1 | The second chapter

1.1 Getting started

We will denote the class of ordinals by On. For X, Y sets we denote by ${}^X Y$ the set of all functions from X to Y. For an infinite cardinal κ, we let H_κ be the set of sets X such that the cardinality of the transitive closure of X is $<\kappa$. ZF^- will denote ZF with the Collection scheme but without the Power Set axiom, following the results of [**?**]. The symbol $\lightning$ will denote a contradiction and $\mathscr{P}(X)$ denotes the power set of X.

A key folklore lemma which we will frequently need when dealing with elementary embeddings existing in generic extensions is the following.

Lemma 1.1.1 (Countable Embedding Absoluteness)**.** *Let $\mathcal{M}, \mathcal{N}$ be transitive and assume that $\mathcal{M}$ is countable. Let $\pi \colon \mathcal{M} \to \mathcal{N}$ be an elementary embedding, $\mathcal{P}$ a transitive class with $\mathcal{M}, \mathcal{N} \in \mathcal{P}$ and*

$$\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC} + \ulcorner \mathcal{M} \text{ is countable} \urcorner,$$

and fix any finite $X \subseteq \mathcal{M}$. Then $\mathcal{P}$ has an elementary embedding $\pi^* \colon \mathcal{M} \to \mathcal{N}$ which agrees with π on X and $\operatorname{crit} \pi = \operatorname{crit} \pi^*$.

Proof. Let $\{a_i \mid i < \omega\} \in \mathcal{P}$ be an enumeration of $\mathcal{M}$ and set $\mathcal{M} \restriction n := \{a_i \mid i < n\}$. Then, in $\mathcal{P}$, build the tree $\mathcal{T}$ of all partial isomorphisms between $\mathcal{M} \restriction n$ and $\mathcal{N}$ for $n < \omega$, ordered by extension. Then $\mathcal{T}$ is illfounded in V by assumption, so it's also illfounded in $\mathcal{P}$ since $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC}$. The branch gives the embedding π^*, and we can ensure that it agrees with π on the critical point and finitely many values by adding these conditions to $\mathcal{T}$. ■

1.2 Strong and supercompact

We start out by defining virtual versions of a variety of large cardinal notions used in this section. We start out with measurables, strongs and supercompacts.

Definition 1.2.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa<\theta$ is...

- **faintly θ-measurable** if, in a forcing extension, there is a transitive class $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta^V\to\mathcal{N}$ with $\operatorname{crit}\pi=\kappa$;
- **faintly θ-strong** if it's faintly θ-measurable, $H_\theta^V\subseteq\mathcal{N}$ and $\pi(\kappa)>\theta$;
- **faintly θ-supercompact** if it's faintly θ-measurable, ${}^{<\theta}\mathcal{N}\subseteq\mathcal{N}$ and $\pi(\kappa)>\theta$.

We further replace "faintly" by **virtually** when $\mathcal{N}\subseteq V$, we attach a **"pre"** if we don't want to assume $\pi(\kappa)>\theta$, and when we don't mention θ we mean that it holds for all regular $\theta>\kappa$. For instance, a faintly prestrong cardinal is a cardinal κ such that for all regular $\theta>\kappa$, κ is faintly θ-measurable with $H_\theta^V\subseteq\mathcal{N}$. ○

We note that even small cardinals can be faintly measurable: we may for instance have a precipitous ideal on ω_1. The "virtually" adverb implies that the cardinals are in fact large cardinals in the usual sense, as the following shows.

Recall from [?] that a cardinal κ is **1-iterable** if to every $A\subseteq\kappa$ there's a transitive $\mathcal{M}\models\mathsf{ZFC}^-$ with $\kappa,A\in\mathcal{M}$ and a weakly amenable $\mathcal{M}$-ultrafilter μ on κ with a wellfounded ultrapower.[1] 1-iterable cardinals are weakly compact limits of weakly compact cardinals.

Proposition 1.2.2 (Virtualised folklore)**.** *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is 1-iterable.*

Proof. (Sketch) Let κ be virtually θ-measurable, witnessed by a forcing $\mathbb{P}$, a transitive $\mathcal{N}\subseteq V$ and an elementary $\pi\colon H_\theta^V\to\mathcal{N}$ with $\pi\in V^{\mathbb{P}}$. If

[1] Also recall that μ is **weakly amenable** if $\mu\cap X\in\mathcal{M}$ for every $X\in\mathcal{M}$ of $\mathcal{M}$-cardinality $\le\kappa$.

κ isn't a strong limit then we have a surjection $\pi(f)\colon \mathscr{P}(\alpha) \to \pi(\kappa)$ with $\operatorname{ran}\pi(f) = \operatorname{ran} f \subseteq \kappa$ for some $\alpha < \kappa$, $\lightning$. Note that we used $\mathcal{N} \subseteq V$ to ensure that $\mathscr{P}(\alpha)^V = \mathscr{P}(\alpha)^{\mathcal{N}}$. The same argument shows that κ is regular. By restricting the generic embedding and using that $\mathscr{P}(\kappa)^V = \mathscr{P}(\kappa)^N$ as $\mathcal{N} \subseteq V$ and $\mathscr{P}(\kappa)^V \subseteq \mathcal{N}$, we get that κ is 1-iterable. ■

Along with the above definition of faintly supercompactness we can also virtualise Magidor's characterisation of supercompact cardinals, which was also one of the original characterisations of the remarkable cardinals in [**?**].

Definition 1.2.3. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-supercompact ala Magidor** if there are $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ such that $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. ○

In the virtual world these two versions of supercompacts remain equivalent, but they also turn out to be equivalent to the virtually strongs:

Theorem 1.2.4 (G.-Schindler)**.** *For an uncountable cardinal κ, the following are equivalent.*[2]

(i) *κ is virtually strong;*

(ii) *κ is virtually supercompact;*

(iii) *κ is virtually supercompact ala Magidor.*

Proof. $(ii) \Rightarrow (i)$ is simply by definition.

$(i) \Rightarrow (iii)$: Fix $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi\colon H^V_{(2^{<\theta})^+} \to \mathcal{M}$ with[3] $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H^V_{(2^{<\theta})^+} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. Since $H^V_\theta, H^{\mathcal{M}}_{\pi(\theta)} \in \mathcal{M}$, Countable Embedding Absoluteness 1.1.1 implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H^V_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ with $\operatorname{crit}\pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H^V_\theta = H^{\mathcal{M}}_\theta$ as $\mathcal{M} \subseteq V$ and $H^V_\theta \subseteq \mathcal{M}$, elementarity of π now implies that $H^V_{(2^{<\theta})^+}$ has ordinals $\bar\kappa < \bar\theta < \kappa$

[2] A cardinal satisfying any/all of these conditions is usually called **remarkable**.

[3] The domain of π is $H^V_{(2^{<\theta})^+}$ to ensure that $H^V_\theta \in \operatorname{dom}\pi$.

and a generic elementary $\sigma\colon H^V_{\bar\theta}\to H^V_\theta$ with $\operatorname{crit}\sigma=\bar\kappa$ and $\sigma(\bar\kappa)=\kappa$. This shows (iii).

$(iii)\Rightarrow(ii)$: Fix $\theta>\kappa$ and $\delta:=(2^{<\theta})^+$. By (iii) there exist ordinals $\bar\kappa<\bar\delta<\kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\delta}\to H^V_\delta$ with $\operatorname{crit}\pi=\bar\kappa$ and $\pi(\bar\kappa)=\kappa$. We will argue that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, so that by elementarity κ is virtually θ-supercompact in H^V_δ and hence also in V by the choice of δ. Consider the restriction

$$\sigma:=\pi\restriction H^V_{\bar\theta}: H^V_{\bar\theta}\to H^V_\theta.$$

Note that H^V_θ is closed under $<\bar\theta$-sequences (and more) in V. Now define

$$X:=\bar\theta{+}1\cup\{x\in H^V_\theta\mid\exists y\in H^V_{\bar\theta}\exists p\in\operatorname{Col}(\omega,H^V_{\bar\theta}): p\Vdash\dot\sigma(\check y)=\check x\}\in V.$$

Note that $|X|=\left|H^V_{\bar\theta}\right|=2^{<\bar\theta}$ and that $\operatorname{ran}\sigma\subseteq X$. Now let $\overline{\mathcal{M}}\prec H^V_\theta$ be such that $X\subseteq\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{\mathcal{M}}$ of size $(2^{<\bar\theta})^{<\bar\theta}=2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{\mathcal{M}}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}|=2^{<\bar\theta}<\bar\delta$, making $\mathcal{M}\in H^V_{\bar\delta}$.

Countable Embedding Absoluteness 1.1.1 then implies that $H^V_{\bar\delta}$ has a generic elementary embedding $\sigma^*\colon H^V_{\bar\theta}\to\mathcal{M}$ with $\operatorname{crit}\sigma^*=\bar\kappa$, showing that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, which is what we wanted to show. ■

Remark 1.2.5. The above proof shows that if κ is virtually $(2^{<\theta})^+$-strong then it's virtually θ-supercompact, and if it's virtually $(2^{<\theta})^+$-supercompact ala Magidor then it's virtually θ-supercompact. It's open whether they are equivalent level-by-level (see Question **??**).

A key difference between the normal large cardinals and the virtual kind is that we don't have a virtual version of the Kunen inconsistency: it's perfectly valid to have an elementary embedding $H^V_\theta\to H^V_\theta$ with θ much larger than the critical point. This becomes important when dealing with the "pre"-versions of the large cardinals. We start with a virtualisation of the α-superstrong cardinals.

Definition 1.2.6. Let θ be a regular uncountable cardinal and and α an ordinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, α)**-superstrong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$[4]. We replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we say that κ is **faintly** α**-superstrong** if it's faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong. ○

Proposition 1.2.7 (N.)**.** *If κ is faintly superstrong then H_κ has a proper class of virtually strong cardinals.*

PROOF. Fix a regular $\theta > \kappa$ and a generic embedding $\pi \colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit} \pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) < \theta$. Then $\pi(\kappa)$ is a V-cardinal, so that $H_{\pi(\kappa)}^V$ thinks that κ is virtually strong. This implies that H_κ^V thinks there is a proper class of virtually strong cardinals, using that $H_\kappa^V \prec H_{\pi(\kappa)}^V$. ■

The following theorem then shows that the only thing stopping prestrongness from being equivalent to strongness is the existence of "Kunen inconsistencies".

Theorem 1.2.8 (N.)**.** *Let θ be an uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-prestrong iff one of the following holds.*

(i) *κ is virtually θ-strong; or*

(ii) *κ is virtually (θ, ω)-superstrong.*

PROOF. ($\Leftarrow$) is trivial, so we show ($\Rightarrow$). Let κ be virtually θ-prestrong. Assume *(i)* fails, meaning that there's a generic extension $V^{\mathbb{P}}$ and an elementary embedding $\pi \in V^{\mathbb{P}}$ such that $\pi \colon H_\theta^V \to \mathcal{N}$ for some transitive $\mathcal{N}$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit} \pi = \kappa$ and $\pi(\kappa) \leq \theta$. Assume $\pi^n(\kappa)$ is defined for all $n < \omega$ and define $\lambda := \sup_{n<\omega} \pi^n(\kappa)$. If $\lambda \leq \theta$ then κ is virtually (θ, ω)-superstrong by definition, so assume that there's some least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$.

This means that κ is virtually ν-strong for every regular $\nu \in (\kappa, \pi^n(\kappa))$, which is a Δ_0-statement in $\{H_{\nu^+}^V\}$ and hence downwards absolute to $H_{\pi^n(\kappa)}^V$.

[4]Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.

This means that κ is virtually strong in $H^V_{\pi^n(\kappa)}$ and also that $\pi^n(\kappa)$ is virtually strong in $H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$ by elementarity, and so in particular virtually θ-strong in $\mathcal{N}$. This means that there's some generic elementary embedding

$$\sigma\colon H^{\mathcal{N}}_\theta \to \mathcal{M}$$

with $H^{\mathcal{N}}_\theta \subseteq \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{N}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. We can now restrict σ to its critical point $\pi^n(\kappa)$ to get that

$$H^V_{\pi^n(\kappa)} = H^{\mathcal{N}}_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))},$$

using that $H^V_\theta = H^{\mathcal{N}}_\theta$ holds as π is a virtual embedding. Since κ is virtually strong in $H^V_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$. In particular, κ is virtually θ-strong in $\mathcal{M}$, and as $H^{\mathcal{M}}_\theta = H^{\mathcal{N}}_\theta = H^V_\theta$, this means that κ is virtually θ-strong in V, contradicting (i). ■

We then get the following consistency result.

Corollary 1.2.9 (N.)**.** *For any uncountable regular θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

PROOF. The above Proposition 1.2.7 and Theorem 1.2.8 show that virtually θ-prestrongs are equiconsistent with virtually θ-strongs. Now note that Countable Embedding Absoluteness 1.1.1 and condensation in L imply that every faintly θ-measurable cardinal is virtually θ-prestrong in L. ■

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H^V_\theta \to H^V_\theta$ with $\operatorname{crit}\pi = \kappa$. We then have the following corollary.

Corollary 1.2.10 (N.)**.** *The following are equivalent:*

(i) For every uncountable cardinal θ, every virtually θ-prestrong cardinal is virtually θ-strong;

(ii) There are no virtually rank-into-rank cardinals.

PROOF. ($\Leftarrow$): Note first that being virtually ω-superstrong is equivalent to being virtually rank-into-rank. Indeed, every virtually rank-into-rank cardinal is virtually ω-superstrong by definition, and if κ is virtually ω-superstrong and $\lambda := \sup_{n<\omega} \pi^n(\kappa)$ then $\pi \restriction H_\lambda^V : H_\lambda^V \to H_\lambda^V$ witnesses that κ is virtually λ-rank-into-rank. The above Theorem 1.2.8 then implies ($\Leftarrow$).

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let (κ, θ) be the lexigraphically least pair such that κ is virtually θ-rank-into-rank, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong then it would be Σ_2-reflecting, so that the statement that there exists a virtually rank-into-rank cardinal would reflect down to H_κ^V, contradicting the minimality of κ. ■

As a final result of this section, we note that the "virtually" adverb *does* yield cardinals different from the faintly ones. This is trivial in general as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

For a slightly more fine-grained distinction let's prepend a **power-** adjective whenever the domain and codomain of the generic elementary embedding have the same subsets of κ. Note that the proof of Lemma 1.2.2 shows that faintly power-measurables are also 1-iterable.

Our separation result is then the following.

Theorem 1.2.11 (G.). *For $\Phi \in \{measurable, prestrong, strong\}$, if κ is virtually Φ then there exist forcing extensions $V[g]$ and $V[h]$ such that*

(i) In $V[g]$, κ is inaccessible and faintly Φ but not faintly power-Φ; and
(ii) In $V[h]$, κ is faintly power-Φ but not virtually Φ.

PROOF. We start with (i). Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\lambda < \kappa$, and let $g \subseteq \mathbb{P}_\kappa$ be V-generic. Note that κ remains inaccessible in $V[g]$. Fix a regular $\theta > \kappa$ and let $\mathbb{Q}_\theta$ be a forcing witnessing that κ is virtually θ-measurable.

Include this argument?

Define and give reference

Since κ is *virtually* measurable we may without loss of generality assume that $\mathbb{Q}_\theta = \mathrm{Col}(\omega, \theta)$ by applying Countable Embedding Absoluteness 1.1.1. Fixing a $V[g]$-generic $h \subseteq \mathbb{Q}_\theta$ we get a transitive $\mathcal{N} \subseteq V$ and in $V[h]$ an elementary embedding

$$\pi\colon H_\theta^V \to \mathcal{N}$$

with $\operatorname{crit}\pi = \kappa$. Let's now work in $V[g][h] = V[h][g] = V[g \times h]$, in which we still have access to π. The lifting criterion is trivial for $\mathbb{P}_\kappa$, so we get an $\mathcal{N}$-generic $\tilde{g} \subseteq \pi(\mathbb{P}_\kappa)$ and an elementary

$$\pi^+\colon H_\theta^{V[g]} \to \mathcal{N}[\tilde{g}]$$

with $\pi \subseteq \pi^+$. Note here that without loss of generality $\pi(\kappa)$ is countable as otherwise we replace $\mathcal{N}$ by a countable hull, so we can indeed construct such a $\tilde{g}$. By elementarity of π it holds that

$$\pi(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \prod_{\lambda \in [\kappa, \pi(\kappa))} \mathrm{Add}(\lambda, 1), \tag{1}$$

so that $\mathcal{N}[\tilde{g}] \not\subseteq V$ as it in particular contains a new subset of κ. If $\Phi =$ measurable then we're done at this point. For $\Phi =$ prestrong we simply note that $g \in \mathcal{N}[\tilde{g}]$ by (1) so that $H_\theta^{V[g]} \subseteq \mathcal{N}[\tilde{g}]$ as well, and since π^+ lifts π it holds that $\pi^+(\kappa) = \pi(\kappa) > \theta$ in the $\Phi =$ strong case.

As for (ii), we simply change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$, which means that $\pi(\mathbb{P}_\kappa)$ doesn't add any subsets of κ and κ thus remains faintly power-Φ. By choosing $\theta > \kappa^+$ it *does* add a subset to κ^+ however, showing that κ is not virtually Φ. ■

In contrast to the above separation result, Theorem 1.3.9 will show that the faintly-virtually distinction vanishes when we're dealing with woodins.

1.3 Woodin and Vopěnka

In this section we will analyse the virtualisations of the woodin and vopěnka cardinals, which can be seen as "boldface" variants of strongs and supercompacts.

Definition 1.3.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, A)**-strong** for a set $A \subseteq H_\theta^V$ if there exists a generic elementary embedding

$$\pi \colon (H_\theta^V, \in, A) \to (\mathcal{M}, \in, B)$$

such that $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta^V \subseteq \mathcal{M}$ and $B \cap H_\theta^V = A$. We say that κ is **faintly** (θ, A)**-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$ and say that κ is **faintly** (θ, A)**-extendible** if $\mathcal{M} = H_\mu^V$ for some V-cardinal μ. We will leave out θ if it holds for all regular $\theta > \kappa$. ◦

Definition 1.3.2. A cardinal δ is **faintly woodin** if given any $A \subseteq H_\delta^V$ there exists a faintly $({<}\delta, A)$-strong cardinal $\kappa < \delta$. ◦

As with the previous definitions, for both of the above two definitions we substitute "faintly" for **virtually** when $\mathcal{M} \subseteq V$, and substitute "strong", "supercompact" and "woodin" for **prestrong**, **presupercompact** and **prewoodin** when we don't require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with the real woodin cardinals, virtually woodin cardinals are mahlo. This contrasts the virtually prewoodins since [**?**], together with Theorem 1.3.9 below, shows can be singular.

Proposition 1.3.3 (Virtualised folklore)**.** *Virtually woodin cardinals are mahlo.*

Proof. Let δ be virtually woodin. Note that δ is a limit of weakly compact cardinals by Proposition 1.2.2, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f \colon \alpha \to \delta$ with $f(0) > \alpha$

and $\alpha < \delta$, and note that f cannot have any closure points. Fix a virtually $({<}\delta, f)$-strong cardinal $\kappa < \delta$; we claim that κ is a closure point for f, which will yield our desired contradiction.

Let $\gamma < \kappa$ and choose a regular $\theta \in (f(\gamma), \delta)$. We then have a generic embedding $\pi\colon (H_\theta^V, \in, f \cap H_\theta^V) \to (\mathcal{N}, \in, f^+)$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and f^+ is a function such that $f^+ \cap H_\theta^V = f \cap H_\theta^V$. But then $f^+(\gamma) = f(\gamma) < \pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma) < \kappa$, making κ a closure point for f, $\lightning$. This shows that δ is inaccessible.

As for mahloness, let $C \subseteq \delta$ be a club and $\kappa < \delta$ a virtually $({<}\delta, C)$-strong cardinal. Let $\theta \in (\min C, \delta)$ and let $\pi\colon H_\theta^V \to \mathcal{N}$ be the associated generic elementary embedding. Then for every $\gamma < \kappa$ there exists an element of C below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an element of C. As κ is regular, this shows that δ is mahlo.
■

The well-known equivalence of the "function definition" and "A-strong" definition of woodin cardinals holds if we restrict ourselves to virtually woodins, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Proposition 1.3.4 (D.-G.-N.). *For an uncountable cardinal δ, the following are equivalent.*

(i) *δ is virtually woodin.*
(ii) *for every $A \subseteq H_\delta^V$ there exists a virtually $({<}\delta, A)$-supercompact $\kappa < \delta$.*
(iii) *for every $A \subseteq H_\delta^V$ there exists a virtually $({<}\delta, A)$-extendible $\kappa < \delta$.*
(iv) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*
(v) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, ${}^{<\pi(f)(\kappa)}\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*

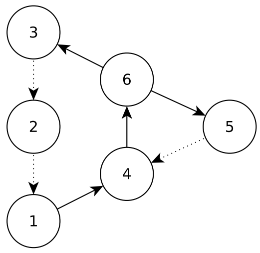

Figure 1.1: Proof strategy of Proposition 1.3.4, dotted lines are trivial implications.

(vi) *for every function* $f\colon \delta \to \delta$ *there are regular cardinals* $\bar\theta < \kappa < \theta < \delta$, *where* κ *is a closure point for* f, *and a generic elementary embedding* $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ *with* $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$ *and* $f \restriction \kappa \in \operatorname{ran}\pi$.

Proof. Firstly note that *(iii)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i)* and *(v)* $\Rightarrow$ *(iv)* are simply by definition.

$\boxed{(i) \Rightarrow (iv)}$ Assume δ is virtually woodin, and fix a function $f\colon \delta \to \delta$. Let $\kappa < \delta$ be virtually $({<}\delta, f)$-strong and let $\theta := \sup_{\alpha \le \kappa} f(\alpha){+}1$. Then there's a generic elementary embedding $\pi\colon (H^V_\theta, \in, f \cap H^V_\theta) \to (\mathcal{M}, \in, f^+)$ where $f^+ \restriction \kappa = f \restriction \kappa$, $\mathcal{M} \subseteq V$ and $\pi(\kappa) > \theta$. We firstly want to show that κ is a closure point for f, so let $\alpha < \kappa$. Then

$$f(\alpha) = f^+(\alpha) = \pi(f)(\alpha) = \pi(f)(\pi(\alpha)) = \pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha < \kappa$. Now, if κ wasn't a closure point for f then, letting $\alpha < \kappa$ be the least such that $f(\alpha) \geq \kappa$,

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument wouldn't work if we had only assumed δ to be virtually prewoodin. Lastly, θ-strongness implies that $H^V_\theta \subseteq \mathcal{M}$, and $\mathcal{M} \subseteq V$ holds by assumption.

$\boxed{(iv) \Rightarrow (vi)}$ Assume *(iv)* holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$. By *(iv)* there's a $\kappa < \delta$ which is a closure point of g and there's a regular $\theta \in (\kappa, \delta)$ and a generic elementary $\pi\colon H^V_\theta \to \mathcal{M}$

with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$. We want to find a regular $\bar\theta < \kappa$ and another elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f(\operatorname{crit}\sigma) = \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\sigma$.

Note that $\mathcal{M} \subseteq V$ and $H_\theta^V \subseteq \mathcal{M}$ implies that $H_\theta^V = H_\theta^{\mathcal{M}}$, so that both H_θ^V and $H_{\pi(\theta)}^{\mathcal{M}}$ are elements of $\mathcal{M}$ (we introduced g to ensure that $\pi(\theta)$ makes sense). An application of Countable Embedding Absoluteness 1.1.1 then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H_\theta^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$ and $\pi(f \restriction \kappa) \in \operatorname{ran}\pi^*$.

By elementarity of π, H_θ^V has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f \restriction \kappa \in \operatorname{ran}\sigma$ and $\bar\theta = f(\operatorname{crit}\sigma)$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta \to \delta$ be given. Define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$, so that by (vi) there exist regular $\bar\kappa < \bar\theta < \kappa < \theta$ such that κ is a closure point for g and there exists a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) = \bar\theta$ and $g \restriction \kappa \in \operatorname{ran}\pi$.

Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 1.2.4 we get a transitive $\mathcal{M} \in H_{g(\bar\kappa)}^V$ closed under ${<}f(\bar\kappa)$-sequences, and $H_{g(\bar\kappa)}^V$ has a generic elementary embedding $\sigma\colon H_{f(\bar\kappa)}^V \to \mathcal{M}$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa > f(\bar\kappa)$. In other words, $\bar\kappa$ is virtually $f(\bar\kappa)$-supercompact in $H_{\bar\theta}^V$. Elementarity of π then implies that κ is virtually $\pi(f)(\kappa)$-supercompact in H_θ^V, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all α such that $(H_\alpha^V, \in, A \cap H_\alpha^V) \prec (H_\delta^V, \in, A)$. Let $f\colon \delta \to \delta$ be given as $f(\alpha) = \langle \alpha_0, \alpha_1 \rangle$ with $\langle -, - \rangle$ being the Gödel pairing function, where α_0 is the first limit of elements of C above α and the α_1's are chosen such that $\{\alpha_1 \mid \alpha < \beta\}$ encodes $A \cap \beta$. This definition makes sense since δ is inaccessible by Proposition 1.2.2.

Let $\kappa < \delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa$, $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$, and $f \restriction \kappa \in \operatorname{ran}\pi$. We claim that $\bar\kappa := \operatorname{crit}\pi$ is virtually $({<}\delta, A)$-extendible. To see this, it suffices by the definition of C to show that

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \models \ulcorner \bar\kappa \text{ is virtually } (A \cap H_\kappa)\text{-extendible}\urcorner, \tag{1}$$

since $\kappa \in C$ because it is a closure point of f. Let $\beta := \min(C - \bar\kappa) < \bar\theta$ and note that β exists as $f(\bar\kappa) = \bar\theta$ so the definition of f says that $\bar\theta$ is a limit of elements of C above $\bar\kappa$. It then holds that $(H^V_{\bar\kappa}, \in, A \cap H^V_{\bar\kappa}) \prec (H^V_\beta, \in, A \cap H^V_\beta)$ as both $\bar\kappa$ and β are elements of C. Since f encodes A in the manner previously described and $\pi^{-1}(f) \upharpoonright \bar\kappa = f \upharpoonright \bar\kappa$, we get that $\pi(A \cap H^V_{\bar\kappa}) = A \cap H^V_\kappa$ and thus

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \prec (H^V_{\pi(\beta)}, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H^V_\beta)$. Now, as $(H^V_\gamma, \in, A \cap H^V_\gamma)$ and $(H^V_{\pi(\gamma)}, \in, A^* \cap H^V_{\pi(\gamma)})$ are elements of $H^V_{\pi(\beta)}$ for every $\gamma < \kappa$, Countable Embedding Absoluteness 1.1.1 implies that $H^V_{\pi(\beta)}$ sees that $\bar\kappa$ is virtually $({<}\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ■

Remark 1.3.5. The above proof shows that the $\mathcal{M} \subseteq V$ assumptions can be replaced by "sufficient" agreement between $\mathcal{M}$ and V: for (i)-(iii) this means that $H^{\mathcal{M}}_\theta = H^V_\theta$ whenever $\mathcal{M}$ is the codomain of a virtual (θ, A)-strong/supercompact/extendible embedding, and in (iv)-(v) this means that $H^{\mathcal{M}}_{\pi(f)(\kappa)} = H^V_{\pi(f)(\kappa)}$. The same thing holds in the "lightface" setting of Theorem 1.2.4.

We will now step away from the woodins for a little bit, and introduce the vopěnkas. In anticipation of the next section we will work with the class-sized version here, but all the following results work equally well for inaccessible virtually vopěnka cardinals[5].

[5]Note however that we have to require inaccessibility here: see [?] for an analysis of the singular virtually vopěnka cardinals.

Definition 1.3.6 (GBC)**.** The **Generic Vopěnka Principle** (gVP) states that for any class C consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$. $\circ$

We will be using a standard variation of gVP involving the following *natural sequences.*

Definition 1.3.7 (GBC)**.** Say that a class function $f\colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. $\circ$

Definition 1.3.8 (GBC)**.** Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ is **natural** if there exists an indexing function $f\colon \mathsf{On} \to \mathsf{On}$ and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$ for every α. Denote this indexing function by $f^{\vec{\mathcal{M}}}$ and the unary relations as $R^{\vec{\mathcal{M}}}_\alpha$. $\circ$

The following theorem is then the main theorem of this section. Firstly it shows that inaccessible cardinals are virtually vopěnka iff they are virtually prewoodin, but also that adding the "virtually" adverb doesn't do anything in this context, in contrast to Theorem 1.2.11.

Theorem 1.3.9 (GBC, D.-G.-N.)**.** *The following are equivalent:*

(i) *gVP holds;*
(ii) *For any natural On-sequence $\vec{\mathcal{M}}$ there exists a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for some $\alpha < \beta$;*
(iii) *On is virtually prewoodin;*
(iv) *On is faintly prewoodin.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume On is faintly prewoodin and fix some On-sequence $\vec{\mathcal{M}} := \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ of structures in a common language. Let κ be $({<}\mathsf{On}, \vec{\mathcal{M}})$-prestrong and fix some regular $\theta > \kappa$ satisfying that $\mathcal{M}_\alpha \in H^V_\theta$

for every $\alpha < \theta$, and fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, \vec{\mathcal{M}}) \to (\mathcal{N}, \in, \mathcal{M}^*)$$

with $H^V_\theta \subseteq \mathcal{N}$ and $\vec{\mathcal{M}} \cap H^V_\theta = \mathcal{M}^* \cap H^V_\theta$. Set $\kappa := \operatorname{crit}\pi$.

We have that $\pi \restriction \mathcal{M}_\kappa\colon\ \mathcal{M}_\kappa \to \mathcal{M}^*_{\pi(\kappa)}$, but we need to reflect this embedding down below θ as we don't know whether $\mathcal{M}^*_{\pi(\kappa)}$ is on the $\vec{\mathcal{M}}$ sequence. Working in the generic extension, we have

$$\mathcal{N} \models \exists\bar\kappa < \pi(\kappa)\exists\dot\sigma \in V^{\operatorname{Col}(\omega,\mathcal{M}^*_{\bar\kappa})}\colon \ulcorner\dot\sigma\colon\ \mathcal{M}^*_{\bar\kappa} \to \mathcal{M}^*_{\pi(\kappa)} \text{ is elementary}\urcorner.$$

Here κ realises $\bar\kappa$ and $\pi \restriction \mathcal{M}_\kappa$ realises σ. Note that $\mathcal{M}^*_\kappa = \mathcal{M}_\kappa$ since we ensured that $\mathcal{M}_\kappa \in H^V_\theta$ and we are assuming that $\vec{\mathcal{M}} \cap H^V_\theta = \mathcal{M}^* \cap H^V_\theta$, so the domain of σ $(= \pi \restriction \mathcal{M}_\kappa)$ *is* $\mathcal{M}^*_\kappa$ — also note that σ exists in a $\operatorname{Col}(\omega, \mathcal{M}_\kappa)$ extension of $\mathcal{N}$ by an application of Countable Embedding Absoluteness 1.1.1. Now elementarity of π implies that

$$H^V_\theta \models \exists\bar\kappa < \kappa\exists\dot\sigma \in V^{\operatorname{Col}(\omega,\mathcal{M}_{\bar\kappa})}\colon \ulcorner\dot\sigma\colon\ \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa \text{ is elementary}\urcorner,$$

which is upwards absolute to V, from which we can conclude that $\sigma\colon\ \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa$ witnesses that gVP holds.

(ii) $\Rightarrow$ *(iii)*: Assume *(ii)* holds and assume that On is not virtually prewoodin, which means that there exists some class A such that there are no virtually A-prestrong cardinals. This allows us to define a function $f\colon \mathsf{On} \to \mathsf{On}$ as $f(\alpha)$ being the least regular $\eta > \alpha$ such that α is not virtually (η, A)-prestrong.

We also define $g\colon \mathsf{On} \to \mathsf{On}$ as taking α to the least strong limit cardinal above α which is a closure point for f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A \cap H^V_{g(\alpha)}$. *(ii)* supplies us with $\alpha < \beta$ and a generic elementary embedding[6]

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}).$$

[6]Note that $V_{g(\alpha)} = H^V_{g(\alpha)}$ since $g(\alpha)$ is a strong limit cardinal.

Since $g(\alpha)$ is a closure point for f it holds that $f(\operatorname{crit}\pi) < g(\alpha)$, so fixing a regular $\theta \in (f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-prestrong, contradicting the definition of f. Hence On is virtually prewoodin. ■

1.4 Weak Vopěnka

We now move to a *weak* variant of gVP, introduced in a category-theoretic context in [**?**]. It starts with the following equivalent characterisation of gVP, which is the virtual analogue of the characterisation shown in [**?**].

Lemma 1.4.1 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an* On*-sequence of first-order structures* $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ *satisfying that*[7]

(i) gVP

(ii) There is not a natural On*-sequence* $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ *satisfying that*
- *there is a generic homomorphism* $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ *for every* $\alpha \leq \beta$*, which is unique in all generic extensions;*
- *there is no generic homomorphism* $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ *for any* $\alpha < \beta$*.*

(iii) There is not a natural On*-sequence* $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ *satisfying that*
- *there is a homomorphism* $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ *in* V *for every* $\alpha \leq \beta$*, which is unique in all generic extensions;*
- *there is no generic homomorphism* $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ *for any* $\alpha < \beta$*.*

Proof. Note that the only difference between *(ii)* and *(iii)* is that the homomorphism exists in V, making *(ii)* $\Rightarrow$ *(iii)* trivial.

(iii) $\Rightarrow$ *(i)*: Assume that gVP fails, meaning by Theorem 1.3.9 that we have a natural On-sequence $\vec{\mathcal{M}}_\alpha$ such that, in every generic extension, there's no homomorphism between any two disctinct $\mathcal{M}_\alpha$'s. Define an On-sequence $\langle \mathcal{N}_\kappa \mid \kappa \in \text{Card}\rangle$ as

$$\mathcal{N}_\kappa := \coprod_{\xi \leq \kappa} \mathcal{M}_\xi = \{(x, \xi) \mid \xi \leq \kappa \wedge \xi \in \text{Card} \wedge x \in \mathcal{M}_\xi\},^{8}$$

with a unary relation R^* given as $R^*(x,\xi)$ iff $\mathcal{M}_\xi \models R(x)$ and a binary relation $\sim^*$ given as $(x,\xi) \sim^* (x',\xi')$ iff $\xi = \xi'$. Whenever we have a homomorphism $f\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ we then get an induced homomorphism $\tilde{f}\colon \mathcal{M}_0 \to \mathcal{M}_\xi$, given as $\tilde{f}(x) := f(x,0)$, where $\xi \leq \kappa$ is given by preservation of $\sim^*$.

[7] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [**?**].

For any two cardinals $\kappa < \lambda$ we have a homomorphism $j_{\kappa\lambda}\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ in V, given as $j_{\kappa\lambda}(x, \xi) := (x, \xi)$. This embedding must also be the *unique* such embedding in all generic extensions, as otherwise we get a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s. Furthermore, there can't be any homomorphism $\mathcal{N}_\lambda \to \mathcal{N}_\kappa$ as that would also imply the existence of a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s.

$(i) \Rightarrow (ii)$: Assume that we have an On-sequence $\vec{\mathcal{M}}_\alpha$ as in the theorem, with generic homomorphisms $j_{\alpha\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ that are unique in all generic extensions for every $\alpha \leq \beta$, with no generic homomorphisms going the other way.

We first note that we can for every $\alpha \leq \beta$ choose the $j_{\alpha\beta}$ in a $\mathrm{Col}(\omega, \mathcal{M}_\alpha)$-extension, by a proof similar to the proof of Lemma 1.1.1 and using the uniqueness of $j_{\alpha\beta}$. Next, fix a proper class $C \subseteq \mathsf{On}$ such that $\alpha \in C$ implies that

$$\sup_{\xi \in C \cap \alpha} |\mathcal{M}_\xi|^V < |\mathcal{M}_\alpha|^V .$$

and note that this implies that $V[g] \models |\mathcal{M}_\xi| < |\mathcal{M}_\alpha|$ for every V-generic $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\xi)$. This means that for every $\alpha \in C$ we may choose some $\eta_\alpha \in \mathcal{M}_\alpha$ which is *not* in the range of any $j_{\xi\alpha}$ for $\xi < \alpha$. But now define first-order structures $\langle \mathcal{N}_\alpha \mid \alpha \in C \rangle$ as $\mathcal{N}_\alpha := (\mathcal{M}_\alpha, \eta_\alpha)$. Then, by our assumption on the $\mathcal{M}_\alpha$'s and construction of the $\mathcal{N}_\alpha$'s, there can be no generic homomorphism between any two distinct $\mathcal{N}_\alpha$, showing that gVP fails. ■

Note that the proof of the above lemma shows that we without loss of generality may assume that the generic homomorphism in (i) exists in V, which we record here:

Lemma 1.4.2 (GBC, Virtualised Adámek-Rosický)**.** *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ satisfying that*[9]

[9] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [?].

(i) *there is a homomorphism* $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ *in* V *for every* $\alpha \leq \beta$*, which is unique in all generic extensions;*

(ii) *there is no generic homomorphism* $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ *for any* $\alpha < \beta$*.*

■

The *weak* version of gVP is then simply "flipping the arrows around" in the above characterisation of gVP.

Definition 1.4.3 (GBC)**. Generic Weak Vopěnka's Principle** (gWVP) states that there does *not* exist an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ such that

- there is a generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\alpha \leq \beta$, which is unique in all generic extensions;
- there is *no* generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for any $\alpha < \beta$.

○

Denoting the corresponding non-generic principle by WVP [?] showed the following.

Theorem 1.4.4 (Wilson)**.** *WVP is equivalent to On being a Woodin cardinal.*

Given our 1.3.9 we may then suspect that in the virtual world these two are equivalent, which indeed turns out to be the case. We will be roughly following the argument in [?], but we have to diverge from it at several points in which they're using the fact that they're working with class-sized elementary embeddings.

Indeed, in that paper they establish a correspondence between elementary embeddings and certain homomorphisms, a correspondence we won't achieve here. Proving that the elementary embeddings we *do* get are non-trivial seems to furthermore require extra assumptions on our structures. Let's begin.

Define for every strong limit cardinal λ and Σ_1-formula φ the relations

$$\begin{aligned} R^\varphi &:= \{x \in V \mid (V, \in) \models \varphi[x]\} \\ R^\varphi_\lambda &:= \{x \subseteq H^V_\lambda \mid \exists y \in R^\varphi : y \cap H^V_\lambda = x\} \end{aligned}$$

and given any class A define the structure

$$\mathscr{P}_{\lambda,A} := (H^V_{\lambda^+}, R^\varphi_\lambda, \{\lambda\}, A \cap H^V_\lambda)_{\varphi \in \Sigma_1}.$$

Say that a homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ is **trivial** if $h(x) \cap H^V_\eta = x \cap H^V_\eta$ for every $x \in H^V_{\lambda^+}$. Note that h can only be trivial if $\eta \leq \lambda$ since $h(\lambda) = \eta$.

Lemma 1.4.5 (GBC, G.-N.)**.** *Let λ be a singular strong limit cardinal, η a strong limit cardinal and $A \subseteq V$ a class. If there exists a non-trivial generic homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ then there's a non-trivial generic elementary embedding*

$$\pi\colon (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that, letting $\nu := \min\{\lambda, \eta\}$, it holds that $H^V_\nu \subseteq \mathcal{M}$, $A \cap H^V_\nu = B \cap H^V_\nu$ and $\operatorname{crit}\pi < \nu$.

Proof. Assume that we have a non-trivial homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ in a forcing extension $V[g]$, define in $V[g]$ the set

$$\mathcal{M}^* := \{\langle b, f\rangle \mid b \in [H_\nu]^{<\omega} \wedge f \in H^V_{\lambda^+} \wedge f\colon H^V_\lambda \to H^V_\lambda\},$$

and define the relation $\in^*$ on $\mathcal{M}^*$ as

$$\langle b_0, f_0\rangle \in^* \langle b_1, f_1\rangle \quad \text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) \in f_1(y)\}).$$

Claim 1.4.5.1. $\in^*$ is wellfounded.

PROOF OF CLAIM. Assume not and let $\cdots \in^* \langle b_1, f_1\rangle \in^* \langle b_0, f_0\rangle$ be an $\in^*$-decreasing chain, which by definition means that, for every $n < \omega$,

$$b_{n+1}b_n \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}). \tag{1}$$

Define the relation $R(v_0, v_1, v_2)$ on H^V_λ as

$$R(X, f, g) \text{ iff } X = \{xy \in [H^V_\lambda]^{<\omega} \mid f(x) \in g(y)\}.$$

This relation is equal to R^φ_λ for some φ, so h moves $\langle X, f, g\rangle \in R^\varphi_\lambda$ to

$$\langle h(X), h(f), h(g)\rangle \in R^\varphi_\eta,$$

meaning that

$$h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}) = \{xy \in [H^V_\eta]^{<\omega} \mid f^*_{n+1}(x) \in f^*_n(y)\}$$

for some f^*_n such that $f^*_n \cap H^V_\eta = h(f_n)$ for all $n < \omega$. But now (1) implies that

$$b_{n+1}b_n \in \{xy \in [H^V_\eta]^{<\omega} \mid f^*_{n+1}(x) \in f^*_n(y)\}$$

and so $h(f_{n+1})(x) = f^*_{n+1}(x) \in f^*_n(y) = h(f_n)(y)$, giving an $\in$-decreasing sequence in $V[g]$ using transitivity of H^V_η, a contradiction!

Hence $\in^*$ is wellfounded. $\dashv$

$\mathcal{M}^*$ is a set, so $\in^*$ is trivially set-like. This means that we can take the transitive collapse $(\mathcal{M}, \in) \cong (\mathcal{M}^*, \in^*)$, and we note that $\mathcal{M} = \{[b, f] \mid \langle b, f\rangle \in \mathcal{M}^*\}$, where $[b, f] := \{[\bar{b}, \bar{f}] \mid \langle \bar{b}, \bar{f}\rangle \in^* \langle b, f\rangle\}$.

We now get a version of Łoś' Theorem whose proof is straight-forward, using that h preserves all Σ_1-relations and that $H^V_\lambda \models \mathsf{ZFC}^-$.

Claim 1.4.5.2. For every formula $\varphi(v_1, \ldots, v_n)$ and every $[b_1, f_1], \ldots, [b_n, f_n] \in \mathcal{M}$ the following are equivalent:

(i) $(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]]$;

(ii) $b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\})$.

Proof of claim. The proof is straightforward, using that h preserves Σ_1-relations. We prove this by induction on φ. If φ is $v_i \in v_j$ then we have that

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow & [b_i, f_i] \in [b_j, f_j] \\
\Leftrightarrow & \langle b_i, f_i \rangle \in^* \langle b_j, f_j \rangle \\
\Leftrightarrow & b_i b_j \in h(\{a_i a_j \in [H_\lambda^V]^{<\omega} \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow & b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow & b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\psi \wedge \chi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow & (\mathcal{M}, \in) \models \psi[[b_1, f_1], \ldots, [b_n, f_n]] \wedge \chi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow & b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \ldots, f_n(a_n)]\}) \cap \\
& \qquad h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \chi[f_1(a_1), \ldots, f_n(a_n)]\}) \\
\Leftrightarrow & b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\neg\psi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow & (\mathcal{M}, \in) \models \neg\psi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow & (\mathcal{M}, \in) \not\models \psi[[b_1, f_1], \ldots, [b_n, f_n]] \\
\Leftrightarrow & b_1 \cdots b_n \notin h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \ldots, f_n(a_n)]\}) \\
\Leftrightarrow & b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\}).
\end{aligned}$$

Finally, if φ is $\exists x\psi$ then

$$\begin{aligned}
&(\mathcal{M},\in)\models\varphi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&(\mathcal{M},\in)\models\exists x\psi[x,[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle\in\mathcal{M}^*\colon(\mathcal{M},\in)\models\psi[[b,f],[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle\in\mathcal{M}^*\colon bb_1\cdots b_n\in h(\{aa_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\psi[f(a),f_1(a_1),\dots,f_n(a_n)]\})\\
\Leftrightarrow&b_1\cdots b_n\in h(\{a_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\varphi[f_1(a_1),\dots,f_n(a_n)]\}).
\end{aligned}$$

This finishes the proof. $\dashv$

Next up, we have the following standard lemma, which implies that $H^V_\eta\subseteq\mathcal{M}$:

Claim 1.4.5.3. For all $y\in H^V_\eta$ we have $y=[\langle y\rangle,\mathrm{pr}]$, where $\mathrm{pr}(\langle x\rangle):=x$.

Proof of claim. We prove this by $\in$-induction on $y\in H^V_\eta$, so suppose that $y'=[\langle y'\rangle,\mathrm{pr}]$ for every $y'\in y$, which implies that $y\subseteq\mathcal{M}$ by transitivity of $\mathcal{M}$. We then get that, for every $[b,f]\in\mathcal{M}$,

$$\begin{aligned}
[b,f]\in[\langle y\rangle,\mathrm{pr}]&\Leftrightarrow b\langle y\rangle\in h(\{a\langle x\rangle\mid f(a)\in\mathrm{pr}(\langle x\rangle)\})\\
&\Leftrightarrow\exists y'\in y\colon b\langle y'\rangle\in h(\{a\langle x\rangle\mid f(a)=x\})\\
&\Leftrightarrow\exists y'\in y\colon[b,f]=[\langle y'\rangle,\mathrm{pr}]=y'\\
&\Leftrightarrow[b,f]\in y,
\end{aligned}$$

showing that $y=[\langle y\rangle,\mathrm{pr}]$. $\dashv$

Now define

$$B:=\{[b,f]\in\mathcal{M}\mid b\in h(\{x\in H^V_\lambda\mid f(x)\in A\})\}.$$

and, in $V[g]$, let $\pi\colon(H^V_\lambda,\in,A\cap H^V_\lambda)\to(\mathcal{M},\in,B)$ be given as $\pi(x):=[\langle\rangle,c_x]$.

Claim 1.4.5.4. π is elementary.

Proof of claim. For $x_1, \ldots, x_n \in H^V_\lambda$ it holds that

$$\begin{aligned}
&(\mathcal{M}, \in, B) \models \varphi[\pi(x_1), \ldots, \pi(x_n)] \\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \varphi[\pi(x_1), \ldots, \pi(x_n)] \\
\Leftrightarrow\ &\langle\rangle \in h(\{\langle\rangle \mid \mathscr{P}_{\lambda,A} \models \varphi[x_1, \ldots, x_n]\}) \\
\Leftrightarrow\ &(H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \models \varphi[x_1, \ldots, x_n]
\end{aligned}$$

and we also get that, for every $x \in H^V_\lambda$,

$$x \in A \Leftrightarrow \langle\rangle \in h(\{a \in H^V_\lambda \mid x \in A\}) \Leftrightarrow \pi(x) \in B,$$

which shows elementarity. ⊣

We next need to show that $B \cap H^V_\nu = A \cap H^V_\nu$, so let $x \in H^V_\nu$. Note that $x = [\langle x\rangle, \mathrm{pr}]$ by Claim 1.4.5.3, which means that

$$x \in B \Leftrightarrow \langle x\rangle \in h(\{\langle y\rangle \in H^V_\lambda \mid y \in A\}) \Leftrightarrow x \in A.$$

The last thing we need to show is that $\operatorname{crit} \pi < \nu$. We start with an analogous result about h.

Claim 1.4.5.5. There exists some $b \in H^V_\nu$ such that $h(b) \neq b$.

Proof of claim. Assume the claim fails. We now have two cases.

$\boxed{\textbf{Case 1: } \lambda \geq \eta}$

By non-triviality of h there's an $x \in H^V_{\lambda^+}$ such that $h(x) \cap H^V_\eta \neq x \cap H^V_\eta$, which means that there exists an $a \in H^V_\eta$ such that $a \in h(x) \Leftrightarrow a \notin x$.

If $a \in x$ then $\{a\} = h(\{a\}) \subseteq h(x)$,[10] making $a \in h(x)$, $\lightning$, so assume instead that $a \in h(x)$. Since η is a strong limit cardinal we may fix a

[10]Note that as h preserves Σ_1 formulas it also preserves singletons and boolean operations.

cardinal $\theta < \eta$ such that $a \in H^V_\theta$ and $H^V_\theta \in H^V_\eta$. We then have that[11]

$$\{a\} \subseteq h(x) \cap H^V_\theta = h(x) \cap h(H^V_\theta) = h(x \cap H^V_\theta) = x \cap H^V_\theta,$$

so that $a \in x$, $\lightning$.

$\boxed{\textbf{Case 2: } \lambda < \eta}$

In this case we are assuming that $h \restriction H^V_\lambda = \mathrm{id}$, but $h(\lambda) = \eta > \lambda$. Since λ is singular we can fix some $\gamma < \lambda$ and a cofinal function $f \colon \gamma \to \lambda$. Define the relation

$$R = \{(\alpha, \beta, \bar\alpha, \bar\beta, g) \mid \ulcorner g \text{ is a cofinal function } g \colon \alpha \to \beta \urcorner \wedge g(\bar\alpha) = \bar\beta\}.$$

Then $R(\gamma, \lambda, \alpha, f(\alpha), f)$ holds by assumption for every $\alpha < \gamma$, so that R holds for some $(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*)$ such that

$$\begin{aligned}(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*) \cap H^V_\eta &= (h(\gamma), h(\lambda), h(\alpha), h(f(\alpha)), h(f)) \\ &= (\gamma, \eta, \alpha, f(\alpha), h(f)),\end{aligned}$$

using our assumption that h fixes every $b \in H^V_\lambda$. Since γ, α and $f(\alpha)$ are transitive and bounded in H^V_λ it holds that $h(\gamma) = \gamma^*$, $h(\alpha) = \alpha^*$ and $h(f(\alpha)) = f(\alpha)^*$. Also, since $\mathrm{dom}(f^*) = \gamma = \mathrm{dom}(f)$ we must in fact have that $f^* = h(f)$. But this means that $h(f) \colon \gamma \to \eta$ is cofinal and $\mathrm{ran}(h(f)) \subseteq \lambda$, a contradiction! $\dashv$

To use the above Claim 1.4.5.5 to conclude anything about π we'll make use of the following standard lemma.

Claim 1.4.5.6. For any $x \in H^V_\lambda$ it holds that $h(x) \cap H^V_\eta = \pi(x) \cap H^V_\eta$.

[11] Note that we're using $\lambda \geq \eta$ here to ensure that $H^V_\theta \in \mathrm{dom}\, h$.

Proof of claim. For any $n < \omega$ and $\langle a_1, \ldots, a_n\rangle \in [H_\eta^V]^n$ we have that

$$\begin{aligned}
&\langle a_1, \ldots, a_n\rangle \in \pi(x)\\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle a_1, \ldots, a_n\rangle \in \pi(x)\\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle [\langle a_1\rangle, \mathrm{pr}], \ldots, [\langle a_n\rangle, \mathrm{pr}]\rangle \in [\langle\rangle, c_x]\\
\Leftrightarrow\ &\langle a_1, \ldots, a_n\rangle \in h(\{\langle x_1, \ldots, x_n\rangle \mid \mathscr{P}_{\lambda,A} \models \langle x_1, \ldots, x_n\rangle \in x\})\\
\Leftrightarrow\ &\langle a_1, \ldots, a_n\rangle \in h(x),
\end{aligned}$$

showing that $h(x) \cap H_\eta^V = \pi(x) \cap H_\eta^V$. ⊣

Now use Claim 1.4.5.5 to fix a $b \in H_\nu^V$ which is moved by h. Claim 1.4.5.6 then implies that

$$\pi(b) \cap H_\eta^V = h(b) \cap H_\eta^V = h(b) \neq b = b \cap H_\eta^V,$$

showing that $\pi(b) \neq b$ and hence $\operatorname{crit}\pi < \nu$. This finishes the proof of the lemma. ■

Theorem 1.4.6 (GBC, G.-N.)**.** *gVP is equivalent to gWVP.*

Proof. ($\Rightarrow$): Assume gVP holds and gWVP fails, and let $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ be an On-sequence of first-order structures such that for every $\alpha \leq \beta$ there exists a generic homomorphism

$$j_{\beta\alpha} \colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$$

in some $V[g]$ which is unique in all generic extensions, with no generic homomorphisms going the other way. Here we may assume, as in the proof of Lemma 1.4.1, that $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\beta)$. We can then find a proper class $C \subseteq \mathsf{On}$ such that $|\mathcal{M}_\alpha|^V < |\mathcal{M}_\beta|^V$ for every $\alpha < \beta$ in C. By gVP there are then $\alpha < \beta$ in C and a generic homomorphism

$$\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta\,.$$

in some $V[h]$, where again we may assume that $h \subseteq \mathrm{Col}(\omega, \mathcal{M}_\alpha)$. But then $\pi \circ j_{\beta\alpha} = \mathrm{id}$ by uniqueness of $j_{\beta\beta} = \mathrm{id}$, which means that $j_{\beta\alpha}$ is injective in $V[g\times h]$ and hence also in $V[g]$. But then $|\mathcal{M}_\beta|^{V[g]} \leq |\mathcal{M}_\alpha|^{V[g]}$, which implies that $|\mathcal{M}_\beta|^V \leq |\mathcal{M}_\alpha|^V$ by the $|\mathcal{M}_\beta|^{+V}$-cc of $\mathrm{Col}(\omega, \mathcal{M}_\beta)$, contradicting the definition of C.

($\Leftarrow$): Assume that gVP fails, which by Theorem 1.3.9 is equivalent to On not being faintly prewoodin. This means that there exists a class A such that there are no faintly A-prestrong cardinals. We can therefore assign to any cardinal κ the least cardinal $f(\kappa) > \kappa$ such that κ is not faintly $(f(\kappa), A)$-prestrong.

Also define a function $g\colon \mathsf{On} \to \mathrm{Card}$ as taking an ordinal α to the least singular strong limit cardinal above α closed under f. Then we're assuming that there's no non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

with $H^V_{g(\alpha)} \subseteq \mathcal{M}$ and $B \cap H^V_{g(\alpha)} = A \cap H^V_{g(\alpha)}$. Assume towards a contradiction that for some α, β there is a non-trivial generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$. Lemma 1.4.5 then gives us a non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that $H^V_\nu \subseteq \mathcal{M}$ with $\nu := \min\{g(\alpha), g(\beta)\}$ and $A \cap H^V_\nu = B \cap H^V_\nu$, a contradiction! Therefore every generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$ is trivial. Since there is a unique trivial homomorphism when $\alpha \geq \beta$ and no trivial homomorphism when $\alpha < \beta$ since $g(\alpha)$ is sent to $g(\beta)$, the sequence of structures

$$\langle \mathscr{P}_{g(\alpha),A} \mid \alpha \in \mathsf{On} \rangle$$

is a counterexample to gWVP, which is what we wanted to show. ■

1.5 Berkeley

Berkeley cardinals was introduced by Woodin at University of California, Berkeley around 1992, and was introduced as a large cardinal candidate that would be inconsistent with ZF. They trivially imply the Kunen inconsistency and are therefore at least inconsistent with ZFC, but that's as far as it currently goes. In the virtual setting the virtually berkeley cardinals, like all the other virtual large cardinals, are simply downwards absolute to L.

It turns out that virtually berkeley cardinals are natural objects, as the main theorem of this section shows that these large cardinals are precisely what separates virtually prewoodins from the virtually woodins, as well as separating virtually vopěnka cardinals from mahlo cardinals.

Definition 1.5.1. Say that a cardinal δ is **virtually proto-berkeley** if for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi < \delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ then δ is **virtually berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C \subseteq \delta$ we say δ is **virtually club berkeley**. ∘

Virtually (proto-)berkeley cardinals turn out to be equivalent to their "boldface" versions, the proof of which is a straight-forward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [**?**].

Proposition 1.5.2 (Virtualised Cutolo)**.** *If δ is virtually proto-berkeley then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with* $\operatorname{crit}\pi < \delta$*. If δ is virtually berkeley then we can furthermore ensure that* $\operatorname{crit}\pi$ *is arbitrarily large below δ.*

Proof. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{\{\langle A, x\rangle \mid x \in \mathcal{M}\}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: a is the first element in the pairs belonging to the set of highest rank, and $\mathcal{M}$ is what remains if we remove the set with the highest rank. But this means that a generic elementary embedding $\pi\colon \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and a, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the wanted conclusion. ■

The following is a straight-forward virtualisation of the usual definition of the vopěnka filter (see e.g. [**?**]).

Definition 1.5.3 (GBC)**.** Define the **virtually vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$. ○

Theorem 1.3.9 shows that this filter is proper iff gVP holds. The proof of Proposition 24.14 in [**?**] also shows that this filter is normal and is proper iff gVP holds. Note that uniformity of filters is non-trivial as we're working with proper classes[12]. Indeed, Theorem 1.5.7 shows that uniformity of this filter is equivalent to there being no virtually berkeley cardinals — the following lemma is the first implication.

Lemma 1.5.4 (GBC, N.)**.** *Assume gVP and that there are no virtually berkeley cardinals. Then the virtually vopěnka filter F on On contains every class club C.*

PROOF. The crucial extra property we get by assuming that there aren't any virtually berkeleys is that F becomes uniform, i.e. contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin F$. Let M be a transitive set with $\delta \subseteq M$ and $\gamma < \delta$ a cardinal. As $(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function

[12] This boils down to the fact that the class club filter is not provably normal in GBC, see asd.

$f\colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\alpha+1, \delta+1)$$

and unary relations $R_\alpha := \langle M, \mathcal{N}_\alpha \rangle$. If $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit}\pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$, then $\pi(R_\alpha) = R_\beta$ implies that $\pi \restriction \mathcal{M}\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi \leq \delta$. We also get that $\operatorname{crit}\pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit}\pi}\colon \mathcal{N}_{\operatorname{crit}\pi} \to \mathcal{N}_{\pi(\operatorname{crit}\pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and hence $\operatorname{crit}\pi > \gamma$ as $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually berkeley, a contradiction. Thus $\operatorname{crit}\pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$.

From here the proof of Lemma 8.11 in [**?**] shows us the wanted. ■

Theorem 1.5.5 (GBC, N.). *If there are no virtually berkeley cardinals then* On *is virtually prewoodin iff* On *is virtually woodin.*

Proof. Assume On is virtually prewoodin, so gVP holds by Theorem 1.3.9 and we can let F be the virtually vopěnka filter. The assumption that there aren't any virtually berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails — we will follow the proof of Theorem 1.3.9.

Failure means that there is some class A and some class club C such that there are no virtually A-prestrong cardinals in C. Since there are no virtually berkeley cardinals, Lemma 1.5.4 imples that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as sending α to the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-prestrong if $\alpha \in C$, and set $f(\alpha) := \alpha$ if $\alpha \notin C$. Also define $g\colon \mathsf{On} \to \mathsf{On}$ as $g(\alpha)$ being the least strong limit cardinal in C above α which is a closure point for f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := \mathrm{Code}(\langle A \cap H^V_{g(\alpha)}, \mathcal{N}_\alpha\rangle)$ and apply gVP to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi\restriction(H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)})\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}),$$

making $\operatorname{crit}\pi$ virtually $(g(\alpha), A)$-prestrong and thus $\operatorname{crit}\pi \notin C$. But as we also get the embedding $\pi\restriction\mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\operatorname{crit}\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \mathrm{Card} \mid (H^V_\kappa, \in, A \cap H^V_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-prestrong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it isn't virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, A \cap H^V_\theta) \to (M, \in, B)$$

with $\operatorname{crit}\pi = \kappa$, $H^V_\theta \subseteq M$, $M \subseteq V$, $A \cap H^V_\theta = B \cap H^V_\theta$ and $\pi(\kappa) < \theta$.

Now $\pi(\kappa)$ is inaccessible, and $(H^V_{\pi(\kappa)}, \in, A \cap H^V_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$ believes that κ is virtually $(A \cap H^V_{\pi(\kappa)})$-strong as in the proof of Theorem 1.2.8, meaning that $(H^V_\kappa, \in, A \cap H^V_\kappa)$ believes that there is a proper class of virtually $(A \cap H^V_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually woodin. ■

Theorem 1.5.6 (GBC, N.)**.** *If there exists a virtually berkeley cardinal δ then gVP holds and On is not mahlo.*

Proof. If On was Mahlo then there would in particular exist an inaccessible cardinal $\kappa > \delta$, but then $H_\kappa^V \models \ulcorner$there exists a virtually berkeley cardinal$\urcorner$, contradicting the incompleteness theorem.

To show gVP we show that On is virtually prewoodin, which is equivalent by Theorem 1.3.9. Fix therefore a class A — we have to show that there exists a virtually A-prestrong cardinal. For every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta \colon (H_\theta^V, \in, A \cap H_\theta^V) \to (H_\theta^V, \in, A \cap H_\theta^V)$$

with $\operatorname{crit} \pi < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-prestrong, making On virtually prewoodin. ■

Theorem 1.5.7 (GBC, N.)**.** *The following are equivalent:*

(i) gVP *implies that* On *is mahlo;*

(ii) On *is virtually prewoodin iff* On *is virtually woodin;*

(iii) There are no virtually berkeley cardinals.

Proof. *(iii)* $\Rightarrow$ *(ii)* is Theorem 1.5.5, and the contraposed version of *(i)* $\Rightarrow$ *(iii)* is Theorem 1.5.6. For *(ii)* $\Rightarrow$ *(i)* note that gVP implies that On is virtually prewoodin by Theorem 1.3.9, which by *(ii)* means that it's virtually woodin and the usual proof shows that virtually woodins are mahlo[13], showing *(i)*. ■

This also immediately implies the following equiconsistency, as virtually berkeley cardinals have strictly larger consistency strength than virtually woodin cardinals.

Corollary 1.5.8 (N.)**.** *The existence of an inaccessible virtually prewoodin cardinal is equiconsistent with the existence of an inaccessible virtually woodin cardinal.* ■

[13] See e.g. Exercise 26.10 in [**?**].

Question 1.7 in [?] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually prestrong cardinal, and Lemmata 2.5 and 2.8 in the same paper also shows that being ω-Erdős is equivalent to being virtually club berkeley and that the least such is also the least virtually berkeley.[14]

Furthermore, they also showed that a non-Σ_2-reflecting virtually prestrong cardinal is equivalent to a virtually prestrong cardinal which isn't virtually strong. We can therefore reformulate their question to the following equivalent question.

Question 1.5.9 (Wilson)**.** If there exists a virtually prestrong cardinal which is not virtually strong, is there then a virtually berkeley cardinal?

[?] showed that their question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem 1.2.8 we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 1.5.10. If there exists a cardinal κ which is virtually (θ, ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually berkeley cardinal?

Our results above at least gives a partially positive result:

Corollary 1.5.11 (N.)**.** *If there exists a virtually A-prestrong cardinal for every class A and there are no virtually strong cardinals, then there exists a virtually berkeley cardinal.*

Proof. The assumption implies by definition that On is virtually prewoodin but not virtually woodin, so Theorem 1.5.7 supplies us with the desired. ∎

[14] Note that this also shows that virtually club berkeley cardinals and virtually berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

The assumption that there is a virtually A-prestrong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 1.5.7 shows that this is necessary, which might lead one to think that the question could have a negative answer.