1 The External Core Model Induction

Introduction.

1.1 HOD mice

Provide overview of this section.

1.1.1 Iteration strategies

At some point we should mention that we adopt John‘s convention of hiding the degree of iteration trees, always taking the maximal possible degree. And that all of our trees are (stacks of) normal trees.

Definition 1.1.2. Let $\vec{\mathcal{T}}$ be a stack of normal trees. We write $\mathrm{lh}(\vec{\mathcal{T}})$ for the length of $\vec{\mathcal{T}}$ and $\mathcal{T}_\alpha$ for the αth tree in $\vec{\mathcal{T}}$, so that

$$\vec{\mathcal{T}} = (\mathcal{T}_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}})).$$

For $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$, $\gamma < \mathrm{lh}(\mathcal{T}_\alpha)$, $\eta < \mathrm{lh}(\mathcal{T}_\beta)$ we let $\mathcal{M}_\gamma^{\mathcal{T}_\alpha}$ be the model with index γ in the tree $\mathcal{T}_\alpha$ and write

$$\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} \colon \mathcal{M}_\gamma^{\mathcal{T}_\alpha} \to \mathcal{M}_\eta^{\mathcal{T}_\beta}$$

for the corresponding embedding, provided it exists.
We also write

$$\pi^{\vec{\mathcal{T}}}_{\alpha,\beta} \colon \mathcal{M}_0^{\mathcal{T}_\alpha} \to \mathcal{M}_0^{\mathcal{T}_\beta}.$$

If $\vec{\mathcal{T}}$ has a last model, i.e. if $\mathrm{lh}(\vec{\mathcal{T}}) = \xi + 1$ and $\mathcal{M}_\infty^{\mathcal{T}_\xi}$ exists, we let $\mathcal{M}_\infty^{\vec{\mathcal{T}}} := \mathcal{M}_\infty^{\mathcal{T}_\xi}$ and $\pi^{\vec{\mathcal{T}}} \colon \mathcal{M}_0^{\mathcal{T}_0} \to \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ be the associated embedding. $\circ$

Definition 1.1.3. Let Σ be an iteration strategy and $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$. We write $\Sigma_{\vec{\mathcal{T}},N}$ for the iteration strategy on N given by

$$\Sigma_{\vec{\mathcal{T}},N}(\vec{\mathcal{U}}) := \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}).$$

We call $\Sigma_{\vec{\mathcal{T}},N}$ the $(\vec{\mathcal{T}}, N)$-*tail strategy*of Σ. ◦

Definition 1.1.4. at the very end we should remove those definitions that we didn't need

Let Σ be an iteration strategy.

(i) Σ has the *Dodd-Jensen property* if for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$ and all $\pi\colon \mathcal{M}_\Sigma \to_{\Sigma_1} N$ we have $\pi^{\vec{\mathcal{T}}}(\alpha) \leq \pi(\alpha)$ for all $\alpha \in o(\mathcal{M}_\Sigma)$.

(ii) Σ has the *positional Dodd-Jensen property* if $\Sigma_{\vec{\mathcal{T}},N}$ has the Dodd-Jensen property for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iii) Σ is *weakly positional* if $\Sigma_{\vec{\mathcal{T}},N} = \Sigma_{\vec{\mathcal{U}},N}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iv) Σ is *positional* if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly positional for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(v) Σ is *weakly commuting* if $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{U}}}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vi) Σ is *commuting* if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly commuting for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vii) Σ is *weakly pullback consistent* if $\Sigma^{\vec{\mathcal{T}}} = \Sigma$ for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(viii) Σ is *pullback consistent* if $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly pullback consistent for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

◦

Definition 1.1.5. If Σ is positional, $\Sigma_{\vec{\mathcal{T}},N}$ doesn't depend on $\vec{\mathcal{T}}$ and hence we simply write Σ_N for this tail strategy.

◦

Definition 1.1.6. An iteration strategy Σ has *branch condensation* (see Figure 1.1) if for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}_\Sigma$ such that

(i) $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ are according to Σ,

(ii) $\vec{\mathcal{U}}$ is a stack of successor length $\gamma + 1$ and $\vec{\mathcal{U}}$'s last component $\mathcal{U}_\gamma$is of limit length,

(iii) $\vec{\mathcal{T}}$ has a last model N such that $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$,

$$\begin{tikzcd}
 & N \\
\mathcal{M}_\Sigma \arrow[ur, "\pi^{\vec{\mathcal{T}}}"] \arrow[dr, "\pi_c^{\vec{\mathcal{U}}}"'] & \\
 & \mathcal{M}_c^{\vec{\mathcal{U}}} \arrow[uu, "\pi"']
\end{tikzcd}$$

Figure 1.1: Branch condensation

(iv) there is some branch c such that $\pi_c^{\vec{\mathcal{U}}}$ exists and for some $\pi\colon \mathcal{M}_c^{\vec{\mathcal{U}}} \to_{\Sigma_1} N$ we have

$$\pi^{\vec{\mathcal{T}}} = \pi \circ \pi_c^{\vec{\mathcal{U}}}.$$

Then $c = \Sigma(\vec{\mathcal{U}})$. $\circ$

Definition 1.1.7. Let $\mathcal{M}$,$\mathcal{N}$ be layered hybrid premice and $\mathcal{T}$,$\mathcal{U}$ be normal trees on $\mathcal{M}$,$\mathcal{N}$ respectively. $(\mathcal{M},\mathcal{T})$ *is a hull of* $(\mathcal{N},\mathcal{U})$ if there are

(i) an embedding, $\pi\colon \mathcal{M} \to_{\Sigma_1} \mathcal{N}$ and

(ii) an order-preserving map $\sigma\colon \operatorname{lh}(\mathcal{T}) \to \operatorname{lh}(\mathcal{U})$

such that

(i) $\alpha \leq_{\mathcal{T}} \beta \iff \sigma(\alpha) \leq_{\mathcal{U}} \sigma(\beta)$

(ii) $[\alpha,\beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}} = \emptyset \iff [\sigma(\alpha),\sigma(\beta)]_{\mathcal{U}} \cap \mathcal{D}^{\mathcal{U}} = \emptyset$,

(iii) $\pi_\alpha\colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_{\sigma(\alpha)}^{\mathcal{U}}$ and $\pi_\alpha(E_\alpha^{\mathcal{T}}) = E_{\sigma(\alpha)}^{\mathcal{U}}$,

(iv) for $\beta < \alpha$ we have $\pi_\alpha \upharpoonright \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1 = \pi_\beta \upharpoonright \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1$,

(v) for $\alpha \leq_{\mathcal{T}} \beta$ with $[\alpha,\beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}}$ we have $\pi_\beta \circ \pi_{\alpha,\beta}^{\mathcal{T}} = \pi_{\sigma(\alpha),\sigma(\beta)}^{\mathcal{U}} \circ \pi_\alpha$,

(vi) if $\beta = \operatorname{pred}_{\mathcal{T}}(\alpha+1)$, then $\sigma(\beta) = \operatorname{pred}_{\mathcal{U}}(\sigma(\alpha+1))$ and $\pi_{\alpha+1}([a,f]_{E_\alpha^{\mathcal{T}}}) = [\pi_\alpha(a), \pi_\beta(f)]_{E_{\sigma(\alpha)}^{\mathcal{T}}}$ and

(vii) $0 \leq_{\mathcal{U}} \sigma(0)$, $[0,\sigma(0)] \cap \mathcal{D}^{\mathcal{U}} = \emptyset$ and $\pi_0 = \pi_{0,\sigma(0)}^{\mathcal{U}} \circ \pi$,

(See Figure 1.2) $\circ$

Definition 1.1.8. Let $\mathcal{M}$,$\mathcal{N}$ be layered hybrid premice and $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ be stacks of normal trees on $\mathcal{M}$,$\mathcal{N}$ respectively. $(\mathcal{M},\vec{\mathcal{T}})$ *is a hull of* $(\mathcal{N},\vec{\mathcal{U}})$ if there are

(i) an order presercing map $\sigma\colon \operatorname{lh}(\vec{\mathcal{T}}) \to \operatorname{lh}(\vec{\mathcal{U}})$,

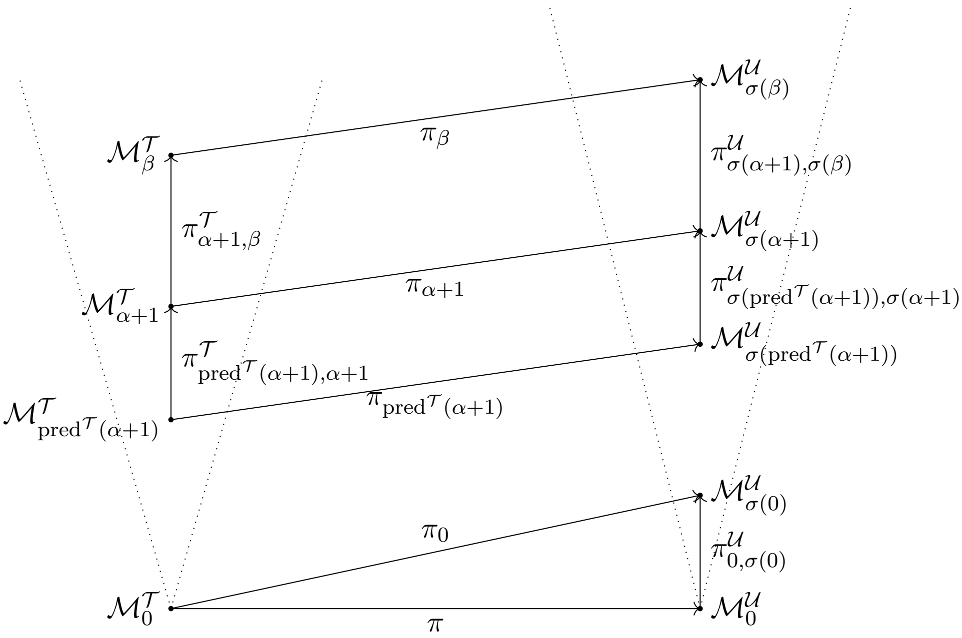

Figure 1.2: $\mathcal{T}$ is a hull of $\mathcal{U}$

(ii) a sequence $(\sigma_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}))$ of order preserving maps $\sigma_\alpha \colon \mathrm{lh}(\mathcal{T}_\alpha) \to \mathrm{lh}(\mathcal{U}_{\sigma(\alpha)})$,

(iii) $(\pi_{\alpha,\beta} \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}) \wedge \beta < \mathrm{lh}(\mathcal{T}_\alpha))$ such that

(a) $\pi_{0,0} = \pi^{\vec{\mathcal{U}}}_{0,\sigma(0)}$ (so that $\pi_{0,0} = \mathrm{id}$ if $\sigma(0) = 0$),

(b) for $\alpha < \mathrm{lh}(\vec{\mathcal{T}})$

$$\pi_{\alpha,0} \colon \mathcal{M}^{\vec{\mathcal{T}}}_{\alpha} \to_{\Sigma_1} \mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}$$

and $(\mathcal{M}^{\vec{\mathcal{T}}}_{\alpha}, \mathcal{T}_\alpha)$ is a $(\pi_{\alpha,0}, \sigma_0)$-hull of $(\mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}, \mathcal{U}_{\sigma(\alpha)})$,

(c) $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$ and $\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)}$ exists, then $\pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))}$ exists and

$$\pi_{\beta,\eta} \circ \pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} = \pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))} \circ \pi_{\alpha,\gamma}.$$

(See Figure 1.3) $\circ$

Definition 1.1.9. Let $\mathcal{M}$ be a layered hybrid premouse and Σ be a (partial) iteration strategy for $\mathcal{M}$. Σ has *hull condensation* if the follwing holds true

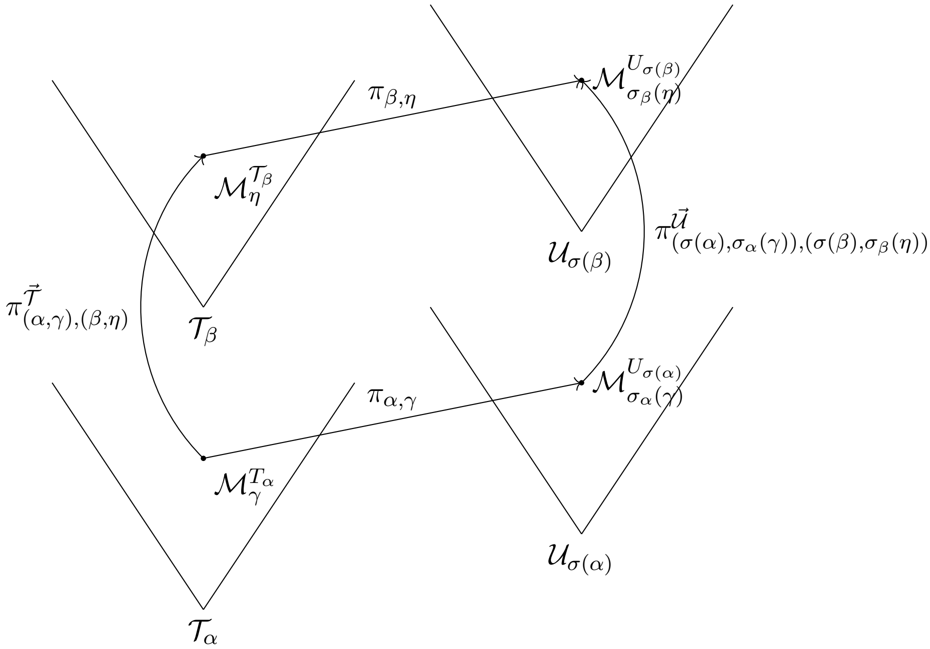

Figure 1.3: $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$

for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}$.
If $\vec{\mathcal{U}}$ is according to Σ and $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$, then $\vec{\mathcal{T}}$ is according to Σ. ∘

Lemma 1.1.10. *Let Σ be an iteration strategy. Then the following hold true.*

(i) If Σ has hull condensation then it is pullback consistent.

(ii) If Σ is positional and pullback consistent then it is commuting.

Proof. See [**?**, Proposition 2.36]. ■

1.1.11 Layered Hybrid Mice

Define strategy mice as a particular kind of hybrid mice, hod mice/pairs and put in positional and commuting in the definition, state comparison. Introduce derived models of hod mice and how they relate to the Solovay hierarchy.
Define Σ-mouse

Definition 1.1.12. Let $\mathcal{M}$ be a transitive set (or structure). We let $o(\mathcal{M}) := \mathcal{M} \cap \mathsf{On}$ be the ordinal height of $\mathcal{M}$. ◦

Definition 1.1.13. Let $\mathcal{M}$ be a (hybrid) premouse and $\alpha \leq o(\mathcal{M})$. We let

(i) $\mathcal{M}||\alpha$ be the initial segment of $\mathcal{M}$ of height α including its top extender and

(ii) $\mathcal{M}|\alpha$ be the passive initial segment of $\mathcal{M}$ of height α, i.e. $\mathcal{M}||\alpha$ but without the top extender.

◦

Definition 1.1.14. Let $\mathcal{M}$ be a $\mathcal{J}$-structure [1] and $\alpha \leq o(\mathcal{M})$. We write $\mathcal{J}_\alpha^{\mathcal{M}}$ for the αth level of $\mathcal{M}$‘s construction. ◦

Definition 1.1.15. A *potential layered hybrid premouse* (over X) is an acceptable $\mathcal{J}$-structure of the form $\mathcal{M} = (J_\alpha^{\vec{E},f}(X); \in, \vec{E}, B, f)$ X such that

(i) $\vec{E}$ is a fine extender sequence (over X),

(ii) f is a function with domain $Y \subseteq \alpha$ such that $f(\gamma)$, for each $\gamma \in Y$, is a shift of an amenable function that typically codes part of an iteration strategy for $\mathcal{M}$,

We will often write $\vec{E}^{\mathcal{M}}, f^{\mathcal{M}}, Y^{\mathcal{M}}$ for $\vec{E}, f, Y$ as above.
If all proper initial segments of $\mathcal{M}$ are sound, we say that $\mathcal{M}$ is a *layered hybrid premouse.* . ◦

In our case, assuming X is a self-well-ordered set, $Y^{\mathcal{M}}$ is determined by the *standard indexing scheme* (see [**?**, Definition 1.18]).
For more details, see [**?**].

Definition 1.1.16. Let Σ be a strategy for a layered hybrid premouse $\mathcal{M}$. For $\alpha \leq o(M)$ we let $\Sigma_{\mathcal{M}|\alpha}$ be the id-pullback iteration strategy on $\mathcal{M}|\alpha$ induced by Σ, i.e. a stack $\vec{\mathcal{T}}$ on $\mathcal{M}|\alpha$ is according to $\Sigma_{\mathcal{M}|\alpha}$ iff id$\vec{\mathcal{T}}$ on $\mathcal{M}$, given by the copy construction via id (see [**?**, 4.1]), is according to Σ. ◦

[1] See [**?**] for the basics on $\mathcal{J}$-structures, premice and their fine structure.

Definition 1.1.17. A *layered strategy premouse* $\mathcal{M}$ is a layered hybrid premouse such that

(i) $f^{\mathcal{M}}(\gamma)$ codes a partial iteration strategy $\Sigma^{\mathcal{M}}_{\gamma}$ for $\mathcal{M}|\gamma$ and
(ii) For $\gamma_0, \gamma_1 \in Y^{\mathcal{M}}$, if $\gamma_0 < \gamma_1$ then $(\Sigma^{\mathcal{M}}_{\gamma_1})_{\mathcal{M}|\gamma_0} \subseteq \Sigma^{\mathcal{M}}_{\gamma_0}$.

We also write $\Sigma^{\mathcal{M}}$ for the the strategy coded by $f^{\mathcal{M}}$. ○

Definition 1.1.18. Let $\mathcal{M}$ be a layered strategy premouse and Σ be an iteration strategy for $\mathcal{M}$. $\mathcal{M}$ is a Σ-*premouse* if $\Sigma^{\mathcal{M}} \subseteq \Sigma$. ○

Definition 1.1.19. Let Σ be an iteration strategy. We write $\mathcal{M}_\Sigma$ for the (layered hybrid) premouse $\mathcal{N}$ such that Σ is an iteration strategy for $\mathcal{N}$. We also let

$$I(\mathcal{M}_\Sigma, \Sigma) := \{(\vec{\mathcal{T}}, N) \mid \vec{\mathcal{T}} \text{ is a stack of normal trees on } \mathcal{M}_\Sigma \text{ according to } \Sigma, \\ \pi^{\vec{\mathcal{T}}} \text{ exist and } N \text{ is the last model of } \vec{\mathcal{T}}\}.$$

and

$$pI(\mathcal{M}_\Sigma, \Sigma) := \{N \mid \exists \vec{\mathcal{T}} \colon (\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)\}. \quad \dashv$$

Definition 1.1.20. A Σ-premouse $\mathcal{M}$ is a Σ-*mouse* if there is a $\omega_1 + 1$-iteration strategy Λ such that all $\mathcal{N} \in pI(\mathcal{M}, \Lambda)$, $\mathcal{N}$ are themselves Σ-premice. ○

Definition 1.1.21. Let a be a transitive self-well-ordered set and let Σ be an iteration strategy with hull-condensation such that $\mathcal{M}_\Sigma \in a$ and let Γ be a pointclass which is closed under Boolean operations and continuous images and preimages. Define the (Γ, Σ)-Lp stack over a recursively as follows:

(i) $\mathrm{Lp}^{\Gamma,\Sigma}_0(a) := a \cup \{a\}$,
(ii) $\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha+1}(a) := \bigcup\{\mathcal{M} \mid \mathcal{M}$ is a sound Σ-mouse over $\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha}(a)$ projecting to $o(\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha}(a))$ and having an iteration strategy in $\Gamma\}$,
(iii) $\mathrm{Lp}^{\Gamma,\Sigma}_{\lambda}(a) := \bigcup_{\alpha<\lambda} \mathrm{Lp}^{\Gamma,\Sigma}(a)$ for limit λ.

We also let $\mathrm{Lp}^{\Gamma,\Sigma}(a) := \mathrm{Lp}^{\Gamma,\Sigma}_1(a)$. ○

1.1.22 HOD Mice

Definition 1.1.23. Suppose $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ is a layered strategic premouse. $\mathcal{P}$ is a HOD-*premouse* [2] provided the following hold:
Let $\lambda = \text{otp}(Y^{\mathcal{P}})$, $(\gamma_\beta \mid \beta < \lambda)$ be the strictly increasing enumeration of $Y^{\mathcal{P}}$ and let, for $\beta < \lambda$, $\mathcal{P}(\beta) := \mathcal{P}||\gamma_\beta$ and moreover $\mathcal{P}(\lambda) := \mathcal{P}$. Then there is a continuous, strictly increasing sequence $(\delta_\beta \mid \beta \leq \lambda)$ of $\mathcal{P}$-cardinals such that

(i) $B = \emptyset$,
(ii) $Y^{\mathcal{P}} \subseteq \delta_\lambda$,
(iii) $(\delta_\beta \mid \beta \leq \lambda)$ is sequence of Woodin cardinals and their limits in $\mathcal{P}$ and
(iv) for all $\beta \leq \lambda$

define strong cut-point

   (a) δ_β is a strong cutpoint of $\mathcal{P}$,
   (b) $\mathcal{P}(\beta) \models \ulcorner\mathsf{ZFC}\text{-Replacement}\urcorner$,
   (c) $\mathcal{P}(\beta) = \mathcal{O}^{\mathcal{P},\omega}_{\delta_\beta}$ [3],
   (d) if β is a limit then $\delta_\beta^{+\mathcal{P}} = \delta_\beta^{+\mathcal{P}(\beta)}$,
   (e) if $\beta < \lambda$ then $f(\gamma_\beta)$ codes a $(o(\mathcal{P}), o(\mathcal{P}))$-strategy, call it $\Sigma^{\mathcal{P}}_\beta$, for $\mathcal{P}(\beta)$ with hull condensation [4],

confirm with Grigor that this is what he had in mind

   (f) if $\alpha < \beta < \lambda$, then $(\Sigma^{\mathcal{P}}_\beta)_{\mathcal{P}(\alpha)} = \Sigma^{\mathcal{P}}_\alpha$,
   (g) if $\beta < \lambda$ and $\eta \in (\delta_\beta, \delta_{\beta+1})$ is a $\mathcal{P}$-successor cardinal, then $\mathcal{P}|\eta$ is a $\Sigma^{\mathcal{P}}_{\gamma_\beta}$-premouse over $\mathcal{P}(\beta)$ which is $(o(\mathcal{P}), o(\mathcal{P}))$-iterable for stacks above δ_β.

(v) $\forall n < \omega \colon \mathcal{P} \models \delta_\lambda^{+n}$ exists and $o(\mathcal{P}) = \sup_{n<\omega}(\delta_\lambda^{+n})^{\mathcal{P}}$.

See Figure 1.4.
We will often write $\delta^{\mathcal{P}}_\beta, \gamma^{\mathcal{P}}_\beta, \lambda^{\mathcal{P}}$ for $\delta_\beta, \gamma_\beta, \lambda$ as above and moreover let $\delta^{\mathcal{P}} := \delta_\lambda$. ◦

include an intutive description of HOD-mice

[2] These are in fact HOD-premice below $\ulcorner\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable$\urcorner$ in [?]. However, since all of our HOD-mice are of this form, we omit this.
[3] see [?, Definition 1.26]
[4] note that $\Sigma^{\mathcal{P}}_\beta \subseteq \mathcal{P}$ is an internal strategy, i.e. only defined on trees that are elements of $\mathcal{P}$

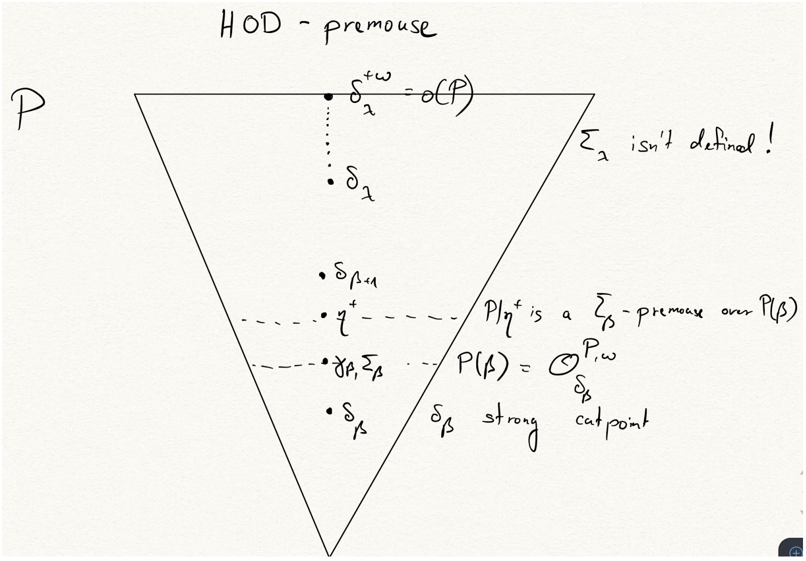

Figure 1.4: HOD-premouse

Definition 1.1.24. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse. We let

$$\mathcal{P}^- = \begin{cases} P|\gamma_{\lambda^{\mathcal{P}}-1} & \text{, if } \lambda^{\mathcal{P}} \text{ is a successor ordinal,} \\ \mathcal{P}|\delta^{\mathcal{P}} & \text{, otherwise.} \end{cases}$$

See **??**

add picture and figure out why we don't just let $\mathcal{P}^- = \mathcal{P}(\gamma^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1})$ in the successor case.

$\circ$

Definition 1.1.25. Let $\mathcal{P}, \mathcal{Q}$ be HOD-premice. We write $\mathcal{P} \trianglelefteq_{\text{HOD}} \mathcal{Q}$ if there is some $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We also write $\mathcal{P} \triangleleft_{\text{HOD}} \mathcal{Q}$ if $\mathcal{P} \trianglelefteq_{\text{HOD}} \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$.
In this case we say that $\mathcal{P}$ is a (proper) HOD-*initial segment* of $\mathcal{Q}$. $\circ$

Definition 1.1.26. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse and $\alpha \leq \lambda^{\mathcal{P}}$.

(i) If $\alpha < \lambda^{\mathcal{P}}$, we let $\Sigma^{\mathcal{P}}_\alpha$ be the internal iteration strategy of $\mathcal{P}(\alpha)$ coded by $f(\alpha)$ and

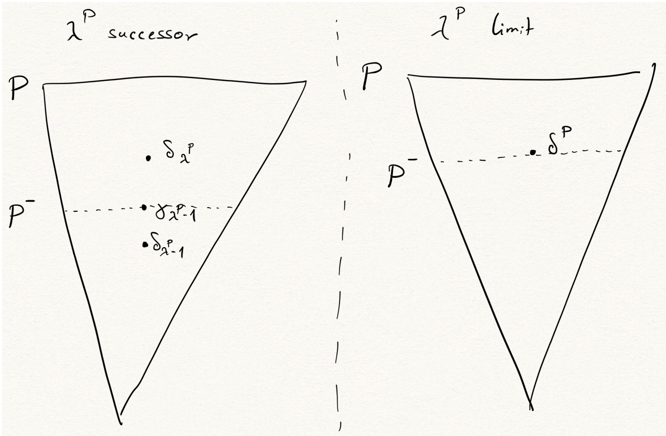

Figure 1.5: $\mathcal{P}^-$

(ii) $\Sigma^{\mathcal{P}}_{<\alpha} := \bigoplus_{\beta<\alpha} \Sigma^{\mathcal{P}}_{\beta}$.

We also let $\Sigma^{\mathcal{P}} := \Sigma^{\mathcal{P}}_{<\lambda^{\mathcal{P}}}$. ○

reference broken

Remark 1.1.27. By the agreement of the internal iteration strategies of HOD-premice (item 4f in subsection 1.1.23), $\Sigma^{\mathcal{P}}_{\alpha}$ already includes all of the information of $\Sigma^{\mathcal{P}}_{<\alpha}$ and can be identified with $\Sigma^{\mathcal{P}}_{<\alpha+1}$.

add reference

Definition 1.1.28. Let $\mathcal{P}$ be a HOD-premouse. Σ is a (κ, λ)-*iteration strategy* for $\mathcal{P}$ if it is a winning strategy for player II in the iteration game $\mathcal{G}(\mathcal{P}, \kappa, \lambda)$ and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, then $\mathcal{Q}$ is a HOD-premouse such that $\Sigma^{\mathcal{Q}} = \Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \cap \mathcal{Q}$. ○

Remark 1.1.29. In particular, $\Sigma^{\mathcal{P}} = \Sigma \cap \mathcal{P}$, i.e. Σ extends the internal iteration strategy of $\mathcal{P}$.

Definition 1.1.30. $(\mathcal{P}, \Sigma)$ is a HOD-*pair* if

(i) $\mathcal{P}$ is a HOD-premouse and

(ii) Σ is a $(\omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{P}$ with hull condensation.

the definition of hod pair is different in both versions of Grigor‘s thesis. Verify that this is the intended one.

○

1.1.31 HOD Analysis

gather all the information we need on HOD – this can be found in Grigor's thesis

Definition 1.1.32. Let $(P, \Sigma), (Q, \Lambda)$ be HOD-pairs. We let $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$, i.e. there is a $(\mathcal{P}, \Sigma)$-iterate $(\mathcal{T}, R)$ and a $(\mathcal{Q}, \Lambda)$-iterate $(\mathcal{U}, S)$ such that

$$\mathcal{R} \trianglelefteq_{\mathrm{HOD}} \mathcal{S} \text{ and } \Sigma_{\mathcal{R},\mathcal{T}} = \Lambda_{\mathcal{R},\mathcal{U}}.$$

We also let $(\mathcal{P}, \Sigma) <_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ and $(\mathcal{Q}, \Lambda) \not\leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.

○

Definition 1.1.33. Let $(\mathcal{P}, \Sigma)$ be a HOD-pair such that Σ has branch condensation and is fullness preserving. We recursively define $\alpha(\mathcal{P}, \Sigma) := |(\mathcal{P}, \Sigma)|_{\leq_{\mathrm{DJ}}} \in \mathsf{On}$ via

$$\begin{aligned} |(\mathcal{P}, \Sigma)|_{\leq_{\mathrm{DJ}}} = \sup\{|(\mathcal{Q}, \Lambda)|_{\leq_{\mathrm{DJ}}} + 1 \mid & (\mathcal{Q}, \Lambda) \text{is a } \mathrm{HOD}\text{-pair such that} \\ & \Lambda \text{ has branch condensation} \\ & \text{and is fullness preserving } \} \end{aligned}$$

○

Remark 1.1.34. As in the case of ordinary premice, $\leq_{\mathrm{DJ}}$ (or rather $<_{\mathrm{DJ}}$) is a wellfounded relation. The interesting question is whether it's total.

Theorem 1.1.35 (Sargsyan)**.** *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are fullness preserving. Then $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ or $(\mathcal{Q}, \Lambda) \leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.*

Proof. [?, Theorem 5.10]. ■

define Code Σ

Theorem 1.1.36 (Sargsyan)**.** *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P},\Sigma),(\mathcal{Q},\Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are Γ-fullness preserving for some pointclass Γ which is closed under continuous images and preimages. Suppose further that there is a good pointclass Γ^* such that $\Gamma\cup\{\mathrm{Code}(\Sigma),\mathrm{Code}(\Lambda)\}\subseteq \underset{\sim}{\Delta}_{\Gamma^*}$. Then $(\mathcal{P},\Sigma)\leq_{\mathrm{DJ}}(\mathcal{Q},\Lambda)$ or $(\mathcal{Q},\Lambda)\leq_{\mathrm{DJ}}(\mathcal{P},\Sigma)$.*

Proof. [?, Theorem 2.33]. ■

Definition 1.1.37. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P},\Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. We let

(i) $\mathcal{F}(\mathcal{P},\Sigma)=\{(\mathcal{Q},\Sigma_{\mathcal{Q}})\mid \mathcal{Q}\in pB(\mathcal{P},\Sigma)\}$ and

(ii) $\mathcal{F}^+(\mathcal{P},\Sigma)=\{(\mathcal{Q},\Sigma_{\mathcal{Q}})\mid \mathcal{Q}\in pI(\mathcal{P},\Sigma)\}$.

∘

Remark 1.1.38. By [?, Corollary 2.44] Σ is commuting, so that $\Sigma_{\mathcal{Q}}$ is indeed well-defined.

Definition 1.1.39. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P},\Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q},\mathcal{R}\in pI(\mathcal{P},\Sigma)\cup pB(\mathcal{P},\Sigma)$. We let $\mathcal{Q}\leq^{\mathcal{P},\Sigma}\mathcal{R}$ if

(i) $\mathcal{Q}\in pI(\mathcal{P},\Sigma)$ and $R\in pI(\mathcal{Q},\Sigma_{Q})$ or

(ii) $\mathcal{Q}\in pB(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Sigma_{\mathcal{Q}})\leq_{\mathrm{DJ}}(\mathcal{R},\Sigma_{\mathcal{R}})$.

∘

Lemma 1.1.40 (Sargsyan)**.** *$\leq^{\mathcal{P},\Sigma}$ is directed.*

Proof. [?, Lemma 4.17]. ■

Definition 1.1.41. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P},\Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q},\mathcal{R}\in \mathrm{pI}(\mathcal{P},\Sigma)\cup\mathrm{pB}(\mathcal{P},\Sigma)$ be such that, for some

$\alpha \leq^{\mathcal{R}}$, $\mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. We let

$$\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q} \to \mathcal{R}(\alpha)$$

be the iteration map given by $\Sigma_{\mathcal{Q}}$.
We let

(i) $\mathcal{M}_\infty(\mathcal{P}, \Sigma) = \operatorname{dirlim}(\mathcal{F}(\mathcal{P}, \Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q}, \mathcal{R} \in pB(\mathcal{P}, \Sigma) \wedge \exists \alpha \leq \lambda^{\mathcal{R}} \mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}}))$ and

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma) = \operatorname{dirlim}(\mathcal{F}(\mathcal{P}, \Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \leq^{\Sigma}_{\mathcal{Q},\mathcal{R}} \mathcal{R})$.

For $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ we let

(i) $\pi^{\Sigma}_{\mathcal{Q},\infty} \colon \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$ and

(ii) $\sigma^{\Sigma}_{\mathcal{R},\infty} \colon \mathcal{R} \to \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$

be the direct limit maps. ◦

Definition 1.1.42. Let $(\mathcal{P}, \Sigma)$ be as above. We let

(i) $\delta_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$,

(ii) $\delta^+_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ and

(iii) $\lambda_\infty(\mathcal{P}, \Sigma) := \lambda^{\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)}$.

◦

Lemma 1.1.43 (Sargsyan). *Let Γ be a pointclass closed under Wadge reducibility. Suppose $(\mathcal{P}, \Sigma)$ is a* HOD*-pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal and Σ has branch condensation and is Γ-fullness preserving. Then*

(i) $\delta_\infty(\mathcal{P}, \Sigma) = \delta^+_\infty(\mathcal{P}, \Sigma)$ *and*

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma) | \delta^+_\infty = \mathcal{M}_\infty(\mathcal{P}, \Sigma)$.

Proof. [?, Lemma 4.18]. ■

We will likely not need the entire theorem and should reduce it to the part that we need once we are done.

Theorem 1.1.44 (Sargsyan). *Assume AD^+, let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Then the following holds:*

(i) If $L(\Gamma, \mathbb{R}) \models \phi$ then for all $(\mathcal{P}, \Sigma) \in \Gamma$ such that $\alpha(\mathcal{P}, \Sigma) < \Omega^\Gamma$ we have, for all $\alpha \leq \alpha(\mathcal{P}, \Sigma)$,

define ϕ

define Ω^Γ

define θ^Γ_α

(a) $\delta_\alpha^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} = \theta^\Gamma_\alpha$ *and*

(b) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}; \in, \vec{E}^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} \restriction \theta^\Gamma_\alpha, \Lambda \restriction \theta^\Gamma_\alpha)$,

where Λ *is the iteration strategy coded by* $f^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}$.

define ψ

(ii) *If* $L(\Gamma,\mathbb{R}) \models \psi$ *then for all* $\alpha \le \Omega^\Gamma$

(a) $\delta_\alpha^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} = \theta^\Gamma_\alpha$ *and*

(b) $\mathcal{M}^+_\infty(\mathcal{P},\Sigma)|\theta^\Gamma_\alpha = (V^{\mathcal{H}}_{\theta^\Gamma_\alpha}; \in, \vec{E}^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)} \restriction \theta^\Gamma_\alpha, \Lambda \restriction \theta^\Gamma_\alpha)$.

(iii) *Suppose* $\Gamma^* \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \models \mathsf{AD}^+$ *and there is a* HOD*-pair* $(\mathcal{P},\Sigma) \in \Gamma^*$ *such that*

(a) Σ *has branch condensation and is* Γ*-fullness preserving,*

(b) $\lambda^{\mathcal{P}}$ *is a successor ordinal,* $\mathrm{Code}(\Sigma_{\mathcal{P}^-}) \in \Gamma$ *and* $L(\Gamma,\mathbb{R}) \models \ulcorner(\mathcal{P},\Sigma_{\mathcal{P}^-})$ *is a suitable pair* $\alpha\urcorner$,

define suitable pair

(c) *there is a sequence* $(B_i \mid i<\omega) \subseteq \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *which guides* Σ *and*

define $\mathbb{B}(..)$ and what it means to be guided

(d) *for any* $B \in \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *there is some* $\mathcal{R} \in \mathrm{pI}(\mathcal{P},\Sigma)$ *such that* $\Sigma_{\mathcal{R}}$ *respects* B.

Then $L(\Gamma,\mathbb{R}) \models \psi$ *and* $\mathcal{M}_\infty(\mathcal{P},\Sigma) = \mathcal{M}^+_\infty(\mathcal{P},\Sigma)$.

define respects B

Proof. [?, Theorem 4.24]. ■

1.2 The tame case

Define

$$\Gamma_0 := \{A \subseteq \mathbb{R} \mid L(A,\mathbb{R}) \models \mathsf{AD} + \Omega = 0\}.$$

Lemma 1.2.1 (DI)**.** $\Gamma_0 = \mathrm{Lp}(\mathbb{R}) \cap \mathscr{P}(\mathbb{R})$.

Proof. ($\supseteq$) Let $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$ and let $A \subseteq \mathbb{R}$ be an element of $\mathcal{M}$. Since $\mathcal{M}$ projects to $\mathbb{R}$ and is sound, we get that A is OD_x for a real x, so that everything in $L(A,\mathbb{R})$ is also ordinal definable in a real as well. Since $\mathrm{Lp}(\mathbb{R}) \models \mathsf{AD}$ we then get that $\mathsf{AD} + \Omega = 0$ holds in $L(A,\mathbb{R})$, making $A \in \Gamma_0$.

Check this proof.

($\subseteq$) Let $A \in \Gamma_0$. Since we're assuming CH we get that $V[g] \models |\mathbb{R}| = \aleph_1^V = \aleph_0$, so fix a generic bijection $b : \omega \to \mathbb{R}^V$ in $V[g]$. Define $a_b \in \mathbb{R}$ as $n \in a_b$ iff $b(n) \in A$. As $L(A,\mathbb{R}) \models \mathsf{AD} + \theta_0 = \Theta$ it holds that A is $\mathrm{OD}_z^{L(A,\mathbb{R})}$

for $z \in \mathbb{R}$, so that

$$A = j(A) \cap \mathbb{R}^V \in \mathrm{OD}_{z,\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}.$$

In particular, as A and $\mathbb{R}^V$ are definable from b and a_b is definable from b, we get that $a_b \in \mathrm{OD}_b^{L(j(A),\mathbb{R}^{V[g]})}$. By MC we then get that there's some b-premouse $\mathcal{M} \in L(j(A),\mathbb{R}^{V[g]})$ projecting to b with $a_b \in \mathcal{M}$ and a Σ such that

$$L(j(A),\mathbb{R}^{V[g]}) \models \ulcorner\Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{M}\urcorner.$$

Why is it that we have to go through b in this fashion? Can't we just use MC and get $\mathcal{N}$ without going through $\mathcal{M}$? Is it because $L(j(A),\mathbb{R}^{V[g]})$ doesn't know that $\mathbb{R}^V$ is countable?

From this $\mathcal{M}$ we can then get an $\mathbb{R}^V$-premouse $\mathcal{N} \in L(j(A),\mathbb{R}^{V[g]})$ projecting to $\mathbb{R}^V$ with $A \in \mathcal{N}$ and

$$L(j(A),\mathbb{R}^{V[g]}) \models \ulcorner\Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{N}\urcorner.$$

Now $\mathcal{N}$ is $\mathrm{OD}_{\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}$, and since we don't have divergent models of AD^+ it holds that, letting $\Theta^{j(A)} := \Theta^{L(j(A),\mathbb{R}^{V[g]})}$,

$$V[g] \models L(j(A),\mathbb{R}) = L(P_{\Theta^{j(A)}}(\mathbb{R})).$$

This means that $\mathcal{N} \in \mathrm{OD}_{\mathbb{R}^V}^{V[g]}$, so that homogeneity of I we get that $\mathcal{N} \in V$. It remains to show that $\mathcal{N} \trianglelefteq \mathrm{Lp}(\mathbb{R}^V)$, meaning that we need to show that $\mathcal{N}$ is countably (ω_1+1)-iterable in V. But letting $\overline{\mathcal{N}} \to \mathcal{N}$ be a countable hull in V we get that $j(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$, so that elementarity of j implies that $\Sigma \restriction V \in V$ is an $\omega_1^{V[g]} = \omega_2^V$-iteration strategy for $\overline{\mathcal{N}}$ and we're done. ■ Why's this?

Is this really this iterable?

Proposition 1.2.2 (DI)**.** $\mathrm{cof}^V(\Theta^{\mathrm{Lp}(\mathbb{R})}) = \omega$.

PROOF.

See Ketchersid's Thesis 3.17 or 7.4.2 in the CMI book. Perhaps we don't need it though, following Wilson's thesis.

■

Theorem 1.2.3. *Let Γ be an inductive-like pointclass. If $\mathcal{M}$ is a suitable quasi-iterable premouse, $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ is closed under recursive join and the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is both total on $\mathcal{M}$ and has the full factors property, then there's a unique Γ-fullness preserving (ω_1, ω_1)-strategy Φ for $\mathcal{M}$ such that, for every quasi-iterate $\mathcal{P}$ of $\mathcal{M}$,*

- *$\mathcal{P}$ is a non-dropping Φ-iterate of $\mathcal{M}$; and*
- *the Φ-iteration map $i : \mathcal{M} \to \mathcal{P}$ equals the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\mathcal{P}}$.*

Let $\Phi_{\mathcal{M}}$ be the unique strategy for $\mathcal{M}$ as in the above theorem. We now improve this to include branch condensation.

The 3d argument is quite similar to the proof of Theorem 7.19 in the outline.

Theorem 1.2.4. *Let Γ be an inductive-like pointclass and assume that $\boldsymbol{\Delta}_\Gamma$ is determined and that Γ-MC holds. Let $\mathcal{M}$ be an ω-suitable quasi-iterable premouse such that $\mathcal{D}(\mathcal{M}) \equiv \mathcal{M}_\Gamma$, let $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ be closed under recursive join, assume $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and that it has the full factors property. Let $\Phi := \Phi_{\mathcal{M}}$. Then there's a $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ has $\mathcal{A}$-condensation, and hence also branch condensation, for every $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$.*

This is the companion of Γ, see Trevor's thesis. I'm not sure if we can find $\mathcal{M}$ like this, however.

Proof. Assume not and fix $A \in \mathrm{Env}(\Gamma)$ such that given any $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ there's a $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ doesn't have A-condensation. Applying this inductively, we get a sequences $\langle \mathcal{Q}^0_n, \mathcal{R}^0_n, \mathcal{T}^0_n, \pi^0_n, \sigma^0_n, j^0_n \mid n < \omega\rangle$ such that

(i) $\mathcal{Q}^0_0 := \mathcal{M}$;

(ii) $\pi^0_n : \mathcal{Q}^0_n \to \mathcal{Q}^0_{n+1}$ is the iteration map through a tree of successor length, according to Φ;

$$\begin{tikzcd}
\mathcal{Q}_0^0 \arrow[rr, "\pi_0^0"] \arrow[dr, "\sigma_0^0"'] \arrow[dd, "\tau_0^0"'] & & \mathcal{Q}_1^0 \arrow[rr, "\pi_1^0"] \arrow[dr, "\sigma_1^0"'] \arrow[dd, "\tau_1^0"'] & & \mathcal{Q}_2^0 \arrow[r, "\pi_2^0"] \arrow[dd, "\tau_2^0"'] & \cdots & \mathcal{Q}_\omega^0 \arrow[dd, "\tau_\omega^0"] \\
& \mathcal{R}_0^0 \arrow[ur, "j_0^0"'] \arrow[dd, "k_0^0"] & & \mathcal{R}_1^0 \arrow[ur, "j_1^0"'] \arrow[dd, "k_1^0"] & & & \\
\mathcal{Q}_0^1 \arrow[rr, "\pi_0^1"] \arrow[dr, "\sigma_0^1"'] \arrow[dd, "\tau_0^1"'] & & \mathcal{Q}_1^1 \arrow[rr, "\pi_1^1"] \arrow[dr, "\sigma_1^1"'] \arrow[dd, "\tau_1^1"'] & & \mathcal{Q}_2^1 \arrow[r, "\pi_2^1"] \arrow[dd, "\tau_2^1"'] & \cdots & \mathcal{Q}_\omega^1 \arrow[dd, "\tau_\omega^1"] \\
& \mathcal{R}_0^1 \arrow[ur, "j_0^1"'] \arrow[dd, "k_0^1"] & & \mathcal{R}_1^1 \arrow[ur, "j_1^1"'] \arrow[dd, "k_1^1"] & & & \\
\mathcal{Q}_0^2 \arrow[rr, "\pi_0^2"] \arrow[dr, "\sigma_0^2"'] \arrow[dd, "\tau_0^2"'] & & \mathcal{Q}_1^2 \arrow[rr, "\pi_1^2"] \arrow[dr, "\sigma_1^2"'] \arrow[dd, "\tau_1^2"'] & & \mathcal{Q}_2^2 \arrow[r, "\pi_2^2"] \arrow[dd, "\tau_2^2"'] & \cdots & \mathcal{Q}_\omega^2 \arrow[dd, "\tau_\omega^2"] \\
& \mathcal{R}_0^2 \arrow[ur, "j_0^2"'] \arrow[dd, "k_0^2"] & & \mathcal{R}_1^2 \arrow[ur, "j_1^2"'] \arrow[dd, "k_1^2"] & & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
& \vdots & & \vdots & & & \\
\mathcal{Q}_0^\omega \arrow[rr, "\pi_0^\omega"] \arrow[dr, "\sigma_0^\omega"'] & & \mathcal{Q}_1^\omega \arrow[rr, "\pi_1^\omega"] \arrow[dr, "\sigma_1^\omega"'] & & \mathcal{Q}_2^\omega \arrow[r, "\pi_2^\omega"] & \cdots & \mathcal{Q}_\omega^\omega \\
& \mathcal{R}_0^\omega \arrow[ur, "j_0^\omega"'] & & \mathcal{R}_1^\omega \arrow[ur, "j_1^\omega"'] & & &
\end{tikzcd}$$

Figure 1.6: The three-dimensional argument in Theorem **??**

(iii) $\sigma_n^0 : \mathcal{Q}_n^0 \to \mathcal{R}_n^0$ an iteration map through a tree of limit length, according to Φ;

(iv) $j_n^0 : \mathcal{R}_n^0 \to \mathcal{Q}_{n+1}^0$ is elementary such that $\pi_n^0 = j_n^0 \circ \sigma_n^0$;

(v) $(j_n^0)^{-1}(\tau_{A,j_n^0(\kappa)}^{\mathcal{Q}_{n+1}^0}) \neq \tau_{A,\kappa}^{\mathcal{R}_n^0}$ for every $\mathcal{R}_n^0$-cardinal $\kappa \geq \delta_0^{\mathcal{R}_n^0}$.

Let $\mathcal{Q}_\omega^0$ be the direct limit of the $\mathcal{Q}_n^0$'s under the π_n^0 maps. Also let $\langle x_n \mid n < \omega\rangle$ enumerate the reals of $\mathcal{M}_\Gamma$ and pick $s \in [\mathsf{On}]^{<\omega}$ and a formula φ such that

$$\forall x \in \mathbb{R}(x \in A \Leftrightarrow \mathcal{M}_\Gamma \models \varphi[x,s]).$$

Our strategy now is now firstly to capture all the x_n's so that the derived models of the resulting structures become equal to $\mathcal{M}_\Gamma$. See Figure **??**.

Perform a genericity iteration of $\mathcal{Q}_0^0$ above $\delta_0^{\mathcal{Q}_0^0}$ to $\mathcal{Q}_0^1$ to make x_0 generic over $\mathcal{Q}_0^1$ at $\delta_1^{\mathcal{Q}_0^1}$, while lifting the genericity iteration tree via the copy construction to the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and picking branches on the genericity iteration tree on $\mathcal{Q}_0^0$ by using $\Phi_{\mathcal{Q}_\omega^0}$ on the lifted tree on $\mathcal{Q}_\omega^0$. Let $\tau_0^0 : \mathcal{Q}_0^0 \to \mathcal{Q}_0^1$ be the genericity iteration map and $\mathcal{W}_0$ the last model of the lifted tree on $\mathcal{Q}_\omega^0$.

Now perform another genericity iteration of the last model of the lifted iteration tree on $\mathcal{R}_0^0$ above its δ_0 to $\mathcal{R}_0^1$ to make x_0 generic over $\mathcal{R}_0^1$ at $\delta_1^{\mathcal{R}_0^1}$, with branches being picked by lifting the iteration tree to $\mathcal{W}_0$ and using the branches according to $\Phi_{\mathcal{W}_0}$. Let $k_0^0 : \mathcal{R}_0^0 \to \mathcal{R}_0^1$ be the iteration embedding, $\sigma_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_0^1$ be the shift of σ_0^0 followed by latter genericity iteration, and $\mathcal{W}_1$ the last model of the lifted tree on $\mathcal{W}_0$.

Do a third genericity iteration of the last model of the lifted stack on $\mathcal{Q}_1^0$ above its δ_0 to $\mathcal{Q}_1^1$ to make x_0 generic at $\delta_1^{\mathcal{Q}_1^1}$, with branches being picked by lifting the tree to $\mathcal{W}_1$ and using branches picked by $\Phi_{\mathcal{W}_1}$. Let $\tau_1^0 : \mathcal{Q}_1^0 \to \mathcal{Q}_1^1$ be the iteration embedding, $j_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_1^1$ be the natural map, and $\pi_0^1 := j_0^1 \circ \sigma_0^1$.

Now continue this process to make x_0 generic over the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and let $\mathcal{Q}_\omega^1$ be the direct limit of the $\mathcal{Q}_n^1$'s under the π_n^1 maps. Then start at $\mathcal{Q}_0^1$ and repeat the same thing to make x_1 generic at the respective δ_2's and so on. Let $\mathcal{Q}_i^\omega$ be the direct limit of the $\mathcal{Q}_i^n$'s under the τ_i^n maps, $\mathcal{R}_i^\omega$

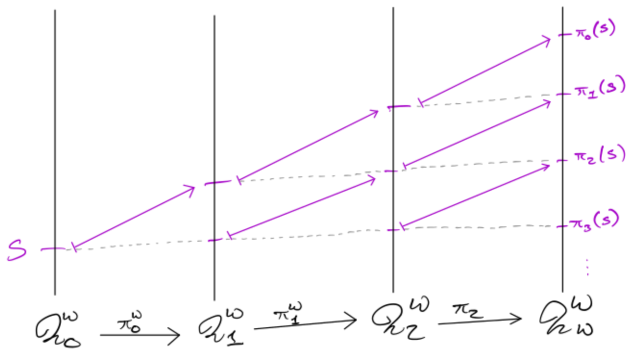

Figure 1.7: The argument in Claim **??**.

the direct limit of the $\mathcal{R}^n_i$'s under the k^n_i maps and $\mathcal{Q}^n_\omega$ the direct limit of the $\mathcal{Q}^n_i$'s under the π^n_i maps.

By construction we get that the π^0_n's and τ^n_ω's are all by Φ and its tails, and that $\mathcal{Q}^\omega_\omega$ is wellfounded and Lp^Γ-full, so that the $\mathcal{Q}^\omega_n$'s and the $\mathcal{R}^\omega_n$'s are also wellfounded and Lp^Γ-full.

Claim 1.2.4.1. There exists some $k < \omega$ such that π^ω_n fixes s for every $n \geq k$.

Proof of claim. It suffices to show that $(\pi^\omega_n(\xi) \mid n < \omega)$ is eventually constant for all $\xi \in s$. Suppose this isn't the case. Fix $\xi \in s$ and a strictly increasing sequence $(i_n \mid n < \omega)$ such that $\pi^\omega_{i_n}(\xi) > \xi$ for all $n < \omega$. For $m < n < \omega$ we then have

$$\pi^\omega_{i_m,\infty}(\xi) = \pi^\omega_{i_n,\infty} \circ \pi^\omega_{i_m,i_n}(\xi) \geq \pi^\omega_{i_n,\infty} \circ \pi^\omega_{i_m}(\xi) > \pi^\omega_{i_n,\infty}(\xi),$$

so that $(\pi^\omega_{i_n}(\xi) \mid n < \omega)$ is a strictly decreasing sequence of ordinals in $\mathcal{Q}^\omega_\omega$ – contradicting its wellfoundedness. See **??**. $\dashv$

Let $k < \omega$ be as in the claim, and note that the j^ω_n's also fix s for $n \geq k$. Since $\mathcal{D}(\mathcal{R}^\omega_n) = \mathcal{M}_\Gamma$ for every $n < \omega$, the $\mathcal{Q}^\omega_n$'s and the $\mathcal{R}^\omega_n$'s have uniform definitions for the term relations for A when $n \geq k$, yielding that j^ω_n pulls back the term relation correctly whenever $n \geq k$, $\lightning$. ■

Theorem 1.2.5 (DI^+)**.** $\mathrm{Lp}(\mathbb{R}) \models \ulcorner$*there's a fullness preserving hod pair below* $\omega_1\urcorner$.

Proof.

Show the above requirements in Wilson's theorem is satisfied? Double check the statement.

■

Theorem 1.2.6 (DI^+)**.** *There is a model M containing all the reals such that $M \models \mathsf{AD}^+ + \theta_0 < \Theta$.*

Proof.

Let $(\mathcal{M}, \Sigma)$ be a fullness preserving hod pair in $\mathrm{Lp}(\mathbb{R})$ given by the above theorem. Then $\Sigma \notin \mathrm{Lp}(\mathbb{R})$ by the proof of 7.4.3 in the CMI book, and in particular $\Sigma \notin \Gamma_0$. Then $M := L(\Sigma, \mathbb{R})$ is the wanted model.

■

1.3 The successor case

Definition 1.3.1. Let $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ be hod pairs below ω_1. We then say that $(\mathcal{Q}, \Lambda)$ *extends* $(\mathcal{P}, \Sigma)$, or is an *extension* of $(\mathcal{P}, \Sigma)$, if there exists some $\alpha < \lambda^{\mathcal{Q}}$ such that

(i) $\mathcal{Q}(\alpha) \in pI(\mathcal{P}, \Sigma)$; and

(ii) $\Sigma_{\mathcal{Q}(\alpha)} = \Lambda_{\mathcal{Q}(\alpha)}$.

We say that $(\mathcal{P}, \Sigma)$ *can be extended* if there exists an extension of $(\mathcal{P}, \Sigma)$. ∘

Theorem 1.3.2 (DI^+)**.** *Every hod pair below ω_1 can be extended.*

Rough steps in the proof:

(i) Show that $M_1^{\sharp,\Sigma}$ exists

(ii) $\mathrm{Lp}^\Sigma(\mathbb{R}) \models \mathsf{AD}^+$ for some appropriate definition of $\mathrm{Lp}^\Sigma(\mathbb{R})$

(iii) The $\Omega > 0$ argument should show that there's an $A \notin \mathrm{Lp}^{\Sigma}(\mathbb{R})$ such that $L(A, \mathbb{R}) \models \mathsf{AD}^{+}$ and $\Sigma <_W A$

(iv) Show $L(A, \mathbb{R})$ then has the desired $(\mathcal{Q}, \Lambda)$ (this step has already been done and can be black boxed)

1.4 The limit case

Theorem 1.4.1 (DI^{+}). *Assume there exists a sequence of hod pairs $(\mathcal{P}_\alpha, \Sigma_\alpha)$ below ω_1 with $(\mathcal{P}_{\alpha+1}, \Sigma_{\alpha+1})$ extending $(\mathcal{P}_\alpha, \Sigma_\alpha)$ for every α. Then either*

(i) There exists a hod pair $(\mathcal{H}, \Lambda)$ below ω_1 such that $\lambda^{\mathcal{H}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$; or

(ii) There exists an $\mathcal{M}$ containing all the reals such that $\mathcal{M} \models \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Rough steps in the proof:

(i) Do the easier countable cofinality case

(ii) Coiterate all the hod pairs to some $(\mathcal{P}, \Sigma)$, which has $\lambda := \lambda^{\mathcal{P}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$

(iii) If λ has non-measurable cofinality then $(\mathcal{P}, \Sigma)$ is the hod pair that we're looking for, so assume this is not the case

(iv) Take the derived model $\mathcal{D}(\mathcal{P}, \lambda)$, which then satisfies $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC} + \Omega = \lambda$, where DC is because λ has uncountable cofinality

This is wrong, as we can't take this derived model. Instead we should form a directed system of all "nice" hod pairs having λ's below $\lambda^{\mathcal{P}}$ and take the Lp-closure of that, which should then be an initial segment of hod; call it $\mathcal{H}$.

(v) Show that $\mathcal{H} \,|\, \delta^{\mathcal{H}}$ is the union of M_∞^α for $\alpha < \lambda$, where M_∞^α is the hod limit of

$$\mathcal{F}_\alpha := \{(\mathcal{Q}, \Psi) \mid \mathrm{Ult}(V, g) \models \ulcorner(\mathcal{Q}, \Psi) \text{ is a hod pair and } \lambda^{\mathcal{Q}} = \alpha\urcorner\}.$$ [5]

Let Φ be the join of the strategies of the M_∞^α's and show that $\mathcal{H} = \mathrm{Lp}_\omega^{\Phi}(\mathcal{H} \,|\, \delta^{\mathcal{H}})$.

(vi) Show that $\mathcal{H} \models \ulcorner \delta^{\mathcal{H}}$ is singular$\urcorner$, since otherwise $\mathcal{D}(\mathcal{H}, \delta^{\mathcal{H}}) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular and we're done.

(vii) We want to construct a strategy Λ for $\mathcal{H}$ such that $(\mathcal{H}, \Lambda)$ is a hod pair below ω_1, as then this is the hod pair that we're looking for.

Definition 1.4.2. Let $(\mathcal{P}, \Sigma)$ be a hod pair. We let

(i) $\mathrm{I}(\mathcal{P}, \Sigma) := \{(\vec{\mathcal{T}}, \mathcal{Q}) \mid \vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ via Σ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists$\}$ be the collection of $(\mathcal{P}, \Sigma)$-*iterates* ,

(ii) $\mathrm{pI}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$,

(iii) $\mathrm{B}(\mathcal{P}, \Sigma) := \{(\mathcal{T}, \mathcal{M}) \mid \mathcal{M} \lhd_{\mathrm{HOD}} \mathcal{Q}$ and $(\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)\}$ be the collection of $(\mathcal{P}, \Sigma)$-*blowups* and

(iv) $\mathrm{pB}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{B}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$.

$\circ$

Definition 1.4.3. Let $(\mathcal{P}, \Sigma)$ be a hod pair and Γ is a pointclass closed under Boolean operations and continuous images and preimages. Then Σ is Γ-*fullness preserving* if for all $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ and $\delta^{\mathcal{Q}}_\alpha < \eta$ which is a strong cutpoint of $\mathcal{Q}(\alpha+1)$ we have

(i) $\mathcal{Q}|\eta^{+\mathcal{Q}(\alpha+1)} = \mathrm{Lp}^{\Gamma, \Sigma_{\mathcal{Q}(\alpha), \vec{\mathcal{T}}}}(\mathcal{Q}|\eta)$ and

(ii) $\mathcal{Q}|\delta^{+\mathcal{Q}}_\alpha = \mathrm{Lp}^{\Gamma, \oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1)}, \vec{\mathcal{T}}}(\mathcal{Q}(\alpha))$.

Provide a motivation for this definition.

Σ is *fullness preserving* iff it is $\mathcal{P}(\mathbb{R})$-fullness preserving.

$\circ$

This will be useful in the proof of the A-condensing lemma.

Lemma 1.4.4. *Let M, N be transitive models of ZFC^- with largest cardinals δ^M, δ^N respectively. Let $\pi \colon M \to N$ be an elementary embedding, $\kappa := \mathrm{crit}(\pi)$ and let E be the long (κ, δ^N)-extender derived from π. Then $N = \mathrm{Ult}(M; E)$ and $\pi = \pi_E$ is the canonical ultrapower embedding.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "\pi"] \arrow[drr, "\pi_E"'] & & N \\
& & \mathrm{Ult}(M;E) \arrow[u, "k"']
\end{tikzcd}$$

$$\begin{tikzcd} & j(\mathcal{H}) \\ \mathcal{H} \arrow[ur, "j"] \arrow[r, "\pi"] & \mathcal{R} \arrow[u, "\tau"'] \end{tikzcd}$$

Figure 1.8: Full Factors Property

where k satisfies $k \restriction \delta^N = \mathrm{id}$. Let $\delta^{\mathrm{Ult}(M;E)}$ be the largest cardinal of $\mathrm{Ult}(M;E)$. By elementarity $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^N$, so that $\delta^{\mathrm{Ult}(M;E)} \leq \delta^N$. If $\delta^{\mathrm{Ult}(M;E)} < \delta^N$, then $k \restriction \delta^N = \mathrm{id}$ yields $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^{\mathrm{Ult}(M;E)} < \delta^N$, which is absurd. Hence $\delta^{\mathrm{Ult}(M;E)} = \delta^N$ and $k \restriction (\delta^{\mathrm{Ult}(M;E)} + 1) = \mathrm{id}$. Since $\delta^{\mathrm{Ult}(M;E)}$ is the largest cardinal of $\mathrm{Ult}(M;E)$, it follows that k doesn't have a critical point. Therefore $k = \mathrm{id}$, $N = \mathrm{Ult}(M;E)$ and $\pi = \pi_E$. ■

Lemma 1.4.5. *$j \restriction \mathcal{H}$ has the* full factors property[6], *meaning that whenever* $\mathcal{R}$

R has to be countable in $V[g]$. How can we ensure that, as this only gives that it has size $\leq \aleph_1$? Do we have to resort to the (long) claim in Grigor's uB paper?

is a hod premouse and there are elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$, then $\mathcal{R}$ is $\Sigma^2_1(j(\Omega)^\tau)$-full.

Proof. Let $\Psi := j(\Omega)^\tau$ and assume the lemma fails, meaning that we have a hod mouse $\mathcal{R}$ and elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$ and $\mathcal{R} \neq \mathrm{Lp}^\Psi_\omega(\mathcal{R} | \delta^\mathcal{R})$, witnessed without loss of generality by an $\mathcal{M} \trianglelefteq \mathrm{Lp}^\Psi(\mathcal{R} | \delta^\mathcal{R})$ such that $\rho(\mathcal{M}) = \delta^\mathcal{R}$ and which is not an initial segment of $\mathcal{R}$.

$$\begin{tikzcd} (\mathcal{H}, \Omega) \arrow[r, "\pi"] \arrow[dr, "j \restriction \mathcal{H}"'] & (\mathcal{R}, \Psi) \arrow[d, "\tau"] \\ & (j(\mathcal{H}), j(\Omega)) \end{tikzcd}$$

[6]This terminology was introduced in [?]; in [?] this was called *weak condensation*.

We can then fix some hod pair $(\S^*, \Lambda^*)$ such that $\tau``\mathcal{R}|\delta^{\mathcal{R}} \subseteq \mathrm{ran}(\pi^{\Lambda^*}_{\S^*,\infty})$, and furthermore let $\xi \leq \lambda^{\S^*}$ be least such that $\tau``\mathcal{R}|\delta^{\mathcal{R}} \subseteq \mathrm{ran}(\pi^{\Lambda^*}_{\S^*(\xi),\infty})$. Lastly let $(\S, \Lambda)$ be an extension of $(\S^*, \Lambda^*)$ such that $\lambda^{\S}$ is a limit ordinal.

Argue why $\S^*$ and $\S$ exist; we should be in the limit case to argue that $\S$ exists.

Let $\sigma : \mathcal{R}|\delta^{\mathcal{R}} \to \S|\delta^{\S}_{\gamma}$, where $\S^*(\xi)$ iterates to $\S(\gamma)$, be given by $\sigma(x) = y$ iff $\tau(x) = \pi^{\Lambda}_{\S(\gamma),\infty}(y)$.

$$\begin{tikzcd}
(\mathcal{R}|\delta^{\mathcal{R}}, \Psi) \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & (j(\mathcal{H})|\delta^{j(\mathcal{H})}, j(\Omega)) \\
& (S|\delta^{\S}_{\gamma}, \bigoplus_{\beta<\gamma} \Lambda_{\S(\beta)}) \arrow[ur, "\pi^{\Lambda}_{\S(\gamma),\infty}"'] &
\end{tikzcd}$$

This should follow from generation of good pointclasses.

We can fix some hod pair $(\S', \Lambda')$ such that

$$L(\Lambda', \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner.$$

This requires us to work in an AD$^+$ model, so we better assume that somewhere.

By coiterating $\S$ and $\S'$ we may assume without loss of generality that $\S = \S'$.

Claim 1.4.5.1. There exists a hod pair $(\mathcal{Q}, \Phi)$ such that $\lambda^{\mathcal{Q}}$ is a limit ordinal and $L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner$.

Proof of claim.

This claim shouldn't be needed, as we should be able to take $\mathcal{Q}$ to be $\S$ in our case, using facts about the Γ-pointclasses. Also ensure that $\mathcal{Q} \supseteq \mathcal{H}$, which is possible as we're stretching by j.

$\dashv$

Fix $(\mathcal{Q}, \Phi)$ as in the claim and let $\mathcal{N}$ be some mouse such that $\mathcal{M} \lhd \mathcal{N}$ and $\mathcal{N}$ has ω many Woodins on top of $\mathcal{M}$.

Explain how this is done. In Grigor's paper he's using that "$j(\eta)$ is closed under hybrid $\mathcal{N}_\omega$-operators". In our measurable cofinality case there might be enough room to get this. Postpone until later, when we have an idea of how much operator closure we have at this point.

Then we get that

Why is $\Gamma(\mathcal{Q},\Phi)$ in $\mathcal{D}(\mathcal{N})$?

$\mathcal{D}(\mathcal{N}) \models \ulcorner L(\Gamma(\mathcal{Q},\Phi),\mathbb{R}) \models \ulcorner \mathcal{M}$ is a Ψ-mouse which isn't an initial segment of $\mathcal{R}\urcorner\urcorner$.

Now throw everything in sight into a countable hull, so that

In $V[g]$, I guess.

$\mathcal{D}(\overline{\mathcal{N}}) \models \ulcorner L(\Gamma(\overline{\mathcal{Q}},\overline{\Phi}),\mathbb{R}) \models \ulcorner \mathcal{M}$ is a $\overline{\Psi}$-mouse which isn't an initial segment of $\mathcal{R}\urcorner\urcorner$.

I think that now $\overline{\mathcal{Q}}$ are taking the role of "$L[\mathcal{T},\mathcal{H}]$", as Grigor's paper seems to indicate that $\mathcal{H} \subseteq \overline{\mathcal{Q}}$.

Now lift π to the ultrapower map π^+ given by the $(\delta^{\mathcal{H}},\delta^{\mathcal{R}})$-extender over $\overline{\mathcal{Q}}$ derived from π, and let $\mathcal{R}^+$ be the ultrapower. Lift also σ,τ to corresponding σ^+,τ^+.

A it hand-wavy.

$$\begin{tikzcd}
(\overline{\mathcal{Q}},\overline{\Phi}) \arrow[r, "\pi^+"] \arrow[dr, "\sigma^+"'] & (\mathcal{R}^+,\Phi^{**}) \arrow[d, "\tau^+"] \\
& (j(\overline{\mathcal{Q}}),\Phi^*)
\end{tikzcd}$$

Let now $\Phi^* := j(\overline{\Phi})$ and $\Phi^{**} := (\Phi^*)^{\tau^+}$, which is then a strategy for $\mathcal{R}^+$. Since $\overline{\Phi} = (\Phi^{**})^{\pi^+}$ we get that

Check this — might be by definition of pullback consistency, which is implied by hull condensation.

In Grigor's uB paper he uses a certain derived model C instead of $\mathcal{D}(\overline{\mathcal{N}})$, but I can't see how they're different from each other. Also, figure out why the following inclusion is true (it's probably folklore).

$$\mathcal{D}(\overline{\mathcal{N}}) \subseteq \mathcal{D}(\mathcal{R}^+,\Phi^{**}),$$

implying that

Note sure what's going on here.

$L(\Gamma(\mathcal{R}^+,\Phi^{**}),\mathbb{R}) \models \ulcorner \mathcal{M}$ is a Ψ-mouse which isn't an initial segment of $\mathcal{R}\urcorner$.

Because $\mathcal{R}^+$ is a Ψ-mouse over $\mathcal{R}|\delta^{\mathcal{R}}$, it follows that

Why's that?

$$\mathcal{D}(\mathcal{R}^+) \models \ulcorner\mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner,$$

I don't see how this last argument works.

which then implies that $\mathcal{M} \in \mathcal{R}^+$, so that $\mathcal{M} \trianglelefteq \mathcal{R}$, a contradiction. ■

Definition 1.4.6. For every $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ define $Q_X := \mathrm{cHull}^{j(\mathcal{H})}(X)$ and let

$$\tau_X \colon Q_X \to j(\mathcal{H})$$

be the uncollapse.

Say that $Y \in \mathcal{P}_{\omega_1}(j(\mathcal{H})|\delta^{j(\mathcal{H})})$ *extends* X if $X \cap j(\delta^{\mathcal{H}}) \subseteq Y$ and in that case let

(i) $\tau_{X,Y} := \tau_{X \cup Y}$,
(ii) $\Phi_{X,Y} := j(\Phi)^{\tau_{X,Y}}$,
(iii) $Q_{X,Y} := Q_{X\cup Y}$ and
(iv) $\pi_{X,Y} \colon Q_X \to Q_{X,Y}$ is the induced embedding given by

$$\pi_{X,Y}(x) = \tau_Y^{-1}(\tau_X(x)).$$

Furthermore define $T_X(A)$ for $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$ as

$$\begin{aligned} T_X(A) : &= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } Q_X \models \varphi[s, A]\} \\ &= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\tau_X(s), \tau_X(A)]\} \end{aligned}$$

and let $T_{X,Y}(A)$ be given as

$$\begin{aligned} T_{X,Y}(A) := \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_{X,Y}}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty}(s), \tau_X(A)], \\ \text{where } \alpha \text{ is least such that } s \in [\delta_\alpha^{Q_{X,Y}}]^{<\omega}\}. \end{aligned}$$

Here $\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty} : Q_{X,Y} \to j(\mathcal{H})|\nu_{X,Y}$

What is $\nu_{X,Y}$?

is given by

Missing! (It will be the iteration into an appropriate level of the directed system leading up to $j(\mathcal{H})$ followed by the direct limit embedding into some initial segment of $j(\mathcal{H})$)

∘

Definition 1.4.7. Let $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ and $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$. Then X is *A-condensing* if $\pi_{X,Y}(T_X(A)) = T_{X,Y}(A)$ for every Y extending X.
We say that X is *condensing* if X is A-condensing for all such A.

We want to show that $j``\mathcal{H}$ is condensing. We first show that it suffices to show that it's α-condensing for every $\alpha < \delta^{\mathcal{H}}$.

Lemma 1.4.8. *If $j``\mathcal{H}$ is $\alpha$-condensing for every $\alpha < \delta^{\mathcal{H}}$ then $j``\mathcal{H}$ is condensing.*

Proof.

Missing!

■

Theorem 1.4.9. *For every $\alpha < \delta^{\mathcal{H}}$ there exists an extension Y of $j``\mathcal{H}$ such that $j``\mathcal{H} \cup Y$ is α-condensing.*

Reduce this to $j``\mathcal{H}$ somehow?

Proof.

update notation

Set $X := j``\mathcal{H}$ and assume the theorem fails. Fix some $\alpha < \delta^{\mathcal{H}}$ such that X is not α-condensing. Fix some Y_0 extending X which witnesses this, meaning that $\pi^X_{Y_0}(T^X_\alpha) \neq T^{X,Y_0}_\alpha$. Since we're also assuming that $\tau^X_{Y_0}$ isn't α-condensing we can find Y_1 extending Y_0 such that $\pi^{Y_0}_{Y_1}(T^{Y_0}_\alpha) \neq T^{Y_0,Y_1}_\alpha$. Continue doing this, generating a sequence $\langle Y_n \mid n < \omega \rangle$ with Y_{n+1} extending Y_n and

$$\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) \neq T^{Y_n,Y_{n+1}}_\alpha \tag{1}$$

for all $n < \omega$. Let $\mathcal{P}_n := Q^X_{Y_n}$, $\pi_{m,n} := \pi^{Y_m}_{Y_n}$ and $\pi_n := \pi_{0,n}$. We want to show that such a sequence can't exist. Towards getting a contradiction we first need to make everything in sight countable, as that will allow us to reason using derived models (the problem is that $j(\mathcal{H})$ is too big, namely it has size $\aleph_1^{V[g]}$).

Using that $\delta^{j(\mathcal{H})}$ has uncountable cofinality we can find $\kappa < \delta^{j(\mathcal{H})}$ such that

$$\kappa = \mathrm{Hull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\mathrm{ran}\,\tau^X_{Y_n} \mid n < \omega\}) \cap \delta^{j(\mathcal{H})}.$$

Set $\mathcal{M} := \mathrm{cHull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\mathrm{ran}\,\tau^X_{Y_n} \mid n < \omega\})$ and note that $\mathcal{M} = j(\mathcal{H})|\kappa^{+j(\mathcal{H})}$. Let $\pi : \mathcal{M} \to j(\mathcal{H})$ be the uncollapse and note that $\mathrm{crit}\,\pi = \kappa$ and that $\kappa = \delta^{\mathcal{M}}$. Define $\iota : \mathcal{H} \to \mathcal{M}$ as $\iota := \pi^{-1} \circ j$ and $\tau_n : \mathcal{P}_n \to \mathcal{M}$ as $\tau_n := \pi^{-1} \circ \tau^X_{Y_n}$. Note that $\mathcal{M}$ is countable in $V[g]$ and is hence an element of $\mathrm{Ult}(V, g)$.

Missing argument

Now define $\mathcal{H}^+$ as the hod limit of iterates of $\mathcal{H}$, so that $\mathcal{H}^+$ is a hod premouse with $\mathcal{H} \lhd_{\mathrm{hod}} \mathcal{H}^+$, $\mathcal{H}^+$ has a strategy Ψ extending Ω such that

Provide more details.

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{H}^+, \Psi).$$

We probably need $\mathcal{H}^+$ to be countable here, so we should probably apply the induced ideal and work in $V[g][h]$.

Also define $(\mathcal{P}^+_n, \Psi_n)$ as $P^+_n := \mathrm{Ult}(\mathcal{H}^+, E_{\pi_n})$, so that we also get that

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{P}^+_n, \Psi_n).$$

Missing argument. This might need that $\mathcal{H}^+, \Psi \restriction V \in V$, but we could probably also just work inside $\mathrm{Ult}(V, g)$, or the second ultrapower, all along.

Now $\mathcal{D}(\mathcal{P}^+_n, \Psi_n)$ has a definition of T^{X,Y_n}_α, so that $\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) = T^{Y_n,Y_{n+1}}_{\pi_{n,n+1}(\alpha)}$. The three-dimensional argument then shows that α must be fixed by $\pi_{n,n+1}$ for some $n < \omega$, so that $X \cup Y_n$ *is* α-condensing, $\lightning$. ■

What is meant by this?

Show this.

Define the strategy Λ for $\mathcal{H}$ and show that $(\mathcal{H}, \Lambda)$ is a hod pair.