1 External core model induction

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1.1 HOD mice

Provide overview of this section.

1.1.1 Iteration strategies

At some point we should mention that we adopt John‘s convention of hiding the degree of iteration trees, always taking the maximal possible degree. And that all of our trees are (stacks of) normal trees.

Definition 1.1. Let $\vec{\mathcal{T}}$ be a stack of normal trees. We write $\mathrm{lh}(\vec{\mathcal{T}})$ for the length of $\vec{\mathcal{T}}$ and $\mathcal{T}_\alpha$ for the α'th tree in $\vec{\mathcal{T}}$, so that

$$\vec{\mathcal{T}} = (\mathcal{T}_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}})).$$

For $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$, $\gamma < \mathrm{lh}(\mathcal{T}_\alpha)$, $\eta < \mathrm{lh}(\mathcal{T}_\beta)$ we let $\mathcal{M}_\gamma^{\mathcal{T}_\alpha}$ be the model with index γ in the tree $\mathcal{T}_\alpha$ and write

$$\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} \colon \mathcal{M}_\gamma^{\mathcal{T}_\alpha} \to \mathcal{M}_\eta^{\mathcal{T}_\beta}$$

for the corresponding embedding, provided it exists. We also write

$$\pi^{\vec{\mathcal{T}}}_{\alpha,\beta} \colon \mathcal{M}_0^{\mathcal{T}_\alpha} \to \mathcal{M}_0^{\mathcal{T}_\beta}.$$

If $\vec{\mathcal{T}}$ has a last model, i.e. if $\mathrm{lh}(\vec{\mathcal{T}}) = \xi + 1$ and $\mathcal{M}_\infty^{\mathcal{T}_\xi}$ exists, we let $\mathcal{M}_\infty^{\vec{\mathcal{T}}} := \mathcal{M}_\infty^{\mathcal{T}_\xi}$ and $\pi^{\vec{\mathcal{T}}} \colon \mathcal{M}_0^{\mathcal{T}_0} \to \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ be the associated embedding. ◦

Definition 1.2. Let Σ be an iteration strategy and $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$. We write $\Sigma_{\vec{\mathcal{T}},N}$ for the iteration strategy on N given by

$$\Sigma_{\vec{\mathcal{T}},N}(\vec{\mathcal{U}}) := \Sigma(\vec{\mathcal{T}}^\frown \vec{\mathcal{U}}).$$

We call $\Sigma_{\vec{\mathcal{T}},N}$ the $(\vec{\mathcal{T}}, N)$**-tail strategy**of Σ. ◦

Definition 1.3. at the very end we should remove those definitions that we didn‘t need

Let Σ be an iteration strategy.

(i) Σ has the **Dodd-Jensen property** if for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$ and all $\pi \colon \mathcal{M}_\Sigma \to_{\Sigma_1} N$ we have $\pi^{\vec{\mathcal{T}}}(\alpha) \leq \pi(\alpha)$ for all $\alpha \in o(\mathcal{M}_\Sigma)$.

(ii) Σ has the **positional Dodd-Jensen property** if $\Sigma_{\vec{\mathcal{T}},N}$ has the Dodd-Jensen property for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iii) Σ is **weakly positional** if $\Sigma_{\vec{\mathcal{T}},N} = \Sigma_{\vec{\mathcal{U}},N}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iv) Σ is **positional** if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly positional for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(v) Σ is **weakly commuting** if $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{U}}}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vi) Σ is **commuting** if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly commuting for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vii) Σ is **weakly pullback consistent** if $\Sigma^{\vec{\mathcal{T}}} = \Sigma$ for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(viii) Σ is **pullback consistent** if $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly pullback consistent for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

◦

$$\begin{tikzcd}
 & N \\
\mathcal{M}_\Sigma \arrow[ur, "\pi^{\vec{\mathcal{T}}}"] \arrow[dr, "\pi_c^{\vec{\mathcal{U}}}"'] & \\
 & \mathcal{M}_c^{\vec{\mathcal{U}}} \arrow[uu, "\pi"']
\end{tikzcd}$$

Figure 1.1: Branch condensation

Definition 1.4. If Σ is positional, $\Sigma_{\vec{\mathcal{T}},N}$ doesn't depend on $\vec{\mathcal{T}}$ and hence we simply write Σ_N for this tail strategy. $\circ$

Definition 1.5. An iteration strategy Σ has **branch condensation** (see Figure 1.1) if for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}_\Sigma$ such that

(i) $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ are according to Σ,

(ii) $\vec{\mathcal{U}}$ is a stack of successor length $\gamma + 1$ and $\vec{\mathcal{U}}$'s last component $\mathcal{U}_\gamma$is of limit length,

(iii) $\vec{\mathcal{T}}$ has a last model N such that $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$,

(iv) there is some branch c such that $\pi_c^{\vec{\mathcal{U}}}$ exists and for some $\pi\colon \mathcal{M}_c^{\vec{\mathcal{U}}} \to_{\Sigma_1} N$ we have

$$\pi^{\vec{\mathcal{T}}} = \pi \circ \pi_c^{\vec{\mathcal{U}}}.$$

Then $c = \Sigma(\vec{\mathcal{U}})$. $\circ$

Definition 1.6. $(\mathcal{M}, \mathcal{T})$ **is a hull of** $(\mathcal{N}, \mathcal{U})$ if there are

(i) an embedding, $\pi\colon \mathcal{M} \to_{\Sigma_1} \mathcal{N}$ and

(ii) an order-preserving map $\sigma\colon \operatorname{lh}(\mathcal{T}) \to \operatorname{lh}(\mathcal{U})$

such that

(i) $\alpha \leq_{\mathcal{T}} \beta \iff \sigma(\alpha) \leq_{\mathcal{U}} \sigma(\beta)$

(ii) $[\alpha, \beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}} = \emptyset \iff [\sigma(\alpha), \sigma(\beta)]_{\mathcal{U}} \cap \mathcal{D}^{\mathcal{U}} = \emptyset$,

(iii) $\pi_\alpha\colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_{\sigma(\alpha)}^{\mathcal{U}}$ and $\pi_\alpha(E_\alpha^{\mathcal{T}}) = E_{\sigma(\alpha)}^{\mathcal{U}}$,

(iv) for $\beta < \alpha$ we have $\pi_\alpha \restriction \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1 = \pi_\beta \restriction \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1$,

(v) for $\alpha \leq_{\mathcal{T}} \beta$ with $[\alpha, \beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}}$ we have $\pi_\beta \circ \pi_{\alpha,\beta}^{\mathcal{T}} = \pi_{\sigma(\alpha),\sigma(\beta)}^{\mathcal{U}} \circ \pi_\alpha$,

(vi) if $\beta = \mathrm{pred}_{\mathcal{T}}(\alpha{+}1)$, then $\sigma(\beta) = \mathrm{pred}_{\mathcal{U}}(\sigma(\alpha{+}1))$ and $\pi_{\alpha+1}([a,f]_{E^{\mathcal{T}}_\alpha}) = [\pi_\alpha(a), \pi_\beta(f)]_{E^{\mathcal{T}}_{\sigma(\alpha)}}$ and

(vii) $0 \leq_{\mathcal{U}} \sigma(0)$, $[0,\sigma(0)] \cap \mathcal{D}^{\mathcal{U}} = \emptyset$ and $\pi_0 = \pi^{\mathcal{U}}_{0,\sigma(0)} \circ \pi$,

(See Figure 1.2) ∘

Definition 1.7. Let $\mathcal{M}$,$\mathcal{N}$ be layered hybrid premice and $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ be stacks of normal trees on $\mathcal{M}$,$\mathcal{N}$ respectively. $(\mathcal{M}, \vec{\mathcal{T}})$ **is a hull of** $(\mathcal{N},\vec{\mathcal{U}})$ if there are

(i) an order presercing map $\sigma\colon \mathrm{lh}(\vec{\mathcal{T}}) \to \mathrm{lh}(\vec{\mathcal{U}})$,

(ii) a sequence $(\sigma_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}))$ of order preserving maps $\sigma_\alpha\colon \mathrm{lh}(\mathcal{T}_\alpha) \to \mathrm{lh}(\mathcal{U}_{\sigma(\alpha)})$,

(iii) $(\pi_{\alpha,\beta} \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}) \wedge \beta < \mathrm{lh}(\mathcal{T}_\alpha))$ such that

  (a) $\pi_{0,0} = \pi^{\vec{\mathcal{U}}}_{0,\sigma(0)}$ (so that $\pi_{0,0} = \mathrm{id}$ if $\sigma(0) = 0$),

  (b) for $\alpha < \mathrm{lh}(\vec{\mathcal{T}})$

$$\pi_{\alpha,0}\colon \mathcal{M}^{\vec{\mathcal{T}}}_\alpha \to_{\Sigma_1} \mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}$$

  and $(\mathcal{M}^{\vec{\mathcal{T}}}_\alpha, \mathcal{T}_\alpha)$ is a $(\pi_{\alpha,0},\sigma_0)$-hull of $(\mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}, \mathcal{U}_{\sigma(\alpha)})$,

  (c) $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$ and $\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)}$ exists, then $\pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))}$ exists and

$$\pi_{\beta,\eta} \circ \pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} = \pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))} \circ \pi_{\alpha,\gamma}.$$

(See Figure 1.3) ∘

Definition 1.8. Let $\mathcal{M}$ be a layered hybrid premouse and Σ be a (partial) iteration strategy for $\mathcal{M}$. Σ has **hull condensation** if the following holds true for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}$.

If $\vec{\mathcal{U}}$ is according to Σ and $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$, then $\vec{\mathcal{T}}$ is according to Σ. ∘

Lemma 1.9. *Let Σ be an iteration strategy. Then the following hold true.*

(i) If Σ has hull condensation then it is pullback consistent.

(ii) If Σ is positional and pullback consistent then it is commuting.

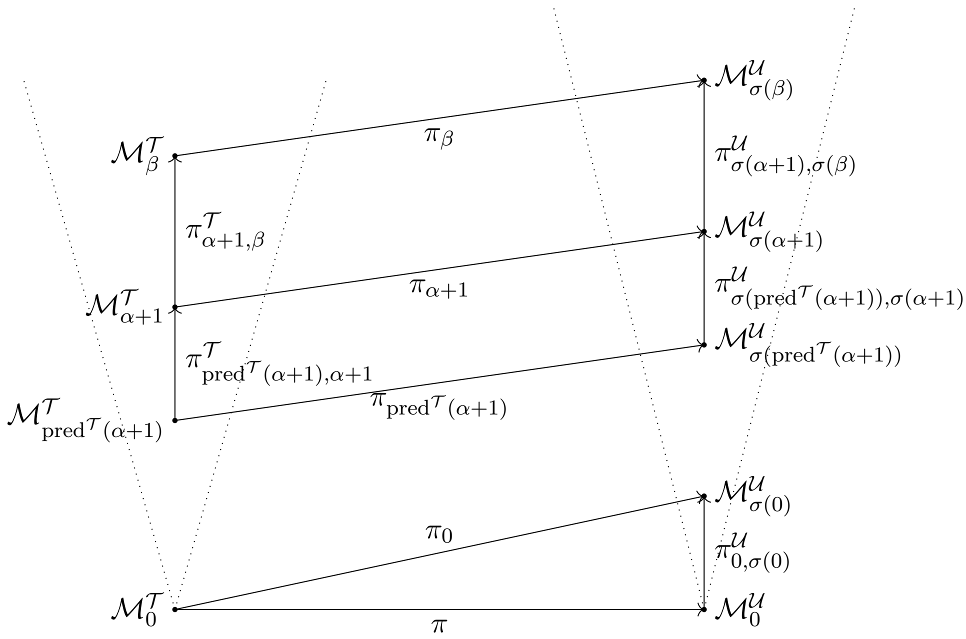

Figure 1.2: $\mathcal{T}$ is a hull of $\mathcal{U}$

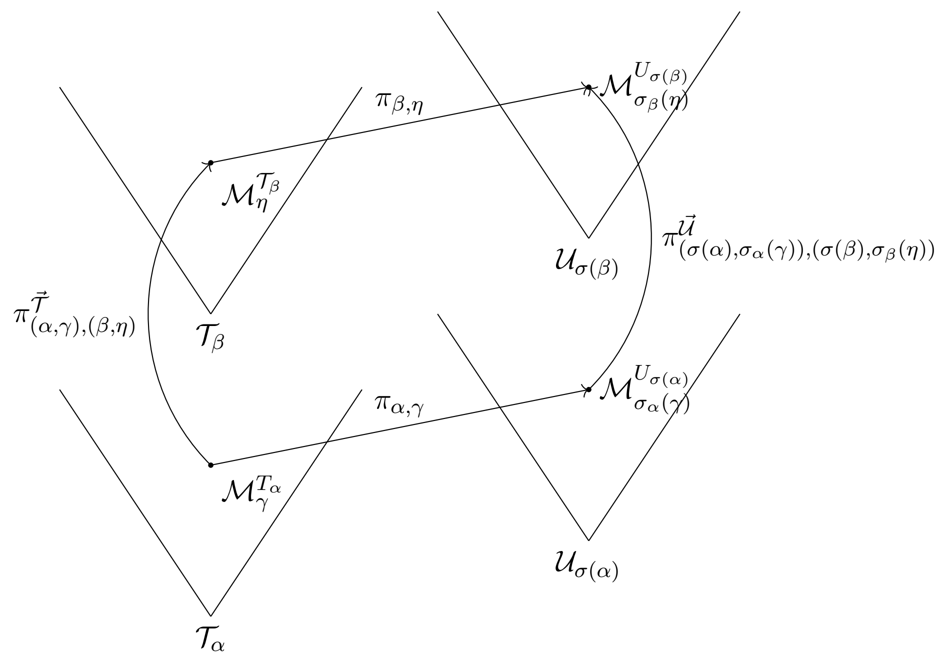

Figure 1.3: $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$

Proof. See [?, Proposition 2.36]. ■

1.1.2 Layered Hybrid Mice

Define strategy mice as a particular kind of hybrid mice, hod mice/pairs and put in positional and commuting in the definition, state comparison. Introduce derived models of hod mice and how they relate to the Solovay hierarchy. Define Σ-mouse

Definition 1.10. Let $\mathcal{M}$ be a transitive set (or structure). We let $o(\mathcal{M}) := \mathcal{M} \cap \mathsf{On}$ be the ordinal height of $\mathcal{M}$. ◦

Definition 1.11. Let $\mathcal{M}$ be a (hybrid) premouse and $\alpha \leq o(\mathcal{M})$. We let

(i) $\mathcal{M}||\alpha$ be the initial segment of $\mathcal{M}$ of height α including its top extender and

(ii) $\mathcal{M}|\alpha$ be the passive initial segment of $\mathcal{M}$ of height α, i.e. $\mathcal{M}||\alpha$ but without the top extender.

◦

Definition 1.12. Let $\mathcal{M}$ be a $\mathcal{J}$-structure [1] and $\alpha \leq o(\mathcal{M})$. We write $\mathcal{J}_\alpha^{\mathcal{M}}$ for the αth level of $\mathcal{M}$'s construction. ◦

Definition 1.13. A **potential layered hybrid premouse** (over X) is an acceptable $\mathcal{J}$-structure of the form $\mathcal{M} = (J_\alpha^{\vec{E},f}(X); \in, \vec{E}, B, f)$ X such that

(i) $\vec{E}$ is a fine extender sequence (over X),

(ii) f is a function with domain $Y \subseteq \alpha$ such that $f(\gamma)$, for each $\gamma \in Y$, is a shift of an amenable function that typically codes part of an iteration strategy for $\mathcal{M}$,

We will often write $\vec{E}^{\mathcal{M}}, f^{\mathcal{M}}, Y^{\mathcal{M}}$ for $\vec{E}, f, Y$ as above. If all proper initial segments of $\mathcal{M}$ are sound, we say that $\mathcal{M}$ is a **layered hybrid premouse**. ◦

[1] See [?] for the basics on $\mathcal{J}$-structures, premice and their fine structure.

In our case, assuming X is a self-well-ordered set, $Y^{\mathcal{M}}$ is determined by the **standard indexing scheme** (see [**?**, Definition 1.18]).

Definition 1.14. Let Σ be a strategy for a layered hybrid premouse $\mathcal{M}$. For $\alpha \leq o(M)$ we let $\Sigma_{\mathcal{M}|\alpha}$ be the id-pullback iteration strategy on $\mathcal{M}|\alpha$ induced by Σ, i.e. a stack $\vec{\mathcal{T}}$ on $\mathcal{M}|\alpha$ is according to $\Sigma_{\mathcal{M}|\alpha}$ iff $\mathrm{id}\vec{\mathcal{T}}$ on $\mathcal{M}$, given by the copy construction via id (see [**?**, 4.1]), is according to Σ. $\circ$

Definition 1.15. A **layered strategy premouse** $\mathcal{M}$ is a layered hybrid premouse such that

(i) $f^{\mathcal{M}}(\gamma)$ codes a partial iteration strategy $\Sigma^{\mathcal{M}}_{\gamma}$ for $\mathcal{M}|\gamma$ and

(ii) For $\gamma_0, \gamma_1 \in Y^{\mathcal{M}}$, if $\gamma_0 < \gamma_1$ then $(\Sigma^{\mathcal{M}}_{\gamma_1})_{\mathcal{M}|\gamma_0} \subseteq \Sigma^{\mathcal{M}}_{\gamma_0}$.

We also write $\Sigma^{\mathcal{M}}$ for the the strategy coded by $f^{\mathcal{M}}$. $\circ$

Definition 1.16. Let $\mathcal{M}$ be a layered strategy premouse and Σ be an iteration strategy for $\mathcal{M}$. $\mathcal{M}$ is a **Σ-premouse** if $\Sigma^{\mathcal{M}} \subseteq \Sigma$. $\circ$

Definition 1.17. Let Σ be an iteration strategy. We write $\mathcal{M}_\Sigma$ for the (layered hybrid) premouse $\mathcal{N}$ such that Σ is an iteration strategy for $\mathcal{N}$. We also let

$I(\mathcal{M}_\Sigma, \Sigma) := \{(\vec{\mathcal{T}}, N) \mid \vec{\mathcal{T}}$ is a stack of normal trees on $\mathcal{M}_\Sigma$ according to Σ,$\pi^{\vec{\mathcal{T}}}$ exist and N is the last mo

and

$$pI(\mathcal{M}_\Sigma, \Sigma) := \{N \mid \exists \vec{\mathcal{T}} \colon (\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)\}. \qquad \dashv$$

$\circ$

Definition 1.18. A Σ-premouse $\mathcal{M}$ is a **Σ-mouse** if there is a $\omega_1 + 1$-iteration strategy Λ such that all $\mathcal{N} \in pI(\mathcal{M}, \Lambda)$, $\mathcal{N}$ are themselves Σ-premice. $\circ$

Definition 1.19. Let a be a transitive self-well-ordered set and let Σ be an iteration strategy with hull-condensation such that $\mathcal{M}_\Sigma \in a$ and let Γ be a pointclass which is closed under Boolean operations and continuous images and preimages. Define the (Γ, Σ)-Lp stack over a recursively as follows:

(i) $\mathrm{Lp}_0^{\Gamma,\Sigma}(a) := a \cup \{a\}$,

(ii) $\mathrm{Lp}_{\alpha+1}^{\Gamma,\Sigma}(a) := \bigcup\{\mathcal{M} \mid \mathcal{M}$ is a sound Σ-mouse over $\mathrm{Lp}_\alpha^{\Gamma,\Sigma}(a)$ projecting to $o(\mathrm{Lp}_\alpha^{\Gamma,\Sigma}(a))$ and

(iii) $\mathrm{Lp}_\lambda^{\Gamma,\Sigma}(a) := \bigcup_{\alpha<\lambda} \mathrm{Lp}^{\Gamma,\Sigma}(a)$ for limit λ.

We also let $\mathrm{Lp}^{\Gamma,\Sigma}(a) := \mathrm{Lp}_1^{\Gamma,\Sigma}(a)$. ◦

1.1.3 Hod mice

Definition 1.20. Suppose $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ is a layered strategic premouse. $\mathcal{P}$ is a HOD-**premouse**[2] provided the following hold: Let $\lambda = \mathrm{otp}(Y^{\mathcal{P}})$, $(\gamma_\beta \mid \beta < \lambda)$ be the strictly increasing enumeration of $Y^{\mathcal{P}}$ and let, for $\beta < \lambda$, $\mathcal{P}(\beta) := \mathcal{P}||\gamma_\beta$ and moreover $\mathcal{P}(\lambda) := \mathcal{P}$. Then there is a continuous, strictly increasing sequence $(\delta_\beta \mid \beta \leq \lambda)$ of $\mathcal{P}$-cardinals such that

(i) $B = \emptyset$,

(ii) $Y^{\mathcal{P}} \subseteq \delta_\lambda$,

(iii) $(\delta_\beta \mid \beta \leq \lambda)$ is sequence of Woodin cardinals and their limits in $\mathcal{P}$ and

(iv) for all $\beta \leq \lambda$

define strong cut-point

(a) δ_β is a strong cutpoint of $\mathcal{P}$,

(b) $\mathcal{P}(\beta) \models \ulcorner\mathsf{ZFC}\text{-Replacement}\urcorner$,

(c) $\mathcal{P}(\beta) = \mathcal{O}_{\delta_\beta}^{\mathcal{P},\omega}$ [3],

(d) if β is a limit then $\delta_\beta^{+\mathcal{P}} = \delta_\beta^{+\mathcal{P}(\beta)}$,

(e) if $\beta < \lambda$ then $f(\gamma_\beta)$ codes a $(o(\mathcal{P}), o(\mathcal{P}))$-strategy, call it $\Sigma_\beta^{\mathcal{P}}$, for $\mathcal{P}(\beta)$ with hull condensation [4],

confirm with Grigor that this is what he had in mind

(f) if $\alpha < \beta < \lambda$, then $(\Sigma_\beta^{\mathcal{P}})_{\mathcal{P}(\alpha)} = \Sigma_\alpha^{\mathcal{P}}$,

(g) if $\beta < \lambda$ and $\eta \in (\delta_\beta, \delta_{\beta+1})$ is a $\mathcal{P}$-successor cardinal, then $\mathcal{P}|\eta$ is a $\Sigma_{\gamma_\beta}^{\mathcal{P}}$-premouse over $\mathcal{P}(\beta)$ which is $(o(\mathcal{P}), o(\mathcal{P}))$-iterable for stacks above δ_β.

(v) $\forall n < \omega\colon \mathcal{P} \models \delta_\lambda^{+n}$ exists and $o(\mathcal{P}) = \sup_{n<\omega}(\delta_\lambda^{+n})^{\mathcal{P}}$.

[2] These are in fact HOD-premice below $\ulcorner\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable$\urcorner$ in [**?**]. However, since all of our HOD-mice are of this form, we omit this.

[3] see [**?**, Definition 1.26]

[4] note that $\Sigma_\beta^{\mathcal{P}} \subseteq \mathcal{P}$ is an internal strategy, i.e. only defined on trees that are elements of $\mathcal{P}$

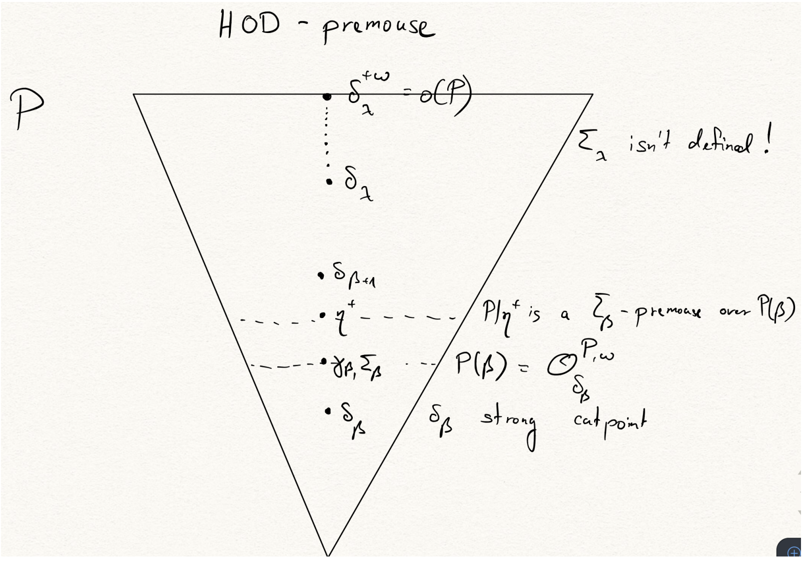

Figure 1.4: HOD premouse

See Figure 1.4. We will often write $\delta_\beta^\mathcal{P}, \gamma_\beta^\mathcal{P}, \lambda^\mathcal{P}$ for $\delta_\beta, \gamma_\beta, \lambda$ as above and moreover let $\delta^\mathcal{P} := \delta_\lambda$. ◦

include an intuitive description of HOD-mice

Definition 1.21. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse. We let

$$\mathcal{P}^- = \begin{cases} P|\gamma_{\lambda^\mathcal{P}-1} & \text{, if } \lambda^\mathcal{P} \text{ is a successor ordinal,} \\ \mathcal{P}|\delta^\mathcal{P} & \text{, otherwise.} \end{cases}$$

See Figure 1.5

add picture and figure out why we don‘t just let $\mathcal{P}^- = \mathcal{P}(\gamma^\mathcal{P}_{\lambda^\mathcal{P}-1})$ in the successor case.

◦

Definition 1.22. Let $\mathcal{P}, \mathcal{Q}$ be HOD-premice. We write $\mathcal{P} \trianglelefteq_{\text{HOD}} \mathcal{Q}$ if there is some $\alpha \leq \lambda^\mathcal{Q}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We also write $\mathcal{P} \triangleleft_{\text{HOD}} \mathcal{Q}$ if $\mathcal{P} \trianglelefteq_{\text{HOD}} \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$.

In this case we say that $\mathcal{P}$ is a (proper) HOD-**initial segment** of $\mathcal{Q}$. ◦

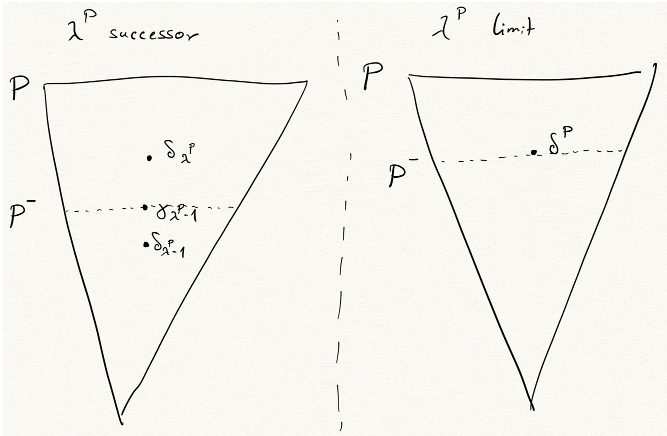

Figure 1.5: $\mathcal{P}^-$

Definition 1.23. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse and $\alpha \leq \lambda^{\mathcal{P}}$.

(i) If $\alpha < \lambda^{\mathcal{P}}$, we let $\Sigma^{\mathcal{P}}_\alpha$ be the internal iteration strategy of $\mathcal{P}(\alpha)$ coded by $f(\alpha)$ and

(ii) $\Sigma^{\mathcal{P}}_{<\alpha} := \bigoplus_{\beta<\alpha} \Sigma^{\mathcal{P}}_\beta$.

We also let $\Sigma^{\mathcal{P}} := \Sigma^{\mathcal{P}}_{<\lambda^{\mathcal{P}}}$. ◦

Remark 1.24. By the agreement of the internal iteration strategies of HOD-premice (item 4f in Theorem 1.20), $\Sigma^{\mathcal{P}}_\alpha$ already includes all of the information of $\Sigma^{\mathcal{P}}_{<\alpha}$ and can be identified with $\Sigma^{\mathcal{P}}_{<\alpha+1}$.

reference broken

Definition 1.25. Let $\mathcal{P}$ be a HOD-premouse. Σ is a (κ, λ)**-iteration strategy** for $\mathcal{P}$ if it is a winning strategy for player II in the iteration game $\mathcal{G}(\mathcal{P}, \kappa, \lambda)$ and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, then $\mathcal{Q}$ is a HOD-premouse such that $\Sigma^{\mathcal{Q}} = \Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \cap \mathcal{Q}$. ◦

add reference

Remark 1.26. In particular, $\Sigma^{\mathcal{P}} = \Sigma \cap \mathcal{P}$, i.e. Σ extends the internal iteration strategy of $\mathcal{P}$.

Definition 1.27. $(\mathcal{P}, \Sigma)$ is a HOD**-pair** if

(i) $\mathcal{P}$ is a HOD-premouse and

(ii) Σ is a $(\omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{P}$ with hull condensation.

the definition of hod pair is different in both versions of Grigor's thesis. Verify that this is the intended one.

○

1.1.4 HOD Analysis

gather all the information we need on HOD – this can be found in Grigor's thesis

Definition 1.28. Let $(P, \Sigma), (Q, \Lambda)$ be HOD-pairs. We let $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$, i.e. there is a $(\mathcal{P}, \Sigma)$-iterate $(\mathcal{T}, R)$ and a $(\mathcal{Q}, \Lambda)$-iterate $(\mathcal{U}, S)$ such that

$$\mathcal{R} \trianglelefteq_{\mathrm{HOD}} \mathcal{S} \text{ and } \Sigma_{\mathcal{R},\mathcal{T}} = \Lambda_{\mathcal{R},\mathcal{U}}.$$

We also let $(\mathcal{P}, \Sigma) <_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ and $(\mathcal{Q}, \Lambda) \not\leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$. ○

Definition 1.29. Let $(\mathcal{P}, \Sigma)$ be a HOD-pair such that Σ has branch condensation and is fullness preserving. We recursively define $\alpha(\mathcal{P}, \Sigma) := |(\mathcal{P}, \Sigma)|_{\leq_{\mathrm{DJ}}} \in \mathsf{On}$ via

$$|(\mathcal{P}, \Sigma)|_{\leq_{\mathrm{DJ}}} = \sup\{|(\mathcal{Q}, \Lambda)|_{\leq_{\mathrm{DJ}}} + 1 \mid (\mathcal{Q}, \Lambda)\text{is a } \mathrm{HOD}\text{-pair such that } \Lambda \text{ has branch condensationand is full}$$

○

Remark 1.30. As in the case of ordinary premice, $\leq_{\mathrm{DJ}}$ (or rather $<_{\mathrm{DJ}}$) is a wellfounded relation. The interesting question is whether it's total.

Theorem 1.31 (Sargsyan)**.** *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are fullness preserving. Then $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ or $(\mathcal{Q}, \Lambda) \leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.*

Proof. [**?**, Theorem 5.10]. ■

Theorem 1.32 (Sargsyan)**.** *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are Γ-fullness preserving for some pointclass Γ which is closed under continuous images and preimages. Suppose further that there is a good pointclass Γ^* such that $\Gamma \cup \{\mathrm{Code}(\Sigma), \mathrm{Code}(\Lambda)\} \subseteq \underset{\sim}{\Delta}_{\Gamma^*}$. Then $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ or $(\mathcal{Q}, \Lambda) \leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.*

define Code Σ

Proof. [?, Theorem 2.33]. ■

Definition 1.33. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. We let

(i) $\mathcal{F}(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \mid \mathcal{Q} \in pB(\mathcal{P}, \Sigma)\}$ and
(ii) $\mathcal{F}^+(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \mid \mathcal{Q} \in pI(\mathcal{P}, \Sigma)\}$.

○

Remark 1.34. By [?, Corollary 2.44] Σ is commuting, so that $\Sigma_{\mathcal{Q}}$ is indeed well-defined.

Definition 1.35. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$. We let $\mathcal{Q} \leq^{\mathcal{P}, \Sigma} \mathcal{R}$ if

(i) $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and $R \in pI(\mathcal{Q}, \Sigma_Q)$ or
(ii) $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \leq_{\mathrm{DJ}} (\mathcal{R}, \Sigma_{\mathcal{R}})$.

○

Lemma 1.36 (Sargsyan)**.** *$\leq^{\mathcal{P}, \Sigma}$ is directed.*

Proof. [?, Lemma 4.17]. ■

Definition 1.37. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-

fullness preserving. Let $\mathcal{Q}, \mathcal{R} \in \mathrm{pI}(\mathcal{P}, \Sigma) \cup \mathrm{pB}(\mathcal{P}, \Sigma)$ be such that, for some $\alpha \leq^{\mathcal{R}}$, $\mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. We let

$$\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q} \to \mathcal{R}(\alpha)$$

be the iteration map given by $\Sigma_{\mathcal{Q}}$. We let

(i) $\mathcal{M}_\infty(\mathcal{P}, \Sigma) = \mathrm{dirlim}(\mathcal{F}(\mathcal{P}, \Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q}, \mathcal{R} \in pB(\mathcal{P}, \Sigma) \wedge \exists \alpha \leq \lambda^{\mathcal{R}} \mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}}))$ and

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma) = \mathrm{dirlim}(\mathcal{F}(\mathcal{P}, \Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \leq^{\Sigma}_{\mathcal{Q},\mathcal{R}} \mathcal{R})$.

For $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ we let

(i) $\pi^{\Sigma}_{\mathcal{Q},\infty} \colon \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$

(ii) $\sigma^{\Sigma}_{\mathcal{R},\infty} \colon \mathcal{R} \to \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$

be the direct limit maps. ○

Definition 1.38. Let $(\mathcal{P}, \Sigma)$ be as above. We let

(i) $\delta_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$,

(ii) $\delta^+_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ and

(iii) $\lambda_\infty(\mathcal{P}, \Sigma) := \lambda^{\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)}$.

○

Lemma 1.39 (Sargsyan)**.** *Let Γ be a pointclass closed under Wadge reducibility. Suppose $(\mathcal{P}, \Sigma)$ is a* HOD*-pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal and Σ has branch condensation and is Γ-fullness preserving. Then*

(i) $\delta_\infty(\mathcal{P}, \Sigma) = \delta^+_\infty(\mathcal{P}, \Sigma)$ *and*

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma) | \delta^+_\infty = \mathcal{M}_\infty(\mathcal{P}, \Sigma)$.

Proof. [**?**, Lemma 4.18]. ■

We will likely not need the entire theorem and should reduce it to the part that we need once we are done.

Theorem 1.40 (Sargsyan)**.** *Assume AD^+, let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Then the following holds:*

(i) If $L(\Gamma, \mathbb{R}) \models \phi$ then for all $(\mathcal{P}, \Sigma) \in \Gamma$ such that $\alpha(\mathcal{P}, \Sigma) < \Omega^\Gamma$ we have, for all $\alpha \leq \alpha(\mathcal{P}, \Sigma)$,

define ϕ

define Ω^Γ

define θ^Γ_α

(a) $\delta_\alpha^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} = \theta^\Gamma_\alpha$ *and*

(b) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}; \in, \vec{E}^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} \restriction \theta^\Gamma_\alpha, \Lambda \restriction \theta^\Gamma_\alpha)$,

where Λ *is the iteration strategy coded by* $f^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}$.

define ψ

(ii) *If* $L(\Gamma,\mathbb{R}) \models \psi$ *then for all* $\alpha \leq \Omega^\Gamma$

(a) $\delta_\alpha^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} = \theta^\Gamma_\alpha$ *and*

(b) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}; \in, \vec{E}^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} \restriction \theta^\Gamma_\alpha, \Lambda \restriction \theta^\Gamma_\alpha)$.

(iii) *Suppose* $\Gamma^* \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \models \mathsf{AD}^+$ *and there is a* HOD*-pair* $(\mathcal{P},\Sigma) \in \Gamma^*$ *such that*

(a) Σ *has branch condensation and is* Γ*-fullness preserving,*

(b) $\lambda^{\mathcal{P}}$ *is a successor ordinal,* $\mathrm{Code}(\Sigma_{\mathcal{P}^-}) \in \Gamma$ *and* $L(\Gamma,\mathbb{R})$ *models that* $(\mathcal{P},\Sigma_{\mathcal{P}^-})$ *is a suitable pair such that* $\alpha(\mathcal{P}^-,\Sigma_{\mathcal{P}^-}) = \alpha$,

define suitable pair

(c) *there is a sequence* $(B_i \mid i < \omega) \subseteq \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *which guides* Σ *and*

define $\mathbb{B}(..)$ and what it means to be guided

(d) *for any* $B \in \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *there is some* $\mathcal{R} \in \mathrm{pI}(\mathcal{P},\Sigma)$ *such that* $\Sigma_{\mathcal{R}}$ *respects* B.

define respects B

Then $L(\Gamma,\mathbb{R}) \models \psi$ *and* $\mathcal{M}_\infty(\mathcal{P},\Sigma) = \mathcal{M}^+_\infty(\mathcal{P},\Sigma)$.

Proof. [?, Theorem 4.24]. ■

1.2 Ω is not zero

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1.3 Ω is not a successor

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1.4 Ω does not have countable cofinality

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1.5 Ω is not singular

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