Virtual Set Theory

PhD Defense

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September 1, 2020

Before we start

Note that I will only include *some* results from my thesis here because of time constraints, and have thus left out entire sections of the thesis.

An overview of the talk

- What is a virtual large cardinal?
- How do the virtuals interact with each other?
- How indestructible are the virtuals?
- How do the virtuals relate to other mathematical objects?
  - Infinite games
  - Ideals

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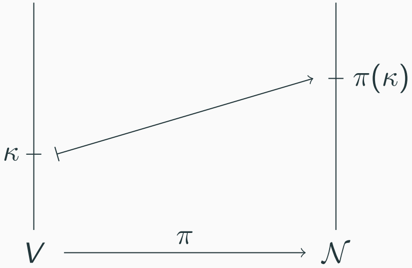
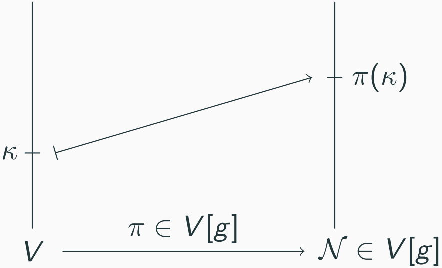
What is a virtual large cardinal?

Many large cardinals are defined as the critical point of an elementary embedding from the universe V into a transitive $\mathcal{N}$.



What is a virtual large cardinal?

We can weaken this large cardinal definition to merely requiring the elementary embedding and target model to exist in a generic extension (of V).



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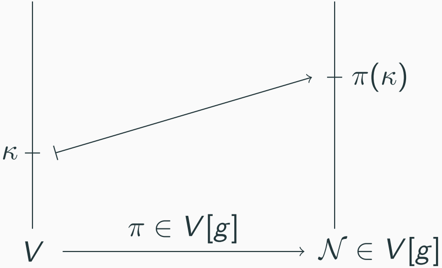
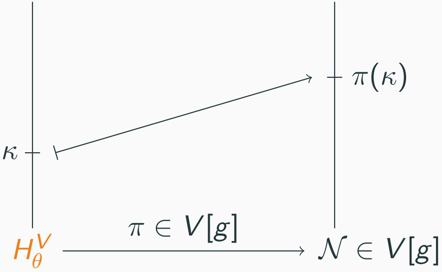


We have two variants of such cardinals.

What is a virtual large cardinal?

The faint large cardinals are the ones where the embedding goes from H_θ^V, for some regular uncountable θ.



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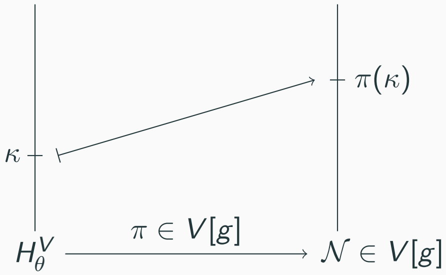

Successor cardinals can be faint large cardinals.

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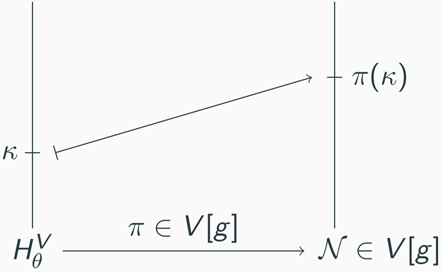
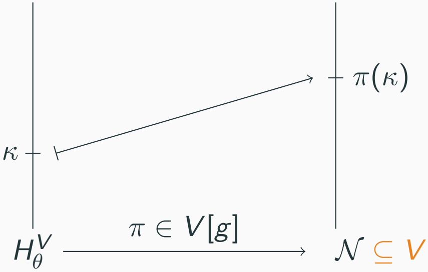


Successor cardinals can be faint large cardinals.

For instance, ω_1 can carry a precipitous ideal.

What is a virtual large cardinal?

The virtual large cardinals are faint large cardinals where the target model is a subset of V.



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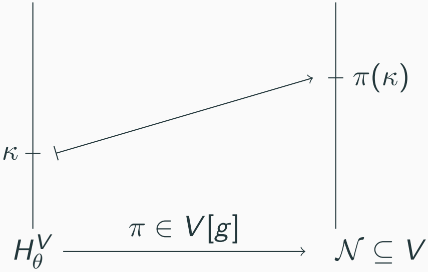

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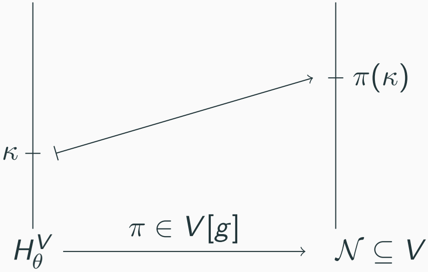

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In particular, weakly compact and inaccessible.

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We attach a pre- prefix to our virtual large cardinals when we do not require this property.

For instance, κ is virtually θ-prestrong if it is the critical point of a generic $\pi\colon H_\theta^V \to \mathcal{N}$, where $\mathcal{N} \subseteq V$ and $H_\theta^V \subseteq \mathcal{N}$.

What is a virtual large cardinal?

All the virtuals (and faints) are weak versions of their “real” counterparts.

- Berkeley

$0 = 1$

- Vopenka
- Supercompact
- Woodin
- Strong
- Measurable

$V = L$

- Virtual
- Weakly compact
- Inaccessible

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How do the virtuals interact with each other?

Most of the results in this section are comprised in a paper joint with Dimopoulos and Gitman, and will be submitted within a couple of weeks or so.

Virtuals behave differently from their real counterparts. The following result from Gitman and Schindler (2018) gave an example of such surprising differences:

Theorem (Gitman-Schindler)

The following are equivalent for an uncountable cardinal κ:

1. κ is virtually supercompact;
2. κ is virtually strong.

How do the virtuals interact with each other?

Strongs and supercompacts

We showed that, in L, virtually measurables are either virtually ω-superstrong or virtually strong. This gives the following consistency result:

Theorem (N.)

The following are equiconsistent for uncountable θ:

1. The existence of a virtually θ-strong cardinal;
2. The existence of a virtually θ-measurable cardinal.

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The reason why we are working in L is that, in L, being virtually measurable is equivalent to being virtually prestrong. This can be seen by condensation.

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Together with the Gitman-Schindler result we thus get that the virtually measurables are equiconsistent with the virtually supercompacts.

Our intuition about the prestrongs having to do with the Kunen inconsistency is confirmed by the following result.

Theorem (N.)

The following are equivalent:

1. There exists an uncountable cardinal θ and a virtually θ-prestrong cardinal which is not virtually θ-strong;
2. There exists a virtually rank-into-rank cardinal.

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κ is virtually rank-into-rank if it is the critical point of a generic embedding $\pi\colon H_\theta^V \to H_\theta^V$.

Moving to the Woodins and Vopenkas, these also *almost* form a "virtual pair":

Theorem (Dimopoulos-Gitman-N.)

The following are equivalent for an inaccessible κ:

1. κ is Vopenka;
2. κ is virtually pre-Woodin;
3. κ is faintly pre-Woodin.

How do the virtuals interact with each other?

Woodins, Vopenkas and Berkeleys

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Letting gVP be the generic analogue of Vopenkas Principle, Gitman and Hamkins (2019) showed the following:

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If $0^\sharp$ exists then gVP holds and On is not Mahlo.

In sharpening the hypothesis, we arrived at the virtual analogue of the Berkeley cardinals.

Definition

A cardinal δ is virtually Berkeley if to every transitive set $\mathcal{M}$ with $\delta \subseteq \mathcal{M}$, there is a generic elementary $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\mathrm{crit}(\pi) < \delta$, and $\mathrm{crit}(\pi)$ can be chosen arbitrarily large below δ.

Gitman and Hamkin's result was then firstly sharpened to these Berkeleys:

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But in analysing the other direction, an interesting result fell out:

Theorem (N.)

If there are no virtually Berkeley cardinals then On is virtually pre-Woodin iff On is virtually Woodin.

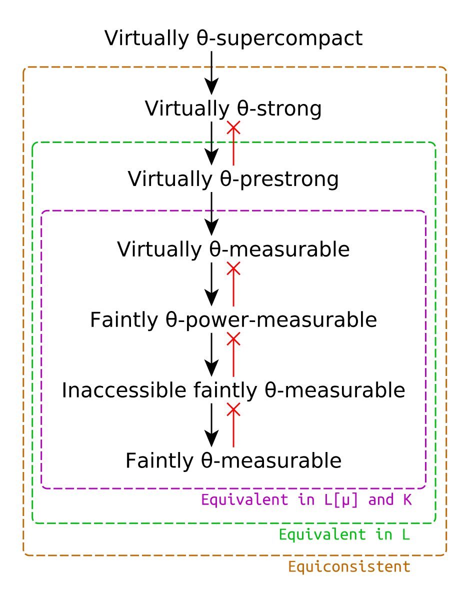
Ultimately this led to the following equivalence, showing Berkeley cardinals to be the optimal hypothesis, as well as showing that the Berkeley cardinals are the natural analogues of the virtually rank-into-rank cardinals:

Theorem (N.)

The following are equivalent:

1. gVP implies that On is Mahlo;
2. On is virtually pre-Woodin iff On is virtually Woodin;
3. There are no virtually Berkeley cardinals.

How do the virtuals interact with each other?



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    - Ideals

How indestructible are the virtuals?

The results in this section is joint with Schlicht and will most likely appear in a joint paper of ours.

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We have seen that the virtuals behave differently from their real counterparts, so we can start asking *how* different.

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Indestructibility properties of small large cardinals have been widely studied[1], so it would be interesting if we could find a virtual analogue of Laver indestructibility.

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After many failed attempts, we strengthened our assumption.

Definition (N.-Schlicht)

κ is generically setwise θ-supercompact if it is faintly θ-supercompact, but where the target model is closed under $<\theta$-sequences *in the generic extension*.

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How indestructible are the virtuals?

We showed that these exhibit many indestructibility properties:

Theorem (N.-Schlicht)

Generically setwise supercompact cardinals κ are indestructible by:

1. Small forcings;
2. Adding κ many Cohen reals;
3. $<\kappa$-directed closed forcings.

Note that we do not require any preparatory forcing.

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We noted firstly that Woodin's countable stationary tower forcing provides an upper consistency bound at a proper class of Woodins, so we at least do not reach the supercompacts.

Usuba then showed the incredibly surprising result that we are in fact still in the consistency realm among the virtuals:

Theorem (Usuba)

If κ is virtually extendible then $\mathrm{Col}(\omega, {<}\kappa)$ forces that ω_1 is generically setwise supercompact.

How indestructible are the virtuals?

Despite their weak consistency strength, these cardinals seem to be quite unnatural:

Theorem (N.-Schlicht)

No cardinal is generically setwise supercompact in either L, K below a measurable, or $L[\mu]$ with μ being a normal ultrafilter.

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Infinite games

The results in this section are included in a joint paper with Welch and is published in the Journal of Symbolic Logic.

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Definition

We say that an $\mathcal{M}$-measure μ on κ is...

- $\mathcal{M}$-normal if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^{\kappa}\mu \colon \triangle\vec{X} \in \mu$;
- genuine if $|\triangle\vec{X}| = \kappa$ for every $\vec{X} \in {}^{\kappa}\mu$;
- normal if $\triangle\vec{X}$ is stationary in κ for every $\vec{X} \in {}^{\kappa}\mu$.

Infinite games

Define the following game $\mathcal{G}^{\theta}_{\gamma}(\kappa)$:

$$\begin{array}{lcccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \dots & & \mathcal{M}_\gamma & \\ \mathrm{II} & & \mu_0 & & \mu_1 & & \dots & & \mu_\gamma \end{array}$$

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These measures are normal when $\alpha < \gamma$, and μ_γ is $\mathcal{M}_\gamma$-normal.

Player II wins iff they can continue playing through all the rounds.

Infinite games

These games lead to the following large cardinals:

Definition

A cardinal κ is γ-Ramsey for $\gamma \leq \kappa$ if player I does not have a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ for all regular $\theta > \kappa$.

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A cardinal κ is γ-Ramsey for $\gamma \leq \kappa$ if player I does not have a winning strategy in $\mathcal{G}_{\gamma}^{\theta}(\kappa)$ for all regular $\theta > \kappa$.

Further, κ is strategic γ-Ramsey if player II *does* have a winning strategy in the game.

We can translate results from Abramson et al (1977) to games and ultrafilters, yielding the following equivalences.

Theorem (Abramson et al.)

For a cardinal $\kappa = \kappa^{<\kappa}$,

- κ is weakly compact iff it is 0-Ramsey;
- κ is weakly ineffable iff it is genuine 0-Ramsey;
- κ is ineffable iff it is normal 0-Ramsey;

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Theorem (N.)

A cardinal κ is completely ineffable iff it is coherently $<\omega$-Ramsey, meaning that it is strategic n-Ramsey for all $n < \omega$, and the strategies agree with each other.

The countable case is when we start making connections back to the virtual large cardinals:

Theorem (N.-Schindler)

In L, the following are equivalent for a cardinal κ:

- κ is strategic ω-Ramsey;
- κ is virtually measurable.

Together with our previous consistency results, we also get the following:

Corollary (N.-Schindler)

The following are equiconsistent:

- The existence of a strategic ω-Ramsey cardinal;
- The existence of a virtually strong cardinal.

As we exceed ω, we make a large jump in consistency strength:

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The proof shows that there is an embedding $\pi\colon V \to \mathcal{N}$ in a $\mathrm{Col}(\omega, 2^\kappa)$-extension of V, from which there is a measurable cardinal in an inner model of that extension.

Welch and Schindler also showed that at the uncountable stage, the strategic Ramseys become measurable in K:

Theorem (Welch)

If $0^\P$ does not exist then every strategic ω_1-Ramsey cardinal is measurable in K.

Theorem (Schindler)

If there is no inner model with a Woodin cardinal then every strategic ω_1-Ramsey cardinal is measurable in K.

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Ideals

The study of embeddings lying in generic extensions started back in Galvin et al (1978) with the study of precipitous ideals.

Having an ideal at hand gives much more control over the embedding, as we can work more directly with the measures, so it is convenient when we can work with such ideals instead of the generic measures.

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Definition (N.)

A poset property $\Phi(\kappa)$ is ideal-absolute if whenever there is a $\Phi(\kappa)$ forcing extension $V[g]$ containing a generic V-measure on κ, then there is an ideal $I \in V$ on κ such that $\mathcal{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(\kappa)$.

Ideals

We first note that a few standard forcing properties are ideal absolute:

Proposition (Folklore)

"κ^+-chain condition" is ideal-absolute.

Theorem (N.)

"$<\lambda$-distributivity" is ideal-absolute for regular $\lambda \in [\omega, \kappa^+]$.

Ideals

The main result of the section is then the following, which builds upon and improves an unpublished result due to Foreman:

Theorem (Foreman-N.)

Let κ and $\lambda \leq \kappa^+$ be regular cardinals. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal ideal *I* such that $\mathcal{P}(\kappa)/I$ is (κ, κ)-distributive and has a dense $<\lambda$-closed subset.

Ideals

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Theorem (Foreman-N.)

Let κ and $\lambda \leq \kappa^+$ be regular cardinals. If player II has a winning strategy in $\mathcal{G}_\lambda^-(\kappa)$ then κ carries a κ-complete normal ideal I such that $\mathcal{P}(\kappa)/I$ is (κ, κ)-distributive and has a dense $<\lambda$-closed subset.

Here $\mathcal{G}_\lambda^-(\kappa)$ is the same game as $\mathcal{G}_\lambda^\theta(\kappa)$ for any θ, but where we do not require the final measure to have a wellfounded ultrapower.

Ideals

Using our previous game-theoretic results, we get the following two corollaries.

Corollary (N.)

"(κ, κ)-distributive $<\lambda$-closed" is ideal-absolute for regular $\lambda \in [\omega, \kappa^+]$.

Corollary (N.)

"$<\lambda$-closed λ-sized" is ideal-absolute for regular $\lambda \in [\omega, \kappa]$ such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$.

The end

Thank you for your attention.