1 Large Cardinals

Since large cardinals came into existence in the beginning of the 20th century, a vast zoo of different types of such have appeared. The aim of this appendix is to act as a reference for the definitions of these as well as the relations between them.

1.1 Inaccessibles

Definition 1.1. A cardinal κ is **regular** if $\operatorname{cof}\kappa = \kappa$; i.e. that there are no $\gamma < \kappa$ with a cofinal function $f\colon \gamma \to \kappa$. κ is a **strong limit** if $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$. If κ is both regular and a strong limit then we say that it is (strongly) **inaccessible**. ○

Proposition 1.2 ([?] Proposition 1.2). *If κ is inaccessible then $(V_\kappa, \in) \models$ ZFC.* ■

Gödel's Second Incompleteness Theorem from [?] then immediately implies the following corollary.

Corollary 1.3. *ZFC can not prove the existence of any inaccessible cardinals. Indeed, not even the consistency of the existence of any inaccessible cardinals can be proven in ZFC.* ■

1.2 Weakly compacts

Definition 1.4. For any function $f\colon A \to B$, a subset $H \subseteq A$ is **homogeneous for f** if $f \upharpoonright H$ is a constant function. ○

Definition 1.5. Let κ and λ be infinite cardinals, γ an ordinal and $n < \omega$. Then the partition relation $\kappa \to (\lambda)^n_\gamma$ holds if to every function $f\colon [\kappa]^n \to \gamma$ there exists a subset $H \subseteq [\kappa]^n$ of size λ which is homogeneous for f. If $\gamma = 2$ then we usually leave it out and simply write $\kappa \to (\lambda)^n$. ○

Definition 1.6. An uncountable cardinal κ is **weakly compact** if $\kappa \to (\kappa)^2$. ○

Theorem 1.7 ([?] Lemma 9.9). *Every weakly compact cardinal is a limit of inaccessible cardinals.* ■

1.3 Ineffables and completely ineffables

Definition 1.8. An uncountable cardinal κ is **ineffable** if to any function $f\colon [\kappa]^2 \to 2$ there exists a *stationary* $H \subseteq [\kappa]^2$ which is homogeneous for f. ○

Ineffable cardinals are weakly compact by definition, and the following theorem from [?] shows that they are strictly stronger.

Theorem 1.9 (Friedman). *Ineffable cardinals are weakly compact limits of weakly compacts.* ■

A way of improving ineffability is to "close under homogeneity", in the sense that if H is homogeneous for $f\colon [\kappa]^2 \to 2$ and $g\colon [H]^2 \to 2$ is any function, then there is a subset of H which is homogeneous for g. To formalise this notion we use the concept of a *stationary class*.

Definition 1.10. For X any set, a collection $\mathcal{R} \subseteq \mathscr{P}(X)$ is a **stationary class** if

- $\mathcal{R} \neq \emptyset$;
- Every $A \in \mathcal{R}$ is a stationary subset of X;
- If $A \in \mathcal{R}$ and $B \supseteq A$ then $B \in \mathcal{R}$. ○

Definition 1.11. An uncountable cardinal κ is **completely ineffable** if there is a stationary class $\mathcal{R} \subseteq \mathscr{P}(\kappa)$ such that for every $A \in \mathcal{R}$ and $f\colon [A]^2 \to 2$ there exists a $H \in \mathcal{R}$ which is homogeneous for f. ○

As suspected, these completely ineffable cardinals are indeed strictly stronger than the ineffables, as the following theorem from [?] shows.

Theorem 1.12 (Abramson et al). *Completely ineffable cardinals are ineffable limits of ineffable cardinals.* ■

1.4 Measurables, strongs and supercompacts

Definition 1.13. For two first-order structures $\mathcal{M}$ and $\mathcal{N}$ with underlying sets M and N, an **elementary embedding** $j\colon \mathcal{M} \to \mathcal{N}$ between them is a function $j\colon M \to N$ such that, for any first-order formula $\varphi(v_1, \dots, v_n)$ and sets $x_1, \dots, x_n \in \mathcal{M}$ it holds that $\mathcal{M} \models \varphi[x_1, \dots, x_n]$ iff $\mathcal{N} \models \varphi[j(x_1), \dots, j(x_n)]$. ∘

As elementary embeddings in particular preserve equality, they are always injective. Identity embeddings are of course always elementary, so we say that an elementary embedding is **non-trivial** if it is not the identity. The following then shows that in most situations these non-trivial embeddings can be associated to a unique ordinal.

Proposition 1.14 ([?] Propostion 5.1). *If $j\colon (\mathcal{M}, \in) \to (\mathcal{N}, \in)$ is an elementary embedding such that $\mathcal{M}$ is transitive and either $\mathcal{N} \subseteq \mathcal{M}$ or $\mathcal{M} \models$ ZFC, then there exists an ordinal $\alpha < o(\mathcal{M})$ moved by j, i.e. that $j(\alpha) \neq \alpha$. We call the least such ordinal the **critical point** of j, and denote it by* $\operatorname{crit} j$. ■

Definition 1.15 (GBC). An uncountable cardinal κ is **measurable** if there exists a transitive class $\mathcal{M}$ and an elementary embedding $j\colon (V, \in) \to (\mathcal{M}, \in)$ with critical point κ. ∘

The measurable cardinals were the first large cardinals shown to "transcend L".

Theorem 1.16 (Scott's Theorem, [?] Corollary 5.5). *L, Gödel's constructible universe, has no measurable cardinals.* ■

Given this result, it's not surprising that the measurables then exceed the strength of the previous large cardinals.

Proposition 1.17. *Measurable cardinals are completely ineffable limits of completely ineffable cardinals.*

Proof. (Sketch) If $j\colon V \to \mathcal{M}$ is a non-trivial elementary embedding then the **derived ultrafilter** $\mu \subseteq \mathscr{P}(\kappa)$ on $\kappa := \operatorname{crit} j$ is defined as $X \in \mu$ iff $\kappa \in j(X)$. Section 5 in [?] shows that it is indeed an ultrafilter and that its ultrapower $\operatorname{Ult}(V, \mu)$

is wellfounded. A reflection argument then shows that we can simply take $\mathcal{R} := \mu$. ■

Definition 1.18 (GBC). An uncountable cardinal κ is **strong** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$ satisfying that $H_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta \colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ. We say that κ is **θ-strong** if the property holds for a specific θ. ○

Proposition 1.19 ([?] 26.6). *Strong cardinals are measurable limits of measurable cardinals.* ■

Definition 1.20 (GBC). An uncountable cardinal κ is **supercompact** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$ satisfying that ${}^{<\theta}\mathcal{M}_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta \colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ. ○

Proposition 1.21. *If κ is supercompact then*

$$V_\kappa \models \ulcorner\text{There exists a proper class of strong cardinals}\urcorner. \tag{1}$$

Proof. (Sketch) By noting that the restrictions of the supercompact embedding is an element of the target model by supercompactness, κ is strong in the target model, so that a reflection argument shows (1). ■

1.5 Woodins and Vopěnkas

Definition 1.22. Let A be any set. An uncountable cardinal κ is **A-strong** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$ satisfying that $H_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta \colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ, such that $A \cap H_\theta = j(A) \cap H_\theta$. ○

Definition 1.23. An uncountable cardinal δ is a **Woodin cardinal** if there to every subset $A \subseteq H_\delta$ exists $\kappa < \delta$ such that $(H_\delta, \in, A) \models \ulcorner\kappa \text{ is } A\text{-strong}\urcorner$. ○

Theorem 1.24 ([?] Theorem 26.14). *The following are equivalent for an uncountable cardinal κ.*

(i) *κ is a Woodin cardinal;*

(ii) *For any $f\colon \kappa \to \kappa$ there exists $\alpha < \kappa$ such that $f[\alpha] \subseteq \alpha$, a transitive $\mathcal{M}$ with $V_{j(f)(\alpha)} \subseteq \mathcal{M}$ and an elementary embedding $j\colon (V, \in) \to (\mathcal{M}, \in)$ with $\operatorname{crit} j = \kappa$.*

Definition 1.25 (GBC). **Vopěnka's Principle (VP)** postulates that to any first-order language $\mathcal{L}$ and proper class $\mathcal{C}$ of $\mathcal{L}$-structures, there exist distinct $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ and an elementary embedding $j\colon\ \mathcal{M} \to \mathcal{N}$. ∘

Definition 1.26. An uncountable cardinal δ is **Vopěnka** if $(V_\delta, \in; V_{\delta+1}) \models \mathsf{VP}$. ∘

The following theorem is from [?].

Theorem 1.27 (Perlmutter). *Vopěnka cardinals are equivalent to cardinals that are "Woodin for supercompactness", meaning a cardinal δ such that to any subset $A \subseteq H_\delta$ there is a cardinal $\kappa < \delta$ such that $(H_\delta, \in, A) \models \ulcorner\kappa$ is A-supercompact$\urcorner$.*[1]

■

1.6 Reinhardts and Kunen inconsistency

Definition 1.28 (GBC). An uncountable cardinal κ is a **Reinhardt cardinal** if there exists an elementary embedding $j\colon (V, \in) \to (V, \in)$ with $\operatorname{crit} j = \kappa$. ∘

Theorem 1.29 (Kunen inconsistency, GBC, [?] Theorem 23.12). *There are no Reinhardt cardinals. Even more, there is no non-trivial elementary $j\colon (V_{\lambda+2}, \in) \to (V_{\lambda+2}, \in)$ for any uncountable cardinal λ.*

When we're dealing with the *virtual* large cardinals in Chapter ?? we show that the property $j(\kappa) > \theta$ is a highly non-trivial assumption. However, when we're not in the virtual world then this is simply automatic.

[1]Here κ is, in analogy with Definition 1.22, A-**supercompact** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$, closed under $<\theta$-sequences, and an elementary $j_\theta\colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ, such that $A \cap H_\theta = j(A) \cap H_\theta$.

Proposition 1.30 ([?] 26.7). *If $j\colon V \to \mathcal{M}_\theta$ witnesses that $\kappa := \operatorname{crit} j$ is a θ-strong cardinal then $j(\kappa) > \theta$.* ■

Note that a crucial part of the proof of the above is Corollary 23.14 in [?], which relies *heavily* on the Kunen inconsistency. There is also the following even stronger version of the Reinhardts.

Definition 1.31. An uncountable cardinal κ is **super Reinhardt** if for all ordinals λ there exists an elementary embedding $j\colon (V, \in) \to (V, \in)$ with $\operatorname{crit} j = \kappa$ and $j(\kappa) > \lambda$. ○

1.7 Berkeleys

Definition 1.32 (GB). An uncountable cardinal δ is a **proto-Berkeley cardinal** if to every transitive *set* $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists an elementary embedding $j\colon (\mathcal{M}, \in) \to (\mathcal{M}, \in)$ with $\operatorname{crit} j < \delta$. ○

Note that if κ is a proto-Berkeley cardinal then every $\lambda > \kappa$ is also proto-Berkeley, which makes it quite an uninteresting notion. But we can isolate the interesting cases, leading to the definition of a Berkeley cardinal. The following is Theorem 2.1.14 in [?].

Theorem 1.33 (Cutolo). *If δ_0 is the least proto-Berkeley cardinal then we can choose the critical point of the embedding to be arbitrarily large below δ_0.* ■

As this property is clearly not preserved upwards, this makes for a good candidate for the large cardinal notion.

Definition 1.34 (GB). A proto-Berkeley cardinal δ is **Berkeley** if we can choose the critical point of the embedding to be arbitrarily large below δ. If we furthermore can choose the critical point as an element of any club $C \subseteq \delta$ then we say that δ is **club Berkeley**. ○

In [?], they furthermore mention that, among the above-mentioned cardinals, the non-trivial relative consistency implications currently known are the following, being Theorem 2.2.1 and 2.2.2 in [?], respectively.

Theorem 1.35 (Cutolo). *Berkeley cardinals are consistency-wise strictly stronger than Reinhardt cardinals.* ■

Theorem 1.36 (Cutolo). *Club Berkeley cardinals are consistency-wise strictly stronger than super Reinhardt cardinals.* ■