1 Preliminaries

1.1 Descriptive set theory

Convention 1.1.1. We will be using the "logician's reals", meaning that $\mathbb{R} := {}^{\omega}\omega$ with the product topology, having the sets $\{x \in \mathbb{R} \mid x \supseteq s\}$ for $s \in {}^{<\omega}\omega$ as a clopen basis.

Definition 1.1.2. Let $A, B \subseteq \mathbb{R}$. We say that A *is Wadge reducible to* B (in symbols $A \leq_W B$) iff there is some continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$A = f^{-1}[B] := \{a \in A \mid f(a) \in B\}.$$

We write $A <_W B$ iff $A \leq_W B$ and $B \not\leq_W A$. ◦

Remark 1.1.3. $\mathbb{R}$ and $\mathbb{R}^n$, for $1 \leq n \leq \omega$ are homeomorphic and we shall often identify them with one another.

Definition 1.1.4. Subseteq of $\mathcal{R}$ are called *pointsets* . Subsets of $\mathcal{P}(\mathbb{R})$ are called *pointclasses* .

◦

Definition 1.1.5. Let Γ be a pointclass. We define

(i) $\exists^{\mathbb{R}}\Gamma := \{A \mid \exists B \in \Gamma \colon A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(x, y) \in B\}\}$,

(ii) $\forall^{\mathbb{R}}\Gamma := \{A \mid \exists B \in \Gamma \colon A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R}(x, y) \in B\}\}$.

◦

Definition 1.1.6. Let Γ be a pointclass. We define

(i) $\breve{\Gamma} := \{\mathbb{R} \setminus A \mid A \in \Gamma\}$,

(ii) $\Delta_\Gamma := \Gamma \cap \breve{\Gamma}$ and

(iii) $\underset{\sim}{\Gamma} := \exists^{\mathbb{R}}\Gamma$.

◦

Lemma 1.1.7 ([?]). *Assume $\mathsf{ZF}+\mathsf{AD}$ and let $A, B \subseteq \mathbb{R}$. Then*

$$A \leq_W B \text{ or } B \leq_W \mathbb{R} \setminus A.$$

Lemma 1.1.8 (Martin-Monk-Wadge). *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. Then $\leq_W$ is wellfounded.*[1]

Remark 1.1.9. When considering $(\mathcal{P}(\mathbb{R}); \leq_W)$ in a $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ context, we will often tacitly identify $A \subseteq \mathbb{R}$ with its complement, making $<_W$ a wellorder.

Definition 1.1.10 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). Let $A \subseteq \mathbb{R}$. Then the *Wadge rank of* A is defined recursively as $|A|_W := \sup\{|B|_W + 1 \mid B <_W A\}$. ◦

Definition 1.1.11. Let X be a set. We write OD_X for the collection of all A for which there is some formula ϕ, ordinals $\alpha_0, \ldots, \alpha_k$ and $x_0, \ldots, x_l \in X$ with

$$A = \{a \mid \phi[a, \alpha_0, \ldots, \alpha_k, x_0, \ldots, x_l]\}. \quad \dashv$$

We write HOD_X for the collection of all A such that $\mathrm{trcl}(\{A\}) \subseteq \mathrm{OD}_X$. If $X = \emptyset$, we will often drop the subscript and simply write OD and HOD for $\mathrm{OD}_\emptyset$ and $\mathrm{HOD}_\emptyset$ respectively.

Definition 1.1.12 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). For $B \subseteq \mathbb{R}$ let

$$\begin{aligned}\theta_B :&= \sup\{|A|_W \mid \exists x \in \mathbb{R}\colon A \in \mathrm{OD}_{\mathbb{R}\cup\{B\}}\}\\ &= \sup\{\alpha \in \mathsf{On} \mid \text{ there is a } \mathrm{OD}_{\mathbb{R}\cup\{B\}}\text{-surjection } f\colon \mathbb{R} \to \alpha\}.\end{aligned}$$

◦

Verify that these two values are in fact identical.

Definition 1.1.13 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). Define the *Solovay sequence* $\langle \theta_\alpha \mid \alpha \leq \Omega\rangle$ as follows:

(i) $\theta_0 := \theta_\emptyset$,

(ii) if there is some B such that $|B|_W = \theta_\alpha$ let $\theta_{\alpha+1} := \theta_B$.[2]

[1]See [?] for a proof.

[2]Since continuous functions are coded by reals, this is independent of the choice of B.

(iii) if α is a limit ordinal, we let $\theta_\alpha := \sup_{\beta<\alpha} \theta_\beta$.

Finally, Ω is the least ordinal such that $\theta_{\alpha+1} = \theta_\alpha$, and $\Theta := \theta_\Omega$. ∘

1.2 Ideals

Definition 1.2.1. Let I be an ideal on a nonempty set Z. Let

(i) $I^+ := \mathcal{P}(Z) \setminus I$,

(ii) for $a, b \in I^+$ let $a \sim_I b$ iff $a \Delta b \in I$,

(iii) $\mathcal{P}(Z)/I := \mathcal{P}(Z)/\sim_I$ is the Boolean algebra with subset inclusion modulo $\sim_I$.

We call $\mathcal{P}(Z)/I$ the *associated forcing* to I. ∘

Definition 1.2.2. If I is an ideal on a cardinal κ and $g \subseteq \mathcal{P}(\kappa)/I$ V-generic then g is a V-ultrafilter on κ in $V[g]$, so that we may take the *generic ultrapower* $\mathrm{Ult}(V, g)$. ∘

Proposition 1.2.3. *Let I be an ideal on a cardinal κ. Then..*

(i) if I is κ-complete then so is any generic ultrafilter;

(ii) if I is normal then so is any generic ultrafilter. ⊣

define κ-complete and normal for an ideal. Add reference.

Definition 1.2.4. Let λ be any cardinal. Then an ideal I on a cardinal κ is...

- *precipitous* if the generic ultrapower is wellfounded;
- λ-*saturated* if the associated forcing has the λ-chain condition;
- λ-*dense* if the associated forcing has a dense subset of size λ. ⊣

Note that λ-dense trivially implies λ^+-saturated. We'll need the following facts about ω_2-saturated ideals on ω_1:

Proposition 1.2.5. *Let I be an ω_2-saturated ideal on ω_1. Then I is precipitous, and letting $j\colon V \to M$ be the generic ultrapower map, it holds that*

Parts (ii)-(v) is Example 4.29 in Foreman's handbook chapter. Perhaps include the proof.

(i) M is closed under ω-sequences in $V[g]$;

(ii) $j(\omega_1^V) = \omega_2^V = \omega_1^{V[g]}$;

(iii) $j(\omega_2^V) \in (\omega_2^V, \omega_3^V)$;

(iv) j *is continuous at* ω_2^V*;*

(v) $j(\omega_n^V) = \omega_n^V = \omega_{n-1}^{V[g]}$ *for all* $n \in [3, \omega]$. $\dashv$

As for the density, we in particular need the following fact:

Proposition 1.2.6. *Let* I *be an* ω_1*-dense ideal on* ω_1*. Then the associated forcing is forcing equivalent to* $\mathrm{Col}(\omega, \omega_1)$*, so in particular it's homogeneous.* ■

PROOF. [**?**, Proposition 10.20]. ■