1 | Set-theoretic connections

1.1 Games

1.1.1 The finite case

In this section we are going to consider properties of the n-Ramsey cardinals for finite n. Note in particular that the $\mathcal{G}_n^\theta(\kappa)$ games are determined, making the "strategic" adjective superfluous in this case. We further note that the θ's are also dispensible in this finite case:

Proposition 1.1.2 (N.). *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n \prec H_\theta$.*

Proof. $\Leftarrow$ is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays $\mathcal{M}_k$ in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \mathrm{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of $\mathcal{M}_k$, and play $\mathcal{M}_k^*$ in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ-responses to the $\mathcal{M}_k^*$'s and let player II play the μ_k's in $\mathcal{G}_n(\kappa)$ as well.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\mathrm{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω-sequence $\vec{f} := \langle f_k \mid k < \omega\rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \mathrm{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\mathrm{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \to H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \operatorname{ran}\pi$.

Now $\bar{A} = A$ for every $A \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the $\bar{f}_k$'s witness that $\mathrm{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\mathrm{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \mathrm{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ's anymore we note that n-Ramseyness can now be described using a Π^1_{2n+2}-formula and normal n-Ramseyness using a Π^1_{2n+3}-formula.

We already have the following characterisations, as proven in [**?**].

Theorem 1.1.3 (Abramson et al.). *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then*

(i) *κ is weakly compact if and only if it is 0-Ramsey;*

(ii) *κ is weakly ineffable if and only if it is genuine 0-Ramsey;*

(iii) *κ is ineffable if and only if it is normal 0-Ramsey.*

PROOF. This is mostly a matter of changing terminology from [**?**] to the current game-theoretic one, so we only show (i). Theorem 1.1.3 in [**?**] shows that κ is weakly compact if and only if every κ-sized collection of subsets of κ is measured by a $<\kappa$-complete measure, in the sense that every $<\kappa$-sequence (in V) of measure one sets has non-empty intersection.

For the $\Rightarrow$ direction we can let player II respond to any $\mathcal{M}_0$ by first getting the $<\kappa$-complete $\mathcal{M}_0$-measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \to \mathrm{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π, which is $\mathcal{M}_0$-normal and good. For $\Leftarrow$, if $X \subseteq \mathscr{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ-model $\mathcal{M}_0 \prec H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play $\mathcal{M}_0$ in $\mathcal{G}_0(\kappa)$ we get some $\mathcal{M}_0$-normal good $\mathcal{M}_0$-measure μ_0 on κ. Since $\mathcal{M}_0$ is closed under $<\kappa$-sequences we get that μ_0 is $<\kappa$-complete. ∎

Indescribability

In this section we aim to prove that n-Ramseys are Π^1_{2n+1}-indescribable and that normal n-Ramseys are Π^1_{2n+2}-indescribable, which will also establish that the hierarchy of alternating n-Ramseys and normal n-Ramseys forms a strict hierarchy. Recall the following definition.

Definition 1.1.4. A cardinal κ is **Π^1_n-indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ◦

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π^1_1-indescribable — see [**?**].

Theorem 1.1.5 (N.). *Every n-Ramsey κ is Π^1_{2n+1}-indescribable for $n < \omega$.*

Proof. Let κ be n-Ramsey and assume that it is not Π^1_{2n+1}-indescribable, witnessed by a Π_{2n+1}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$ and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$-many quantifiers occuring in φ in $(n+1)$-many steps. We will here describe the first two steps with the remaining steps following the same pattern.

$\boxed{\textbf{First step.}}$ Write $\varphi(v) \equiv \forall v_1\psi(v, v_1)$ for a Σ_{2n}-formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A^{(0)}_\alpha \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let $\mathcal{M}_0$ be a weak κ-model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good $\mathcal{M}_0$-normal $\mathcal{M}_0$-measure μ_0 on κ, using the 0-Ramseyness of κ. Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \text{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. $\mathcal{M}_0$-normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, $\mathcal{M}_0$-normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\text{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}]\urcorner. \tag{1}$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0-formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

$\boxed{\textbf{Second step.}}$ Write $\psi(v, v_1) \equiv \exists v_2\forall v_3\chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$-formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3\chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \tag{2}$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, v_3]. \tag{3}$$

Fix witnesses $A^{(1)}_\alpha \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S^{(0)}_\alpha := \{\xi < \kappa \mid A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S^{(0)}_\alpha \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and $\mathcal{M}_0$-normality of μ_0 then implies that $S^{(0)}_\alpha \in \mu_0$ is equivalent to

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ-model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an $\mathcal{M}_1$-normal $\mathcal{M}_1$-measure on κ, using the 1-Ramseyness of κ, so that $\mathcal{M}_1$-normality of μ_1 yields that $\triangle\vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle\vec{S}^{(0)}$ if and only if $A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A^{(0)}_\xi = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \mathrm{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A^{(1)}_\alpha$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha].$$

Now this, paired with the above observation regarding $\triangle\vec{S}^{(0)}$, means that for every $\alpha \in \triangle\vec{S}^{(0)} \cap \mathrm{Lim}$ we have that

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha],$$

so that $\mathcal{M}_1$-normality of μ_1 and Łoś' lemma implies that

$$\mathrm{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg\chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]\urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$-many steps, ending with a Δ_0-formula $\phi(v, v_1, \ldots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\mathrm{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n)}]\urcorner$. But now absoluteness of $\neg\phi$ means that $V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n)}]$, contradicting (4). ■

Note that this is optimal, as n-Ramseyness can be described by a Π^1_{2n+2}-formula. As a corollary we then immediately get the following.

Corollary 1.1.6 (N.)**.** *Every $<\omega$-Ramsey cardinal is Δ^2_0-indescribable.* ■

The second indescribability result concerns the normal n-Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π^1_2-indescribable — see [**?**].

Theorem 1.1.7 (N.)**.** *Every normal n-Ramsey κ is Π^1_{2n+2}-indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 1.1.5, as we would end up with a Π^1_1 statement in an ultrapower, and as Π^1_1 statements are not upwards absolute in general we would not be able to get our contradiction.

Proof. Let κ be normal n-Ramsey and assume that it is not Π^1_{2n+2}-indescribable, witnessed by a Π_{2n+2}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n-Ramsey to perform the same $n+1$ steps as in the proof of Theorem 1.1.5. This gives us a Σ_1-formula $\phi(v, v_1, \ldots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \cdots, \mathcal{A}^{(n)}\rangle$, $\langle B^{(0)}, \ldots, B^{(n-1)}\rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n\rangle$

of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{1}$$

and, for μ_n-many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, A^{(n)}_\alpha].$$

Now form $S^{(n)}_\alpha \in \mu_n$ as in the proof of Theorem 1.1.5. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 1.1.5 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for $k < n$), but we can now use normality[1] of μ_n to ensure that we *do* have that $\triangle\vec{S}^{(n)}$ is stationary in κ. This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \tag{2}$$

Now note that since κ is inaccessible it is Σ^1_1-indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of [?] shows that the set of reflection points of Σ^1_1-formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \dots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n-Ramseyness can be described by a Π^1_{2n+3}-formula. In particular this then means that every $(n+1)$-Ramsey is a normal n-Ramsey stationary limit of normal n-Ramseys, and every normal n-Ramsey is an n-Ramsey stationary limit of n-Ramseys, making the hierarchy of alternating n-Ramseys and normal n-Ramseys a strict hierarchy.

[1] Recall that this is stronger than just requiring it to be $\mathcal{M}_n$-normal — we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.

Downwards absoluteness to L

The following proof is basically the proof of Theorem 4.1.1 in [**?**].

Theorem 1.1.8 (N.). *Genuine- and normal n-Ramseys are downwards absolute to L, for every $n < \omega$.*

Proof. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ-model — we want to find a genuine $\mathcal{M}$-measure inside L. By assumption we *can* find such a measure μ in V; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa \rangle \in L$ of $\mathscr{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

Claim 1.1.8.1. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

Proof of claim. Let $\vec{B}$ be the **μ-positive part** of $\vec{A}$, meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle \vec{B}$ has size κ, so we can pick $\delta \in \triangle \vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L. $\dashv$

But now Lemma 4.1.2 in [**?**] shows that there is a Π_1 formula $\varphi(v)$ such that, given any non-zero ordinal ζ, $V_{\zeta+1} \models \varphi[A]$ if and only if ζ is a regular cardinal and A is a non-constructible subset of ζ. If we therefore assume that $T \notin L$ then $V_{\kappa+1} \models \varphi[T]$, which by Π^1_1-indescribability of κ means that there exists some $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle\mu)^L = \triangle\mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V. The cases where κ is a genuine- or normal n-Ramsey cardinal is analogous. ■

Since $(n+1)$-Ramseys are normal n-Ramseys we then immediately get the following.

Corollary 1.1.9 (N.). *Every $(n+1)$-Ramsey is normal n-Ramsey in L, for every $n < \omega$. In particular, $<\omega$-Ramseys are downwards absolute to L.* ■

Complete ineffability

In this section we provide a characterisation of the *completely ineffable* cardinals in terms of the α-Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$-Ramsey cardinals, namely the *coherent $<\omega$-Ramseys* as defined in **??**. Note that a coherent $<\omega$-Ramsey is precisely a cardinal satisfying the ω-filter property, as defined in [**?**].

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is very closely related to the proof of Theorem 1.1.7 (the indescribability of normal n-Ramseys), but the main difference is that we want everything to occur locally inside our weak κ-models.

Theorem 1.1.10 (N.)**.** *Every coherent $<\omega$-Ramsey is a stationary limit of $<\omega$-Ramseys.*

Proof. Let κ be coherent $<\omega$-Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 \prec H_\theta$ be a weak κ-model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ-model $\mathcal{M} \prec H_\theta$ with an $\mathcal{M}$-normal $\mathcal{M}$-measure $\mu := \bigcup_{n<\omega} \mu_n$ on κ.

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } {<}\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by $\mathcal{M}$-normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k+1$ steps as in the proof of Theorem 1.1.7 with $\varphi(\xi)$ being $\ulcorner \xi$ is k-Ramsey$\urcorner$, so that we get a weak κ-model $\mathcal{M}_{k+1} \prec H_\theta$, an $\mathcal{M}_{k+1}$-normal $\mathcal{M}_{k+1}$-measure $\tilde{\mu}_{k+1}$ on κ, a Σ_1-formula $\varphi(v, v_1, v_2, \ldots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \ldots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \ldots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \ldots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \tag{2}$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg\varphi[\xi, A^{(0)}_\xi, B^{(0)} \cap V_\xi, A^{(1)}_\xi, B^{(1)} \cap V_\xi, \dots, A^{(k-1)}_\xi, B^{(k-1)} \cap V_\xi, A^{(k)}_\xi], \tag{3}$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \mathrm{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 1.1.5.

Since κ in particular is Σ^1_1-indescribable, Lemma 3.4.3 of [**?**] implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ-model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n+1)$-steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of $\mathcal{M}_{k+1}$. In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ, $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A^{(i)}_\xi \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa \rangle$ and note that since $\vec{S}$ is definable it is an element of $\mathcal{M}_{k+2}$ as well. Then $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $\triangle\vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \triangle\vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \triangle\vec{S}$ we see, as in the proof of Theorem 1.1.5, that

$$V_{\zeta+1} \models \varphi[\zeta, A^{(0)}_\zeta, B^{(0)} \cap V_\zeta, A^{(1)}_\zeta, B^{(1)} \cap V_\zeta, \dots, A^{(k)}_\zeta]$$

since $\triangle\vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} \prec H_\theta$ we have that $\mathcal{M}$ is correct about stationary subsets of κ, meaning that κ is a stationary limit of $<\omega$-Ramseys. ■

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

Definition 1.1.11. A collection $R \subseteq \mathscr{P}(\kappa)$ is a **stationary class** if

(i) $R \neq \emptyset$;
(ii) every $A \in R$ is stationary in κ;
(iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

○

Definition 1.1.12. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$ there is an $H \in R$ homogeneous for f. ◦

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in [**?**].

Theorem 1.1.13 (N.)**.** *A cardinal κ is completely ineffable if and only if it is coherent $<\omega$-Ramsey.*

Proof. ($\Leftarrow$): Assume κ is coherent $<\omega$-Ramsey, witnessed by strategies $\langle \tau_n \mid n < \omega \rangle$. Let $f : [\kappa]^2 \to 2$ be arbitrary and form the sequence $\langle A^f_\alpha \mid \alpha < \kappa \rangle$ as

$$A^f_\alpha := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let $\mathcal{M}_f$ be a transitive weak κ-model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated $\mathcal{M}_f$-measure on κ given by τ_0.[2] 1-Ramseyness of κ ensures that μ_f is normal, meaning $\triangle\mu_f$ is stationary in κ. Define a new sequence $\vec{B}^f$ as the μ_f-positive part of $\vec{A}^f$.[3] Then $B^f_\alpha \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\triangle\vec{B}^f$ is stationary.

Let now $\mathcal{M}'_f$ be a new transitive weak κ-model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an $\mathcal{M}'_f$-measure $\mu'_f \supseteq \mu_f$ on κ. Then $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \in \mu_f\}$ and $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \notin \mu_f\}$ are both elements of $\mathcal{M}'_f$, so one of them is in μ'_f; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f.

Now let $g : [H_f]^2 \to 2$ be arbitrary and again form

$$A^g_\alpha := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ-model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$-measure $\mu_{f,g} \supseteq \mu'_f$ on κ. As before we then get a

[2] Technically we would have to require that $\mathcal{M}_f \prec H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0, but note that we could simply get a measure on $\mathrm{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to $\mathcal{M}_f$. We will use this throughout the proof.

[3] The *μ-positive part* was defined in Claim 1.1.8.1.

stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g. We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists \vec{f} : H_{\vec{f}} \subseteq A\},$$

where the $\vec{f}$'s range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \to 2$ and $f_{k+1} : [H_{f_k}] \to 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \to 2$, we can find $H \in R$ which is homogeneous for f. Indeed, if we let $\vec{f}$ be such that $H_{\vec{f}} \subseteq A$, which exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

($\Rightarrow$): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n-Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play $\mathcal{M}_0$ and enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A^0_\alpha \mid \alpha < \kappa\rangle$ such that $A^0_\xi \subseteq A^0_\zeta$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^0_\xi$. Let $<^\alpha_{\text{lex}}$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f, using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \restriction \alpha \mid \xi \in H_0 \wedge \xi > \alpha\rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least such η, which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C.

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \restriction \zeta = r_\gamma \restriction \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \restriction \zeta$ holds for every $\gamma \in H_0$, $\gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^0_\xi$ iff $\gamma \in A^0_\xi$. Define thus the

$\mathcal{M}_0$-measure μ_0 on κ as

$$\begin{aligned}\mu_0(A^0_\xi) = 1 \quad &\text{iff} \quad (\forall\beta \in H_0 \cap C)(\beta > \xi \to \beta \in A^0_\xi) \\ &\text{iff} \quad (\exists\beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A^0_\xi),\end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa\rangle$ be the μ_0-positive part of $\vec{A}^0$, it is also simple to check that $H_0 \cap C \subseteq \triangle\vec{B}$, making μ_0 normal and hence also both $\mathcal{M}_0$-normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n-Ramsey and let $\langle \mathcal{M}_0, \mu_0, \dots, \mathcal{M}_n, \mu_n, \mathcal{M}_{n+1}\rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}$ as $\vec{A}^{n+1} = \langle A^{n+1}_\xi \mid \xi < \kappa\rangle$, again satisfying that $\xi \le \zeta$ whenever $A^{n+1}_\xi \subseteq A^{n+1}_\zeta$, but also such that given any $\xi < \kappa$ there are $\zeta, \zeta' \in (\xi, \kappa)$ satisfying that $A^{n+1}_\zeta \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and $A^{n+1}_{\zeta'} \in (\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}) - \mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n \in R$ and C be club in κ such that $H_n \cap C \subseteq \triangle\mu_n$, which exist by our inductive assumption. For $\alpha < \kappa$ define $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^{n+1}_\xi$, and define a colouring $f : [H_n]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise}\end{cases}$$

As $H_n \in R$ there is an $H_{n+1} \in R$ homogeneous for f. Just as before, define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least $\eta \in [\alpha, \kappa)$ such that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_{n+1}$ with $\eta \le \beta < \gamma$, and let D be the club of fixed points of g. As above we get that given any $\alpha, \gamma \in H_{n+1} \cap D$ with $\alpha \le \gamma$ and $\xi < \alpha$, $\alpha \in A^{n+1}_\xi$ iff $\gamma \in A^{n+1}_\xi$. Define then the $\mathcal{M}_{n+1}$-measure μ_{n+1} on κ as

$$\begin{aligned}\mu_{n+1}(A^{n+1}_\xi) = 1 \quad &\text{iff} \quad (\forall\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \to \beta \in A^{n+1}_\xi) \\ &\text{iff} \quad (\exists\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A^{n+1}_\xi).\end{aligned}$$

Then $H_{n+1} \cap D \cap C \subseteq \triangle\mu_{n+1}$, making μ_{n+1} normal, $\mathcal{M}_{n+1}$-normal and good, just as before. It remains to show that $\mu_n \subseteq \mu_{n+1}$. Let thus $A \in \mu_n$ be given, and say $A = A^{n+1}_\xi = A^n_\eta$, where $\vec{A}^n$ was the enumeration of $\mathscr{P}(\kappa)\cap\mathcal{M}_n$ used at the n'th stage. Then by definition of μ_n we get that for every $\beta \in H_n \cap C$ with $\beta > \eta$, $\beta \in A^n_\eta$. We need to show that

$$(\exists\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A^{n+1}_\xi)$$

holds. But here we can simply pick a $\beta > \max(\xi, \eta)$ with $\beta \in H_{n+1}\cap D\cap C \subseteq H_n \cap C$. This shows that $\mu_n \subseteq \mu_{n+1}$, making κ $(n+1)$-Ramsey and thus inductively also coherent $<\omega$-Ramsey. ■

1.1.14 The countable case

This section covers the (strategic) γ-Ramsey cardinals whenever γ has countable cofinality. This case is special because, as mentioned in Section **??**, we cannot ensure that the final measure is countably complete and so the existence of winning strategies in the $\mathcal{G}^\theta_\gamma(\kappa)$ *might* depend on θ, in contrast with the uncountable cofinality case; see e.g. Question **??**.

Fix ref

[Strategic] ω-Ramsey cardinals

Remove the virtual definitions and basic results if they're redundant

We now move to the strategic ω-Ramsey cardinals and their relationship to the (non-strategic) ω-Ramseys. For this we define a new addition to the family of *virtual cardinals* from [**?**], the *virtually measurable cardinals.*

Definition 1.1.15. A cardinal κ is **virtually measurable** if for every regular $\nu > \kappa$ there exists a transitive M and a forcing $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there exists an elementary embedding $j : H^V_\nu \to M$ with $\operatorname{crit} j = \kappa$. ○

We'll need the following well-known lemmata; see Lemma 7.1 in [**?**] and Lemma 3.1 in [**?**] for their proofs.

Lemma 1.1.16 (Ancient Kunen Lemma)**.** *Let $M \models \mathsf{ZFC}^-$ and $j : M \to N$ an elementary embedding with critical point κ such that $\kappa + 1 \subseteq M \subseteq N$. Assume that $X \in M$ has M-cardinality κ. Then $j \restriction X \in N$.* ■

Lemma 1.1.17 (Absoluteness of embeddings on countable structures). *Let M be a countable first-order structure and $j : M \to N$ an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of) ZFC such that M is countable in W and $N \in W$, then for any finite subset of M, W has some elementary embedding $j^* : M \to N$, which agrees with j on that subset. Moreover, if both M and N are transitive $\in$-structures and j has a critical point, we can also assume that* $\operatorname{crit}(j^*) = \operatorname{crit}(j)$. ■

Theorem 1.1.18 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. Then κ is generically θ-measurable iff player II has a winning strategy in $\mathcal{C}^\theta_\omega(\kappa)$.*

PROOF. ($\Leftarrow$) : Fix a winning strategy σ for player II in $\mathcal{C}^\theta_\omega(\kappa)$. Let $g \subseteq \operatorname{Col}(\omega, H^V_\theta)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega \rangle$ of weak κ-models $\mathcal{M}_n \prec H^V_\theta$ such that $H^V_\theta \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^V_θ-normal H^V_θ-measure μ on κ.

We claim that $\operatorname{Ult}(H^V_\theta, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \theta$ such that $g_n \in H^V_\theta$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f \colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \ldots, M_{f_n}, \mu_{f_n} \rangle$ is a partial play of $\mathcal{C}^\theta_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $C^\theta_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\operatorname{Ult}(H^V_\theta, \mu)$ is wellfounded, so that the ultrapower embedding $\pi \colon H^V_\theta \to \operatorname{Ult}(H^V_\theta, \mu)$ witnesses that κ is generically θ-measurable.

($\Rightarrow$) : Assume that κ is generically θ-measurable. Let $\mathbb{P}$ be a forcing $\dot{\mu}$ a $\mathbb{P}$-name for an H_θ^V-normal H_θ^V-measure on κ and $\dot{\pi}$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}_\omega^\theta(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f_i^n \mid i < k\rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \check{\mu} \cap \mathcal{M}_n = \check{\mu}_n \cap \forall i < \check{k} \colon \dot{\pi}(\check{f}_i^n)(\check{\kappa}) = \check{\alpha}_i^n \urcorner,$$

with $\mu_n, \alpha_i^n \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f_{i_n}^{k_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check{\alpha}_{i_{n+1}}^{k_{n+1}} < \check{\alpha}_{i_n}^{k_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

Here's a near-analogous result for the $\mathcal{G}_\omega^\theta(\kappa)$ game from [**?**], with a proof added for completeness.

Theorem 1.1.19 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$, and if player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$ then κ is generically θ-power-measurable. In particular, $\mathcal{G}_\omega^\theta(\kappa)^L \sim \mathcal{C}_\omega^\theta(\kappa)^L$.*

Proof. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi \colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a name $\dot{\mu}$ and a $\mathbb{P}$-condition p such that

$$p \Vdash \ulcorner \dot{\mu} \text{ is a weakly amenable } \check{H}_\theta\text{-normal } \check{H}_\theta\text{-measure with a wellfounded ultrapower} \urcorner.$$

We now define a strategy σ for player II in $\mathcal{G}_\omega^\theta(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H_\theta^V$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n \colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \leq p$, $p_n \leq p_k$

for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot{\mu} \cap \check{\mathcal{M}}_n = \check{\mu}_n \cap \check{\mu}_n = \dot{\mu} \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma **??** we get that $\pi \restriction \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega} \mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot{\pi}(\check{g}_{n+1})(\check{\kappa}) < \dot{\pi}(\check{g}_n)(\check{\kappa}) \urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check{\pi}_{n+1}(\check{g}_{n+1})(\check{\kappa}) < \check{\pi}_n(\check{g}_n)(\check{\kappa}) \urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy. ■

We get the following immediate corollary.

Corollary 1.1.20 (N.-Schindler)**.** *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ■

Note also that the proof of Theorem **??** shows that whenever κ is strategic ω-Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H_ν^V-normal H_ν-measure on κ.

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω-Ramsey cardinals and ω-Ramsey cardinals lie, namely that strategic ω-Ramseys are equiconsistent with *remarkables* and ω-Ramseys are strictly below. Theorem 4.8 of [?] showed that 2-iterables are limits of remarkables, and our Propositions ?? and 1.2.20 shows that ω-Ramseys are limits of 1-iterables, so that the strategic ω-Ramseys and the ω-Ramseys both lie strictly between the 2-iterables and 1-iterables. It was shown in [?] that ω-Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by [?], and [?] showed the following two equivalent formulations.

Definition 1.1.21. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

(i) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H_\lambda^V \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_\nu^V \to M$ with critical point κ and $\pi(\kappa) > \lambda$;

(ii) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^\lambda M \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_\nu^V \to M$ with critical point κ and $\pi(\kappa) > \lambda$.

$\circ$

Theorem 1.1.22 (N.)**.** *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models$ ⌜there is a proper class of virtually measurables⌝. In particular, the two notions are equiconsistent.*

Proof. Virtually measurables are downwards absolute to L by Lemma 1.1.17, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H_\lambda^V \subseteq M$ and forcing poset $\mathbb{P}$ it holds that, in $V^{\mathbb{P}}$, there's no elementary embedding $\pi : H_\nu^V \to M$ with $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν-measurable to fix a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's an elementary $\pi : H_\nu^V \to M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H_\nu^V = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we're assuming that κ isn't remarkable. Then by restricting the generic embedding to H^V_κ we get that $H^V_\kappa \prec H^M_{\pi(\kappa)} = H^V_{\pi(\kappa)}$, using that $\pi(\kappa) < \nu$ and $H^V_\nu = H^M_\nu$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H^V_ν since $\pi(\kappa) < \nu$, and as $H^V_\nu \prec_1 V$ we get that $\pi(\kappa)$ *is* a cardinal. But then, again using that $H_{\pi(\kappa)} \prec_1 V$, κ is virtually measurable in $H^V_{\pi(\kappa)}$ since being virtually measurable is Π_2. This means that for every $\xi < \kappa$ it holds that

$$H^V_{\pi(\kappa)} \models \exists\alpha > \xi : \ulcorner\alpha \text{ is virtually measurable}\urcorner,$$

implying that $H^V_\kappa \models \ulcorner$There is a proper class of virtually measurables$\urcorner$. ■

Now Theorem 1.1.22 and Corollary 1.1.20 yield the following immediate corollary.

Corollary 1.1.23 (N.-Schindler)**.** *Let κ be strategic ω-Ramsey. Then either κ is remarkable in L or otherwise*

$$L_\kappa \models \ulcorner \textit{there is a proper class of strategic } \omega\textit{-Ramseys}\urcorner.$$

In particular, the two notions are equiconsistent. ■

Now, using these results we show that the strategic ω-Ramseys have strictly stronger consistency strength than the ω-Ramseys.

Theorem 1.1.24 (N.)**.** *Remarkable cardinals are strategic ω-Ramsey limits of ω-Ramsey cardinals.*

Proof. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ-sequences and a generic elementary embedding $\pi : H^V_\nu \to M$ for some $\nu > 2^\kappa$. We will show that κ is ω-Ramsey in M. Note that remarkables are clearly virtually measurable, and thus by Theorem **??** also strategic ω-Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}^\theta_\omega(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}^\theta_\omega(\kappa)^M$. Since M is closed under 2^κ-sequences it means that $\mathscr{P}(\mathscr{P}(\kappa)) \subseteq M$

and thus that M contains all possible filters on κ. We let player II follow τ, which produces a play $\sigma * \tau$ in which player II wins. But all player II's moves are in $\mathscr{P}(\mathscr{P}(\kappa))$ and hence in M, and as M is furthermore closed under ω-sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω-Ramsey in M.

This also implies that κ is a limit of ω-Ramseys in H_ν. But as κ is remarkable it holds that $H_\kappa \prec_2 V$, in analogy with the same property for strongs and supercompacts, and as being ω-Ramsey is a Π_2-notion this means that κ *is* a limit of ω-Ramseys. ■

This immediately yields the following corollary.

Corollary 1.1.25 (N.-Schindler)**.** *If κ is a strategic ω-Ramsey cardinal then*

$$L_\kappa \models \ulcorner \textit{there is a proper class of } \omega\textit{-Ramseys} \urcorner. \qquad \dashv$$

When we have a winning strategy

Theorem 1.1.26 (Schindler-N.)**.** *Let $\kappa < \theta$ be regular cardinals. Then κ is generically θ-measurable iff player II has a winning strategy in $\mathcal{C}^\theta_\omega(\kappa)$.*

PROOF. $(\Leftarrow)$: Fix a winning strategy σ for player II in $\mathcal{C}^\theta_\omega(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, H^V_\theta)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega \rangle$ of weak κ-models $\mathcal{M}_n \prec H^V_\theta$ such that $H^V_\theta \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^V_θ-normal H^V_θ-measure μ on κ.

We claim that $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \theta$ such that $g_n \in H^V_\theta$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f \colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_\mathcal{T} \cdots <_\mathcal{T} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \ldots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{C}^\theta_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $C^\theta_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so that the ultrapower embedding $\pi\colon H^V_\theta \to \mathrm{Ult}(H^V_\theta, \mu)$ witnesses that κ is generically θ-measurable.

($\Rightarrow$) : Assume that κ is generically θ-measurable. Let $\mathbb{P}$ be a forcing $\dot\mu$ a $\mathbb{P}$-name for an H^V_θ-normal H^V_θ-measure on κ and $\dot\pi$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}^\theta_\omega(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f^n_i \mid i < k\rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \check\mu \cap \mathcal{M}_n = \check\mu_n \cap \forall i < \check k\colon \dot\pi(\check f^n_i)(\check\kappa) = \check\alpha^n_i \urcorner,$$

with $\mu_n, \alpha^n_i \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f^{k_n}_{i_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check\alpha^{k_{n+1}}_{i_{n+1}} < \check\alpha^{k_n}_{i_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

Here's a near-analogous result for the $\mathcal{G}^\theta_\omega(\kappa)$ game from [?], with a proof added for completeness.

Theorem 1.1.27 (Schindler-N.)**.** *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$, and if player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$ then κ is generically θ-power-measurable. In particular, $\mathcal{G}^\theta_\omega(\kappa)^L \sim \mathcal{C}^\theta_\omega(\kappa)^L$.*

Proof. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually

θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi\colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a name $\dot\mu$ and a $\mathbb{P}$-condition p such that

$$p \Vdash \ulcorner \dot\mu \text{ is a weakly amenable } \check H_\theta\text{-normal } \check H_\theta\text{-measure with a wellfounded ultrapower}\urcorner.$$

We now define a strategy σ for player II in $\mathcal{G}_\omega^\theta(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H_\theta^V$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n\colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot\mu \cap \check{\mathcal{M}}_n = \check\mu_n \cap \check\mu_n = \dot\mu \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma **??** we get that $\pi \restriction \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n\colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega} \mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot\pi(\check g_{n+1})(\check\kappa) < \dot\pi(\check g_n)(\check\kappa)\urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check\pi_{n+1}(\check g_{n+1})(\check\kappa) < \check\pi_n(\check g_n)(\check\kappa)\urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy. ■

1.1.28 The general case

Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from [?] and investigate further connections between these and the α-Ramsey cardinals. First, a definition.

Definition 1.1.29 (Gitman)**.** A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ. If furthermore $\mathcal{M} \prec H_{\kappa^+}$ then we say that κ is **super Ramsey**. ◦

Note that since the model $\mathcal{M}$ in question is a *κ-model* it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$-sequences. [?] shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M} \models \mathsf{ZFC}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ.

Now, a first connection between the α-Ramseys and the strongly- and super Ramseys is the result in [?] that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

Theorem 1.1.30 (N.-Welch)**.** *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

Proof. Let κ be strongly Ramsey and let $\mathcal{M} \models \mathsf{ZFC}$ be a transitive κ-model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^{\mathcal{M}}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of $\mathcal{M}$ by weak amenability of μ. As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside $\mathcal{M}$, so $\mathcal{M}$ sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \to \mathcal{N}$ be the induced embedding, κ-powerset preservation of j_μ implies that μ is also a weakly amenable $\mathcal{N}$-normal $\mathcal{N}$-measure on κ. This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in $\mathcal{N}$, entailing that it is a stationary limit of almost fully Ramseys in $\mathcal{M}$. But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in [?], so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and $\mathcal{M}$ being transitive implies that κ really *is* a stationary limit of almost fully Ramseys. ■

Downwards absoluteness to K

Lastly, we consider the question of whether the α-Ramseys are downwards absolute to K, which turns out to at least be true in many cases. The below Theorem 1.1.32 then also answers Question 9.4 from [?] in the positive, asking whether α-Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of $0^\P$.

Definition 1.1.31. $0^\P$ is "the sharp for a strong cardinal", meaning the minimal sound active mouse $\mathcal{M}$ with $\mathcal{M} \,l\, \text{crit}(\dot{F}^{\mathcal{M}}) \models \ulcorner$There exists a strong cardinal$\urcorner$, with $\dot{F}^{\mathcal{M}}$ being the top extender of $\mathcal{M}$. ○

Theorem 1.1.32 (N.-Welch)**.** *Assume $0^\P$ does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ-Ramsey. Then $K \models$ $\ulcorner\kappa$ is a λ-Ramsey cardinal$\urcorner$.*

Proof. Note first that $\kappa^{+K} = \kappa^+$ by [?], since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ-models $\mathcal{M} \prec K l \kappa^+$. We can then define a strategy $\tilde\sigma$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde\sigma(\emptyset) := \text{Hull}^{H_{\kappa^+}}(K l \kappa \cup \sigma(\emptyset))$. Assuming now that $\langle \tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$, we have two cases. If $\tilde\mu_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle \mathcal{M}_\alpha \mid \alpha < \gamma\rangle$ be the corresponding models played in

$\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde\sigma(\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha<\gamma\rangle) := \mathrm{Hull}^{H_{\kappa^+}}(Kl\kappa\cup\sigma(\langle\mathcal{M}_\alpha,\tilde\mu_\alpha\mid\alpha<\gamma\rangle)),$$

and otherwise let $\tilde\sigma$ play arbitrarily. As κ is λ-Ramsey (in V) there exists a play $\langle\tilde{\mathcal{M}}_\alpha,\tilde\mu_\alpha\mid\alpha\leq\lambda\rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda\cap Kl\kappa^+\prec Kl\kappa^+$ so let $\mathcal{N}$ be the transitive collapse of $\tilde{\mathcal{M}}_\lambda\cap Kl\kappa^+$. But if $j:\mathcal{N}\to Kl\kappa^+$ is the uncollapse then $\operatorname{crit} j$ is both an $\mathcal{N}$-cardinal and also $>\kappa$ because we ensured that $Kl\kappa\subseteq\mathcal{N}$. This means that $j=\mathrm{id}$ because κ is the largest $\mathcal{N}$-cardinal by elementarity in $Kl\kappa^+$, so that $\tilde{\mathcal{M}}_\lambda\cap Kl\kappa^+=\mathcal{N}$ is a transitive elementary substructure of $Kl\kappa^+$, making it an initial segment of K.

Now, since $\mu:=\tilde\mu_\lambda$ is a countably complete weakly amenable $Klo(\mathcal{N})$-measure[4], the "beaver argument"[5] shows that $\mu\in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu\cap\mathcal{N}\in K$ whenever player I plays $\mathcal{N}$. Since $\mu=\tilde\mu_\lambda$ we also have that $\mu\cap\mathcal{M}_\alpha=\tilde\mu_\alpha\cap\mathcal{M}_\alpha$, so that σ will eventually play $\mathcal{N}$, making τ win against σ.[6] ■

Note that the only thing we used $\operatorname{cof}\lambda>\omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine- or normal α-Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

Corollary 1.1.33 (N.-Welch)**.** *Assume $0^\P$ does not exist and let α be any limit ordinal. Then every genuine- and every normal α-Ramsey cardinal is downwards absolute to K. In particular, if α is a limit of limit ordinals then every $<\alpha$-Ramsey cardinal is downwards absolute to K as well.* ■

Indiscernible games

We now move to the strategic versions of the α-Ramsey hierarchy. The first thing we want to do is define *α-very Ramsey cardinals*, introduced in [?],

[4]Here we use that $\mathcal{N}\lhd K$.

[5]See Lemmata 7.3.7–7.3.9 and 8.3.4 in [?] for this argument.

[6]Note that τ is not necessarily a winning strategy — all we know is that it is winning against this particular strategy σ.

and show the tight connection between these and the strategic α-Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 1.2.22.

Definition 1.1.34. A **good set of indiscernibles** for a structure $\mathcal{M}$ is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for $\mathcal{M}$ such that $\mathcal{M} \restriction \iota \prec \mathcal{M}$ for any $\iota \in I$. $\circ$

Definition 1.1.35 (Sharpe-Welch)**.** Define the **indiscernible game** $G^I_\gamma(\kappa)$ in γ many rounds as follows

$$\begin{array}{lccccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \text{II} & & I_0 & & I_1 & & I_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha$ is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for $\mathcal{M}_\alpha$ and the I_α's are $\subseteq$-decreasing. Player II wins iff they can continue playing through all the rounds. $\circ$

Definition 1.1.36 (Sharpe-Welch)**.** A cardinal κ is **γ-very Ramsey** if player II has a winning strategy in the game $G^I_\gamma(\kappa)$. $\circ$

The next couple of results concerns the connection between the strategic α-Ramseys and the α-very Ramseys. We start with the following.

Theorem 1.1.37 (N.)**.** *Every $(\omega{+}1)$-Ramsey is an ω-very Ramsey stationary limit of ω-very Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey. We will describe a winning strategy for player II in the indiscernible game $G^I_\omega(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G^I_\omega(\kappa)$ then let player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ play

$$\mathcal{H}_0 := \text{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) \prec H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ which starts off with $\mathcal{H}_0$ and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}_0^{(\alpha)} := \mathcal{M}_\alpha$ and $\mu_0^{(\alpha)} := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{cccccccccc} \mathrm{I} & \mathcal{H}_0^{(0)} & & \cdots & & \mathcal{H}_0^{(\omega)} & & \mathcal{H}_0^{(\omega+1)} & \\ \mathrm{II} & & \mu_0^{(0)} & & \cdots & & \mu_0^{(\omega)} & & \mu_0^{(\omega+1)} \end{array}$$

By the choice of player I's strategy we get that $\mu_0^{(\omega)}$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ (it's even normal). Now Lemma 2.9 of [?] gives us a set of good indiscernibles $I_0 \in \mu_0^{(\omega)}$ for $\mathcal{M}_0$, as $\mathcal{M}_0 \in \mathcal{H}_0^{(\omega)}$ and $\mu_0^{(\omega)}$ is a countably complete weakly amenable $\mathcal{H}_0^{(\omega)}$-normal $\mathcal{H}_0^{(\omega)}$-measure on κ. Let player II play I_0 in $G_\omega^I(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G_\omega^I(\kappa)$.

$$\begin{array}{cccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 \\ \mathrm{II} & & I_0 & \end{array}$$

Since $\mu_0^{(\omega)} = \bigcup_n \mu_0^{(n)}$ we must have that $I_0 \in \mu_0^{(n_0)}$ for some $n_0 < \omega$. In the $(n_0{+}1)$'st round of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}_0^{(k)}, \mu_0^{(k)} \mid k \leq n_0 \rangle\}) \prec H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ thus looks like the following so far.

$$\begin{array}{cccccccc} \mathrm{I} & \mathcal{H}_0^{(0)} & & \cdots & & \mathcal{H}_0^{(n_0)} & & \mathcal{H}_1 \\ \mathrm{II} & & \mu_0^{(0)} & & \cdots & & \mu_0^{(n_0)} & \end{array}$$

Now player II in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ is not losing at round n_0, so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I_1' \in \mu_1^{(n_1)}$ of good indiscernibles for $\mathcal{M}_1$, where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G_\omega^I(\kappa)$

play $I_1 := I_0 \cap I_1' \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G_\omega^I(\kappa)$, making κ ω-very Ramsey.

As for showing that κ is a stationary limit of ω-very Ramseys, let $\mathcal{M} \prec H_{\kappa^+}$ be a weak κ-model with a weakly amenable countably complete $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ, which exists by Theorem 1.2.26 as κ is $(\omega+1)$-Ramsey. Then by elementarity $\mathcal{M} \models \ulcorner\kappa$ is ω-very Ramsey$\urcorner$ and since κ being ω-very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ-ultrapower, meaning that κ is a stationary limit of ω-very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

Theorem 1.1.38 (N.). *For limit ordinals α, every coherent $<\omega\alpha$-Ramsey is $\omega\alpha$-very Ramsey.*

Proof. This is basically the same proof as the proof of Theorem 1.1.37. We do the "going-back" trick in ω-chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}_{\omega\alpha}^{\kappa^+}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$-Ramseyness. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω-chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 1.1.37. ■

As for going from the α-very Ramseys to the strategic α-Ramseys we got the following.

Theorem 1.1.39 (N.). *For γ any ordinal, every coherent $<\gamma$-very Ramsey*[7] *is coherent $<\gamma$-Ramsey.*[8]

[7]Here the coherency again just means that the winning strategies σ_α for player II in $G_\alpha^I(\kappa)$ are $\subseteq$-increasing.

[8]Here a "coherent $<\gamma$-very Ramsey cardinal" is defined from γ-very Ramseys in the same way as coherent $<\gamma$-Ramsey cardinals is defined from γ-Ramseys. When γ is a limit ordinal then coherent $<\gamma$-very Ramseys are precisely the same as γ-very Ramseys, so this is solely to "subtract one" when γ is a successor ordinal — i.e. a coherent $<(\gamma+1)$-very Ramsey cardinal is the same thing as a γ-very Ramsey cardinal.

Proof. The reason why we work with $<\gamma$-Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ-very Ramsey and let τ be a winning strategy for player II in $G^I_\zeta(\kappa)$. Let $\mathcal{M}_\alpha \prec H_\theta$ be any move by player I in the α'th round of $\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in $\mathcal{M}_\alpha$ and form now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G^I_\zeta(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for $\mathcal{N}_\alpha$ such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in [**?**] we get a structure $\mathcal{P}_\alpha$ with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a $\mathcal{P}_\alpha$-measure $\tilde{\mu}_\alpha$ on κ, generated by I_α.[9] Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α's are generated by the I_α's, the μ_α's are $\subseteq$-increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$-Ramsey. ■

The following result is then a direct corollary of Theorems 1.1.38 and 1.1.39.

Corollary 1.1.40 (N.)**.** *For limit ordinals α, κ is $\omega\alpha$-very Ramsey iff it is coherent $<\omega\alpha$-Ramsey. In particular, κ is λ-very Ramsey iff it is strategic λ-Ramsey for any λ with uncountable cofinality.* ■

We can now use this equivalence to transfer results from the α-very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in [**?**]. A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in [**?**]. We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in [**?**].

Theorem 1.1.41 (Sharpe-Welch)**.** *A cardinal is completely Ramsey if and only if it is ω-very Ramsey.* ■

[9] By *generated* here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α.

This, together with Theorem 1.1.37, immediately yields the following strengthening of Theorem 1.2.26.

Corollary 1.1.42 (N.). *Every $(\omega+1)$-Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* ■

The above Theorem 1.1.39 also yields the following consequence.

Corollary 1.1.43 (N.). *Every completely Ramsey cardinal is completely ineffable.*

Proof. From Theorem 1.1.41 we have that being completely Ramsey is equivalent to being ω-very Ramsey, so the above Theorem 1.1.39 then yields that a completely Ramsey cardinal is coherent $<\omega$-Ramsey, which we saw in Theorem 1.1.13 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 1.1.40 yields that strategic ω_1-Ramsey cardinals are ω_1-very Ramsey, and Theorem 3.50 in [?] states that ω_1-very Ramseys are measurable in the core model K, assuming $0^\P$ doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

Theorem 1.1.44 (Welch). *Assuming $0^\P$ doesn't exist, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Let κ be strategic ω_1-Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \mathrm{Col}(\omega_1, \kappa^+)$ is V-generic. Since $\mathrm{Col}(\omega_1, \kappa^+)$ is ω-closed, V and $V[g]$ have the same countable sequences of V, so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V.

Now let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of regular K-cardinals cofinal in κ^+, let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \mathrm{Hull}^{H_\theta}(Kl\kappa_\alpha) \prec H_\theta$ and player II follow τ. This results in a countably complete weakly amenable

K-measure μ_{ω_1}, which the "beaver argument"[10] then shows is actually an element of K, making κ measurable in K. ■

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1-Ramsey cardinal is also measurable in Steel's core model below a Woodin, a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from [**?**].[11]

Theorem 1.1.45 (Schindler)**.** *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ, and let $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.* ■

Theorem 1.1.46 (Schindler)**.** *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of weak κ-models $M_\alpha \prec H^V_\theta$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H^V_{\kappa^+} \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\mathrm{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Let $j : M_{\omega_1} \to \mathrm{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H^V_{\kappa^+} \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ with N transitive.

[10] See Lemmata 7.3.7–7.3.9 and 8.3.4 in [**?**] for this argument.

[11] That paper assumes the existence of a measurable as well, but by [**?**] we can omit that here.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 1.1.45 applies, meaning that $\pi \restriction K : K \to K^N$ is an iteration map with critical point κ, making κ measurable in K. ■

1.2 Ideals

Definition 1.2.1. A poset property[12] $\Phi(\kappa)$ is **ideal-absolute** if whenever κ satisfies that there's a $\Phi(\kappa)$ forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure μ on κ, then there's an ideal I on κ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(v)$. ○

Note that this is *almost* saying that $\Phi(\kappa)$ ideally measurables are equivalent to $\Phi(\kappa)$ generically ∞-measurables, but the only difference is that these definitions require well-foundedness of the above M.

Also note that ω-distributive generically θ_0-measurable cardinals are equivalent to ω-distributive generically θ_1-measurable cardinals for all regular $\theta_0, \theta_1 \in \infty \cup \{\infty\}$ since wellfoundedness becomes automatic, so in this case we will simply write "ω-distributive generically measurable".

Note that the ideally measurables aren't equiconsistent with the generically- and virtually measurables, since the ideally measurable cardinals are ideally ∞-measurable and are therefore equiconsistent with a measurable cardinal. Because of this proposition we will refrain from using the "ideally ∞-measurable" terminology and only use "ideally measurable" from now on.

Add proof?

We *do* get an equiconsistency at the critical level though, as Theorem 2.11 of [**?**] shows that if κ is generically critical then it's ideally critical in $L^{\mathrm{Col}(\omega,<\kappa)}$.

Definition 1.2.2. Let κ be a regular cardinal, $\mathbb{P}$ a poset and $\dot{\mu}$ a $\mathbb{P}$-name for a V-normal V-measure on κ. Then the **induced ideal** is

$$\mathcal{I}(\mathbb{P}, \dot{\mu}) := \{X \subseteq \kappa \mid \big|\big|\check{X} \in \dot{\mu}\big|\big|_{\mathcal{B}(\mathbb{P})} = 0\},$$

where $\mathcal{B}(\mathbb{P})$ is the boolean completion of $\mathbb{P}$. ○

Note that if the generic measure μ is furthermore V-normal then $\mathcal{I}(\mathbb{P}, \dot{\mu})$ is also normal.

[12] Examples of these are having the κ-chain condition, being κ-closed, κ-distributive, κ-Knaster, κ-sized and so on.

1.2.3 κ^+-chain condition

Other chain conditions?

Theorem 1.2.4 (Folklore)**.** *"The κ^+-chain condition" is ideal-absolute.*

PROOF. Assume $\mathbb{P}$ has the κ^+-chain condition such that there's a $\mathbb{P}$-name $\dot\mu$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot\mu)$ — we will show that $\mathscr{P}(\kappa)/I$ has the κ^+-chain condition. Assume not and let $\langle X_\alpha \mid \alpha < \kappa^+\rangle$ be an antichain of $\mathscr{P}(\kappa)/I$, which by normality of I we may assume is pairwise almost disjoint. But this then makes $\langle ||\check X_\alpha \in \dot\mu||_{\mathcal{B}(\mathbb{P})} \mid \alpha < \kappa^+\rangle$ an antichain of $\mathbb{P}$ of size κ^+, $\lightning$. ■

1.2.5 $<\lambda$-distributivity

Recall that an ideal I on some κ is ω-distributive if and only if it's precipitous[13], so that carrying an ω-distributive ideal coincides with our definition of *ideally measurable*.

Theorem 1.2.6 (N.)**.** *"$<\lambda$-distributivity" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*

PROOF. Assume that $\mathbb{P}$ is a $<\lambda$-distributive forcing such that there exists a $\mathbb{P}$-name $\dot\mu$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot\mu)$ — we'll show that $\mathscr{P}(\kappa)/I$ is $<\lambda$-distributive. Let $\mathcal{T} \subseteq (\mathscr{P}(\kappa)/I)^{<\lambda}$ be an unrooted tree of height $<\lambda$ such that every level $\mathcal{T}_\alpha$ is a maximal antichain. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of branches of $\mathcal{T}$. Now define a corresponding tree $\mathcal{T}^* \subseteq \mathbb{P}^{<\lambda}$ as

Do this in terms of $\prec$-chains of antichains instead.

$$\mathcal{T}^*_\alpha := \{||\check X \in \dot\mu||_{\mathcal{B}(\mathbb{P})} \mid X \in \mathcal{T}_\alpha\}.$$

Note that every level $\mathcal{T}^*_\alpha$ is an antichain in $\mathbb{P}$. They're also maximal, because if $p \in \mathbb{P}$ was incompatible with every condition in $\mathcal{T}^*_\alpha$ then, letting $X := \bigcap \mathcal{T}_\alpha$, we have that p is compatible with $||\check X \in \dot\mu||_{\mathcal{B}(\mathbb{P})}$, so that $X \in I^+$. But X is incompatible with everything in $\mathcal{T}_\alpha$, contradicting that $\mathcal{T}_\alpha$ is maximal.

[13] See [?] and [?].

By $<\lambda$-distributivity of $\mathbb{P}$ we get an antichain $\mathcal{A}^*$ consisting of limit points of branches of $\mathcal{T}^*$. But note that for every $p \in \mathcal{A}^*$ it holds that $p \leq ||\Delta b_p \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$ with b_p being the branch of $\mathcal{T}^*$ with limit p,[14] so that $\Delta b_p \in I^+$. Now $\mathcal{A} := \{\Delta b_p \mid p \in \mathcal{A}^*\}$ gives us a maximal antichain consisting of limit points of branches of $\mathcal{T}$. ■

1.2.7 (κ,κ)-distributivity & $<\lambda$-closure

In this section we will prove a slightly stronger version of the following unpublished result by Foreman:

Theorem 1.2.8 (Foreman)**.** *Let κ be a regular cardinal such that $2^\kappa = \kappa^+$, and let $\lambda \leq \kappa^+$ be an infinite successor cardinal. If player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$ then κ carries a κ-complete normal precipitous ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ has a dense $<\lambda$-closed subset of size κ^+.*

Theorem 1.2.9 (Foreman-N.)**.** *Let κ be a regular cardinal and $\lambda \leq \kappa^+$ be regular infinite. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is (κ,κ)-distributive and has a dense $<\lambda$-closed subset of size κ^+.*

Before we start the proof, let us note that the only difference between the two theorems is that we are requiring neither $2^\kappa = \kappa^+$ nor that λ is a successor cardinal. The proof strategy is similar to the original proof, but with some more technical details to ensure these strengthenings.

Proof. Set $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$ if $2^\kappa > \kappa^+$ and $\mathbb{P} := \{\emptyset\}$ otherwise. If κ is measurable then the dual ideal to the measure on κ satisfies all of the wanted properties, so assume that κ is not measurable. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \kappa^+ \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \kappa^+$;
- $\mathcal{N}_{\gamma+1} \prec H^V_{\kappa^+}$ is a κ-model for every $\gamma < \kappa^+$;
- $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$ for limit ordinals $\delta < \kappa^+$;

[14] Here we're using that all branches have length $<\kappa^+$, by choice of λ.

- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for $\gamma < \beta < \kappa^+$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\kappa^+} \mathcal{N}_\gamma$.

Define now the auxilliary game $\mathcal{G}(\kappa)$ of length λ as follows.

I	α_0		α_1		$\cdots$
II		$p_0, \mathcal{M}_0, \mu_0, Y_0$		$p_1, \mathcal{M}_1, \mu_1, Y_1$	$\cdots$

Here $\langle \alpha_\gamma \mid \gamma < \lambda \rangle$ is an increasing continuous sequence of ordinals bounded in κ^+, $\vec{p}_\gamma$ is a decreasing sequence of $\mathbb{P}$-conditions satisfying that

$$p_\gamma \Vdash \ulcorner \check{\mathcal{M}}_\gamma = \pi(\check{\alpha}_\gamma) \wedge \check{\mu}_\gamma \text{ is a } \check{\mathcal{M}}_\gamma\text{-normal } \check{\mathcal{M}}_\gamma\text{-measure on } \check{\kappa} \urcorner$$

such that $Y_\gamma = \Delta_{\xi<\kappa} X_\xi^{\mu_\gamma}$, where $\vec{X}_\xi^{\mu_\gamma} \in H_{\kappa^+}^V$ is the $<_{\kappa^+}$-least enumeration of μ_γ.[15] We require that the μ_γ's are $\subseteq$-increasing, and player II wins iff she can continue playing throughout all λ rounds. Let $\mu_\lambda := \bigcup_{\xi<\lambda} \mu_\xi$ be the **final measure** of the play.

To every limit ordinal $\eta < \kappa^+$ define the **restricted auxilliary game** $\mathcal{G}(\kappa) \restriction \eta$ in which player I is only allowed to play ordinals $<\eta$. Note that a strategy τ for player II is winning in $\mathcal{G}(\kappa)$ if and only if it's winning in $\mathcal{G}(\kappa) \restriction \eta$ for all $\eta < \kappa^+$, simply because all sequences of ordinals played by player I are bounded in κ^+.

Note that μ_λ is precisely the tail measure on κ defined by the Y_γ's; i.e. that $X \in \mu_\lambda$ iff there exists a $\delta < \lambda$ such that $|Y_\delta - X| < \kappa$. From this it's simple to see that $\mathcal{G}(\kappa)$ is equivalent to $\mathcal{G}_\lambda^-(\kappa)$, so player II has a winning strategy τ_0 in $\mathcal{G}(\kappa)$.

For any winning strategy τ in $\mathcal{G}(\kappa) \restriction \eta$ and to every partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ, define the associated **hopeless ideal**[16]

$$I_p^\tau \restriction \eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa) \restriction \eta, \\ X \text{ is } \textit{not} \text{ in the final measure}\}$$

Claim 1.2.9.1. Every hopeless ideal $I_p^\tau \restriction \eta$ is normal and (κ, κ)-distributive.

[15] We use that $\mathbb{P}$ is κ-closed to get the p_γ's as well as to ensure that $\mathcal{M}_\gamma, \mu_\gamma \in V$.
[16] This terminology is due to Matt Foreman.

Proof of claim. For normality, if $\langle Z_\gamma \mid \gamma < \kappa \rangle$ is a sequence of elements of I_p^τ such that $Z := \nabla_\gamma Z_\gamma$ is I_p^τ-positive, then there exists a play of $\mathcal{G}(\kappa) \restriction \eta$ in which player II follows τ such that Z lies in the final measure. If we let player I play sufficiently large ordinals in $\mathcal{G}(\kappa) \restriction \eta$ we may assume that $\langle Z_\gamma \mid \gamma < \kappa \rangle$ is a subset and an element of the final model as well, meaning that one of the Z_γ's also lies in the final measure, $\lightning$.

We now show (κ, κ)-distributivity. Let $\mathcal{U} \subseteq \mathscr{P}(\kappa)/I_p^\tau$ be an unrooted tree of height κ such that every level $\mathcal{U}_\alpha$ is a maximal antichain of size $\leq \kappa$. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of branches of $\mathcal{U}$. Pick $X \in \mathcal{U}$ and let p be a play of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ with limit model $\mathcal{M}$ and limit measure μ, such that $X \in \mu$.

By letting player I in p play sufficiently large ordinals, we may assume that $\mathcal{U} \subseteq \mathcal{M}$, using that $|\mathcal{U}| \leq \kappa$, and also that $b_X := \mathcal{U} \cap \mu \in \mathcal{M}$. This means that $d_X := \Delta b_X \in \mathscr{P}(\kappa)/I_p^\tau$ is a limit point of the branch b_X through $\mathcal{U}$, so that $\mathcal{A} := \{d_X \mid X \in \mathcal{U}\}$ is a maximal antichain of limit points of branches of $\mathcal{U}$, making $\mathscr{P}(\kappa)/I_p^\tau$ (κ, κ)-distributive. $\dashv$

Fix some limit ordinal $\eta < \kappa^+$. We will recursively construct a tree $\mathcal{T}^\eta$ of height λ which consists of subsets $X \subseteq \kappa$, ordered by reverse inclusion. During the construction of the tree we will inductively maintain the following properties of $\mathcal{T}^\eta \restriction \alpha$ for $\alpha \leq \lambda$:

- Tree strategy: For every $\gamma < \alpha$ there is a winning strategy τ_γ^η for player II in $\mathcal{G}(\kappa) \restriction \eta$ such that for every $\beta < \gamma$, the β'th move by τ_γ^η is an element of $\mathcal{T}_\beta^\eta$ and τ_γ^η is consistent with τ_β^η for the first β-many rounds.
- Unique pre-history: Given any $\beta < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$ there's a unique partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ_β^η ending with Y — we define $I_Y^\tau := I_p^\tau$ for τ being any winning strategy for player II in $\mathcal{G}(\kappa) \restriction \eta$ satisfying that p is consistent with τ_β^η.

- Cofinally many responds: Let $\beta+1<\alpha$ and $Y\in\mathcal{T}^\eta_\beta$, and set p to be the unique partial play of $\mathcal{G}(\kappa)\restriction\eta$ given by the unique pre-history of Y. Then the $\mathcal{T}^\eta$-successors of Y consists of player II's τ^η_β-responds to τ^η_β-partial plays extending p such that player I's last move in these partial plays are cofinal in η.[17]
- Positivity: If $\beta<\alpha$ and $Y\in\mathcal{T}^\eta_\beta$ then Y is $I_X^{\tau^\eta_\gamma}$-positive for every $\gamma<\beta$ and every $X\in\mathcal{T}^\eta\restriction\gamma+1$ with $X\leq_{\mathcal{T}^\eta}Y$.[18]
- Almost disjointness property: Every level $\mathcal{T}^\eta_\beta$ consists of pairwise almost disjoint sets.[19]
- Hopeless ideal coherence: $I_{\langle\rangle}^{\tau^\eta_\beta}\cap\mathscr{P}(Y)=I_Y^{\tau^\eta_\beta}\cap\mathscr{P}(Y)$ for every $\beta<\alpha$ and $Y\in\mathcal{T}^\eta_\beta$.

Note that what we're really aiming for is achieving the hopeless ideal coherence, since that enables us to ensure that if $X,Y\in\mathcal{T}^\eta$ and $X\subseteq Y$ then really $X\geq_{\mathcal{T}^\eta}Y$ — i.e. that we "catch" both X and Y in the same play of $\mathcal{G}(\kappa)\restriction\eta$. The rest of the properties are inductive properties we need to ensure this.

Set $\mathcal{T}^\eta_0:=\{\kappa\}$. Assume that we've built $\mathcal{T}^\eta\restriction\alpha+1$ satisfying the inductive assumptions[20] and let $Y\in\mathcal{T}^\eta_\alpha$ — we need to specify what the $\mathcal{T}^\eta$-successors of Y are. Since κ is weakly compact and not measurable it holds by Proposition 6.4 in [?] that $\mathrm{sat}(I_Y^{\tau^\eta_\alpha})\geq\kappa^+$, so we can fix a maximal antichain $\langle X^Y_\gamma\mid\gamma<\eta\rangle$ of $I_Y^{\tau^\eta_\alpha}$-positive sets. By κ-completeness of $I_Y^{\tau^\eta_\alpha}$ we can by Exercise 22.1 in [?] even ensure that all of the X^Y_γ's are pairwise disjoint.

To every $\gamma<\eta$ we fix a partial play p of even length of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ^η_α such that the last ordinal β^Y_γ in p played by player I is greater than or equal to γ and X^Y_γ has measure one with respect to the last measure in p. We then define the $\mathcal{T}^\eta$-successors of Y to be player II's τ^η_α-responses to the

[17]The reason why we're dealing with the *restricted* auxilliary games is to achieve this property.
[18]This actually follows from the cofinally many responds, but we include it here for transparency.
[19]Two subsets $X,Y\subseteq\kappa$ are *almost disjoint* if $|X\cap Y|<\kappa$.
[20]In particular, we assume that τ^η_α is defined.

β_γ's (which are subsets of the X^Y_γ's modulo a bounded set and are therefore pairwise almost disjoint).

For limit stages $\delta < \lambda$ we apply τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ to get $\mathcal{T}^\eta_\delta$.

We now have to check that the inductive assumptions still hold; let's start with the tree strategy. Assume that we have a partial play p of length $2 \cdot \alpha + 1$ of $\mathcal{G}(\kappa) \restriction \eta$, i.e. the last move in p is by player II, consistent with τ^η_α; write ξ_p for player I's last move in p and Y_p for player II's response to ξ_p, which is also the last move in p. We can then pick a $\zeta < \eta$ such that $\beta^{Y_p}_\zeta > \xi_p$ by the cofinally many responds property and let $\tau^\eta_{\alpha+1}(p)$ be player II's τ^η_α-response to the partial play leading up to $\beta^{Y_p}_\zeta$. After this $(\alpha+1)$'th round we just set $\tau^\eta_{\alpha+1}$ to follow τ_0. It's clear that $\tau^\eta_{\alpha+1}$ satisfies the required properties.

Before we move on to checking the remaining inductive assumptions, let's pause to get some intuition about the tree strategies. In the definition of $\tau^\eta_{\alpha+1}$ above, we took a partial play consistent with τ^η_α, applied τ_0 for a while, took note of player II's last τ_0-response and then included *only that* response in our new $\tau^\eta_{\alpha+1}$ partial play. This means that to every τ^η_α-partial play there's an ostensibly much longer τ_0-partial play into which τ^η_α embeds; so we can look at the τ^η_α-partial plays as being "collapsed" τ_0-partial plays.

Given the above tree strategy, $\mathcal{T}^\eta_{\alpha+1}$ clearly satisfies the cofinally many responds property and the positivity property, simply by construction. For the unique pre-history, let $Y \in \mathcal{T}^\eta_{\alpha+1}$ and assume it has two distinct immediate $\mathcal{T}^\eta$-predecessors $Z_0, Z_1 \in \mathcal{T}^\eta_\alpha$. But then $Y \subseteq Z_0 \cap Z_1$ and Y is $I^{\tau^\eta_\alpha}_{Z_0}$-positive by the positivity assumption, contradicting that Z_0 and Z_1 are almost disjoint by the almost disjointness property. Given the unique pre-history we then also get the almost disjointness property.

Claim 1.2.9.2. $\mathcal{T}^\eta \restriction \alpha + 2$ satisfies the hopeless ideal coherence property.

Proof of claim. Let $Y \in \mathcal{T}^\eta_{\alpha+1}$ — we have to show that

$$I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y). \tag{1}$$

It's clear that $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \subseteq I^{\tau^\eta_{\alpha+1}}_Y$, so let $Z \in I^{\tau^\eta_{\alpha+1}}_Y \cap \mathscr{P}(Y)$ and assume for a contradiction that Z is $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle}$-positive. Letting $\vec{\alpha}_\xi * \vec{Y}_\xi$ be a play of $\mathcal{G}(\kappa)\restriction\eta$ consistent with $\tau^\eta_{\alpha+1}$ such that Z is in the final measure, the definition of $\tau^\eta_{\alpha+1}$ yields that $Y_\alpha \in \mathcal{T}^\eta_{\alpha+1}$. As $Z \in I^{\tau^\eta_{\alpha+1}}_Y$ we have to assume that $Y \neq Y_\alpha$, so that the almost disjointness property implies that

$$|Y \cap Y_\alpha| < \kappa, \tag{2}$$

By the choice of $\vec{\alpha}_\xi * \vec{Y}_\xi$ there's some $\delta \in (\alpha, \lambda)$ such that $|Y_\delta - Z| < \kappa$, i.e. that Y_δ is a subset of Z modulo a bounded set, since the Y_α's generate the final measure of the play. But then $Y_\delta \subseteq Y_\alpha$ by the rules of $\mathcal{G}(\kappa)\restriction\eta$, and also that $|Y_\delta - Y| < \kappa$ since $Z \subseteq Y$. But this means that $Y \cap Y_\alpha$ is $I^{\tau^\eta_{\alpha+1}}_Y$-positive since Y_δ is, contradicting (2). This shows (1). $\dashv$

This finishes the construction of $\mathcal{T}^\eta_{\alpha+1}$. For limit levels $\delta < \lambda$ we define τ^η_δ as simply applying τ_0 to the branches of $\mathcal{T}^\eta\restriction\delta$ — showing that the inductive assumptions hold at $\mathcal{T}^\eta_\delta$ is analogous to the above arguments, so we're now done with the construction of $\mathcal{T}^\eta$. Let $\tau^\eta := \bigcup_{\alpha<\lambda} \tau^\eta_\alpha \restriction {}^{<\alpha}H_{\kappa^+}$ and define[21] $\mathcal{I}^\eta := I^{\tau^\eta}_{\langle\rangle}$.

Now note that $\mathcal{I}^{\eta+1} \subseteq \mathcal{I}^\eta$ and $\mathcal{T}^\eta \subseteq \mathcal{T}^{\eta+1}$ for every $\eta < \kappa^+$ — set $\mathcal{I} := \bigcap_{\eta<\kappa^+} \mathcal{I}^\eta$ and $\mathcal{T} := \bigcup_{\eta<\kappa^+} \mathcal{T}^\eta$. We showed that all hopeless ideals are κ-complete, normal and (κ,κ)-distributive, so this holds in particular for the $\mathcal{I}^\eta$'s and thus also for $\mathcal{I}$.

We claim that $\mathcal{T}$ is dense in $\mathscr{P}(\kappa)/\mathcal{I}$.[22] Let X be an $\mathcal{I}$-positive set, making it $\mathcal{I}^\eta$-positive for some $\eta < \kappa^+$, meaning that there's a play $\vec{\alpha}_\gamma * \tau^\eta$ of $\mathcal{G}(\kappa)\restriction\eta$ such that X is in the final measure, which means that $|Y_\delta - X| < \kappa$ for some large $\delta < \lambda$ and in particular that $Y_\delta - X \in \mathcal{I}$. But $Y_\delta \in \mathcal{T}^\eta \subseteq \mathcal{T}$ by definition of τ^η, which shows that $\mathcal{T}$ is dense.

It remains to show that $\mathcal{T}$ is $<\lambda$-closed. If $\lambda = \omega$ then this is trivial, so assume that $\lambda \geq \omega_1$. Let $\beta < \lambda$ and let $\langle Z_\alpha \mid \alpha < \beta\rangle$ be a $\subseteq$-decreasing sequence of elements $Z_\alpha \in \mathcal{T}$. We can fix some $\eta < \kappa^+$ such that $Z_\alpha \in \mathcal{T}^\eta$

[21] Note that the tree strategy property above ensures that the strategies *do* line up, so that τ^η is a well-defined strategy as well.

[22] This means that given any $\mathcal{I}$-positive set X there's a $Y \in \mathcal{T}$ such that $Y - X \in \mathcal{I}$.

for every $\alpha < \beta$ by regularity of κ^+, and since the Z_α's are $\subseteq$-decreasing they must also be $\leq_{\mathcal{T}^\eta}$-increasing by the hopeless ideal coherence for $\mathcal{T}^\eta$[23].

Let $\tilde{Z} \in \mathcal{T}^\eta$ be player II's τ^η-response to the unique partial play of $\mathcal{G}(\kappa) \restriction \eta$ corresponding to the branch containing the Z_α's, and pick $Z \in \mathcal{T}^\eta$ such that $\left|Z - \tilde{Z}\right| < \kappa$ and $Z \geq_{\mathcal{T}^\eta} Z_\alpha$ for all $\alpha < \beta$, again by the density claim and the hopeless ideal coherence. Then Z witnesses $<\lambda$-closure of $\mathcal{T}$.[24] ■

Theorem 1.2.10 (N.). *Let κ be a regular cardinal and $\lambda \in [\omega_1, \kappa^+]$ be regular. Then the following are equivalent:*

(i) *κ is $<\lambda$-closed generically power-measurable;*

(ii) *κ is $<\lambda$-closed ideally power-measurable;*

(iii) *κ is (κ, κ)-distributive $<\lambda$-closed generically measurable;*

(iv) *κ is (κ, κ)-distributive $<\lambda$-closed ideally measurable;*

(v) *Player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$.*

Proof. $(v) \Rightarrow (iv)$ is Theorem 1.2.9 above[25] and $(iv) \Rightarrow (iii) + (ii)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial, so we show $(i) \Rightarrow (v)$.

Assume κ is $<\lambda$-closed generically power-measurable, so there's a $<\lambda$-closed forcing $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there exists a transitive class N and a κ-powerset preserving elementary embedding $\pi \colon V \to N$. Write μ for the induced weakly amenable V-normal V-measure on κ. Now, back in V, define a strategy σ for player II in $G_\lambda(\kappa)$ as follows.

Whenever player I plays some model M_α then we let player II respond with a filter μ_α such that, for some $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash \ulcorner \check{\mu}_\alpha = \dot{\mu} \cap \check{M}_\alpha \urcorner$ — such a filter exists because μ is weakly amenable. We require the p_α's to be decreasing, which is possible by $<\lambda$-closure. Now, all the μ_α's are clearly M_α-normal M_α-measures on κ, which makes σ a winning strategy. ■

Ignoring wellfoundedness we get the same equivalence in the $\lambda = \omega$ case.

[23] This is the only place in which we're using hopeless ideal coherence.
[24] We're using that λ is regular to get Z.
[25] Here wellfoundedness of the generic ultrapower is automatic since λ has uncountable cofinality.

Corollary 1.2.11 (N.). *Let κ be a regular cardinal. Then the following are equivalent:*[26]

(i) *There exists a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a weakly amenable V-normal V-measure on κ;*

(ii) *There exists a (κ,κ)-distributive forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure on κ;*

(iii) *κ carries a normal (κ,κ)-distributive ideal;*

(iv) *Player II has a winning strategy in $\mathcal{G}^-_\omega(\kappa)$;*

(v) *κ is completely ineffable.*

Proof. $(iv) \Leftrightarrow (v)$ was shown in [**?**], and $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. $(i) \Rightarrow (iv)$ is as $(i) \Rightarrow (v)$ in Theorem 1.2.10, and $(iv) \Rightarrow (iii)$ is Theorem 1.2.9. ■

Corollary 1.2.12. *"(κ,κ)-distributive $<\lambda$-closed" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■

1.2.13 λ-density & $<\lambda$-closure

Can we get κ-complete below somehow? In this case, when $\lambda < \kappa$, κ cannot be inaccessible and cannot be a successor cardinal, by Kunen's "Saturated Ideals" paper.

Theorem 1.2.14 (N.). *Let κ and $\lambda \leq \kappa^+$ be regular infinite cardinals such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. If player II has a winning strategy in $\mathcal{C}^-_\lambda(\kappa)$ then κ carries a λ-complete ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $Add(\lambda, 1)$.*

Proof. If $\lambda = \kappa^+$ then we're done by Theorem 1.2.9, since $\mathcal{G}_{\kappa^+}(\kappa)$ is equivalent to $\mathcal{C}_{\kappa^+}(\kappa)$, so assume that $\lambda \leq \kappa$. We follow the proof of

[26] Points (i) and (ii) look a lot like what a definition of generically power-measurable and (κ,κ)-distributive ideally measurable *should* be, but here we're not requiring the ultrapowers to be well-founded, so that would be stretching the definition of being measurable.

Theorem 1.2.9 closely. Set $\mathbb{P} := \mathrm{Col}(\lambda, 2^\kappa)$. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \lambda \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \lambda$;
- $\kappa{+}1 \subseteq \mathcal{N}_\gamma$ and $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for every $\gamma < \lambda$;
- If $\delta < \lambda$ is a limit ordinal then $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$, $\mathcal{N}_\delta \prec H_{\kappa^+}$ and $\mathcal{N}_\delta \models \mathsf{ZFC}^-$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for all $\gamma < \beta < \lambda$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\lambda} \mathcal{N}_\gamma$.

Define the auxilliary game $\mathcal{G}(\kappa)$ as in the proof of Theorem 1.2.9 but where player I plays ordinals $\alpha_\eta < \lambda$ and where we use the above $\mathcal{N}_\gamma$'s. Here we only need $<\lambda$-closure of $\mathbb{P}$ to get an equivalence between $\mathcal{G}(\kappa)$ and $\mathcal{C}^-_\lambda(\kappa)$, since $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for all $\gamma < \lambda$.

To every limit ordinal $\eta < \lambda$ we define the restricted auxilliary game $\mathcal{G}(\kappa) \restriction \eta$ as in the proof of Theorem 1.2.9, and to every winning strategy τ in $\mathcal{G}(\kappa) \restriction \eta$ and partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ define the associated **hopeless ideal**[27]

$$I^\tau_p \restriction \eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction \eta,\ X \text{ is } \textit{not} \text{ in the final measure}\}.$$

As in the proof of Claim 1.2.9.1 we get that every hopeless ideal is λ-complete.

Now, if κ is measurable then we trivially get the conclusion,[28] so assume κ isn't measurable. Then $\mathrm{sat}(\kappa) \geq \lambda$ since $2^{<\theta} < \kappa$ for every $\theta < \lambda$,[29] so that we can continue exactly as in the proof of Theorem 1.2.9 to construct (λ-sized) trees $\mathcal{T}^\eta$ and winning strategies τ^η for all limit ordinals $\eta < \lambda$ such that, setting $\mathcal{I} := \bigcap_{\eta<\lambda} I^{\tau^\eta}_{\langle\rangle}$ and $\mathcal{T} := \bigcup_{\eta<\lambda} \mathcal{T}^\eta$, $\mathcal{T}$ is a dense $<\lambda$-closed subset of $\mathscr{P}(\kappa)/\mathcal{I}$ of size λ, so that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $\mathrm{Add}(\lambda, 1)$. ∎

[27] This terminology is due to Matt Foreman.
[28] Take $\mathcal{I}(\mathrm{Add}(\lambda,1), \check{\mu})$ for μ the measure on κ.
[29] See Proposition 16.4 in [**?**].

Corollary 1.2.15 (N.). *Let κ and $\lambda \in [\omega_1, \kappa^+]$ be regular such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. Then the following are equivalent:*

(i) *κ is $<\lambda$-closed generically measurable;*
(ii) *κ is $<\lambda$-closed ideally measurable;*
(iii) *κ is $<\lambda$-closed λ-sized generically measurable;*
(iv) *κ is $<\lambda$-closed λ-sized ideally measurable;*
(v) *Player II has a winning strategy in $\mathcal{C}_\lambda(\kappa)$.*

Proof. $(iv) \Rightarrow (iii) + (ii)$, $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ all trivial, and $(i) \Rightarrow (v)$ is like $(i) \Rightarrow (v)$ in Theorem 1.2.10, and $(v) \Rightarrow (iv)$ is Theorem 1.2.14. ∎

Again, if we ignore wellfoundedness then we get the same equivalence in the $\lambda = \omega$ case:

Corollary 1.2.16 (N.). *Let κ be regular infinite. Then:*

(i) *Player II has a winning strategy in $\mathcal{C}^-_\omega(\kappa)$; and*
(ii) *κ carries an ideal I such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to $Add(\omega, 1)$.*

Proof. Player II has a winning strategy in $\mathcal{C}^-_\omega(\kappa)$ as we're simply measuring finitely many sets without any demand for wellfoundedness, showing (i). Since $2^{<n} < \kappa$ for all $n < \omega$ as κ is infinite, Theorem 1.2.14 then implies (ii). ∎

Corollary 1.2.17. *"$<\lambda$-closed λ-sized" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ∎

(ω, α)-Ramsey cardinals

A natural generalisation of the γ-Ramsey definition is to require more iterability of the last measure. Of course, by Proposition **??** we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when $\operatorname{cof} \gamma > \omega$ so the next definition is only interesting whenever $\operatorname{cof} \gamma = \omega$.

Definition 1.2.18 (N.)**.** Let α, β be ordinals. Then a cardinal κ is **(α, β)-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ for all regular $\theta > \kappa$.[30] ◦

Definition 1.2.19 (Gitman)**.** A cardinal κ is **α-iterable** if for every $A \subseteq \kappa$ there exists a *transitive* weak κ-model $\mathcal{M}$ with $A \in \mathcal{M}$ and an α-good $\mathcal{M}$-measure μ on $\mathcal{M}$. ◦

Proposition 1.2.20. *If $\beta > 0$ then every (α, β)-Ramsey is a β-iterable stationary limit of β-iterables.*

Proof. Let $(\mathcal{M}, \in, \mu)$ be a result of a play of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$ in which player II won. Then the transitive collapse of $(\mathcal{M}, \in, \mu)$ witnesses that κ is β-iterable, since μ is β-good by definition of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$.

That κ is β-iterable is reflected to some H_θ, so let now $(\mathcal{N}, \in, \nu)$ be a result of a play of $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ in which player II won. Then $\mathcal{N} \prec H_\theta$, so that κ is also β-iterable in $\mathcal{N}$. Since being β-iterable is witnessed by a subset of κ and $\beta > 0$ implies[31] that we get a κ-powerset preserving $j : \mathcal{N} \to \mathcal{P}$, $\mathcal{P}$ also thinks that κ is β-iterable, making κ a stationary limit of β-iterables by elementarity. ■

We now move towards Theorem 1.2.24 which gives an upper consistency bound for the (ω, α)-Ramseys. We first recall a few definitions and a folklore lemma.

Definition 1.2.21. For an infinite ordinal α, a cardinal κ is **α-Erdős** for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \to \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c; i.e. that $|c``[H]^n| \leq 1$ holds for every $n < \omega$. ◦

Definition 1.2.22. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$. ◦

[30] Note that an α-Ramsey cardinal is the same as an $(\alpha, 0)$-Ramsey cardinal.
[31] Recall that β-good for $\beta > 0$ in particular implies weak amenability.

Lemma 1.2.23 (Folklore)**.** *Let κ be α-Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure $\mathcal{M}$ in a countable language $\mathcal{L}$ with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

Proof. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all $\mathcal{L}$-formulas and define $c : [C]^{<\omega} \to \kappa$ as follows. For an increasing sequence $\alpha_1 < \cdots < \alpha_{2n} \in C$ let

$$\begin{aligned} c(\{\alpha_1, \dots, \alpha_{2n}\}) := \text{ the least } \lambda < \alpha_1 \text{ such that} \\ \exists \delta_1 < \cdots \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \dots, \delta_k \rangle \wedge \\ \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}] \end{aligned}$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α-Erdős we get a homogeneous $I \in [C]^\alpha$ for c; i.e. that $|c``[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \dots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \dots, \alpha_{2n} \in I$, as otherwise there exists an $m < \omega$ and $\delta_1 < \cdots \delta_k$ such that for any $\alpha_1 < \ldots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}]. \tag{$\dagger$}$$

But then simply pick $\alpha_1 < \ldots \alpha_{2n} < \alpha'_1 < \cdots < \alpha'_{2n}$ so that both $\{\alpha_1, \dots, \alpha_{2n}\}$ and $\{\alpha'_1, \dots, \alpha'_{2n}\}$ witnesses ($\dagger$); then either $\{\alpha_1, \dots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \dots, \alpha_n, \alpha'_{n+1}, \dots, \alpha'_{2n}\}$ also witnesses that ($\dagger$) fails, $\lightning$. ■

Theorem 1.2.24 (N.)**.** *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α-Erdős cardinal is a limit of (ω, α)-Ramsey cardinals.*

Proof. Let κ be α-Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 1.2.23 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I. We will show that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in [**?**] it suffices to find a weak ι-model $\mathcal{M} \prec H_\theta$ and an α-good $\mathcal{M}$-measure on ι. Define

$$\mathcal{M} := \mathrm{Hull}^{H_\theta}(\iota \cup I) \prec H_\theta$$

and let $\pi : I \to I$ be the right-shift map. Since I is remarkable, I $(= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \ldots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι-powerset preserving we get that $\mathcal{M} \prec H_\theta$ is a weak ι-model satisfying ZFC^- with a 1-good $\mathcal{M}$-measure μ_j on ι. Furthermore, as we can linearly iterate $\mathcal{M}$ simply by applying j we get an α-iteration of $\mathcal{M}$ since there are α-many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \ldots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\eta, \alpha)$. If we repeat this procedure for all regular $\theta > \kappa$ we get by the pidgeon hole principle that κ is a limit of (ω, α)-Ramsey cardinals. ■

As Theorem 4.5 in [?] shows that $(\alpha{+}1)$-iterable cardinals have α-Erdős cardinals below them for $\alpha \geq \omega$ additively closed, this shows that the (ω, α)-Ramseys form a strict hierarchy. Further, as α-Erdős cardinals are consistent with $V = L$ when $\alpha < \omega_1^L$ and ω_1-iterable cardinals aren't consistent with $V = L$, we also get that (ω, α)-Ramsey cardinals are consistent with $V = L$ if $\alpha < \omega_1^L$ and that they aren't if $\alpha = \omega_1$.

[Strategic] $(\omega{+}1)$-Ramsey cardinals

The next step is then to consider $(\omega{+}1)$-Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in [?] and in the proof of Lemma 1.3 in [?] — see also [?] and [?].

Theorem 1.2.25 (Dodd, Mitchell)**.** *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model $\mathcal{M}$ such that there exists a weakly amenable countably complete $\mathcal{M}$-measure on κ.* ■

The following theorem then supplies us with a lower bound for the strength of the (ω+1)-Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 1.1.37, but we include this Ramsey lower bound as well for completeness.

Theorem 1.2.26 (N.). *Every* (ω+1)*-Ramsey cardinal is a Ramsey limit of Ramseys.*

PROOF. Let κ be (ω+1)-Ramsey and $A \subseteq \kappa$. Let σ be a strategy for player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A \in \mathcal{M}_0$ and $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Then σ isn't winning as κ is (ω+1)-Ramsey, so we may fix a play $\sigma * \vec{\mu}_\alpha$ of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable $\mathcal{M}_\omega$-measure on κ, and by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 1.2.25.

Since κ is Ramsey, $\mathcal{M}_\omega \models \ulcorner\kappa$ is Ramsey$\urcorner$ as well. Letting $j : \mathcal{M}_\omega \to \mathcal{N}$ be the κ-powerset preservering embedding induced by μ_ω, we also get that $\mathcal{N} \models \ulcorner\kappa$ is Ramsey$\urcorner$ by κ-powerset preservation. This then implies that κ *is* a stationary limit of Ramsey cardinals inside $\mathcal{M}_\omega$, and thus also in V by elementarity. ∎

As for the *consistency* strength of the strategic (ω+1)-Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal — see e.g. p. 249 in [**?**].

Theorem 1.2.27 (N.). *If* κ *is a strategic* (ω+1)*-Ramsey cardinal then, in* $V^{\mathrm{Col}(\omega,2^\kappa)}$*, there's a transitive class* N *and an elementary embedding* $j : V \to N$ *with* $\mathrm{crit}\, j = \kappa$*. In particular, the existence of a strategic* (ω+1)*-Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

PROOF. Set $\mathbb{P} := \mathrm{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$. Let $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name of an ω-sequence $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak

κ-models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n \prec H^V_{\kappa^+}$ and $\mathscr{P}(\kappa)^V \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, and let $\dot\mu$ be a $\mathbb{P}$-name for the ω-sequence of σ-responses to the $\mathcal{M}_n$'s in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)^V$.

Assume that there's a $\mathbb{P}$-condition p which forces the generic ultrapower $\mathrm{Ult}(V, \bigcup_n \dot\mu_n)$ to be illfounded, meaning that we can fix a $\mathbb{P}$-name $\dot f$ for an ω-sequence $\langle f_n \mid n < \omega\rangle$ such that

$$p \Vdash \dot X_n := \{\alpha < \kappa \mid \dot f_{n+1}(\alpha) < \dot f_n(\alpha)\} \in \bigcup_{n<\omega} \dot\mu_n.$$

Now, in V, we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} \prec H_\theta$ such that $\dot{\mathcal{M}}, \dot\mu, \dot f, H^V_{\kappa^+}, \sigma, p \in \mathcal{N}$. We can find an $\mathcal{N}$-generic $g \subseteq \mathbb{P}^{\mathcal{N}}$ in V with $p \in g$ since $\mathcal{N}$ is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}^g_n * \dot\mu^g_n$ is a play of $\mathcal{G}^{\kappa^+}_\omega(\kappa)^V$ which is according to σ, meaning that $\bigcup_{n<\omega} \dot\mu^g_n$ is normal and in particular countably complete (in V). Then $\bigcap_{n<\omega} \dot X^g_n \neq \emptyset$, but if $\alpha \in \bigcap_{n<\omega} \dot X^g_n$ then $\langle \dot f^g_n(\alpha) \mid n < \omega\rangle$ is a strictly decreasing ω-sequence of ordinals, $\lightning$. This means that $\mathrm{Ult}(V, \bigcup_n \mu_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in [?]. ■

The above Theorem 1.2.27 then answers Question 9.2 in [?] in the negative, asking if λ-Ramseys are strategic λ-Ramseys for uncountable cardinals λ, as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.