M^{α}_{∞}'s and show that $\mathcal{H} = \mathrm{Lp}^{\Phi}_{\omega}(\mathcal{H}\,|\delta^{\mathcal{H}})$.

(vi) Show that $\mathcal{H} \models \ulcorner\delta^{\mathcal{H}}$ is singular$\urcorner$, since otherwise $\mathcal{D}(\mathcal{H}, \delta^{\mathcal{H}}) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular and we're done.

(vii) We want to construct a strategy Λ for $\mathcal{H}$ such that $(\mathcal{H}, \Lambda)$ is a hod pair below ω_1, as then this is the hod pair that we're looking for.

Definition 8.19. Let $(\mathcal{P}, \Sigma)$ be a hod pair. We let

(i) $\mathrm{I}(\mathcal{P}, \Sigma) := \{(\vec{\mathcal{T}}, \mathcal{Q}) \mid \vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ via Σ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$exists$\}$ be the collection of $(\mathcal{P}, \Sigma)$**-iterates**,

(ii) $\mathrm{pI}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$

(iii) $\mathrm{B}(\mathcal{P}, \Sigma) := \{(\mathcal{T}, \mathcal{M}) \mid \mathcal{M} \lhd_{\mathrm{HOD}} \mathcal{Q}$ and $(\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)\}$ be the collection of $(\mathcal{P}, \Sigma)$**-blowups** and

(iv) $\mathrm{pB}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{B}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$.

$\circ$

Definition 8.20. Let $(\mathcal{P}, \Sigma)$ be a hod pair and Γ is a pointclass closed under Boolean operations and continuous images and preimages. Then Σ is **Γ-fullness preserving** if for all $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ and $\delta^{\mathcal{Q}}_{\alpha} < \eta$ which is a strong cutpoint of $\mathcal{Q}(\alpha + 1)$ we have

(i) $\mathcal{Q}|\eta^{+\mathcal{Q}(\alpha+1)} = \mathrm{Lp}^{\Gamma, \Sigma_{\mathcal{Q}(\alpha)}, \vec{\mathcal{T}}}(\mathcal{Q}|\eta)$ and

(ii) $\mathcal{Q}|\delta^{+\mathcal{Q}}_{\alpha} = \mathrm{Lp}^{\Gamma, \oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1)}, \vec{\mathcal{T}}}(\mathcal{Q}(\alpha))$.

Σ is **fullness preserving** iff it is $\mathcal{P}(\mathbb{R})$-fullness preserving.

Provide a motivation for this definition.

$\circ$

$$\begin{tikzcd} & j(\mathcal{H}) \\ \mathcal{H} \arrow[ur, "j"] \arrow[r, "\pi"] & \mathcal{R} \arrow[u, "\tau"'] \end{tikzcd}$$

Figure 8.3: Full Factors Property

Lemma 8.21. *This will be useful in the proof of the A-condensing lemma.*

Let M, N be transitive models of ZFC$^-$ with largest cardinals δ^M, δ^N respectively. Let $\pi\colon M \to N$ be an elementary embedding, $\kappa := \mathrm{crit}(\pi)$ and let E be the long (κ, δ^N)-extender derived from π. Then $N = \mathrm{Ult}(M; E)$ and $\pi = \pi_E$ is the canonical ultrapower embedding.

Proof. We have the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\pi_E"] & N \\ & \mathrm{Ult}(M;E) \arrow[u, "k"'] \end{tikzcd}$$

where k satisfies $k \restriction \delta^N = \mathrm{id}$. Let $\delta^{\mathrm{Ult}(M;E)}$ be the largest cardinal of $\mathrm{Ult}(M;E)$. By elementarity $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^N$, so that $\delta^{\mathrm{Ult}(M;E)} \leq \delta^N$. If $\delta^{\mathrm{Ult}(M;E)} < \delta^N$, then $k \restriction \delta^N = \mathrm{id}$ yields $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^{\mathrm{Ult}(M;E)} < \delta^N$, which is absurd. Hence $\delta^{\mathrm{Ult}(M;E)} = \delta^N$ and $k \restriction (\delta^{\mathrm{Ult}(M;E)} + 1) = \mathrm{id}$. Since $\delta^{\mathrm{Ult}(M;E)}$ is the largest cardinal of $\mathrm{Ult}(M;E)$, it follows that k doesn't have a critical point. Therefore $k = \mathrm{id}$, $N = \mathrm{Ult}(M;E)$ and $\pi = \pi_E$. ■

Lemma 8.22. *$j \restriction \mathcal{H}$ has the **full factors property**[2], meaning that whenever $\mathcal{R}$*

[2] This terminology was introduced in [Wilson, 2012]; in [Sargsyan, 2015a] this was called *weak condensation*.

R has to be countable in $V[g]$. How can we ensure that, as this only gives that it has size $\leq \aleph_1$? Do we have to resort to the (long) claim in Grigor's uB paper?

is a hod premouse and there are elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$, then $\mathcal{R}$ is $\Sigma^2_1(j(\Omega)^\tau)$-full.

Proof. Let $\Psi := j(\Omega)^\tau$ and assume the lemma fails, meaning that we have a hod mouse $\mathcal{R}$ and elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$ and $\mathcal{R} \neq \mathrm{Lp}^\Psi_\omega(\mathcal{R}\,|\,\delta^{\mathcal{R}})$, witnessed without loss of generality by an $\mathcal{M} \trianglelefteq \mathrm{Lp}^\Psi(\mathcal{R}\,|\,\delta^{\mathcal{R}})$ such that $\rho(\mathcal{M}) = \delta^{\mathcal{R}}$ and which is not an initial segment of $\mathcal{R}$.

$$\begin{tikzcd}
(\mathcal{H}, \Omega) \arrow[r, "\pi"] \arrow[dr, "j \restriction \mathcal{H}"'] & (\mathcal{R}, \Psi) \arrow[d, "\tau"] \\
& (j(\mathcal{H}), j(\Omega))
\end{tikzcd}$$

We can then fix some hod pair $(\mathcal{S}^*, \Lambda^*)$ such that $\tau``\mathcal{R}\,|\,\delta^{\mathcal{R}} \subseteq \mathrm{ran}(\pi^{\Lambda^*}_{\mathcal{S}^*,\infty})$, and furthermore let $\xi \leq \lambda^{\mathcal{S}^*}$ be least such that $\tau``\mathcal{R}\,|\,\delta^{\mathcal{R}} \subseteq \mathrm{ran}(\pi^{\Lambda^*}_{\mathcal{S}^*(\xi),\infty})$. Lastly let $(\mathcal{S}, \Lambda)$ be an extension of $(\mathcal{S}^*, \Lambda^*)$ such that $\lambda^{\mathcal{S}}$ is a limit ordinal.

Argue why $\mathcal{S}^*$ and $\mathcal{S}$ exist; we should be in the limit case to argue that $\mathcal{S}$ exists.

Let $\sigma : \mathcal{R}\,|\,\delta^{\mathcal{R}} \to \mathcal{S}|\delta^{\mathcal{S}}_\gamma$, where $\mathcal{S}^*(\xi)$ iterates to $\mathcal{S}(\gamma)$, be given by $\sigma(x) = y$ iff $\tau(x) = \pi^{\Lambda}_{\mathcal{S}(\gamma),\infty}(y)$.

$$\begin{tikzcd}
(\mathcal{R}\,|\,\delta^{\mathcal{R}}, \Psi) \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & (j(\mathcal{H})|\delta^{j(\mathcal{H})}, j(\Omega)) \\
& (\mathcal{S}|\delta^{\mathcal{S}}_\gamma, \bigoplus_{\beta<\gamma} \Lambda_{\mathcal{S}(\beta)}) \arrow[ur, "\pi^{\Lambda}_{\mathcal{S}(\gamma),\infty}"'] &
\end{tikzcd}$$

This should follow from generation of good pointclasses.

We can fix some hod pair $(\mathcal{S}', \Lambda')$ such that

$$L(\Lambda', \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner.$$

This requires us to work in an AD^+ model, so we better assume that somewhere.

By coiterating $\mathcal{S}$ and $\mathcal{S}'$ we may assume without loss of generality that $\mathcal{S} = \mathcal{S}'$.

Claim 8.23. There exists a hod pair $(\mathcal{Q}, \Phi)$ such that $\lambda^{\mathcal{Q}}$ is a limit ordinal and $L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M}$ is a Ψ-mouse$\urcorner$.

Proof of claim.

This claim shouldn't be needed, as we should be able to take $\mathcal{Q}$ to be $\mathcal{S}$ in our case, using facts about the Γ-pointclasses. Also ensure that $\mathcal{Q} \supseteq \mathcal{H}$, which is possible as we're stretching by j.

$\dashv$

Fix $(\mathcal{Q}, \Phi)$ as in the claim and let $\mathcal{N}$ be some mouse such that $\mathcal{M} \lhd \mathcal{N}$ and $\mathcal{N}$ has ω many Woodins on top of $\mathcal{M}$.

Explain how this is done. In Grigor's paper he's using that "$j(\eta)$ is closed under hybrid $\mathcal{N}_\omega$-operators". In our measurable cofinality case there might be enough room to get this. Postpone until later, when we have an idea of how much operator closure we have at this point.

Then we get that

Why is $\Gamma(\mathcal{Q}, \Phi)$ in $\mathcal{D}(\mathcal{N})$?

$\mathcal{D}(\mathcal{N}) \models \ulcorner L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M}$ is a Ψ-mouse which isn't an initial segment of $\mathcal{R}\urcorner\urcorner$.

Now throw everything in sight into a countable hull, so that In $V[g]$, I guess.

$\mathcal{D}(\overline{\mathcal{N}}) \models \ulcorner L(\Gamma(\overline{\mathcal{Q}}, \overline{\Phi}), \mathbb{R}) \models \ulcorner \mathcal{M}$ is a $\overline{\Psi}$-mouse which isn't an initial segment of $\mathcal{R}\urcorner\urcorner$.

I think that now $\overline{\mathcal{Q}}$ are taking the role of "$L[\mathcal{T}, \mathcal{H}]$", as Grigor's paper seems to indicate that $\mathcal{H} \subseteq \overline{\mathcal{Q}}$.

Now lift π to the ultrapower map π^+ given by the $(\delta^{\mathcal{H}}, \delta^{\mathcal{R}})$-extender over $\overline{\mathcal{Q}}$ derived from π, and let $\mathcal{R}^+$ be the ultrapower. Lift also σ, τ to corresponding σ^+, τ^+. A it hand-wavy.

$$\begin{tikzcd}
(\overline{\mathcal{Q}}, \overline{\Phi}) \arrow[r, "\pi^+"] \arrow[dr, "\sigma^+"'] & (\mathcal{R}^+, \Phi^{**}) \arrow[d, "\tau^+"] \\
 & (j(\overline{\mathcal{Q}}), \Phi^*)
\end{tikzcd}$$

Let now $\Phi^* := j(\overline{\Phi})$ and $\Phi^{**} := (\Phi^*)^{\tau^+}$, which is then a strategy for $\mathcal{R}^+$. Since $\overline{\Phi} = (\Phi^{**})^{\pi^+}$ we get that

Check this — might be by definition of pullback consistency, which is implied by hull condensation.

In Grigor's uB paper he uses a certain derived model C instead of $\mathcal{D}(\overline{\mathcal{N}})$, but I can't see how they're different from each other. Also, figure out why the following inclusion is true (it's probably folklore).

$$\mathcal{D}(\overline{\mathcal{N}}) \subseteq \mathcal{D}(\mathcal{R}^+, \Phi^{**}),$$

Note sure what's going on here.

implying that

$$L(\Gamma(\mathcal{R}^+, \Phi^{**}), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner.$$

Why's that?

Because $\mathcal{R}^+$ is a Ψ-mouse over $\mathcal{R}|\delta^{\mathcal{R}}$, it follows that

$$\mathcal{D}(\mathcal{R}^+) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner,$$

I don't see how this last argument works.

which then implies that $\mathcal{M} \in \mathcal{R}^+$, so that $\mathcal{M} \trianglelefteq \mathcal{R}$, a contradiction. ■

Definition 8.24. For every $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ define $Q_X := \mathrm{cHull}^{j(\mathcal{H})}(X)$ and let

$$\tau_X \colon Q_X \to j(\mathcal{H})$$

be the uncollapse.

Say that $Y \in \mathcal{P}_{\omega_1}(j(\mathcal{H})|\delta^{j(\mathcal{H})})$ **extends** X if $X \cap j(\delta^{\mathcal{H}}) \subseteq Y$ and in that case let

(i) $\tau_{X,Y} := \tau_{X \cup Y}$,

(ii) $\Phi_{X,Y} := j(\Phi)^{\tau_{X,Y}}$,

(iii) $Q_{X,Y} := Q_{X \cup Y}$ and

(iv) $\pi_{X,Y} \colon Q_X \to Q_{X,Y}$ is the induced embedding given by

$$\pi_{X,Y}(x) = \tau_Y^{-1}(\tau_X(x)).$$

Furthermore define $T_X(A)$ for $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$ as

$$\begin{aligned} T_X(A) :&= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } Q_X \models \varphi[s, A]\} \\ &= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\tau_X(s), \tau_X(A)]\} \end{aligned}$$

and let $T_{X,Y}(A)$ be given as

$$\begin{aligned} T_{X,Y}(A) := \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_{X,Y}}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty}(s), \tau_X(A)], \\ \text{where } \alpha \text{ is least such that } s \in [\delta_\alpha^{Q_{X,Y}}]^{<\omega}\}. \end{aligned}$$

Here $\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty} : Q_{X,Y} \to j(\mathcal{H})|\nu_{X,Y}$ is given by

What is $\nu_{X,Y}$?

Missing! (It will be the iteration into an appropriate level of the directed system leading up to $j(\mathcal{H})$ followed by the direct limit embedding into some initial segment of $j(\mathcal{H})$)

$\circ$

Definition 8.25. Let $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ and $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$. Then X is **A-condensing** if $\pi_{X,Y}(T_X(A)) = T_{X,Y}(A)$ for every Y extending X. We say that X is **condensing** if X is A-condensing for all such A. $\circ$

We want to show that $j``\mathcal{H}$ is condensing. We first show that it suffices to show that it's α-condensing for every $\alpha < \delta^{\mathcal{H}}$.

Lemma 8.26. *If $j``\mathcal{H}$ is $\alpha$-condensing for every $\alpha < \delta^{\mathcal{H}}$ then $j``\mathcal{H}$ is condensing.*

Proof.

Missing!

■

Theorem 8.27. *For every $\alpha < \delta^{\mathcal{H}}$ there exists an extension Y of $j``\mathcal{H}$ such that $j``\mathcal{H} \cup Y$ is α-condensing.*

Reduce this to $j``\mathcal{H}$ somehow?

Proof. Set $X := j``\mathcal{H}$ and assume the theorem fails. Fix some $\alpha < \delta^{\mathcal{H}}$ such that X is not α-condensing. Fix some Y_0 extending X which witnesses this, meaning that $\pi^X_{Y_0}(T^X_\alpha) \neq T^{X,Y_0}_\alpha$. Since we're also assuming that $\tau^X_{Y_0}$ isn't α-condensing we can find Y_1 extending Y_0 such that $\pi^{Y_0}_{Y_1}(T^{Y_0}_\alpha) \neq T^{Y_0,Y_1}_\alpha$. Continue doing this, generating a sequence $\langle Y_n \mid n < \omega \rangle$ with Y_{n+1} extending Y_n and

$$\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) \neq T^{Y_n,Y_{n+1}}_\alpha \tag{1}$$

for all $n < \omega$. Let $\mathcal{P}_n := Q^X_{Y_n}$, $\pi_{m,n} := \pi^{Y_m}_{Y_n}$ and $\pi_n := \pi_{0,n}$. We want to show that such a sequence can't exist. Towards getting a contradiction we first need to make everything in sight countable, as that will allow us to reason using derived models (the problem is that $j(\mathcal{H})$ is too big, namely it has size $\aleph_1^{V[g]}$).

Using that $\delta^{j(\mathcal{H})}$ has uncountable cofinality we can find $\kappa < \delta^{j(\mathcal{H})}$ such that

$$\kappa = \mathrm{Hull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\mathrm{ran}\, \tau^X_{Y_n} \mid n < \omega\}) \cap \delta^{j(\mathcal{H})}.$$

Set $\mathcal{M} := \mathrm{cHull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\mathrm{ran}\, \tau^X_{Y_n} \mid n < \omega\})$ and note that $\mathcal{M} = j(\mathcal{H})|\kappa^{+j(\mathcal{H})}$. Let $\pi : \mathcal{M} \to j(\mathcal{H})$ be the uncollapse and note that $\mathrm{crit}\, \pi = \kappa$ and that $\kappa = \delta^{\mathcal{M}}$. Define $\iota : \mathcal{H} \to \mathcal{M}$ as $\iota := \pi^{-1} \circ j$ and $\tau_n : \mathcal{P}_n \to \mathcal{M}$ as $\tau_n := \pi^{-1} \circ \tau^X_{Y_n}$. Note that $\mathcal{M}$ is countable in $V[g]$ and is hence an element of $\mathrm{Ult}(V, g)$.

Missing argument

Now define $\mathcal{H}^+$ as the hod limit of iterates of $\mathcal{H}$, so that $\mathcal{H}^+$ is a hod premouse with $\mathcal{H} \lhd_{\mathrm{hod}} \mathcal{H}^+$, $\mathcal{H}^+$ has a strategy Ψ extending Ω such that

Provide more details.

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{H}^+, \Psi).$$

We probably need $\mathcal{H}^+$ to be countable here, so we should probably apply the induced ideal and work in $V[g][h]$.

Also define $(\mathcal{P}^+_n, \Psi_n)$ as $P^+_n := \mathrm{Ult}(\mathcal{H}^+, E_{\pi_n})$, so that we also get that

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{P}^+_n, \Psi_n).$$

Missing argument. This might need that $\mathcal{H}^+, \Psi \restriction V \in V$, but we could probably also just work inside $\mathrm{Ult}(V, g)$, or

Now $\mathcal{D}(\mathcal{P}_n^+, \Psi_n)$ has a definition of T_α^{X,Y_n}, so that $\pi_{Y_{n+1}}^{Y_n}(T_\alpha^{Y_n}) = T_{\pi_{n,n+1}(\alpha)}^{Y_n,Y_{n+1}}$. The three-dimensional argument then shows that α must be fixed by $\pi_{n,n+1}$ for some $n < \omega$, so that $X \cup Y_n$ *is* α-condensing, $\lightning$. ■

What is meant by this?

Show this.

Define the strategy Λ for $\mathcal{H}$ and show that $(\mathcal{H}, \Lambda)$ is a hod pair.

9 | Forcing direction

Have a look at Trevor's thesis; he's doing something similar.

In this section we will prove the following unpublished theorem by Woodin.

Theorem 9.1 (Woodin)**.** *Assume $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Then there is a generic extension of V satisfying DI^+.*

Assume thus that $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Missing proof.

10 | Further questions

10.1 Section

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Part III

Appendices

A | Large Cardinals

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B | Forcing

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The following lemma is from [Lücke and Schlicht, 2014]

Lemma B.1. *Let λ be an infinite cardinal, $\mathcal{M} \models \mathsf{ZF}^-$ a transitive model, $\mathbb{P} \in \mathcal{M}$ a λ^+-cc forcing notion and $g \subseteq \mathbb{P}$ an $\mathcal{M}$-generic filter. Then $V \models {}^\lambda\mathcal{M} \subseteq \mathcal{M}$ implies that $V[g] \models {}^\lambda\mathcal{M} \subseteq \mathcal{M}$.*

Proof. Work in $V[g]$. Let $c := \langle c_\alpha \mid \alpha < \lambda \rangle$ be a λ-sequence such that $c_\alpha \in \mathcal{M}[g]$ for every $\alpha < \lambda$. Fix for every $\alpha < \lambda$ a $\mathbb{P}$-name $\dot{c}_\alpha$ such that $\dot{c}_\alpha^g = c_\alpha$. Also let $\dot{a}$ be a $\mathbb{P}$-name with $\dot{a}^g = \langle \dot{c}_\alpha \mid \alpha < \lambda \rangle$ and choose $p \in g$ such that $V \models \ulcorner p \Vdash \forall \alpha < \check{\lambda} \colon \dot{a}(\alpha) \in \mathcal{M}^{\mathbb{P}} \urcorner$.

Now, working in V, there is for each $\alpha < \lambda$ a maximal antichain A_α below p such that every $q \in A_\alpha$ decides $\dot{a}(\alpha)$; i.e., $q \Vdash \ulcorner \dot{a}(\alpha) = \check{x} \urcorner$ for some $x \in \mathcal{M}$. Define now

$$\sigma := \{((\alpha, x), q) \mid \alpha \in \lambda \wedge q \in A_\alpha \wedge q \Vdash \ulcorner \dot{a}(\alpha) = \check{x} \urcorner\}.$$

Then $p \Vdash \ulcorner \sigma = \dot{a} \urcorner$. Note that $|\sigma| \leq \lambda$, since $|A_\alpha| \leq \lambda$ for each $\alpha < \lambda$. Thus $\sigma \in \mathcal{M}$. Now it holds that

$$V[g] \models \ulcorner \langle \dot{c}_\alpha \mid \alpha < \lambda \rangle = \dot{a}^g = \sigma^g \in \mathcal{M}[g] \urcorner,$$

and we can compute $c = \langle c_\alpha \mid \alpha < \lambda \rangle = \langle \dot{c}_\alpha^g \mid \alpha < \lambda \rangle$ from $\langle \dot{c}_\alpha \mid \alpha < \lambda \rangle$ and g, so $c \in \mathcal{M}[g]$ by Replacement. ∎

C | Core model theory

Show the beaver argument, Lemmata 7.3.7–7.3.9 and 8.3.4 in [Zeman, 2001]

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D | Infinite game theory

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Two games G_1 and G_2 are **equivalent** if player I has a winning strategy in G_1 iff they have one in G_2, and player II has a winning strategy in G_1 iff they have one in G_2.

E | Ideals

Definition E.1. Let I be an ideal on a nonempty set Z. Let

(i) $I^+ := \mathcal{P}(Z) \setminus I$,

(ii) for $a, b \in I^+$ let $a \sim_I b$ iff $a \Delta b \in I$,

(iii) $\mathcal{P}(Z)/I := \mathcal{P}(Z)/{\sim_I}$ is the Boolean algebra with subset inclusion modulo $\sim_I$.

We call $\mathcal{P}(Z)/I$ the *associated forcing* to I. ○

Definition E.2. If I is an ideal on a cardinal κ and $g \subseteq \mathcal{P}(\kappa)/I$ V-generic then g is a V-ultrafilter on κ in $V[g]$, so that we may take the *generic ultrapower* $\mathrm{Ult}(V, g)$. ○

Proposition E.3. *Let I be an ideal on a cardinal κ. Then..*

(i) if I is κ-complete then so is any generic ultrafilter;

(ii) if I is normal then so is any generic ultrafilter. ⊣

define κ-complete and normal for an ideal. Add reference.

Definition E.4. Let λ be any cardinal. Then an ideal I on a cardinal κ is...

- *precipitous* if the generic ultrapower is wellfounded;
- *λ-saturated* if the associated forcing has the λ-chain condition;
- *λ-dense* if the associated forcing has a dense subset of size λ. ⊣

Note that λ-dense trivially implies λ^+-saturated. We'll need the following facts about ω_2-saturated ideals on ω_1:

Parts (ii)-(v) is Example 4.29 in Foreman's handbook chapter. Perhaps include the proof.

Proposition E.5. *Let I be an ω_2-saturated ideal on ω_1. Then I is precipitous, and letting $j\colon V \to M$ be the generic ultrapower map, it holds that*

(i) M is closed under ω-sequences in $V[g]$;

(ii) $j(\omega_1^V) = \omega_2^V = \omega_1^{V[g]}$;

(iii) $j(\omega_2^V) \in (\omega_2^V, \omega_3^V)$*;*

(iv) j *is continuous at* ω_2^V*;*

(v) $j(\omega_n^V) = \omega_n^V = \omega_{n-1}^{V[g]}$ *for all* $n \in [3, \omega]$*.* $\dashv$

As for the density, we in particular need the following fact:

Proposition E.6. *Let* I *be an* ω_1*-dense ideal on* ω_1*. Then the associated forcing is forcing equivalent to* $\mathrm{Col}(\omega, \omega_1)$*, so in particular it's homogeneous.* ■

Proof. [Kanamori, 2008, Proposition 10.20]. ■

F | Descriptive Set Theory

Convention F.1. We will be using the "logician's reals", meaning that $\mathbb{R} := {}^{\omega}\omega$ with the product topology, having the sets $\{x \in \mathbb{R} \mid x \supseteq s\}$ for $s \in {}^{<\omega}\omega$ as a clopen basis.

Definition F.2. Let $A, B \subseteq \mathbb{R}$. We say that A *is Wadge reducible to* B (in symbols $A \leq_W B$) iff there is some continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$A = f^{-1}[B] := \{a \in A \mid f(a) \in B\}.$$

We write $A <_W B$ iff $A \leq_W B$ and $B \not\leq_W A$. ○

Remark F.3. $\mathbb{R}$ and $\mathbb{R}^n$, for $1 \leq n \leq \omega$ are homeomorphic and we shall often identify them with one another.

Definition F.4. Subseteq of $\mathcal{R}$ are called *pointsets* . Subsets of $\mathcal{P}(\mathbb{R})$ are called *pointclasses* .

○

Definition F.5. Let Γ be a pointclass. We define

(i) $\exists^{\mathbb{R}}\Gamma := \{A \mid \exists B \in \Gamma\colon A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(x, y) \in B\}\}$,
(ii) $\forall^{\mathbb{R}}\Gamma := \{A \mid \exists B \in \Gamma\colon A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R}(x, y) \in B\}\}$.

○

Definition F.6. Let Γ be a pointclass. We define

(i) $\breve{\Gamma} := \{\mathbb{R} \setminus A \mid A \in \Gamma\}$,
(ii) $\Delta_\Gamma := \Gamma \cap \breve{\Gamma}$ and
(iii) $\utilde{\Gamma} := \exists^{\mathbb{R}}\Gamma$.

○

Lemma F.7 ([Wadge, 1972]). *Assume $\mathsf{ZF}+\mathsf{AD}$ and let $A, B \subseteq \mathbb{R}$. Then*

$$A \leq_W B \text{ or } B \leq_W \mathbb{R} \setminus A.$$

Lemma F.8 (Martin-Monk-Wadge). *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. Then $\leq_W$ is wellfounded.*[1]

Remark F.9. When considering $(\mathcal{P}(\mathbb{R}); \leq_W)$ in a $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ context, we will often tacitly identify $A \subseteq \mathbb{R}$ with its complement, making $<_W$ a wellorder.

Definition F.10 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). Let $A \subseteq \mathbb{R}$. Then the *Wadge rank of* A is defined recursively as $|A|_W := \sup\{|B|_W + 1 \mid B <_W A\}$. ○

Definition F.11. Let X be a set. We write OD_X for the collection of all A for which there is some formula ϕ, ordinals $\alpha_0, \dots, \alpha_k$ and $x_0, \dots, x_l \in X$ with

$$A = \{a \mid \phi[a, \alpha_0, \dots, \alpha_k, x_0, \dots, x_l]\}. \qquad \dashv$$

We write HOD_X for the collection of all A such that $\mathrm{trcl}(\{A\}) \subseteq \mathrm{OD}_X$. If $X = \emptyset$, we will often drop the subscript and simply write OD and HOD for $\mathrm{OD}_\emptyset$ and $\mathrm{HOD}_\emptyset$ respectively.

Definition F.12 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). For $B \subseteq \mathbb{R}$ let

$$\begin{aligned} \theta_B :&= \sup\{|A|_W \mid \exists x \in \mathbb{R} \colon A \in \mathrm{OD}_{\mathbb{R}\cup\{B\}}\} \\ &= \sup\{\alpha \in \mathsf{On} \mid \text{ there is a } \mathrm{OD}_{\mathbb{R}\cup\{B\}}\text{-surjection } f \colon \mathbb{R} \to \alpha\}. \end{aligned}$$

○

Verify that these two values are in fact identical.

Definition F.13 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). Define the *Solovay sequence* $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ as follows:

(i) $\theta_0 := \theta_\emptyset$,

(ii) if there is some B such that $|B|_W = \theta_\alpha$ let $\theta_{\alpha+1} := \theta_B$.[2]

[1] See [Larson, 2019] for a proof.

[2] Since continuous functions are coded by reals, this is independent of the choice of B.

(iii) if α is a limit ordinal, we let $\theta_\alpha := \sup_{\beta<\alpha} \theta_\beta$.

Finally, Ω is the least ordinal such that $\theta_{\alpha+1} = \theta_\alpha$, and $\Theta := \theta_\Omega$. ∘

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