Thesis title

Dan Saattrup Nielsen

School of Mathematics

University of Bristol

A dissertation submitted to the University of Bristol in accordance with the requirements for award of the degree of Doctor of Philosophy in the Faculty of Science

Month 2020

Abstract

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

xx Month 2020

SIGNED: ______________________________

Contents

Introduction

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

1 The first chapter

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

1.1 Setting the scene

In this section we will recall a handful of definitions concerning Ramsey-like cardinals, as well as define the α-Ramsey cardinals for arbitrary ordinals α. We start out with the models and measures that we are going to consider.

Definition 1.1.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa+1 \subseteq \mathcal{M}$ and $(\mathcal{M}, \in) \models \mathsf{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$, $\mathcal{M}$ is a **κ-model**.[1] $\circ$

Recall that μ is an **$\mathcal{M}$-measure** if $(\mathcal{M}, \in, \mu) \models \ulcorner \mu$ is a κ-complete ultrafilter on $\kappa \urcorner$.

Definition 1.1.2. Let $\mathcal{M}$ be a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

[1] Note that our (weak) κ-models do not have to be transitive, in contrast to the models considered in [Gitman, 2011] and [Gitman and Welch, 2011]. Not requiring the models to be transitive was introduced in [Holy and Schlicht, 2018].

- **weakly amenable** if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with $\mathcal{M}$-cardinality κ;
- **countably complete** if $\bigcap \vec{X} \neq \emptyset$ for every ω-sequence $\vec{X} \in {}^\omega\mu$;
- **$\mathcal{M}$-normal** if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \triangle\vec{X} \in \mu$;
- **genuine** if $|\triangle\vec{X}| = \kappa$ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- **normal** if $\triangle\vec{X}$ is stationary in κ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- 0-**good**, or simply **good**, if it has a well-founded ultrapower;
- α-**good** for $\alpha > 0$ if it is weakly amenable and has α-many well-founded iterates.

$\circ$

Note that a genuine $\mathcal{M}$-measure is $\mathcal{M}$-normal and countably complete, and a countably complete weakly amenable $\mathcal{M}$-measure is α-good for all ordinals α. We'll use the fact shown in [Holy and Schlicht, 2018] that an $\mathcal{M}$-measure μ is normal iff $\triangle\vec{X}$ is stationary for some enumeration $\vec{X} = \langle X_\alpha \mid \alpha < \kappa\rangle$ of μ. We are also going to use the following alternative characterisation of weak amenability.

Proposition 1.1.3 (Folklore)**.** *Let $\mathcal{M}$ be a weak κ-model, μ an $\mathcal{M}$-measure and $j : \mathcal{M} \to \mathcal{N}$ the associated ultrapower embedding. Then μ is weakly amenable if and only if j is κ-**powerset preserving**, meaning that $\mathcal{M} \cap \mathscr{P}(\kappa) = \mathcal{N} \cap \mathscr{P}(\kappa)$.* $\blacksquare$

The α-Ramsey cardinals in [Holy and Schlicht, 2018] are based upon the following game.[2]

Definition 1.1.4 (Holy-Schlicht)**.** For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, a limit ordinal $\gamma \leq \kappa$ and a regular cardinal $\theta > \kappa$ define the game $wfG^\theta_\gamma(\kappa)$ of length γ as follows.

$$\begin{array}{llllllllll} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \text{II} & & \mu_0 & & \mu_1 & & \mu_2 & & \cdots \end{array}$$

[2]Unless otherwise stated, every game considered will be a game with perfect information between two players I and II. For a formal framework modelling these games, see e.g. [Kanamori, 2008].

Here $\mathcal{M}_\alpha \prec H_\theta$ is a κ-model and μ_α is a filter for all $\alpha < \gamma$, such that μ_α is an $\mathcal{M}_\alpha$-measure, the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing and $\langle \mathcal{M}_\xi \mid \xi < \alpha\rangle, \langle \mu_\xi \mid \xi < \alpha\rangle \in \mathcal{M}_\alpha$ for every $\alpha < \gamma$. Letting $\mu := \bigcup_{\alpha<\gamma} \mu_\alpha$ and $\mathcal{M} := \bigcup_{\alpha<\gamma} \mathcal{M}_\alpha$, player II wins iff μ is an $\mathcal{M}$-normal good $\mathcal{M}$-measure. ○

Recall that two games G_1 and G_2 are **equivalent** if player I has a winning strategy in G_1 iff they have one in G_2, and player II has a winning strategy in G_1 iff they have one in G_2. [Holy and Schlicht, 2018] showed that the games $wfG_\gamma^{\theta_0}(\kappa)$ and $wfG_\gamma^{\theta_1}(\kappa)$ are equivalent for any γ with $\operatorname{cof}\gamma \neq \omega$ and any regular $\theta_0, \theta_1 > \kappa$. We will be working with a variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I but more of player II. It will turn out that this change of game is innocuous, as Proposition 1.1.6 will show that they are equivalent.

Definition 1.1.5 (Holy-Schlicht-N.)**.** Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following game $\mathcal{G}_\gamma^\theta(\kappa, \zeta)$ with $(\gamma{+}1)$-many rounds:

$$\begin{array}{lcccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & & \mathcal{M}_\gamma & \\ \mathrm{II} & & \mu_0 & & \mu_1 & & \cdots & & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leq \gamma$, μ_α is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$, μ_γ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure and the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore require that $\mathcal{M}_\alpha = \bigcup_{\xi<\alpha} \mathcal{M}_\xi$, $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$ and that μ_α is ζ-good. Player II wins iff they could continue to play throughout all $(\gamma{+}1)$-many rounds. ○

For convenience we will write $\mathcal{G}_\gamma^\theta(\kappa)$ for the game $\mathcal{G}_\gamma^\theta(\kappa, 0)$, and $\mathcal{G}_\gamma(\kappa)$ for $\mathcal{G}_\gamma^\theta(\kappa)$ whenever $\operatorname{cof}\gamma \neq \omega$, as again the existence of winning strategies in these games doesn't depend upon a specific θ. Note that we assume that $\kappa = \kappa^{<\kappa}$ is uncountable in the definition of the games that we're considering, so this is a standing assumption throughout the paper, whenever any one of the above two games are considered.

Proposition 1.1.6 (Holy-Schlicht-N.). *$\mathcal{G}^\theta_\gamma(\kappa)$, $\mathcal{G}^\theta_\gamma(\kappa,1)$ and $wfG^\theta_\gamma(\kappa)$ are all equivalent for all limit ordinals $\gamma \leq \kappa$, and $\mathcal{G}^\theta_\gamma(\kappa,\zeta)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$ whenever* $\operatorname{cof}\gamma > \omega$ *and $\zeta \in \mathsf{On}$.*

Proof. We start by showing the latter statement, so assume that $\operatorname{cof}\gamma > \omega$. Consider now the auxilliary game, call it $\mathcal{G}$, which is exactly like $\mathcal{G}^\theta_\gamma(\kappa,0)$, but where we also require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ and $\langle\mathcal{M}_\xi \mid \xi \leq \alpha\rangle, \langle\mu_\xi \mid \xi \leq \alpha\rangle \in \mathcal{M}_{\alpha+1}$ for every $\alpha < \gamma$.

Claim 1.1.6.1. $\mathcal{G}$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$.

Proof of claim. If player I has a winning strategy in $\mathcal{G}$ then they also have one in $\mathcal{G}^\theta_\gamma(\kappa)$, by doing exactly the same. Analogously, if player II has a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have one in $\mathcal{G}$. If player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ then we can construct a winning strategy σ' in $\mathcal{G}$, which is defined as follows. Fix some $\alpha \leq \gamma$ and, writing $\vec{\mathcal{M}}_\xi := \langle\mathcal{M}_\xi \mid \xi \leq \alpha\rangle$ and $\vec{\mu}_\xi := \langle\mu_\xi \mid \xi \leq \alpha\rangle$, we set

$$\sigma'(\langle\mathcal{M}_\xi,\mu_\xi \mid \xi \leq \alpha\rangle) := \operatorname{Hull}^{H_\theta}(\sigma(\langle\mathcal{M}_\xi,\mu_\xi \mid \xi \leq \alpha\rangle) \cup {}^\omega\mathcal{M}_\alpha \cup \{\vec{\mathcal{M}}_\xi,\vec{\mu}_\xi\}),$$

i.e. that we're simply throwing in the sequences into our models and making sure that we're still an elementary substructure of H_θ. This new strategy σ' is clearly winning. Assuming now that τ is a winning strategy for player II in $\mathcal{G}$, we define a winning strategy τ' for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ by letting $\tau'(\langle\mathcal{M}_\xi,\mu_\xi \mid \xi \leq \alpha\rangle)$ be the result of throwing in the appropriate sequences into the models $\mathcal{M}_\xi$, applying τ to get a measure, and intersecting that measure with $\mathcal{M}_\alpha$ to get an $\mathcal{M}_\alpha$-measure. $\dashv$

Now, letting $\mathcal{M}_\gamma$ be the final model of a play of $\mathcal{G}$, $\operatorname{cof}\gamma > \omega$ implies that any ω-sequence $\vec{X} \in \mathcal{M}_\gamma$ really is a sequence of elements from some $\mathcal{M}_\xi$ for $\xi < \gamma$, so that $\vec{X} \in \mathcal{M}_{\xi+1}$ by definition of $\mathcal{G}$, making $\mathcal{M}_\gamma$ closed under ω-sequences and thus also μ_γ countably complete. Since γ is a limit ordinal and the models contain the previous measures and models as elements, the proof of e.g. Theorem 5.6 in [Holy and Schlicht, 2018] shows that μ_γ is also weakly amenable, making it ζ-good for all ordinals ζ.

Now we deal with the first statement, so fix a limit ordinal γ. Firstly $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa, 1)$ as above, since both are equivalent to the auxilliary game $\mathcal{G}$ when γ is a limit ordinal. So it remains to show that $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $wfG^\theta_\gamma(\kappa)$. If player I has a winning strategy σ in $wfG^\theta_\gamma(\kappa)$ then define a winning strategy σ' for player I in $\mathcal{G}^\theta_\gamma(\kappa)$ as

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) := \sigma(\langle \mathcal{M}_0, \mu_0\rangle^\frown\langle \mathcal{M}_{\xi+1}, \mu_{\xi+1} \mid \xi + 1 \leq \alpha\rangle)$$

and for limit ordinals $\alpha \leq \gamma$ set $\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi < \alpha\rangle) := \bigcup_{\xi<\alpha} \mathcal{M}_\xi$; i.e. they simply follow the same strategy as in $wfG^\theta_\gamma(\kappa)$ but plugs in unions at limit stages. Likewise, if player II had a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have a winning strategy in $wfG^\theta_\gamma(\kappa)$, this time just by skipping the limit steps in $\mathcal{G}^\theta_\gamma(\kappa)$.

Now assume that player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ and that player I *doesn't* have a winning strategy in $wfG^\theta_\gamma(\kappa)$. Then define a strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$ as follows. Let $s = \langle \mathcal{M}_\alpha, \mu_\alpha \mid \alpha \leq \eta\rangle$ be a partial play of $wfG^\theta_\gamma(\kappa)$ and let s' be the modified version of s in which we have 'inserted' unions at limit steps, just as in the above paragraph. We can assume that every μ_α in s' is good and $\mathcal{M}_\alpha$-normal as otherwise player II has already lost and player I can play anything. Now, we want to show that s' is a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$. All the models in s are κ-models, so in particular weak κ-models.

Claim 1.1.6.2. Every μ_α in s' is normal.

Proof of claim. Assume without loss of generality that $\alpha = \eta$. Let player I play any legal response $\mathcal{M}$ to s in $wfG^\theta_\gamma(\kappa)$ (such a response always exists). If player II can't respond then player I has a winning strategy by simply following $s^\frown\langle\mathcal{M}\rangle$, $\lightning$, so player II *does* have a response μ to $s^\frown \mathcal{M}$. But now the rules of $wfG^\theta_\gamma(\kappa)$ ensures that $\mu_\eta \in \mathcal{M}$, so since

$$(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \ulcorner \triangle\vec{X} \text{ is stationary in } \kappa\urcorner,$$

we then also get that $\mathcal{M} \models \ulcorner \triangle\mu_\eta$ is stationary in $\kappa\urcorner$ since $\mu_\eta \subseteq \mu$, so elementarity of $\mathcal{M}$ in H_θ implies that $\triangle\mu_\eta$ really *is* stationary in κ, making μ_η normal. $\dashv$

This makes s' a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$, so we may form the weak κ-model $\tilde{\mathcal{M}}_\eta := \sigma(s')$. Now let $\mathcal{M}_\eta \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_\eta \subseteq \mathcal{M}_\eta$ and $s \in \mathcal{M}_\eta$ and set $\sigma'(s) := \mathcal{M}_\eta$. This defines the strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$, which is winning since the winning condition for the two games is the same for γ a limit.[3]

Next, assume that player II has a winning strategy τ in $wfG^\theta_\gamma(\kappa)$. We recursively define a strategy $\tilde{\tau}$ for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ as follows. If $\tilde{\mathcal{M}}_0$ is the first move by player I in $\mathcal{G}^\theta_\gamma(\kappa)$, let $\mathcal{M}_0 \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_0 \subseteq \mathcal{M}_0$, making $\mathcal{M}_0$ a valid move for player I in $wfG^\theta_\gamma(\kappa)$. Write $\mu_0 := \tau(\langle\mathcal{M}_0\rangle)$ and then set $\tilde{\tau}(\langle\tilde{\mathcal{M}}_0\rangle)$ to be $\tilde{\mu}_0 := \mu_0 \cap \tilde{\mathcal{M}}_0$, which again is normal by the same trick as above, making $\tilde{\mu}_0$ a legal move for player II in $\mathcal{G}^\theta_\gamma(\kappa)$. Successor stages $\alpha+1$ in the construction are analogous, but we also make sure that $\langle\mathcal{M}_\xi \mid \xi < \alpha+1\rangle, \langle\mu_\xi \mid \xi < \alpha+1\rangle \in \mathcal{M}_{\alpha+1}$. At limit stages τ outputs unions, as is required by the rules of $\mathcal{G}^\theta_\gamma(\kappa)$. Since the union of all the μ_α's is good as τ is winning, $\tilde{\mu}_\gamma := \bigcup_{\alpha<\gamma}\tilde{\mu}_\alpha$ is good as well, making $\tilde{\tau}$ winning and we are done. ■

We now arrive at the definitions of the cardinals we will be considering. They were in [Holy and Schlicht, 2018] only defined for γ being a cardinal, but given the above result we generalise it to all ordinals γ.

Definition 1.1.7. Let κ be a cardinal and $\gamma \leq \kappa$ an ordinal. Then κ is **γ-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. We furthermore say that κ is **strategic γ-Ramsey** if player II *does* have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. Define **(strategic) genuine γ-Ramseys** and **(strategic) normal γ-Ramseys**

[3] More precisely, that σ is winning in $\mathcal{G}^\theta_\gamma(\kappa)$ means that there's a sequence $\langle f_n : \kappa \to \kappa \mid n < \omega\rangle$ with the f_n's all being elements of the last model $\tilde{\mathcal{M}}_\gamma$, witnessing the illfoundedness of the ultrapower. But then all these functions will also be elements of the union of the $\mathcal{M}_\alpha$'s, since we ensured that $\mathcal{M}_\alpha \supseteq \tilde{\mathcal{M}}_\alpha$ in the construction above, making the ultrapower of $\bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$ by $\bigcup_{\alpha<\gamma}\mu_\alpha$ illfounded as well.

analogously, but where we require the last measure μ_γ to be genuine and normal, respectively. ◦

Definition 1.1.8 (N.). A cardinal κ is **$<\gamma$-Ramsey** if it is α-Ramsey for every $\alpha < \gamma$, **almost fully Ramsey** if it is $<\kappa$-Ramsey and **fully Ramsey** if it is κ-Ramsey. Further, say that κ is **coherent $<\gamma$-Ramsey** if it's strategic α-Ramsey for every $\alpha < \gamma$ and that there exists a choice of winning strategies τ_α in $\mathcal{G}_\alpha(\kappa)$ for player II satisfying that $\tau_\alpha \subseteq \tau_\beta$ whenever $\alpha < \beta$. In other words, there is a single strategy τ for player II in $\mathcal{G}_\gamma(\kappa)$ such that τ is a winning strategy for player II in $\mathcal{G}_\alpha(\kappa)$ for every $\alpha < \gamma$.[4] ◦

This is not the original definition of (strategic) γ-Ramsey cardinals however, as this involved elementary embeddings between weak κ-models – but as the following theorem of [Holy and Schlicht, 2018] shows, the two definitions coincide whenever γ is a regular cardinal.

Theorem 1.1.9 (Holy-Schlicht). *For regular cardinals λ, a cardinal κ is λ-Ramsey iff for arbitrarily large $\theta > \kappa$ and every $A \subseteq \kappa$ there is a weak κ-model $\mathcal{M} \prec H_\theta$ with $\mathcal{M}^{<\lambda} \subseteq \mathcal{M}$ and $A \in \mathcal{M}$ with an $\mathcal{M}$-normal 1-good $\mathcal{M}$-measure μ on κ.* ■

1.2 The finite case

In this section we are going to consider properties of the n-Ramsey cardinals for finite n. Note in particular that the $\mathcal{G}_n^\theta(\kappa)$ games are determined, making the "strategic" adjective superfluous in this case. We further note that the θ's are also dispensible in this finite case:

Proposition 1.2.1 (N.). *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n \prec H_\theta$.*

[4] Note that, with this terminology, "coherent" is a stronger notion than "strategic". We could've called the cardinals *coherent strategic $<\gamma$-Ramseys*, but we opted for brevity instead.

Proof. $\Leftarrow$ is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays $\mathcal{M}_k$ in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \mathrm{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of $\mathcal{M}_k$, and play $\mathcal{M}_k^*$ in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ-responses to the $\mathcal{M}_k^*$'s and let player II play the μ_k's in $\mathcal{G}_n(\kappa)$ as well.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\mathrm{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω-sequence $\vec{f} := \langle f_k \mid k < \omega\rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \mathrm{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\mathrm{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \to H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \mathrm{ran}\,\pi$.

Now $\bar{A} = A$ for every $A \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the $\bar{f}_k$'s witness that $\mathrm{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\mathrm{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \mathrm{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ's anymore we note that n-Ramseyness can now be described using a Π^1_{2n+2}-formula and normal n-Ramseyness using a Π^1_{2n+3}-formula.

We already have the following characterisations, as proven in [Abramson et al., 1977].

Theorem 1.2.2 (Abramson et al.). *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then*

(i) κ is weakly compact if and only if it is 0-Ramsey;

(ii) κ is weakly ineffable if and only if it is genuine 0-Ramsey;

(iii) κ is ineffable if and only if it is normal 0-Ramsey.

Proof. This is mostly a matter of changing terminology from [Abramson et al., 1977] to the current game-theoretic one, so we only show (i). Theorem 1.1.3 in [Abramson et al., 1977] shows that κ is weakly compact if and only if every κ-sized collection of subsets of κ is measured by a $<\kappa$-complete measure, in the sense that every $<\kappa$-sequence (in V) of measure one sets has non-empty intersection.

For the $\Rightarrow$ direction we can let player II respond to any $\mathcal{M}_0$ by first getting the $<\kappa$-complete $\mathcal{M}_0$-measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \to \mathrm{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π, which is $\mathcal{M}_0$-normal and good. For $\Leftarrow$, if $X \subseteq \mathscr{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ-model $\mathcal{M}_0 \prec H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play $\mathcal{M}_0$ in $\mathcal{G}_0(\kappa)$ we get some $\mathcal{M}_0$-normal good $\mathcal{M}_0$-measure μ_0 on κ. Since $\mathcal{M}_0$ is closed under $<\kappa$-sequences we get that μ_0 is $<\kappa$-complete. ■

Indescribability

In this section we aim to prove that n-Ramseys are Π^1_{2n+1}-indescribable and that normal n-Ramseys are Π^1_{2n+2}-indescribable, which will also establish that the hierarchy of alternating n-Ramseys and normal n-Ramseys forms a strict hierarchy. Recall the following definition.

Definition 1.2.3. A cardinal κ is **Π^1_n-indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ∘

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π^1_1-indescribable — see [Abramson et al., 1977].

Theorem 1.2.4 (N.). *Every n-Ramsey κ is Π^1_{2n+1}-indescribable for $n < \omega$.*

Proof. Let κ be n-Ramsey and assume that it is not Π^1_{2n+1}-indescribable, witnessed by a Π_{2n+1}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$ and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$-many quantifiers occuring in φ in $(n+1)$-many steps. We will here describe the first two steps with the remaining steps following the same pattern.

First step. Write $\varphi(v) \equiv \forall v_1 \psi(v, v_1)$ for a Σ_{2n}-formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A^{(0)}_\alpha \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let $\mathcal{M}_0$ be a weak κ-model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good $\mathcal{M}_0$-normal $\mathcal{M}_0$-measure μ_0 on κ, using the 0-Ramseyness of κ. Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \mathrm{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. $\mathcal{M}_0$-normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, $\mathcal{M}_0$-normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}]\urcorner. \tag{1}$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0-formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

$\boxed{\textbf{Second step.}}$ Write $\psi(v, v_1) \equiv \exists v_2 \forall v_3 \chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$-formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3 \chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \tag{2}$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, v_3]. \tag{3}$$

Fix witnesses $A^{(1)}_\alpha \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S^{(0)}_\alpha := \{\xi < \kappa \mid A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S^{(0)}_\alpha \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and $\mathcal{M}_0$-normality of μ_0 then implies that $S^{(0)}_\alpha \in \mu_0$ is equivalent to

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ-model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an $\mathcal{M}_1$-normal $\mathcal{M}_1$-measure on κ, using the 1-Ramseyness of κ, so that $\mathcal{M}_1$-normality of μ_1 yields that $\triangle\vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle\vec{S}^{(0)}$ if and only if $A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A^{(0)}_\xi = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \mathrm{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A^{(1)}_\alpha$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha].$$

Now this, paired with the above observation regarding $\triangle\vec{S}^{(0)}$, means that for every $\alpha \in \triangle\vec{S}^{(0)} \cap \mathrm{Lim}$ we have that

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha],$$

so that $\mathcal{M}_1$-normality of μ_1 and Łoś' lemma implies that

$$\mathrm{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg\chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]\urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$-many steps, ending with a Δ_0-formula $\phi(v, v_1, \dots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\mathrm{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]\urcorner$. But now absoluteness of $\neg\phi$ means that $V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]$, contradicting (4). ∎

Note that this is optimal, as n-Ramseyness can be described by a Π^1_{2n+2}-formula. As a corollary we then immediately get the following.

Corollary 1.2.5 (N.). *Every $<\omega$-Ramsey cardinal is Δ^2_0-indescribable.* ∎

The second indescribability result concerns the normal n-Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π^1_2-indescribable — see [Abramson et al., 1977].

Theorem 1.2.6 (N.). *Every normal n-Ramsey κ is Π^1_{2n+2}-indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 1.2.4, as we would end up with a Π^1_1 statement in an ultrapower, and as Π^1_1 statements are not upwards absolute in general we would not be able to get our contradiction.

PROOF. Let κ be normal n-Ramsey and assume that it is not Π^1_{2n+2}-indescribable, witnessed by a Π_{2n+2}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n-Ramsey to perform the same $n+1$ steps as in the proof of Theorem 1.2.4. This gives us a Σ_1-formula $\phi(v, v_1, \dots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \cdots, \mathcal{A}^{(n)} \rangle$, $\langle B^{(0)}, \dots, B^{(n-1)} \rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n \rangle$ of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{1}$$

and, for μ_n-many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, A^{(n)}_\alpha].$$

Now form $S^{(n)}_\alpha \in \mu_n$ as in the proof of Theorem 1.2.4. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 1.2.4 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for $k < n$), but we can now use normality[5] of μ_n to ensure that we *do* have that $\triangle\vec{S}^{(n)}$ is stationary in κ. This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \tag{2}$$

[5] Recall that this is stronger than just requiring it to be $\mathcal{M}_n$-normal — we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.

Now note that since κ is inaccessible it is Σ^1_1-indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of [Abramson et al., 1977] shows that the set of reflection points of Σ^1_1-formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \ldots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n-Ramseyness can be described by a Π^1_{2n+3}-formula. In particular this then means that every $(n+1)$-Ramsey is a normal n-Ramsey stationary limit of normal n-Ramseys, and every normal n-Ramsey is an n-Ramsey stationary limit of n-Ramseys, making the hierarchy of alternating n-Ramseys and normal n-Ramseys a strict hierarchy.

Downwards absoluteness to L

The following proof is basically the proof of Theorem 4.1.1 in [Abramson et al., 1977].

Theorem 1.2.7 (N.). *Genuine- and normal n-Ramseys are downwards absolute to L, for every $n < \omega$.*

PROOF. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ-model — we want to find a genuine $\mathcal{M}$-measure inside L. By assumption we *can* find such a measure μ in V; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa \rangle \in L$ of $\mathscr{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

Claim 1.2.7.1. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

PROOF OF CLAIM. Let $\vec{B}$ be the **μ-positive part** of $\vec{A}$, meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle \vec{B}$ has size κ, so we can pick $\delta \in \triangle \vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L. $\dashv$

But now Lemma 4.1.2 in [Abramson et al., 1977] shows that there is a Π_1 formula $\varphi(v)$ such that, given any non-zero ordinal ζ, $V_{\zeta+1} \models \varphi[A]$ if and only if ζ is a regular cardinal and A is a non-constructible subset of ζ. If we therefore assume that $T \notin L$ then $V_{\kappa+1} \models \varphi[T]$, which by Π^1_1-indescribability of κ means that there exists some $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle\mu)^L = \triangle\mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V. The cases where κ is a genuine- or normal n-Ramsey cardinal is analogous. ■

Since $(n+1)$-Ramseys are normal n-Ramseys we then immediately get the following.

Corollary 1.2.8 (N.). *Every $(n+1)$-Ramsey is normal n-Ramsey in L, for every $n < \omega$. In particular, $<\omega$-Ramseys are downwards absolute to L.* ■

Complete ineffability

In this section we provide a characterisation of the *completely ineffable* cardinals in terms of the α-Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$-Ramsey cardinals, namely the *coherent* $<\omega$*-Ramseys* as defined in 1.1.8. Note that a coherent $<\omega$-Ramsey is precisely a cardinal satisfying the ω-filter property, as defined in [Holy and Schlicht, 2018].

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is very closely related to the proof of Theorem 1.2.6 (the indescribability of normal n-Ramseys), but the main difference is that we want everything to occur locally inside our weak κ-models.

Theorem 1.2.9 (N.). *Every coherent* $<\omega$*-Ramsey is a stationary limit of* $<\omega$*-Ramseys.*

Proof. Let κ be coherent $<\omega$-Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 \prec H_\theta$ be a weak κ-model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ-model $\mathcal{M} \prec H_\theta$ with an $\mathcal{M}$-normal $\mathcal{M}$-measure $\mu := \bigcup_{n<\omega} \mu_n$ on κ.

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } {<}\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by $\mathcal{M}$-normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k+1$ steps as in the proof of Theorem 1.2.6 with $\varphi(\xi)$ being $\ulcorner \xi$ is k-Ramsey$\urcorner$, so that we get a weak κ-model $\mathcal{M}_{k+1} \prec H_\theta$, an $\mathcal{M}_{k+1}$-normal $\mathcal{M}_{k+1}$-measure $\tilde{\mu}_{k+1}$ on κ, a Σ_1-formula $\varphi(v, v_1, v_2, \dots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \dots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \dots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \tag{2}$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg\varphi[\xi, A^{(0)}_\xi, B^{(0)} \cap V_\xi, A^{(1)}_\xi, B^{(1)} \cap V_\xi, \dots, A^{(k-1)}_\xi, B^{(k-1)} \cap V_\xi, A^{(k)}_\xi], \tag{3}$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \mathrm{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 1.2.4.

Since κ in particular is Σ^1_1-indescribable, Lemma 3.4.3 of [Abramson et al., 1977] implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ-model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n+1)$-steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of $\mathcal{M}_{k+1}$. In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ, $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A^{(i)}_\xi \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa \rangle$ and note that since $\vec{S}$ is definable it is an element of $\mathcal{M}_{k+2}$ as well. Then $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $\triangle\vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \triangle\vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \triangle\vec{S}$ we see, as in the proof of Theorem 1.2.4, that

$$V_{\zeta+1} \models \varphi[\zeta, A^{(0)}_\zeta, B^{(0)} \cap V_\zeta, A^{(1)}_\zeta, B^{(1)} \cap V_\zeta, \dots, A^{(k)}_\zeta]$$

since $\triangle\vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} \prec H_\theta$ we have that $\mathcal{M}$ is correct about stationary subsets of κ, meaning that κ is a stationary limit of $<\omega$-Ramseys. ■

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

Definition 1.2.10. A collection $R \subseteq \mathscr{P}(\kappa)$ is a **stationary class** if

(i) $R \neq \emptyset$;
(ii) every $A \in R$ is stationary in κ;
(iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

∘

Definition 1.2.11. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$ there is an $H \in R$ homogeneous for f. ∘

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in [Abramson et al., 1977].

Theorem 1.2.12 (N.). *A cardinal κ is completely ineffable if and only if it is coherent $<\omega$-Ramsey.*

Proof. ($\Leftarrow$): Assume κ is coherent $<\omega$-Ramsey, witnessed by strategies $\langle \tau_n \mid n < \omega \rangle$. Let $f : [\kappa]^2 \to 2$ be arbitrary and form the sequence $\langle A^f_\alpha \mid \alpha < \kappa \rangle$ as

$$A^f_\alpha := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let $\mathcal{M}_f$ be a transitive weak κ-model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated $\mathcal{M}_f$-measure on κ given by τ_0.[6] 1-Ramseyness of κ ensures that μ_f is normal, meaning $\triangle \mu_f$ is stationary in κ. Define a new sequence $\vec{B}^f$ as the μ_f-positive part of $\vec{A}^f$.[7] Then $B^f_\alpha \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\triangle \vec{B}^f$ is stationary.

Let now $\mathcal{M}'_f$ be a new transitive weak κ-model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an $\mathcal{M}'_f$-measure $\mu'_f \supseteq \mu_f$ on κ. Then $\triangle \vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \in \mu_f\}$ and $\triangle \vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \notin \mu_f\}$ are both elements of $\mathcal{M}'_f$, so one of them is in μ'_f; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f.

Now let $g : [H_f]^2 \to 2$ be arbitrary and again form

$$A^g_\alpha := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ-model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$-measure $\mu_{f,g} \supseteq \mu'_f$ on κ. As before we then get a

[6] Technically we would have to require that $\mathcal{M}_f \prec H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0, but note that we could simply get a measure on $\mathrm{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to $\mathcal{M}_f$. We will use this throughout the proof.

[7] The *μ-positive part* was defined in Claim 1.2.7.1.

stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g. We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists \vec{f} : H_{\vec{f}} \subseteq A\},$$

where the $\vec{f}$'s range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \to 2$ and $f_{k+1} : [H_{f_k}] \to 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \to 2$, we can find $H \in R$ which is homogeneous for f. Indeed, if we let $\vec{f}$ be such that $H_{\vec{f}} \subseteq A$, which exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

($\Rightarrow$): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n-Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play $\mathcal{M}_0$ and enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A^0_\alpha \mid \alpha < \kappa\rangle$ such that $A^0_\xi \subseteq A^0_\zeta$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^0_\xi$. Let $<^\alpha_{\text{lex}}$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f, using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \restriction \alpha \mid \xi \in H_0 \wedge \xi > \alpha\rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least such η, which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C.

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \restriction \zeta = r_\gamma \restriction \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \restriction \zeta$ holds for every $\gamma \in H_0$, $\gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^0_\xi$ iff $\gamma \in A^0_\xi$. Define thus the

$\mathcal{M}_0$-measure μ_0 on κ as

$$\begin{aligned}\mu_0(A^0_\xi) = 1 \quad &\text{iff} \quad (\forall\beta \in H_0 \cap C)(\beta > \xi \to \beta \in A^0_\xi)\\ &\text{iff} \quad (\exists\beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A^0_\xi),\end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa\rangle$ be the μ_0-positive part of $\vec{A}^0$, it is also simple to check that $H_0 \cap C \subseteq \triangle\vec{B}$, making μ_0 normal and hence also both $\mathcal{M}_0$-normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n-Ramsey and let $\langle \mathcal{M}_0, \mu_0, \dots, \mathcal{M}_n, \mu_n, \mathcal{M}_{n+1}\rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}$ as $\vec{A}^{n+1} = \langle A^{n+1}_\xi \mid \xi < \kappa\rangle$, again satisfying that $\xi \leq \zeta$ whenever $A^{n+1}_\xi \subseteq A^{n+1}_\zeta$, but also such that given any $\xi < \kappa$ there are $\zeta, \zeta' \in (\xi, \kappa)$ satisfying that $A^{n+1}_\zeta \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and $A^{n+1}_{\zeta'} \in (\mathscr{P}(\kappa) \cap \mathcal{M}_{n+1}) - \mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n \in R$ and C be club in κ such that $H_n \cap C \subseteq \triangle\mu_n$, which exist by our inductive assumption. For $\alpha < \kappa$ define $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^{n+1}_\xi$, and define a colouring $f : [H_n]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta)\\ 1 & \text{otherwise}\end{cases}$$

As $H_n \in R$ there is an $H_{n+1} \in R$ homogeneous for f. Just as before, define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least $\eta \in [\alpha, \kappa)$ such that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_{n+1}$ with $\eta \leq \beta < \gamma$, and let D be the club of fixed points of g. As above we get that given any $\alpha, \gamma \in H_{n+1} \cap D$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^{n+1}_\xi$ iff $\gamma \in A^{n+1}_\xi$. Define then the $\mathcal{M}_{n+1}$-measure μ_{n+1} on κ as

$$\begin{aligned}\mu_{n+1}(A^{n+1}_\xi) = 1 \quad &\text{iff} \quad (\forall\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \to \beta \in A^{n+1}_\xi)\\ &\text{iff} \quad (\exists\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A^{n+1}_\xi).\end{aligned}$$

Then $H_{n+1} \cap D \cap C \subseteq \triangle\mu_{n+1}$, making μ_{n+1} normal, $\mathcal{M}_{n+1}$-normal and good, just as before. It remains to show that $\mu_n \subseteq \mu_{n+1}$. Let thus $A \in \mu_n$ be given, and say $A = A^{n+1}_\xi = A^n_\eta$, where $\vec{A}^n$ was the enumeration of $\mathscr{P}(\kappa)\cap\mathcal{M}_n$ used at the n'th stage. Then by definition of μ_n we get that for every $\beta \in H_n \cap C$ with $\beta > \eta$, $\beta \in A^n_\eta$. We need to show that

$$(\exists\beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A^{n+1}_\xi)$$

holds. But here we can simply pick a $\beta > \max(\xi, \eta)$ with $\beta \in H_{n+1}\cap D\cap C \subseteq H_n \cap C$. This shows that $\mu_n \subseteq \mu_{n+1}$, making κ $(n+1)$-Ramsey and thus inductively also coherent $<\omega$-Ramsey. ■

1.3 The countable case

This section covers the (strategic) γ-Ramsey cardinals whenever γ has countable cofinality. This case is special because, as mentioned in Section 1.1, we cannot ensure that the final measure is countably complete and so the existence of winning strategies in the $\mathcal{G}^\theta_\gamma(\kappa)$ *might* depend on θ, in contrast with the uncountable cofinality case; see e.g. Question **??**.

[Strategic] ω-Ramsey cardinals

We now move to the strategic ω-Ramsey cardinals and their relationship to the (non-strategic) ω-Ramseys. For this we define a new addition to the family of *virtual cardinals* from [Gitman and Schindler, 2015], the *virtually measurable cardinals.*

Definition 1.3.1. A cardinal κ is **virtually measurable** if for every regular $\nu > \kappa$ there exists a transitive M and a forcing $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there exists an elementary embedding $j : H^V_\nu \to M$ with $\operatorname{crit} j = \kappa$. ◦

We'll need the following well-known lemmata; see Lemma 7.1 in [Holy and Schlicht, 2018] and Lemma 3.1 in [Gitman and Schindler, 2015] for their proofs.

Lemma 1.3.2 (Ancient Kunen Lemma)**.** *Let $M \models \mathsf{ZFC}^-$ and $j : M \to N$ an elementary embedding with critical point κ such that $\kappa + 1 \subseteq M \subseteq N$. Assume that $X \in M$ has M-cardinality κ. Then $j \restriction X \in N$.* ■

Lemma 1.3.3 (Absoluteness of embeddings on countable structures)**.** *Let M be a countable first-order structure and $j : M \to N$ an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of) ZFC such that M is countable in W and $N \in W$, then for any finite subset of M, W has some elementary embedding $j^* : M \to N$, which agrees with j on that subset. Moreover, if both M and N are transitive $\in$-structures and j has a critical point, we can also assume that* $\operatorname{crit}(j^*) = \operatorname{crit}(j)$. ■

Theorem 1.3.4 (Schindler-N.)**.** *Every virtually measurable cardinal is strategic ω-Ramsey, and every strategic ω-Ramsey cardinal is virtually measurable in L.*

Proof. Let κ be virtually measurable and fix a regular $\nu > \kappa$, a transitive M, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi : H_\nu^V \to M$ with $\operatorname{crit}\pi = \kappa$. Fix a name $\dot\mu$ and a $\mathbb{P}$-condition p such that[8]

$$p \Vdash \ulcorner \dot\mu \text{ is a 1-good } \check H_\nu\text{-normal } \check H_\nu\text{-measure}\urcorner$$

We now define a strategy σ for player II in $\mathcal{G}_\omega^\nu(\kappa)$ as follows. Whenever player I plays a weak κ-model $M_n \prec H_\nu^V$, player II fixes $p_n \in \mathbb{P}$, an M_n-measure μ_n and a function $\pi_n : M_n \to V$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot\mu \cap \check M_n = \check\mu_n \wedge \check\pi_n = \dot\pi \restriction \check M_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma 1.3.2 we get that $\pi \restriction M_n \in M \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to M_n with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ is good. Assume it's not, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n : \kappa \to M_\omega := \bigcup_{n<\omega} M_n$ such that $g_n \in M_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

[8] Recall that an M-measure μ is **1-good** if it's weakly amenable and $\operatorname{Ult}(M, \mu)$ is well-founded.

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in M_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot\pi(\check g_{n+1})(\check\kappa) < \dot\pi(\check g_n)(\check\kappa)\urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check\pi_{n+1}(\check g_{n+1})(\check\kappa) < \check\pi_n(\check g_n)(\check\kappa)\urcorner, \tag{3}$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω is good, making σ a winning strategy and thus also making κ strategic ω-Ramsey since ν was arbitrary.

Next, let κ be strategic ω-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}^\nu_\omega(\kappa)$ for a regular $\nu > \kappa$. Let $g \subseteq \mathrm{Col}(\omega, H^L_\nu)$ be V-generic and in $V[g]$ fix an elementary chain $\langle L_{\kappa_n} \mid n < \omega\rangle$ of weak κ-models $L_{\kappa_n} \prec H^L_\nu$ such that $H^L_\nu \subseteq \bigcup_{n<\omega} L_{\kappa_n}$, using that ν is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^L_ν-normal H^L_ν-measure μ on κ.

Claim 1.3.4.1. $\mathrm{Ult}(H^L_\nu, \mu)$ is well-founded.

Proof of claim. Assume for a contradiction that $\mathrm{Ult}(H^L_\nu, \mu)$ is ill-founded, witnessed by a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n : \kappa \to \nu$ such that $g_n \in H^L_\nu$ and $\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu$. Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f : \kappa \to \nu$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{G}^\nu_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now, the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of well-foundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $\mathcal{G}^\nu_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H^L_\nu, \mu)$ is well-founded. $\dashv$

Let $j : H^L_\nu \to \mathrm{Ult}(H^L_\nu, \mu) \cong M$ be the ultrapower embedding followed by the transitive collapse, so that $M = L_\alpha$ for some α by elementarity. Let now

$h \subseteq \mathrm{Col}(\omega, \kappa^{+L})^L$ be L-generic, so that H^L_ν is countable in $L[h]$ and (trivially) $M \in L[h]$. By Lemma 1.3.3 we then get that there's an elementary embedding $j^* : H^L_\nu \to M$ in $L[h]$ with critical point κ. Since we also have that $M \in L$ and as ν was arbitrary, this makes κ virtually measurable in L. ■

We get the following immediate corollary.

Corollary 1.3.5 (Schindler-N.). *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ■

Note also that the proof of Theorem 1.3.4 shows that whenever κ is strategic ω-Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H^V_ν-normal H_ν-measure on κ.

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω-Ramsey cardinals and ω-Ramsey cardinals lie, namely that strategic ω-Ramseys are equiconsistent with *remarkables* and ω-Ramseys are strictly below. Theorem 4.8 of [Gitman and Welch, 2011] showed that 2-iterables are limits of remarkables, and our Propositions 1.1.6 and 1.3.13 shows that ω-Ramseys are limits of 1-iterables, so that the strategic ω-Ramseys and the ω-Ramseys both lie strictly between the 2-iterables and 1-iterables. It was shown in [Holy and Schlicht, 2018] that ω-Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by [Schindler, 2000], and [Gitman and Schindler, 2015] showed the following two equivalent formulations.

Definition 1.3.6. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

(i) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H^V_\lambda \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H^V_\nu \to M$ with critical point κ and $\pi(\kappa) > \lambda$;

(ii) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^{\lambda}M \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_{\nu}^{V} \to M$ with critical point κ and $\pi(\kappa) > \lambda$.

$\circ$

Theorem 1.3.7 (N.). *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models \ulcorner$there is a proper class of virtually measurables$\urcorner$. In particular, the two notions are equiconsistent.*

Proof. Virtually measurables are downwards absolute to L by Lemma 1.3.3, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H^V_\lambda \subseteq M$ and forcing poset $\mathbb{P}$ it holds that, in $V^{\mathbb{P}}$, there's no elementary embedding $\pi : H^V_\nu \to M$ with $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν-measurable to fix a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's an elementary $\pi : H^V_\nu \to M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H^V_\nu = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we're assuming that κ isn't remarkable. Then by restricting the generic embedding to H^V_κ we get that $H^V_\kappa \prec H^M_{\pi(\kappa)} = H^V_{\pi(\kappa)}$, using that $\pi(\kappa) < \nu$ and $H^V_\nu = H^M_\nu$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H^V_ν since $\pi(\kappa) < \nu$, and as $H^V_\nu \prec_1 V$ we get that $\pi(\kappa)$ *is* a cardinal. But then, again using that $H_{\pi(\kappa)} \prec_1 V$, κ is virtually measurable in $H^V_{\pi(\kappa)}$ since being virtually measurable is Π_2. This means that for every $\xi < \kappa$ it holds that

$$H^V_{\pi(\kappa)} \models \exists\alpha > \xi : \ulcorner\alpha \text{ is virtually measurable}\urcorner,$$

implying that $H^V_\kappa \models \ulcorner$There is a proper class of virtually measurables$\urcorner$. ■

Now Theorem 1.3.7 and Corollary 1.3.5 yield the following immediate corollary.

Corollary 1.3.8 (Schindler-N.). *Let κ be strategic ω-Ramsey. Then either κ is remarkable in L or otherwise $L_\kappa \models \ulcorner$there is a proper class of strategic ω-Ramseys$\urcorner$. In particular, the two notions are equiconsistent.* ■

Now, using these results we show that the strategic ω-Ramseys have strictly stronger consistency strength than the ω-Ramseys.

Theorem 1.3.9 (N.). *Remarkable cardinals are strategic ω-Ramsey limits of ω-Ramsey cardinals.*

Proof. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ-sequences and a generic elementary embedding $\pi : H_\nu^V \to M$ for some $\nu > 2^\kappa$. We will show that κ is ω-Ramsey in M. Note that remarkables are clearly virtually measurable, and thus by Theorem 1.3.4 also strategic ω-Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}_\omega^\theta(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}_\omega^\theta(\kappa)^M$. Since M is closed under 2^κ-sequences it means that $\mathscr{P}(\mathscr{P}(\kappa)) \subseteq M$ and thus that M contains all possible filters on κ. We let player II follow τ, which produces a play $\sigma * \tau$ in which player II wins. But all player II's moves are in $\mathscr{P}(\mathscr{P}(\kappa))$ and hence in M, and as M is furthermore closed under ω-sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω-Ramsey in M.

This also implies that κ is a limit of ω-Ramseys in H_ν. But as κ is remarkable it holds that $H_\kappa \prec_2 V$, in analogy with the same property for strongs and supercompacts, and as being ω-Ramsey is a Π_2-notion this means that κ *is* a limit of ω-Ramseys. ■

This immediately yields the following corollary.

Corollary 1.3.10 (Schindler-N.). *If κ is a strategic ω-Ramsey cardinal then*

$$L_\kappa \models \ulcorner \textit{there is a proper class of } \omega\textit{-Ramseys} \urcorner. \qquad \dashv$$

(ω, α)-Ramsey cardinals

A natural generalisation of the γ-Ramsey definition is to require more iterability of the last measure. Of course, by Proposition 1.1.6 we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when $\operatorname{cof}\gamma > \omega$ so the next definition is only interesting whenever $\operatorname{cof}\gamma = \omega$.

Definition 1.3.11 (N.)**.** Let α, β be ordinals. Then a cardinal κ is (α, β)**-Ramsey** if player I does not have a winning strategy in $\mathcal{G}_\alpha^\theta(\kappa, \beta)$ for all regular $\theta > \kappa$.[9] $\circ$

Definition 1.3.12 (Gitman)**.** A cardinal κ is α**-iterable** if for every $A \subseteq \kappa$ there exists a *transitive* weak κ-model $\mathcal{M}$ with $A \in \mathcal{M}$ and an α-good $\mathcal{M}$-measure μ on $\mathcal{M}$. $\circ$

Proposition 1.3.13. *If $\beta > 0$ then every (α, β)-Ramsey is a β-iterable stationary limit of β-iterables.*

Proof. Let $(\mathcal{M}, \in, \mu)$ be a result of a play of $\mathcal{G}_\alpha^{\kappa^+}(\kappa, \beta)$ in which player II won. Then the transitive collapse of $(\mathcal{M}, \in, \mu)$ witnesses that κ is β-iterable, since μ is β-good by definition of $\mathcal{G}_\alpha^{\kappa^+}(\kappa, \beta)$.

That κ is β-iterable is reflected to some H_θ, so let now $(\mathcal{N}, \in, \nu)$ be a result of a play of $\mathcal{G}_\alpha^\theta(\kappa, \beta)$ in which player II won. Then $\mathcal{N} \prec H_\theta$, so that κ is also β-iterable in $\mathcal{N}$. Since being β-iterable is witnessed by a subset of κ and $\beta > 0$ implies[10] that we get a κ-powerset preserving $j : \mathcal{N} \to \mathcal{P}$, $\mathcal{P}$ also thinks that κ is β-iterable, making κ a stationary limit of β-iterables by elementarity. ■

We now move towards Theorem 1.3.17 which gives an upper consistency bound for the (ω, α)-Ramseys. We first recall a few definitions and a folklore lemma.

Definition 1.3.14. For an infinite ordinal α, a cardinal κ is α**-Erdős** for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \to \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c; i.e. that $|c``[H]^n| \leq 1$ holds for every $n < \omega$. $\circ$

Definition 1.3.15. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$. $\circ$

[9] Note that an α-Ramsey cardinal is the same as an $(\alpha, 0)$-Ramsey cardinal.
[10] Recall that β-good for $\beta > 0$ in particular implies weak amenability.

Lemma 1.3.16 (Folklore)**.** *Let κ be α-Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure $\mathcal{M}$ in a countable language $\mathcal{L}$ with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

Proof. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all $\mathcal{L}$-formulas and define $c : [C]^{<\omega} \to \kappa$ as follows. For an increasing sequence $\alpha_1 < \cdots < \alpha_{2n} \in C$ let

$$c(\{\alpha_1, \ldots, \alpha_{2n}\}) := \text{the least } \lambda < \alpha_1 \text{ such that } \exists \delta_1 < \cdots \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \ldots, \delta_k \rangle \wedge \\ \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}]$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α-Erdős we get a homogeneous $I \in [C]^\alpha$ for c; i.e. that $|c``[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \ldots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \ldots, \alpha_{2n} \in I$, as otherwise there exists an $m < \omega$ and $\delta_1 < \cdots \delta_k$ such that for any $\alpha_1 < \ldots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}]. \tag{$\dagger$}$$

But then simply pick $\alpha_1 < \ldots \alpha_{2n} < \alpha'_1 < \cdots < \alpha'_{2n}$ so that both $\{\alpha_1, \ldots, \alpha_{2n}\}$ and $\{\alpha'_1, \ldots, \alpha'_{2n}\}$ witnesses (†); then either $\{\alpha_1, \ldots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \ldots, \alpha_n, \alpha'_{n+1}, \ldots, \alpha'_{2n}\}$ also witnesses that (†) fails, $\lightning$. ■

Theorem 1.3.17 (N.)**.** *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α-Erdős cardinal is a limit of (ω, α)-Ramsey cardinals.*

Proof. Let κ be α-Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 1.3.16 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I. We will show that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in [Holy and Schlicht, 2018] it suffices to find a weak ι-model $\mathcal{M} \prec H_\theta$ and an α-good $\mathcal{M}$-measure on ι. Define

$$\mathcal{M} := \operatorname{Hull}^{H_\theta}(\iota \cup I) \prec H_\theta$$

and let $\pi : I \to I$ be the right-shift map. Since I is remarkable, I $(= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \ldots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι-powerset preserving we get that $\mathcal{M} \prec H_\theta$ is a weak ι-model satisfying ZFC^- with a 1-good $\mathcal{M}$-measure μ_j on ι. Furthermore, as we can linearly iterate $\mathcal{M}$ simply by applying j we get an α-iteration of $\mathcal{M}$ since there are α-many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \ldots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\eta, \alpha)$. If we repeat this procedure for all regular $\theta > \kappa$ we get by the pidgeon hole principle that κ is a limit of (ω, α)-Ramsey cardinals. ■

As Theorem 4.5 in [Gitman and Schindler, 2015] shows that $(\alpha{+}1)$-iterable cardinals have α-Erdős cardinals below them for $\alpha \geq \omega$ additively closed, this shows that the (ω, α)-Ramseys form a strict hierarchy. Further, as α-Erdős cardinals are consistent with $V = L$ when $\alpha < \omega_1^L$ and ω_1-iterable cardinals aren't consistent with $V = L$, we also get that (ω, α)-Ramsey cardinals are consistent with $V = L$ if $\alpha < \omega_1^L$ and that they aren't if $\alpha = \omega_1$.

[Strategic] $(\omega{+}1)$-Ramsey cardinals

The next step is then to consider $(\omega{+}1)$-Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in [Mitchell, 1979] and in the proof of Lemma 1.3 in [Donder et al., 1981] — see also [Dodd, 1982] and [Gitman, 2011].

Theorem 1.3.18 (Dodd, Mitchell)**.** *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model $\mathcal{M}$ such that there exists a weakly amenable countably complete $\mathcal{M}$-measure on κ.* ■

The following theorem then supplies us with a lower bound for the strength of the $(\omega{+}1)$-Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 1.4.9, but we include this Ramsey lower bound as well for completeness.

Theorem 1.3.19 (N.)**.** *Every $(\omega{+}1)$-Ramsey cardinal is a Ramsey limit of Ramseys.*

PROOF. Let κ be $(\omega{+}1)$-Ramsey and $A \subseteq \kappa$. Let σ be a strategy for player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A \in \mathcal{M}_0$ and $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Then σ isn't winning as κ is $(\omega{+}1)$-Ramsey, so we may fix a play $\sigma * \vec{\mu}_\alpha$ of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable $\mathcal{M}_\omega$-measure on κ, and by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 1.3.18.

Since κ is Ramsey, $\mathcal{M}_\omega \models \ulcorner\kappa$ is Ramsey$\urcorner$ as well. Letting $j : \mathcal{M}_\omega \to \mathcal{N}$ be the κ-powerset preservering embedding induced by μ_ω, we also get that $\mathcal{N} \models \ulcorner\kappa$ is Ramsey$\urcorner$ by κ-powerset preservation. This then implies that κ *is* a stationary limit of Ramsey cardinals inside $\mathcal{M}_\omega$, and thus also in V by elementarity. ■

As for the *consistency* strength of the strategic $(\omega{+}1)$-Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal — see e.g. p. 249 in [Kanamori and Magidor, 1978].

Theorem 1.3.20 (N.)**.** *If κ is a strategic $(\omega{+}1)$-Ramsey cardinal then, in $V^{\mathrm{Col}(\omega,2^\kappa)}$, there's a transitive class N and an elementary embedding $j : V \to N$ with* $\mathrm{crit}\, j = \kappa$*. In particular, the existence of a strategic $(\omega{+}1)$-Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

Proof. Set $\mathbb{P} := \mathrm{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$. Let $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name of an ω-sequence $\langle\mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n \prec H^V_{\kappa^+}$ and $\mathscr{P}(\kappa)^V \subseteq \bigcup_{n<\omega}\mathcal{M}_n$, and let $\dot\mu$ be a $\mathbb{P}$-name for the ω-sequence of σ-responses to the $\mathcal{M}_n$'s in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)^V$.

Assume that there's a $\mathbb{P}$-condition p which forces the generic ultrapower $\mathrm{Ult}(V, \bigcup_n \dot\mu_n)$ to be illfounded, meaning that we can fix a $\mathbb{P}$-name $\dot f$ for an ω-sequence $\langle f_n \mid n < \omega\rangle$ such that

$$p \Vdash \dot X_n := \{\alpha < \kappa \mid \dot f_{n+1}(\alpha) < \dot f_n(\alpha)\} \in \bigcup_{n<\omega} \dot\mu_n.$$

Now, in V, we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} \prec H_\theta$ such that $\dot{\mathcal{M}}, \dot\mu, \dot f, H^V_{\kappa^+}, \sigma, p \in \mathcal{N}$. We can find an $\mathcal{N}$-generic $g \subseteq \mathbb{P}^{\mathcal{N}}$ in V with $p \in g$ since $\mathcal{N}$ is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}^g_n * \dot\mu^g_n$ is a play of $\mathcal{G}^{\kappa^+}_\omega(\kappa)^V$ which is according to σ, meaning that $\bigcup_{n<\omega}\dot\mu^g_n$ is normal and in particular countably complete (in V). Then $\bigcap_{n<\omega}\dot X^g_n \neq \emptyset$, but if $\alpha \in \bigcap_{n<\omega}\dot X^g_n$ then $\langle \dot f^g_n(\alpha) \mid n < \omega\rangle$ is a strictly decreasing ω-sequence of ordinals, $\lightning$. This means that $\mathrm{Ult}(V, \bigcup_n \mu_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in [Kellner and Shelah, 2011]. ■

The above Theorem 1.3.20 then answers Question 9.2 in [Holy and Schlicht, 2018] in the negative, asking if λ-Ramseys are strategic λ-Ramseys for uncountable cardinals λ, as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.

1.4 The general case

Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from [Gitman, 2011] and investigate further connections between these and the α-Ramsey cardinals. First, a definition.

Definition 1.4.1 (Gitman)**.** A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M}$ with a weakly amenable

$\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ. If furthermore $\mathcal{M} \prec H_{\kappa^+}$ then we say that κ is **super Ramsey**. $\circ$

Note that since the model $\mathcal{M}$ in question is a *κ-model* it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$-sequences. [Gitman, 2011] shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M} \models \mathsf{ZFC}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ.

Now, a first connection between the α-Ramseys and the strongly- and super Ramseys is the result in [Holy and Schlicht, 2018] that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

Theorem 1.4.2 (N.-W.). *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

Proof. Let κ be strongly Ramsey and let $\mathcal{M} \models \mathsf{ZFC}$ be a transitive κ-model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^{\mathcal{M}}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of $\mathcal{M}$ by weak amenability of μ. As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside $\mathcal{M}$, so $\mathcal{M}$ sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \to \mathcal{N}$ be the induced embedding, κ-powerset preservation of j_μ implies that μ is also a weakly amenable $\mathcal{N}$-normal $\mathcal{N}$-measure on κ. This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in $\mathcal{N}$, entailing that it is a stationary limit of almost fully Ramseys in $\mathcal{M}$. But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in [Holy and Schlicht, 2018], so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and $\mathcal{M}$ being transitive implies that κ really *is* a stationary limit of almost fully Ramseys. ■

Downwards absoluteness to K

Lastly, we consider the question of whether the α-Ramseys are downwards absolute to K, which turns out to at least be true in many cases. The below Theorem 1.4.4 then also answers Question 9.4 from [Holy and Schlicht, 2018] in the positive, asking whether α-Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of $0^{\P}$.

Definition 1.4.3. $0^{\P}$ is "the sharp for a strong cardinal", meaning the minimal sound active mouse $\mathcal{M}$ with $\mathcal{M} \,l\, \mathrm{crit}(\dot{F}^{\mathcal{M}}) \models \ulcorner$There exists a strong cardinal$\urcorner$, with $\dot{F}^{\mathcal{M}}$ being the top extender of $\mathcal{M}$. ○

Theorem 1.4.4 (N.-W.)**.** *Assume $0^{\P}$ does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ-Ramsey. Then $K \models \ulcorner\kappa$ is a λ-Ramsey cardinal$\urcorner$.*

Proof. Note first that $\kappa^{+K} = \kappa^+$ by [Schindler, 1997], since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ-models $\mathcal{M} \prec Kl\kappa^+$. We can then define a strategy $\tilde{\sigma}$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde{\sigma}(\emptyset) := \mathrm{Hull}^{H_{\kappa^+}}(Kl\kappa \cup \sigma(\emptyset))$. Assuming now that $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma\rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$, we have two cases. If $\tilde{\mu}_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle \mathcal{M}_\alpha \mid \alpha < \gamma\rangle$ be the corresponding models played in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde{\sigma}(\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma\rangle) := \mathrm{Hull}^{H_{\kappa^+}}(Kl\kappa \cup \sigma(\langle \mathcal{M}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma\rangle)),$$

and otherwise let $\tilde{\sigma}$ play arbitrarily. As κ is λ-Ramsey (in V) there exists a play $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha \leq \lambda\rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda \cap Kl\kappa^+ \prec Kl\kappa^+$ so let $\mathcal{N}$ be the transitive collapse of $\tilde{\mathcal{M}}_\lambda \cap Kl\kappa^+$. But if $j : \mathcal{N} \to Kl\kappa^+$ is the uncollapse then $\mathrm{crit}\, j$ is both an $\mathcal{N}$-cardinal and also $> \kappa$ because we ensured that $Kl\kappa \subseteq \mathcal{N}$. This means that $j = \mathrm{id}$ because κ is the largest $\mathcal{N}$-cardinal by elementarity in $K|\kappa^+$, so that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ = \mathcal{N}$ is a transitive elementary substructure of $K|\kappa^+$, making it an initial segment of K.

Now, since $\mu := \tilde{\mu}_\lambda$ is a countably complete weakly amenable $K|o(\mathcal{N})$-measure[11], the "beaver argument"[12] shows that $\mu \in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu \cap \mathcal{N} \in K$ whenever player I plays $\mathcal{N}$. Since $\mu = \tilde{\mu}_\lambda$ we also have that $\mu \cap \mathcal{M}_\alpha = \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$, so that σ will eventually play $\mathcal{N}$, making τ win against σ.[13] ■

Note that the only thing we used $\operatorname{cof}\lambda > \omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine- or normal α-Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

Corollary 1.4.5 (N.-W.). *Assume $0^\P$ does not exist and let α be any limit ordinal. Then every genuine- and every normal α-Ramsey cardinal is downwards absolute to K. In particular, if α is a limit of limit ordinals then every $<\alpha$-Ramsey cardinal is downwards absolute to K as well.* ■

Indiscernible games

We now move to the strategic versions of the α-Ramsey hierarchy. The first thing we want to do is define *α-very Ramsey cardinals*, introduced in [Sharpe and Welch, 2011], and show the tight connection between these and the strategic α-Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 1.3.15.

Definition 1.4.6. A **good set of indiscernibles** for a structure $\mathcal{M}$ is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for $\mathcal{M}$ such that $\mathcal{M}|\iota \prec \mathcal{M}$ for any $\iota \in I$. ○

[11] Here we use that $\mathcal{N} \lhd K$.
[12] See Lemmata 7.3.7–7.3.9 and 8.3.4 in [Zeman, 2002] for this argument.
[13] Note that τ is not necessarily a winning strategy — all we know is that it is winning against this particular strategy σ.

Definition 1.4.7 (Sharpe-W.)**.** Define the **indiscernible game** $G^I_\gamma(\kappa)$ in γ many rounds as follows

$$\begin{array}{lcccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \text{II} & & I_0 & & I_1 & & I_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha$ is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for $\mathcal{M}_\alpha$ and the I_α's are $\subseteq$-decreasing. Player II wins iff they can continue playing through all the rounds. ○

Definition 1.4.8 (Sharpe-W.)**.** A cardinal κ is **γ-very Ramsey** if player II has a winning strategy in the game $G^I_\gamma(\kappa)$. ○

The next couple of results concerns the connection between the strategic α-Ramseys and the α-very Ramseys. We start with the following.

Theorem 1.4.9 (N.)**.** *Every $(\omega{+}1)$-Ramsey is an ω-very Ramsey stationary limit of ω-very Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey. We will describe a winning strategy for player II in the indiscernible game $G^I_\omega(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G^I_\omega(\kappa)$ then let player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ play

$$\mathcal{H}_0 := \text{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) \prec H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ which starts off with $\mathcal{H}_0$ and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}^{(\alpha)}_0 := \mathcal{M}_\alpha$ and $\mu^{(\alpha)}_0 := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{lcccccccc} \text{I} & \mathcal{H}^{(0)}_0 & & \cdots & & \mathcal{H}^{(\omega)}_0 & & \mathcal{H}^{(\omega+1)}_0 & \\ \text{II} & & \mu^{(0)}_0 & & \cdots & & \mu^{(\omega)}_0 & & \mu^{(\omega+1)}_0 \end{array}$$

By the choice of player I's strategy we get that $\mu_0^{(\omega)}$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ (it's even normal). Now Lemma 2.9 of [Sharpe and Welch, 2011] gives us a set of good indiscernibles $I_0 \in \mu_0^{(\omega)}$ for $\mathcal{M}_0$, as $\mathcal{M}_0 \in \mathcal{H}_0^{(\omega)}$ and $\mu_0^{(\omega)}$ is a countably complete weakly amenable $\mathcal{H}_0^{(\omega)}$-normal $\mathcal{H}_0^{(\omega)}$-measure on κ. Let player II play I_0 in $G_\omega^I(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G_\omega^I(\kappa)$.

I	$\mathcal{M}_0$		$\mathcal{M}_1$
II		I_0	

Since $\mu_0^{(\omega)} = \bigcup_n \mu_0^{(n)}$ we must have that $I_0 \in \mu_0^{(n_0)}$ for some $n_0 < \omega$. In the $(n_0{+}1)$'st round of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}_0^{(k)}, \mu_0^{(k)} \mid k \leq n_0 \rangle\}) \prec H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ thus looks like the following so far.

I	$\mathcal{H}_0^{(0)}$		$\cdots$		$\mathcal{H}_0^{(n_0)}$		$\mathcal{H}_1$
II		$\mu_0^{(0)}$		$\cdots$		$\mu_0^{(n_0)}$	

Now player II in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ is not losing at round n_0, so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I_1' \in \mu_1^{(n_1)}$ of good indiscernibles for $\mathcal{M}_1$, where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G_\omega^I(\kappa)$ play $I_1 := I_0 \cap I_1' \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G_\omega^I(\kappa)$, making κ ω-very Ramsey.

As for showing that κ is a stationary limit of ω-very Ramseys, let $\mathcal{M} \prec H_{\kappa^+}$ be a weak κ-model with a weakly amenable countably complete $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ, which exists by Theorem 1.3.19 as κ is $(\omega{+}1)$-Ramsey. Then by elementarity $\mathcal{M} \models \ulcorner\kappa$ is ω-very Ramsey$\urcorner$ and since κ being ω-very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ-ultrapower, meaning that κ is a stationary limit of ω-very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

Theorem 1.4.10 (N.). *For limit ordinals α, every coherent $<\omega\alpha$-Ramsey is $\omega\alpha$-very Ramsey.*

Proof. This is basically the same proof as the proof of Theorem 1.4.9. We do the "going-back" trick in ω-chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}^{\kappa^+}_{\omega\alpha}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$-Ramseyness. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω-chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 1.4.9. ■

As for going from the α-very Ramseys to the strategic α-Ramseys we got the following.

Theorem 1.4.11 (N.). *For γ any ordinal, every coherent $<\gamma$-very Ramsey*[14] *is coherent $<\gamma$-Ramsey.*[15]

Proof. The reason why we work with $<\gamma$-Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ-very Ramsey and let τ be a winning strategy for player II in $G^I_\zeta(\kappa)$. Let $\mathcal{M}_\alpha \prec H_\theta$ be any move by player I in the α'th round of

[14] Here the coherency again just means that the winning strategies σ_α for player II in $G^I_\alpha(\kappa)$ are $\subseteq$-increasing.

[15] Here a "coherent $<\gamma$-very Ramsey cardinal" is defined from γ-very Ramseys in the same way as coherent $<\gamma$-Ramsey cardinals is defined from γ-Ramseys. When γ is a limit ordinal then coherent $<\gamma$-very Ramseys are precisely the same as γ-very Ramseys, so this is solely to "subtract one" when γ is a successor ordinal — i.e. a coherent $<(\gamma + 1)$-very Ramsey cardinal is the same thing as a γ-very Ramsey cardinal.

$\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in $\mathcal{M}_\alpha$ and form now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G^I_\zeta(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for $\mathcal{N}_\alpha$ such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in [Sharpe and Welch, 2011] we get a structure $\mathcal{P}_\alpha$ with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a $\mathcal{P}_\alpha$-measure $\tilde{\mu}_\alpha$ on κ, generated by I_α.[16] Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α's are generated by the I_α's, the μ_α's are $\subseteq$-increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$-Ramsey. ■

The following result is then a direct corollary of Theorems 1.4.10 and 1.4.11.

Corollary 1.4.12 (N.)**.** *For limit ordinals α, κ is $\omega\alpha$-very Ramsey iff it is coherent $<\omega\alpha$-Ramsey. In particular, κ is λ-very Ramsey iff it is strategic λ-Ramsey for any λ with uncountable cofinality.* ■

We can now use this equivalence to transfer results from the α-very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in [Feng, 1990]. A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in [Sharpe and Welch, 2011]. We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in [Sharpe and Welch, 2011].

Theorem 1.4.13 (Sharpe-W.)**.** *A cardinal is completely Ramsey if and only if it is ω-very Ramsey.* ■

This, together with Theorem 1.4.9, immediately yields the following strengthening of Theorem 1.3.19.

[16]By *generated* here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α.

Corollary 1.4.14 (N.)**.** *Every* $(\omega{+}1)$*-Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* ■

The above Theorem 1.4.11 also yields the following consequence.

Corollary 1.4.15 (N.)**.** *Every completely Ramsey cardinal is completely ineffable.*

Proof. From Theorem 1.4.13 we have that being completely Ramsey is equivalent to being ω-very Ramsey, so the above Theorem 1.4.11 then yields that a completely Ramsey cardinal is coherent $<\omega$-Ramsey, which we saw in Theorem 1.2.12 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 1.4.12 yields that strategic ω_1-Ramsey cardinals are ω_1-very Ramsey, and Theorem 3.50 in [Sharpe and Welch, 2011] states that ω_1-very Ramseys are measurable in the core model K, assuming $0^{¶}$ doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

Theorem 1.4.16 (W.)**.** *Assuming* $0^{¶}$ *doesn't exist, every strategic* ω_1*-Ramsey cardinal is measurable in* K*.*

Proof. Let κ be strategic ω_1-Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \mathrm{Col}(\omega_1, \kappa^+)$ is V-generic. Since $\mathrm{Col}(\omega_1, \kappa^+)$ is ω-closed, V and $V[g]$ have the same countable sequences of V, so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V.

Now let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of regular K-cardinals cofinal in κ^+, let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \mathrm{Hull}^{H_\theta}(K l \kappa_\alpha) \prec H_\theta$ and player II follow τ. This results in a countably complete weakly amenable K-measure μ_{ω_1}, which the "beaver argument"[17] then shows is actually an element of K, making κ measurable in K. ■

[17] See Lemmata 7.3.7–7.3.9 and 8.3.4 in [Zeman, 2002] for this argument.

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1-Ramsey cardinal is also measurable in Steel's core model below a Woodin, a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from [Schindler, 2006].[18]

Theorem 1.4.17 (Schindler). *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ, and let $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.* ■

Theorem 1.4.18 (Schindler). *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ of weak κ-models $M_\alpha \prec H^V_\theta$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H^V_{\kappa^+} \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\mathrm{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Let $j : M_{\omega_1} \to \mathrm{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H^V_{\kappa^+} \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ with N transitive.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 1.4.17 applies, meaning that $\pi \restriction K : K \to K^N$ is an iteration map with critical point κ, making κ

[18] That paper assumes the existence of a measurable as well, but by [Jensen and Steel, 2013] we can omit that here.

measurable in K. ■

2 | The second chapter

2.1 Getting started

We will denote the class of ordinals by On. For X, Y sets we denote by ${}^XY$ the set of all functions from X to Y. For an infinite cardinal κ, we let H_κ be the set of sets X such that the cardinality of the transitive closure of X is $<\kappa$. ZF^- will denote ZF with the Collection scheme but without the Power Set axiom, following the results of [**?**]. The symbol $\lightning$ will denote a contradiction and $\mathscr{P}(X)$ denotes the power set of X.

A key folklore lemma which we will frequently need when dealing with elementary embeddings existing in generic extensions is the following.

Lemma 2.1.1 (Countable Embedding Absoluteness)**.** *Let $\mathcal{M}, \mathcal{N}$ be transitive and assume that $\mathcal{M}$ is countable. Let $\pi\colon \mathcal{M} \to \mathcal{N}$ be an elementary embedding, $\mathcal{P}$ a transitive class with $\mathcal{M}, \mathcal{N} \in \mathcal{P}$ and*

$$\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC} + \ulcorner \mathcal{M} \text{ is countable} \urcorner,$$

and fix any finite $X \subseteq \mathcal{M}$. Then $\mathcal{P}$ has an elementary embedding $\pi^*\colon \mathcal{M} \to \mathcal{N}$ which agrees with π on X and $\operatorname{crit}\pi = \operatorname{crit}\pi^*$.

Proof. Let $\{a_i \mid i < \omega\} \in \mathcal{P}$ be an enumeration of $\mathcal{M}$ and set $\mathcal{M} \restriction n := \{a_i \mid i < n\}$. Then, in $\mathcal{P}$, build the tree $\mathcal{T}$ of all partial isomorphisms between $\mathcal{M} \restriction n$ and $\mathcal{N}$ for $n < \omega$, ordered by extension. Then $\mathcal{T}$ is illfounded in V by assumption, so it's also illfounded in $\mathcal{P}$ since $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC}$. The branch gives the embedding π^*, and we can ensure that it agrees with π on the critical point and finitely many values by adding these conditions to $\mathcal{T}$. ■

2.2 Strong and supercompact

We start out by defining virtual versions of a variety of large cardinal notions used in this section. We start out with measurables, strongs and supercompacts.

Definition 2.2.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is...

- **faintly θ-measurable** if, in a forcing extension, there is a transitive class $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$;
- **faintly θ-strong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$;
- **faintly θ-supercompact** if it's faintly θ-measurable, ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

We further replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we attach a **"pre"** if we don't want to assume $\pi(\kappa) > \theta$, and when we don't mention θ we mean that it holds for all regular $\theta > \kappa$. For instance, a faintly prestrong cardinal is a cardinal κ such that for all regular $\theta > \kappa$, κ is faintly θ-measurable with $H_\theta^V \subseteq \mathcal{N}$. ∘

We note that even small cardinals can be faintly measurable: we may for instance have a precipitous ideal on ω_1. The "virtually" adverb implies that the cardinals are in fact large cardinals in the usual sense, as the following shows.

Recall from [Gitman and Schindler, 2015] that a cardinal κ is **1-iterable** if to every $A \subseteq \kappa$ there's a transitive $\mathcal{M} \models \mathsf{ZFC}^-$ with $\kappa, A \in \mathcal{M}$ and a weakly amenable $\mathcal{M}$-ultrafilter μ on κ with a wellfounded ultrapower.[1] 1-iterable cardinals are weakly compact limits of weakly compact cardinals.

Proposition 2.2.2 (Virtualised folklore)**.** *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is 1-iterable.*

Proof. (Sketch) Let κ be virtually θ-measurable, witnessed by a forcing $\mathbb{P}$, a transitive $\mathcal{N} \subseteq V$ and an elementary $\pi\colon H_\theta^V \to \mathcal{N}$ with $\pi \in V^{\mathbb{P}}$. If

[1] Also recall that μ is **weakly amenable** if $\mu \cap X \in \mathcal{M}$ for every $X \in \mathcal{M}$ of $\mathcal{M}$-cardinality $\leq \kappa$.

κ isn't a strong limit then we have a surjection $\pi(f)\colon \mathscr{P}(\alpha) \to \pi(\kappa)$ with $\operatorname{ran}\pi(f) = \operatorname{ran} f \subseteq \kappa$ for some $\alpha < \kappa$, $\lightning$. Note that we used $\mathcal{N} \subseteq V$ to ensure that $\mathscr{P}(\alpha)^V = \mathscr{P}(\alpha)^{\mathcal{N}}$. The same argument shows that κ is regular. By restricting the generic embedding and using that $\mathscr{P}(\kappa)^V = \mathscr{P}(\kappa)^N$ as $\mathcal{N} \subseteq V$ and $\mathscr{P}(\kappa)^V \subseteq \mathcal{N}$, we get that κ is 1-iterable. ■

Along with the above definition of faintly supercompactness we can also virtualise Magidor's characterisation of supercompact cardinals, which was also one of the original characterisations of the remarkable cardinals in [Schindler, 1997].

Definition 2.2.3. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-supercompact ala Magidor** if there are $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ such that $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. ○

In the virtual world these two versions of supercompacts remain equivalent, but they also turn out to be equivalent to the virtually strongs:

Theorem 2.2.4 (G.-Schindler)**.** *For an uncountable cardinal κ, the following are equivalent.*[2]

(i) *κ is virtually strong;*
(ii) *κ is virtually supercompact;*
(iii) *κ is virtually supercompact ala Magidor.*

Proof. $(ii) \Rightarrow (i)$ is simply by definition.

$(i) \Rightarrow (iii)$: Fix $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi\colon H^V_{(2^{<\theta})^+} \to \mathcal{M}$ with[3] $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H^V_{(2^{<\theta})^+} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. Since $H^V_\theta, H^{\mathcal{M}}_{\pi(\theta)} \in \mathcal{M}$, Countable Embedding Absoluteness 2.1.1 implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H^V_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ with $\operatorname{crit}\pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H^V_\theta = H^{\mathcal{M}}_\theta$ as $\mathcal{M} \subseteq V$ and $H^V_\theta \subseteq \mathcal{M}$, elementarity of π now implies that $H^V_{(2^{<\theta})^+}$ has ordinals $\bar\kappa < \bar\theta < \kappa$

[2] A cardinal satisfying any/all of these conditions is usually called **remarkable**.
[3] The domain of π is $H^V_{(2^{<\theta})^+}$ to ensure that $H^V_\theta \in \operatorname{dom}\pi$.

and a generic elementary $\sigma\colon H^V_{\bar\theta} \to H^V_\theta$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa$. This shows (iii).

$(iii) \Rightarrow (ii)$: Fix $\theta > \kappa$ and $\delta := (2^{<\theta})^+$. By (iii) there exist ordinals $\bar\kappa < \bar\delta < \kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\delta} \to H^V_\delta$ with $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. We will argue that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, so that by elementarity κ is virtually θ-supercompact in H^V_δ and hence also in V by the choice of δ. Consider the restriction

$$\sigma := \pi \restriction H^V_{\bar\theta} : H^V_{\bar\theta} \to H^V_\theta.$$

Note that H^V_θ is closed under $<\bar\theta$-sequences (and more) in V. Now define

$$X := \bar\theta{+}1 \cup \{x \in H^V_\theta \mid \exists y \in H^V_{\bar\theta} \exists p \in \operatorname{Col}(\omega, H^V_{\bar\theta})\colon p \Vdash \dot\sigma(\check y) = \check x\} \in V.$$

Note that $|X| = \left|H^V_{\bar\theta}\right| = 2^{<\bar\theta}$ and that $\operatorname{ran}\sigma \subseteq X$. Now let $\overline{\mathcal{M}} \prec H^V_\theta$ be such that $X \subseteq \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{\mathcal{M}}$ of size $(2^{<\bar\theta})^{<\bar\theta} = 2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{\mathcal{M}}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}| = 2^{<\bar\theta} < \bar\delta$, making $\mathcal{M} \in H^V_{\bar\delta}$.

Countable Embedding Absoluteness 2.1.1 then implies that $H^V_{\bar\delta}$ has a generic elementary embedding $\sigma^*\colon H^V_{\bar\theta} \to \mathcal{M}$ with $\operatorname{crit}\sigma^* = \bar\kappa$, showing that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, which is what we wanted to show. ■

Remark 2.2.5. The above proof shows that if κ is virtually $(2^{<\theta})^+$-strong then it's virtually θ-supercompact, and if it's virtually $(2^{<\theta})^+$-supercompact ala Magidor then it's virtually θ-supercompact. It's open whether they are equivalent level-by-level (see Question **??**).

A key difference between the normal large cardinals and the virtual kind is that we don't have a virtual version of the Kunen inconsistency: it's perfectly valid to have an elementary embedding $H^V_\theta \to H^V_\theta$ with θ much larger than the critical point. This becomes important when dealing with the "pre"-versions of the large cardinals. We start with a virtualisation of the α-superstrong cardinals.

Definition 2.2.6. Let θ be a regular uncountable cardinal and and α an ordinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, α)**-superstrong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$[4]. We replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we say that κ is **faintly** α**-superstrong** if it's faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong. ◦

Proposition 2.2.7 (N.)**.** *If κ is faintly superstrong then H_κ has a proper class of virtually strong cardinals.*

Proof. Fix a regular $\theta > \kappa$ and a generic embedding $\pi \colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) < \theta$. Then $\pi(\kappa)$ is a V-cardinal, so that $H_{\pi(\kappa)}^V$ thinks that κ is virtually strong. This implies that H_κ^V thinks there is a proper class of virtually strong cardinals, using that $H_\kappa^V \prec H_{\pi(\kappa)}^V$. ■

The following theorem then shows that the only thing stopping prestrongness from being equivalent to strongness is the existence of "Kunen inconsistencies".

Theorem 2.2.8 (N.)**.** *Let θ be an uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-prestrong iff one of the following holds.*

(i) κ is virtually θ-strong; or

(ii) κ is virtually (θ, ω)-superstrong.

Proof. ($\Leftarrow$) is trivial, so we show ($\Rightarrow$). Let κ be virtually θ-prestrong. Assume (i) fails, meaning that there's a generic extension $V^{\mathbb{P}}$ and an elementary embedding $\pi \in V^{\mathbb{P}}$ such that $\pi \colon H_\theta^V \to \mathcal{N}$ for some transitive $\mathcal{N}$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) \leq \theta$. Assume $\pi^n(\kappa)$ is defined for all $n < \omega$ and define $\lambda := \sup_{n<\omega} \pi^n(\kappa)$. If $\lambda \leq \theta$ then κ is virtually (θ, ω)-superstrong by definition, so assume that there's some least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$.

This means that κ is virtually ν-strong for every regular $\nu \in (\kappa, \pi^n(\kappa))$, which is a Δ_0-statement in $\{H_{\nu^+}^V\}$ and hence downwards absolute to $H_{\pi^n(\kappa)}^V$.

[4] Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.

This means that κ is virtually strong in $H^V_{\pi^n(\kappa)}$ and also that $\pi^n(\kappa)$ is virtually strong in $H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$ by elementarity, and so in particular virtually θ-strong in $\mathcal{N}$. This means that there's some generic elementary embedding

$$\sigma\colon H^{\mathcal{N}}_\theta \to \mathcal{M}$$

with $H^{\mathcal{N}}_\theta \subseteq \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{N}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. We can now restrict σ to its critical point $\pi^n(\kappa)$ to get that

$$H^V_{\pi^n(\kappa)} = H^{\mathcal{N}}_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))},$$

using that $H^V_\theta = H^{\mathcal{N}}_\theta$ holds as π is a virtual embedding. Since κ is virtually strong in $H^V_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$. In particular, κ is virtually θ-strong in $\mathcal{M}$, and as $H^{\mathcal{M}}_\theta = H^{\mathcal{N}}_\theta = H^V_\theta$, this means that κ is virtually θ-strong in V, contradicting (i). ■

We then get the following consistency result.

Corollary 2.2.9 (N.). *For any uncountable regular θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

Proof. The above Proposition 2.2.7 and Theorem 2.2.8 show that virtually θ-prestrongs are equiconsistent with virtually θ-strongs. Now note that Countable Embedding Absoluteness 2.1.1 and condensation in L imply that every faintly θ-measurable cardinal is virtually θ-prestrong in L. ■

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H^V_\theta \to H^V_\theta$ with $\operatorname{crit}\pi = \kappa$. We then have the following corollary.

Corollary 2.2.10 (N.). *The following are equivalent:*

(i) For every uncountable cardinal θ, every virtually θ-prestrong cardinal is virtually θ-strong;

(ii) There are no virtually rank-into-rank cardinals.

PROOF. ($\Leftarrow$): Note first that being virtually ω-superstrong is equivalent to being virtually rank-into-rank. Indeed, every virtually rank-into-rank cardinal is virtually ω-superstrong by definition, and if κ is virtually ω-superstrong and $\lambda := \sup_{n<\omega} \pi^n(\kappa)$ then $\pi \upharpoonright H_\lambda^V : H_\lambda^V \to H_\lambda^V$ witnesses that κ is virtually λ-rank-into-rank. The above Theorem 2.2.8 then implies ($\Leftarrow$).

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let (κ, θ) be the lexigraphically least pair such that κ is virtually θ-rank-into-rank, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong then it would be Σ_2-reflecting, so that the statement that there exists a virtually rank-into-rank cardinal would reflect down to H_κ^V, contradicting the minimality of κ. ■

As a final result of this section, we note that the "virtually" adverb *does* yield cardinals different from the faintly ones. This is trivial in general as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

For a slightly more fine-grained distinction let's prepend a **power-** adjective whenever the domain and codomain of the generic elementary embedding have the same subsets of κ. Note that the proof of Lemma 2.2.2 shows that faintly power-measurables are also 1-iterable.

Our separation result is then the following.

Theorem 2.2.11 (G.). *For $\Phi \in \{measurable, prestrong, strong\}$, if κ is virtually Φ then there exist forcing extensions $V[g]$ and $V[h]$ such that*

(i) In $V[g]$, κ is inaccessible and faintly Φ but not faintly power-Φ; and

(ii) In $V[h]$, κ is faintly power-Φ but not virtually Φ.

PROOF. We start with (i). Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\lambda < \kappa$, and let $g \subseteq \mathbb{P}_\kappa$ be V-generic. Note that κ remains inaccessible in $V[g]$. Fix a regular $\theta > \kappa$ and let $\mathbb{Q}_\theta$ be a forcing witnessing that κ is virtually θ-measurable.

Since κ is *virtually* measurable we may without loss of generality assume that $\mathbb{Q}_\theta = \mathrm{Col}(\omega, \theta)$ by applying Countable Embedding Absoluteness 2.1.1. Fixing a $V[g]$-generic $h \subseteq \mathbb{Q}_\theta$ we get a transitive $\mathcal{N} \subseteq V$ and in $V[h]$ an elementary embedding

Include this argument?

$$\pi \colon H_\theta^V \to \mathcal{N}$$

with $\operatorname{crit} \pi = \kappa$. Let's now work in $V[g][h] = V[h][g] = V[g \times h]$, in which we still have access to π. The lifting criterion is trivial for $\mathbb{P}_\kappa$, so we get an $\mathcal{N}$-generic $\tilde{g} \subseteq \pi(\mathbb{P}_\kappa)$ and an elementary

Define and give reference

$$\pi^+ \colon H_\theta^{V[g]} \to \mathcal{N}[\tilde{g}]$$

with $\pi \subseteq \pi^+$. Note here that without loss of generality $\pi(\kappa)$ is countable as otherwise we replace $\mathcal{N}$ by a countable hull, so we can indeed construct such a $\tilde{g}$. By elementarity of π it holds that

$$\pi(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \prod_{\lambda \in [\kappa, \pi(\kappa))} \mathrm{Add}(\lambda, 1), \tag{1}$$

so that $\mathcal{N}[\tilde{g}] \not\subseteq V$ as it in particular contains a new subset of κ. If $\Phi =$ measurable then we're done at this point. For $\Phi =$ prestrong we simply note that $g \in \mathcal{N}[\tilde{g}]$ by (1) so that $H_\theta^{V[g]} \subseteq \mathcal{N}[\tilde{g}]$ as well, and since π^+ lifts π it holds that $\pi^+(\kappa) = \pi(\kappa) > \theta$ in the $\Phi =$ strong case.

As for (ii), we simply change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$, which means that $\pi(\mathbb{P}_\kappa)$ doesn't add any subsets of κ and κ thus remains faintly power-Φ. By choosing $\theta > \kappa^+$ it *does* add a subset to κ^+ however, showing that κ is not virtually Φ. ■

In contrast to the above separation result, Theorem 2.3.9 will show that the faintly-virtually distinction vanishes when we're dealing with woodins.

2.3 Woodin and Vopěnka

In this section we will analyse the virtualisations of the woodin and vopěnka cardinals, which can be seen as “boldface” variants of strongs and supercompacts.

Definition 2.3.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, A)**-strong** for a set $A \subseteq H_\theta^V$ if there exists a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, A) \to (\mathcal{M}, \in, B)$$

such that $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta^V \subseteq \mathcal{M}$ and $B \cap H_\theta^V = A$. We say that κ is **faintly** (θ, A)**-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$ and say that κ is **faintly** (θ, A)**-extendible** if $\mathcal{M} = H_\mu^V$ for some V-cardinal μ. We will leave out θ if it holds for all regular $\theta > \kappa$. ○

Definition 2.3.2. A cardinal δ is **faintly woodin** if given any $A \subseteq H_\delta^V$ there exists a faintly $({<}\delta, A)$-strong cardinal $\kappa < \delta$. ○

As with the previous definitions, for both of the above two definitions we substitute “faintly” for **virtually** when $\mathcal{M} \subseteq V$, and substitute “strong”, “supercompact” and “woodin” for **prestrong**, **presupercompact** and **prewoodin** when we don't require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with the real woodin cardinals, virtually woodin cardinals are mahlo. This contrasts the virtually prewoodins since [**?**], together with Theorem 2.3.9 below, shows can be singular.

Proposition 2.3.3 (Virtualised folklore)**.** *Virtually woodin cardinals are mahlo.*

Proof. Let δ be virtually woodin. Note that δ is a limit of weakly compact cardinals by Proposition 2.2.2, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f\colon \alpha \to \delta$ with $f(0) > \alpha$

and $\alpha < \delta$, and note that f cannot have any closure points. Fix a virtually $(<\delta, f)$-strong cardinal $\kappa < \delta$; we claim that κ is a closure point for f, which will yield our desired contradiction.

Let $\gamma < \kappa$ and choose a regular $\theta \in (f(\gamma), \delta)$. We then have a generic embedding $\pi\colon (H_\theta^V, \in, f \cap H_\theta^V) \to (\mathcal{N}, \in, f^+)$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and f^+ is a function such that $f^+ \cap H_\theta^V = f \cap H_\theta^V$. But then $f^+(\gamma) = f(\gamma) < \pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma) < \kappa$, making κ a closure point for f, $\lightning$. This shows that δ is inaccessible.

As for mahloness, let $C \subseteq \delta$ be a club and $\kappa < \delta$ a virtually $(<\delta, C)$-strong cardinal. Let $\theta \in (\min C, \delta)$ and let $\pi\colon H_\theta^V \to \mathcal{N}$ be the associated generic elementary embedding. Then for every $\gamma < \kappa$ there exists an element of C below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an element of C. As κ is regular, this shows that δ is mahlo.
■

The well-known equivalence of the "function definition" and "A-strong" definition of woodin cardinals holds if we restrict ourselves to virtually woodins, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Proposition 2.3.4 (D.-G.-N.). *For an uncountable cardinal δ, the following are equivalent.*

- *(i)* *δ is virtually woodin.*
- *(ii)* *for every $A \subseteq H_\delta^V$ there exists a virtually $(<\delta, A)$-supercompact $\kappa < \delta$.*
- *(iii)* *for every $A \subseteq H_\delta^V$ there exists a virtually $(<\delta, A)$-extendible $\kappa < \delta$.*
- *(iv)* *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*
- *(v)* *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, ${}^{<\pi(f)(\kappa)}\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*

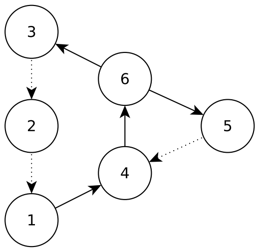

Figure 2.1: Proof strategy of Proposition 2.3.4, dotted lines are trivial implications.

(vi) *for every function* $f\colon \delta \to \delta$ *there are regular cardinals* $\bar\theta < \kappa < \theta < \delta$, *where* κ *is a closure point for* f, *and a generic elementary embedding* $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ *with* $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$ *and* $f \restriction \kappa \in \operatorname{ran}\pi$.

Proof. Firstly note that *(iii)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i)* and *(v)* $\Rightarrow$ *(iv)* are simply by definition.

$\boxed{(i) \Rightarrow (iv)}$ Assume δ is virtually woodin, and fix a function $f\colon \delta \to \delta$. Let $\kappa < \delta$ be virtually $({<}\delta, f)$-strong and let $\theta := \sup_{\alpha\leq\kappa} f(\alpha)+1$. Then there's a generic elementary embedding $\pi\colon (H^V_\theta, \in, f \cap H^V_\theta) \to (\mathcal{M}, \in, f^+)$ where $f^+ \restriction \kappa = f \restriction \kappa$, $\mathcal{M} \subseteq V$ and $\pi(\kappa) > \theta$. We firstly want to show that κ is a closure point for f, so let $\alpha < \kappa$. Then

$$f(\alpha) = f^+(\alpha) = \pi(f)(\alpha) = \pi(f)(\pi(\alpha)) = \pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha < \kappa$. Now, if κ wasn't a closure point for f then, letting $\alpha < \kappa$ be the least such that $f(\alpha) \geq \kappa$,

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument wouldn't work if we had only assumed δ to be virtually prewoodin. Lastly, θ-strongness implies that $H^V_\theta \subseteq \mathcal{M}$, and $\mathcal{M} \subseteq V$ holds by assumption.

$\boxed{(iv) \Rightarrow (vi)}$ Assume *(iv)* holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$. By *(iv)* there's a $\kappa < \delta$ which is a closure point of g and there's a regular $\theta \in (\kappa, \delta)$ and a generic elementary $\pi\colon H^V_\theta \to \mathcal{M}$

with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$. We want to find a regular $\bar\theta < \kappa$ and another elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f(\operatorname{crit}\sigma) = \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\sigma$.

Note that $\mathcal{M} \subseteq V$ and $H_\theta^V \subseteq \mathcal{M}$ implies that $H_\theta^V = H_\theta^{\mathcal{M}}$, so that both H_θ^V and $H_{\pi(\theta)}^{\mathcal{M}}$ are elements of $\mathcal{M}$ (we introduced g to ensure that $\pi(\theta)$ makes sense). An application of Countable Embedding Absoluteness 2.1.1 then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H_\theta^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$ and $\pi(f \restriction \kappa) \in \operatorname{ran}\pi^*$.

By elementarity of π, H_θ^V has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f \restriction \kappa \in \operatorname{ran}\sigma$ and $\bar\theta = f(\operatorname{crit}\sigma)$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta \to \delta$ be given. Define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$, so that by (vi) there exist regular $\bar\kappa < \bar\theta < \kappa < \theta$ such that κ is a closure point for g and there exists a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) = \bar\theta$ and $g \restriction \kappa \in \operatorname{ran}\pi$.

Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 2.2.4 we get a transitive $\mathcal{M} \in H_{g(\bar\kappa)}^V$ closed under $<f(\bar\kappa)$-sequences, and $H_{g(\bar\kappa)}^V$ has a generic elementary embedding $\sigma\colon H_{f(\bar\kappa)}^V \to \mathcal{M}$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa > f(\bar\kappa)$. In other words, $\bar\kappa$ is virtually $f(\bar\kappa)$-supercompact in $H_{\bar\theta}^V$. Elementarity of π then implies that κ is virtually $\pi(f)(\kappa)$-supercompact in H_θ^V, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all α such that $(H_\alpha^V, \in, A \cap H_\alpha^V) \prec (H_\delta^V, \in, A)$. Let $f\colon \delta \to \delta$ be given as $f(\alpha) = \langle \alpha_0, \alpha_1 \rangle$ with $\langle -, - \rangle$ being the Gödel pairing function, where α_0 is the first limit of elements of C above α and the α_1's are chosen such that $\{\alpha_1 \mid \alpha < \beta\}$ encodes $A \cap \beta$. This definition makes sense since δ is inaccessible by Proposition 2.2.2.

Let $\kappa < \delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa$, $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$, and $f \restriction \kappa \in \operatorname{ran}\pi$. We claim that $\bar\kappa := \operatorname{crit}\pi$ is virtually $(<\delta, A)$-extendible. To see this, it suffices by the definition of C to show that

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \models \ulcorner \bar\kappa \text{ is virtually } (A \cap H_\kappa)\text{-extendible}\urcorner, \tag{1}$$

since $\kappa \in C$ because it is a closure point of f. Let $\beta := \min(C - \bar\kappa) < \bar\theta$ and note that β exists as $f(\bar\kappa) = \bar\theta$ so the definition of f says that $\bar\theta$ is a limit of elements of C above $\bar\kappa$. It then holds that $(H^V_{\bar\kappa}, \in, A \cap H^V_{\bar\kappa}) \prec (H^V_\beta, \in, A \cap H^V_\beta)$ as both $\bar\kappa$ and β are elements of C. Since f encodes A in the manner previously described and $\pi^{-1}(f) \restriction \bar\kappa = f \restriction \bar\kappa$, we get that $\pi(A \cap H^V_{\bar\kappa}) = A \cap H^V_\kappa$ and thus

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \prec (H^V_{\pi(\beta)}, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H^V_\beta)$. Now, as $(H^V_\gamma, \in, A \cap H^V_\gamma)$ and $(H^V_{\pi(\gamma)}, \in, A^* \cap H^V_{\pi(\gamma)})$ are elements of $H^V_{\pi(\beta)}$ for every $\gamma < \kappa$, Countable Embedding Absoluteness 2.1.1 implies that $H^V_{\pi(\beta)}$ sees that $\bar\kappa$ is virtually $({<}\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ■

Remark 2.3.5. The above proof shows that the $\mathcal{M} \subseteq V$ assumptions can be replaced by "sufficient" agreement between $\mathcal{M}$ and V: for (i)-(iii) this means that $H^{\mathcal{M}}_\theta = H^V_\theta$ whenever $\mathcal{M}$ is the codomain of a virtual (θ, A)-strong/supercompact/extendible embedding, and in (iv)-(v) this means that $H^{\mathcal{M}}_{\pi(f)(\kappa)} = H^V_{\pi(f)(\kappa)}$. The same thing holds in the "lightface" setting of Theorem 2.2.4.

We will now step away from the woodins for a little bit, and introduce the vopěnkas. In anticipation of the next section we will work with the class-sized version here, but all the following results work equally well for inaccessible virtually vopěnka cardinals[5].

[5] Note however that we have to require inaccessibility here: see [?] for an analysis of the singular virtually vopěnka cardinals.

Definition 2.3.6 (GBC)**.** The **Generic Vopěnka Principle** (gVP) states that for any class C consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$. ∘

We will be using a standard variation of gVP involving the following *natural sequences*.

Definition 2.3.7 (GBC)**.** Say that a class function $f\colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. ∘

Definition 2.3.8 (GBC)**.** Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ is **natural** if there exists an indexing function $f\colon \mathsf{On} \to \mathsf{On}$ and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$ for every α. Denote this indexing function by $f^{\vec{\mathcal{M}}}$ and the unary relations as $R^{\vec{\mathcal{M}}}_\alpha$. ∘

The following theorem is then the main theorem of this section. Firstly it shows that inaccessible cardinals are virtually vopěnka iff they are virtually prewoodin, but also that adding the "virtually" adverb doesn't do anything in this context, in contrast to Theorem 2.2.11.

Theorem 2.3.9 (GBC, D.-G.-N.)**.** *The following are equivalent:*

(i) *gVP holds;*
(ii) *For any natural On-sequence $\vec{\mathcal{M}}$ there exists a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for some $\alpha < \beta$;*
(iii) *On is virtually prewoodin;*
(iv) *On is faintly prewoodin.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume On is faintly prewoodin and fix some On-sequence $\vec{\mathcal{M}} := \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ of structures in a common language. Let κ be $({<}\mathsf{On}, \vec{\mathcal{M}})$-prestrong and fix some regular $\theta > \kappa$ satisfying that $\mathcal{M}_\alpha \in H^V_\theta$

for every $\alpha < \theta$, and fix a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, \vec{\mathcal{M}}) \to (\mathcal{N}, \in, \mathcal{M}^*)$$

with $H_\theta^V \subseteq \mathcal{N}$ and $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$. Set $\kappa := \operatorname{crit}\pi$.

We have that $\pi \restriction \mathcal{M}_\kappa\colon \mathcal{M}_\kappa \to \mathcal{M}^*_{\pi(\kappa)}$, but we need to reflect this embedding down below θ as we don't know whether $\mathcal{M}^*_{\pi(\kappa)}$ is on the $\vec{\mathcal{M}}$ sequence. Working in the generic extension, we have

$$\mathcal{N} \models \exists\bar\kappa < \pi(\kappa)\exists\dot\sigma \in V^{\mathrm{Col}(\omega,\mathcal{M}^*_{\bar\kappa})} \colon \ulcorner \dot\sigma\colon \mathcal{M}^*_{\bar\kappa} \to \mathcal{M}^*_{\pi(\kappa)} \text{ is elementary}\urcorner.$$

Here κ realises $\bar\kappa$ and $\pi \restriction \mathcal{M}_\kappa$ realises σ. Note that $\mathcal{M}^*_\kappa = \mathcal{M}_\kappa$ since we ensured that $\mathcal{M}_\kappa \in H_\theta^V$ and we are assuming that $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$, so the domain of σ ($= \pi \restriction \mathcal{M}_\kappa$) *is* $\mathcal{M}^*_\kappa$ — also note that σ exists in a $\mathrm{Col}(\omega, \mathcal{M}_\kappa)$ extension of $\mathcal{N}$ by an application of Countable Embedding Absoluteness 2.1.1. Now elementarity of π implies that

$$H_\theta^V \models \exists\bar\kappa < \kappa\exists\dot\sigma \in V^{\mathrm{Col}(\omega,\mathcal{M}_{\bar\kappa})} \colon \ulcorner \dot\sigma\colon \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa \text{ is elementary}\urcorner,$$

which is upwards absolute to V, from which we can conclude that $\sigma\colon \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa$ witnesses that gVP holds.

$(ii) \Rightarrow (iii)$: Assume (ii) holds and assume that On is not virtually prewoodin, which means that there exists some class A such that there are no virtually A-prestrong cardinals. This allows us to define a function $f\colon \mathsf{On} \to \mathsf{On}$ as $f(\alpha)$ being the least regular $\eta > \alpha$ such that α is not virtually (η, A)-prestrong.

We also define $g\colon \mathsf{On} \to \mathsf{On}$ as taking α to the least strong limit cardinal above α which is a closure point for f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A \cap H^V_{g(\alpha)}$. (ii) supplies us with $\alpha < \beta$ and a generic elementary embedding[6]

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}).$$

[6] Note that $V_{g(\alpha)} = H^V_{g(\alpha)}$ since $g(\alpha)$ is a strong limit cardinal.

Since $g(\alpha)$ is a closure point for f it holds that $f(\operatorname{crit}\pi) < g(\alpha)$, so fixing a regular $\theta \in (f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-prestrong, contradicting the definition of f. Hence On is virtually prewoodin. ■

2.4 Weak Vopěnka

We now move to a *weak* variant of gVP, introduced in a category-theoretic context in [**?**]. It starts with the following equivalent characterisation of gVP, which is the virtual analogue of the characterisation shown in [**?**].

Lemma 2.4.1 (GBC, Virtualised Adámek-Rosický)**.** *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that*[7]

(i) *gVP*

(ii) *There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that*
- *there is a generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

(iii) *There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that*
- *there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

Proof. Note that the only difference between *(ii)* and *(iii)* is that the homomorphism exists in V, making *(ii)* $\Rightarrow$ *(iii)* trivial.

(iii) $\Rightarrow$ *(i)*: Assume that gVP fails, meaning by Theorem 2.3.9 that we have a natural On-sequence $\vec{\mathcal{M}}_\alpha$ such that, in every generic extension, there's no homomorphism between any two disctinct $\mathcal{M}_\alpha$'s. Define an On-sequence $\langle \mathcal{N}_\kappa \mid \kappa \in \mathrm{Card}\rangle$ as

$$\mathcal{N}_\kappa := \coprod_{\xi \leq \kappa} \mathcal{M}_\xi = \{(x,\xi) \mid \xi \leq \kappa \wedge \xi \in \mathrm{Card} \wedge x \in \mathcal{M}_\xi\},^{8}$$

with a unary relation R^* given as $R^*(x,\xi)$ iff $\mathcal{M}_\xi \models R(x)$ and a binary relation $\sim^*$ given as $(x,\xi) \sim^* (x',\xi')$ iff $\xi = \xi'$. Whenever we have a homomorphism $f\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ we then get an induced homomorphism $\tilde{f}\colon \mathcal{M}_0 \to \mathcal{M}_\xi$, given as $\tilde{f}(x) := f(x,0)$, where $\xi \leq \kappa$ is given by preservation of $\sim^*$.

[7] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [**?**].

For any two cardinals $\kappa < \lambda$ we have a homomorphism $j_{\kappa\lambda} \colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ in V, given as $j_{\kappa\lambda}(x, \xi) := (x, \xi)$. This embedding must also be the *unique* such embedding in all generic extensions, as otherwise we get a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s. Furthermore, there can't be any homomorphism $\mathcal{N}_\lambda \to \mathcal{N}_\kappa$ as that would also imply the existence of a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s.

$(i) \Rightarrow (ii)$: Assume that we have an On-sequence $\vec{\mathcal{M}}_\alpha$ as in the theorem, with generic homomorphisms $j_{\alpha\beta} \colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ that are unique in all generic extensions for every $\alpha \leq \beta$, with no generic homomorphisms going the other way.

We first note that we can for every $\alpha \leq \beta$ choose the $j_{\alpha\beta}$ in a $\mathrm{Col}(\omega, \mathcal{M}_\alpha)$-extension, by a proof similar to the proof of Lemma 2.1.1 and using the uniqueness of $j_{\alpha\beta}$. Next, fix a proper class $C \subseteq \mathsf{On}$ such that $\alpha \in C$ implies that

$$\sup_{\xi \in C \cap \alpha} |\mathcal{M}_\xi|^V < |\mathcal{M}_\alpha|^V .$$

and note that this implies that $V[g] \models |\mathcal{M}_\xi| < |\mathcal{M}_\alpha|$ for every V-generic $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\xi)$. This means that for every $\alpha \in C$ we may choose some $\eta_\alpha \in \mathcal{M}_\alpha$ which is *not* in the range of any $j_{\xi\alpha}$ for $\xi < \alpha$. But now define first-order structures $\langle \mathcal{N}_\alpha \mid \alpha \in C \rangle$ as $\mathcal{N}_\alpha := (\mathcal{M}_\alpha, \eta_\alpha)$. Then, by our assumption on the $\mathcal{M}_\alpha$'s and construction of the $\mathcal{N}_\alpha$'s, there can be no generic homomorphism between any two distinct $\mathcal{N}_\alpha$, showing that gVP fails. ■

Note that the proof of the above lemma shows that we without loss of generality may assume that the generic homomorphism in (i) exists in V, which we record here:

Lemma 2.4.2 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ satisfying that*[9]

[9] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [?].

(i) there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;

(ii) there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.

■

The *weak* version of gVP is then simply “flipping the arrows around” in the above characterisation of gVP.

Definition 2.4.3 (GBC)**. Generic Weak Vopěnka's Principle** (gWVP) states that there does *not* exist an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ such that

- there is a generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\alpha \leq \beta$, which is unique in all generic extensions;
- there is *no* generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for any $\alpha < \beta$.

○

Denoting the corresponding non-generic principle by WVP [**?**] showed the following.

Theorem 2.4.4 (Wilson)**.** *WVP is equivalent to On being a Woodin cardinal.*

Given our 2.3.9 we may then suspect that in the virtual world these two are equivalent, which indeed turns out to be the case. We will be roughly following the argument in [**?**], but we have to diverge from it at several points in which they're using the fact that they're working with class-sized elementary embeddings.

Indeed, in that paper they establish a correspondence between elementary embeddings and certain homomorphisms, a correspondence we won't achieve here. Proving that the elementary embeddings we *do* get are non-trivial seems to furthermore require extra assumptions on our structures. Let's begin.

Define for every strong limit cardinal λ and Σ_1-formula φ the relations

$$R^\varphi := \{x \in V \mid (V, \in) \models \varphi[x]\}$$
$$R^\varphi_\lambda := \{x \subseteq H^V_\lambda \mid \exists y \in R^\varphi : y \cap H^V_\lambda = x\}$$

and given any class A define the structure

$$\mathscr{P}_{\lambda,A} := (H^V_{\lambda^+}, R^\varphi_\lambda, \{\lambda\}, A \cap H^V_\lambda)_{\varphi \in \Sigma_1}.$$

Say that a homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ is **trivial** if $h(x) \cap H^V_\eta = x \cap H^V_\eta$ for every $x \in H^V_{\lambda^+}$. Note that h can only be trivial if $\eta \leq \lambda$ since $h(\lambda) = \eta$.

Lemma 2.4.5 (GBC, G.-N.). *Let λ be a singular strong limit cardinal, η a strong limit cardinal and $A \subseteq V$ a class. If there exists a non-trivial generic homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ then there's a non-trivial generic elementary embedding*

$$\pi\colon (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that, letting $\nu := \min\{\lambda, \eta\}$, it holds that $H^V_\nu \subseteq \mathcal{M}$, $A \cap H^V_\nu = B \cap H^V_\nu$ and $\operatorname{crit} \pi < \nu$.

Proof. Assume that we have a non-trivial homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ in a forcing extension $V[g]$, define in $V[g]$ the set

$$\mathcal{M}^* := \{\langle b, f\rangle \mid b \in [H_\nu]^{<\omega} \wedge f \in H^V_{\lambda^+} \wedge f\colon H^V_\lambda \to H^V_\lambda\},$$

and define the relation $\in^*$ on $\mathcal{M}^*$ as

$$\langle b_0, f_0\rangle \in^* \langle b_1, f_1\rangle \quad \text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) \in f_1(y)\}).$$

Claim 2.4.5.1. $\in^*$ is wellfounded.

PROOF OF CLAIM. Assume not and let $\cdots \in^* \langle b_1, f_1\rangle \in^* \langle b_0, f_0\rangle$ be an $\in^*$-decreasing chain, which by definition means that, for every $n < \omega$,

$$b_{n+1}b_n \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}). \tag{1}$$

Define the relation $R(v_0, v_1, v_2)$ on H^V_λ as

$$R(X, f, g) \text{ iff } X = \{xy \in [H^V_\lambda]^{<\omega} \mid f(x) \in g(y)\}.$$

This relation is equal to R^φ_λ for some φ, so h moves $\langle X, f, g\rangle \in R^\varphi_\lambda$ to

$$\langle h(X), h(f), h(g)\rangle \in R^\varphi_\eta,$$

meaning that

$$h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}) = \{xy \in [H^V_\eta]^{<\omega} \mid f^*_{n+1}(x) \in f^*_n(y)\}$$

for some f^*_n such that $f^*_n \cap H^V_\eta = h(f_n)$ for all $n < \omega$. But now (1) implies that

$$b_{n+1}b_n \in \{xy \in [H^V_\eta]^{<\omega} \mid f^*_{n+1}(x) \in f^*_n(y)\}$$

and so $h(f_{n+1})(x) = f^*_{n+1}(x) \in f^*_n(y) = h(f_n)(y)$, giving an $\in$-decreasing sequence in $V[g]$ using transitivity of H^V_η, a contradiction!

Hence $\in^*$ is wellfounded. $\dashv$

$\mathcal{M}^*$ is a set, so $\in^*$ is trivially set-like. This means that we can take the transitive collapse $(\mathcal{M}, \in) \cong (\mathcal{M}^*, \in^*)$, and we note that $\mathcal{M} = \{[b, f] \mid \langle b, f\rangle \in \mathcal{M}^*\}$, where $[b, f] := \{[\bar{b}, \bar{f}] \mid \langle \bar{b}, \bar{f}\rangle \in^* \langle b, f\rangle\}$.

We now get a version of Łoś' Theorem whose proof is straight-forward, using that h preserves all Σ_1-relations and that $H^V_\lambda \models \mathsf{ZFC}^-$.

Claim 2.4.5.2. For every formula $\varphi(v_1, \ldots, v_n)$ and every $[b_1, f_1], \ldots, [b_n, f_n] \in \mathcal{M}$ the following are equivalent:

(i) $(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]]$;

(ii) $b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\})$.

Proof of claim. The proof is straightforward, using that h preserves Σ_1-relations. We prove this by induction on φ. If φ is $v_i \in v_j$ then we have that

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow &[b_i, f_i] \in [b_j, f_j] \\
\Leftrightarrow &\langle b_i, f_i \rangle \in^* \langle b_j, f_j \rangle \\
\Leftrightarrow &b_i b_j \in h(\{a_i a_j \in [H_\lambda^V]^{<\omega} \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\psi \wedge \chi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \models \psi[[b_1, f_1], \ldots, [b_n, f_n]] \wedge \chi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \ldots, f_n(a_n)]\}) \cap \\
&\qquad h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \chi[f_1(a_1), \ldots, f_n(a_n)]\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\neg\psi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \models \neg\psi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \not\models \psi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow &b_1 \cdots b_n \notin h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \ldots, f_n(a_n)]\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\}).
\end{aligned}$$

Finally, if φ is $\exists x\psi$ then

$$\begin{aligned}
&(\mathcal{M},\in)\models\varphi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&(\mathcal{M},\in)\models\exists x\psi[x,[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle\in\mathcal{M}^*\colon(\mathcal{M},\in)\models\psi[[b,f],[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle\in\mathcal{M}^*\colon bb_1\cdots b_n\in h(\{aa_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\psi[f(a),f_1(a_1),\dots,f_n(a_n)]\})\\
\Leftrightarrow&b_1\cdots b_n\in h(\{a_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\varphi[f_1(a_1),\dots,f_n(a_n)]\}).
\end{aligned}$$

This finishes the proof. ⊣

Next up, we have the following standard lemma, which implies that $H_\eta^V\subseteq\mathcal{M}$:

Claim 2.4.5.3. For all $y\in H_\eta^V$ we have $y=[\langle y\rangle,\mathrm{pr}]$, where $\mathrm{pr}(\langle x\rangle):=x$.

Proof of claim. We prove this by $\in$-induction on $y\in H_\eta^V$, so suppose that $y'=[\langle y'\rangle,\mathrm{pr}]$ for every $y'\in y$, which implies that $y\subseteq\mathcal{M}$ by transitivity of $\mathcal{M}$. We then get that, for every $[b,f]\in\mathcal{M}$,

$$\begin{aligned}
[b,f]\in[\langle y\rangle,\mathrm{pr}]&\Leftrightarrow b\langle y\rangle\in h(\{a\langle x\rangle\mid f(a)\in\mathrm{pr}(\langle x\rangle)\})\\
&\Leftrightarrow\exists y'\in y\colon b\langle y'\rangle\in h(\{a\langle x\rangle\mid f(a)=x\})\\
&\Leftrightarrow\exists y'\in y\colon[b,f]=[\langle y'\rangle,\mathrm{pr}]=y'\\
&\Leftrightarrow[b,f]\in y,
\end{aligned}$$

showing that $y=[\langle y\rangle,\mathrm{pr}]$. ⊣

Now define

$$B:=\{[b,f]\in\mathcal{M}\mid b\in h(\{x\in H_\lambda^V\mid f(x)\in A\})\}.$$

and, in $V[g]$, let $\pi\colon(H_\lambda^V,\in,A\cap H_\lambda^V)\to(\mathcal{M},\in,B)$ be given as $\pi(x):=[\langle\rangle,c_x]$.

Claim 2.4.5.4. π is elementary.

Proof of claim. For $x_1, \dots, x_n \in H^V_\lambda$ it holds that

$$\begin{aligned} & (\mathcal{M}, \in, B) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\ \Leftrightarrow\ & (\mathcal{M}, \in) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\ \Leftrightarrow\ & \langle\rangle \in h(\{\langle\rangle \mid \mathscr{P}_{\lambda,A} \models \varphi[x_1, \dots, x_n]\}) \\ \Leftrightarrow\ & (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \models \varphi[x_1, \dots, x_n] \end{aligned}$$

and we also get that, for every $x \in H^V_\lambda$,

$$x \in A \Leftrightarrow \langle\rangle \in h(\{a \in H^V_\lambda \mid x \in A\}) \Leftrightarrow \pi(x) \in B,$$

which shows elementarity. ⊣

We next need to show that $B \cap H^V_\nu = A \cap H^V_\nu$, so let $x \in H^V_\nu$. Note that $x = [\langle x\rangle, \mathrm{pr}]$ by Claim 2.4.5.3, which means that

$$x \in B \Leftrightarrow \langle x\rangle \in h(\{\langle y\rangle \in H^V_\lambda \mid y \in A\}) \Leftrightarrow x \in A.$$

The last thing we need to show is that $\operatorname{crit} \pi < \nu$. We start with an analogous result about h.

Claim 2.4.5.5. There exists some $b \in H^V_\nu$ such that $h(b) \neq b$.

Proof of claim. Assume the claim fails. We now have two cases.

$\boxed{\textbf{Case 1: } \lambda \geq \eta}$

By non-triviality of h there's an $x \in H^V_{\lambda^+}$ such that $h(x) \cap H^V_\eta \neq x \cap H^V_\eta$, which means that there exists an $a \in H^V_\eta$ such that $a \in h(x) \Leftrightarrow a \notin x$.

If $a \in x$ then $\{a\} = h(\{a\}) \subseteq h(x)$,[10] making $a \in h(x)$, $\lightning$, so assume instead that $a \in h(x)$. Since η is a strong limit cardinal we may fix a

[10]Note that as h preserves Σ_1 formulas it also preserves singletons and boolean operations.

cardinal $\theta < \eta$ such that $a \in H^V_\theta$ and $H^V_\theta \in H^V_\eta$. We then have that[11]

$$\{a\} \subseteq h(x) \cap H^V_\theta = h(x) \cap h(H^V_\theta) = h(x \cap H^V_\theta) = x \cap H^V_\theta,$$

so that $a \in x$, $\lightning$.

$\boxed{\textbf{Case 2: } \lambda < \eta}$

In this case we are assuming that $h \restriction H^V_\lambda = \mathrm{id}$, but $h(\lambda) = \eta > \lambda$. Since λ is singular we can fix some $\gamma < \lambda$ and a cofinal function $f\colon \gamma \to \lambda$. Define the relation

$$R = \{(\alpha, \beta, \bar\alpha, \bar\beta, g) \mid \ulcorner g \text{ is a cofinal function } g\colon \alpha \to \beta \urcorner \wedge g(\bar\alpha) = \bar\beta\}.$$

Then $R(\gamma, \lambda, \alpha, f(\alpha), f)$ holds by assumption for every $\alpha < \gamma$, so that R holds for some $(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*)$ such that

$$\begin{aligned}(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*) \cap H^V_\eta &= (h(\gamma), h(\lambda), h(\alpha), h(f(\alpha)), h(f)) \\ &= (\gamma, \eta, \alpha, f(\alpha), h(f)),\end{aligned}$$

using our assumption that h fixes every $b \in H^V_\lambda$. Since γ, α and $f(\alpha)$ are transitive and bounded in H^V_λ it holds that $h(\gamma) = \gamma^*$, $h(\alpha) = \alpha^*$ and $h(f(\alpha)) = f(\alpha)^*$. Also, since $\mathrm{dom}(f^*) = \gamma = \mathrm{dom}(f)$ we must in fact have that $f^* = h(f)$. But this means that $h(f)\colon \gamma \to \eta$ is cofinal and $\mathrm{ran}(h(f)) \subseteq \lambda$, a contradiction! $\dashv$

To use the above Claim 2.4.5.5 to conclude anything about π we'll make use of the following standard lemma.

Claim 2.4.5.6. For any $x \in H^V_\lambda$ it holds that $h(x) \cap H^V_\eta = \pi(x) \cap H^V_\eta$.

[11] Note that we're using $\lambda \geq \eta$ here to ensure that $H^V_\theta \in \mathrm{dom}\, h$.

PROOF OF CLAIM. For any $n < \omega$ and $\langle a_1, \ldots, a_n \rangle \in [H_\eta^V]^n$ we have that

$$\begin{aligned}
&\langle a_1, \ldots, a_n \rangle \in \pi(x) \\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle a_1, \ldots, a_n \rangle \in \pi(x) \\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle [\langle a_1 \rangle, \mathrm{pr}], \ldots, [\langle a_n \rangle, \mathrm{pr}] \rangle \in [\langle\rangle, c_x] \\
\Leftrightarrow\ &\langle a_1, \ldots, a_n \rangle \in h(\{\langle x_1, \ldots, x_n \rangle \mid \mathscr{P}_{\lambda,A} \models \langle x_1, \ldots, x_n \rangle \in x\}) \\
\Leftrightarrow\ &\langle a_1, \ldots, a_n \rangle \in h(x),
\end{aligned}$$

showing that $h(x) \cap H_\eta^V = \pi(x) \cap H_\eta^V$. ⊣

Now use Claim 2.4.5.5 to fix a $b \in H_\nu^V$ which is moved by h. Claim 2.4.5.6 then implies that

$$\pi(b) \cap H_\eta^V = h(b) \cap H_\eta^V = h(b) \neq b = b \cap H_\eta^V,$$

showing that $\pi(b) \neq b$ and hence $\operatorname{crit} \pi < \nu$. This finishes the proof of the lemma. ■

Theorem 2.4.6 (GBC, G.-N.)**.** *gVP is equivalent to gWVP.*

PROOF. ($\Rightarrow$): Assume gVP holds and gWVP fails, and let $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ be an On-sequence of first-order structures such that for every $\alpha \leq \beta$ there exists a generic homomorphism

$$j_{\beta\alpha} \colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$$

in some $V[g]$ which is unique in all generic extensions, with no generic homomorphisms going the other way. Here we may assume, as in the proof of Lemma 2.4.1, that $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\beta)$. We can then find a proper class $C \subseteq \mathsf{On}$ such that $|\mathcal{M}_\alpha|^V < |\mathcal{M}_\beta|^V$ for every $\alpha < \beta$ in C. By gVP there are then $\alpha < \beta$ in C and a generic homomorphism

$$\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta\,.$$

in some $V[h]$, where again we may assume that $h \subseteq \mathrm{Col}(\omega, \mathcal{M}_\alpha)$. But then $\pi \circ j_{\beta\alpha} = \mathrm{id}$ by uniqueness of $j_{\beta\beta} = \mathrm{id}$, which means that $j_{\beta\alpha}$ is injective in $V[g\times h]$ and hence also in $V[g]$. But then $|\mathcal{M}_\beta|^{V[g]} \leq |\mathcal{M}_\alpha|^{V[g]}$, which implies that $|\mathcal{M}_\beta|^V \leq |\mathcal{M}_\alpha|^V$ by the $|\mathcal{M}_\beta|^{+V}$-cc of $\mathrm{Col}(\omega, \mathcal{M}_\beta)$, contradicting the definition of C.

($\Leftarrow$): Assume that gVP fails, which by Theorem 2.3.9 is equivalent to On not being faintly prewoodin. This means that there exists a class A such that there are no faintly A-prestrong cardinals. We can therefore assign to any cardinal κ the least cardinal $f(\kappa) > \kappa$ such that κ is not faintly $(f(\kappa), A)$-prestrong.

Also define a function $g\colon \mathsf{On} \to \mathrm{Card}$ as taking an ordinal α to the least singular strong limit cardinal above α closed under f. Then we're assuming that there's no non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

with $H^V_{g(\alpha)} \subseteq \mathcal{M}$ and $B \cap H^V_{g(\alpha)} = A \cap H^V_{g(\alpha)}$. Assume towards a contradiction that for some α, β there is a non-trivial generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$. Lemma 2.4.5 then gives us a non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that $H^V_\nu \subseteq \mathcal{M}$ with $\nu := \min\{g(\alpha), g(\beta)\}$ and $A \cap H^V_\nu = B \cap H^V_\nu$, a contradiction! Therefore every generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$ is trivial. Since there is a unique trivial homomorphism when $\alpha \geq \beta$ and no trivial homomorphism when $\alpha < \beta$ since $g(\alpha)$ is sent to $g(\beta)$, the sequence of structures

$$\langle \mathscr{P}_{g(\alpha),A} \mid \alpha \in \mathsf{On}\rangle$$

is a counterexample to gWVP, which is what we wanted to show. ■

2.5 Berkeley

Berkeley cardinals was introduced by Woodin at University of California, Berkeley around 1992, and was introduced as a large cardinal candidate that would be inconsistent with ZF. They trivially imply the Kunen inconsistency and are therefore at least inconsistent with ZFC, but that's as far as it currently goes. In the virtual setting the virtually berkeley cardinals, like all the other virtual large cardinals, are simply downwards absolute to L.

It turns out that virtually berkeley cardinals are natural objects, as the main theorem of this section shows that these large cardinals are precisely what separates virtually prewoodins from the virtually woodins, as well as separating virtually vopěnka cardinals from mahlo cardinals.

Definition 2.5.1. Say that a cardinal δ is **virtually proto-berkeley** if for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi < \delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ then δ is **virtually berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C \subseteq \delta$ we say δ is **virtually club berkeley**. ∘

Virtually (proto-)berkeley cardinals turn out to be equivalent to their "boldface" versions, the proof of which is a straight-forward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [**?**].

Proposition 2.5.2 (Virtualised Cutolo)**.** *If δ is virtually proto-berkeley then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with* $\operatorname{crit}\pi < \delta$*. If δ is virtually berkeley then we can furthermore ensure that* $\operatorname{crit}\pi$ *is arbitrarily large below δ.*

Proof. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{\{\langle A, x\rangle \mid x \in \mathcal{M}\}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: a is the first element in the pairs belonging to the set of highest rank, and $\mathcal{M}$ is what remains if we remove the set with the highest rank. But this means that a generic elementary embedding $\pi\colon \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and a, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the wanted conclusion. ■

The following is a straight-forward virtualisation of the usual definition of the vopěnka filter (see e.g. [Kanamori, 2008]).

Definition 2.5.3 (GBC)**.** Define the **virtually vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$. ∘

Theorem 2.3.9 shows that this filter is proper iff gVP holds. The proof of Proposition 24.14 in [Kanamori, 2008] also shows that this filter is normal and is proper iff gVP holds. Note that uniformity of filters is non-trivial as we're working with proper classes[12]. Indeed, Theorem 2.5.7 shows that uniformity of this filter is equivalent to there being no virtually berkeley cardinals — the following lemma is the first implication.

Lemma 2.5.4 (GBC, N.)**.** *Assume gVP and that there are no virtually berkeley cardinals. Then the virtually vopěnka filter F on On contains every class club C.*

Proof. The crucial extra property we get by assuming that there aren't any virtually berkeleys is that F becomes uniform, i.e. contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin F$. Let M be a transitive set with $\delta \subseteq M$ and $\gamma < \delta$ a cardinal. As $(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function

[12] This boils down to the fact that the class club filter is not provably normal in GBC, see asd.

$f\colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\alpha+1, \delta+1)$$

and unary relations $R_\alpha := \langle M, \mathcal{N}_\alpha\rangle$. If $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit}\pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$, then $\pi(R_\alpha) = R_\beta$ implies that $\pi \restriction \mathcal{M}\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi \leq \delta$. We also get that $\operatorname{crit}\pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit}\pi}\colon \mathcal{N}_{\operatorname{crit}\pi} \to \mathcal{N}_{\pi(\operatorname{crit}\pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and hence $\operatorname{crit}\pi > \gamma$ as $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually berkeley, a contradiction. Thus $\operatorname{crit}\pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$.

From here the proof of Lemma 8.11 in [Jech, 2003] shows us the wanted.

■

Theorem 2.5.5 (GBC, N.). *If there are no virtually berkeley cardinals then* On *is virtually prewoodin iff* On *is virtually woodin.*

Proof. Assume On is virtually prewoodin, so gVP holds by Theorem 2.3.9 and we can let F be the virtually vopěnka filter. The assumption that there aren't any virtually berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails — we will follow the proof of Theorem 2.3.9.

Failure means that there is some class A and some class club C such that there are no virtually A-prestrong cardinals in C. Since there are no virtually berkeley cardinals, Lemma 2.5.4 imples that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as sending α to the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-prestrong if $\alpha \in C$, and set $f(\alpha) := \alpha$ if $\alpha \notin C$. Also define $g\colon \mathsf{On} \to \mathsf{On}$ as $g(\alpha)$ being the least strong limit cardinal in C above α which is a closure point for f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := \mathrm{Code}(\langle A \cap H^V_{g(\alpha)}, \mathcal{N}_\alpha\rangle)$ and apply gVP to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi \restriction (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)})\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}),$$

making $\operatorname{crit}\pi$ virtually $(g(\alpha), A)$-prestrong and thus $\operatorname{crit}\pi \notin C$. But as we also get the embedding $\pi \restriction \mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\operatorname{crit}\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \mathrm{Card} \mid (H^V_\kappa, \in, A \cap H^V_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-prestrong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it isn't virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, A \cap H^V_\theta) \to (M, \in, B)$$

with $\operatorname{crit}\pi = \kappa$, $H^V_\theta \subseteq M$, $M \subseteq V$, $A \cap H^V_\theta = B \cap H^V_\theta$ and $\pi(\kappa) < \theta$.

Now $\pi(\kappa)$ is inaccessible, and $(H^V_{\pi(\kappa)}, \in, A \cap H^V_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$ believes that κ is virtually $(A \cap H^V_{\pi(\kappa)})$-strong as in the proof of Theorem 2.2.8, meaning that $(H^V_\kappa, \in, A \cap H^V_\kappa)$ believes that there is a proper class of virtually $(A \cap H^V_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually woodin. ■

Theorem 2.5.6 (GBC, N.)**.** *If there exists a virtually berkeley cardinal δ then gVP holds and On is not mahlo.*

PROOF. If On was Mahlo then there would in particular exist an inaccessible cardinal $\kappa > \delta$, but then $H_\kappa^V \models \ulcorner$there exists a virtually berkeley cardinal$\urcorner$, contradicting the incompleteness theorem.

To show gVP we show that On is virtually prewoodin, which is equivalent by Theorem 2.3.9. Fix therefore a class A — we have to show that there exists a virtually A-prestrong cardinal. For every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta \colon (H_\theta^V, \in, A \cap H_\theta^V) \to (H_\theta^V, \in, A \cap H_\theta^V)$$

with $\operatorname{crit} \pi < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-prestrong, making On virtually prewoodin. ■

Theorem 2.5.7 (GBC, N.). *The following are equivalent:*

(i) gVP *implies that* On *is mahlo;*
(ii) On *is virtually prewoodin iff* On *is virtually woodin;*
(iii) There are no virtually berkeley cardinals.

PROOF. *(iii)* $\Rightarrow$ *(ii)* is Theorem 2.5.5, and the contraposed version of *(i)* $\Rightarrow$ *(iii)* is Theorem 2.5.6. For *(ii)* $\Rightarrow$ *(i)* note that gVP implies that On is virtually prewoodin by Theorem 2.3.9, which by *(ii)* means that it's virtually woodin and the usual proof shows that virtually woodins are mahlo[13], showing *(i)*. ■

This also immediately implies the following equiconsistency, as virtually berkeley cardinals have strictly larger consistency strength than virtually woodin cardinals.

Corollary 2.5.8 (N.). *The existence of an inaccessible virtually prewoodin cardinal is equiconsistent with the existence of an inaccessible virtually woodin cardinal.* ■

[13]See e.g. Exercise 26.10 in [Kanamori, 2008].

Question 1.7 in [?] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually prestrong cardinal, and Lemmata 2.5 and 2.8 in the same paper also shows that being ω-Erdős is equivalent to being virtually club berkeley and that the least such is also the least virtually berkeley.[14]

Furthermore, they also showed that a non-Σ_2-reflecting virtually prestrong cardinal is equivalent to a virtually prestrong cardinal which isn't virtually strong. We can therefore reformulate their question to the following equivalent question.

Question 2.5.9 (Wilson)**.** If there exists a virtually prestrong cardinal which is not virtually strong, is there then a virtually berkeley cardinal?

[?] showed that their question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem 2.2.8 we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 2.5.10. If there exists a cardinal κ which is virtually (θ, ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually berkeley cardinal?

Our results above at least gives a partially positive result:

Corollary 2.5.11 (N.)**.** *If there exists a virtually A-prestrong cardinal for every class A and there are no virtually strong cardinals, then there exists a virtually berkeley cardinal.*

PROOF. The assumption implies by definition that On is virtually prewoodin but not virtually woodin, so Theorem 2.5.7 supplies us with the desired. ■

[14] Note that this also shows that virtually club berkeley cardinals and virtually berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

The assumption that there is a virtually A-prestrong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 2.5.7 shows that this is necessary, which might lead one to think that the question could have a negative answer.

A | Appendix title

A.1 Section title

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna.

Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Bibliography

[Abramson et al., 1977] Abramson, F. G., Harrington, L. A., Kleinberg, E. M., and Zwicker, W. S. (1977). Flipping properties: a unifying thread in the theory of large cardinals. *Annals of Mathematical Logic*, 12:25–58.

[Dodd, 1982] Dodd, A. J. (1982). The core model. *London Mathematical Society Lecture Note Series*, 61.

[Donder et al., 1981] Donder, D., Jensen, R. B., and Koppelberg, B. J. (1981). Some applications of the core model. In *Set theory and model theory*, pages 55–97. Springer.

[Feng, 1990] Feng, Q. (1990). A hierarchy of Ramsey cardinals. *Annals of Pure and Applied Logic*, 49(2):257–277.

[Foreman, 1983] Foreman, M. (1983). Games played on boolean algebras. *The Journal of Symbolic Logic*, 48(3):714–723.

[Galvin et al., 1978] Galvin, F., Jech, T. J., and Magidor, M. (1978). An ideal game. *The Journal of Symbolic Logic*, 43:284–292.

[Gitman, 2011] Gitman, V. (2011). Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540.

[Gitman and Schindler, 2015] Gitman, V. and Schindler, R. (2015). Virtual large cardinals. *The Proceedings of the Logic Colloquium.*

[Gitman and Welch, 2011] Gitman, V. and Welch, P. (2011). Ramsey-like cardinals II. *The Journal of Symbolic Logic*, 76(2):541–560.

[Holy and Schlicht, 2018] Holy, P. and Schlicht, P. (2018). A hierarchy of Ramsey-like cardinals. *Fundamenta Mathematicae*, 242:49–74.

[Jech, 2003] Jech, T. (2003). *Set Theory.* Springer-Verlag, millenium edition.

[Jensen and Steel, 2013] Jensen, R. and Steel, J. (2013). K without the measurable. *The Journal of Symbolic Logic*, 78(3):708–734.

[Kanamori, 2008] Kanamori, A. (2008). *The Higher Infinite: Large cardinals in set theory from their beginnings.* Springer Science & Business Media.

[Kanamori and Magidor, 1978] Kanamori, A. and Magidor, M. (1978). The evolution of large cardinal axioms in set theory. In *Higher set theory*, pages 99–275. Springer.

[Kellner and Shelah, 2011] Kellner, J. and Shelah, S. (2011). More on the pressing down game. *Archive for Mathematical Logic*, 50(3-4):477–501.

[Mitchell, 1979] Mitchell, W. J. (1979). Ramsey cardinals and constructibility. *The Journal of Symbolic Logic*, 44(2):260–266.

[Schindler, 1997] Schindler, R. (1997). Weak covering at large cardinals. *Mathematical Logic Quarterly*, 43:22–28.

[Schindler, 2006] Schindler, R. (2006). Iterates of the core model. *The Journal of Symbolic Logic*, 71(1):241–251.

[Schindler, 2000] Schindler, R.-D. (2000). Proper forcing and remarkable cardinals. *The Bulletin of Symbolic Logic*, 6(2):176–184.

[Sharpe and Welch, 2011] Sharpe, I. and Welch, P. D. (2011). Greatly Erds cardinals with some generalizations to the Chang and Ramsey properties. *Annals of Pure and Applied Logic*, 162:863–902.

[Zeman, 2002] Zeman, M. (2002). Inner models and large cardinals. In *Series in Logic and its Applications*, volume 5. de Gruyter, Berlin, New York.