1 Set-theoretic connections

Moving away from the pure theory of the virtual large cardinals from Chapter ??, we now move to connections between these large cardinals and common set-theoretic objects of study: filters, games and ideals.

1.1 Filters & Games

This section covers the content in the paper [?], which started out as a further analysis of the results in [?] and somewhat surprisingly we ended up in the realm of virtual large cardinals. As is custom in Mathematics, we will pretend that this was the goal all along.

We will in this section be dealing with many properties of $\mathcal{M}$-measures[1], so we start with a couple of definitions.

Definition 1.1. Let κ be a cardinal, $\mathcal{M}$ a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

- $\mathcal{M}$**-normal** if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \triangle\vec{X} \in \mu$;
- **genuine** if $|\triangle\vec{X}| = \kappa$ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- **normal** if $\triangle\vec{X}$ is stationary in κ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- 0-**good**, or simply **good**, if it has a well-founded ultrapower;
- α-**good** for $\alpha > 0$ if it is weakly amenable and has α-many well-founded iterates.

$\circ$

Note that a genuine $\mathcal{M}$-measure is $\mathcal{M}$-normal and countably complete, and a countably complete weakly amenable $\mathcal{M}$-measure is α-good for all ordinals α.

We will also be employing the following well-known result regarding set-sized embeddings.

[1] See the beginning of Chapter ?? for the definitions of weak κ-models $\mathcal{M}$ and their associated $\mathcal{M}$-measures.

Lemma 1.2 (Ancient Kunen Lemma). *Let κ be regular, $\mathcal{M},\mathcal{N}$ weak κ-models, $\theta \in (\kappa, o(\mathcal{M}))$ a regular $\mathcal{M}$-cardinal, and $\pi\colon \mathcal{M} \to \mathcal{N}$ an elementary embedding with $\operatorname{crit}\pi = \kappa$ and $H_\theta^{\mathcal{M}} \subseteq \mathcal{N}$. Then for every $X \in H_\theta^{\mathcal{M}}$ with $\operatorname{card}^{\mathcal{M}}(X) = \kappa$ it holds that $\pi \restriction X \in \mathcal{N}$.*

Proof. Let $f\colon \kappa \to X$, $f \in \mathcal{M}$, be a bijection and note that $\pi(x) = \pi(f)(f^{-1}(x))$ for all $x \in X$, so it suffices that $f, \pi(f) \in \mathcal{N}$, which is true since $f \in H_\theta^{\mathcal{M}} \subseteq \mathcal{N}$. ■

In [?] they provide the following characterisation of the normal measures.

Lemma 1.3 (Holy-Schlicht). *Let $\mathcal{M}$ be a weak κ-model and μ and $\mathcal{M}$-measure. Then μ is normal iff $\triangle\vec{X}$ is stationary for some enumeration $\vec{X}$ of μ.*

Proof. $(\Rightarrow)$ is trivial, so assume that $\vec{X}$ is an enumeration of μ such that $\triangle\vec{X}$ is stationary. Let $\vec{Y} \in {}^\kappa\mu$ be a κ sequence and define $g\colon \kappa \to \kappa$ such that $Y_\alpha = X_{g(\alpha)}$ for $\alpha < \kappa$. Letting $C_g \subseteq \kappa$ be the club of closure points of g we get that $\triangle\vec{X} \cap C_g \subseteq \triangle\vec{Y} \cap C_g$, making $\triangle\vec{Y}$ stationary. ■

The α-Ramsey cardinals in [?] are based upon the following game[2].

Definition 1.4 (Holy-Schlicht). For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, a limit ordinal $\gamma \leq \kappa$ and a regular cardinal $\theta > \kappa$ define the game $wfG_\gamma^\theta(\kappa)$ of length γ as follows.

$$\begin{array}{c|cccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \text{II} & & \mu_0 & & \mu_1 & & \mu_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a κ-model and μ_α is a filter for all $\alpha < \gamma$, such that μ_α is an $\mathcal{M}_\alpha$-measure, the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing and $\langle \mathcal{M}_\xi \mid \xi < \alpha\rangle, \langle \mu_\xi \mid \xi < \alpha\rangle \in \mathcal{M}_\alpha$ for every $\alpha < \gamma$. Letting $\mu := \bigcup_{\alpha<\gamma} \mu_\alpha$ and $\mathcal{M} := \bigcup_{\alpha<\gamma} \mathcal{M}_\alpha$, player II wins iff μ is an $\mathcal{M}$-normal good $\mathcal{M}$-measure. ◦

We will also be using the following fact from [?, Lemma 3.3], that the games $wfG_\gamma^\theta(\kappa)$ do not depend upon the values of θ:

[2] See Appendix ?? for a refresher on infinite game theory.

Lemma 1.5 (Holy-Schlicht). *Let γ be a limit ordinal with* $\operatorname{cof}\gamma \neq \omega$. *Then $wfG_\gamma^{\theta_0}(\kappa)$ and $wfG_\gamma^{\theta_1}(\kappa)$ are equivalent for any regular $\theta_0, \theta_1 > \kappa$.* ■

We will be working with a variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I but more of player II. It will turn out that this change of game is innocuous, as Proposition 1.8 will show that they are equivalent.

Definition 1.6 (Holy-N.-Schlicht). Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following game $\mathcal{G}_\gamma^\theta(\kappa, \zeta)$ with $(\gamma{+}1)$-many rounds:

I	$\mathcal{M}_0$		$\mathcal{M}_1$		$\cdots$		$\mathcal{M}_\gamma$	
II		μ_0		μ_1		$\cdots$		μ_γ

Here $\mathcal{M}_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leq \gamma$, μ_α is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$, μ_γ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure and the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore require that $\mathcal{M}_\alpha = \bigcup_{\xi<\alpha} \mathcal{M}_\xi$, $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$ and that μ_α is ζ-good. Player II wins iff they could continue to play throughout all $(\gamma{+}1)$-many rounds. ∘

For convenience we will write $\mathcal{G}_\gamma^\theta(\kappa)$ for the game $\mathcal{G}_\gamma^\theta(\kappa, 0)$, and $\mathcal{G}_\gamma(\kappa)$ for $\mathcal{G}_\gamma^\theta(\kappa)$ whenever $\operatorname{cof}\gamma \neq \omega$, as again the existence of winning strategies in these games doesn't depend upon a specific θ. Note that we assume that $\kappa = \kappa^{<\kappa}$ is uncountable in the definition of the games that we're considering, so this is a standing assumption throughout this section, whenever any one of the above two games are considered.

Definition 1.7. Define the **Cohen game** $\mathcal{C}_\gamma^\theta(\kappa)$ as $\mathcal{G}_\gamma^\theta(\kappa)$ but where we require that $|\mathcal{M}_\alpha - H_\kappa| < \gamma$ for every $\alpha < \gamma$, i.e. that we only allow player I to add $<\gamma$ new elements to the models in each round, and where we only require $\mathcal{M}_\alpha \models \mathsf{ZFC}^-$ and $\mathcal{M}_\alpha \prec H_\theta$ for $\alpha \leq \gamma$ limit.[3]

Also define the **weak Cohen game** $\mathcal{C}_\gamma^-(\kappa)$ in analogy with $\mathcal{G}_\gamma^-(\kappa)$. ∘

[3] $\mathcal{C}_\omega^\theta(\kappa)$ is similar to the $H(F, \lambda)$-games in [?].

Proposition 1.8 (N.). *Assume $\gamma^{\aleph_0} = \gamma$ and let κ be regular. Then $\mathcal{C}^-_\gamma(\kappa)$ is equivalent to $\mathcal{C}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. In particular, if CH holds then $\mathcal{C}^-_{\omega_1}(\kappa)$ is equivalent to $\mathcal{C}^\theta_{\omega_1}(\kappa)$ for all regular $\theta > \kappa$.*

Proof. The assumption that $\gamma^{\aleph_0} = \gamma$ allows us to ensure that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ for all $\alpha < \gamma$. If player I has a winning strategy in $\mathcal{C}^\theta_\gamma(\kappa)$ for some regular $\theta > \kappa$ then they still win if we require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ (since they're only enlargening their models, making it even harder for player II to win), in which case the final measure μ_γ is countably complete and hence automatically has a wellfounded ultrapower.

If player II has a winning strategy in $\mathcal{C}^-_\gamma(\kappa)$ then they still win if player I plays $\mathcal{M}_\alpha$ such that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$, again ensuring that μ_γ has a wellfounded ultrapower. ■

Proposition 1.9 (Holy-N.-Schlicht). *$\mathcal{G}^\theta_\gamma(\kappa)$, $\mathcal{G}^\theta_\gamma(\kappa, 1)$ and $wfG^\theta_\gamma(\kappa)$ are all equivalent for all limit ordinals $\gamma \leq \kappa$, and $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$ whenever $\operatorname{cof}\gamma > \omega$ and $\zeta \in On$.*

Proof. We start by showing the latter statement, so assume that $\operatorname{cof}\gamma > \omega$. Consider now the auxilliary game, call it $\mathcal{G}$, which is exactly like $\mathcal{G}^\theta_\gamma(\kappa, 0)$, but where we also require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ and $\langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle, \langle \mu_\xi \mid \xi \leq \alpha\rangle \in \mathcal{M}_{\alpha+1}$ for every $\alpha < \gamma$.

Claim 1.10. $\mathcal{G}$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$.

Proof of claim. If player I has a winning strategy in $\mathcal{G}$ then they also have one in $\mathcal{G}^\theta_\gamma(\kappa)$, by doing exactly the same. Analogously, if player II has a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have one in $\mathcal{G}$. If player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ then we can construct a winning strategy σ' in $\mathcal{G}$, which is defined as follows. Fix some $\alpha \leq \gamma$ and, writing $\vec{\mathcal{M}}_\xi := \langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle$ and $\vec{\mu}_\xi := \langle \mu_\xi \mid \xi \leq \alpha\rangle$, we set

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) := \mathrm{Hull}^{H_\theta}(\sigma(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) \cup {}^\omega\mathcal{M}_\alpha \cup \{\vec{\mathcal{M}}_\xi, \vec{\mu}_\xi\}),$$

i.e. that we're simply throwing in the sequences into our models and making sure that we're still an elementary substructure of H_θ. This new strategy σ' is clearly winning. Assuming now that τ is a winning strategy for player II in $\mathcal{G}$, we define a winning strategy τ' for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ by letting $\tau'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle)$ be the result of throwing in the appropriate sequences into the models $\mathcal{M}_\xi$, applying τ to get a measure, and intersecting that measure with $\mathcal{M}_\alpha$ to get an $\mathcal{M}_\alpha$-measure. $\dashv$

Now, letting $\mathcal{M}_\gamma$ be the final model of a play of $\mathcal{G}$, $\operatorname{cof}\gamma > \omega$ implies that any ω-sequence $\vec{X} \in \mathcal{M}_\gamma$ really is a sequence of elements from some $\mathcal{M}_\xi$ for $\xi < \gamma$, so that $\vec{X} \in \mathcal{M}_{\xi+1}$ by definition of $\mathcal{G}$, making $\mathcal{M}_\gamma$ closed under ω-sequences and thus also μ_γ countably complete. Since γ is a limit ordinal and the models contain the previous measures and models as elements, the proof of e.g. Theorem 5.6 in [?] shows that μ_γ is also weakly amenable, making it ζ-good for all ordinals ζ.

Now we deal with the first statement, so fix a limit ordinal γ. Firstly $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa, 1)$ as above, since both are equivalent to the auxilliary game $\mathcal{G}$ when γ is a limit ordinal. So it remains to show that $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $wfG^\theta_\gamma(\kappa)$. If player I has a winning strategy σ in $wfG^\theta_\gamma(\kappa)$ then define a winning strategy σ' for player I in $\mathcal{G}^\theta_\gamma(\kappa)$ as

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle) := \sigma(\langle \mathcal{M}_0, \mu_0 \rangle^\frown \langle \mathcal{M}_{\xi+1}, \mu_{\xi+1} \mid \xi + 1 \leq \alpha \rangle)$$

and for limit ordinals $\alpha \leq \gamma$ set $\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi < \alpha \rangle) := \bigcup_{\xi<\alpha} \mathcal{M}_\xi$; i.e. they simply follow the same strategy as in $wfG^\theta_\gamma(\kappa)$ but plugs in unions at limit stages. Likewise, if player II had a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have a winning strategy in $wfG^\theta_\gamma(\kappa)$, this time just by skipping the limit steps in $\mathcal{G}^\theta_\gamma(\kappa)$.

Now assume that player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ and that player I *doesn't* have a winning strategy in $wfG^\theta_\gamma(\kappa)$. Then define a strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$ as follows. Let $s = \langle \mathcal{M}_\alpha, \mu_\alpha \mid \alpha \leq \eta \rangle$ be a partial play of $wfG^\theta_\gamma(\kappa)$ and let s' be the modified version of s in which we have 'inserted' unions at limit steps, just as in the above paragraph. We can assume that every μ_α in s' is good and $\mathcal{M}_\alpha$-normal as otherwise player II has already lost and player I can play anything. Now, we want to show that s' is a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$. All the models in s are κ-models, so in particular weak κ-models.

Claim 1.11. Every μ_α in s' is normal.

PROOF OF CLAIM. Assume without loss of generality that $\alpha = \eta$. Let player I play any legal response $\mathcal{M}$ to s in $wfG^\theta_\gamma(\kappa)$ (such a response always exists). If player II can't respond then player I has a winning strategy by simply following $s^\frown\langle\mathcal{M}\rangle$, $\lightning$, so player II *does* have a response μ to $s^\frown\mathcal{M}$. But now the rules of $wfG^\theta_\gamma(\kappa)$ ensures that $\mu_\eta \in \mathcal{M}$, so since

$$(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \ulcorner \triangle \vec{X} \text{ is stationary in } \kappa \urcorner,$$

we then also get that $\mathcal{M} \models \ulcorner \triangle\mu_\eta$ is stationary in $\kappa\urcorner$ since $\mu_\eta \subseteq \mu$, so elementarity of $\mathcal{M}$ in H_θ implies that $\triangle\mu_\eta$ really *is* stationary in κ, making μ_η normal. $\dashv$

This makes s' a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$, so we may form the weak κ-model $\tilde{\mathcal{M}}_\eta := \sigma(s')$. Now let $\mathcal{M}_\eta \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_\eta \subseteq \mathcal{M}_\eta$ and $s \in \mathcal{M}_\eta$ and set $\sigma'(s) := \mathcal{M}_\eta$. This defines the strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$, which is winning since the winning condition for the two games is the same for γ a limit.[4]

Next, assume that player II has a winning strategy τ in $wfG^\theta_\gamma(\kappa)$. We recursively define a strategy $\tilde{\tau}$ for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ as follows. If $\tilde{\mathcal{M}}_0$ is the first move by player I in $\mathcal{G}^\theta_\gamma(\kappa)$, let $\mathcal{M}_0 \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_0 \subseteq \mathcal{M}_0$, making $\mathcal{M}_0$ a valid move for player I in $wfG^\theta_\gamma(\kappa)$. Write $\mu_0 := \tau(\langle\mathcal{M}_0\rangle)$ and then set $\tilde{\tau}(\langle\tilde{\mathcal{M}}_0\rangle)$ to be $\tilde{\mu}_0 := \mu_0 \cap \tilde{\mathcal{M}}_0$, which again is normal by the same trick as above, making $\tilde{\mu}_0$ a legal move for player II in $\mathcal{G}^\theta_\gamma(\kappa)$. Successor stages $\alpha + 1$ in the construction are analogous, but we also make sure that $\langle\mathcal{M}_\xi \mid \xi < \alpha+1\rangle, \langle\mu_\xi \mid \xi < \alpha+1\rangle \in \mathcal{M}_{\alpha+1}$. At limit stages τ outputs unions, as is required by the rules of $\mathcal{G}^\theta_\gamma(\kappa)$. Since the union of all the μ_α's is good as τ is winning, $\tilde{\mu}_\gamma := \bigcup_{\alpha<\gamma}\tilde{\mu}_\alpha$ is good as well, making $\tilde{\tau}$ winning and we are done. ■

[4]More precisely, that σ is winning in $\mathcal{G}^\theta_\gamma(\kappa)$ means that there's a sequence $\langle f_n : \kappa \to \kappa \mid n < \omega\rangle$ with the f_n's all being elements of the last model $\tilde{\mathcal{M}}_\gamma$, witnessing the illfoundedness of the ultrapower. But then all these functions will also be elements of the union of the $\mathcal{M}_\alpha$'s, since we ensured that $\mathcal{M}_\alpha \supseteq \tilde{\mathcal{M}}_\alpha$ in the construction above, making the ultrapower of $\bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$ by $\bigcup_{\alpha<\gamma}\mu_\alpha$ illfounded as well.

We now arrive at the definitions of the cardinals we will be considering. They were in [?] only defined for γ being a cardinal, but given the above result we generalise it to all ordinals γ.

DEFINITION 1.12. Let κ be a cardinal and $\gamma \leq \kappa$ an ordinal. Then κ is **γ-Ramsey** if player I does not have a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ for all regular $\theta > \kappa$. We furthermore say that κ is **strategic γ-Ramsey** if player II *does* have a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ for all regular $\theta > \kappa$.

Define **(strategic) genuine γ-Ramseys** and **(strategic) normal γ-Ramseys** analogously, but where we require the last measure μ_γ to be genuine and normal, respectively. ∘

DEFINITION 1.13 (N.). A cardinal κ is **$<\gamma$-Ramsey** if it is α-Ramsey for every $\alpha < \gamma$, **almost fully Ramsey** if it is $<\kappa$-Ramsey and **fully Ramsey** if it is κ-Ramsey.

Further, say that κ is **coherent $<\gamma$-Ramsey** if it's strategic α-Ramsey for every $\alpha < \gamma$ and that there exists a choice of winning strategies τ_α in $\mathcal{G}_\alpha(\kappa)$ for player II satisfying that $\tau_\alpha \subseteq \tau_\beta$ whenever $\alpha < \beta$. In other words, there is a single strategy τ for player II in $\mathcal{G}_\gamma(\kappa)$ such that τ is a winning strategy for player II in $\mathcal{G}_\alpha(\kappa)$ for every $\alpha < \gamma$.[5] ∘

This is not the original definition of (strategic) γ-Ramsey cardinals however, as this involved elementary embeddings between weak κ-models – but as the following theorem of [?] shows, the two definitions coincide whenever γ is a regular cardinal.

THEOREM 1.14 (Holy-Schlicht). *For regular cardinals λ, a cardinal κ is λ-Ramsey iff for arbitrarily large $\theta > \kappa$ and every $A \subseteq \kappa$ there is a weak κ-model $\mathcal{M} \prec H_\theta$ with $\mathcal{M}^{<\lambda} \subseteq \mathcal{M}$ and $A \in \mathcal{M}$ with an $\mathcal{M}$-normal 1-good $\mathcal{M}$-measure μ on κ.* ■

1.1.1 The finite case

In this section we are going to consider properties of the n-Ramsey cardinals for finite n. Note in particular that the $\mathcal{G}_n^\theta(\kappa)$ games are determined, making the "strate-

[5] Note that, with this terminology, "coherent" is a stronger notion than "strategic". We could've called the cardinals *coherent strategic $<\gamma$-Ramseys*, but we opted for brevity instead.

gic" adjective superfluous in this case. We further note that the θ's are also dispensible in this finite case:

Proposition 1.15 (N.). *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n \prec H_\theta$.*

Proof. $\Leftarrow$ is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays $\mathcal{M}_k$ in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \mathrm{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of $\mathcal{M}_k$, and play $\mathcal{M}_k^*$ in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ-responses to the $\mathcal{M}_k^*$'s and let player II play the μ_k's in $\mathcal{G}_n(\kappa)$ as well.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\mathrm{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω-sequence $\vec{f} := \langle f_k \mid k < \omega\rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \mathrm{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\mathrm{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \to H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \operatorname{ran}\pi$.

Now $\bar{A} = A$ for every $A \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the $\bar{f}_k$'s witness that $\mathrm{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\mathrm{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \mathrm{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ's anymore we note that n-Ramseyness can now be described using a Π^1_{2n+2}-formula and normal n-Ramseyness using a Π^1_{2n+3}-formula.

We have the following characterisations, as proven in [?].

Theorem 1.16 (Abramson et al.). *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then*

(i) κ is weakly compact if and only if it is 0-Ramsey;

(ii) κ is weakly ineffable if and only if it is genuine 0-Ramsey;

(iii) κ is ineffable if and only if it is normal 0-Ramsey.

Proof. This is mostly just changing the terminology in [?] to the current game-theoretic one, so we only show (i).

Theorem 1.1.3 in [?] shows that κ is weakly compact if and only if every κ-sized collection of subsets of κ is measured by a $<\kappa$-complete measure, in the sense that every $<\kappa$-sequence (in V) of measure one sets has non-empty intersection.

For the $\Rightarrow$ direction we can let player II respond to any $\mathcal{M}_0$ by first getting the $<\kappa$-complete $\mathcal{M}_0$-measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \to \mathrm{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π, which is $\mathcal{M}_0$-normal and good. For $\Leftarrow$, if $X \subseteq \mathscr{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ-model $\mathcal{M}_0 \prec H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play $\mathcal{M}_0$ in $\mathcal{G}_0(\kappa)$ we get some $\mathcal{M}_0$-normal good $\mathcal{M}_0$-measure μ_0 on κ. Since $\mathcal{M}_0$ is closed under $<\kappa$-sequences we get that μ_0 is $<\kappa$-complete. ■

Indescribability

In this section we aim to prove that n-Ramseys are Π^1_{2n+1}-indescribable and that normal n-Ramseys are Π^1_{2n+2}-indescribable, which will also establish that the hierarchy of alternating n-Ramseys and normal n-Ramseys forms a strict hierarchy. Recall the following definition.

Definition 1.17. A cardinal κ is **Π^1_n-indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ∘

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π^1_1-indescribable – see [?].

Theorem 1.18 (N.). *Every n-Ramsey κ is Π^1_{2n+1}-indescribable for $n < \omega$.*

Proof. Let κ be n-Ramsey and assume that it is not Π^1_{2n+1}-indescribable, witnessed by a Π_{2n+1}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$ and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$-many quantifiers occuring in φ in $(n+1)$-many steps. We will here describe the first two steps with the remaining steps following the same pattern.

$\boxed{\text{First step.}}$ Write $\varphi(v) \equiv \forall v_1 \psi(v, v_1)$ for a Σ_{2n}-formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A^{(0)}_\alpha \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let $\mathcal{M}_0$ be a weak κ-model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good $\mathcal{M}_0$-normal $\mathcal{M}_0$-measure μ_0 on κ, using the 0-Ramseyness of κ. Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \mathrm{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. $\mathcal{M}_0$-normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, $\mathcal{M}_0$-normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}]\urcorner. \tag{1}$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0-formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

$\boxed{\textbf{Second step.}}$ Write $\psi(v, v_1) \equiv \exists v_2 \forall v_3 \chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$-formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3 \chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \tag{2}$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, v_3]. \tag{3}$$

Fix witnesses $A^{(1)}_\alpha \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S^{(0)}_\alpha := \{\xi < \kappa \mid A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S^{(0)}_\alpha \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and $\mathcal{M}_0$-normality of μ_0 then implies that $S^{(0)}_\alpha \in \mu_0$ is equivalent to

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ-model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an $\mathcal{M}_1$-normal $\mathcal{M}_1$-measure on κ, using the 1-Ramseyness of κ, so that $\mathcal{M}_1$-normality of μ_1 yields that $\triangle\vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle\vec{S}^{(0)}$ if and only if $A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A^{(0)}_\xi = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \mathrm{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A^{(1)}_\alpha$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha].$$

Now this, paired with the above observation regarding $\triangle\vec{S}^{(0)}$, means that for every $\alpha \in \triangle\vec{S}^{(0)} \cap \mathrm{Lim}$ we have that

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha],$$

so that $\mathcal{M}_1$-normality of μ_1 and Łoś' lemma implies that

$$\mathrm{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg\chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]\urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$-many steps, ending with a Δ_0-formula $\phi(v, v_1, \dots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\mathrm{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]\urcorner$. But now absoluteness of $\neg\phi$ means that $V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]$, contradicting (4). ■

Note that this is optimal, as n-Ramseyness can be described by a Π^1_{2n+2}-formula. As a corollary we then immediately get the following.

Corollary 1.19 (N.). *Every $<\omega$-Ramsey cardinal is Δ^2_0-indescribable.* ■

The second indescribability result concerns the normal n-Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π^1_2-indescribable – see [?].

Theorem 1.20 (N.). *Every normal n-Ramsey κ is Π^1_{2n+2}-indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 1.17, as we would end up with a Π^1_1 statement in an ultrapower, and as Π^1_1 statements are not upwards absolute in general we would not be able to get our contradiction.

Proof. Let κ be normal n-Ramsey and assume that it is not Π^1_{2n+2}-indescribable, witnessed by a Π_{2n+2}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n-Ramsey to perform the same $n+1$ steps as in the proof of Theorem 1.17. This gives us a Σ_1-formula $\phi(v, v_1, \ldots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \cdots, \mathcal{A}^{(n)} \rangle$, $\langle B^{(0)}, \ldots, B^{(n-1)} \rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n \rangle$ of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{1}$$

and, for μ_n-many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \ldots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, A^{(n)}_\alpha].$$

Now form $S^{(n)}_\alpha \in \mu_n$ as in the proof of Theorem 1.17. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 1.17 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for $k < n$), but we can now use normality[6] of μ_n to ensure that we *do* have that $\triangle\vec{S}^{(n)}$ is stationary in κ. This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \ldots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \tag{2}$$

[6] Recall that this is stronger than just requiring it to be $\mathcal{M}_n$-normal – we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.

Now note that since κ is inaccessible it is Σ^1_1-indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of [?] shows that the set of reflection points of Σ^1_1-formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \dots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n-Ramseyness can be described by a Π^1_{2n+3}-formula. In particular this then means that every $(n+1)$-Ramsey is a normal n-Ramsey stationary limit of normal n-Ramseys, and every normal n-Ramsey is an n-Ramsey stationary limit of n-Ramseys, making the hierarchy of alternating n-Ramseys and normal n-Ramseys a strict hierarchy.

Downwards absoluteness to L

Our absoluteness result below, Theorem 1.21, is inspired by arguments in [?], and uses the following lemma from that paper.

Lemma 1.21 (Abramson et al). *There is a Π^1_1 formula $\varphi(A)$ such that, for any ordinal α, $(V_\alpha, V_{\alpha+1}) \models \varphi[A]$ iff α is a regular cardinal and A is a non-constructible subset of α.*[7] ■

Theorem 1.22 (N.). *Genuine- and normal n-Ramseys are downwards absolute to L, for every $n < \omega$.*

Proof. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ-model – we want to find a genuine $\mathcal{M}$-measure inside L. By assumption we *can* find such a measure μ in V; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa\rangle \in L$ of $\mathscr{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

Claim 1.23. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

[7] This appears as Lemma 4.1.2 in [?].

PROOF OF CLAIM. Let $\vec{B}$ be the μ-**positive part** of $\vec{A}$, meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle\vec{B}$ has size κ, so we can pick $\delta \in \triangle\vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L. $\dashv$

Now let φ be the Π^1_1 formula given by Lemma 1.20. If we therefore assume that $T \notin L$ then $(V_\kappa, V_{\kappa+1}) \models \varphi[T]$, which by Π^1_1-indescribability of κ means that there exists some $\alpha < \kappa$ such that $(V_\alpha, V_{\alpha+1}) \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle\mu)^L = \triangle\mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V. The cases where κ is a genuine- or normal n-Ramsey cardinal is analogous. ■

Since $(n+1)$-Ramseys are normal n-Ramseys we then immediately get the following.

COROLLARY 1.24 (N.). *Every $(n+1)$-Ramsey is normal n-Ramsey in L, for every $n < \omega$. In particular, $<\omega$-Ramseys are downwards absolute to L.* ■

Complete ineffability

In this section we provide a characterisation of the *completely ineffable* cardinals[8] in terms of the α-Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$-Ramsey cardinals, namely the *coherent $<\omega$-Ramseys* as defined in 1.12. Note that a coherent $<\omega$-Ramsey is precisely a cardinal satisfying the ω-filter property, as defined in [?].

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is closely related to the proof of Theorem 1.19 (the indescribability of normal n-Ramseys), but the main difference is that we want everything to occur locally inside our weak κ-models. We'll need another lemma from [?].

[8]See Appendix ?? for a definition of the completely ineffable cardinals.

Lemma 1.25 (Abramson et al). *Let κ be inaccessible, $X \subseteq \kappa$ and φ a Σ^1_1-formula such that $(V_\kappa, \in, X) \models \varphi[X]$. Then*

$$\{\alpha < \kappa \mid (V_\alpha, \in, X \cap V_\alpha) \models \varphi[X \cap V_\alpha]\}$$

is a club. ■

Theorem 1.26 (N.). *Every coherent $<\omega$-Ramsey is a stationary limit of $<\omega$-Ramseys.*

Proof. Let κ be coherent $<\omega$-Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 \prec H_\theta$ be a weak κ-model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ-model $\mathcal{M} \prec H_\theta$ with an $\mathcal{M}$-normal $\mathcal{M}$-measure $\mu := \bigcup_{n<\omega} \mu_n$ on κ.

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } {<}\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by $\mathcal{M}$-normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k+1$ steps as in the proof of Theorem 1.19 with $\varphi(\xi)$ being $\ulcorner \xi \text{ is } k\text{-Ramsey} \urcorner$, so that we get a weak κ-model $\mathcal{M}_{k+1} \prec H_\theta$, an $\mathcal{M}_{k+1}$-normal $\mathcal{M}_{k+1}$-measure $\tilde{\mu}_{k+1}$ on κ, a Σ_1-formula $\varphi(v, v_1, v_2, \dots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \dots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \dots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \tag{2}$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg\varphi[\xi, A^{(0)}_\xi, B^{(0)} \cap V_\xi, A^{(1)}_\xi, B^{(1)} \cap V_\xi, \dots, A^{(k-1)}_\xi, B^{(k-1)} \cap V_\xi, A^{(k)}_\xi], \tag{3}$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \mathrm{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 1.17.

Since κ in particular is Σ^1_1-indescribable, Lemma 1.24 implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ-model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n+1)$-steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of $\mathcal{M}_{k+1}$. In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ, $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A_\xi^{(i)} \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa \rangle$ and note that since $\vec{S}$ is definable it is an element of $\mathcal{M}_{k+2}$ as well. Then $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $\triangle\vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \triangle\vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \triangle\vec{S}$ we see, as in the proof of Theorem 1.17, that

$$V_{\zeta+1} \models \varphi[\zeta, A_\zeta^{(0)}, B^{(0)} \cap V_\zeta, A_\zeta^{(1)}, B^{(1)} \cap V_\zeta, \dots, A_\zeta^{(k)}]$$

since $\triangle\vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} \prec H_\theta$ we have that $\mathcal{M}$ is correct about stationary subsets of κ, meaning that κ is a stationary limit of $<\omega$-Ramseys. ■

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

Definition 1.27. A collection $R \subseteq \mathscr{P}(\kappa)$ is a **stationary class** if

(i) $R \neq \emptyset$;
(ii) every $A \in R$ is stationary in κ;
(iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

○

Definition 1.28. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$ there is an $H \in R$ homogeneous for f. ○

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in [?].

Theorem 1.29 (N.). *A cardinal κ is completely ineffable if and only if it is coherent $<\omega$-Ramsey.*

Proof. ($\Leftarrow$): Assume κ is coherent $<\omega$-Ramsey, witnessed by strategies $\langle\tau_n \mid n < \omega\rangle$. Let $f : [\kappa]^2 \to 2$ be arbitrary and form the sequence $\langle A^f_\alpha \mid \alpha < \kappa\rangle$ as

$$A^f_\alpha := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let $\mathcal{M}_f$ be a transitive weak κ-model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated $\mathcal{M}_f$-measure on κ given by τ_0.[9] 1-Ramseyness of κ ensures that μ_f is normal, meaning $\triangle\mu_f$ is stationary in κ. Define a new sequence $\vec{B}^f$ as the μ_f-positive part of $\vec{A}^f$.[10] Then $B^f_\alpha \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\triangle\vec{B}^f$ is stationary.

Let now $\mathcal{M}'_f$ be a new transitive weak κ-model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an $\mathcal{M}'_f$-measure $\mu'_f \supseteq \mu_f$ on κ. Then $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \in \mu_f\}$ and $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \notin \mu_f\}$ are both elements of $\mathcal{M}'_f$, so one of them is in μ'_f; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f.

Now let $g : [H_f]^2 \to 2$ be arbitrary and again form

$$A^g_\alpha := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ-model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$-measure $\mu_{f,g} \supseteq \mu'_f$ on κ. As before we then get a stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g. We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists\vec{f} : H_{\vec{f}} \subseteq A\},$$

where the $\vec{f}$'s range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \to 2$ and $f_{k+1} : [H_{f_k}] \to 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \to 2$, we can find $H \in R$ which is homogeneous for f. Indeed, if we let $\vec{f}$ be such that $H_{\vec{f}} \subseteq A$, which exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

[9] Technically we would have to require that $\mathcal{M}_f \prec H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0, but note that we could simply get a measure on $\mathrm{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to $\mathcal{M}_f$. We will use this throughout the proof.

[10] The *μ-positive part* was defined in Claim 1.22.

($\Rightarrow$): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n-Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play $\mathcal{M}_0$ and enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A^0_\alpha \mid \alpha < \kappa \rangle$ such that $A^0_\xi \subseteq A^0_\zeta$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^0_\xi$. Let $<^\alpha_{\text{lex}}$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f, using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \restriction \alpha \mid \xi \in H_0 \wedge \xi > \alpha \rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least such η, which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C.

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \restriction \zeta = r_\gamma \restriction \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \restriction \zeta$ holds for every $\gamma \in H_0$, $\gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^0_\xi$ iff $\gamma \in A^0_\xi$. Define thus the $\mathcal{M}_0$-measure μ_0 on κ as

$$\begin{aligned} \mu_0(A^0_\xi) = 1 \quad &\text{iff} \quad (\forall \beta \in H_0 \cap C)(\beta > \xi \to \beta \in A^0_\xi) \\ &\text{iff} \quad (\exists \beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A^0_\xi), \end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa \rangle$ be the μ_0-positive part of $\vec{A}^0$, it is also simple to check that $H_0 \cap C \subseteq \triangle \vec{B}$, making μ_0 normal and hence also both $\mathcal{M}_0$-normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n-Ramsey and let $\langle \mathcal{M}_0, \mu_0, \dots, \mathcal{M}_n, \mu_n, \mathcal{M}_{n+1} \rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}$ as $\vec{A}^{n+1} = \langle A^{n+1}_\xi \mid \xi < \kappa \rangle$, again satisfying that $\xi \leq \zeta$ whenever $A^{n+1}_\xi \subseteq A^{n+1}_\zeta$, but also such that given any $\xi < \kappa$ there are $\zeta, \zeta' \in (\xi, \kappa)$ satisfying that $A^{n+1}_\zeta \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and $A^{n+1}_{\zeta'} \in (\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}) - \mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n \in R$ and C be club in κ such that $H_n \cap C \subseteq \triangle\mu_n$, which exist by our inductive assumption. For $\alpha < \kappa$ define $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^{n+1}_\xi$, and define a colouring $f : [H_n]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

As $H_n \in R$ there is an $H_{n+1} \in R$ homogeneous for f. Just as before, define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least $\eta \in [\alpha, \kappa)$ such that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_{n+1}$ with $\eta \leq \beta < \gamma$, and let D be the club of fixed points of g. As above we get that given any $\alpha, \gamma \in H_{n+1} \cap D$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^{n+1}_\xi$ iff $\gamma \in A^{n+1}_\xi$. Define then the $\mathcal{M}_{n+1}$-measure μ_{n+1} on κ as

$$\begin{aligned} \mu_{n+1}(A^{n+1}_\xi) = 1 \quad &\text{iff} \quad (\forall\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \to \beta \in A^{n+1}_\xi) \\ &\text{iff} \quad (\exists\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A^{n+1}_\xi). \end{aligned}$$

Then $H_{n+1} \cap D \cap C \subseteq \triangle\mu_{n+1}$, making μ_{n+1} normal, $\mathcal{M}_{n+1}$-normal and good, just as before. It remains to show that $\mu_n \subseteq \mu_{n+1}$. Let thus $A \in \mu_n$ be given, and say $A = A^{n+1}_\xi = A^n_\eta$, where $\vec{A}^n$ was the enumeration of $\mathscr{P}(\kappa) \cap \mathcal{M}_n$ used at the n'th stage. Then by definition of μ_n we get that for every $\beta \in H_n \cap C$ with $\beta > \eta$, $\beta \in A^n_\eta$. We need to show that

$$(\exists\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A^{n+1}_\xi)$$

holds. But here we can simply pick a $\beta > \max(\xi, \eta)$ with $\beta \in H_{n+1} \cap D \cap C \subseteq H_n \cap C$. This shows that $\mu_n \subseteq \mu_{n+1}$, making κ $(n+1)$-Ramsey and thus inductively also coherent $<\omega$-Ramsey. ■

1.1.2 The countable case

This section covers the (strategic) γ-Ramsey cardinals whenever γ has countable cofinality. This case is special because, as we cannot ensure that the final measure in $\mathcal{G}^\theta_\gamma(\kappa)$ is countably complete and so the existence of winning strategies *might* depend on θ, in contrast with the uncountable cofinality case.

[Strategic] ω-Ramsey cardinals

We now move to the strategic ω-Ramsey cardinals and their relationship to the (non-strategic) ω-Ramseys.

Theorem 1.30 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. Then κ is faintly θ-measurable iff player II has a winning strategy in $\mathcal{C}^\theta_\omega(\kappa)$.*

Proof. $(\Leftarrow)$: Fix a winning strategy σ for player II in $\mathcal{C}^\theta_\omega(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, H^V_\theta)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \prec H^V_\theta$ such that $H^V_\theta \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^V_θ-normal H^V_θ-measure μ on κ.

We claim that $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n \colon \kappa \to \theta$ such that $g_n \in H^V_\theta$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f \colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_\mathcal{T} \cdots <_\mathcal{T} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{C}^\theta_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $C^\theta_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so that the ultrapower embedding $\pi \colon H^V_\theta \to \mathrm{Ult}(H^V_\theta, \mu)$ witnesses that κ is faintly θ-measurable.

$(\Rightarrow)$: Assume that κ is faintly θ-measurable. Let $\mathbb{P}$ be a forcing $\dot\mu$ a $\mathbb{P}$-name for an H^V_θ-normal H^V_θ-measure on κ and $\dot\pi$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}^\theta_\omega(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f_i^n \mid i < k \rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \check{\mu} \cap \mathcal{M}_n = \check{\mu}_n \cap \forall i < \check{k} \colon \dot{\pi}(\check{f}_i^n)(\check{\kappa}) = \check{\alpha}_i^n \urcorner,$$

with $\mu_n, \alpha_i^n \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f_{i_n}^{k_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check{\alpha}_{i_{n+1}}^{k_{n+1}} < \check{\alpha}_{i_n}^{k_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

We note that the above Theorem along with our results from Chapter ?? shows that winning the Cohen games doesn't guarantee weak compactness.

Corollary 1.31 (N.). *Let κ be inaccessible.*

(i) If player II wins $\mathcal{C}_\omega^\theta(\kappa)$ for all regular $\theta > \kappa$ then κ is not necessarily weakly compact;

(ii) If player II wins $\mathcal{C}_\kappa(\kappa)$ then κ is weakly compact.

Proof. The first claim is directly by Proposition ?? and Theorem 1.29, and the second claim is because the hypothesis implies that player II wins $\mathcal{G}_0(\kappa)$ so that inaccesibility of κ makes κ weakly compact – see e.g. [?] for this characterisation of weak compactness. ■

Here's a near-analogous result of Theorem 1.29 for the $\mathcal{G}_\omega^\theta(\kappa)$ game.

Theorem 1.32 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$, and if player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$ then κ is faintly θ-power-measurable. In particular, $\mathcal{G}_\omega^\theta(\kappa)^L \sim \mathcal{C}_\omega^\theta(\kappa)^L$.*

Proof. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi \colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a name $\dot{\mu}$ and a $\mathbb{P}$-condition p such that

$p \Vdash \ulcorner \dot{\mu}$ is a weakly amenable $\check{H}_\theta$-normal $\check{H}_\theta$-measure with a wellfounded ultrapower$\urcorner$.

We now define a strategy σ for player II in $\mathcal{G}^\theta_\omega(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H^V_\theta$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n\colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot{\mu} \cap \check{\mathcal{M}}_n = \check{\mu}_n \cap \check{\mu}_n = \dot{\mu} \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma ?? we get that $\pi \restriction \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n\colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega} \mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot{\pi}(\check{g}_{n+1})(\check{\kappa}) < \dot{\pi}(\check{g}_n)(\check{\kappa}) \urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check{\pi}_{n+1}(\check{g}_{n+1})(\check{\kappa}) < \check{\pi}_n(\check{g}_n)(\check{\kappa}) \urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy. ■

We get the following immediate corollary.

Corollary 1.33 (N.-Schindler). *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the* *existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ■

Note also that the proof of Theorem 1.31 shows that whenever κ is strategic ω-Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H_ν^V-normal H_ν-measure on κ.

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω-Ramsey cardinals and ω-Ramsey cardinals lie, namely that strategic ω-Ramseys are equiconsistent with *remarkables* and ω-Ramseys are strictly below. Theorem 4.8 of [?] showed that 2-iterables are limits of remarkables, and our Propositions 1.8 and 1.40 shows that ω-Ramseys are limits of 1-iterables, so that the strategic ω-Ramseys and the ω-Ramseys both lie strictly between the 2-iterables and 1-iterables. It was shown in [?] that ω-Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by [?], and [?] showed the following two equivalent formulations.

Definition 1.34. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

(i) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H_\lambda^V \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_\nu^V \to M$ with critical point κ and $\pi(\kappa) > \lambda$;
(ii) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^\lambda M \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_\nu^V \to M$ with critical point κ and $\pi(\kappa) > \lambda$.

○

Theorem 1.35 (N.). *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models \ulcorner$there is a proper class of virtually measurables$\urcorner$. In particular, the two notions are equiconsistent.*

Proof. Virtually measurables are downwards absolute to L by Lemma ??, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H_\lambda^V \subseteq M$ and forcing poset $\mathbb{P}$ it holds that, in $V^{\mathbb{P}}$, there's no elementary embedding $\pi : H_\nu^V \to M$ with $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν-measurable to fix a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's an elementary $\pi : H_\nu^V \to M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H_\nu^V = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we're assuming that κ isn't remarkable. Then by restricting the generic embedding to H_κ^V we get that $H_\kappa^V \prec H_{\pi(\kappa)}^M = H_{\pi(\kappa)}^V$, using that $\pi(\kappa) < \nu$ and $H_\nu^V = H_\nu^M$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H_ν^V since $\pi(\kappa) < \nu$, and as $H_\nu^V \prec_1 V$ we get that $\pi(\kappa)$ *is* a cardinal. But then, again using that $H_{\pi(\kappa)} \prec_1 V$, κ is virtually measurable in $H_{\pi(\kappa)}^V$ since being virtually measurable is Π_2. This means that for every $\xi < \kappa$ it holds that

$$H_{\pi(\kappa)}^V \models \exists\alpha > \xi : \ulcorner \alpha \text{ is virtually measurable} \urcorner,$$

implying that $H_\kappa^V \models \ulcorner$There is a proper class of virtually measurables$\urcorner$. ■

Now Theorem 1.34 and Corollary 1.32 yield the following immediate corollary.

Corollary 1.36 (N.-Schindler). *Let κ be strategic ω-Ramsey. Then either κ is remarkable in L or otherwise*

$$L_\kappa \models \ulcorner \textit{there is a proper class of strategic } \omega\textit{-Ramseys} \urcorner.$$

In particular, the two notions are equiconsistent. ■

Now, using these results we show that the strategic ω-Ramseys have strictly stronger consistency strength than the ω-Ramseys.

Theorem 1.37 (N.). *Remarkable cardinals are strategic ω-Ramsey limits of ω-Ramsey cardinals.*

Proof. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ-sequences and a generic elementary embedding $\pi : H_\nu^V \to M$ for some $\nu > 2^\kappa$. We will show that κ is

ω-Ramsey in M. Note that remarkables are clearly virtually measurable, and thus by Theorem 1.31 also strategic ω-Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}^\theta_\omega(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}^\theta_\omega(\kappa)^M$. Since M is closed under 2^κ-sequences it means that $\mathscr{P}(\mathscr{P}(\kappa)) \subseteq M$ and thus that M contains all possible filters on κ. We let player II follow τ, which produces a play $\sigma * \tau$ in which player II wins. But all player II's moves are in $\mathscr{P}(\mathscr{P}(\kappa))$ and hence in M, and as M is furthermore closed under ω-sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω-Ramsey in M.

This also implies that κ is a limit of ω-Ramseys in H_ν. But as κ is remarkable it holds that $H_\kappa \prec_2 V$, in analogy with the same property for strongs and supercompacts, and as being ω-Ramsey is a Π_2-notion this means that κ *is* a limit of ω-Ramseys. ■

This immediately yields the following corollary.

Corollary 1.38 (N.-Schindler). *If κ is a strategic ω-Ramsey cardinal then*

$$L_\kappa \models \ulcorner\text{there is a proper class of } \omega\text{-Ramseys}\urcorner. \qquad \dashv$$

(ω, α)-Ramsey cardinals

A natural generalisation of the γ-Ramsey definition is to require more iterability of the last measure. Of course, by Proposition 1.8 we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when cof $\gamma > \omega$ so the next definition is only interesting whenever cof $\gamma = \omega$.

Definition 1.39 (N.). Let α, β be ordinals. Then a cardinal κ is (α, β)-**Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ for all regular $\theta > \kappa$.[11] ∘

Definition 1.40 (Gitman). A cardinal κ is α-**iterable** if for every $A \subseteq \kappa$ there exists a *transitive* weak κ-model $\mathcal{M}$ with $A \in \mathcal{M}$ and an α-good $\mathcal{M}$-measure μ on $\mathcal{M}$. ∘

[11] Note that an α-Ramsey cardinal is the same as an $(\alpha, 0)$-Ramsey cardinal.

Proposition 1.41. *If $\beta > 0$ then every (α, β)-Ramsey is a β-iterable stationary limit of β-iterables.*

Proof. Let $(\mathcal{M}, \in, \mu)$ be a result of a play of $\mathcal{G}_\alpha^{\kappa^+}(\kappa, \beta)$ in which player II won. Then the transitive collapse of $(\mathcal{M}, \in, \mu)$ witnesses that κ is β-iterable, since μ is β-good by definition of $\mathcal{G}_\alpha^{\kappa^+}(\kappa, \beta)$.

That κ is β-iterable is reflected to some H_θ, so let now $(\mathcal{N}, \in, \nu)$ be a result of a play of $\mathcal{G}_\alpha^\theta(\kappa, \beta)$ in which player II won. Then $\mathcal{N} \prec H_\theta$, so that κ is also β-iterable in $\mathcal{N}$. Since being β-iterable is witnessed by a subset of κ and $\beta > 0$ implies[12] that we get a κ-powerset preserving $j : \mathcal{N} \to \mathcal{P}$, $\mathcal{P}$ also thinks that κ is β-iterable, making κ a stationary limit of β-iterables by elementarity. ■

We now move towards Theorem 1.44 which gives an upper consistency bound for the (ω, α)-Ramseys. We first recall a few definitions and a folklore lemma.

Definition 1.42. For an infinite ordinal α, a cardinal κ is α-**Erdős** for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \to \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c; i.e. that $|c``[H]^n| \leq 1$ holds for every $n < \omega$. ○

Definition 1.43. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$.[13] ○

Lemma 1.44 (Folklore). *Let κ be α-Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure $\mathcal{M}$ in a countable language $\mathcal{L}$ with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

[12] Recall that β-good for $\beta > 0$ in particular implies weak amenability.
[13] Note that this terminology is not at all related to remarkable *cardinals*.

Proof. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all $\mathcal{L}$-formulas and define $c : [C]^{<\omega} \to \kappa$ as follows. For an increasing sequence $\alpha_1 < \cdots < \alpha_{2n} \in C$ let

$$\begin{aligned} c(\{\alpha_1, \ldots, \alpha_{2n}\}) := \text{the least } \lambda < \alpha_1 \text{ such that} \\ \exists \delta_1 < \cdots \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \ldots, \delta_k \rangle \wedge \\ \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}] \end{aligned}$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α-Erdős we get a homogeneous $I \in [C]^\alpha$ for c; i.e. that $|c``[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \ldots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \ldots, \alpha_{2n} \in I$, as otherwise there exists an $m < \omega$ and $\delta_1 < \cdots \delta_k$ such that for any $\alpha_1 < \ldots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}]. \tag{†}$$

But then simply pick $\alpha_1 < \ldots \alpha_{2n} < \alpha'_1 < \cdots < \alpha'_{2n}$ so that both $\{\alpha_1, \ldots, \alpha_{2n}\}$ and $\{\alpha'_1, \ldots, \alpha'_{2n}\}$ witnesses (†); then either $\{\alpha_1, \ldots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \ldots, \alpha_n, \alpha'_{n+1}, \ldots, \alpha'_{2n}\}$ also witnesses that (†) fails, $\lightning$. ■

Theorem 1.45 (N.). *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α-Erdős cardinal is a limit of (ω, α)-Ramsey cardinals.*

Proof. Let κ be α-Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 1.43 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I. We will show that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in [?] it suffices to find a weak ι-model $\mathcal{M} \prec H_\theta$ and an α-good $\mathcal{M}$-measure on ι. Define

$$\mathcal{M} := \text{Hull}^{H_\theta}(\iota \cup I) \prec H_\theta$$

and let $\pi : I \to I$ be the right-shift map. Since I is remarkable, I $(= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \dots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \dots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι-powerset preserving we get that $\mathcal{M} \prec H_\theta$ is a weak ι-model satisfying ZFC^- with a 1-good $\mathcal{M}$-measure μ_j on ι. Furthermore, as we can linearly iterate $\mathcal{M}$ simply by applying j we get an α-iteration of $\mathcal{M}$ since there are α-many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \dots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \dots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\eta, \alpha)$. If we repeat this procedure for all regular $\theta > \kappa$ we get by the pidgeon hole principle that κ is a limit of (ω, α)-Ramsey cardinals. ■

As Theorem 4.5 in [?] shows that $(\alpha{+}1)$-iterable cardinals have α-Erdős cardinals below them for $\alpha \geq \omega$ additively closed, this shows that the (ω, α)-Ramseys form a strict hierarchy. Further, as α-Erdős cardinals are consistent with $V = L$ when $\alpha < \omega_1^L$ and ω_1-iterable cardinals aren't consistent with $V = L$, we also get that (ω, α)-Ramsey cardinals are consistent with $V = L$ if $\alpha < \omega_1^L$ and that they aren't if $\alpha = \omega_1$.

[Strategic] $(\omega{+}1)$-Ramsey cardinals

The next step is then to consider $(\omega{+}1)$-Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in [?] and in the proof of Lemma 1.3 in [?] – see also [?] and [?].

Theorem 1.46 (Dodd, Mitchell). *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model $\mathcal{M}$ such that there exists a weakly amenable countably complete $\mathcal{M}$-measure on κ.* ■

The following theorem then supplies us with a lower bound for the strength of the $(\omega{+}1)$-Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 1.56, but we include this Ramsey lower bound as well for completeness.

Theorem 1.47 (N.). *Every $(\omega{+}1)$-Ramsey cardinal is a Ramsey limit of Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey and $A \subseteq \kappa$. Let σ be a strategy for player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A \in \mathcal{M}_0$ and $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Then σ isn't winning as κ is $(\omega{+}1)$-Ramsey, so we may fix a play $\sigma * \vec{\mu}_\alpha$ of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable $\mathcal{M}_\omega$-measure on κ, and by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 1.45.

Since κ is Ramsey, $\mathcal{M}_\omega \models \ulcorner\kappa$ is Ramsey$\urcorner$ as well. Letting $j : \mathcal{M}_\omega \to \mathcal{N}$ be the κ-powerset preservering embedding induced by μ_ω, we also get that $\mathcal{N} \models \ulcorner\kappa$ is Ramsey$\urcorner$ by κ-powerset preservation. This then implies that κ *is* a stationary limit of Ramsey cardinals inside $\mathcal{M}_\omega$, and thus also in V by elementarity. ■

As for the *consistency* strength of the strategic $(\omega{+}1)$-Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal – see e.g. p. 249 in [?].

Theorem 1.48 (N.). *If κ is a strategic $(\omega{+}1)$-Ramsey cardinal then, in $V^{\mathrm{Col}(\omega,2^\kappa)}$, there's a transitive class N and an elementary embedding $j : V \to N$ with $\operatorname{crit} j = \kappa$. In particular, the existence of a strategic $(\omega{+}1)$-Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

Proof. Set $\mathbb{P} := \mathrm{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$. Let $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name of an ω-sequence $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n \prec H^V_{\kappa^+}$ and $\mathscr{P}(\kappa)^V \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, and let $\dot{\mu}$ be a $\mathbb{P}$-name for the ω-sequence of σ-responses to the $\mathcal{M}_n$'s in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)^V$.

Assume that there's a $\mathbb{P}$-condition p which forces the generic ultrapower $\mathrm{Ult}(V, \bigcup_n \dot{\mu}_n)$ to be illfounded, meaning that we can fix a $\mathbb{P}$-name $\dot{f}$ for an ω-sequence $\langle f_n \mid n <$

$\omega\rangle$ such that

$$p \Vdash \dot{X}_n := \{\alpha < \kappa \mid \dot{f}_{n+1}(\alpha) < \dot{f}_n(\alpha)\} \in \bigcup_{n<\omega} \dot{\mu}_n.$$

Now, in V, we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} \prec H_\theta$ such that $\dot{\mathcal{M}}, \dot{\mu}, \dot{f}, H^V_{\kappa^+}, \sigma, p \in \mathcal{N}$. We can find an $\mathcal{N}$-generic $g \subseteq \mathbb{P}^{\mathcal{N}}$ in V with $p \in g$ since $\mathcal{N}$ is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}^g_n * \dot{\mu}^g_n$ is a play of $\mathcal{G}^{\kappa^+}_\omega(\kappa)^V$ which is according to σ, meaning that $\bigcup_{n<\omega} \dot{\mu}^g_n$ is normal and in particular countably complete (in V). Then $\bigcap_{n<\omega} \dot{X}^g_n \neq \emptyset$, but if $\alpha \in \bigcap_{n<\omega} \dot{X}^g_n$ then $\langle \dot{f}^g_n(\alpha) \mid n < \omega\rangle$ is a strictly decreasing ω-sequence of ordinals, $\lightning$. This means that $\mathrm{Ult}(V, \bigcup_n \mu_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in [?]. ■

The above Theorem 1.47 then answers Question 9.2 in [?] in the negative, asking if λ-Ramseys are strategic λ-Ramseys for uncountable cardinals λ, as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.

1.1.3 The general case

Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from [?] and investigate further connections between these and the α-Ramsey cardinals. First, a definition.

Definition 1.49 (Gitman). A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ. If furthermore $\mathcal{M} \prec H_{\kappa^+}$ then we say that κ is **super Ramsey**. ○

Note that since the model $\mathcal{M}$ in question is a *κ-model* it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$-sequences. [?] shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M} \models \mathsf{ZFC}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ.

Now, a first connection between the α-Ramseys and the strongly- and super Ramseys is the result in [?] that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

Theorem 1.50 (N.-Welch). *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

Proof. Let κ be strongly Ramsey and let $\mathcal{M} \models \mathsf{ZFC}$ be a transitive κ-model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^{\mathcal{M}}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of $\mathcal{M}$ by weak amenability of μ. As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside $\mathcal{M}$, so $\mathcal{M}$ sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \to \mathcal{N}$ be the induced embedding, κ-powerset preservation of j_μ implies that μ is also a weakly amenable $\mathcal{N}$-normal $\mathcal{N}$-measure on κ. This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in $\mathcal{N}$, entailing that it is a stationary limit of almost fully Ramseys in $\mathcal{M}$. But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in [?], so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and $\mathcal{M}$ being transitive implies that κ really *is* a stationary limit of almost fully Ramseys. ■

Downwards absoluteness to K

Lastly, we consider the question of whether the α-Ramseys are downwards absolute to K, which turns out to at least be true in many cases. The below Theorem 1.51 then also answers Question 9.4 from [?] in the positive, asking whether α-Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of $0^\P$.

Definition 1.51. $0^\P$ is "the sharp for a strong cardinal", meaning the minimal sound active mouse $\mathcal{M}$ with $\mathcal{M} \,|\, \text{crit}(\dot{F}^{\mathcal{M}}) \models \ulcorner$There exists a strong cardinal$\urcorner$, with $\dot{F}^{\mathcal{M}}$ being the top extender of $\mathcal{M}$. ○

Theorem 1.52 (N.-Welch). *Assume $0^\P$ does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ-Ramsey. Then $K \models \ulcorner\kappa$ is a λ-Ramsey cardinal$\urcorner$.*

Proof. Note first that $\kappa^{+K} = \kappa^+$ by [?], since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ-models $\mathcal{M} \prec K|\kappa^+$. We can then define a strategy $\tilde\sigma$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde\sigma(\emptyset) := \text{Hull}^{H_{\kappa^+}}(K|\kappa \cup \sigma(\emptyset))$. Assuming now that $\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$, we have two cases. If $\tilde\mu_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle\mathcal{M}_\alpha \mid \alpha < \gamma\rangle$ be the corresponding models played in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde\sigma(\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle) := \text{Hull}^{H_{\kappa^+}}(K|\kappa \cup \sigma(\langle\mathcal{M}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle)),$$

and otherwise let $\tilde\sigma$ play arbitrarily. As κ is λ-Ramsey (in V) there exists a play $\langle\tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha \leq \lambda\rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ \prec K|\kappa^+$ so let $\mathcal{N}$ be the transitive collapse of $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+$. But if $j : \mathcal{N} \to K|\kappa^+$ is the uncollapse then crit j is both an $\mathcal{N}$-cardinal and also $> \kappa$ because we ensured that $K|\kappa \subseteq \mathcal{N}$. This means that $j = \text{id}$ because κ is the largest $\mathcal{N}$-cardinal by elementarity in $K|\kappa^+$, so that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ = \mathcal{N}$ is a transitive elementary substructure of $K|\kappa^+$, making it an initial segment of K.

Now, since $\mu := \tilde\mu_\lambda$ is a countably complete weakly amenable $K|o(\mathcal{N})$-measure[14], the "beaver argument"[15] shows that $\mu \in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu \cap \mathcal{N} \in K$ whenever player I plays $\mathcal{N}$. Since $\mu = \tilde\mu_\lambda$ we also have that $\mu \cap \mathcal{M}_\alpha = \tilde\mu_\alpha \cap \mathcal{M}_\alpha$, so that σ will eventually play $\mathcal{N}$, making τ win against σ.[16] ■

[14]Here we use that $\mathcal{N} \lhd K$.

[15]See Appendix ?? for details regarding the beaver argument.

[16]Note that τ is not necessarily a winning strategy – all we know is that it is winning against this particular strategy σ.

Note that the only thing we used $\operatorname{cof}\lambda > \omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine- or normal α-Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

Corollary 1.53 (N.-Welch). *Assume $0^\P$ does not exist and let α be any limit ordinal. Then every genuine- and every normal α-Ramsey cardinal is downwards absolute to K. In particular, if α is a limit of limit ordinals then every $<\alpha$-Ramsey cardinal is downwards absolute to K as well.* ■

Indiscernible games

We now move to the strategic versions of the α-Ramsey hierarchy. The first thing we want to do is define *α-very Ramsey cardinals*, introduced in [?], and show the tight connection between these and the strategic α-Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 1.42.

Definition 1.54. A **good set of indiscernibles** for a structure $\mathcal{M}$ is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for $\mathcal{M}$ such that $\mathcal{M} \,|\iota \prec \mathcal{M}$ for any $\iota \in I$. ○

Definition 1.55 (Sharpe-Welch). Define the **indiscernible game** $G^I_\gamma(\kappa)$ in γ many rounds as follows

$$\begin{array}{lccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots \\ \text{II} & & I_0 & & I_1 & & I_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha$ is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for $\mathcal{M}_\alpha$ and the I_α's are $\subseteq$-decreasing. Player II wins iff they can continue playing through all the rounds. ○

Definition 1.56 (Sharpe-Welch). A cardinal κ is γ-**very Ramsey** if player II has a winning strategy in the game $G^I_\gamma(\kappa)$. ○

The next couple of results concerns the connection between the strategic α-Ramseys and the α-very Ramseys. We start with the following.

THEOREM 1.57 (N.). *Every $(\omega+1)$-Ramsey is an ω-very Ramsey stationary limit of ω-very Ramseys.*

PROOF. Let κ be $(\omega+1)$-Ramsey. We will describe a winning strategy for player II in the indiscernible game $G^I_\omega(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G^I_\omega(\kappa)$ then let player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ play

$$\mathcal{H}_0 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) \prec H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ which starts off with $\mathcal{H}_0$ and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}^{(\alpha)}_0 := \mathcal{M}_\alpha$ and $\mu^{(\alpha)}_0 := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{lccccccc} \mathrm{I} & \mathcal{H}^{(0)}_0 & & \cdots & & \mathcal{H}^{(\omega)}_0 & & \mathcal{H}^{(\omega+1)}_0 & \\ \mathrm{II} & & \mu^{(0)}_0 & & \cdots & & \mu^{(\omega)}_0 & & \mu^{(\omega+1)}_0 \end{array}$$

By the choice of player I's strategy we get that $\mu^{(\omega)}_0$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ (it's even normal). Now Lemma 2.9 of [?] gives us a set of good indiscernibles $I_0 \in \mu^{(\omega)}_0$ for $\mathcal{M}_0$, as $\mathcal{M}_0 \in \mathcal{H}^{(\omega)}_0$ and $\mu^{(\omega)}_0$ is a countably complete weakly amenable $\mathcal{H}^{(\omega)}_0$-normal $\mathcal{H}^{(\omega)}_0$-measure on κ. Let player II play I_0 in $G^I_\omega(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G^I_\omega(\kappa)$.

$$\begin{array}{lccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 \\ \mathrm{II} & & I_0 & \end{array}$$

Since $\mu^{(\omega)}_0 = \bigcup_n \mu^{(n)}_0$ we must have that $I_0 \in \mu^{(n_0)}_0$ for some $n_0 < \omega$. In the (n_0+1)'st round of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}^{(k)}_0, \mu^{(k)}_0 \mid k \leq n_0 \rangle\}) \prec H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ thus looks like the following so far.

$$\begin{array}{lccccccc} \text{I} & \mathcal{H}_0^{(0)} & & \cdots & & \mathcal{H}_0^{(n_0)} & & \mathcal{H}_1 \\ \text{II} & & \mu_0^{(0)} & & \cdots & & \mu_0^{(n_0)} & \end{array}$$

Now player II in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ is not losing at round n_0, so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I_1' \in \mu_1^{(n_1)}$ of good indiscernibles for $\mathcal{M}_1$, where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G^I_\omega(\kappa)$ play $I_1 := I_0 \cap I_1' \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G^I_\omega(\kappa)$, making κ ω-very Ramsey.

As for showing that κ is a stationary limit of ω-very Ramseys, let $\mathcal{M} \prec H_{\kappa^+}$ be a weak κ-model with a weakly amenable countably complete $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ, which exists by Theorem 1.46 as κ is $(\omega{+}1)$-Ramsey. Then by elementarity $\mathcal{M} \models \ulcorner\kappa$ is ω-very Ramsey$\urcorner$ and since κ being ω-very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ-ultrapower, meaning that κ is a stationary limit of ω-very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

Theorem 1.58 (N.). *For limit ordinals α, every coherent $<\omega\alpha$-Ramsey is $\omega\alpha$-very Ramsey.*

Proof. This is basically the same proof as the proof of Theorem 1.56. We do the "going-back" trick in ω-chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}^{\kappa^+}_{\omega\alpha}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$-Ramseyness. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω-chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 1.56. ■

As for going from the α-very Ramseys to the strategic α-Ramseys we got the following.

Theorem 1.59 (N.). *For γ any ordinal, every coherent $<\gamma$-very Ramsey*[17] *is coherent $<\gamma$-Ramsey.*[18]

Proof. The reason why we work with $<\gamma$-Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ-very Ramsey and let τ be a winning strategy for player II in $G^I_\zeta(\kappa)$. Let $\mathcal{M}_\alpha \prec H_\theta$ be any move by player I in the α'th round of $\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in $\mathcal{M}_\alpha$ and form now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G^I_\zeta(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for $\mathcal{N}_\alpha$ such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in [?] we get a structure $\mathcal{P}_\alpha$ with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a $\mathcal{P}_\alpha$-measure $\tilde{\mu}_\alpha$ on κ, generated by I_α.[19] Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α's are generated by the I_α's, the μ_α's are $\subseteq$-increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$-Ramsey. ■

The following result is then a direct corollary of Theorems 1.57 and 1.58.

Corollary 1.60 (N.). *For limit ordinals α, κ is $\omega\alpha$-very Ramsey iff it is coherent $<\omega\alpha$-Ramsey. In particular, κ is λ-very Ramsey iff it is strategic λ-Ramsey for any λ with uncountable cofinality.* ■

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[17] Here the coherency again just means that the winning strategies σ_α for player II in $G^I_\alpha(\kappa)$ are $\subseteq$-increasing.

[18] Here a "coherent $<\gamma$-very Ramsey cardinal" is defined from γ-very Ramseys in the same way as coherent $<\gamma$-Ramsey cardinals is defined from γ-Ramseys. When γ is a limit ordinal then coherent $<\gamma$-very Ramseys are precisely the same as γ-very Ramseys, so this is solely to "subtract one" when γ is a successor ordinal – i.e. a coherent $<(\gamma+1)$-very Ramsey cardinal is the same thing as a γ-very Ramsey cardinal.

[19] By *generated* here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α.

We can now use this equivalence to transfer results from the α-very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in [?]. A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in [?]. We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in [?].

Theorem 1.61 (Sharpe-Welch). *A cardinal is completely Ramsey if and only if it is ω-very Ramsey.* ■

This, together with Theorem 1.56, immediately yields the following strengthening of Theorem 1.46.

Corollary 1.62 (N.). *Every $(\omega{+}1)$-Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* ■

The above Theorem 1.58 also yields the following consequence.

Corollary 1.63 (N.). *Every completely Ramsey cardinal is completely ineffable.*

Proof. From Theorem 1.60 we have that being completely Ramsey is equivalent to being ω-very Ramsey, so the above Theorem 1.58 then yields that a completely Ramsey cardinal is coherent $<\omega$-Ramsey, which we saw in Theorem 1.28 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 1.59 yields that strategic ω_1-Ramsey cardinals are ω_1-very Ramsey, and Theorem 3.50 in [?] states that ω_1-very Ramseys are measurable in the core model K, assuming $0^\P$ doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

Theorem 1.64 (Welch). *Assuming $0^\P$ doesn't exist, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Let κ be strategic ω_1-Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \text{Col}(\omega_1, \kappa^+)$ is V-generic. Since $\text{Col}(\omega_1, \kappa^+)$

is ω-closed, V and $V[g]$ have the same countable sequences of V, so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V.

Now let $\langle\kappa_\alpha \mid \alpha < \omega_1\rangle$ be an increasing sequence of regular K-cardinals cofinal in κ^+, let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \mathrm{Hull}^{H_\theta}(K|\kappa_\alpha) \prec H_\theta$ and player II follow τ. This results in a countably complete weakly amenable K-measure μ_{ω_1}, which the "beaver argument"[20] then shows is actually an element of K, making κ measurable in K. ■

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1-Ramsey cardinal is also measurable in Steel's core model below a Woodin[21], a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from [?].[22]

Theorem 1.65 (Schindler). *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ, and let $\pi : V \to \mathrm{Ult}(V,\mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \upharpoonright K$.* ■

Theorem 1.66 (Schindler). *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ of weak κ-models $M_\alpha \prec H^V_\theta$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H^V_{\kappa^+} \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\mathrm{Col}(\omega_1, 2^\kappa)$ is $<\!\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Let $j : M_{\omega_1} \to \mathrm{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ

[20]See Appendix ?? for details regarding the beaver argument.
[21]See Appendix ??.
[22]That paper assumes the existence of a measurable as well, but by [?] we can omit that here.

countably complete in $V[g]$. As we also ensured that $H^V_{\kappa^+} \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ with N transitive.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 1.64 applies, meaning that $\pi \restriction K : K \to K^N$ is an iteration map with critical point κ, making κ measurable in K. ■

1.2 Ideals

Historically, the idea of considering elementary embeddings existing only in generic extensions has been around for a while, but it all started as an analysis of *ideals*. *Precipitous ideals* were introduced in [?] and further analysed in [?], being ideals that give rise to wellfounded generic ultrapowers[23].

In this section we will introduce the *ideally measurable cardinals*, essentially just switching perspective from the ideals themselves to the cardinals they are on. We then proceed to show how these cardinals relate to "pure" generic cardinals, being proper class versions of the faintly measurable cardinals that we have considered throughout Chapter ??. We start with a definition of the latter.

Definition 1.67 (GBC). A cardinal κ is **generically measurable** if there is a generic extension $V[g]$, a transitive class $\mathcal{N} \subseteq V[g]$ and a generic elementary embedding $\pi\colon V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$. ○

Note that, trivially, every generically measurable cardinal is faintly measurable. The corresponding ideal version of this is then the following.

Definition 1.68. A cardinal κ is **ideally measurable** if there exists an ideal $\mathcal{I}$ on θ such that the generic ultrapower $\mathrm{Ult}(V, \mathcal{I})$ is wellfounded in $V^{\mathbb{P}}$ for $\mathbb{P} := \mathscr{P}^V(\kappa)/\mathcal{I}$. ○

It should also be noted that [?] generalised the concept of ideally measurables to *ideally strong cardinals* by introducing the concept of *ideal extenders* to capture the strongness properties.

[23] See Appendix ?? for some preliminaries on these concepts.

Throughout this section we will be interested in how properties of the *forcings* affect the large cardinal structure of a critical point of a generic embedding. We thus define the following.

Definition 1.69. Let θ be a regular uncountable cardinal, $\kappa < \theta$ a cardinal and $\Phi(\kappa)$ a poset property[24]. Then κ is $\Phi(\kappa)$ **faintly θ-measurable** if it is faintly θ-measurable, witnessed by a forcing poset satisfying $\Phi(\kappa)$. Similarly, κ is $\Phi(\kappa)$ **generically measurable** if it is generically measurable with the associated forcing satisfying $\Phi(\kappa)$. ○

Note that ω-distributive faintly θ-measurable cardinals are equivalent to ω-distributive generically measurable cardinals for all regular θ since wellfoundedness becomes automatic.

Definition 1.70. A poset property $\Phi(\kappa)$ is **ideal-absolute** if whenever κ satisfies that there's a $\Phi(\kappa)$ forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure μ on κ, then there's an ideal I on κ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(v)$. ○

Note that this is *almost* saying that $\Phi(\kappa)$ ideally measurables are equivalent to $\Phi(\kappa)$ generically measurables, but the only difference is that these definitions require wellfoundedness of the target model.

A typical ideal that we will be utilising is the following.

Definition 1.71. Let κ be a regular cardinal, $\mathbb{P}$ a poset and $\dot\mu$ a $\mathbb{P}$-name for a V-normal V-measure on κ. Then the **induced ideal** is

$$\mathcal{I}(\mathbb{P},\dot\mu) := \{X \subseteq \kappa \mid ||\check{X} \in \dot\mu||_{\mathcal{B}(\mathbb{P})} = 0\},$$

where $\mathcal{B}(\mathbb{P})$ is the boolean completion of $\mathbb{P}$. ○

[24]Examples of these are having the κ-chain condition, being κ-closed, κ-distributive, κ-Knaster, κ-sized and so on. Formally speaking, $\Phi(\kappa)$ is a first-order formula $\varphi(\kappa,\mathbb{P})$ which is true iff $\mathbb{P}$ is a poset, κ is a cardinal and some first-order formula $\psi(\kappa,\mathbb{P})$ is true.

Note that if the generic measure μ is V-normal then $\mathcal{I}(\mathbb{P}, \dot{\mu})$ is also normal. This ideal will witness our first ideal-absoluteness result, which is a simple rephrasing of a folklore result.

Theorem 1.72 (Folklore). *"The κ^+-chain condition" is ideal-absolute.*

Proof. Assume $\mathbb{P}$ has the κ^+-chain condition such that there's a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot{\mu})$ – we will show that $\mathscr{P}(\kappa)/I$ has the κ^+-chain condition. Assume not and let $\langle X_\alpha \mid \alpha < \kappa^+\rangle$ be an antichain of $\mathscr{P}(\kappa)/I$, which by normality of I we may assume is pairwise almost disjoint. But this then makes $\langle ||\check{X}_\alpha \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid \alpha < \kappa^+\rangle$ an antichain of $\mathbb{P}$ of size κ^+, $\lightning$. ■

We next move to distributivity. This property is especially interesting in the context of our generic large cardinals, as an ideal I on some cardinal κ is ω-distributive precisely if it's precipitous[25], so that carrying an ω-distributive ideal coincides with our definition of *ideally measurable*.

Theorem 1.73 (N.). *"$<\lambda$-distributivity" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*

Proof. Assume that $\mathbb{P}$ is a $<\lambda$-distributive forcing such that there exists a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $\mathcal{I} := \mathcal{I}(\mathbb{P}, \dot{\mu})$ – we'll show that $\mathscr{P}(\kappa)/\mathcal{I}$ is $<\lambda$-distributive.

Let $\gamma < \lambda$ and let $\vec{\mathcal{A}}$ be a γ-sequence of maximum antichains $\mathcal{A}_\alpha \subseteq \mathscr{P}(\kappa)/\mathcal{I}$ such that $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ for $\alpha \leq \beta$. We have to show that there's a maximal antichain $\mathcal{A}$ which refines all the antichains in $\vec{\mathcal{A}}$.

Now define for every $\alpha < \gamma$ the sets

$$\mathcal{A}^*_\alpha := \{||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid X \in \mathcal{A}_\alpha\}.$$

Note that $\mathcal{A}^*_\alpha$ is an antichain in $\mathbb{P}$. They're also maximal, because if $p \in \mathbb{P}$ was incompatible with every condition in $\mathcal{A}^*_\alpha$ then, letting $X := \bigcap \mathcal{A}_\alpha$, we have that p is compatible with $||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$, so that $X \in \mathcal{I}^+$. But X is incompatible with everything in $\mathcal{A}_\alpha$, contradicting that $\mathcal{A}_\alpha$ is maximal.

[25]See [?] and [?].

By $<\lambda$-distributivity of $\mathbb{P}$ we get an antichain $\mathcal{A}^*$ which refines all the antichains in $\vec{\mathcal{A}}^*$. But note that for every $p \in \mathcal{A}^*$, if we define $s_p(\alpha)$ to be the unique $a \in \mathcal{A}_\alpha$ such that $p \leq a$, then it holds that $p \leq ||\Delta s_p \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$,[26] so that $\Delta b_p \in \mathcal{I}^+$. Now $\mathcal{A} := \{\Delta b_p \mid p \in \mathcal{A}^*\}$ gives us a maximal antichain consisting of limit points of branches of $\mathcal{T}$. ■

The main technical result of this section is then the following. In an unpublished paper, Foreman proved the following.

Theorem 1.74 (Foreman). *Let κ be a regular cardinal such that $2^\kappa = \kappa^+$, and let $\lambda \leq \kappa^+$ be an infinite successor cardinal. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal precipitous ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ has a dense $<\lambda$-closed subset of size κ^+.*

Here we improve that result by not relying on the CH-assumption, reaching the conclusion for all regular infinite λ and also showing (κ, κ)-distributivity of the ideal forcing. The argument follows the same overall structure as the original, with more technicalities to achieve the stronger result.

Theorem 1.75 (Foreman-N.). *Let κ be a regular cardinal and $\lambda \leq \kappa^+$ be regular infinite. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is (κ, κ)-distributive and has a dense $<\lambda$-closed subset of size κ^+.*

Proof. Set $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$ if $2^\kappa > \kappa^+$ and $\mathbb{P} := \{\emptyset\}$ otherwise. If κ is measurable then the dual ideal to the measure on κ satisfies all of the wanted properties, so assume that κ is not measurable. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \kappa^+ \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \kappa^+$;
- $\mathcal{N}_{\gamma+1} \prec H^V_{\kappa^+}$ is a κ-model for every $\gamma < \kappa^+$;
- $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$ for limit ordinals $\delta < \kappa^+$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for $\gamma < \beta < \kappa^+$;

[26] Here we're using that $\lambda \leq \kappa^+$ to ensure that the diagonal intersection is in the measure.

- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\kappa^+} \mathcal{N}_\gamma$.

Define now the auxilliary game $\mathcal{G}(\kappa)$ of length λ as follows.

I	α_0		α_1		$\cdots$	
II		$p_0, \mathcal{M}_0, \mu_0, Y_0$		$p_1, \mathcal{M}_1, \mu_1, Y_1$		$\cdots$

Here $\langle \alpha_\gamma \mid \gamma < \lambda \rangle$ is an increasing continuous sequence of ordinals bounded in κ^+, $\vec{p}_\gamma$ is a decreasing sequence of $\mathbb{P}$-conditions satisfying that

$$p_\gamma \Vdash \ulcorner \check{\mathcal{M}}_\gamma = \pi(\check{\alpha}_\gamma) \wedge \check{\mu}_\gamma \text{ is a } \check{\mathcal{M}}_\gamma\text{-normal } \check{\mathcal{M}}_\gamma\text{-measure on } \check{\kappa} \urcorner$$

such that $Y_\gamma = \Delta_{\xi<\kappa} X_\xi^{\mu_\gamma}$, where $\vec{X}_\xi^{\mu_\gamma} \in H_{\kappa^+}^V$ is the $<_{\kappa^+}$-least enumeration of μ_γ.[27] We require that the μ_γ's are $\subseteq$-increasing, and player II wins iff she can continue playing throughout all λ rounds. Let $\mu_\lambda := \bigcup_{\xi<\lambda} \mu_\xi$ be the **final measure** of the play.

To every limit ordinal $\eta < \kappa^+$ define the **restricted auxilliary game** $\mathcal{G}(\kappa) \upharpoonright \eta$ in which player I is only allowed to play ordinals $<\eta$. Note that a strategy τ for player II is winning in $\mathcal{G}(\kappa)$ if and only if it's winning in $\mathcal{G}(\kappa) \upharpoonright \eta$ for all $\eta < \kappa^+$, simply because all sequences of ordinals played by player I are bounded in κ^+.

Note that μ_λ is precisely the tail measure on κ defined by the Y_γ's; i.e. that $X \in \mu_\lambda$ iff there exists a $\delta < \lambda$ such that $|Y_\delta - X| < \kappa$. From this it's simple to see that $\mathcal{G}(\kappa)$ is equivalent to $\mathcal{G}_\lambda^-(\kappa)$, so player II has a winning strategy τ_0 in $\mathcal{G}(\kappa)$.

For any winning strategy τ in $\mathcal{G}(\kappa) \upharpoonright \eta$ and to every partial play p of $\mathcal{G}(\kappa) \upharpoonright \eta$ consistent with τ, define the associated **hopeless ideal**[28]

$$\begin{aligned} I_p^\tau \upharpoonright \eta := \{X \subseteq \kappa \mid & \text{ For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa) \upharpoonright \eta, \\ & X \text{ is } \textit{not} \text{ in the final measure}\} \end{aligned}$$

Claim 1.76. Every hopeless ideal $I_p^\tau \upharpoonright \eta$ is normal and (κ, κ)-distributive.

PROOF OF CLAIM. For normality, if $\langle Z_\gamma \mid \gamma < \kappa \rangle$ is a sequence of elements of I_p^τ such that $Z := \nabla_\gamma Z_\gamma$ is I_p^τ-positive, then there exists a play of $\mathcal{G}(\kappa) \upharpoonright \eta$ in

[27]We use that $\mathbb{P}$ is κ-closed to get the p_γ's as well as to ensure that $\mathcal{M}_\gamma, \mu_\gamma \in V$.
[28]This terminology is due to Matt Foreman.

which player II follows τ such that Z lies in the final measure. If we let player I play sufficiently large ordinals in $\mathcal{G}(\kappa)\restriction\eta$ we may assume that $\langle Z_\gamma \mid \gamma < \kappa\rangle$ is a subset and an element of the final model as well, meaning that one of the Z_γ's also lies in the final measure, $\lightning$.

We now show (κ,κ)-distributivity. Let $\mathcal{U} \subseteq \mathscr{P}(\kappa)/I^\tau_p$ be an unrooted tree of height κ such that every level $\mathcal{U}_\alpha$ is a maximal antichain of size $\leq \kappa$. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of branches of $\mathcal{U}$. Pick $X \in \mathcal{U}$ and let p be a play of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ with limit model $\mathcal{M}$ and limit measure μ, such that $X \in \mu$.

By letting player I in p play sufficiently large ordinals, we may assume that $\mathcal{U} \subseteq \mathcal{M}$, using that $|\mathcal{U}| \leq \kappa$, and also that $b_X := \mathcal{U} \cap \mu \in \mathcal{M}$. This means that $d_X := \Delta b_X \in \mathscr{P}(\kappa)/I^\tau_p$ is a limit point of the branch b_X through $\mathcal{U}$, so that $\mathcal{A} := \{d_X \mid X \in \mathcal{U}\}$ is a maximal antichain of limit points of branches of $\mathcal{U}$, making $\mathscr{P}(\kappa)/I^\tau_p$ (κ,κ)-distributive. $\dashv$

Fix some limit ordinal $\eta < \kappa^+$. We will recursively construct a tree $\mathcal{T}^\eta$ of height λ which consists of subsets $X \subseteq \kappa$, ordered by reverse inclusion. During the construction of the tree we will inductively maintain the following properties of $\mathcal{T}^\eta\restriction\alpha$ for $\alpha \leq \lambda$:

- Tree strategy: For every $\gamma < \alpha$ there is a winning strategy τ^η_γ for player II in $\mathcal{G}(\kappa)\restriction\eta$ such that for every $\beta < \gamma$, the β'th move by τ^η_γ is an element of $\mathcal{T}^\eta_\beta$ and τ^η_γ is consistent with τ^η_β for the first β-many rounds.
- Unique pre-history: Given any $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$ there's a unique partial play p of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ^η_β ending with Y – we define $I^\tau_Y := I^\tau_p$ for τ being any winning strategy for player II in $\mathcal{G}(\kappa)\restriction\eta$ satisfying that p is consistent with τ^η_β.
- Cofinally many responds: Let $\beta + 1 < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$, and set p to be the unique partial play of $\mathcal{G}(\kappa)\restriction\eta$ given by the unique pre-history of Y. Then the $\mathcal{T}^\eta$-successors of Y consists of player II's τ^η_β-responds to τ^η_β-partial plays extending p such that player I's last move in these partial plays are cofinal in η.[29]

[29]The reason why we're dealing with the *restricted* auxilliary games is to achieve this property.

- POSITIVITY: If $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$ then Y is $I^{\tau^\eta_\gamma}_X$-positive for every $\gamma < \beta$ and every $X \in \mathcal{T}^\eta \restriction \gamma + 1$ with $X \leq_{\mathcal{T}^\eta} Y$.[30]
- ALMOST DISJOINTNESS PROPERTY: Every level $\mathcal{T}^\eta_\beta$ consists of pairwise almost disjoint sets.[31]
- HOPELESS IDEAL COHERENCE: $I^{\tau^\eta_\beta}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_\beta}_Y \cap \mathscr{P}(Y)$ for every $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$.

Note that what we're really aiming for is achieving the hopeless ideal coherence, since that enables us to ensure that if $X, Y \in \mathcal{T}^\eta$ and $X \subseteq Y$ then really $X \geq_{\mathcal{T}^\eta} Y$ – i.e. that we "catch" both X and Y in the same play of $\mathcal{G}(\kappa) \restriction \eta$. The rest of the properties are inductive properties we need to ensure this.

Set $\mathcal{T}^\eta_0 := \{\kappa\}$. Assume that we've built $\mathcal{T}^\eta \restriction \alpha + 1$ satisfying the inductive assumptions[32] and let $Y \in \mathcal{T}^\eta_\alpha$ – we need to specify what the $\mathcal{T}^\eta$-successors of Y are. Since κ is weakly compact and not measurable it holds by Proposition 6.4 in [?] that $\text{sat}(I^{\tau^\eta_\alpha}_Y) \geq \kappa^+$, so we can fix a maximal antichain $\langle X^Y_\gamma \mid \gamma < \eta \rangle$ of $I^{\tau^\eta_\alpha}_Y$-positive sets. By κ-completeness of $I^{\tau^\eta_\alpha}_Y$ we can by Exercise 22.1 in [?] even ensure that all of the X^Y_γ's are pairwise disjoint.

To every $\gamma < \eta$ we fix a partial play p of even length of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ^η_α such that the last ordinal β^Y_γ in p played by player I is greater than or equal to γ and X^Y_γ has measure one with respect to the last measure in p. We then define the $\mathcal{T}^\eta$-successors of Y to be player II's τ^η_α-responses to the β_γ's (which are subsets of the X^Y_γ's modulo a bounded set and are therefore pairwise almost disjoint).

For limit stages $\delta < \lambda$ we apply τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ to get $\mathcal{T}^\eta_\delta$.

We now have to check that the inductive assumptions still hold; let's start with the tree strategy. Assume that we have a partial play p of length $2 \cdot \alpha + 1$ of $\mathcal{G}(\kappa) \restriction \eta$, i.e. the last move in p is by player II, consistent with τ^η_α; write ξ_p for player I's last move in p and Y_p for player II's response to ξ_p, which is also the last move in p. We can then pick a $\zeta < \eta$ such that $\beta^{Y_p}_\zeta > \xi_p$ by the cofinally many responds property and let $\tau^\eta_{\alpha+1}(p)$ be player II's τ^η_α-response to the partial play leading up to $\beta^{Y_p}_\zeta$. After this $(\alpha + 1)$'th round we just set $\tau^\eta_{\alpha+1}$ to follow τ_0. It's clear that $\tau^\eta_{\alpha+1}$ satisfies the required properties.

[30] This actually follows from the cofinally many responds, but we include it here for transparency.

[31] Two subsets $X, Y \subseteq \kappa$ are *almost disjoint* if $|X \cap Y| < \kappa$.

[32] In particular, we assume that τ^η_α is defined.

Before we move on to checking the remaining inductive assumptions, let's pause to get some intuition about the tree strategies. In the definition of $\tau^\eta_{\alpha+1}$ above, we took a partial play consistent with τ^η_α, applied τ_0 for a while, took note of player II's last τ_0-response and then included *only that* response in our new $\tau^\eta_{\alpha+1}$ partial play. This means that to every τ^η_α-partial play there's an ostensibly much longer τ_0-partial play into which τ^η_α embeds; so we can look at the τ^η_α-partial plays as being "collapsed" τ_0-partial plays.

Given the above tree strategy, $\mathcal{T}^\eta_{\alpha+1}$ clearly satisfies the cofinally many responds property and the positivity property, simply by construction. For the unique pre-history, let $Y \in \mathcal{T}^\eta_{\alpha+1}$ and assume it has two distinct immediate $\mathcal{T}^\eta$-predecessors $Z_0, Z_1 \in \mathcal{T}^\eta_\alpha$. But then $Y \subseteq Z_0 \cap Z_1$ and Y is $I^{\tau^\eta_\alpha}_{Z_0}$-positive by the positivity assumption, contradicting that Z_0 and Z_1 are almost disjoint by the almost disjointness property. Given the unique pre-history we then also get the almost disjointness property.

Claim 1.77. $\mathcal{T}^\eta \restriction \alpha + 2$ satisfies the hopeless ideal coherence property.

Proof of claim. Let $Y \in \mathcal{T}^\eta_{\alpha+1}$ – we have to show that

$$I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y). \tag{1}$$

It's clear that $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \subseteq I^{\tau^\eta_{\alpha+1}}_{Y}$, so let $Z \in I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y)$ and assume for a contradiction that Z is $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle}$-positive. Letting $\vec{\alpha}_\xi * \vec{Y}_\xi$ be a play of $\mathcal{G}(\kappa) \restriction \eta$ consistent with $\tau^\eta_{\alpha+1}$ such that Z is in the final measure, the definition of $\tau^\eta_{\alpha+1}$ yields that $Y_\alpha \in \mathcal{T}^\eta_{\alpha+1}$. As $Z \in I^{\tau^\eta_{\alpha+1}}_{Y}$ we have to assume that $Y \neq Y_\alpha$, so that the almost disjointness property implies that

$$|Y \cap Y_\alpha| < \kappa, \tag{2}$$

By the choice of $\vec{\alpha}_\xi * \vec{Y}_\xi$ there's some $\delta \in (\alpha, \lambda)$ such that $|Y_\delta - Z| < \kappa$, i.e. that Y_δ is a subset of Z modulo a bounded set, since the Y_α's generate the final measure of the play. But then $Y_\delta \subseteq Y_\alpha$ by the rules of $\mathcal{G}(\kappa) \restriction \eta$, and also that $|Y_\delta - Y| < \kappa$ since $Z \subseteq Y$. But this means that $Y \cap Y_\alpha$ is $I^{\tau^\eta_{\alpha+1}}_{Y}$-positive since

Y_δ is, contradicting (2). This shows (1). $\dashv$

This finishes the construction of $\mathcal{T}^\eta_{\alpha+1}$. For limit levels $\delta < \lambda$ we define τ^η_δ as simply applying τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ – showing that the inductive assumptions hold at $\mathcal{T}^\eta_\delta$ is analogous to the above arguments, so we're now done with the construction of $\mathcal{T}^\eta$. Let $\tau^\eta := \bigcup_{\alpha<\lambda} \tau^\eta_\alpha \restriction {}^{<\alpha}H_{\kappa^+}$ and define[33] $\mathcal{I}^\eta := I^{\tau^\eta}_{\langle\rangle}$.

Now note that $\mathcal{I}^{\eta+1} \subseteq \mathcal{I}^\eta$ and $\mathcal{T}^\eta \subseteq \mathcal{T}^{\eta+1}$ for every $\eta < \kappa^+$ – set $\mathcal{I} := \bigcap_{\eta<\kappa^+} \mathcal{I}^\eta$ and $\mathcal{T} := \bigcup_{\eta<\kappa^+} \mathcal{T}^\eta$. We showed that all hopeless ideals are κ-complete, normal and (κ, κ)-distributive, so this holds in particular for the $\mathcal{I}^\eta$'s and thus also for $\mathcal{I}$.

We claim that $\mathcal{T}$ is dense in $\mathscr{P}(\kappa)/\mathcal{I}$.[34] Let X be an $\mathcal{I}$-positive set, making it $\mathcal{I}^\eta$-positive for some $\eta < \kappa^+$, meaning that there's a play $\vec{\alpha}_\gamma * \tau^\eta$ of $\mathcal{G}(\kappa) \restriction \eta$ such that X is in the final measure, which means that $|Y_\delta - X| < \kappa$ for some large $\delta < \lambda$ and in particular that $Y_\delta - X \in \mathcal{I}$. But $Y_\delta \in \mathcal{T}^\eta \subseteq \mathcal{T}$ by definition of τ^η, which shows that $\mathcal{T}$ is dense.

It remains to show that $\mathcal{T}$ is $<\lambda$-closed. If $\lambda = \omega$ then this is trivial, so assume that $\lambda \geq \omega_1$. Let $\beta < \lambda$ and let $\langle Z_\alpha \mid \alpha < \beta \rangle$ be a $\subseteq$-decreasing sequence of elements $Z_\alpha \in \mathcal{T}$. We can fix some $\eta < \kappa^+$ such that $Z_\alpha \in \mathcal{T}^\eta$ for every $\alpha < \beta$ by regularity of κ^+, and since the Z_α's are $\subseteq$-decreasing they must also be $\leq_{\mathcal{T}^\eta}$-increasing by the hopeless ideal coherence for $\mathcal{T}^\eta$[35].

Let $\tilde{Z} \in \mathcal{T}^\eta$ be player II's τ^η-response to the unique partial play of $\mathcal{G}(\kappa) \restriction \eta$ corresponding to the branch containing the Z_α's, and pick $Z \in \mathcal{T}^\eta$ such that $\left|Z - \tilde{Z}\right| < \kappa$ and $Z \geq_{\mathcal{T}^\eta} Z_\alpha$ for all $\alpha < \beta$, again by the density claim and the hopeless ideal coherence. Then Z witnesses $<\lambda$-closure of $\mathcal{T}$.[36] ∎

With a bit more work we can from this result then derive the following equivalences.

Corollary 1.78 (N.). *Let κ be a regular cardinal and $\lambda \in [\omega_1, \kappa^+]$ be regular. Then the following are equivalent:*

(i) κ is $<\lambda$-closed faintly power-measurable;

[33]Note that the tree strategy property above ensures that the strategies *do* line up, so that τ^η is a well-defined strategy as well.

[34]This means that given any $\mathcal{I}$-positive set X there's a $Y \in \mathcal{T}$ such that $Y - X \in \mathcal{I}$.

[35]This is the only place in which we're using hopeless ideal coherence.

[36]We're using that λ is regular to get Z.

(ii) κ *is* $<\lambda$*-closed ideally power-measurable;*

(iii) κ *is* (κ,κ)*-distributive* $<\lambda$*-closed faintly measurable;*

(iv) κ *is* (κ,κ)*-distributive* $<\lambda$*-closed ideally measurable;*

(v) Player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$*.*

Proof. $(v) \Rightarrow (iv)$ is Theorem 1.74 above[37] and $(iv) \Rightarrow (iii) + (ii)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial, so we show $(i) \Rightarrow (v)$.

Assume κ is $<\lambda$-closed faintly power-measurable, so there's a $<\lambda$-closed forcing $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there exists a transitive class N and a κ-powerset preserving elementary embedding $\pi\colon V \to N$. Write μ for the induced weakly amenable V-normal V-measure on κ. Now, back in V, define a strategy σ for player II in $G_\lambda(\kappa)$ as follows.

Whenever player I plays some model M_α then we let player II respond with a filter μ_α such that, for some $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash \ulcorner \check{\mu}_\alpha = \dot{\mu} \cap \check{M}_\alpha \urcorner$ – such a filter exists because μ is weakly amenable. We require the p_α's to be decreasing, which is possible by $<\lambda$-closure. Now, all the μ_α's are clearly M_α-normal M_α-measures on κ, which makes σ a winning strategy. ■

Note that the above results all relied on λ being uncountable to achieve wellfoundedness of the generic ultrapower. If we simply ignore this wellfoundedness aspect then we get the following similar equivalence in the $\lambda = \omega$ case, which then also includes completely ineffable cardinals.

Corollary 1.79 (N.). *Let* κ *be a regular cardinal. Then the following are equivalent:*[38]

(i) There exists a forcing poset $\mathbb{P}$ *such that, in* $V^{\mathbb{P}}$*, there's a weakly amenable* V*-normal* V*-measure on* κ*;*

(ii) There exists a (κ,κ)*-distributive forcing poset* $\mathbb{P}$ *such that, in* $V^{\mathbb{P}}$*, there's a* V*-normal* V*-measure on* κ*;*

(iii) κ *carries a normal* (κ,κ)*-distributive ideal;*

(iv) Player II has a winning strategy in $\mathcal{G}^-_\omega(\kappa)$*;*

[37] Here wellfoundedness of the generic ultrapower is automatic since λ has uncountable cofinality.

[38] Points (i) and (ii) look a lot like the definition of faintly power-measurable and (κ,κ)-distributive ideally measurable, but here we're not requiring the ultrapowers to be well-founded, so that would be stretching the definition of being measurable.

(v) κ *is completely ineffable.*

Proof. $(iv) \Leftrightarrow (v)$ was shown in Theorem 1.28, and $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. $(i) \Rightarrow (iv)$ is as $(i) \Rightarrow (v)$ in Corollary 1.77, and $(iv) \Rightarrow (iii)$ is Theorem 1.74. ■

As an immediate consequence we then get another ideal-absoluteness result.

Corollary 1.80. *"(κ,κ)-distributive $<\lambda$-closed" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■

We get the following similar results for the $\mathcal{C}$-games[39].

Theorem 1.81 (N.). *Let κ and $\lambda \leq \kappa^+$ be regular infinite cardinals such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. If player II has a winning strategy in $\mathcal{C}^-_\lambda(\kappa)$ then κ carries a λ-complete ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to Add$(\lambda, 1)$.*

Proof. If $\lambda = \kappa^+$ then we're done by Theorem 1.74, since $\mathcal{G}_{\kappa^+}(\kappa)$ is equivalent to $\mathcal{C}_{\kappa^+}(\kappa)$, so assume that $\lambda \leq \kappa$. We follow the proof of Theorem 1.74 closely. Set $\mathbb{P} := \mathrm{Col}(\lambda, 2^\kappa)$. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \lambda\rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \lambda$;
- $\kappa{+}1 \subseteq \mathcal{N}_\gamma$ and $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for every $\gamma < \lambda$;
- If $\delta < \lambda$ is a limit ordinal then $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$, $\mathcal{N}_\delta \prec H_{\kappa^+}$ and $\mathcal{N}_\delta \models$ ZFC^-;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for all $\gamma < \beta < \lambda$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\lambda} \mathcal{N}_\gamma$.

Define the auxilliary game $\mathcal{G}(\kappa)$ as in the proof of Theorem 1.74 but where player I plays ordinals $\alpha_\eta < \lambda$ and where we use the above $\mathcal{N}_\gamma$'s. Here we only need $<\lambda$-closure of $\mathbb{P}$ to get an equivalence between $\mathcal{G}(\kappa)$ and $\mathcal{C}^-_\lambda(\kappa)$, since $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for all $\gamma < \lambda$.

[39] Theorem 1.80 is the reason for naming the $\mathcal{C}$-games "Cohen games".

To every limit ordinal $\eta < \lambda$ we define the restricted auxilliary game $\mathcal{G}(\kappa) \restriction \eta$ as in the proof of Theorem 1.74, and to every winning strategy τ in $\mathcal{G}(\kappa) \restriction \eta$ and partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ define the associated **hopeless ideal**[40]

$$\begin{aligned} I_p^\tau \restriction \eta := \{X \subseteq \kappa \mid &\text{ For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa) \restriction \eta, \\ &X \text{ is } \textit{not} \text{ in the final measure}\}. \end{aligned}$$

As in the proof of Claim 1.75 we get that every hopeless ideal is λ-complete.

Now, if κ is measurable then we trivially get the conclusion,[41] so assume κ isn't measurable. Then $\operatorname{sat}(\kappa) \geq \lambda$ since $2^{<\theta} < \kappa$ for every $\theta < \lambda$,[42] so that we can continue exactly as in the proof of Theorem 1.74 to construct (λ-sized) trees $\mathcal{T}^\eta$ and winning strategies τ^η for all limit ordinals $\eta < \lambda$ such that, setting $\mathcal{I} := \bigcap_{\eta<\lambda} I_{\langle\rangle}^{\tau^\eta}$ and $\mathcal{T} := \bigcup_{\eta<\lambda} \mathcal{T}^\eta$, $\mathcal{T}$ is a dense $<\lambda$-closed subset of $\mathscr{P}(\kappa)/\mathcal{I}$ of size λ, so that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $\operatorname{Add}(\lambda, 1)$. ■

Corollary 1.82 (N.). *Let κ and $\lambda \in [\omega_1, \kappa^+]$ be regular such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. Then the following are equivalent:*

(i) κ is $<\lambda$-closed faintly measurable;
(ii) κ is $<\lambda$-closed ideally measurable;
(iii) κ is $<\lambda$-closed λ-sized faintly measurable;
(iv) κ is $<\lambda$-closed λ-sized ideally measurable;
(v) Player II has a winning strategy in $\mathcal{C}_\lambda(\kappa)$.

Proof. $(iv) \Rightarrow (iii) + (ii)$, $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ all trivial, and $(i) \Rightarrow (v)$ is like $(i) \Rightarrow (v)$ in Corollary 1.77, and $(v) \Rightarrow (iv)$ is Theorem 1.80. ■

Again, if we ignore wellfoundedness then we get the same equivalence in the $\lambda = \omega$ case:

Corollary 1.83 (N.). *Let κ be regular infinite. Then:*

(i) Player II has a winning strategy in $\mathcal{C}_\omega^-(\kappa)$; and

[40] This terminology is due to Matt Foreman.
[41] Take $\mathcal{I}(\operatorname{Add}(\lambda, 1), \check{\mu})$ for μ the measure on κ.
[42] See Proposition 16.4 in [?].

(ii) κ *carries an ideal* I *such that* $\mathscr{P}(\kappa)/I$ *is forcing equivalent to* $Add(\omega, 1)$*.*

Proof. Player II has a winning strategy in $\mathcal{C}^-_\omega(\kappa)$ as we're simply measuring finitely many sets without any demand for wellfoundedness, showing (i). Since $2^{<n} < \kappa$ for all $n < \omega$ as κ is infinite, Theorem 1.80 then implies (ii). ■

Corollary 1.84. *"$<\lambda$-closed λ-sized" is ideal-absolute for all regular* $\lambda \in [\omega, \kappa^+]$.
■