1 Forcing

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The following lemma is from [**?**]

Lemma 1.1. *Let λ be an infinite cardinal, $\mathcal{M}$ a transitive model with $\mathsf{On} \subseteq \mathcal{M}$, $\mathbb{P} \in \mathcal{M}$ a forcing notion and $g \subseteq \mathbb{P}$ an $\mathcal{M}$-generic filter. Then $V \models {}^{\lambda}\mathcal{M} \subseteq \mathcal{M}$ implies that $V[g] \models {}^{\lambda}\mathcal{M} \subseteq \mathcal{M}$.*

Proof. Work in $V[g]$. Let $c := \langle c_\alpha \mid \alpha < \lambda \rangle$ be a λ-sequence such that $c_\alpha \in \mathcal{M}[g]$ for every $\alpha < \lambda$. Fix for every $\alpha < \lambda$ a $\mathbb{P}$-name $\dot{c}_\alpha$ such that $\dot{c}_\alpha^g = c_\alpha$. Also let $\dot{a}$ be a $\mathbb{P}$-name with $\dot{a}^g = \langle \dot{c}_\alpha \mid \alpha < \lambda \rangle$ and choose $p \in g$ such that

$$V \models \ulcorner p \Vdash \forall \alpha < \check{\lambda} \colon \dot{a}(\alpha) \in \mathcal{M}^{\mathbb{P}} \urcorner.$$

Now, working in V, there is for each $\alpha < \lambda$ a maximal antichain A_α below p such that every $q \in A_\alpha$ decides $\dot{a}(\alpha)$; i.e., $q \Vdash \ulcorner \dot{a}(\alpha) = \check{x} \urcorner$ for some $x \in \mathcal{M}$.

Define now

$$\sigma := \{((\alpha, x), q) \mid \alpha \in \lambda \wedge q \in A_\alpha \wedge q \Vdash \ulcorner \dot{a}(\alpha) = \check{x} \urcorner\}.$$

Then $p \Vdash \ulcorner \sigma = \dot{a} \urcorner$. Note that $|\sigma| \leq \lambda$, since $|A_\alpha| \leq \lambda$ for each $\alpha < \lambda$. Thus $\sigma \in \mathcal{M}$.

Now, going back to $V[g]$ again, it holds that $\langle \dot{c}_\alpha \mid \alpha < \lambda \rangle = \dot{a}^g = \sigma^g \in \mathcal{M}[g]$. But we can compute $c = \langle c_\alpha \mid \alpha < \lambda \rangle = \langle \dot{c}_\alpha^g \mid \alpha < \lambda \rangle$ from $\langle \dot{c}_\alpha \mid \alpha < \lambda \rangle$ and g, so that $c \in \mathcal{M}[g]$ by Replacement. ■