

1 | LARGE CARDINALS

Since large cardinals came into existence in the beginning of the 20th century, a vast zoo of different types of such have appeared. The aim of this appendix is to act as a reference for the definitions of these as well as the relations between them.

1.1 INACCESSIBLES

DEFINITION 1.1. A cardinal κ is **regular** if $\text{cof } \kappa = \kappa$; i.e. that there are no $\gamma < \kappa$ with a cofinal function $f: \gamma \rightarrow \kappa$. κ is a **strong limit** if $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$. If κ is both regular and a strong limit then we say that it is (strongly) **inaccessible**. ◦

PROPOSITION 1.2 ([?] Proposition 1.2). *If κ is inaccessible then $(V_\kappa, \in) \models \text{ZFC}$.* ■

Gödel's Second Incompleteness Theorem from [?] then immediately implies the following corollary.

COROLLARY 1.3. *ZFC can not prove the existence of any inaccessible cardinals. Indeed, not even the consistency of the existence of any inaccessible cardinals can be proven in ZFC.* ■

1.2 WEAKLY COMPACTS

DEFINITION 1.4. For any function $f: A \rightarrow B$, a subset $H \subseteq A$ is **homogeneous for f** if $f \upharpoonright H$ is a constant function. ◦

DEFINITION 1.5. Let κ and λ be infinite cardinals, γ an ordinal and $n < \omega$. Then the partition relation $\kappa \rightarrow (\lambda)_\gamma^n$ holds if to every function $f: [\kappa]^n \rightarrow \gamma$ there exists a subset $H \subseteq [\kappa]^n$ of size λ which is homogeneous for f . If $\gamma = 2$ then we usually leave it out and simply write $\kappa \rightarrow (\lambda)^n$. ◦

DEFINITION 1.6. An uncountable cardinal κ is **weakly compact** if $\kappa \rightarrow (\kappa)^2$. ◦

THEOREM 1.7 ([?] Lemma 9.9). *Every weakly compact cardinal is a limit of inaccessible cardinals.* ■

1.3 INEFFABLES AND COMPLETELY INEFFABLES

DEFINITION 1.8. An uncountable cardinal κ is **ineffable** if to any function $f: [\kappa]^2 \rightarrow 2$ there exists a *stationary* $H \subseteq [\kappa]^2$ which is homogeneous for f . ○

Ineffable cardinals are weakly compact by definition, and the following theorem from [?] shows that they are strictly stronger.

THEOREM 1.9 (Friedman). *Ineffable cardinals are weakly compact limits of weakly compacts.* ■

A way of improving ineffability is to “close under homogeneity”, in the sense that if H is homogeneous for $f: [\kappa]^2 \rightarrow 2$ and $g: [H]^2 \rightarrow 2$ is any function, then there is a subset of H which is homogeneous for g . To formalise this notion we use the concept of a *stationary class*.

DEFINITION 1.10. For X any set, a collection $\mathcal{R} \subseteq \mathcal{P}(X)$ is a **stationary class** if

- $\mathcal{R} \neq \emptyset$;
- Every $A \in \mathcal{R}$ is a stationary subset of X ;
- If $A \in \mathcal{R}$ and $B \supseteq A$ then $B \in \mathcal{R}$. ○

DEFINITION 1.11. An uncountable cardinal κ is **completely ineffable** if there is a stationary class $\mathcal{R} \subseteq \mathcal{P}(\kappa)$ such that for every $A \in \mathcal{R}$ and $f: [A]^2 \rightarrow 2$ there exists a $H \in \mathcal{R}$ which is homogeneous for f . ○

As suspected, these completely ineffable cardinals are indeed strictly stronger than the ineffables, as the following theorem from [?] shows.

THEOREM 1.12 (Abramson et al). *Completely ineffable cardinals are ineffable limits of ineffable cardinals.* ■

1.4 MEASURABLES, STRONGS AND SUPERCOMPACTS

DEFINITION 1.13. For two first-order structures \mathcal{M} and \mathcal{N} with underlying sets M and N , an **elementary embedding** $j: \mathcal{M} \rightarrow \mathcal{N}$ between them is a function $j: M \rightarrow N$ such that, for any first-order formula $\varphi(v_1, \dots, v_n)$ and sets $x_1, \dots, x_n \in \mathcal{M}$ it holds that $\mathcal{M} \models \varphi[x_1, \dots, x_n]$ iff $\mathcal{N} \models \varphi[j(x_1), \dots, j(x_n)]$. \circ

As elementary embeddings in particular preserve equality, they are always injective. Identity embeddings are of course always elementary, so we say that an elementary embedding is **non-trivial** if it is not the identity. The following then shows that in most situations these non-trivial embeddings can be associated to a unique ordinal.

PROPOSITION 1.14 ([?] Propostion 5.1). *If $j: (\mathcal{M}, \in) \rightarrow (\mathcal{N}, \in)$ is an elementary embedding such that \mathcal{M} is transitive and either $\mathcal{N} \subseteq \mathcal{M}$ or $\mathcal{M} \models \text{ZFC}$, then there exists an ordinal $\alpha < o(\mathcal{M})$ moved by j , i.e. that $j(\alpha) \neq \alpha$. We call the least such ordinal the **critical point** of j , and denote it by $\text{crit } j$.* ■

DEFINITION 1.15 (GBC). An uncountable cardinal κ is **measurable** if there exists a transitive class \mathcal{M} and an elementary embedding $j: (V, \in) \rightarrow (\mathcal{M}, \in)$ with critical point κ . \circ

The measurable cardinals were the first large cardinals shown to “transcend L ”.

THEOREM 1.16 (Scott’s Theorem, [?] Corollary 5.5). *L , Gödel’s constructible universe, has no measurable cardinals.* ■

Given this result, it’s not surprising that the measurables then exceed the strength of the previous large cardinals.

PROPOSITION 1.17. *Measurable cardinals are completely ineffable limits of completely ineffable cardinals.*

PROOF. (Sketch) If $j: V \rightarrow \mathcal{M}$ is a non-trivial elementary embedding then the **derived ultrafilter** $\mu \subseteq \mathcal{P}(\kappa)$ on $\kappa := \text{crit } j$ is defined as $X \in \mu$ iff $\kappa \in j(X)$. Section 5 in [?] shows that it is indeed an ultrafilter and that its ultrapower $\text{Ult}(V, \mu)$

is wellfounded. A reflection argument then shows that we can simply take $\mathcal{R} := \mu$.

■

DEFINITION 1.18 (GBC). An uncountable cardinal κ is **strong** if there to every cardinal $\theta > \kappa$ exists a transitive class \mathcal{M}_θ satisfying that $H_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta: (V, \in) \rightarrow (\mathcal{M}_\theta, \in)$ with critical point κ . We say that κ is θ -**strong** if the property holds for a specific θ . ◻

Ref

PROPOSITION 1.19. *Strong cardinals are measurable limits of measurable cardinals.* ■

DEFINITION 1.20 (GBC). An uncountable cardinal κ is **supercompact** if there to every cardinal $\theta > \kappa$ exists a transitive class \mathcal{M}_θ satisfying that ${}^{<\theta}\mathcal{M}_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta: (V, \in) \rightarrow (\mathcal{M}_\theta, \in)$ with critical point κ . ◻

Ref

PROPOSITION 1.21. *Supercompact cardinals are strong limits of strong cardinals.* ■

1.5 WOODINS AND VOPĚNKAS

DEFINITION 1.22. Let A be any set. An uncountable cardinal κ is **A -strong** if there to every cardinal $\theta > \kappa$ exists a transitive class \mathcal{M}_θ satisfying that $H_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta: (V, \in) \rightarrow (\mathcal{M}_\theta, \in)$ with critical point κ , such that $A \cap H_\theta = j_\theta(A) \cap H_\theta$. ◻

DEFINITION 1.23. An uncountable cardinal δ is a **Woodin cardinal** if there to every subset $A \subseteq H_\delta$ exists $\kappa < \delta$ such that $(H_\delta, \in, A) \models \ulcorner \kappa \text{ is } A\text{-strong} \urcorner$. ◻

THEOREM 1.24 ([?] Theorem 26.14). *The following are equivalent for an uncountable cardinal κ .*

- (i) κ is a Woodin cardinal;
- (ii) For any $f: \kappa \rightarrow \kappa$ there exists $\alpha < \kappa$ such that $f[\alpha] \subseteq \alpha$, a transitive \mathcal{M} with $V_{j(f)(\alpha)} \subseteq \mathcal{M}$ and an elementary embedding $j: (V, \in) \rightarrow (\mathcal{M}, \in)$ with $\text{crit } j = \kappa$.

DEFINITION 1.25 (GBC). **Vopěnka's Principle (VP)** postulates that to any first-order language \mathcal{L} and proper class \mathcal{C} of \mathcal{L} -structures, there exist distinct $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ and an elementary embedding $j: \mathcal{M} \rightarrow \mathcal{N}$. \circ

DEFINITION 1.26. An uncountable cardinal δ is **Vopěnka** if $(V_\delta, \in; V_{\delta+1}) \models \text{VP}$. \circ

Include the result about “woodin-for-supercompactness equals vopenka”

1.6 REINHARDTS AND KUNEN INCONSISTENCY

DEFINITION 1.27 (GBC). An uncountable cardinal κ is a **Reinhardt cardinal** if there exists an elementary embedding $j: (V, \in) \rightarrow (V, \in)$ with $\text{crit } j = \kappa$. \circ

THEOREM 1.28 (Kunen inconsistency, GBC, [?] Theorem 23.12). *There are no Reinhardt cardinals. Even more, there is no non-trivial elementary $j: (V_{\lambda+2}, \in) \rightarrow (V_{\lambda+2}, \in)$ for any uncountable cardinal λ .*

When we're dealing with the *virtual* large cardinals in Chapter ?? we show that the property $j(\kappa) > \theta$ is a highly non-trivial assumption. However, when we're not in the virtual world then this is simply automatic.

PROPOSITION 1.29. *If $j: V \rightarrow \mathcal{M}_\theta$ witnesses that $\kappa := \text{crit } j$ is a θ -strong cardinal then $j(\kappa) > \theta$.*

PROOF. (Sketch) If $j(\kappa) \leq \theta$ then

Missing argument or ref

■

1.7 BERKELEYS

DEFINITION 1.30 (GB). An uncountable cardinal δ is a **proto-Berkeley cardinal** if to every transitive set \mathcal{M} such that $\delta \subseteq \mathcal{M}$ there exists an elementary embedding $j: (\mathcal{M}, \in) \rightarrow (\mathcal{M}, \in)$ with $\text{crit } j < \delta$. \circ

Note that if κ is a proto-Berkeley cardinal then every $\lambda > \kappa$ is also proto-Berkeley, which makes it quite an uninteresting notion. But we can isolate the interesting cases, leading to the definition of a Berkeley cardinal.

Show that the least proto-Berkeley is Berkeley

DEFINITION 1.31 (GB). A proto-Berkeley cardinal δ is **Berkeley** if we can choose the critical point of the embedding to be arbitrarily large below δ . If we furthermore can choose the critical point as an element of any club $C \subseteq \delta$ then we say that δ is **club Berkeley**. \circ

Results on relations between the berkeleys and club berkeleys