1 Inner model direction

1.1 Determinacy in mice from DI

Proposition 1.1 (Folklore?). *If ω_1 carries a saturated ideal then mouse reflection holds at ω_1.*

Proof. Let $\mathcal{N}$ be a mouse operator defined on HC and fix some $x \in H_{\omega_2}$; we want to show that $\mathcal{N}(x)$ is defined. Let $j : V \to M$ be the generic ultrapower with $\operatorname{crit} j = \omega_1^V$ and note that $j(\omega_1^V) = \omega_1^M = \omega_1^{V[g]} = \omega_2^V$ by saturation of the ideal. This means in particular that $\mathsf{HC} \prec H_{\omega_2}^M$. Since

$$\mathsf{HC} \models \ulcorner \mathcal{N}(y) \text{ exists for all sets } y \urcorner$$

we get that $H_{\omega_2}^M$ believes the same is true. But $H_{\omega_2}^V \subseteq H_{\omega_2}^M$ since $\operatorname{crit} j = \omega_1^V$, so that in particular $H_{\omega_2}^M$ believes that $x^\sharp$ exists. Since M is closed under ω-sequences in $V[g]$ by Proposition **??**, we get that $x^\sharp$ exists in $V[g]$ and hence also in V as set forcing can't add sharps. ■

Prove this or give a reference.

Proposition 1.2 (Folklore?). *If ω_1 carries a precipitous ideal then HC is closed under sharps. If the ideal is furthermore saturated then H_{ω_2} is closed under sharps.*

Proof. Proposition 1.1 gives the latter statement if we show the former, so fix an $x \in \mathsf{HC}$ and let $j : V \to M$ be the generic ultrapower from a precipitous ideal on ω_1^V. Since $j(x) = x$ we get that $j : L[x] \to L[x]$ with $\operatorname{crit} j > \operatorname{rk} x$, implying that $x^\sharp$ exists in the generic extension. But set forcing can't add sharps so $x^\sharp$ exists in V as well. ■

Add argument or reference.

Definition 1.3. Let $j : V \to M$ be an elementary embedding in some $V[g]$ and let F be a model operator. Then F is **j-radiant** if it condenses well, determines itself on generic extensions and satisfies the **extension property**, which says that $F \subseteq j(F)$ and $j(F) \restriction \mathsf{HC}^{V[g]}$ is definable in $V[g]$. $\circ$

Lemma 1.4 (DI)**.** *M_1^F is total on H_{ω_2} for any j-radiant model operator F on H_{ω_2}.*

Proof. We want to use the hybrid core model dichotomy **??**, but the problem is that F is not total. We solve this by going to a smaller model; the model $W := L^F_{\omega_2^V}(\mathbb{R})$ will be a first attempt (note that $\mathbb{R} \in \operatorname{dom} F$ as we're assuming CH). To be able to apply the dichotomy in a model we need it to satisfy ZFC. The following claim is the first step towards this.

Claim 1.5. Given any real x, $L^F_{\omega_2}(x) \models \ulcorner\omega_1^V$ is inaccessible$\urcorner$.

Proof of claim. Letting $j : V \to M$ be the generic elementary embedding, note that j doesn't move x, so that

$$j \restriction L^F_{\omega_2^V}(x) : L^F_{\omega_2^V}(x) \to L^{j(F)}_{\omega_2^M}(x).$$

Since F has the extension property, $L^{j(F)}_{\omega_2^M}(x)$ is just an end-extension of $L^F_{\omega_2^V}(x)$. In particular ω_1^V is still a cardinal in there, meaning that, for every $\alpha < \omega_1^V$,

$$L^{j(F)}_{\omega_1^M}(x) \models \ulcorner\text{there's a cardinal} > \alpha\urcorner.$$

By elementarity this makes ω_1^V a limit cardinal in $L^F_{\omega_2^V}(x)$ and by GCH in $L^F_{\omega_2^V}(x)$ it's inaccessible. $\dashv$

This claim is now transferred to M, and as $\mathbb{R}^V$ is a real from the point of view of M, we get that

$$L^{j(F)}_{\omega_2^M}(\mathbb{R}^V) \models \ulcorner \omega_1^M \text{ is inaccessible} \urcorner.$$

Noting that $\omega_1^M = \omega_2^V$ and again using the extension property of F, we get that $W \models \mathsf{ZF}$. We don't get choice in W as it doesn't contain a wellorder of the reals, so we we'll work with $W[h]$ instead, where $h \subseteq \mathrm{Col}(\omega_1, \mathbb{R})^W$ is W-generic. Since we're assuming CH we get that $g \in V$, making $W[h] \in V$ as well, $W[h]$ is still closed under F since F determines itself on generic extensions, and $W[h] \models \mathsf{ZFC}$.

We can now apply the hybrid core model dichotomy **??** inside $W[h]$ to conclude that, for every real x, either $K^F(x)^{W[h]}$ exists or $M_1^F(x)$ exists (note that (ω_1, ω_1)-iterability is absolute between $W[h]$ and V since $W[h]$ contains all the reals). Since mouse reflection holds at ω_1 by Proposition 1.1 if the latter conclusion held at all reals x then we would also get that M_1^F is total on H_{ω_2} and we'd be done. So assume $K := K^F(x)^{W[h]}$ exists.

Claim 1.6. $j(K) \in V$.

Proof of claim. This is where we'll be using homogeneity of our ideal. Firstly K is definable in $W[h]$ and thus also in W by homogeneity of $\mathrm{Col}(\omega_1, \mathbb{R})$, so that $j(K)$ is definable in $j(W)$. But $j(W)$ is definable in $V[g]$ as the unique $j(F)$-premouse over $\mathbb{R}$ of height ω_1, making $j(K)$ definable in $V[g]$ with $j(F) \restriction \mathsf{HC}$ as a parameter. But $j(F) \restriction \mathsf{HC}$ is definable in $V[g]$ since F satisfies the extension property, so homogeneity of our ideal implies that $j(F) \in V$ and hence $j(K) \in V$ as well. $\dashv$

This claim also implies that ω_1^V is inaccessible in K, as if it wasn't, say $\omega_1^V = \lambda^{+K}$, then $\omega_2^V = j(\omega_1^V) = j(\lambda)^{+j(K)} = \lambda^{+j(K)}$, so that ω_2^V isn't a cardinal in V, $\lightning$.

We then also get that $(\omega_1^V)^{+j(K)} < \omega_2^V$, since if they were equal then elementarity would imply that ω_1^V was a successor in K, $\lightning$.

Since $K|\omega_1^V = j(K)|\omega_1^V$, elementarity and the above implies that

$$j^2(K)|(\omega_1^V)^{+j^2(K)} = j(K)|(\omega_1^V)^{+j(K)},$$

which makes sense as $j(K) \in V$.

Let now E be the (ω_1^V, ω_2^V)-extender derived from $j \restriction j(K)$, and note that $E \restriction \alpha \in M$ for every $\alpha < \omega_2^V = \omega_1^M$ as M is closed under countable sequences in $V[g]$.

Claim 1.7. $E \restriction \alpha$ is on the $j(K)$-sequence for every $\alpha < \omega_2^V$.

Why is this sufficient?

Proof of claim. We need to show that

$$j(W) \models \ulcorner \langle\langle j(K), \mathrm{Ult}(j(K), E \restriction \alpha)\rangle, \alpha\rangle \text{ is } \mathsf{On}\text{-iterable}\urcorner.$$

What kind of reflection?

Assume not. Then by reflection we get, in $j(W)$, a countable $\overline{K}$ and an elementary $\sigma : \overline{K} \to \mathrm{Ult}(j(K), E \restriction \alpha)$ with $\sigma \restriction \alpha = \mathrm{id}$ and $\langle\langle j(K), \overline{K}\rangle, \alpha\rangle$ isn't ω_1-iterable.

Let $k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$ be the factor map with $k \restriction \alpha = \mathrm{id}$ and define $\psi := k \circ \sigma : \overline{K} \to j^2(K)$, so that $(k \circ \sigma) \restriction \alpha = \mathrm{id}$. We have both ψ and $\overline{K}$ in M, which is the generic ultrapower $\mathrm{Ult}(V, g)$, so we also get that $\psi = [\vec{\psi}_\xi]_g$, $\overline{K} = [\vec{K}_\xi]_g$ and $\alpha = [\vec{\alpha}_\xi]_g$. We need to show that

For g-almost every $\xi < \omega_1^V$ it holds that $W \models \ulcorner\langle\langle K, K_\xi\rangle, \alpha_\xi\rangle$ is ω_1-iterable$\urcorner$

By Łoś' Lemma we have that, in V and hence also in $V[g]$, there are embeddings $\psi_\xi : K_\xi \to j(K)$ with $\psi_\xi \restriction \alpha_\xi = \mathrm{id}$ for g-almost every $\xi < \omega_1^V$. As $j(W)$ is closed under countable sequences in $V[g]$ it sees that the K_ξ's are countable, so that an application of absoluteness of wellfoundedness shows that $j(W)$ also has elementary embeddings $\psi^*_\xi : K_\xi \to j(K)$ with $\psi^*_\xi \restriction \alpha_\xi$.

Include this argument perhaps.

But $j(K) = K^{j(F)}(x)^{j(W[h])}$, so $j(W[h])$ sees that $\langle\langle K, K_\xi\rangle, \alpha_\xi\rangle$ is ω_1-iterable, which is therefore also true in W since $W \cap \mathbb{R} \subseteq \mathbb{R}^{V[g]} = j(W[h]) \cap \mathbb{R}$. ⊣

Our desired contradiction is then showing that K has a Shelah cardinal, which is impossible. Let $f : \omega_1^V \to \omega_1^V$ be a function in $j(K)$ and pick some $\alpha \in (j(f)(\kappa), \omega_2^V)$. Letting

Insert argument?

$$k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$$

be the factor map, we get that $\operatorname{crit} k \geq \alpha$ by coherence of extenders on the K-sequence and hence that $i_{E\restriction\alpha}(f)(\omega_1^V) < \alpha$ as well. This shows that ω_1^V is Shelah in $j(K)$ and hence K has a Shelah cardinal by elementarity, ↯. ■

Theorem 1.8 (DI)**.** $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R}) \models$ *AD for all "nice"* Γ *and* Σ.

Specify niceness.

Proof.

Show that all the operators occuring in the $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R})$ induction are j-radiant.

■

1.2 Ω is not zero

Collapse all these Ω sections into one, where the abstract results are moved into the internal/external CMI chapters

Define

$$\Gamma_0 := \{A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \models \mathsf{AD} + \Omega = 0\}.$$

Lemma 1.9 (DI)**.** $\Gamma_0 = \mathrm{Lp}(\mathbb{R}) \cap \mathscr{P}(\mathbb{R})$.

Proof. ($\supseteq$) Let $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$ and let $A \subseteq \mathbb{R}$ be an element of $\mathcal{M}$. Since $\mathcal{M}$ projects to $\mathbb{R}$ and is sound, we get that A is OD_x for a real x, so that everything in $L(A, \mathbb{R})$ is also ordinal definable in a real as well. Since $\mathrm{Lp}(\mathbb{R}) \models \mathsf{AD}$ we then get that $\mathsf{AD} + \Omega = 0$ holds in $L(A, \mathbb{R})$, making $A \in \Gamma_0$.

Check this proof.

($\subseteq$) Let $A \in \Gamma_0$. Since we're assuming CH we get that $V[g] \models |\mathbb{R}| = \aleph_1^V = \aleph_0$, so fix a generic bijection $b : \omega \to \mathbb{R}^V$ in $V[g]$. Define $a_b \in \mathbb{R}$ as

$n \in a_b$ iff $b(n) \in A$. As $L(A,\mathbb{R}) \models \mathsf{AD} + \theta_0 = \Theta$ it holds that A is $\mathrm{OD}_z^{L(A,\mathbb{R})}$ for $z \in \mathbb{R}$, so that

$$A = j(A) \cap \mathbb{R}^V \in \mathrm{OD}_{z,\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}.$$

In particular, as A and $\mathbb{R}^V$ are definable from b and a_b is definable from b, we get that $a_b \in \mathrm{OD}_b^{L(j(A),\mathbb{R}^{V[g]})}$. By MC we then get that there's some b-premouse $\mathcal{M} \in L(j(A),\mathbb{R}^{V[g]})$ projecting to b with $a_b \in \mathcal{M}$ and a Σ such that

$$L(j(A),\mathbb{R}^{V[g]}) \models \ulcorner\Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{M}\urcorner.$$

Why is it that we have to go through b in this fashion? Can't we just use MC and get $\mathcal{N}$ without going through $\mathcal{M}$? Is it because $L(j(A),\mathbb{R}^{V[g]})$ doesn't know that $\mathbb{R}^V$ is countable?

From this $\mathcal{M}$ we can then get an $\mathbb{R}^V$-premouse $\mathcal{N} \in L(j(A),\mathbb{R}^{V[g]})$ projecting to $\mathbb{R}^V$ with $A \in \mathcal{N}$ and

$$L(j(A),\mathbb{R}^{V[g]}) \models \ulcorner\Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{N}\urcorner.$$

Now $\mathcal{N}$ is $\mathrm{OD}_{\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}$, and since we don't have divergent models of AD^+ it holds that, letting $\Theta^{j(A)} := \Theta^{L(j(A),\mathbb{R}^{V[g]})}$,

$$V[g] \models L(j(A),\mathbb{R}) = L(P_{\Theta^{j(A)}}(\mathbb{R})).$$

This means that $\mathcal{N} \in \mathrm{OD}_{\mathbb{R}^V}^{V[g]}$, so that homogeneity of I we get that $\mathcal{N} \in V$. It remains to show that $\mathcal{N} \trianglelefteq \mathrm{Lp}(\mathbb{R}^V)$, meaning that we need to show that $\mathcal{N}$ is countably (ω_1+1)-iterable in V. But letting $\overline{\mathcal{N}} \to \mathcal{N}$ be a countable hull in V we get that $j(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$, so that elementarity of j implies that $\Sigma \restriction V \in V$ is an $\omega_1^{V[g]} = \omega_2^V$-iteration strategy for $\overline{\mathcal{N}}$ and we're done. ■

Why's this?

Is this really this iterable?

Proposition 1.10 (DI)**.** $\mathrm{cof}^V(\Theta^{\mathrm{Lp}(\mathbb{R})}) = \omega$.

Proof.

See Ketchersid's Thesis 3.17 or 7.4.2 in the CMI book. Perhaps we don't need it though, following Wilson's thesis.

■

Theorem 1.11. *Let Γ be an inductive-like pointclass. If $\mathcal{M}$ is a suitable quasi-iterable premouse, $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ is closed under recursive join and the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is both total on $\mathcal{M}$ and has the full factors property, then there's a unique Γ-fullness preserving (ω_1, ω_1)-strategy Φ for $\mathcal{M}$ such that, for every quasi-iterate $\mathcal{P}$ of $\mathcal{M}$,*

- *$\mathcal{P}$ is a non-dropping Φ-iterate of $\mathcal{M}$; and*
- *the Φ-iteration map $i : \mathcal{M} \to \mathcal{P}$ equals the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\mathcal{P}}$.*

Let $\Phi_{\mathcal{M}}$ be the unique strategy for $\mathcal{M}$ as in the above theorem. We now improve this to include branch condensation.

The 3d argument is quite similar to the proof of Theorem 7.19 in the outline.

Theorem 1.12. *Let Γ be an inductive-like pointclass and assume that $\mathbf{\Delta}_\Gamma$ is determined and that Γ-MC holds. Let $\mathcal{M}$ be an ω-suitable quasi-iterable premouse such that $\mathcal{D}(\mathcal{M}) \equiv \mathcal{M}_\Gamma$, let $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ be closed under recursive join, assume $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and that it has the full factors property. Let $\Phi := \Phi_{\mathcal{M}}$. Then there's a $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ has $\mathcal{A}$-condensation, and hence also branch condensation, for every $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$.*

This is the companion of Γ, see Trevor's thesis. I'm not sure if we can find $\mathcal{M}$ like this, however.

Proof. Assume not and fix $A \in \mathrm{Env}(\Gamma)$ such that given any $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ there's a $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ doesn't have A-condensation. Applying this inductively, we get a sequences $\langle \mathcal{Q}^0_n, \mathcal{R}^0_n, \mathcal{T}^0_n, \pi^0_n, \sigma^0_n, j^0_n \mid n < \omega \rangle$ such that

(i) $\mathcal{Q}^0_0 := \mathcal{M}$;

(ii) $\pi^0_n : \mathcal{Q}^0_n \to \mathcal{Q}^0_{n+1}$ is the iteration map through a tree of successor length, according to Φ;

$$
\begin{tikzcd}
\mathcal{Q}_0^0 \arrow[rr, "\pi_0^0"] \arrow[dr, "\sigma_0^0"'] \arrow[dd, "\tau_0^0"'] & & \mathcal{Q}_1^0 \arrow[rr, "\pi_1^0"] \arrow[dr, "\sigma_1^0"'] \arrow[dd, "\tau_1^0"'] & & \mathcal{Q}_2^0 \arrow[r, "\pi_2^0"] \arrow[dd, "\tau_2^0"'] & \cdots & \mathcal{Q}_\omega^0 \arrow[dd, "\tau_\omega^0"'] \\
& \mathcal{R}_0^0 \arrow[ur, "j_0^0"'] \arrow[dd, "k_0^0"] & & \mathcal{R}_1^0 \arrow[ur, "j_1^0"'] \arrow[dd, "k_1^0"] & & & \\
\mathcal{Q}_0^1 \arrow[rr, "\pi_0^1"] \arrow[dr, "\sigma_0^1"'] \arrow[dd, "\tau_0^1"'] & & \mathcal{Q}_1^1 \arrow[rr, "\pi_1^1"] \arrow[dr, "\sigma_1^1"'] \arrow[dd, "\tau_1^1"'] & & \mathcal{Q}_2^1 \arrow[r, "\pi_2^1"] \arrow[dd, "\tau_2^1"'] & \cdots & \mathcal{Q}_\omega^1 \arrow[dd, "\tau_\omega^1"'] \\
& \mathcal{R}_0^1 \arrow[ur, "j_0^1"'] \arrow[dd, "k_0^1"] & & \mathcal{R}_1^1 \arrow[ur, "j_1^1"'] \arrow[dd, "k_1^1"] & & & \\
\mathcal{Q}_0^2 \arrow[rr, "\pi_0^2"] \arrow[dr, "\sigma_0^2"'] \arrow[dd, "\tau_0^2"'] & & \mathcal{Q}_1^2 \arrow[rr, "\pi_1^2"] \arrow[dr, "\sigma_1^2"'] \arrow[dd, "\tau_1^2"'] & & \mathcal{Q}_2^2 \arrow[r, "\pi_2^2"] \arrow[dd, "\tau_2^2"'] & \cdots & \mathcal{Q}_\omega^2 \arrow[dd, "\tau_\omega^2"'] \\
& \mathcal{R}_0^2 \arrow[dd, "k_0^2"] \arrow[ur, "j_0^2"'] & & \mathcal{R}_1^2 \arrow[dd, "k_1^2"] \arrow[ur, "j_1^2"'] & & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
& \vdots & & \vdots & & & \\
\mathcal{Q}_0^\omega \arrow[rr, "\pi_0^\omega"] \arrow[dr, "\sigma_0^\omega"'] & & \mathcal{Q}_1^\omega \arrow[rr, "\pi_1^\omega"] \arrow[dr, "\sigma_1^\omega"'] & & \mathcal{Q}_2^\omega \arrow[r, "\pi_2^\omega"] & \cdots & \mathcal{Q}_\omega^\omega \\
& \mathcal{R}_0^\omega \arrow[ur, "j_0^\omega"'] & & \mathcal{R}_1^\omega \arrow[ur, "j_1^\omega"'] & & &
\end{tikzcd}
$$

Figure 1.1: The three-dimensional argument in Theorem 1.12

(iii) $\sigma_n^0 : \mathcal{Q}_n^0 \to \mathcal{R}_n^0$ an iteration map through a tree of limit length, according to Φ;

(iv) $j_n^0 : \mathcal{R}_n^0 \to \mathcal{Q}_{n+1}^0$ is elementary such that $\pi_n^0 = j_n^0 \circ \sigma_n^0$;

(v) $(j_n^0)^{-1}(\tau_{A,j_n^0(\kappa)}^{\mathcal{Q}_{n+1}^0}) \neq \tau_{A,\kappa}^{\mathcal{R}_n^0}$ for every $\mathcal{R}_n^0$-cardinal $\kappa \geq \delta_0^{\mathcal{R}_n^0}$.

Let $\mathcal{Q}_\omega^0$ be the direct limit of the $\mathcal{Q}_n^0$'s under the π_n^0 maps. Also let $\langle x_n \mid n < \omega\rangle$ enumerate the reals of $\mathcal{M}_\Gamma$ and pick $s \in [\mathsf{On}]^{<\omega}$ and a formula φ such that

$$\forall x \in \mathbb{R}(x \in A \Leftrightarrow \mathcal{M}_\Gamma \models \varphi[x, s]).$$

Our strategy now is now firstly to capture all the x_n's so that the derived models of the resulting structures become equal to $\mathcal{M}_\Gamma$. See Figure 1.1.

Perform a genericity iteration of $\mathcal{Q}_0^0$ above $\delta_0^{\mathcal{Q}_0^0}$ to $\mathcal{Q}_0^1$ to make x_0 generic over $\mathcal{Q}_0^1$ at $\delta_1^{\mathcal{Q}_0^1}$, while lifting the genericity iteration tree via the copy construction to the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and picking branches on the genericity iteration tree on $\mathcal{Q}_0^0$ by using $\Phi_{\mathcal{Q}_\omega^0}$ on the lifted tree on $\mathcal{Q}_\omega^0$. Let $\tau_0^0 : \mathcal{Q}_0^0 \to \mathcal{Q}_0^1$ be the genericity iteration map and $\mathcal{W}_0$ the last model of the lifted tree on $\mathcal{Q}_\omega^0$.

Now perform another genericity iteration of the last model of the lifted iteration tree on $\mathcal{R}_0^0$ above its δ_0 to $\mathcal{R}_0^1$ to make x_0 generic over $\mathcal{R}_0^1$ at $\delta_1^{\mathcal{R}_0^1}$, with branches being picked by lifting the iteration tree to $\mathcal{W}_0$ and using the branches according to $\Phi_{\mathcal{W}_0}$. Let $k_0^0 : \mathcal{R}_0^0 \to \mathcal{R}_0^1$ be the iteration embedding, $\sigma_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_0^1$ be the shift of σ_0^0 followed by latter genericity iteration, and $\mathcal{W}_1$ the last model of the lifted tree on $\mathcal{W}_0$.

Do a third genericity iteration of the last model of the lifted stack on $\mathcal{Q}_1^0$ above its δ_0 to $\mathcal{Q}_1^1$ to make x_0 generic at $\delta_1^{\mathcal{Q}_1^1}$, with branches being picked by lifting the tree to $\mathcal{W}_1$ and using branches picked by $\Phi_{\mathcal{W}_1}$. Let $\tau_1^0 : \mathcal{Q}_1^0 \to \mathcal{Q}_1^1$ be the iteration embedding, $j_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_1^1$ be the natural map, and $\pi_0^1 := j_0^1 \circ \sigma_0^1$.

Now continue this process to make x_0 generic over the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and let $\mathcal{Q}_\omega^1$ be the direct limit of the $\mathcal{Q}_n^1$'s under the π_n^1 maps. Then start at $\mathcal{Q}_0^1$ and repeat the same thing to make x_1 generic at the respective δ_2's and so on. Let $\mathcal{Q}_i^\omega$ be the direct limit of the $\mathcal{Q}_i^n$'s under the τ_i^n maps, $\mathcal{R}_i^\omega$

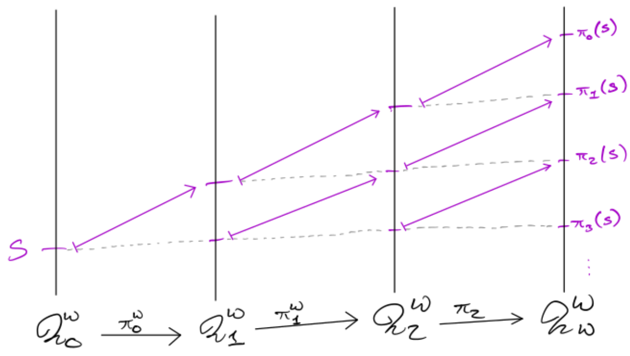

Figure 1.2: The argument in Claim 1.13.

the direct limit of the $\mathcal{R}_i^n$'s under the k_i^n maps and $\mathcal{Q}_\omega^n$ the direct limit of the $\mathcal{Q}_i^n$'s under the π_i^n maps.

By construction we get that the π_n^0's and τ_ω^n's are all by Φ and its tails, and that $\mathcal{Q}_\omega^\omega$ is wellfounded and Lp^Γ-full, so that the $\mathcal{Q}_n^\omega$'s and the $\mathcal{R}_n^\omega$'s are also wellfounded and Lp^Γ-full.

Claim 1.13. There exists some $k < \omega$ such that π_n^ω fixes s for every $n \geq k$.

Proof of claim. It suffices to show that $(\pi_n^\omega(\xi) \mid n < \omega)$ is eventually constant for all $\xi \in s$. Suppose this isn‘t the case. Fix $\xi \in s$ and a strictly increasing sequence $(i_n \mid n < \omega)$ such that $\pi_{i_n}^\omega(\xi) > \xi$ for all $n < \omega$. For $m < n < \omega$ we then have

$$\pi_{i_m,\infty}^\omega(\xi) = \pi_{i_n,\infty}^\omega \circ \pi_{i_m,i_n}^\omega(\xi) \geq \pi_{i_n,\infty}^\omega \circ \pi_{i_m}^\omega(\xi) > \pi_{i_n,\infty}^\omega(\xi),$$

so that $(\pi_{i_n}^\omega(\xi) \mid n < \omega)$ is a strictly decreasing sequence of ordinals in $\mathcal{Q}_\omega^\omega$ – contradicting its wellfoundedness. See Figure 1.2. $\dashv$

Let $k < \omega$ be as in the claim, and note that the j_n^ω's also fix s for $n \geq k$. Since $\mathcal{D}(\mathcal{R}_n^\omega) = \mathcal{M}_\Gamma$ for every $n < \omega$, the $\mathcal{Q}_n^\omega$'s and the $\mathcal{R}_n^\omega$'s have uniform definitions for the term relations for A when $n \geq k$, yielding that j_n^ω pulls back the term relation correctly whenever $n \geq k$, $\lightning$. ■

Theorem 1.14 (DI^+)**.** $\mathrm{Lp}(\mathbb{R}) \models \ulcorner$*there's a fullness preserving hod pair below* $\omega_1\urcorner$.

Proof.

Show the above requirements in Wilson's theorem is satisfied? Double check the statement.

■

Theorem 1.15 (DI^+)**.** *There is a model* M *containing all the reals such that* $M \models \mathsf{AD}^+ + \theta_0 < \Theta$.

Proof.

Let $(\mathcal{M}, \Sigma)$ be a fullness preserving hod pair in $\mathrm{Lp}(\mathbb{R})$ given by the above theorem. Then $\Sigma \notin \mathrm{Lp}(\mathbb{R})$ by the proof of 7.4.3 in the CMI book, and in particular $\Sigma \notin \Gamma_0$. Then $M := L(\Sigma, \mathbb{R})$ is the wanted model.

■

1.3 Ω is not a successor

Definition 1.16. Let $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ be hod pairs below ω_1. We then say that $(\mathcal{Q}, \Lambda)$ **extends** $(\mathcal{P}, \Sigma)$, or is an **extension** of $(\mathcal{P}, \Sigma)$, if there exists some $\alpha < \lambda^{\mathcal{Q}}$ such that

(i) $\mathcal{Q}(\alpha) \in pI(\mathcal{P}, \Sigma)$; and

(ii) $\Sigma_{\mathcal{Q}(\alpha)} = \Lambda_{\mathcal{Q}(\alpha)}$.

We say that $(\mathcal{P}, \Sigma)$ **can be extended** if there exists an extension of $(\mathcal{P}, \Sigma)$.

○

Theorem 1.17 (DI^+)**.** *Every hod pair below* ω_1 *can be extended.*

Rough steps in the proof:

(i) Show that $M_1^{\sharp,\Sigma}$ exists

(ii) $\mathrm{Lp}^{\Sigma}(\mathbb{R}) \models \mathsf{AD}^+$ for some appropriate definition of $\mathrm{Lp}^{\Sigma}(\mathbb{R})$

(iii) The $\Omega > 0$ argument should show that there's an $A \notin \mathrm{Lp}^{\Sigma}(\mathbb{R})$ such that $L(A, \mathbb{R}) \models \mathsf{AD}^+$ and $\Sigma <_W A$

(iv) Show $L(A, \mathbb{R})$ then has the desired $(\mathcal{Q}, \Lambda)$ (this step has already been done and can be black boxed)

1.4 Ω does not have countable cofinality

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1.5 Ω is not singular

Theorem 1.18 (DI^+)**.** *Assume there exists a sequence of hod pairs $(\mathcal{P}_\alpha, \Sigma_\alpha)$ below ω_1 with $(\mathcal{P}_{\alpha+1}, \Sigma_{\alpha+1})$ extending $(\mathcal{P}_\alpha, \Sigma_\alpha)$ for every α. Then either*

(i) There exists a hod pair $(\mathcal{H}, \Lambda)$ below ω_1 such that $\lambda^{\mathcal{H}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$; or

(ii) There exists an $\mathcal{M}$ containing all the reals such that $\mathcal{M} \models \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Rough steps in the proof:

(i) Do the easier countable cofinality case

(ii) Coiterate all the hod pairs to some $(\mathcal{P}, \Sigma)$, which has $\lambda := \lambda^{\mathcal{P}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$

(iii) If λ has non-measurable cofinality then $(\mathcal{P}, \Sigma)$ is the hod pair that we're looking for, so assume this is not the case

(iv) Take the derived model $\mathcal{D}(\mathcal{P}, \lambda)$, which then satisfies $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}+\Omega = \lambda$, where DC is because λ has uncountable cofinality

This is wrong, as we can't take this derived model. Instead we should form a directed system of all "nice" hod pairs having λ's below $\lambda^{\mathcal{P}}$ and take the Lp-closure of that, which should then be an initial segment of hod; call it $\mathcal{H}$.

(v) Show that $\mathcal{H}\,|\delta^{\mathcal{H}}$ is the union of M_∞^α for $\alpha < \lambda$, where M_∞^α is the hod limit of

$$\mathcal{F}_\alpha := \{(\mathcal{Q}, \Psi) \mid \mathrm{Ult}(V, g) \models \ulcorner(\mathcal{Q}, \Psi) \text{ is a hod pair and } \lambda^{\mathcal{Q}} = \alpha\urcorner\}.$$ [1]

Let Φ be the join of the strategies of the M_∞^α's and show that $\mathcal{H} = \mathrm{Lp}_\omega^\Phi(\mathcal{H}\,|\delta^{\mathcal{H}})$.

(vi) Show that $\mathcal{H} \models \ulcorner\delta^{\mathcal{H}}$ is singular$\urcorner$, since otherwise $\mathcal{D}(\mathcal{H}, \delta^{\mathcal{H}}) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular and we're done.

(vii) We want to construct a strategy Λ for $\mathcal{H}$ such that $(\mathcal{H}, \Lambda)$ is a hod pair below ω_1, as then this is the hod pair that we're looking for.

Definition 1.19. Let $(\mathcal{P}, \Sigma)$ be a hod pair. We let

(i) $\mathrm{I}(\mathcal{P}, \Sigma) := \{(\vec{\mathcal{T}}, \mathcal{Q}) \mid \vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ via Σ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$exists$\}$ be the collection of $(\mathcal{P}, \Sigma)$**-iterates**,

(ii) $\mathrm{pI}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$

(iii) $\mathrm{B}(\mathcal{P}, \Sigma) := \{(\mathcal{T}, \mathcal{M}) \mid \mathcal{M} \lhd_{\mathrm{HOD}} \mathcal{Q}$ and $(\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)\}$ be the collection of $(\mathcal{P}, \Sigma)$**-blowups** and

(iv) $\mathrm{pB}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{B}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$.

∘

Definition 1.20. Let $(\mathcal{P}, \Sigma)$ be a hod pair and Γ is a pointclass closed under Boolean operations and continuous images and preimages. Then Σ is Γ**-fullness preserving** if for all $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ and $\delta_\alpha^{\mathcal{Q}} < \eta$ which is a strong cutpoint of $\mathcal{Q}(\alpha + 1)$ we have

(i) $\mathcal{Q}|\eta^{+\mathcal{Q}(\alpha+1)} = \mathrm{Lp}^{\Gamma, \Sigma_{\mathcal{Q}(\alpha), \vec{\mathcal{T}}}}(\mathcal{Q}|\eta)$ and

(ii) $\mathcal{Q}|\delta_\alpha^{+\mathcal{Q}} = \mathrm{Lp}^{\Gamma, \oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1)}, \vec{\mathcal{T}}}(\mathcal{Q}(\alpha))$.

Σ is **fullness preserving** iff it is $\mathcal{P}(\mathbb{R})$-fullness preserving.

Provide a motivation for this definition.

∘

$$\begin{tikzcd} & j(\mathcal{H}) \\ \mathcal{H} \arrow[ur, "j"] \arrow[r, "\pi"] & \mathcal{R} \arrow[u, "\tau"'] \end{tikzcd}$$

Figure 1.3: Full Factors Property

Lemma 1.21. *This will be useful in the proof of the A-condensing lemma.*

Let M, N be transitive models of ZFC^- with largest cardinals δ^M, δ^N respectively. Let $\pi\colon M \to N$ be an elementary embedding, $\kappa := \mathrm{crit}(\pi)$ and let E be the long (κ, δ^N)-extender derived from π. Then $N = \mathrm{Ult}(M; E)$ and $\pi = \pi_E$ is the canonical ultrapower embedding.

Proof. We have the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\pi_E"'] & N \\ & \mathrm{Ult}(M;E) \arrow[u, "k"'] \end{tikzcd}$$

where k satisfies $k \restriction \delta^N = \mathrm{id}$. Let $\delta^{\mathrm{Ult}(M;E)}$ be the largest cardinal of $\mathrm{Ult}(M; E)$. By elementarity $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^N$, so that $\delta^{\mathrm{Ult}(M;E)} \leq \delta^N$. If $\delta^{\mathrm{Ult}(M;E)} < \delta^N$, then $k \restriction \delta^N = \mathrm{id}$ yields $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^{\mathrm{Ult}(M;E)} < \delta^N$, which is absurd. Hence $\delta^{\mathrm{Ult}(M;E)} = \delta^N$ and $k \restriction (\delta^{\mathrm{Ult}(M;E)} + 1) = \mathrm{id}$. Since $\delta^{\mathrm{Ult}(M;E)}$ is the largest cardinal of $\mathrm{Ult}(M; E)$, it follows that k doesn‘t have a critical point. Therefore $k = \mathrm{id}$, $N = \mathrm{Ult}(M; E)$ and $\pi = \pi_E$. ■

Lemma 1.22. *$j \restriction \mathcal{H}$ has the* ***full factors property***[2]*, meaning that whenever $\mathcal{R}$*

R has to be countable in $V[g]$. How can we ensure that, as this only gives that it has size $\leq \aleph_1$? Do we have to resort to the (long) claim in Grigor's uB paper?

[2]This terminology was introduced in [?]; in [?] this was called *weak condensation*.

is a hod premouse and there are elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$, then $\mathcal{R}$ is $\Sigma^2_1(j(\Omega)^\tau)$-full.

Proof. Let $\Psi := j(\Omega)^\tau$ and assume the lemma fails, meaning that we have a hod mouse $\mathcal{R}$ and elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$ and $\mathcal{R} \neq \mathrm{Lp}^\Psi_\omega(\mathcal{R}\,|\delta^\mathcal{R})$, witnessed without loss of generality by an $\mathcal{M} \trianglelefteq \mathrm{Lp}^\Psi(\mathcal{R}\,|\delta^\mathcal{R})$ such that $\rho(\mathcal{M}) = \delta^\mathcal{R}$ and which is not an initial segment of $\mathcal{R}$.

$$\begin{tikzcd}
(\mathcal{H},\Omega) \arrow[r, "\pi"] \arrow[dr, "j\restriction\mathcal{H}"'] & (\mathcal{R},\Psi) \arrow[d, "\tau"] \\
& (j(\mathcal{H}), j(\Omega))
\end{tikzcd}$$

We can then fix some hod pair $(\mathcal{S}^*, \Lambda^*)$ such that $\tau``\mathcal{R}\,|\delta^\mathcal{R} \subseteq \mathrm{ran}(\pi^{\Lambda^*}_{\mathcal{S}^*,\infty})$, and furthermore let $\xi \leq \lambda^{\mathcal{S}^*}$ be least such that $\tau``\mathcal{R}\,|\delta^\mathcal{R} \subseteq \mathrm{ran}(\pi^{\Lambda^*}_{\mathcal{S}^*(\xi),\infty})$. Lastly let $(\mathcal{S}, \Lambda)$ be an extension of $(\mathcal{S}^*, \Lambda^*)$ such that $\lambda^\mathcal{S}$ is a limit ordinal.

Argue why $\mathcal{S}^*$ and $\mathcal{S}$ exist; we should be in the limit case to argue that $\mathcal{S}$ exists.

Let $\sigma : \mathcal{R}\,|\delta^\mathcal{R} \to \mathcal{S}|\delta^\mathcal{S}_\gamma$, where $\mathcal{S}^*(\xi)$ iterates to $\mathcal{S}(\gamma)$, be given by $\sigma(x) = y$ iff $\tau(x) = \pi^\Lambda_{\mathcal{S}(\gamma),\infty}(y)$.

$$\begin{tikzcd}
(\mathcal{R}\,|\delta^\mathcal{R}, \Psi) \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & (j(\mathcal{H})|\delta^{j(\mathcal{H})}, j(\Omega)) \\
& (\mathcal{S}|\delta^\mathcal{S}_\gamma, \bigoplus_{\beta<\gamma} \Lambda_{\mathcal{S}(\beta)}) \arrow[ur, "\pi^\Lambda_{\mathcal{S}(\gamma),\infty}"'] &
\end{tikzcd}$$

We can fix some hod pair $(\mathcal{S}', \Lambda')$ such that

This should follow from generation of good pointclasses.

$$L(\Lambda', \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner.$$

By coiterating $\mathcal{S}$ and $\mathcal{S}'$ we may assume without loss of generality that $\mathcal{S} = \mathcal{S}'$.

This requires us to work in an AD$^+$ model, so we better assume that somewhere.

Claim 1.23. There exists a hod pair $(\mathcal{Q}, \Phi)$ such that $\lambda^\mathcal{Q}$ is a limit ordinal and $L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner$.

Proof of claim.

This claim shouldn't be needed, as we should be able to take $\mathcal{Q}$ to be $\mathcal{S}$ in our case, using facts about the Γ-pointclasses. Also ensure that $\mathcal{Q} \supseteq \mathcal{H}$, which is possible as we're stretching by j.

⊣

Fix $(\mathcal{Q}, \Phi)$ as in the claim and let $\mathcal{N}$ be some mouse such that $\mathcal{M} \lhd \mathcal{N}$ and $\mathcal{N}$ has ω many Woodins on top of $\mathcal{M}$.

Explain how this is done. In Grigor's paper he's using that "$j(\eta)$ is closed under hybrid $\mathcal{N}_\omega$-operators". In our measurable cofinality case there might be enough room to get this. Postpone until later, when we have an idea of how much operator closure we have at this point.

Then we get that

Why is $\Gamma(\mathcal{Q}, \Phi)$ in $\mathcal{D}(\mathcal{N})$?

$$\mathcal{D}(\mathcal{N}) \models \ulcorner L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner\urcorner.$$

In $V[g]$, I guess.

Now throw everything in sight into a countable hull, so that

$$\mathcal{D}(\overline{\mathcal{N}}) \models \ulcorner L(\Gamma(\overline{\mathcal{Q}}, \overline{\Phi}), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \overline{\Psi}\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner\urcorner.$$

I think that now $\overline{\mathcal{Q}}$ are taking the role of "$L[\mathcal{T}, \mathcal{H}]$", as Grigor's paper seems to indicate that $\mathcal{H} \subseteq \overline{\mathcal{Q}}$.

A it hand-wavy.

Now lift π to the ultrapower map π^+ given by the $(\delta^{\mathcal{H}}, \delta^{\mathcal{R}})$-extender over $\overline{\mathcal{Q}}$ derived from π, and let $\mathcal{R}^+$ be the ultrapower. Lift also σ, τ to corresponding σ^+, τ^+.

$$\begin{tikzcd}
(\overline{\mathcal{Q}}, \overline{\Phi}) \arrow[r, "\pi^+"] \arrow[dr, "\sigma^+"'] & (\mathcal{R}^+, \Phi^{**}) \arrow[d, "\tau^+"] \\
 & (j(\overline{\mathcal{Q}}), \Phi^*)
\end{tikzcd}$$

Let now $\Phi^* := j(\overline{\Phi})$ and $\Phi^{**} := (\Phi^*)^{\tau^+}$, which is then a strategy for $\mathcal{R}^+$. Since $\overline{\Phi} = (\Phi^{**})^{\pi^+}$ we get that

Check this — might be by definition of pullback consistency, which is implied by hull condensation.

In Grigor's uB paper he uses a certain derived model C instead of $\mathcal{D}(\overline{\mathcal{N}})$, but I can't see how they're different from each other. Also, figure out why the following inclusion is true (it's probably folklore).

$$\mathcal{D}(\overline{\mathcal{N}}) \subseteq \mathcal{D}(\mathcal{R}^+, \Phi^{**}),$$

implying that

Note sure what's going on here.

$L(\Gamma(\mathcal{R}^+, \Phi^{**}), \mathbb{R}) \models \ulcorner\mathcal{M}$ is a Ψ-mouse which isn't an initial segment of $\mathcal{R}\urcorner$.

Because $\mathcal{R}^+$ is a Ψ-mouse over $\mathcal{R}|\delta^{\mathcal{R}}$, it follows that

Why's that?

$$\mathcal{D}(\mathcal{R}^+) \models \ulcorner\mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner,$$

which then implies that $\mathcal{M} \in \mathcal{R}^+$, so that $\mathcal{M} \trianglelefteq \mathcal{R}$, a contradiction. ■

I don't see how this last argument works.

Definition 1.24. For every $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ define $Q_X := \mathrm{cHull}^{j(\mathcal{H})}(X)$ and let

$$\tau_X \colon Q_X \to j(\mathcal{H})$$

be the uncollapse.

Say that $Y \in \mathcal{P}_{\omega_1}(j(\mathcal{H})|\delta^{j(\mathcal{H})})$ **extends** X if $X \cap j(\delta^{\mathcal{H}}) \subseteq Y$ and in that case let

(i) $\tau_{X,Y} := \tau_{X \cup Y}$,
(ii) $\Phi_{X,Y} := j(\Phi)^{\tau_{X,Y}}$,
(iii) $Q_{X,Y} := Q_{X \cup Y}$ and
(iv) $\pi_{X,Y} \colon Q_X \to Q_{X,Y}$ is the induced embedding given by

$$\pi_{X,Y}(x) = \tau_Y^{-1}(\tau_X(x)).$$

Furthermore define $T_X(A)$ for $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$ as

$$\begin{aligned} T_X(A) :&= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } Q_X \models \varphi[s, A]\} \\ &= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\tau_X(s), \tau_X(A)]\} \end{aligned}$$

and let $T_{X,Y}(A)$ be given as

$$T_{X,Y}(A) := \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_{X,Y}}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty}(s), \tau_X(A)],$$
$$\text{where } \alpha \text{ is least such that } s \in [\delta^{Q_{X,Y}}_\alpha]^{<\omega}\}.$$

What is $\nu_{X,Y}$?

Here $\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty} : Q_{X,Y} \to j(\mathcal{H})|\nu_{X,Y}$ is given by

Missing! (It will be the iteration into an appropriate level of the directed system leading up to $j(\mathcal{H})$ followed by the direct limit embedding into some initial segment of $j(\mathcal{H})$)

$\circ$

Definition 1.25. Let $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ and $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$. Then X is ***A*-condensing** if $\pi_{X,Y}(T_X(A)) = T_{X,Y}(A)$ for every Y extending X.
We say that X is **condensing** if X is A-condensing for all such A. $\circ$

We want to show that $j``\mathcal{H}$ is condensing. We first show that it suffices to show that it's α-condensing for every $\alpha < \delta^{\mathcal{H}}$.

Lemma 1.26. *If $j``\mathcal{H}$ is $\alpha$-condensing for every $\alpha < \delta^{\mathcal{H}}$ then $j``\mathcal{H}$ is condensing.*

Proof.

Missing!

■

Reduce this to $j``\mathcal{H}$ somehow?

Theorem 1.27. *For every $\alpha < \delta^{\mathcal{H}}$ there exists an extension Y of $j``\mathcal{H}$ such that $j``\mathcal{H} \cup Y$ is α-condensing.*

Proof. Set $X := j``\mathcal{H}$ and assume the theorem fails. Fix some $\alpha < \delta^{\mathcal{H}}$ such that X is not α-condensing. Fix some Y_0 extending X which witnesses this, meaning that $\pi^X_{Y_0}(T^X_\alpha) \neq T^{X,Y_0}_\alpha$. Since we're also assuming that $\tau^X_{Y_0}$ isn't α-condensing we can find Y_1 extending Y_0 such that $\pi^{Y_0}_{Y_1}(T^{Y_0}_\alpha) \neq T^{Y_0,Y_1}_\alpha$. Continue doing this, generating a sequence $\langle Y_n \mid n < \omega\rangle$ with Y_{n+1} extending Y_n and

$$\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) \neq T^{Y_n,Y_{n+1}}_\alpha \tag{1}$$

for all $n < \omega$. Let $\mathcal{P}_n := Q^X_{Y_n}$, $\pi_{m,n} := \pi^{Y_m}_{Y_n}$ and $\pi_n := \pi_{0,n}$. We want to show that such a sequence can't exist. Towards getting a contradiction we first need to make everything in sight countable, as that will allow us to reason using derived models (the problem is that $j(\mathcal{H})$ is too big, namely it has size $\aleph_1^{V[g]}$).

Using that $\delta^{j(\mathcal{H})}$ has uncountable cofinality we can find $\kappa < \delta^{j(\mathcal{H})}$ such that

$$\kappa = \mathrm{Hull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\operatorname{ran} \tau^X_{Y_n} \mid n < \omega\}) \cap \delta^{j(\mathcal{H})}.$$

Set $\mathcal{M} := \mathrm{cHull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\operatorname{ran} \tau^X_{Y_n} \mid n < \omega\})$ and note that $\mathcal{M} = j(\mathcal{H})|\kappa^{+j(\mathcal{H})}$. Let $\pi : \mathcal{M} \to j(\mathcal{H})$ be the uncollapse and note that $\operatorname{crit}\pi = \kappa$ and that $\kappa = \delta^{\mathcal{M}}$. Define $\iota : \mathcal{H} \to \mathcal{M}$ as $\iota := \pi^{-1} \circ j$ and $\tau_n : \mathcal{P}_n \to \mathcal{M}$ as $\tau_n := \pi^{-1} \circ \tau^X_{Y_n}$. Note that $\mathcal{M}$ is countable in $V[g]$ and is hence an element of $\mathrm{Ult}(V, g)$.

Now define $\mathcal{H}^+$ as the hod limit of iterates of $\mathcal{H}$, so that $\mathcal{H}^+$ is a hod premouse with $\mathcal{H} \lhd_{\mathrm{hod}} \mathcal{H}^+$, $\mathcal{H}^+$ has a strategy Ψ extending Ω such that

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{H}^+, \Psi).$$

Also define $(\mathcal{P}^+_n, \Psi_n)$ as $P^+_n := \mathrm{Ult}(\mathcal{H}^+, E_{\pi_n})$, so that we also get that

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{P}^+_n, \Psi_n).$$

Now $\mathcal{D}(\mathcal{P}^+_n, \Psi_n)$ has a definition of T^{X,Y_n}_α, so that $\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) = T^{Y_n,Y_{n+1}}_{\pi_{n,n+1}(\alpha)}$. The three-dimensional argument then shows that α must be fixed by $\pi_{n,n+1}$

Missing argument

Provide more details.

We probably need $\mathcal{H}^+$ to be countable here, so we should probably apply the induced ideal and work in $V[g][h]$.

Missing argument. This might need that $\mathcal{H}^+, \Psi \restriction V \in V$, but we could probably also just work inside $\mathrm{Ult}(V, g)$, or the second ultrapower, all along.

for some $n < \omega$, so that $X \cup Y_n$ *is* α-condensing, $\lightning$. ■

Define the strategy Λ for $\mathcal{H}$ and show that $(\mathcal{H}, \Lambda)$ is a hod pair.