1 Filters & Games

Moving away from the pure theory of the virtual large cardinals from Chapter ??, we now move to connections between these large cardinals and common set-theoretic objects of study. In this chapter those objects are filters and games, with the next chapter dealing with connections to ideals. This chapter covers the content of the paper [?], which started out as a further analysis of the results in [?] and somewhat surprisingly we ended up in the realm of virtual large cardinals.

We will in this section be dealing with many properties of $\mathcal{M}$-measures[1], so we start with a couple of definitions.

Definition 1.1. Let κ be a cardinal, $\mathcal{M}$ a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is...

- $\mathcal{M}$**-normal** if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \triangle\vec{X} \in \mu$;
- **genuine** if $|\triangle\vec{X}| = \kappa$ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- **normal** if $\triangle\vec{X}$ is stationary in κ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- 0-**good**, or simply **good**, if it has a well-founded ultrapower when applied to $\mathcal{M}$;
- α-**good** for $\alpha > 0$ if it is weakly amenable and has α-many well-founded iterated ultrapowers when applied to $\mathcal{M}$. ○

We emphasise that the main difference between $\mathcal{M}$-normality and normality (and genuineness) is that the former is *local* and the latter are *global*.

Question: Are normal measures always $\mathcal{M}$-normal?

We note a few basic relations between these properties.

Proposition 1.2. *Let κ be a cardinal, $\mathcal{M}$ a weak κ-model. Then*

(i) Every genuine $\mathcal{M}$-measure on κ is countably complete;

(ii) Every countably complete weakly amenable $\mathcal{M}$-measure on κ is α-good for all ordinals α.

[1] See Section ?? for the definitions of weak κ-models $\mathcal{M}$ and their associated $\mathcal{M}$-measures.

Proof. (i): Let μ be a genuine $\mathcal{M}$-measure on κ. To show countable completeness, let $\vec{X} \in {}^{\omega}\mu$ be an ω-sequence of measure one sets and define a κ-sequence $\vec{Y} \in {}^{\kappa}\mu$ as $Y_n := X_n$ for $n < \omega$ and $Y_\alpha := \kappa$ for $\alpha \in [\omega, \kappa)$. Then $\left|\triangle\vec{Y}\right| = \kappa$ as μ is genuine, so letting $\alpha \in \triangle\vec{Y} - \omega$ we get that $\alpha \in \bigcap\vec{X}$, making μ countably complete.

(ii): Now let μ be a countably complete weakly amenable $\mathcal{M}$-measure on κ. Firstly note that countable completeness implies that the ultrapower $\mathrm{Ult}(\mathcal{M}, \mu)$ is well-founded. Next, weak amenability implies that $X := \{\alpha < \kappa \mid X_\alpha \in \mu\} \in \mathcal{M}$ for every $\vec{X} \in {}^{\kappa}\mu \cap \mathcal{M}$ since we can rewrite the set as

$$X = \{\alpha < \kappa \mid X_\alpha \in \{X_\alpha \mid \alpha < \kappa\} \cap \mu\}$$

and weak amenability ensures that $\{X_\alpha \mid \alpha < \kappa\} \cap \mu \in \mathcal{M}$. From this we can form iterated ultrapowers as in Chapter 19 of [?], which will all be well-founded by countable completeness of the measure. ■

In [?] they provide the following characterisation of the normal measures.

Lemma 1.3 (Holy-Schlicht). *Let $\mathcal{M}$ be a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is normal iff $\triangle\vec{X}$ is stationary for some enumeration $\vec{X}$ of μ.*

Proof. $(\Rightarrow)$ is trivial since $\left|\vec{X}\right| = |\mu| \leq |\mathcal{M}| = \kappa$, so assume that $\vec{X}$ is an enumeration of μ such that $\triangle\vec{X}$ is stationary. Let $\vec{Y} \in {}^{\kappa}\mu$ be a κ-sequence and define $g\colon \kappa \to \kappa$ such that $Y_\alpha = X_{g(\alpha)}$ for $\alpha < \kappa$. Letting $C_g \subseteq \kappa$ be the club of closure points of g we get that $\triangle\vec{X} \cap C_g \subseteq \triangle\vec{Y} \cap C_g$, making $\triangle\vec{Y}$ stationary. ■

We next move on to the games. All of our games will be two-player games with perfect information; see e.g. [?, Chapter 27] for an introduction to set-theoretic game theory. We will also, mostly for convenience, use the following *game equivalence* notion.

Definition 1.4. Two games $\mathcal{G}_0$ and $\mathcal{G}_1$ are said to be **game equivalent**, or simply **equivalent**, if player I has a winning strategy in $\mathcal{G}_0$ iff they have one in $\mathcal{G}_1$, and player II has a winning strategy in $\mathcal{G}_0$ iff they have one in $\mathcal{G}_1$. We will also denote such an equivalence as $\mathcal{G}_0 \sim \mathcal{G}_1$. $\circ$

The following is a game which was introduced in [?] and led to their notion of *α-Ramsey cardinals*.

Definition 1.5 (Holy-Schlicht). For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, a regular cardinal $\gamma \leq \kappa$ and a regular cardinal $\theta > \kappa$ define the game $wfG^\theta_\gamma(\kappa)$ of length γ as follows.

$$\begin{array}{lcccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \text{II} & & \mu_0 & & \mu_1 & & \mu_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a κ-model and μ_α is an $\mathcal{M}_\alpha$-measure, the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing and $\langle \mathcal{M}_\xi \mid \xi < \alpha \rangle, \langle \mu_\xi \mid \xi < \alpha \rangle \in \mathcal{M}_\alpha$ for every $\alpha < \gamma$. Letting $\mu := \bigcup_{\alpha<\gamma} \mu_\alpha$ and $\mathcal{M} := \bigcup_{\alpha<\gamma} \mathcal{M}_\alpha$, player II wins iff μ is an $\mathcal{M}$-normal good $\mathcal{M}$-measure. $\circ$

We will also be using the following fact from [?, Lemma 3.3], that the games $wfG^\theta_\gamma(\kappa)$ do not depend upon the values of θ.

Lemma 1.6 (Holy-Schlicht). *For a fixed κ and γ, $wfG^{\theta_0}_\gamma(\kappa)$ and $wfG^{\theta_1}_\gamma(\kappa)$ are equivalent for any regular $\theta_0, \theta_1 > \kappa$.* $\blacksquare$

See the proof of Proposition ?? below for an idea of the proof strategy of this lemma.

We will be working with the following variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I and more of player II. It will turn out that this change of game is innocuous, as Proposition ?? will show that they are (game) equivalent.

Definition 1.7 (Holy-N.-Schlicht). Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following **filter game** $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ with $(\gamma+1)$-many rounds.

$$\begin{array}{lcccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & & \mathcal{M}_\gamma & \\ \text{II} & & \mu_0 & & \mu_1 & & \cdots & & \mu_\gamma \end{array}$$