1 Virtual large cardinals

In this chapter we investigate the properties of virtual versions of well-known large cardinals, including measurables, strongs, supercompacts, Woodins and Vopěnkas. This entails firstly analysing the relationships between them, and secondly looking at more general properties in terms of their behaviour in core models as well as their indestructibility. This virtual perspective also allows us to analyse virtualised versions of large cardinals that are otherwise inconsistent with ZFC, such as the Berkeley cardinals.

Before we start, we will briefly cover a few standard definitions and lemmata that we will be using freely throughout the chapter. When we're dealing with embeddings between set-sized structures, we will usually be interested in structures of the following form.

Definition 1.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa+1 \subseteq \mathcal{M}$ and $(\mathcal{M}, \in) \models \mathsf{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$, $\mathcal{M}$ is a **κ-model**.[1] $\circ$

Embeddings between these weak κ-models can equivalently be phrased in terms of ultrafilters, or *measures*. Recall that μ is an **$\mathcal{M}$-measure** if

$$(\mathcal{M}, \in, \mu) \models \ulcorner \mu \text{ is a } \kappa\text{-complete ultrafilter on } \kappa \urcorner.$$

Some common properties of such measures are the following.

Definition 1.2. For a weak κ-model $\mathcal{M}$, an $\mathcal{M}$-measure μ is...

- **weakly amenable** if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with $\mathrm{Card}^{\mathcal{M}}(x) = \kappa$;
- **countably complete** if $\bigcap \vec{X} \neq \emptyset$ for every ω-sequence $\vec{X} \in {}^{\omega}\mu$. $\circ$

[1] Note that our (weak) κ-models do not have to be transitive, in contrast to the models considered in [?] and [?]. Not requiring the models to be transitive was introduced in [?].

Weak amenability can equivalently be phrased in terms of a property concerning only the embedding.

Proposition 1.3 (Folklore). *Let $\mathcal{M}$ be a weak κ-model, μ an $\mathcal{M}$-measure and $j : \mathcal{M} \to \mathcal{N}$ the associated ultrapower embedding. Then μ is weakly amenable if and only if j is κ-**powerset preserving**, meaning that $\mathcal{M} \cap \mathscr{P}(\kappa) = \mathcal{N} \cap \mathscr{P}(\kappa)$.* ■

1.1 Strongs & supercompacts

We start out with measurables, strongs and supercompacts. Their (non-virtual) definitions can be found in Section ??.

Definition 1.4. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is...

- **faintly θ-measurable** if, in a forcing extension, there is a transitive class $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$;
- **faintly θ-strong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$;
- **faintly θ-supercompact** if it's faintly θ-measurable, ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

We further replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we attach a "**pre**" if we don't assume that $\pi(\kappa) > \theta$, and we will leave out θ when it holds for all regular $\theta > \kappa$. ○

As a quick example of this terminology, a *faintly prestrong cardinal* is a cardinal κ such that for all regular $\theta > \kappa$, κ is faintly θ-measurable with $H_\theta^V \subseteq \mathcal{N}$.

Observe that whenever we have a virtual large cardinal that has its defining property for all regular θ, we can assume that the target of the embedding is an element of the ground model V and not just a subset of V. Suppose, for instance, that κ is virtually measurable and fix a regular $\theta > \kappa$ and set $\lambda := (2^{<\theta})^+$. Take a generic elementary embedding $\pi : H_\lambda \to \mathcal{M}_\lambda$ witnessing that κ is virtually λ-measurable. The restriction $\pi \restriction H_\theta\colon H_\theta \to \pi(H_\theta)$ witnesses that κ is virtually θ-measurable and the target model $\mathcal{M}_\theta := \pi(H_\theta)$ is in V because $\mathcal{M}_\lambda \subseteq V$ by assumption. Thus, the weaker assumption that the target model $\mathcal{M}_\theta \subseteq V$ only affects level-by-level virtual large cardinals. Indeed, as we will see in later sections, even further weakening the assumption $\mathcal{M}_\theta \subseteq V$ to $H_\theta = H_\theta^{\mathcal{M}_\theta}$ in the definition of virtually strong (or supercompact) cardinals yields the same notion (again, we do not know whether this holds level-by-level).

We note that even small cardinals can be faintly measurable: we may for instance have a precipitous ideal on ω_1; see [?, Theorem 22.33]. The "virtually" adverb implies that the cardinals are in fact large cardinals in the usual sense, as Proposition 1.5 below shows.

Proposition 1.5 (Virtualised folklore). *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is 1-iterable.*

Proof. Let κ be virtually θ-measurable, witnessed by a forcing $\mathbb{P}$, a transitive $\mathcal{N} \subseteq V$ and an elementary $\pi\colon H_\theta^V \to \mathcal{N}$ with $\pi \in V^{\mathbb{P}}$. If κ isn't a strong limit then we have a surjection $\pi(f)\colon \mathscr{P}(\alpha) \to \pi(\kappa)$ with $\operatorname{ran}\pi(f) = \operatorname{ran} f \subseteq \kappa$ for some $\alpha < \kappa$, $\lightning$. Note that we used $\mathcal{N} \subseteq V$ to ensure that $\mathscr{P}(\alpha)^V = \mathscr{P}(\alpha)^{\mathcal{N}}$. The same argument shows that κ is regular. By restricting the generic embedding and using that $\mathscr{P}(\kappa)^V = \mathscr{P}(\kappa)^N$ as $\mathcal{N} \subseteq V$ and $\mathscr{P}(\kappa)^V \subseteq \mathcal{N}$, we get that κ is 1-iterable. ■

Along with the above definition of faintly supercompactness we can also virtualise Magidor's characterisation of supercompact cardinals[2], which was one of the original characterisations of the remarkable cardinals in [?].

Definition 1.6. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-Magidor-supercompact** if there are $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ such that $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. ◦

Gitman and Schindler observed in [?] that Schindler's remarkable cardinals are precisely the virtually supercompacts, and indeed surprisingly, they are also precisely the virtually strongs, which in turn makes virtually strongs and virtually supercompacts equivalent. We give the proof of the equivalences, which was omitted in [?], here.

[2] See Appendix ?? for the non-virtual version of this characterisation.

Theorem 1.7 (Gitman-Schindler). *For an uncountable cardinal κ, the following are equivalent.*[3]

(i) *κ is virtually strong;*

(ii) *κ is virtually supercompact;*

(iii) *κ is virtually Magidor-supercompact.*

Proof. $(ii) \Rightarrow (i)$ is simply by definition.

$(i) \Rightarrow (iii)$: Fix $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi\colon H^V_{(2^{<\theta})^+} \to \mathcal{M}$ with[4] $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H^V_{(2^{<\theta})^+} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. Since $H^V_\theta, H^{\mathcal{M}}_{\pi(\theta)} \in \mathcal{M}$, Countable Embedding Absoluteness ?? implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H^V_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ with $\operatorname{crit}\pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H^V_\theta = H^{\mathcal{M}}_\theta$ as $\mathcal{M} \subseteq V$ and $H^V_\theta \subseteq \mathcal{M}$, elementarity of π now implies that $H^V_{(2^{<\theta})^+}$ has ordinals $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\sigma\colon H^V_{\bar\theta} \to H^V_\theta$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa$. This shows (iii).

$(iii) \Rightarrow (ii)$: Fix $\theta > \kappa$ and $\delta := (2^{<\theta})^+$. By (iii) there exist ordinals $\bar\kappa < \bar\delta < \kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\delta} \to H^V_\delta$ with $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. We will argue that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, so that by elementarity κ is virtually θ-supercompact in H^V_δ and hence also in V by the choice of δ. Consider the restriction

$$\sigma := \pi \restriction H^V_{\bar\theta} : H^V_{\bar\theta} \to H^V_\theta .$$

Note that H^V_θ is closed under $<\bar\theta$-sequences (and more) in V. Now define

$$X := \bar\theta{+}1 \cup \{x \in H^V_\theta \mid \exists y \in H^V_{\bar\theta}\, \exists p \in \operatorname{Col}(\omega, H^V_{\bar\theta}) \colon p \Vdash \dot\sigma(\check y) = \check x\} \in V.$$

Note that $|X| = \left|H^V_{\bar\theta}\right| = 2^{<\bar\theta}$ and that $\operatorname{ran}\sigma \subseteq X$. Now let $\overline{\mathcal{M}} \prec H^V_\theta$ be such that $X \subseteq \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{\mathcal{M}}$ of size $(2^{<\bar\theta})^{<\bar\theta} = 2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{\mathcal{M}}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}| = 2^{<\bar\theta} < \bar\delta$, making $\mathcal{M} \in H^V_{\bar\delta}$.

Countable Embedding Absoluteness ?? then implies that $H^V_{\bar\delta}$ has a generic elementary embedding $\sigma^*\colon H^V_{\bar\theta} \to \mathcal{M}$ with $\operatorname{crit}\sigma^* = \bar\kappa$, showing that $\bar\kappa$ is virtually

[3]A cardinal satisfying any/all of these conditions is usually called **remarkable**.

[4]The domain of π is $H^V_{(2^{<\theta})^+}$ to ensure that $H^V_\theta \in \operatorname{dom}\pi$.

$\bar{\theta}$-supercompact in $H^V_{\bar{\delta}}$, which is what we wanted to show. ■

Remark 1.8. The above proof in fact shows something stronger: if κ is virtually $(2^{<\theta})^+$-strong then it is virtually θ-supercompact, and if it's virtually $(2^{<\theta})^+$-Magidor-supercompact then it's virtually θ-supercompact. It's open whether they are equivalent level-by-level (see Question ??).

As a corollary of the proof, we obtain the following weaker characterization of virtually strong cardinals.

Proposition 1.9. *A cardinal κ is virtually strong if and only if for every $\theta > \kappa$ in a forcing extension, there is a transitive set $\mathcal{N}$ and an elementary embedding $\pi : H_\theta \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$ and $H_\theta = H^{\mathcal{N}}_\theta$.* ■

A key difference between the normal large cardinals and the virtual kind is that we don't have a virtual version of the Kunen inconsistency[5]: it's perfectly valid to have a generic elementary embedding $H^V_\theta \to H^V_\theta$ with θ much larger than the critical point.

Proposition 1.10 (Folklore). *If $0^\sharp$ exists then there are inaccessible cardinals $\kappa < \theta$ such that, in a generic extension of L, there is an elementary embedding $\pi\colon L_\theta \to L_\theta$. In other words, π witnesses a strong failure of the virtualised Kunen inconsistency.*

Proof. From $0^\sharp$ we get an elementary embedding $j\colon L \to L$. Let $C \subseteq \mathsf{On}$ be the proper class club of limit points of j above $\operatorname{crit} j$, which then contains an inaccessible cardinal θ as there are stationarily many such. Restrict j to $\pi := j \restriction L_\theta \to \mathcal{N}$ and note that $\mathcal{N} = L_\theta$ by condensation and because θ is a limit point of j. Let $\kappa := \operatorname{crit}\pi$. Now an application of Countable Embedding Absoluteness ?? shows that a generic extension of L contains an elementary embedding $\tilde{\pi} : L_\theta \to L_\theta$ with $\operatorname{crit}\tilde{\pi} = \kappa$. ■

[5]See Appendix ?? for the Kunen inconsistency.

This becomes important when dealing with the "pre"-versions of the large cardinals. We next move to a virtualisation of the α-superstrong cardinals.

Definition 1.11. Let θ be a regular uncountable cardinal and and α an ordinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, α)**-superstrong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$[6]. We replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we say that κ is **faintly** α**-superstrong** if it's faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong.[7] ○

As in the non-virtual case, the virtually superstrongs supercede the virtually strongs in consistency strength. Note that this then also implies that the superstrongs are stronger than the virtually supercompacts, which is *not* the case outside the virtual world.

Proposition 1.12 (N.). *If κ is faintly superstrong then H_κ has a proper class of virtually strong cardinals.*

Proof. Fix a regular $\theta > \kappa$ and a generic embedding $\pi \colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit} \pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) < \theta$. Then $\pi(\kappa)$ is a V-cardinal, so that $H_{\pi(\kappa)}^V$ thinks that κ is virtually strong. This implies that H_κ^V thinks there is a proper class of virtually strong cardinals, using that $H_\kappa^V \prec H_{\pi(\kappa)}^V$. ■

The following theorem then shows that the only thing stopping prestrongness from being equivalent to strongness is the existence of "Kunen inconsistencies".

Theorem 1.13 (N.). *Let θ be an uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-prestrong iff either*

(i) κ is virtually θ-strong; or

(ii) κ is virtually (θ, ω)-superstrong.

Proof. ($\Leftarrow$) is trivial, so we show ($\Rightarrow$). Let κ be virtually θ-prestrong. Assume (i) fails, meaning that there is a generic elementary embedding $\pi \colon H_\theta \to \mathcal{N}$ for some transitive $\mathcal{N} \subseteq V$ with $H_\theta \subseteq \mathcal{N}$, $\operatorname{crit} \pi = \kappa$ and $\pi(\kappa) \leq \theta$.

[6] Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.

[7] Note that the conventions stated here are different from the ones in Definition 1.4.

First, assume that there is some $n < \omega$ such that $\pi^n(\kappa) = \theta$. The proof of Proposition 1.12 shows that κ is virtually strong in $H_{\pi(\kappa)}$. It follows that $\pi(\kappa)$ is virtually strong in $H_{\pi^2(\kappa)}$ by elementarity, and by applying elementarity repeatedly, we get that $\pi^n(\kappa) = \theta$ is virtually strong in $\mathcal{N}$. Note that the condition $\pi^n(\kappa) = \theta$ implies that θ is inaccessible in $\mathcal{N}$, and hence a limit cardinal there. In particular, θ is virtually $\delta = (\theta^+)^{\mathcal{N}}$-strong in $\mathcal{N}$, and so $\mathcal{N}$ has a generic elementary embedding $\sigma : H_\delta^{\mathcal{N}} \to \mathcal{M}$ with $\operatorname{crit}\sigma = \theta$ and $H_\theta^V \subseteq H_\delta^{\mathcal{N}} \subseteq \mathcal{M}$. Thus, $H_\theta^V \prec H_{\sigma(\theta)}^{\mathcal{M}}$, from which it follows that κ is virtually strong in $H_{\sigma(\theta)}^{\mathcal{M}}$, and, in particular, virtually θ-strong. But $H_{\sigma(\theta)}^{\mathcal{M}}$ must be correct about this since $H_\theta^{\mathcal{M}} = H_\theta^{\mathcal{N}} = H_\theta$. But then κ is actually virtually θ-strong, contradicting our assumption that (i) fails.

Next, assume that there is a least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$. Since $\pi(\kappa) \leq \theta$, we have as before that κ is virtually strong in $H_{\pi(\kappa)}$. Since, by elementarity, $H_{\pi^i(\kappa)} \prec H_{\pi^{i+1}(\kappa)}$, we have that κ is virtually strong in $H_{\pi^n(\kappa)}$. Applying elementarity to the statement that κ is virtually strong in $H_{\pi(\kappa)}$, we also get that $\pi^n(\kappa)$ is virtually strong in $H_{\pi^{n+1}(\kappa)}^{\mathcal{N}}$. This means that there is some generic elementary embedding $\sigma\colon H_\theta \to \mathcal{M}$ with $H_\theta \subseteq \mathcal{M}$, $\mathcal{M} \subseteq H_{\pi^{n+1}(\kappa)}^{\mathcal{N}}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. Thus, by elementarity, we get $H_{\pi^n(\kappa)} \prec H_{\sigma(\pi^n(\kappa))}^{\mathcal{M}}$. Since, as we already argued, κ is virtually strong in $H_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H_{\sigma(\pi^n(\kappa))}^{\mathcal{M}}$, and as $H_\theta^{\mathcal{M}} = H_\theta^{\mathcal{N}} = H_\theta$, this means that κ is actually virtually θ-strong, contradicting our assumption that (i) fails.

Finally, assume $\pi^n(\kappa) < \theta$ for all $n < \omega$ and let $\lambda = \sup_{n<\omega} \pi^n(\kappa)$. Since $\lambda \leq \theta$, we have that κ is virtually (θ, ω)-superstrong by definition. ■

We then get the following consistency result.

Corollary 1.14 (N.). *For any uncountable regular θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

Proof. The above Proposition 1.12 and Theorem 1.13 show that virtually θ-prestrongs are equiconsistent with virtually θ-strongs. Now note that Countable Embedding Absoluteness ?? and condensation in L imply that every faintly θ-measurable cardinal is virtually θ-prestrong in L. ■

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_\theta^V \to H_\theta^V$ with $\operatorname{crit}\pi = \kappa$. We firstly note that the virtually ω-superstrongs coincide with the virtually rank-into-ranks.

Proposition 1.15 (N.). *A regular uncountable cardinal κ is virtually ω-superstrong iff it is virtually rank-into-rank.*

Proof. If κ is virtually ω-superstrong, witnessed by a generic embedding $\pi\colon H_\theta^V \to \mathcal{N}$, then $\lambda := \sup_{n<\omega} \pi^n(\kappa)$ is well-defined. By restricting π to $\pi \restriction H_\lambda^V : H_\lambda^V \to H_\lambda^V$ we get a witness to κ being virtually λ-rank-into-rank.

Conversely, if κ is θ-rank-into-rank, witnessed by a generic embedding $\pi\colon H_\theta^V \to H_\theta^V$, then one readily checks that π also witnesses that κ is virtually ω-superstrong. ■

We then have the following corollary.

Corollary 1.16 (N.). *The following are equivalent.*

(i) For every regular uncountable cardinal θ, every virtually θ-prestrong cardinal is virtually θ-strong.

(ii) There are no virtually rank-into-rank cardinals.

Proof. ($\Leftarrow$): Note first that being virtually ω-superstrong is equivalent to being virtually rank-into-rank. Indeed, every virtually rank-into-rank cardinal is virtually ω-superstrong by definition, and if κ is virtually ω-superstrong, as witnessed by a generic elementary embedding $\pi : H_\theta \to \mathcal{M}$ with $\pi^\omega(\kappa) \leq \theta$, and we let $\lambda = \pi^\omega(\kappa)$, then restriction $\pi\colon H_\lambda \to H_\lambda$ witnesses that κ is virtually rank-into-rank. The above Theorem 1.13 then implies ($\Leftarrow$).

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal, then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let $\langle \kappa, \theta \rangle$ be the lexicographically least pair such that κ is virtually rank-into-rank as witnessed by a generic embedding $\pi : H_\theta \to H_\theta$, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong, then we would have a generic elementary embedding $\pi^* : H_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) > \theta$, and $\mathcal{M} \subseteq V$. By Countable Embedding Absoluteness ??, $\mathcal{M}$ sees that

κ virtually rank-into-rank, but then, using elementarity, this reflects down below κ, showing that the pair $\langle\kappa,\theta\rangle$ could not have been least. ■

1.2 Woodins & Vopěnkas

In this section we will analyse the virtualisations of the Woodin and Vopěnka cardinals, which can be seen as "boldface" variants of strongs and supercompacts.

Definition 1.17. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, A)**-strong** for a set $A \subseteq H_\theta^V$ if there exists a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, A) \to (\mathcal{M}, \in, B)$$

with $\mathcal{M}$ transitive, such that $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta^V \subseteq \mathcal{M}$ and $B \cap H_\theta^V = A$. We say that κ is **faintly** (θ, A)**-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$ and say that κ is **faintly** (θ, A)**-extendible** if $\mathcal{M} = H_\mu^V$ for some V-cardinal μ. We will leave out θ if it holds for all regular $\theta > \kappa$. ○

Definition 1.18. A cardinal δ is **faintly Woodin** if given any $A \subseteq H_\delta^V$ there exists a faintly $(<\delta, A)$-strong cardinal $\kappa < \delta$. ○

As with the previous definitions, for both of the above two definitions we substitute "faintly" for **virtually** when $\mathcal{M} \subseteq V$, and substitute "strong", "supercompact" and "Woodin" for **prestrong**, **presupercompact** and **pre-Woodin** when we don't require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with the real Woodin cardinals, virtually Woodin cardinals are Mahlo. This contrasts the virtually pre-Woodins since [?], together with Theorem 1.26 below, shows that they can be singular.

Proposition 1.19 (Virtualised folklore). *Virtually Woodin cardinals are Mahlo.*

Proof. Let δ be virtually Woodin. Note that δ is a limit of weakly compact cardinals by Proposition 1.5, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f\colon\alpha\to\delta$ with $f(0)>\alpha$ and $\alpha<\delta$, and note that f cannot have any closure points. Fix a virtually $({<}\delta,f)$-strong cardinal $\kappa<\delta$; we claim that κ is a closure point for f, which will yield our desired contradiction.

Let $\gamma<\kappa$ and choose a regular $\theta\in(f(\gamma),\delta)$. We then have a generic embedding $\pi\colon(H_\theta^V,\in,f\cap H_\theta^V)\to(\mathcal{N},\in,f^+)$ with $H_\theta^V\subseteq\mathcal{N}$, $\mathcal{N}\subseteq V$, $\operatorname{crit}\pi=\kappa$, $\pi(\kappa)>\theta$ and f^+ is a function such that $f^+\cap H_\theta^V=f\cap H_\theta^V$. But then $f^+(\gamma)=f(\gamma)<\pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma)<\kappa$, making κ a closure point for f, $\lightning$. This shows that δ is inaccessible.

As for Mahloness, let $C\subseteq\delta$ be a club and $\kappa<\delta$ a virtually $({<}\delta,C)$-strong cardinal. Let $\theta\in(\min C,\delta)$ and let $\pi\colon H_\theta^V\to\mathcal{N}$ be the associated generic elementary embedding. Then for every $\gamma<\kappa$ there exists an element of C below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an element of C. As κ is regular, this shows that δ is Mahlo. ■

The well-known equivalence of the "function definition" and "A-strong" definition of Woodin cardinals[8] holds if we restrict ourselves to *virtually* Woodins, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Proposition 1.20 (Dimopoulos-Gitman-N.). *For an uncountable cardinal δ, the following are equivalent.*

(i) δ is virtually Woodin.

(ii) for every $A\subseteq H_\delta^V$ there exists a virtually $({<}\delta,A)$-supercompact $\kappa<\delta$.

(iii) for every $A\subseteq H_\delta^V$ there exists a virtually $({<}\delta,A)$-extendible $\kappa<\delta$.

(iv) for every function $f\colon\delta\to\delta$ there are regular cardinals $\kappa<\theta<\delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V\to\mathcal{M}$ such that $\operatorname{crit}\pi=\kappa$, $H_\theta^V\subseteq\mathcal{M}$, $\mathcal{M}\subseteq V$ and $\theta=\pi(f\restriction\kappa)(\kappa)$.

(v) for every function $f\colon\delta\to\delta$ there are regular cardinals $\kappa<\theta<\delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V\to\mathcal{M}$ such that $\operatorname{crit}\pi=\kappa$, ${}^{<\pi(f)(\kappa)}\mathcal{M}\subseteq\mathcal{M}$, $\mathcal{M}\subseteq V$ and $\theta=\pi(f\restriction\kappa)(\kappa)$.

(vi) for every function $f\colon\delta\to\delta$ there are regular cardinals $\bar\theta<\kappa<\theta<\delta$, where κ is a closure point for f, and a generic elementary embedding $\pi\colon H_{\bar\theta}^V\to H_\theta^V$ with $\pi(\operatorname{crit}\pi)=\kappa$, $f(\operatorname{crit}\pi)=\bar\theta$ and $f\restriction\kappa\in\operatorname{ran}\pi$.

[8] See Appendix ?? for this characterisation of (non-virtual) Woodin cardinals.

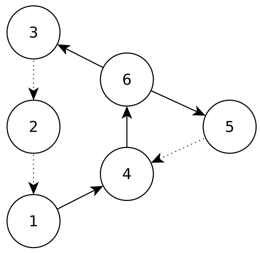

Figure 1.1: Proof strategy of Proposition 1.20, dotted lines are trivial implications.

Proof. Firstly note that $(iii) \Rightarrow (ii) \Rightarrow (i)$ and $(v) \Rightarrow (iv)$ are simply by definition.

$\boxed{(i) \Rightarrow (iv)}$ Assume δ is virtually Woodin, and fix a function $f\colon \delta \to \delta$. Let $\kappa < \delta$ be virtually $({<}\delta, f)$-strong and let $\theta < \delta$ be a regular cardinal such that $\sup_{\alpha\leq\kappa} f(\alpha) < \theta$. Then there is a generic elementary embedding $\pi\colon (H_\theta, \in, f \cap H_\theta) \to (\mathcal{M}, \in, f^+)$ such that $H_\theta \subseteq \mathcal{M}$, $f \cap H_\theta = f^+ \cap H_\theta$, $\mathcal{M} \subseteq V$, and $\pi(\kappa) > \theta$. Note that, by our choice of θ, $f \restriction \kappa \in H_\theta^V$ and $\pi(f \restriction \kappa)(\kappa) = f^+(\kappa) = f(\kappa) < \theta$.

So it suffices to show that κ is a closure point for f. Let $\alpha < \kappa$. Then

$$f(\alpha) = f^+(\alpha) = \pi(f \restriction \kappa)(\alpha) = \pi(f \restriction \kappa)(\pi(\alpha)) = \pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha < \kappa$. Now, if κ was not a closure point of f then, letting $\alpha < \kappa$ be the least such that $f(\alpha) \geq \kappa$, we have

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument would not work if we had only assumed δ to be virtually pre-Woodin.

$\boxed{(iv) \Rightarrow (vi)}$ Assume (iv) holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<\gamma_\alpha})^+$, where γ_α is the least regular cardinal above $|f(\alpha)|$. By (iv) there is a $\kappa < \delta$ which is a closure point of g (and so also a closure point of f), and there is a regular $\lambda \in (\kappa, \delta)$ for which there is a generic elementary embedding $\pi\colon H_\lambda \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$, $H_\lambda \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$, and $\pi(g \restriction \kappa)(\kappa) < \lambda$.

Let θ be the least regular cardinal above $|\pi(f \restriction \kappa)(\kappa)|$, and note that $H_\theta \in H_\lambda$ by our definition of g. Thus, both H_θ and $H^{\mathcal{M}}_{\pi(\theta)}$ are elements of $\mathcal{M}$. An application of Countable Embedding Absoluteness ?? then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H_\theta^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$, $\pi(f\restriction\kappa) \in \operatorname{ran}\pi^*$, and $\pi(f\restriction\kappa)(\kappa) < \theta$. By elementarity of π, H_θ has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta} \to H_\theta$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f\restriction\kappa \in \operatorname{ran}\sigma$ and $f(\operatorname{crit}\sigma) < \bar\theta$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta\to\delta$ be given. Define $g\colon\delta\to\delta$ as $g(\alpha) := \langle (2^{<\gamma_\alpha})^+, f(\alpha)\rangle$, where γ_α is the least regular cardinal above $|f(\alpha)|$. In particular, g codes f. By (vi) there exist regular $\bar\kappa < \bar\lambda < \kappa < \lambda$ such that κ is a closure point of g (so also a closure point of f) and there exists a generic elementary embedding $\pi\colon H_{\bar\lambda} \to H_\lambda$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) < \bar\lambda$, and $g\restriction\kappa \in \operatorname{ran}\pi$. Since f is definable from g and $g\restriction\kappa\in\operatorname{ran}\pi$, it follows that $f\restriction\kappa\in\operatorname{ran}\pi$. So let $\pi(\bar f) = f\restriction\kappa$ with $\bar f\colon\bar\kappa\to\bar\kappa$. Now observe that $\bar f = f\restriction\bar\kappa$ since for $\alpha<\bar\kappa$, we have $f(\alpha) = \pi(\bar f)(\alpha) = \pi(\bar f(\alpha)) = \bar f(\alpha)$.

Let $\bar\theta$ be the least regular cardinal above $|f(\bar\kappa)|$. By the definition of g, we have $H_{\bar\theta} \in H_{\bar\lambda}$. Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 1.7 we get that $H_{\bar\lambda}$ has a generic elementary embedding $\sigma\colon H_{\bar\theta}\to\mathcal{M}$ with $\mathcal{M}$ closed under $<\bar\theta$-sequences from V, $\operatorname{crit}\sigma = \bar\kappa$, $\sigma(\bar\kappa) > \bar\theta$, and $\sigma(\bar f)(\bar\kappa) < \bar\theta$. Let $\pi(\bar\theta) = \theta$ and $\pi(\mathcal{M}) = \mathcal{N}$. Now by elementarity of π, we get that there is a generic elementary embedding $\sigma^* : H_\theta \to \mathcal{N}$ with $\operatorname{crit}\sigma^* = \kappa$, $\sigma^*(\kappa) > \theta$, and $\sigma^*(\pi(\bar f))(\kappa) = \sigma^*(f\restriction\kappa)(\kappa) < \theta$.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all α such that

$$(H_\alpha, \in, A\cap H_\alpha) \prec (H_\delta, \in, A).$$

Let $f\colon\delta\to\delta$ be given as $f(\alpha) := \langle\gamma_0^\alpha, \gamma_1^\alpha\rangle$, where γ_0^α is the first limit point of C above α and the γ_1^α are chosen such that $\{\gamma_1^\alpha \mid \alpha<\beta\}$ encodes $A\cap\beta$. This definition makes sense since δ is inaccessible by Proposition 1.5.

Let $\kappa<\delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa < \theta$ and a generic elementary embedding $\pi\colon H_{\bar\theta}\to H_\theta$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) < \bar\theta$, and $f\restriction\kappa\in\operatorname{ran}\pi$. Let $\bar\kappa = \operatorname{crit}\pi$. We claim that $\bar\kappa$ is virtually $({<}\delta, A)$-extendible. Since $\kappa\in C$ because it is a closure point of f, it suffices by the definition of C to show that

$$(H_\kappa, \in, A\cap H_\kappa) \models \ulcorner\bar\kappa \text{ is virtually } (A\cap H_\kappa)\text{-extendible}\urcorner. \tag{1}$$

Let β be the least element of C above $\bar{\kappa}$ but below $\bar{\theta}$, and note that β exists as $f(\bar{\kappa}) < \bar{\theta}$, and the definition of f says that the first coordinate of $f(\bar{\kappa})$ is a limit point of C above $\bar{\kappa}$. It then holds that

$$(H_{\bar{\kappa}}, \in, A \cap H_{\bar{\kappa}}) \prec (H_\beta, \in, A \cap H_\beta)$$

as both $\bar{\kappa}$ and β are elements of C. Since f encodes A in the manner previously described and $\pi(f \upharpoonright \bar{\kappa}) = f \upharpoonright \kappa$, we get that $\pi(A \cap H_{\bar{\kappa}}) = A \cap H_\kappa$, and thus

$$(H_\kappa, \in, A \cap H_\kappa) \prec (H_{\pi(\beta)}, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H_\beta)$. Now, as $(H_\gamma, \in, A \cap H_\gamma)$ and $(H_{\pi(\gamma)}, \in, A^* \cap H_{\pi(\gamma)})$ are elements of $H_{\pi(\beta)}$ for every $\gamma < \kappa$, Countable Embedding Absoluteness ?? implies that $H_{\pi(\beta)}$ sees that $\bar{\kappa}$ is virtually $(<\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ■

As a corollary of the proof, we now have an analogue of Proposition 1.9 for virtually Woodin cardinals.

Proposition 1.21. *A cardinal δ is virtually Woodin if and only if, for every $A \subseteq H_\delta$, there is a cardinal κ satisfying the weakening of virtual $(<\delta, A)$-strongness where $H_\theta = H_\theta^{\mathcal{N}}$ holds in place of $\mathcal{N} \subseteq V$, with $\mathcal{N}$ being the target of the generic embedding.* ■

Proposition 1.22. *Faintly Woodin cardinals are virtually Woodin.*

Proof. Using Proposition 1.21, it suffices to observe that if $\pi : (H_\theta, \in, A) \to (\mathcal{M}, \in, B)$ is a faintly (θ, A)-strong embedding such that A codes the sequence of H_λ for $\lambda < \theta$, then $H_\theta = H_\theta^{\mathcal{M}}$. ■

We will now step away from the Woodins for a little bit, and introduce the Vopěnkas. In anticipation of the next section we will work with the class-sized version here, but all the following results work equally well for inaccessible virtually Vopěnka cardinals[9].

Definition 1.23 (GBC). The **Generic Vopěnka Principle** (gVP) states that for any class C consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$. ○

We will be using a standard variation of gVP involving the following *natural sequences*.

Definition 1.24 (GBC). Say that a class function $f\colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. ○

Definition 1.25 (GBC). Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ is **natural** if there exists an indexing function $f\colon \mathsf{On} \to \mathsf{On}$ and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$ for every α. Denote this indexing function by $f^{\vec{\mathcal{M}}}$ and the unary relations as $R^{\vec{\mathcal{M}}}_\alpha$. ○

The following Theorem 1.26 is then the main theorem of this section. Firstly it shows that inaccessible cardinals are virtually Vopěnka iff they are virtually pre-Woodin, but also that adding the "virtually" adverb doesn't do anything in this context, in contrast to Theorem 1.55.

Theorem 1.26 (GBC, Dimopoulos-Gitman-N.). *The following are equivalent:*

(i) *gVP holds;*

(ii) *For any natural On-sequence $\vec{\mathcal{M}}$ there exists a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for some $\alpha < \beta$;*

(iii) *On is virtually pre-Woodin;*

(iv) *On is faintly pre-Woodin.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume On is faintly pre-Woodin and fix some On-sequence $\vec{\mathcal{M}} := \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ of structures in a common language. Let κ be $(<\mathsf{On}, \vec{\mathcal{M}})$-

[9] Note however that we have to require inaccessibility here: see [?] for an analysis of the singular virtually Vopěnka cardinals.

prestrong and fix some regular $\theta > \kappa$ satisfying that $\mathcal{M}_\alpha \in H_\theta^V$ for every $\alpha < \theta$, and fix a generic elementary embedding

$$\pi \colon (H_\theta^V, \in, \vec{\mathcal{M}}) \to (\mathcal{N}, \in, \mathcal{M}^*)$$

with $H_\theta^V \subseteq \mathcal{N}$ and $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$. Set $\kappa := \operatorname{crit} \pi$.

We have that $\pi \restriction \mathcal{M}_\kappa \colon \mathcal{M}_\kappa \to \mathcal{M}^*_{\pi(\kappa)}$, but we need to reflect this embedding down below θ as we don't know whether $\mathcal{M}^*_{\pi(\kappa)}$ is on the $\vec{\mathcal{M}}$ sequence. Working in the generic extension, we have

$$\mathcal{N} \models \exists \bar{\kappa} < \pi(\kappa) \exists \dot{\sigma} \in V^{\mathrm{Col}(\omega, \mathcal{M}^*_{\bar{\kappa}})} \colon \ulcorner \dot{\sigma} \colon \mathcal{M}^*_{\bar{\kappa}} \to \mathcal{M}^*_{\pi(\kappa)} \text{ is elementary} \urcorner.$$

Here κ realises $\bar{\kappa}$ and $\pi \restriction \mathcal{M}_\kappa$ realises σ. Note that $\mathcal{M}^*_\kappa = \mathcal{M}_\kappa$ since we ensured that $\mathcal{M}_\kappa \in H_\theta^V$ and we are assuming that $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$, so the domain of σ ($= \pi \restriction \mathcal{M}_\kappa$) *is* $\mathcal{M}^*_\kappa$ – also note that σ exists in a $\mathrm{Col}(\omega, \mathcal{M}_\kappa)$ extension of $\mathcal{N}$ by an application of Countable Embedding Absoluteness ??. Now elementarity of π implies that

$$H_\theta^V \models \exists \bar{\kappa} < \kappa \exists \dot{\sigma} \in V^{\mathrm{Col}(\omega, \mathcal{M}_{\bar{\kappa}})} \colon \ulcorner \dot{\sigma} \colon \mathcal{M}_{\bar{\kappa}} \to \mathcal{M}_\kappa \text{ is elementary} \urcorner,$$

which is upwards absolute to V, from which we can conclude that $\sigma \colon \mathcal{M}_{\bar{\kappa}} \to \mathcal{M}_\kappa$ witnesses that gVP holds.

$(ii) \Rightarrow (iii)$: Assume (ii) holds and assume that On is not virtually pre-Woodin, which means that there exists some class A such that there are no virtually A-prestrong cardinals. This allows us to define a function $f \colon \mathsf{On} \to \mathsf{On}$ as $f(\alpha)$ being the least regular $\eta > \alpha$ such that α is not virtually (η, A)-prestrong.

We also define $g \colon \mathsf{On} \to \mathsf{On}$ as taking α to the least strong limit cardinal above α which is a closure point for f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A \cap H^V_{g(\alpha)}$. (ii) supplies us with $\alpha < \beta$ and a generic elementary embedding[10]

$$\pi \colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}).$$

[10]Note that $V_{g(\alpha)} = H^V_{g(\alpha)}$ since $g(\alpha)$ is a strong limit cardinal.

Since $g(\alpha)$ is a closure point for f it holds that $f(\operatorname{crit}\pi) < g(\alpha)$, so fixing a regular $\theta \in (f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-prestrong, contradicting the definition of f. Hence On is virtually pre-Woodin. ■

1.2.1 Weak Vopěnka

We now move to a *weak* variant of gVP, introduced in a category-theoretic context in [?]. It starts with the following equivalent characterisation of gVP, which is the virtual analogue of the characterisation shown in [?].

Lemma 1.27 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < On \rangle$ satisfying that*[11]

(i) gVP

(ii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < On \rangle$ satisfying that
- *there is a generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

(iii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < On \rangle$ satisfying that
- *there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

Proof. Note that the only difference between (ii) and (iii) is that the homomorphism exists in V, making $(ii) \Rightarrow (iii)$ trivial.

$(iii) \Rightarrow (i)$: Assume that gVP fails, meaning by Theorem 1.26 that we have a natural On-sequence $\vec{\mathcal{M}}_\alpha$ such that, in every generic extension, there's no homomorphism between any two disctinct $\mathcal{M}_\alpha$'s. Define an On-sequence $\langle \mathcal{N}_\kappa \mid \kappa \in \text{Card} \rangle$ as

$$\mathcal{N}_\kappa := \coprod_{\xi \leq \kappa} \mathcal{M}_\xi = \{(x, \xi) \mid \xi \leq \kappa \wedge \xi \in \text{Card} \wedge x \in \mathcal{M}_\xi\},$$ [12]

with a unary relation R^* given as $R^*(x, \xi)$ iff $\mathcal{M}_\xi \models R(x)$ and a binary relation $\sim^*$ given as $(x, \xi) \sim^* (x', \xi')$ iff $\xi = \xi'$. Whenever we have a homomorphism

[11] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [?].

$f\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ we then get an induced homomorphism $\tilde{f}\colon \mathcal{M}_0 \to \mathcal{M}_\xi$, given as $\tilde{f}(x) := f(x, 0)$, where $\xi \leq \kappa$ is given by preservation of $\sim^*$.

For any two cardinals $\kappa < \lambda$ we have a homomorphism $j_{\kappa\lambda}\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ in V, given as $j_{\kappa\lambda}(x, \xi) := (x, \xi)$. This embedding must also be the *unique* such embedding in all generic extensions, as otherwise we get a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s. Furthermore, there can't be any homomorphism $\mathcal{N}_\lambda \to \mathcal{N}_\kappa$ as that would also imply the existence of a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s.

$(i) \Rightarrow (ii)$: Assume that we have an On-sequence $\vec{\mathcal{M}}_\alpha$ as in the theorem, with generic homomorphisms $j_{\alpha\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ that are unique in all generic extensions for every $\alpha \leq \beta$, with no generic homomorphisms going the other way.

We first note that we can for every $\alpha \leq \beta$ choose the $j_{\alpha\beta}$ in a $\mathrm{Col}(\omega, \mathcal{M}_\alpha)$-extension, by a proof similar to the proof of Lemma ?? and using the uniqueness of $j_{\alpha\beta}$. Next, fix a proper class $C \subseteq \mathsf{On}$ such that $\alpha \in C$ implies that

$$\sup_{\xi \in C \cap \alpha} |\mathcal{M}_\xi|^V < |\mathcal{M}_\alpha|^V.$$

and note that this implies that $V[g] \models |\mathcal{M}_\xi| < |\mathcal{M}_\alpha|$ for every V-generic $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\xi)$. This means that for every $\alpha \in C$ we may choose some $\eta_\alpha \in \mathcal{M}_\alpha$ which is *not* in the range of any $j_{\xi\alpha}$ for $\xi < \alpha$. But now define first-order structures $\langle \mathcal{N}_\alpha \mid \alpha \in C \rangle$ as $\mathcal{N}_\alpha := (\mathcal{M}_\alpha, \eta_\alpha)$. Then, by our assumption on the $\mathcal{M}_\alpha$'s and construction of the $\mathcal{N}_\alpha$'s, there can be no generic homomorphism between any two distinct $\mathcal{N}_\alpha$, showing that gVP fails. ■

Note that the proof of the above lemma shows that we without loss of generality may assume that the generic homomorphism in (i) exists in V, which we record here:

Lemma 1.28 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ satisfying that*[13]

(i) there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;

[13] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [?].

(ii) there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$. ■

The *weak* version of gVP is then simply "flipping the arrows around" in the above characterisation of gVP.

Definition 1.29 (GBC). **Generic Weak Vopěnka's Principle** (gWVP) states that there does *not* exist an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ such that

- there is a generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\alpha \leq \beta$, which is unique in all generic extensions;
- there is *no* generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for any $\alpha < \beta$.

∘

We start by showing that gWVP is indeed a weaker version of gVP.

Proposition 1.30. *gVP implies gWVP.*

Proof. Assume gVP holds and gWVP fails, and let $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ be an On-sequence of first-order structures such that for every $\alpha \leq \beta$ there exists a generic homomorphism

$$j_{\beta\alpha} \colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$$

in some $V[g]$ which is unique in all generic extensions, with no generic homomorphisms going the other way. Here we may assume, as in the proof of Lemma 1.27, that $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\beta)$. We can then find a proper class $C \subseteq \mathsf{On}$ such that $|\mathcal{M}_\alpha|^V < |\mathcal{M}_\beta|^V$ for every $\alpha < \beta$ in C. By gVP there are then $\alpha < \beta$ in C and a generic homomorphism

$$\pi \colon \mathcal{M}_\alpha \to \mathcal{M}_\beta .$$

in some $V[h]$, where again we may assume that $h \subseteq \mathrm{Col}(\omega, \mathcal{M}_\alpha)$. But then $\pi \circ j_{\beta\alpha} = \mathrm{id}$ by uniqueness of $j_{\beta\beta} = \mathrm{id}$, which means that $j_{\beta\alpha}$ is injective in $V[g \times h]$ and hence also in $V[g]$. But then $|\mathcal{M}_\beta|^{V[g]} \leq |\mathcal{M}_\alpha|^{V[g]}$, which implies that $|\mathcal{M}_\beta|^V \leq |\mathcal{M}_\alpha|^V$ by the $|\mathcal{M}_\beta|^{+V}$-cc of $\mathrm{Col}(\omega, \mathcal{M}_\beta)$, contradicting the definition of C. ■

Denoting the corresponding non-generic principle by WVP [?] showed the following.

Theorem 1.31 (Wilson). *WVP is equivalent to On being a Woodin cardinal.*

Given our 1.26 we may then suspect that in the virtual world these two are equivalent, which turns out to *almost* be the case. We will be roughly following the argument in [?], but we have to diverge from it at several points in which they're using the fact that they're working with class-sized elementary embeddings.

Indeed, in that paper they establish a correspondence between elementary embeddings and certain homomorphisms, a correspondence we won't achieve here. Proving that the elementary embeddings we *do* get are non-trivial seems to furthermore require extra assumptions on our structures. Let's begin.

Define for every strong limit cardinal λ and Σ_1-formula φ the relations

$$R^\varphi := \{x \in V \mid (V, \in) \models \varphi[x]\}$$
$$R^\varphi_\lambda := \{x \subseteq H^V_\lambda \mid \exists y \in R^\varphi \colon y \cap H^V_\lambda = x\}$$

and given any class A define the structure

$$\mathscr{P}_{\lambda,A} := (H^V_{\lambda^+}, R^\varphi_\lambda, \{\lambda\}, A \cap H^V_\lambda)_{\varphi \in \Sigma_1}.$$

Say that a homomorphism $h \colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ is **trivial** if $h(x) \cap H^V_\eta = x \cap H^V_\eta$ for every $x \in H^V_{\lambda^+}$. Note that h can only be trivial if $\eta \leq \lambda$ since $h(\lambda) = \eta$.

Lemma 1.32 (GBC, Gitman-N.). *Let λ be a singular strong limit cardinal, η a strong limit cardinal and $A \subseteq V$ a class. If there exists a non-trivial generic homomorphism $h \colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ then there's a non-trivial generic elementary embedding*

$$\pi \colon (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$$

for some $\mathcal{M}$ such that, letting $\nu := \min\{\lambda, \eta\}$, it holds that $H^V_\nu = H^{\mathcal{M}}_\nu$, $A \cap H^V_\nu = B \cap H^V_\nu$ and $\operatorname{crit} \pi < \nu$.

Proof. Assume that we have a non-trivial homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ in a forcing extension $V[g]$, define in $V[g]$ the set

$$\mathcal{M}^* := \{\langle b, f\rangle \mid b \in [H_\nu]^{<\omega} \wedge f \in H^V_{\lambda^+} \wedge f\colon H^V_\lambda \to H^V_\lambda\},$$

and define the standard relations $\in^*$ and $=^*$ on $\mathcal{M}^*$ as

$$\begin{aligned}\langle b_0, f_0\rangle \in^* \langle b_1, f_1\rangle \quad &\text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) \in f_1(y)\})\\ \langle b_0, f_0\rangle =^* \langle b_1, f_1\rangle \quad &\text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) = f_1(y)\})\end{aligned}$$

Let $\mathcal{M} := \mathcal{M}^* / =^*$, and also call $\in^*$ the induced relation on $\mathcal{M}$, which is clearly well-defined. We then get a version of Łoś' Theorem, using that h preserves all Σ_1-relations and that $H^V_\lambda \models \mathsf{ZFC}^-$.

Claim 1.33. For every formula $\varphi(v_1, \dots, v_n)$ and every $[b_1, f_1], \dots, [b_n, f_n] \in \mathcal{M}$ the following are equivalent:

(i) $(\mathcal{M}, \in^*) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]$;

(ii) $b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\})$.

Proof of claim. The proof is straightforward, using that h preserves Σ_1-relations. We prove this by induction on φ. If φ is $v_i \in v_j$ then we have that

$$\begin{aligned}&(\mathcal{M}, \in^*) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]\\ \Leftrightarrow &\langle b_i, f_i\rangle \in^* \langle b_j, f_j\rangle\\ \Leftrightarrow &b_i b_j \in h(\{a_i a_j \in [H^V_\lambda]^{<\omega} \mid f_i(a_i) \in f_j(a_j)\})\\ \Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid f_i(a_i) \in f_j(a_j)\})\\ \Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).\end{aligned}$$

The cases where φ is $\psi \wedge \chi$ or $\neg\psi$ is straightforward. If φ is $\exists x\psi$ then

$$\begin{aligned}&(\mathcal{M}, \in^*) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]\\ \Leftrightarrow &\exists \langle b, f\rangle \in \mathcal{M}^*\colon (\mathcal{M}, \in) \models \psi[\langle b, f\rangle, \langle b_1, f_1\rangle, \dots, \langle b_n, f_n\rangle]\\ \Leftrightarrow &\exists \langle b, f\rangle \in \mathcal{M}^*\colon bb_1 \cdots b_n \in h(\{aa_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f(a), f_1(a_1), \dots, f_n(a_n)]\})\\ \Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}),\end{aligned}$$

finishing the proof. $\dashv$

Note that we haven't shown that $(\mathcal{M}, \in^*)$ is wellfounded, and indeed it might not be. However, the following claim will show that $(H_\nu^V, \in)$ is isomorphic to a rank-initial segment of $(\mathcal{M}^*, \in^*)$, giving wellfoundedness up to that point at least. Define the function $\chi\colon (H_\nu^V, \in) \to (\mathcal{M}^*, \in^*)$ as $\chi(a) := [\langle a\rangle, \mathrm{pr}]$, where $\mathrm{pr}(\langle x\rangle) := x$.

Claim 1.34. For every $[a, f] \in \mathcal{M}$ and $b \in H_\nu^V$,

$$[a, f] \in^* \chi(b) \quad\Leftrightarrow\quad \exists c \in H_\nu^V : [a, f] = \chi(c).$$

Proof of claim. We have that

$$\begin{aligned}
[a, f] \in^* \chi(b) = [\langle b\rangle, \mathrm{pr}] &\Leftrightarrow a\langle b\rangle \in h(\{x\langle y\rangle \mid f(x) \in y\}) \\
&\Leftrightarrow a\langle b\rangle \in h(\{x\langle y\rangle \mid \exists z \in y\colon f(x) = z\}) \\
&\Leftrightarrow \exists c \in b\colon a\langle c\rangle \in h(\{x\langle z\rangle \mid f(x) = z\}) \\
&\Leftrightarrow \exists c \in b\colon [a, f] = [\langle c\rangle, \mathrm{pr}] = \chi(c),
\end{aligned}$$

yielding the wanted. $\dashv$

This claim implies that by taking the transitive collapse of $\operatorname{ran}\chi \subseteq \mathcal{M}$ we may assume that $H_\nu^V = H_\nu^{\mathcal{M}}$. Now define

$$B := \{[b, f] \in \mathcal{M} \mid b \in h(\{x \in H_\lambda^V \mid f(x) \in A\})\}.$$

and, in $V[g]$, let $\pi\colon (H_\lambda^V, \in, A \cap H_\lambda^V) \to (\mathcal{M}, \in, B)$ be given as $\pi(x) := [\langle\rangle, c_x]$.

Claim 1.35. π is elementary.

Proof of claim. For $x_1, \dots, x_n \in H_\lambda^V$ it holds that

$$\begin{aligned}
(\mathcal{M}, \in^*, B) \models \varphi[\pi(x_1), \dots, \pi(x_n)] &\Leftrightarrow\ (\mathcal{M}, \in^*) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\
&\Leftrightarrow\ \langle\rangle \in h(\{\langle\rangle \mid \mathscr{P}_{\lambda,A} \models \varphi[x_1, \dots, x_n]\}) \\
&\Leftrightarrow\ (H_{\lambda^+}^V, \in, A \cap H_\lambda^V) \models \varphi[x_1, \dots, x_n]
\end{aligned}$$

and we also get that, for every $x \in H^V_\lambda$,

$$x \in A \Leftrightarrow \langle\rangle \in h(\{a \in H^V_\lambda \mid x \in A\}) \Leftrightarrow \pi(x) \in B,$$

which shows elementarity. $\dashv$

We next need to show that $B \cap H^V_\nu = A \cap H^V_\nu$, so let $x \in H^V_\nu$. Note that $x = [\langle x\rangle, \mathrm{pr}]$ by Claim 1.34 and the observation proceeding it, which means that

$$x \in B \Leftrightarrow \langle x\rangle \in h(\{\langle y\rangle \in H^V_\lambda \mid y \in A\}) \Leftrightarrow x \in A.$$

The last thing we need to show is that $\operatorname{crit}\pi < \nu$. We start with an analogous result about h.

Claim 1.36. There exists some $b \in H^V_\nu$ such that $h(b) \neq b$.

Proof of claim. Assume the claim fails. We now have two cases.

$\boxed{\textbf{Case 1: } \lambda \geq \eta}$

By non-triviality of h there's an $x \in \mathscr{P}^V(H^V_\lambda)$ such that $h(x) \neq x \cap H^V_\eta$, which means that there exists an $a \in H^V_\eta$ such that $a \in h(x) \Leftrightarrow a \notin x$.

If $a \in x$ then $\{a\} = h(\{a\}) \subseteq h(x)$,[14] making $a \in h(x)$, $\lightning$, so assume instead that $a \in h(x)$. Since η is a strong limit cardinal we may fix a cardinal $\theta < \eta$ such that $a \in H^V_\theta$ and $H^V_\theta \in H^V_\eta$. We then have that[15]

$$\{a\} \subseteq h(x) \cap H^V_\theta = h(x) \cap h(H^V_\theta) = h(x \cap H^V_\theta) = x \cap H^V_\theta,$$

so that $a \in x$, $\lightning$.

$\boxed{\textbf{Case 2: } \lambda < \eta}$

In this case we are assuming that $h \restriction H^V_\lambda = \mathrm{id}$, but $h(\lambda) = \eta > \lambda$. Since λ is singular we can fix some $\gamma < \lambda$ and a cofinal function $f\colon \gamma \to \lambda$. Define the

[14] Note that as h preserves Σ_1 formulas it also preserves singletons and boolean operations.
[15] Note that we're using $\lambda \geq \eta$ here to ensure that $H^V_\theta \in \operatorname{dom} h$.

relation

$$R = \{(\alpha, \beta, \bar{\alpha}, \bar{\beta}, g) \mid \ulcorner g \text{ is a cofinal function } g\colon \alpha \to \beta \urcorner \wedge g(\bar{\alpha}) = \bar{\beta}\}.$$

Then $R(\gamma, \lambda, \alpha, f(\alpha), f)$ holds by assumption for every $\alpha < \gamma$, so that R holds for some $(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*)$ such that

$$\begin{aligned}(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*) \cap H_\eta^V &= (h(\gamma), h(\lambda), h(\alpha), h(f(\alpha)), h(f)) \\ &= (\gamma, \eta, \alpha, f(\alpha), h(f)),\end{aligned}$$

using our assumption that h fixes every $b \in H_\lambda^V$. Since γ, α and $f(\alpha)$ are transitive and bounded in H_λ^V it holds that $h(\gamma) = \gamma^*$, $h(\alpha) = \alpha^*$ and $h(f(\alpha)) = f(\alpha)^*$. Also, since $\operatorname{dom}(f^*) = \gamma = \operatorname{dom}(f)$ we must in fact have that $f^* = h(f)$. But this means that $h(f)\colon \gamma \to \eta$ is cofinal and $\operatorname{ran}(h(f)) \subseteq \lambda$, a contradiction! ⊣

To use the above Claim 1.36 to conclude anything about π we'll make use of the following standard lemma.

Claim 1.37. For any $x \in H_\lambda^V$ it holds that $h(x) = \pi(x) \cap H_\eta^V$.

Proof of claim. For any $n < \omega$ and $\langle a_1, \dots, a_n\rangle \in [H_\eta^V]^n$ we have that

$$\begin{aligned}&\langle a_1, \dots, a_n\rangle \in \pi(x) \\ \Leftrightarrow\ &(\mathcal{M}, \in) \models \langle a_1, \dots, a_n\rangle \in \pi(x) \\ \Leftrightarrow\ &(\mathcal{M}, \in) \models \langle [\langle a_1\rangle, \mathrm{pr}], \dots, [\langle a_n\rangle, \mathrm{pr}]\rangle \in [\langle\rangle, c_x] \\ \Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(\{\langle x_1, \dots, x_n\rangle \mid \mathscr{P}_{\lambda,A} \models \langle x_1, \dots, x_n\rangle \in x\}) \\ \Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(x),\end{aligned}$$

showing that $h(x) = \pi(x) \cap H_\eta^V$. ⊣

Now use Claim 1.36 to fix a $b \in H^V_\nu$ which is moved by h. Claim 1.37 then implies that

$$\pi(b) \cap H^V_\eta = h(b) \cap H^V_\eta = h(b) \neq b = b \cap H^V_\eta,$$

showing that $\pi(b) \neq b$ and hence $\operatorname{crit} \pi < \nu$. This finishes the proof of the lemma. ■

Theorem 1.38 (GBC, Gitman-N.). *gWVP holds iff On is W-virtually pre-Woodin.*

Proof. ($\Leftarrow$) is just observing that the virtualisation of the argument in [?] that WVP holds if On is Woodin works in the W-virtually pre-Woodin case, so we only give a brief sketch.

Assume On is W-virtually pre-Woodin and let $\vec{\mathcal{M}}$ be a counterexample to gWVP, so that we in V we have homomorphisms $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for all $\alpha \leq \beta$. Work in some generic extension $V[g]$, fix a W-virtually $\vec{\mathcal{M}}$-prestrong cardinal κ and let $\theta \gg \kappa$ be such that $\mathcal{M}_{\kappa+1} \in H^V_\theta$. Letting $\pi\colon (H^V_\theta, \in) \to (\mathcal{M}, \in^*)$ be the corresponding embedding we get that $\mathcal{M}_{\kappa+1} = \pi(\vec{\mathcal{M}})_{\kappa+1}$, so that

$$\pi \restriction \mathcal{M}_\kappa \colon (\mathcal{M}_\kappa, \in) \to (\pi(\mathcal{M}_\kappa), \in^*) = (\pi(\vec{\mathcal{M}})_{\pi(\kappa)}, \in^*).$$

But then the choice of θ and elementarity of π we get that $\mathcal{M}$ has a homomorphism

$$h\colon (\pi(\vec{\mathcal{M}})_{\pi(\kappa)}, \in^*) \to (\pi(\vec{\mathcal{M}})_{\kappa+1}, \in^*) = (\mathcal{M}_{\kappa+1}, \in),$$

making $h \circ (\pi \restriction \mathcal{M}_\kappa)\colon (\mathcal{M}_\kappa, \in) \to (\mathcal{M}_{\kappa+1}, \in)$ a counterexample to gWVP.

($\Rightarrow$): Assume that On is not W-virtually pre-Woodin. This means that there exists a class A such that there are no W-virtually A-prestrong cardinals. We can therefore assign to any cardinal κ the least cardinal $f(\kappa) > \kappa$ such that κ is not W-virtually $(f(\kappa), A)$-prestrong.

Also define a function $g\colon \mathsf{On} \to \mathrm{Card}$ as taking an ordinal α to the least singular strong limit cardinal above α closed under f. Then we're assuming that there's no non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

with $H^V_{g(\alpha)} \subseteq \mathcal{M}$ and $B \cap H^V_{g(\alpha)} = A \cap H^V_{g(\alpha)}$. Assume towards a contradiction that for some α, β there is a non-trivial generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$. Lemma 1.32 then gives us a non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that $H^V_\nu \subseteq \mathcal{M}$ with $\nu := \min\{g(\alpha), g(\beta)\}$ and $A \cap H^V_\nu = B \cap H^V_\nu$, a contradiction! Therefore every generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$ is trivial. Since there is a unique trivial homomorphism when $\alpha \geq \beta$ and no trivial homomorphism when $\alpha < \beta$ since $g(\alpha)$ is sent to $g(\beta)$, the sequence of structures

$$\langle \mathscr{P}_{g(\alpha),A} \mid \alpha \in \mathsf{On} \rangle$$

is a counterexample to gWVP, which is what we wanted to show. ■

1.3 Berkeleys

We next move to the higher realms of the virtual large cardinal hierarchy, and study cardinals whose non-virtual versions are inconsistent with ZFC.

In the virtual setting the virtually Berkeley cardinals, like all the other virtual large cardinals, are simply downwards absolute to L. It turns out that virtually Berkeley cardinals are natural objects, as the main theorem of this section, Theorem 1.46, shows that these large cardinals are precisely what separates virtually pre-Woodins from the virtually Woodins, as well as separating virtually Vopěnka cardinals from Mahlo cardinals.

Definition 1.39. Say that a cardinal δ is **virtually proto-Berkeley** if for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} \pi < \delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ then δ is **virtually Berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C \subseteq \delta$ we say δ is **virtually club Berkeley**. ◦

Virtually (proto-)Berkeley cardinals turn out to be equivalent to their "boldface" versions, the proof of which is a straightforward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [?].

Proposition 1.40 (Virtualised Cutolo). *If δ is virtually proto-Berkeley then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\pi < \delta$. If δ is virtually Berkeley then we can furthermore ensure that $\operatorname{crit}\pi$ is arbitrarily large below δ.*

Proof. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{\{\langle A, x\rangle \mid x \in \mathcal{M}\}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: a is the first element in the pairs belonging to the set of highest rank, and $\mathcal{M}$ is what remains if we remove the set with the highest rank. But this means that a generic elementary embedding $\pi\colon \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and a, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the wanted conclusion. ■

The following is a straightforward virtualisation of the usual definition of the Vopěnka filter (see e.g. [?]).

Definition 1.41 (GBC). Define the **virtually Vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$. ◦

Theorem 1.26 shows that $\emptyset$ is in the virtually Vopěnka filter iff gVP fails, in analogy with the non-virtual case. Normality also holds in the virtual context, as the following proof shows.

Lemma 1.42 (GBC, Virtualised folklore). *The virtually Vopěnka filter is a normal filter.*

Proof. Let F be the virtually Vopěnka filter. We first show that F is actually a filter. If $X \in F$ and $Y \supseteq X$ then $Y \in F$ simply by definition of F. If $X, Y \in F$, witnessed by natural sequences $\vec{\mathcal{M}}$ and $\vec{\mathcal{N}}$, then $X \cap Y \in F$ as well, witnessed by the natural sequence $\vec{\mathcal{P}}$ induced by the indexing function $f^{\vec{\mathcal{P}}} := \max(f^{\vec{\mathcal{M}}}, f^{\vec{\mathcal{N}}})$ and unary relations $R^{\vec{\mathcal{P}}}_\alpha := \text{Code}(\langle R^{\vec{\mathcal{M}}}_\alpha, R^{\vec{\mathcal{N}}}_\alpha\rangle)$. Indeed, if $\pi\colon \mathcal{P}_\alpha \to \mathcal{P}_\beta$ is a generic elementary embedding with critical point μ then μ is also the critical point of both $\pi \upharpoonright \mathcal{M}_\alpha\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ and $\pi \upharpoonright \mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$.

For normality, let $X \in F^+$ be F-positive, where we recall that this means that $X \cap C \neq \emptyset$ for every $C \in F$, and let $f\colon X \to \mathsf{On}$ be regressive. We want to show that f is constant on an F-positive set.

Assume this fails, meaning that there are natural sequences $\vec{\mathcal{M}}^\gamma$ for γ such that for any generic elementary $\pi\colon \mathcal{M}^\gamma_\alpha \to \mathcal{M}^\gamma_\beta$ satisfies that $f(\text{crit}\,\pi) \neq \gamma$. Define a new natural sequence $\vec{\mathcal{N}}$ as induced by the indexing function $g\colon \mathsf{On} \to \mathsf{On}$ given as $g(\alpha) := \sup_{\gamma<\alpha} \text{rk}\,\mathcal{M}^\gamma_\alpha + \omega$ and unary relations $R^{\vec{\mathcal{N}}}_\alpha$ given as

$$R^{\vec{\mathcal{N}}}_\alpha := \text{Code}(\langle\langle \mathcal{M}^\gamma_\alpha \mid \gamma < \alpha\rangle, f \upharpoonright \alpha\rangle).$$

Now since X is F-positive there exists a generic elementary embedding $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ with $\text{crit}\,\pi \in X$. As $f(\text{crit}\,\pi) < \text{crit}\,\pi$ we get that $\pi(f(\text{crit}\,\pi)) = f(\text{crit}\,\pi)$, so that we have a generic elementary embedding

$$\pi \upharpoonright \mathcal{M}^{f(\text{crit}\,\pi)}_\alpha\colon \mathcal{M}^{f(\text{crit}\,\pi)}_\alpha \to \mathcal{M}^{f(\text{crit}\,\pi)}_\beta,$$

but this contradicts the definition of $\vec{\mathcal{M}}^{f(\text{crit}\,\pi)}$! Thus F is normal. ■

The reason why we are being careful in showing all these analogous properties for the virtual Vopěnka filter is that not all properties carry over. Indeed, note that uniformity of filters is non-trivial as we're working with proper classes[16], and we will see in Theorem 1.46 shows that uniformity of this filter is equivalent to there being no virtually Berkeley cardinals – the following lemma is the first implication.

[16] This boils down to the fact that the class club filter is not provably normal in GBC, see [?]

Lemma 1.43 (GBC, N.). *Assume gVP and that there are no virtually Berkeley cardinals. Then the virtually Vopěnka filter F on On contains every class club C.*

Proof. The crucial extra property we get by assuming that there aren't any virtually Berkeleys is that F becomes uniform, i.e. contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin F$. Let M be a transitive set with $\delta \subseteq M$ and $\gamma < \delta$ a cardinal. As $(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function $f\colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\alpha + 1, \delta + 1)$$

and unary relations $R_\alpha := \langle M, \mathcal{N}_\alpha \rangle$. If $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit} \pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$, then $\pi(R_\alpha) = R_\beta$ implies that $\pi \restriction \mathcal{M}\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} \pi \leq \delta$. We also get that $\operatorname{crit} \pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit} \pi}\colon \mathcal{N}_{\operatorname{crit} \pi} \to \mathcal{N}_{\pi(\operatorname{crit} \pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and hence $\operatorname{crit} \pi > \gamma$ as $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually Berkeley, a contradiction. Thus $\operatorname{crit} \pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$.

Note that the class $C_0 \subseteq \mathsf{On}$ of limit ordinals is in F, since it's the diagonal intersection of the tails $(\alpha + 1, \mathsf{On})$. Now let $C \subseteq \mathsf{On}$ be a class club, and let $C = \{a_\alpha \mid \alpha < \mathsf{On}\}$ be its increasing enumeration. Then $C \supseteq C_0 \cap \triangle_{\alpha < \mathsf{On}} (a_\alpha, \mathsf{On})$, implying that $C \in F$. ■

Theorem 1.44 (GBC, N.). *If there are no virtually Berkeley cardinals then On is virtually pre-Woodin iff On is virtually Woodin.*

Proof. Assume On is virtually pre-Woodin, so gVP holds by Theorem 1.26 and we can let F be the virtually Vopěnka filter. The assumption that there aren't any virtually Berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails – we will follow the proof of Theorem 1.26.

Failure means that there is some class A and some class club C such that there are no virtually A-prestrong cardinals in C. Since there are no virtually Berkeley cardinals, Lemma 1.43 imples that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as sending α to the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-prestrong if $\alpha \in C$, and set $f(\alpha) := \alpha$ if $\alpha \notin C$. Also define $g\colon \mathsf{On} \to \mathsf{On}$ as $g(\alpha)$ being the least strong limit cardinal in C above α which is a closure point for f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := \mathrm{Code}(\langle A \cap H^V_{g(\alpha)}, \mathcal{N}_\alpha\rangle)$ and apply gVP to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi\restriction(H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)})\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}),$$

making $\operatorname{crit}\pi$ virtually $(g(\alpha), A)$-prestrong and thus $\operatorname{crit}\pi \notin C$. But as we also get the embedding $\pi\restriction\mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\operatorname{crit}\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \mathrm{Card} \mid (H^V_\kappa, \in, A \cap H^V_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-prestrong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it isn't virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, A \cap H^V_\theta) \to (M, \in, B)$$

with $\operatorname{crit}\pi = \kappa$, $H^V_\theta \subseteq M$, $M \subseteq V$, $A \cap H^V_\theta = B \cap H^V_\theta$ and $\pi(\kappa) < \theta$.

Now $\pi(\kappa)$ is inaccessible, and $(H^V_{\pi(\kappa)}, \in, A \cap H^V_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$ believes that κ is virtually $(A \cap H^V_{\pi(\kappa)})$-strong as in the proof of Theorem 1.13, meaning that $(H^V_\kappa, \in, A \cap H^V_\kappa)$ believes that there is a proper class of virtually $(A \cap H^V_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually Woodin. ■

Theorem 1.45 (GBC, N.). *If there exists a virtually Berkeley cardinal δ then gVP holds and On is not Mahlo.*

Proof. If On was Mahlo then there would in particular exist an inaccessible cardinal $\kappa > \delta$, but then $H_\kappa^V \models \ulcorner$there exists a virtually berkeley cardinal$\urcorner$, contradicting the incompleteness theorem.

To show gVP we show that On is virtually pre-Woodin, which is equivalent by Theorem 1.26. Fix therefore a class A – we have to show that there exists a virtually A-prestrong cardinal. For every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta\colon (H_\theta^V, \in, A \cap H_\theta^V) \to (H_\theta^V, \in, A \cap H_\theta^V)$$

with $\operatorname{crit}\pi < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-prestrong, making On virtually pre-Woodin. ■

Theorem 1.46 (GBC, N.). *The following are equivalent:*

(i) gVP implies that On is Mahlo;

(ii) On is virtually pre-Woodin iff On is virtually Woodin;

(iii) There are no virtually Berkeley cardinals.

Proof. $(iii) \Rightarrow (ii)$ is Theorem 1.44, and the contraposed version of $(i) \Rightarrow (iii)$ is Theorem 1.45. For $(ii) \Rightarrow (i)$ note that gVP implies that On is virtually pre-Woodin by Theorem 1.26, which by (ii) means that it's virtually Woodin and the usual proof shows that virtually Woodins are Mahlo[17], showing (i). ■

This also immediately implies the following equiconsistency, as virtually Berkeley cardinals have strictly larger consistency strength than virtually Woodin cardinals.

[17]See e.g. Exercise 26.10 in [?].

Corollary 1.47 (N.). *The existence of an inaccessible virtually pre-Woodin cardinal is equiconsistent with the existence of an inaccessible virtually Woodin cardinal.* ■

1.4 Behaviour in core models

Most of the cardinals turn out to downwards absolute to most inner models, including L:

Proposition 1.48. *For any regular uncountable cardinal θ, faintly θ-measurable cardinals are downwards absolute to any transitive class $\mathcal{U} \subseteq V$ satisfying $\mathsf{ZF}^- + \mathsf{DC}$.*

Proof. Let κ be faintly θ-measurable, witnessed by a forcing poset $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there's a transitive $\mathcal{M}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a transitive class $\mathcal{U} \subseteq V$ which satisfies $\mathsf{ZF}^- + \mathsf{DC}$. Restricting the embedding to $\pi \restriction H_\theta^{\mathcal{U}}\colon H_\theta^{\mathcal{U}} \to \mathcal{N}$ we can now apply the Countable Absoluteness Lemma ?? to $\pi \restriction H_\theta^{\mathcal{U}}$ to get that there exists an embedding $\pi^*\colon H_\theta^{\mathcal{U}} \to \mathcal{N}^*$ in a generic extension of U, making κ faintly θ-measurable in $\mathcal{U}$. ■

Theorem 1.49 (N.). *Let θ be a regular uncountable cardinal.*

(i) $L \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-prestrongs*$\urcorner$.

(ii) *Assume that $L[\mu]$ exists. It then holds that*
$L[\mu] \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-measurables*$\urcorner$.

(iii) *Assume there is no inner model with a Woodin. It then holds that*
$K \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-measurables*$\urcorner$.

Proof. For (i) simply note that if $\pi\colon L_\theta \to \mathcal{N}$ is a generic elementary embedding with $\mathcal{N}$ transitive, then by condensation we have that $\mathcal{N} = L_\gamma$ for some $\gamma \geq \theta$, so that π also witnesses the virtual θ-prestrongness of $\operatorname{crit}\pi$.

(ii): Assume that $V = L[\mu]$ for notational simplicity and let κ be faintly θ-measurable, witnessed by a generic elementary embedding $\pi\colon L_\theta[\mu] \to \mathcal{N}$ existing in some generic extension $V[g]$. By condensation we get that $\mathcal{N} = L_\gamma[\overline{\mu}]$ for some

$\gamma \geq \theta$ and $\overline{\mu} \in V[g]$, but we're not guaranteed that $\overline{\mu} \in V$ here. Let λ be the unique measurable cardinal of $V = L[\mu]$.

Note that $\bar{\mu}$ is a measure on $\pi(\lambda) \geq \lambda$. If $\pi(\lambda) = \lambda$ then $L[\mu] = L[\bar{\mu}]$ by [?, Theorem 20.10] and we trivially get that $\mathcal{N} \subseteq V$. Assume thus that $\pi(\lambda) > \lambda$, which implies that $L[\bar{\mu}]$ is an internal iterate of $L[\mu]$ by [?, Theorem 20.12]. In particular it then holds that $L[\bar{\mu}] \subseteq L[\mu]$, so again we get that $\mathcal{N} \subseteq V$.

(iii): Assume that $V = K = L[\mathcal{E}]$ and fix a faintly θ-measurable cardinal κ, witnessed by a generic embedding $\pi\colon L_\theta[\mathcal{E}] \to \mathcal{N} = L_\gamma[\overline{\mathcal{E}}]$ in some generic extension $V[g]$. Now coiterate $L[\mathcal{E}]$ with $L[\overline{\mathcal{E}}]$, and denote the last models by $\mathcal{P}$ and $\mathcal{Q}$. Since $K = K^{V[g]}$ and as K is universal we get that $\mathcal{Q} \trianglelefteq \mathcal{P}$. Then the $L[\overline{\mathcal{E}}]$-to-$\mathcal{Q}$ branch did not drop, giving us an iteration embedding $i\colon L[\overline{\mathcal{E}}] \to \mathcal{Q}$.

Note that $\operatorname{crit} i \geq \kappa$ as $\overline{\mathcal{E}}$ is simply the pointwise image of $\mathcal{E}$ under π, so nothing below κ is touched and is therefore not used in the comparison either. This means that $\operatorname{crit}(i \circ \pi) = \kappa$, so that $(i \circ \pi)\colon L_\theta[\mathcal{E}] \to \mathcal{Q}$ witnesses that κ is virtually θ-measurable, since $\mathcal{Q} \trianglelefteq \mathcal{P}$ implies that $\mathcal{Q} \subseteq K$. ■

Note that the proofs of (ii) and (iii) above do not show that κ is virtually θ-prestrong, as it might still be the case that $\bar{\mu} \neq \mu$ or $\bar{\mathcal{E}} \neq \mathcal{E}$, so we cannot conclude that $L_\theta[\mu] \subseteq L_\theta[\bar{\mu}]$ or $L_\theta[\mathcal{E}] \subseteq L_\theta[\overline{\mathcal{E}}]$. It might still hold however; see Question ??.

1.5 Separation results

Having proving many positive results about the relations between the virtual large cardinals in the previous sections, this section is dedicated to the negatives. More precisely, we will aim to *separate* many of the defined notions (potentially under suitable large cardinal assumptions).

Our first separation result is that the virtuals form a level-by-level hierarchy.

Theorem 1.50 (N.). *Let $\alpha < \kappa$ and assume that κ is faintly $\kappa^{+\alpha+2}$-measurable. Then*

$$L_\kappa \models \ulcorner \textit{There's a proper class of } \lambda \textit{ which are virtually } \lambda^{+\alpha+1}\textit{-strong}\urcorner.$$

Proof. Write $\theta := \kappa^{+\alpha+1}$. Then by Theorem 1.13 we get that either κ is faintly θ^+-strong in L or otherwise, in particular, L_κ thinks that there's a proper class of remarkables. In the second case we also get that L_κ thinks that there's a proper class of λ such that λ is virtually $\lambda^{+\alpha+1}$-strong and we'd be done, so assume the first case. Then $L_\kappa \prec_2 L_{\theta^+}$, so define for each $\xi < \kappa$ the sentence ψ_ξ as

$$\psi_\xi :\equiv \exists\lambda < \xi \colon \ulcorner\lambda \text{ is virtually } \lambda^{+\alpha+1}\text{-strong}\urcorner.$$

Then ψ_ξ is $\Sigma_2(\{\alpha, \xi\})$ since being virtually β-strong is a $\Delta_2(\{\beta\})$-statement. As $L_{\theta^+} \models \psi_\xi$ for all $\xi < \kappa$ we also get that $L_\kappa \models \psi_\xi$ for all $\xi < \kappa$, which is what we wanted to show. ■

As we are only assuming κ to be *faintly* measurable in the above, this also shows that the faintly $\kappa^{+\alpha+1}$-measurable cardinals κ form a strict hierarchy whenever $\alpha < \kappa$.

A separation result in a similar vein is the following, showing that it is consistent to have an inaccessible faintly measurable cardinal which is not weakly compact.

Proposition 1.51 (N.). *Assuming κ is measurable, there's a generic extension of V in which κ is inaccessible and faintly measurable, but not weakly compact.*

Proof. Let $\mathbb{P}$ be the forcing notion that adds a κ-Suslin tree $\mathcal{T}$. By [?] it then holds that $\mathbb{P} * \mathcal{T} \cong \mathrm{Add}(\kappa^+, 1)$, a $<\kappa^+$-closed forcing, which preserves the measurability of κ. Further, the $\mathbb{P}$ forcing is shown to preserve the inaccesibility of κ, making κ inaccesible and faintly measurable in $V[g]$. Lastly, it cannot be weakly compact in $V[g]$ because $\mathcal{T}$ is a κ-tree without a branch, by definition. ■

Next, we show that the virtuals are in fact different from the faints. This is trivial in general as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

A key ingredient is that virtually κ^+-cardinals are Π^2_1-indescribable, whose proof is identical to the standard proof in [?] that measurable cardinals are Π^2_1-indescribable. It should be noted that we crucially need the "virtual" property for the proof to go through. Using this indescribability fact, the proof of the following theorem is precisely the same as Hamkins' Proposition 8.2 in [?].

Theorem 1.52 (Virtualised Hamkins). *Assuming κ is a κ^{++}-tall cardinal,*[18] *there's a forcing extension of V in which κ is inaccessible but not virtually κ^+-measurable, and becomes measurable in an $Add(\kappa^+, 1)$-generic extension.* ■

This then gives us our first separation result.

Corollary 1.53 (N.). *Assuming κ is a κ^{++}-tall cardinal, it's consistent that κ is faintly measurable but not virtually κ^+-measurable.*

Proof. By the above Theorem 1.56 we may assume that κ is not virtually κ^+-measurable but that it's measurable in $V^{\mathbb{P}}$ for $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$, so that κ is κ-closed κ^+-sized faintly ∞-measurable. ■

For a slightly more fine-grained distinction let's define an intermediate large cardinal between the faintly and virtual.

Definition 1.54. Let $\kappa < \theta$ be infinite regular cardinals. Say that κ is **faintly θ-power-Φ** for $\Phi \in \{\text{measurable, prestrong, strong}\}$ if it is faintly θ-Φ, witnessed by an embedding $\pi\colon H_\theta^V \to \mathcal{N}$, and $\mathscr{P}^V(\kappa) = \mathscr{P}^{\mathcal{N}}(\kappa)$. ∘

Note that the proof of Lemma 1.5 shows that faintly power-measurables are also 1-iterable and so in particular weakly compact. Our separation result is then the following.

Theorem 1.55 (Gitman-N.). *For $\Phi \in \{measurable, prestrong, strong\}$, if κ is virtually Φ, then there exist forcing extensions $V[G]$ and $V[G^*]$ such that*

(i) in $V[G]$, κ is inaccessible and faintly Φ, but not faintly power-Φ, and

(ii) in $V[G^*]$, κ is faintly power-Φ, but not virtually κ^{++}-prestrong.

Proof. We start with (i). Let us assume that κ is virtually measurable. This implies, in particular, that for every regular $\theta > \kappa$, we have generic embeddings $\pi : H_\theta \to \mathcal{M}$ with $\operatorname{crit} \pi = \kappa$ such that $\mathcal{M} \in V$. Thus, by Proposition ??, we can assume that the generic embedding π exists in a $\mathrm{Col}(\omega, H_\theta)$-extension.

[18] Recall that κ is **κ^{++}-tall** if there's an elementary embedding $j\colon V \to M$ with $\operatorname{crit} j = \kappa$, ${}^\kappa M \subseteq M$ and $j(\kappa) > \kappa^{++}$.

Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\alpha < \kappa$, and let $G \subseteq \mathbb{P}_\kappa$ be V-generic. Standard computations show that $\mathbb{P}_\kappa$ preserves all inaccessible cardinals.

Fix a regular $\theta \gg \kappa$ and let $h \subseteq \mathrm{Col}(\omega, H_\theta)$ be $V[G]$-generic. In $V[h]$, we must have an elementary embedding $\pi : H_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$ and $\mathcal{M} \in V$, and we can assume without loss that $\mathcal{M}$ is countable. Obviously, $\pi \in V[G][h]$. Working in $V[G][h]$, we will now lift π to an elementary embedding on $H_\theta[G]$. To ensure that such a lift exists, it suffices to find in $V[G][h]$ an $\mathcal{M}$-generic filter for $\pi(\mathbb{P}_\kappa)$ containing $\pi''G$[19]. Observe first that $\pi''G = G$ since the critical point of π is κ and we can assume that $\mathbb{P}_\kappa \subseteq V_\kappa$. Next, observe that $\pi(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing beyond κ. Since $\mathcal{M}[G]$ is countable, we can build an $\mathcal{M}[G]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ in $V[G][h]$. Thus, $G * G_{\text{tail}}$ is $\mathcal{M}$-generic for $\pi(\mathbb{P}_\kappa)$, and so we can lift π to $\pi : H_\theta[G] \to \mathcal{M}[G][G_{\text{tail}}]$. Since θ was chosen arbitrarily, we have just shown that κ is faintly measurable in $V[G]$.

Now suppose that κ is faintly power-measurable in $V[G]$. Fix regular $\theta < \bar\theta$ and a generic elementary embedding $\sigma : H_{\bar\theta}[G] \to \mathcal{N}$ with $\operatorname{crit}\sigma = \kappa$ and $\mathscr{P}(\kappa)^{V[G]} = \mathscr{P}(\kappa)^{\mathcal{N}}$. By elementarity, $H^{\mathcal{N}}_{\sigma(\theta)} = \sigma(H_\theta)[\sigma(G)]$ is a forcing extension of $K = \sigma(H_\theta)$ by $\sigma(G) = G * \bar G_{\text{tail}} \subseteq \sigma(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \bar{\mathbb{P}}_{\text{tail}}$. Thus, we have the restrictions $\sigma : H_\theta \to K$ and $\sigma : H_\theta[G] \to K[G][\bar G_{\text{tail}}]$. Let us argue that $\mathscr{P}^{V[G]}(\kappa) \subseteq \mathscr{P}^{K[G]}(\kappa)$, and hence we have equality. Suppose $A \subseteq \kappa$ in $V[G]$ and let $\dot A$ be a nice $\mathbb{P}_\kappa$-name for A, which can be coded by a subset of κ. Since $\operatorname{crit}\sigma = \kappa$, we have that $\dot A \in K$, and hence $A = \dot A_G \in K[G]$. But now it follows that the $K[G]$-generic for $\mathrm{Add}(\kappa, 1)$, the forcing at stage κ in $\sigma(\mathbb{P}_\kappa)$, cannot be in $V[G]$. Thus, we have reached a contradiction, showing that κ cannot be faintly power-measurable in $V[G]$.

If $\Phi =$ measurable, then we are done at this point. For $\Phi =$ prestrong we simply note that $G \in \mathcal{M}[G * G_{\text{tail}}]$ so that $H^{V[G]}_\theta \subseteq \mathcal{N}[G * G_{\text{tail}}]$ as well, and since we lifted π, we still have $\pi(\kappa) > \theta$ in the $\Phi =$ strong case.

For (ii), we change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$ and call the resulting forcing $\mathbb{P}^*_\kappa$. Let $G^* \subseteq \mathbb{P}^*_\kappa$ be V-generic. We verify that κ is faintly-Φ as above by lifting an embedding $\pi : H_\theta \to \mathcal{M}$, with $\mathcal{M} \subseteq V$, to $\pi : H_\theta[G^*] \to \mathcal{M}[G^*][G_{\text{tail}}]$ in a collapse extension $V[G^*][h]$. The lifted embedding

[19]This standard lemma is referred to in the literature as the **lifting criterion**.

is κ-powerset preserving because a subset of κ from $\mathcal{M}[G^*][G_{\text{tail}}]$ has to already be in $\mathcal{M}[G^*]$ as $\mathbb{P}_{\text{tail}}$ is $\leq\kappa$-closed, and $\mathcal{M}[G^*] \subseteq V[G^*]$. So it remains to show that κ is not virtually κ^{++}-prestrong. Suppose it is and fix a generic embedding $\sigma : H_{\kappa^{++}}[G^*] \to K[G^*][G_{\text{tail}}]$ with $\operatorname{crit}\sigma = \kappa$ and $H_{\kappa^{++}}[G^*] \subseteq K[G^*][G_{\text{tail}}]$. It follows that the generic subset of κ^+ added at stage κ^+ by the tail forcing must be $V[G^*]$-generic, which is contradictory. ■

Starting from much stronger hypothesis, it can be shown that a power-measurable cardinal need not even be virtually κ^+-measurable. Now, first, we observe that virtually θ-measurable cardinals κ are Π^2_1-indescribable for all $\theta > \kappa$. The proof is identical to the standard Hanf-Scott proof that measurable cardinals are Π^2_1-indescribable; see e.g. Proposition 6.5 in [?]. It should be noted that we crucially need the "virtual" property for the proof to go through. Using this indescribability fact, the proof of the following theorem is precisely the same as Hamkins' Proposition 8.2 in [?].

Theorem 1.56 (Virtualised Hamkins). *Assuming κ is a κ^{++}-tall cardinal,*[20] *there is a forcing extension in which κ is not virtually κ^+-measurable, but becomes measurable in a further $Add(\kappa^+, 1)$-generic extension.*

This then immediately gives the separation result.

Corollary 1.57. *Assuming κ is a κ^{++}-tall cardinal, it is consistent that κ is faintly power-measurable, but not virtually κ^+-measurable.*

In contrast to the above separation result, we will show in Proposition 1.22 and Theorem 1.26 that the faint-virtual distinction vanishes when we are dealing with virtual pre-Woodin or Woodin cardinals.

Our next separation result is concerning the virtually prestrong and virtually strong cardinals.

[20] Recall that κ is κ^{++}-**tall** if there is an elementary embedding $j\colon V \to M$ with $\operatorname{crit} j = \kappa$, ${}^{\kappa}M \subseteq M$ and $j(\kappa) > \kappa^{++}$.

Corollary 1.58 (N.). *There exists a virtually rank-into-rank cardinal iff there is an uncountable cardinal θ and a virtually θ-prestrong cardinal which is not virtually θ-strong.*

Proof. ($\Leftarrow$) is directly from the above Proposition 1.15 and Theorem 1.13.

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let (κ, θ) be the lexicographically least pair such that κ is virtually θ-rank-into-rank, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong then it would be Σ_2-reflecting, so that the statement that there exists a virtually rank-into-rank cardinal would reflect down to H_κ^V, contradicting the minimality of κ. ■

Figure 1.2 summarises the separation results along with the results from Section 1.4. Note that it *might* be the case that virtually θ-measurables are always virtually θ-prestrong (and hence also equivalent in $L[\mu]$ and K below a Woodin cardinal); see Question ??.

1.6 Indestructibility

It is well-known that supercompact cardinals κ can be made indestructible by all $<\kappa$-directed closed forcings by a suitable *Laver preparatory forcing*, which is the main theorem in the seminal paper [?]. A natural question, then, is whether similar results hold for the faintly and virtual versions. We noted in Proposition 1.5 that the virtuals are weakly compact, so the following theorem from [?] shows that the consistency strength of indestructible virtual supercompacts is very large, potentially even in the realm of supercompacts themselves.

Theorem 1.59 (Schindler). *The consistency strength of a weakly compact cardinal κ which is indestructible by $<\kappa$-directed closed forcing is larger than the consistency strength of a proper class of strong cardinals and a proper class of Woodin cardinals.*

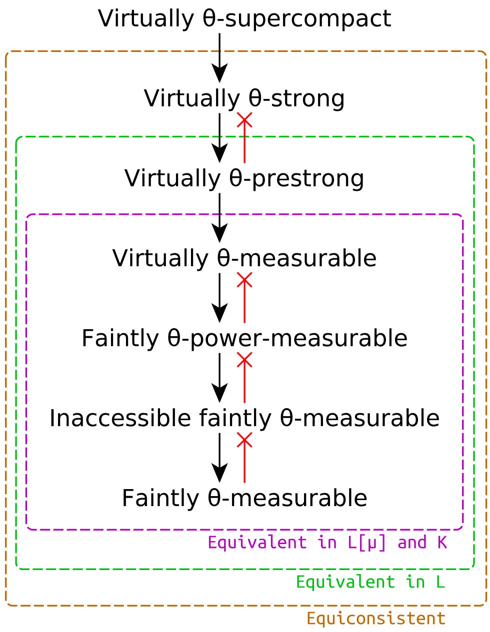

Figure 1.2: Direct implications between virtuals, where the red lines with crosses indicate that ZFC doesn't prove the reverse implication.

This gets close to resolving the question about the indestructible virtuals, so what about the faintly supercompact cardinals? To make things a bit easier for ourselves, let us make the notion a bit stronger.

Definition 1.60. Fix uncountable cardinals $\kappa < \theta$. Then κ is **generically setwise θ-supercompact** if there exists a generic extension $V[g]$, a transitive $\mathcal{N} \in V[g]$ and a generic elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$, $\pi \in V[g]$, with $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and $V[g] \models {}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$. If it holds for all $\theta > \kappa$ then we say that κ is **generically setwise supercompact**. ○

Note that the only difference between a generically setwise θ-supercompact cardinal and a virtually θ-supercompact cardinal is that the former is closed under sequences *in the generic extension*, where the latter is only closed under sequences in V; i.e., that $V \cap {}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$.

Ostensibly this seems to be an incredibly strong notion, as the target model now inherits a lot of structure from the generic extension. A first stab at an upper consistency bound could be to note that if there exists a proper class of Woodin cardinals then ω_1 is generically setwise supercompact. This can be shown using the countable stationary tower, see [?].

But, surprisingly, the following result from [?] shows that they can exist in L.

Theorem 1.61 (Usuba). *If κ is virtually extendible then $\mathrm{Col}(\omega, {<}\kappa)$ forces that ω_1 is generically setwise supercompact.* ■

It turns out that this slightly stronger notion *does* have indestructiblity properties. We warm up by firstly showing that they are indestructible by small forcing.

Proposition 1.62 (N.-Schlicht). *Generic setwise supercompactness of κ is indestructible for forcing notions of size $< \kappa$.*

Proof. Fix a forcing $\mathbb{P}$ of size ${<}\kappa$ and assume without loss of generality that $\mathbb{P} \in H^V_\kappa$, and fix also a cardinal $\theta > \kappa$. Using the setwise supercompactness of κ we may fix a forcing $\mathbb{Q}$ and a V-generic $h \subseteq \mathbb{Q}$ such that in $V[h]$ we have an elementary $\pi\colon M := H^V_\theta \to \mathcal{N}$ in $V[h]$ with $\mathcal{N}$ ${<}\theta$-closed.

Let $g \subseteq \mathbb{P}$ be $V[h]$-generic and work in $V[g \times h]$. By the lifting criterion we get a lift $\tilde{\pi}\colon M[g] \to \mathcal{N}[g]$ of π. If κ is a limit cardinal, then we may choose a cardinal $\lambda < \kappa$ such that $\mathbb{P} \in H^V_\lambda$. Since $\mathbb{P}$ has the λ^+-cc in V we get that $\pi(\mathbb{P}) = \mathbb{P}$ has the λ^+-cc in $\mathcal{N}$ and hence in $V[h]$ as well, making $\mathcal{N}[g]$ ${<}\theta$-closed by Lemma ?? and we're done.

If $\kappa = \nu^+$, then there are no cardinals between ν and $\pi(\kappa)$ in $\mathcal{N}$ and hence $|\theta| \leq \nu$. Thus it suffices to show that $\mathcal{N}[g]$ is ν-closed. Since $\pi(\mathbb{P}) = \mathbb{P}$ has size $\leq\nu$ in V, it has the $\nu^{+V[h]}$-cc in $V[h]$. Therefore $\mathcal{N}[g]$ is again ${<}\theta$-closed by Lemma ??. ■

Next, we show that these generic setwise supercompact cardinals κ *are* in fact indestructible for ${<}\kappa$-directed closed forcings, without having to do any preparation forcing at all.

Theorem 1.63 (N.-Schlicht). *Generic setwise supercompactness of κ is indestructible for ${<}\kappa$-directed closed forcings.*

Proof. Suppose that κ is generically setwise supercompact, $\mathbb{P}$ is a $<\kappa$-directed closed forcing and g is $\mathbb{P}$-generic over V. We'll show that κ is generically setwise supercompact in $V[g]$.

In V fix a regular $\theta > \kappa$ such that $\mathbb{P} \in H_\theta^V$, and let $\mathbb{Q}$ be the forcing given by the definition of setwise supercompactness. Let h be $\mathbb{Q}$-generic over $V[g]$. Let $\pi \colon H_\theta^V \to \mathcal{N}$ be as in the definition of generically setwise supercompactness, so that $\pi \in V[h] \subseteq V[g \times h]$. Work in $V[g \times h]$.

We may assume that $\theta = \theta^{<\theta}$ holds, as otherwise we just replace $\mathbb{Q}$ with $\mathbb{Q} * \mathrm{Col}(\theta, \theta^{<\theta})$ – we retain the $<\theta$-closure of $\mathcal{N}$ because $\mathrm{Col}(\theta, \theta^{<\theta})$ is $<\theta$-closed. We can further assume that $|\mathcal{N}| = \theta^{<\theta} = \theta$, as otherwise we can take a hull of $\mathcal{N}$ containing $\mathrm{ran}(\pi)$ and recursively close under $<\theta$-sequences, ending up with a $<\theta$-closed elementary substructure $\mathcal{H} \prec \mathcal{N}$ containing $\mathrm{ran}(\pi)$ – now replace $\mathcal{N}$ by the transitive collapse of $\mathcal{H}$.

Claim 1.64. There's a $\pi(\mathbb{P})$-generic filter $\tilde{g}$ over $\mathcal{N}$ that extends $\pi[g]$.

Proof of claim. Since $\mathcal{N}$ is (in particular) $|\mathbb{P}|$-closed in $V[h]$ and $\mathbb{P}$ is trivially $|\mathbb{P}|^+$-cc, Lemma ?? implies that $\mathcal{N}$ is still $|\mathbb{P}|$-closed in $V[g \times h]$. As below, we thus still get that $\pi[g] \in \mathcal{N}$. Now work in $V[h]$, where we have full $<\theta$-closure of $\mathcal{N}$.

Since $\pi(\mathbb{P})$ is directed, there's a condition $q \leq \pi[g]$ in $\pi(\mathbb{P})$. Using the fact that $|\mathcal{N}| = \theta$ and $\pi(\mathbb{P})$ is $<\theta$-closed, we can construct a $\pi(\mathbb{P})$-generic filter $\tilde{g}$ over $\mathcal{N}$ with $q \in g$.[21] Then $\tilde{g}$ is as required. $\dashv$

Since we now have that $\pi[g] \subseteq \tilde{g}$ by the claim, the lifting criterion implies that we can lift π to $\tilde{\pi} \colon H_\theta^V[g] \to \mathcal{N}[\tilde{g}]$.

It thus remains to see that $\mathcal{N}[\tilde{g}]$ is $<\theta$-closed. To see this, take a sequence $\vec{x} = \langle x_i \mid i < \gamma \rangle$ with $\gamma < \theta$ and $x_i \in \mathcal{N}[\tilde{g}]$ and find names $\sigma_i \in \mathcal{N}$ with $\sigma_i^{\tilde{g}} = x_i$ for all $i < \gamma$. Since ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ we have that $\vec{\sigma} = \langle \sigma_i \mid i < \gamma \rangle \in \mathcal{N}$, and from $\vec{\sigma}$ we obtain a canonical name $\vec{\sigma}^\bullet \in \mathcal{N}$ with $\vec{\sigma}^{\bullet\tilde{g}} = \vec{x} \in \mathcal{N}[\tilde{g}]$. ■

[21] Namely, enumerate the dense subsets of $\pi(\mathbb{P})$ that are elements of $\mathcal{N}$ in order type θ and use the fact that the initial segments of the sequence, and of the corresponding sequence of conditions that we construct, are in $\mathcal{N}$.

Investigating further, we also show indestructibility for some forcings that do not fall into the above-mentioned categories.

Proposition 1.65 (N.-Schlicht). *Generic setwise supercompactness of a regular cardinal κ is indestructible for* $\mathrm{Add}(\omega,\kappa)$. *If κ is a successor cardinal then it is also indestructible for* $\mathrm{Col}(\omega,{<}\kappa)$.

Proof. Let g be $\mathrm{Add}(\omega,\kappa)$-generic over V. In V fix a regular $\theta>\kappa$ and let $\mathbb{Q}$ be the forcing given by the definition of generic setwise supercompactness. Let h be $\mathbb{Q}$-generic over $V[g]$ and work in $V[g\times h]$.

Let $\pi\colon H^V_\theta\to\mathcal{N}$ be as in the definition of generically setwise supercompactness. Moreover, let $\tilde g$ be $\mathrm{Add}(\omega,\pi(\kappa))$-generic over $V[g\times h]$. Since $\pi[g]=g$, the lifting criterion allows us to extend π to some $\tilde\pi\colon H^V_\theta[g]\to\mathcal{N}[g\times\tilde g]$. To show that $\mathcal{N}[g\times\tilde g]$ is ${<}\theta$-closed in $V[g\times h\times\tilde g]$, it suffices that $\mathrm{Add}(\omega,\pi(\kappa))$ has the ccc by Lemma ??.

For $\mathrm{Col}(\omega,{<}\kappa)$, we proceed similarly. Assume that $\kappa=\nu^+$. Take $\mathrm{Col}(\omega,{<}\kappa)$-, $\mathbb{Q}$- and $\mathrm{Col}(\omega,{<}\pi(\kappa))$-generic filters g, h and $\tilde g$. π and $\mathcal{N}$ are as above. Since $\nu<\kappa<\theta<\pi(\kappa)$ and there are no cardinals between ν and $\pi(\kappa)$ (in $\mathcal{N}$ and thus also in $V[h]$), ${<}\theta$-closure means ν-closure (in any model containing $V[h]$). By Lemma ??, it's thus sufficient to know that $\mathrm{Col}(\omega,{<}\pi(\kappa))$ has the ν^+-cc in $V[g\times h]$. This is because $\pi(\kappa)=\nu^{+\mathcal{N}}\leq\nu^{+V[g\times h]}$. ■

Usuba's Theorem 1.61 shows that the *consistency strength* of these generically setwise supercompact cardinals is small, but do they appear naturally anywhere? The following result shows that we cannot find any in neither L nor $L[\mu]$.

Proposition 1.66 (N.-Schlicht). *No cardinal κ is generically setwise supercompact in neither L nor $L[\mu]$ with μ being a normal ultrafilter.*

Proof. Assume first that $V=L$ and that κ is generically setwise supercompact. Let g be a generic filter and $\pi\colon L_\theta\to\mathcal{N}$ an embedding in $V[g]$ with $\pi\restriction L_{\kappa^{+L}}\in\mathcal{N}$. Then $\mathcal{N}=L_\alpha$ for some α by condensation and thus $\pi\restriction H_{\kappa^{+L}}\in L$. But this would induce a ${<}\kappa$-complete ultrafilter on κ, contradicting $V=L$.

Assume now that $V = L[\mu]$ and that κ is generically setwise supercompact, witnessed by a generic embedding $\pi\colon L_\theta[\mu] \to L_\alpha[\bar{\mu}]$. In particular this means that $\pi \restriction L_{\kappa^{+L[\mu]}}[\mu] \in L_\alpha[\bar{\mu}]$. If $\operatorname{crit}\mu < \kappa$ then $\mu = \bar{\mu}$ and $\mathscr{P}^{L[\mu]}(\kappa) = \mathscr{P}^{L[\bar{\mu}]}$, so that both $\pi(\kappa)$ and κ are now measurable cardinals in $L[\bar{\mu}]$, contradicting [?, Lemma 20.2]. So $\operatorname{crit}\mu \geq \kappa$.

If $\pi(\operatorname{crit}\mu) > \operatorname{crit}\mu$ then by [?, Theorem 20.12] we get that $L[\bar{\mu}]$ is an iterate of $L[\mu]$. But iteration embeddings preserve the subsets of their critical point, so again we have that $\mathscr{P}^{L[\mu]}(\kappa) = \mathscr{P}^{L[\bar{\mu}]}$ and we get the same contradiction as before.

Lastly, if $\operatorname{crit}\mu > \kappa$ and $\pi(\operatorname{crit}\mu) = \operatorname{crit}\mu$ then $\mu = \bar{\mu}$ by [?, Theorem 20.10], so we get a contradiction as in the $\operatorname{crit}\mu < \kappa$ ■