1 Internal core model induction

1.1 Operators

Define model operator, (hybrid) mouse operator, tameness

1.2 Mouse witness equivalence

Definition 1.1. asd

Define coarse (k, U, x)-Woodin pairs

$\circ$

Definition 1.2. Let F be a total condensing operator and let α be an ordinal. Then the **coarse mouse witness condition at α with F**, written $W^*_\alpha(F)$, states that given any scaled-co-scaled $U \subseteq \mathbb{R}$ whose associated sequences of prewellorderings are elements of $\mathrm{Lp}^F_\alpha(\mathbb{R})$, we have for every $k < \omega$ and $x \in \mathbb{R}$ a coarse (k, U, x)-Woodin pair (N, Σ) with $\Sigma \restriction \mathsf{HC} \in \mathrm{Lp}^F_\alpha(\mathbb{R})$. $\circ$

Check if this is a reasonable definition.

Theorem 1.3 (Hybrid witness equivalence)**.** *Let $\theta > 0$ be a cardinal, $g \subseteq \mathrm{Col}(\omega, {<}\theta)$ V-generic, $\mathbb{R}^g := \bigcup_{\alpha<\theta} \mathbb{R}^{V[g \restriction \alpha]}$, F a total radiant operator and α a critical ordinal of $\mathrm{Lp}^F(\mathbb{R}^g)$. Assume that*

$$\mathrm{Lp}^F(\mathbb{R}^g) \models \mathsf{DC} + \ulcorner W^*_\beta(F) \textit{ holds for all } \beta \leq \alpha \urcorner.$$

Then there is a hybrid mouse operator $\mathcal{N} \in V$ on $H_{\aleph_1^{V[g]}}$ such that

$$\mathrm{Lp}^F(\mathbb{R}^g) \models W^*_{\alpha+1}(F) \quad \textit{iff} \quad V \models \ulcorner M^{\mathcal{N}}_n \textit{ is total on } H_{\aleph_1^{V[g]}} \textit{ for all } n < \omega \urcorner$$

Furthermore, if $\theta < \aleph_1^V$ then we only need to assume that F is total and condensing.

Be more explicit about what the given operator $\mathcal{N}$ looks like.

1.3 Core models

Define $K(x)$, $K^F(x)$, $\mathfrak{C}(X)$

1.4 Core model dichotomy

Lemma 1.4. *Let θ be a regular uncountable cardinal or $\theta = \infty$ and let $\mathcal{N}$ be a tame hybrid mouse operator on H_θ which relativises well. Then $\mathcal{N}$ is countably iterable iff it's (θ,θ)-iterable, guided by $\mathcal{N}$. Furthermore, for every $x \in H_\theta$, if $M_1^{\mathcal{N}}(x)$ exists and is countably iterable, then it's also (θ,θ)-iterable, guided by $\mathcal{N}$.*

Change this to model operators; perhaps change parts of the proof and/or assumptions needed.

Proof. Fix $x \in H_\theta$. We first show that $\mathcal{N}(x)$ is (θ,θ)-iterable. Let $\mathcal{T} \in H_\theta$ be a normal tree of limit length on $\mathcal{N}(x)$. Let $\eta \gg \operatorname{rk}(\mathcal{T})$ and let

$$\mathcal{H} := \operatorname{cHull}^{H_\eta}(\{x, \mathcal{N}(x), \mathcal{T}\})$$

with uncollapse $\pi\colon \mathcal{H} \to H_\eta$. Set $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran}\pi$. Note that $\overline{\mathcal{N}(x)} = \mathcal{N}(\overline{x})$ since $\mathcal{N}$ relativises well. Now $\overline{\mathcal{T}}$ is a normal, countable iteration tree on $\mathcal{N}(\overline{x})$ and hence our iteration strategy yields a wellfounded cofinal branch $\overline{b} \in V$ for $\overline{\mathcal{T}}$. Note that $\overline{\mathcal{Q}} := \mathcal{Q}(\overline{b}, \overline{\mathcal{T}})$ exists, since if $\overline{b}$ drops then there's nothing to do, and otherwise we have that

$$\rho_1(\mathcal{M}_{\overline{b}}^{\overline{\mathcal{T}}}) = \rho_1(\mathcal{N}(\overline{x})) = \operatorname{rk}\overline{x} < \delta(\overline{\mathcal{T}}),$$

Why is that?

so $\delta(\overline{\mathcal{T}})$ is not definably Woodin over $\mathcal{M}_{\overline{b}}^{\overline{\mathcal{T}}}$.

Claim 1.5. $\overline{\mathcal{Q}} \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$

Proof of claim. If $\overline{\mathcal{Q}} = \mathcal{M}(\overline{\mathcal{T}})$ then the claim is trivial, so assume that $\mathcal{M}(\overline{\mathcal{T}}) \triangleleft \overline{\mathcal{Q}}$. Note that $\overline{\mathcal{Q}} \trianglelefteq M_{\overline{b}}^{\overline{\mathcal{T}}}$ by definition of $\mathcal{Q}$-structures, and that $M_{\overline{b}}^{\overline{\mathcal{T}}}$ satisfies (2) of the definition of relativises well, meaning that

Define this, cut-point and $\mathcal{M}_b^{\mathcal{T}}$

$$M_{\bar{b}}^{\overline{\mathcal{T}}} \models \ulcorner \forall\eta\forall\zeta > \eta : \text{if } \eta \text{ is a cutpoint then } M_{\bar{b}}^{\overline{\mathcal{T}}}|\zeta \not\models \varphi_{\mathcal{N}}[\bar{x}, p_{\mathcal{N}}]\urcorner. \qquad (1)$$

This statement is Π^1_2 and $\overline{\mathcal{Q}}$ is Π^1_2-correct since it contains a Woodin cardinal, so that $\mathcal{Q}$ satisfies the statement as well. Since $\mathcal{N}$ is tame we get that $\delta(\overline{\mathcal{T}})$ is a cutpoint of $\overline{\mathcal{Q}}$, so that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ is *not* a proper initial segment of $\overline{\mathcal{Q}}$. Further, as we're assuming that both $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}}$ are $(\omega_1{+}1)$-iterable above $\delta(\overline{\mathcal{T}})$ the same thing holds for $\overline{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\overline{\mathcal{T}}}$, so that we can compare $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ with $\overline{\mathcal{Q}}$ (in V). Let

$$(\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})), \overline{\mathcal{Q}}) \rightsquigarrow (\mathcal{P}, \mathcal{R})$$

be the result of the coiteration. We claim that $\mathcal{R} \trianglelefteq \mathcal{P}$. Suppose $\mathcal{P} \triangleleft \mathcal{R}$. Then there is no drop in $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ and in fact $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P}$ since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ projects to $\delta(\overline{\mathcal{T}})$. Furthermore, as we established that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ isn't a proper initial segment of $\overline{\mathcal{Q}}$ it can't be a proper initial segment of $\mathcal{R}$ either, as the coiteration is above $\delta(\overline{\mathcal{T}})$. But we're assuming that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P} \triangleleft \mathcal{R}$, a contradiction. So $\mathcal{R} \trianglelefteq \mathcal{P}$.

Since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\overline{\mathcal{Q}}$ agree up to $\delta(\overline{\mathcal{T}})$ and there is no drop $\overline{\mathcal{Q}} \rightsquigarrow \mathcal{R}$ we have that $\overline{\mathcal{Q}} = \mathcal{R}$. If $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ doesn't move either we're done, so assume not. Let F be the first exit extender of $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ in the coiteration. We have $\mathrm{lh}(F) \leq o(\overline{\mathcal{Q}})$, $\overline{\mathcal{Q}} \trianglelefteq \mathcal{P}$ and $\mathrm{lh}(F)$ is a cardinal in $\mathcal{P}$. Define this

As $\overline{\mathcal{Q}}$ is $\delta(\overline{\mathcal{T}})$-sound and projects to $\delta(\overline{\mathcal{T}})$ it follows that $J(\overline{\mathcal{Q}}|\,\mathrm{lh}(F))$ collapses $\mathrm{lh}(F)$, so it has to be the case that $\overline{\mathcal{Q}}|\,\mathrm{lh}(F) = \mathcal{P}$ and thus $o(\mathcal{P}) = \mathrm{lh}(F)$. But this means that $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ even though we assumed that $\mathcal{N}(\mathcal{M}(\mathcal{T})) \rightsquigarrow \mathcal{P}$ moved, a contradiction. $\dashv$

Now, in a sufficiently large collapsing extension extension of $\mathcal{H}$, $\bar{b}$ is the unique cofinal, wellfounded branch of $\overline{\mathcal{T}}$ such that $\mathcal{Q}(\bar{b}, \overline{\mathcal{T}}) \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ exists. Hence, by the homogeneity of $\mathrm{Col}(\omega, \theta)$, $\bar{b} \in H$. By elementarity there is a unique cofinal, wellfounded branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$. This proves that M is (uniquely) On-iterable and virtually the same argument yields the iterability of M via successor-many stacks of normal trees.

To show that M is fully iterable, it remains to be seen that the unique iteration strategy (guided by $\mathcal{N}$) of M outlined above leads to wellfounded direct limits for stacks of normal trees on M of limit length. Let λ be a limit ordinal and $\vec{\mathcal{T}} = (\mathcal{T}_i \mid i < \lambda)$ a stack according to our iteration strategy. Suppose $\lim_{i<\lambda} \mathcal{M}_\infty^{\mathcal{T}_i}$ is illfounded.

Redefine $\eta \gg \operatorname{rk}(\vec{\mathcal{T}})$, $\mathcal{H} := \operatorname{cHull}^{H_\eta}(\{x, M, \vec{\mathcal{T}}\})$ and $\pi : \mathcal{H} \to H_\eta$ the uncollapse, again with $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran}\pi$. By elementarity we get that $\mathcal{H} \models \ulcorner \lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i}$ is illfounded$\urcorner$. But $\overline{\vec{\mathcal{T}}}$ is countable and according to the iteration strategy guided by $\mathcal{N}$, so that

$$V \models \ulcorner \lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i} \text{ is wellfounded.}\urcorner$$

Now note that $(\lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^{\mathcal{H}} = (\lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^V$ and wellfoundedness is absolute between $\mathcal{H}$ and V, a contradiction.

Now assume that $M_1^{\mathcal{N}}(x)$ exists for some $x \in H_\theta$, and that it's countably iterable. We then do exactly the same thing as with $\mathcal{N}(x)$ *except* that in the claim we replace (1) with

$$\overline{\mathcal{Q}} \models \forall \eta (\overline{\mathcal{Q}}|\eta \not\models \ulcorner \delta(\overline{\mathcal{T}}) \text{ is not Woodin}\urcorner),$$

so that if $\mathcal{P} \lhd \mathcal{R}$ then $\delta(\overline{\mathcal{T}})$ is still Woodin in $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$, contradicting the defining property of $M_1^{\mathcal{N}}(x)$ (and thus also of $\mathcal{R}$). The rest of the proof is a copy of the above. ■

Theorem 1.6 (Hybrid core model dichotomy). *Let θ be a $\beth$-fixed point or $\theta = \infty$, and let F be a tame model operator on H_θ that condenses well. Let $x \in H_\theta$. Then either:*

I don't think tame is needed here, as we're only indexing extenders at F-initial segments

(i) The core model $K^F(x)|\theta$ exists and is (θ,θ)-iterable; or

(ii) $M_1^F(x)$ exists and is (θ,θ)-iterable.

Proof. Assume first that $K^{c,F}(x)|\theta$ reaches a premouse which isn't F-small; let $\mathcal{N}_\xi$ be the first part of the construction witnessing this. Then $\mathfrak{C}(\mathcal{N}_\xi) = M_1^F(x)$, and by Lemma **??** it suffices to show that $M_1^F(x)$ is countably iterable.

Insert argument?

Show that $M_1^F(x)$ is countably iterable.

We can thus assume that $K^{c,F}(x)|\theta$ is F-small. Note that if $K^{c,F}(x)|\theta$ has a Woodin cardinal then because the model is F-closed we contradict F-smallness, so the model has no Woodin cardinals either, making it (θ,θ)-iterable.

Let $\kappa < \theta$ be any uncountable cardinal and let $\Omega := \beth_\kappa(\kappa)^+$. Note that $\Omega < \theta$ since we assumed that θ is a $\beth$-fixed point and $\kappa < \theta$. If Ω is a limit cardinal in $K^{c,F}(x)|\theta$ then let $\mathcal{S} := \mathrm{Lp}(K^{c,F}(x)|\Omega)$ and otherwise let $\mathcal{S} := K^{c,F}(x)|\Omega$. Then by Lemma 3.3 of [**?**] we get that $\mathcal{S}$ is countably iterable, with largest cardinal Ω in the "limit cardinal case".

This also means that Ω isn't Woodin in $L[\mathcal{S}]$, as it's trivial in the case where Ω is a successor cardinal of $K^{c,F}(x)|\theta$ by our case assumption, and in the "limit cardinal case" it also holds since

$$K^{c,F}(x)|\Omega^{+K^{c,F}(x)|\theta} \subseteq \mathcal{S}.$$

By [**?**] and [**?**] this means that we can build $K^F(x)|\kappa$, as the only places they use that there's no inner model with a Woodin are to guarantee that $K^{c,F}(x)|\Omega$ exists and has no Woodin cardinals, and in Lemma 4.27 of [**?**] in which they only require that Ω isn't Woodin in $L[\mathcal{S}]$.

As $\kappa < \theta$ was arbitrary we then get that $K^F(x)|\theta$ exists. Note that $K^F(x)|\theta$ has no Woodin cardinals either and is F-small, so that $\mathcal{Q}$-structures trivially exist, making it (θ,θ)-iterable. ■