1 Further questions

Here we record many open questions related to the content of the preceeding chapters, broadly separated by topic.

1.1 Berkeleys

Question 1.7 in [?] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually prestrong cardinal, and Lemmata 2.5 and 2.8 in the same paper also shows that being ω-Erdős is equivalent to being virtually club berkeley and that the least such is also the least virtually berkeley.[1]

Furthermore, they also showed that a non-Σ_2-reflecting virtually prestrong cardinal is equivalent to a virtually prestrong cardinal which isn't virtually strong. We can therefore reformulate their question to the following equivalent question.

Question 1.1 (Wilson). If there exists a virtually prestrong cardinal which is not virtually strong, is there then a virtually berkeley cardinal?

[?] showed that their question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem ?? we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 1.2. If there exists a cardinal κ which is virtually (θ, ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually berkeley cardinal?

Theorem ?? from Chapter ?? at least gives a partially positive result, noting that the assumption by definition implies that On is virtually prewoodin but not virtually woodin.

[1]Note that this also shows that virtually club berkeley cardinals and virtually berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

Corollary 1.3 (N.). *If there exists a virtually A-prestrong cardinal for every class A and there are no virtually strong cardinals, then there exists a virtually berkeley cardinal.* ■

The assumption that there is a virtually A-prestrong cardinal for every class A in the above corollary may seem a bit strong, but Theorem ?? shows that this is necessary, which might lead one to think that the question could have a negative answer.

1.2 Relations between virtuals

The analysis in Chapter ?? showed several implication and separation results between the virtual large cardinals. A few of these relations remain open, however.

Question 1.4. Are virtually θ-strong cardinals, virtually θ-supercompacts and virtually θ-Magidor-supercompacts all equivalent, for any uncountable regular cardinal θ?

Question 1.5. Let θ be an uncountable cardinal.

(i) Is every virtually θ-measurable cardinal also virtually θ-prestrong? What if we assume $V = L[\mu]$ or $V = K$, with K being the core model below a woodin cardinal?
(ii) Is every virtually θ-strong cardinal virtually θ-supercompact? Are they equiconsistent?

1.3 Indestructibility

Our original goal concerning indestructibility was to see what indestructibility properties the faintly supercompacts have, whether any analogy with the supercompact cardinals holds. This still remains open.

Question 1.6. Do faintly supercompact cardinals have indestructibility properties? For instance, if κ is faintly supercompact, does it remain supercompact after forcing with $\mathrm{Add}(\kappa, 1)$?

We proved several indestructibility properties of the ostensibly stronger notion of *generically setwise supercompacts*, and several questions then arise concerning the nature of these cardinals.

Question 1.7. What's the consistency strength of the generically setwise supercompact cardinals? The best upper bound is a virtually extendible, as given by Usuba's Theorem ??, and a lower bound is the trivial faintly supercompact one. What if we require the cardinal to be inaccessible?

Question 1.8. Is it consistent to have a faintly supercompact cardinal which isn't generically setwise supercompact?

Question 1.9. Assume there exists no inner model with a woodin cardinal. Can there then exist generically setwise supercompact cardinals in K?

1.4 Games and small embeddings

Our results in Chapter ?? provide answers to the following questions, which were posed in [?].

(i) If γ is an uncountable cardinal and the challenger does not have a winning strategy in the game $\mathcal{G}^\theta_\gamma(\kappa)$, does it follow that the judge has one?

(ii) If $\omega \leq \alpha \leq \kappa$, are α-Ramsey cardinals downwards absolute to the Dodd-Jensen core model?

(iii) Does 2-iterability imply ω-Ramseyness, or conversely?

(iv) Does κ having the strategic κ-filter property have the consistency strength of a measurable cardinal?

Here the "challenger" is player I and the "judge" is player II, so this is asking if every γ-Ramsey is strategic γ-Ramsey, when γ is an uncountable cardinal. Theorem ?? therefore gives a negative answer to (i) for all uncountable ordinals γ. Theorem ?? and Corollary ?? answer (ii) positively, for α-Ramseys with α having uncountable cofinality, and for $<\alpha$-Ramseys when α is a limit of limit ordinals. Note that (ii) in the $\alpha = \omega$ case was answered positively in [?].

As for (iii), it's mentioned in [?] that Gitman has showed that ω-Ramseys are not in general 2-iterable by showing that 2-iterables have strictly stronger consistency strength than the ω-Ramseys, which also follows from Theorem ?? and Theorem 4.8 in [?]. Corollary ?? shows that ω-Ramsey cardinals are Δ^2_0-indescribable, and as 2-iterables are (at least) Π^1_3-definable it holds that any 2-iterable ω-Ramsey cardinal is a limit of 2-iterables, so that in general 2-iterables can't be ω-Ramsey either, answering (iii) in the negative. Lastly, Theorem ?? gives a positive answer to (iv).

We conjecture the following two questions to be true. The first is a direct analogue to Theorem ??, and the latter is a suspected analogy between the genuine n-Ramsey cardinals and the weakly ineffable cardinals.

Question 1.10. If κ is faintly θ-power-measurable, does player II then have a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$?

Question 1.11. Are genuine n-Ramsey cardinals limits of n-Ramsey cardinals? We conjecture this to be true, in analogy with the weakly ineffables being limits of weakly compacts. Since "weakly ineffable = Π^1_1-indescribability + subtlety", this might involve some notion of "n-iterated subtlety". The difference here is that n-Ramseys cannot be *equivalent* to Π^1_{2n+1}-indescribables for consistency reasons, so there is some work to be done.

We showed in Theorem ??, see also Corollary ?? that completely ineffable cardinals could be characterised in terms of player II having a winning strategy in $\mathcal{G}^-_\omega(\kappa)$. This lends itself to the following question.

Question 1.12. Are there higher analogues of ineffability which are equivalent to player II having a winning strategy in $\mathcal{G}^-_\alpha(\kappa)$ for $\alpha > \omega$?

1.5 Ideal absoluteness

One can ask of any poset property whether it is ideal-absolute, but we choose to only highlight one particular property here. We saw in Corollary ?? that $<\lambda$-closed faintly power-measurables "corresponds to" (κ, κ)-distributive $<\lambda$-closed forcings, and in Corollary ?? that completely ineffable cardinals "corresponds to" (κ, κ)-distributive forcings. In an attempt to find the forcing that corresponds to the faintly power-measurables, we arrive at the following question.

Question 1.13. For κ a regular cardinal, are the following equivalent?

(i) κ is faintly power-measurable;

(ii) κ is ideally power-measurable;

(iii) κ is (κ, κ)-distributive ω-distributive faintly measurable;

(iv) κ is (κ, κ)-distributive ω-distributive ideally measurable;

(v) Player II has a winning strategy in $\mathcal{G}_\omega(\kappa)$.