# 1 IDEAL ABSOLUTENESS

Historically, the idea of considering elementary embeddings existing only in generic extensions has been around for a while, but it all started as an analysis of *ideals*. *Precipitous ideals* were introduced in [?] and further analysed in [?], being ideals that give rise to wellfounded generic ultrapowers.

In this chapter we will introduce the *ideally measurable cardinals*, essentially just switching perspective from the ideals themselves to the cardinals they are on. We then proceed to show how these cardinals relate to "pure" generic cardinals, being proper class versions of the faintly measurable cardinals that we have considered throughout Chapter ??. We start with a definition of the latter.

**DEFINITION 1.1** (GBC). A cardinal $\kappa$ is **generically measurable** if there is a generic extension $V[g]$, a transitive class $\mathcal{N} \subseteq V[g]$ and a generic elementary embedding $\pi\colon V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$. ○

Note that, trivially, every generically measurable cardinal is faintly measurable. The corresponding ideal version of this is then the following.

**DEFINITION 1.2.** A cardinal $\kappa$ is **ideally measurable** if there exists an ideal $\mathcal{I}$ on $\theta$ such that the generic ultrapower $\operatorname{Ult}(V, \mathcal{I})$ is wellfounded in $V^{\mathbb{P}}$ for $\mathbb{P} := \mathscr{P}^V(\kappa)/\mathcal{I}$. ○

It should also be noted that [?] generalised the concept of ideally measurables to *ideally strong cardinals* by introducing the concept of *ideal extenders* to capture the strongness properties.

Throughout this chapter we will be interested in how properties of the *forcings* affect the large cardinal structure of a critical point of a generic embedding. We thus define the following.

**Definition 1.3.** Let $\theta$ be a regular uncountable cardinal, $\kappa < \theta$ a cardinal and $\Phi(\kappa)$ a poset property[1]. Then $\kappa$ is $\Phi(\kappa)$ **faintly $\theta$-measurable** if it is faintly $\theta$-measurable, witnessed by a forcing poset satisfying $\Phi(\kappa)$. Similarly, $\kappa$ is $\Phi(\kappa)$ **generically measurable** if it is generically measurable with the associated forcing satisfying $\Phi(\kappa)$. ◦

Note that $\omega$-distributive faintly $\theta$-measurable cardinals are equivalent to $\omega$-distributive generically measurable cardinals for all regular $\theta$ since wellfoundedness becomes automatic.

**Definition 1.4.** A poset property $\Phi(\kappa)$ is **ideal-absolute** if whenever $\kappa$ satisfies that there's a $\Phi(\kappa)$ forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a $V$-normal $V$-measure $\mu$ on $\kappa$, then there's an ideal $I$ on $\kappa$ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(v)$. ◦

Note that this is *almost* saying that $\Phi(\kappa)$ ideally measurables are equivalent to $\Phi(\kappa)$ generically measurables, but the only difference is that these definitions require wellfoundedness of the target model.

A typical ideal that we will be utilising is the following.

**Definition 1.5.** Let $\kappa$ be a regular cardinal, $\mathbb{P}$ a poset and $\dot{\mu}$ a $\mathbb{P}$-name for a $V$-normal $V$-measure on $\kappa$. Then the **induced ideal** is

$$\mathcal{I}(\mathbb{P}, \dot{\mu}) := \{X \subseteq \kappa \mid ||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} = 0\},$$

where $\mathcal{B}(\mathbb{P})$ is the boolean completion of $\mathbb{P}$. ◦

Note that if the generic measure $\mu$ is $V$-normal then $\mathcal{I}(\mathbb{P}, \dot{\mu})$ is also normal. This ideal will witness our first ideal-absoluteness result, which is a simple rephrasing of a folklore result.

**Theorem 1.6** (Folklore). *"The $\kappa^+$-chain condition" is ideal-absolute.*

[1]Examples of these are having the $\kappa$-chain condition, being $\kappa$-closed, $\kappa$-distributive, $\kappa$-Knaster, $\kappa$-sized and so on. Formally speaking, $\Phi(\kappa)$ is a first-order formula $\varphi(\kappa, \mathbb{P})$ which is true iff $\mathbb{P}$ is a poset, $\kappa$ is a cardinal and some first-order formula $\psi(\kappa, \mathbb{P})$ is true.

Proof. Assume $\mathbb{P}$ has the $\kappa^+$-chain condition such that there's a $\mathbb{P}$-name $\dot{\mu}$ for a $V$-normal $V$-measure on $\kappa$. Let $I := \mathcal{I}(\mathbb{P}, \dot{\mu})$ – we will show that $\mathscr{P}(\kappa)/I$ has the $\kappa^+$-chain condition. Assume not and let $\langle X_\alpha \mid \alpha < \kappa^+\rangle$ be an antichain of $\mathscr{P}(\kappa)/I$, which by normality of $I$ we may assume is pairwise almost disjoint. But this then makes $\langle ||\check{X}_\alpha \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid \alpha < \kappa^+\rangle$ an antichain of $\mathbb{P}$ of size $\kappa^+$, $\lightning$. ■

We next move to distributivity. This property is especially interesting in the context of our generic large cardinals, as an ideal $I$ on some cardinal $\kappa$ is $\omega$-distributive precisely if it's precipitous[2], so that carrying an $\omega$-distributive ideal coincides with our definition of *ideally measurable*.

**Theorem 1.7** (N.). *"$<\lambda$-distributivity" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*

Proof. Assume that $\mathbb{P}$ is a $<\lambda$-distributive forcing such that there exists a $\mathbb{P}$-name $\dot{\mu}$ for a $V$-normal $V$-measure on $\kappa$. Let $\mathcal{I} := \mathcal{I}(\mathbb{P}, \dot{\mu})$ – we'll show that $\mathscr{P}(\kappa)/\mathcal{I}$ is $<\lambda$-distributive.

Let $\gamma < \lambda$ and let $\vec{\mathcal{A}}$ be a $\gamma$-sequence of maximum antichains $\mathcal{A}_\alpha \subseteq \mathscr{P}(\kappa)/\mathcal{I}$ such that $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ for $\alpha \leq \beta$. We have to show that there's a maximal antichain $\mathcal{A}$ which refines all the antichains in $\vec{\mathcal{A}}$.

Now define for every $\alpha < \gamma$ the sets

$$\mathcal{A}^*_\alpha := \{||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid X \in \mathcal{A}_\alpha\}.$$

Note that $\mathcal{A}^*_\alpha$ is an antichain in $\mathbb{P}$. They're also maximal, because if $p \in \mathbb{P}$ was incompatible with every condition in $\mathcal{A}^*_\alpha$ then, letting $X := \bigcap \mathcal{A}_\alpha$, we have that $p$ is compatible with $||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$, so that $X \in \mathcal{I}^+$. But $X$ is incompatible with everything in $\mathcal{A}_\alpha$, contradicting that $\mathcal{A}_\alpha$ is maximal.

By $<\lambda$-distributivity of $\mathbb{P}$ we get an antichain $\mathcal{A}^*$ which refines all the antichains in $\vec{\mathcal{A}}^*$. But note that for every $p \in \mathcal{A}^*$, if we define $s_p(\alpha)$ to be the unique $a \in \mathcal{A}_\alpha$ such that $p \leq a$, then it holds that $p \leq ||\Delta s_p \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$,[3] so that $\Delta b_p \in \mathcal{I}^+$. Now $\mathcal{A} := \{\Delta b_p \mid p \in \mathcal{A}^*\}$ gives us a maximal antichain consisting of limit points of branches of $\mathcal{T}$. ■

[2] See [?] and [?].

[3] Here we're using that $\lambda \leq \kappa^+$ to ensure that the diagonal intersection is in the measure.

In an unpublished paper, Foreman proved the following.

**Theorem 1.8** (Foreman). *Let $\kappa$ be a regular cardinal such that $2^\kappa = \kappa^+$, and let $\lambda \leq \kappa^+$ be an infinite successor cardinal. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then $\kappa$ carries a $\kappa$-complete normal precipitous ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ has a dense $<\lambda$-closed subset of size $\kappa^+$.*

Here we improve that result by not relying on the CH-assumption, reaching the conclusion for all regular infinite $\lambda$ and also showing $(\kappa,\kappa)$-distributivity of the ideal forcing. The argument follows the same overall structure as the original, with more technicalities to achieve the stronger result.

**Theorem 1.9** (Foreman-N.). *Let $\kappa$ be a regular cardinal and $\lambda \leq \kappa^+$ be regular infinite. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then $\kappa$ carries a $\kappa$-complete normal ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is $(\kappa,\kappa)$-distributive and has a dense $<\lambda$-closed subset of size $\kappa^+$.*

Proof. Set $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$ if $2^\kappa > \kappa^+$ and $\mathbb{P} := \{\emptyset\}$ otherwise. If $\kappa$ is measurable then the dual ideal to the measure on $\kappa$ satisfies all of the wanted properties, so assume that $\kappa$ is not measurable. Fix a wellordering $<_{\kappa^+}$ of $H_{\kappa^+}$ and a $\mathbb{P}$-name $\pi$ for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \kappa^+\rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \kappa^+$;
- $\mathcal{N}_{\gamma+1} \prec H^V_{\kappa^+}$ is a $\kappa$-model for every $\gamma < \kappa^+$;
- $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$ for limit ordinals $\delta < \kappa^+$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for $\gamma < \beta < \kappa^+$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\kappa^+} \mathcal{N}_\gamma$.

Define now the auxilliary game $\mathcal{G}(\kappa)$ of length $\lambda$ as follows.

| | | | | | |
|---|---|---|---|---|---|
| I | $\alpha_0$ | | $\alpha_1$ | | $\cdots$ |
| II | | $p_0, \mathcal{M}_0, \mu_0, Y_0$ | | $p_1, \mathcal{M}_1, \mu_1, Y_1$ | $\cdots$ |

Here $\langle\alpha_\gamma \mid \gamma<\lambda\rangle$ is an increasing continuous sequence of ordinals bounded in $\kappa^+$, $\vec{p}_\gamma$ is a decreasing sequence of $\mathbb{P}$-conditions satisfying that

$$p_\gamma \Vdash \ulcorner \check{\mathcal{M}}_\gamma = \pi(\check{\alpha}_\gamma) \wedge \check{\mu}_\gamma \text{ is a } \check{\mathcal{M}}_\gamma\text{-normal } \check{\mathcal{M}}_\gamma\text{-measure on } \check{\kappa}\urcorner$$

such that $Y_\gamma = \Delta_{\xi<\kappa} X_\xi^{\mu_\gamma}$, where $\vec{X}_\xi^{\mu_\gamma} \in H_{\kappa^+}^V$ is the $<_{\kappa^+}$-least enumeration of $\mu_\gamma$.[4] We require that the $\mu_\gamma$'s are $\subseteq$-increasing, and player II wins iff she can continue playing throughout all $\lambda$ rounds. Let $\mu_\lambda := \bigcup_{\xi<\lambda}\mu_\xi$ be the **final measure** of the play.

To every limit ordinal $\eta<\kappa^+$ define the **restricted auxilliary game** $\mathcal{G}(\kappa)\restriction\eta$ in which player I is only allowed to play ordinals $<\eta$. Note that a strategy $\tau$ for player II is winning in $\mathcal{G}(\kappa)$ if and only if it's winning in $\mathcal{G}(\kappa)\restriction\eta$ for all $\eta<\kappa^+$, simply because all sequences of ordinals played by player I are bounded in $\kappa^+$.

Note that $\mu_\lambda$ is precisely the tail measure on $\kappa$ defined by the $Y_\gamma$'s; i.e. that $X\in\mu_\lambda$ iff there exists a $\delta<\lambda$ such that $|Y_\delta - X|<\kappa$. From this it's simple to see that $\mathcal{G}(\kappa)$ is equivalent to $\mathcal{G}_\lambda^-(\kappa)$, so player II has a winning strategy $\tau_0$ in $\mathcal{G}(\kappa)$.

For any winning strategy $\tau$ in $\mathcal{G}(\kappa)\restriction\eta$ and to every partial play $p$ of $\mathcal{G}(\kappa)\restriction\eta$ consistent with $\tau$, define the associated **hopeless ideal**[5]

$$I_p^\tau\restriction\eta := \{X\subseteq\kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction\eta,\ X \text{ is } \textit{not} \text{ in the final measure}\}$$

*Claim* 1.10. Every hopeless ideal $I_p^\tau\restriction\eta$ is normal and $(\kappa,\kappa)$-distributive.

Proof of claim. For normality, if $\langle Z_\gamma \mid \gamma<\kappa\rangle$ is a sequence of elements of $I_p^\tau$ such that $Z := \nabla_\gamma Z_\gamma$ is $I_p^\tau$-positive, then there exists a play of $\mathcal{G}(\kappa)\restriction\eta$ in which player II follows $\tau$ such that $Z$ lies in the final measure. If we let player I play sufficiently large ordinals in $\mathcal{G}(\kappa)\restriction\eta$ we may assume that $\langle Z_\gamma \mid \gamma<\kappa\rangle$ is a subset and an element of the final model as well, meaning that one of the $Z_\gamma$'s also lies in the final measure, $\lightning$.

We now show $(\kappa,\kappa)$-distributivity. Let $\mathcal{U}\subseteq\mathscr{P}(\kappa)/I_p^\tau$ be an unrooted tree of height $\kappa$ such that every level $\mathcal{U}_\alpha$ is a maximal antichain of size $\leq\kappa$. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of

[4] We use that $\mathbb{P}$ is $\kappa$-closed to get the $p_\gamma$'s as well as to ensure that $\mathcal{M}_\gamma,\mu_\gamma\in V$.
[5] This terminology is due to Matt Foreman.

branches of $\mathcal{U}$. Pick $X \in \mathcal{U}$ and let $p$ be a play of $\mathcal{G}(\kappa) \upharpoonright \eta$ consistent with $\tau$ with limit model $\mathcal{M}$ and limit measure $\mu$, such that $X \in \mu$.

By letting player I in $p$ play sufficiently large ordinals, we may assume that $\mathcal{U} \subseteq \mathcal{M}$, using that $|\mathcal{U}| \leq \kappa$, and also that $b_X := \mathcal{U} \cap \mu \in \mathcal{M}$. This means that $d_X := \Delta b_X \in \mathscr{P}(\kappa)/I_p^\tau$ is a limit point of the branch $b_X$ through $\mathcal{U}$, so that $\mathcal{A} := \{d_X \mid X \in \mathcal{U}\}$ is a maximal antichain of limit points of branches of $\mathcal{U}$, making $\mathscr{P}(\kappa)/I_p^\tau$ $(\kappa, \kappa)$-distributive. $\dashv$

Fix some limit ordinal $\eta < \kappa^+$. We will recursively construct a tree $\mathcal{T}^\eta$ of height $\lambda$ which consists of subsets $X \subseteq \kappa$, ordered by reverse inclusion. During the construction of the tree we will inductively maintain the following properties of $\mathcal{T}^\eta \upharpoonright \alpha$ for $\alpha \leq \lambda$:

- Tree strategy: For every $\gamma < \alpha$ there is a winning strategy $\tau_\gamma^\eta$ for player II in $\mathcal{G}(\kappa) \upharpoonright \eta$ such that for every $\beta < \gamma$, the $\beta$'th move by $\tau_\gamma^\eta$ is an element of $\mathcal{T}_\beta^\eta$ and $\tau_\gamma^\eta$ is consistent with $\tau_\beta^\eta$ for the first $\beta$-many rounds.
- Unique pre-history: Given any $\beta < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$ there's a unique partial play $p$ of $\mathcal{G}(\kappa) \upharpoonright \eta$ consistent with $\tau_\beta^\eta$ ending with $Y$ – we define $I_Y^\tau := I_p^\tau$ for $\tau$ being any winning strategy for player II in $\mathcal{G}(\kappa) \upharpoonright \eta$ satisfying that $p$ is consistent with $\tau_\beta^\eta$.
- Cofinally many responds: Let $\beta + 1 < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$, and set $p$ to be the unique partial play of $\mathcal{G}(\kappa) \upharpoonright \eta$ given by the unique pre-history of $Y$. Then the $\mathcal{T}^\eta$-successors of $Y$ consists of player II's $\tau_\beta^\eta$-responds to $\tau_\beta^\eta$-partial plays extending $p$ such that player I's last move in these partial plays are cofinal in $\eta$.[6]
- Positivity: If $\beta < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$ then $Y$ is $I_X^{\tau_\gamma^\eta}$-positive for every $\gamma < \beta$ and every $X \in \mathcal{T}^\eta \upharpoonright \gamma + 1$ with $X \leq_{\mathcal{T}^\eta} Y$.[7]
- Almost disjointness property: Every level $\mathcal{T}_\beta^\eta$ consists of pairwise almost disjoint sets.[8]

[6] The reason why we're dealing with the *restricted* auxilliary games is to achieve this property.
[7] This actually follows from the cofinally many responds, but we include it here for transparency.
[8] Two subsets $X, Y \subseteq \kappa$ are *almost disjoint* if $|X \cap Y| < \kappa$.

- Hopeless ideal coherence: $I^{\tau^\eta_\beta}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_\beta}_Y \cap \mathscr{P}(Y)$ for every $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$.

Note that what we're really aiming for is achieving the hopeless ideal coherence, since that enables us to ensure that if $X, Y \in \mathcal{T}^\eta$ and $X \subseteq Y$ then really $X \geq_{\mathcal{T}^\eta} Y$ – i.e. that we "catch" both $X$ and $Y$ in the same play of $\mathcal{G}(\kappa) \restriction \eta$. The rest of the properties are inductive properties we need to ensure this.

Set $\mathcal{T}^\eta_0 := \{\kappa\}$. Assume that we've built $\mathcal{T}^\eta \restriction \alpha + 1$ satisfying the inductive assumptions[9] and let $Y \in \mathcal{T}^\eta_\alpha$ – we need to specify what the $\mathcal{T}^\eta$-successors of $Y$ are. Since $\kappa$ is weakly compact and not measurable it holds by Proposition 6.4 in [?] that $\mathrm{sat}(I^{\tau^\eta_\alpha}_Y) \geq \kappa^+$, so we can fix a maximal antichain $\langle X^Y_\gamma \mid \gamma < \eta \rangle$ of $I^{\tau^\eta_\alpha}_Y$-positive sets. By $\kappa$-completeness of $I^{\tau^\eta_\alpha}_Y$ we can by Exercise 22.1 in [?] even ensure that all of the $X^Y_\gamma$'s are pairwise disjoint.

To every $\gamma < \eta$ we fix a partial play $p$ of even length of $\mathcal{G}(\kappa) \restriction \eta$ consistent with $\tau^\eta_\alpha$ such that the last ordinal $\beta^Y_\gamma$ in $p$ played by player I is greater than or equal to $\gamma$ and $X^Y_\gamma$ has measure one with respect to the last measure in $p$. We then define the $\mathcal{T}^\eta$-successors of $Y$ to be player II's $\tau^\eta_\alpha$-responses to the $\beta_\gamma$'s (which are subsets of the $X^Y_\gamma$'s modulo a bounded set and are therefore pairwise almost disjoint).

For limit stages $\delta < \lambda$ we apply $\tau_0$ to the branches of $\mathcal{T}^\eta \restriction \delta$ to get $\mathcal{T}^\eta_\delta$.

We now have to check that the inductive assumptions still hold; let's start with the tree strategy. Assume that we have a partial play $p$ of length $2 \cdot \alpha + 1$ of $\mathcal{G}(\kappa) \restriction \eta$, i.e. the last move in $p$ is by player II, consistent with $\tau^\eta_\alpha$; write $\xi_p$ for player I's last move in $p$ and $Y_p$ for player II's response to $\xi_p$, which is also the last move in $p$. We can then pick a $\zeta < \eta$ such that $\beta^{Y_p}_\zeta > \xi_p$ by the cofinally many responds property and let $\tau^\eta_{\alpha+1}(p)$ be player II's $\tau^\eta_\alpha$-response to the partial play leading up to $\beta^{Y_p}_\zeta$. After this $(\alpha + 1)$'th round we just set $\tau^\eta_{\alpha+1}$ to follow $\tau_0$. It's clear that $\tau^\eta_{\alpha+1}$ satisfies the required properties.

Before we move on to checking the remaining inductive assumptions, let's pause to get some intuition about the tree strategies. In the definition of $\tau^\eta_{\alpha+1}$ above, we took a partial play consistent with $\tau^\eta_\alpha$, applied $\tau_0$ for a while, took note of player II's last $\tau_0$-response and then included *only that* response in our new $\tau^\eta_{\alpha+1}$ partial play. This means that to every $\tau^\eta_\alpha$-partial play there's an ostensibly much longer

[9] In particular, we assume that $\tau^\eta_\alpha$ is defined.

$\tau_0$-partial play into which $\tau_\alpha^\eta$ embeds; so we can look at the $\tau_\alpha^\eta$-partial plays as being "collapsed" $\tau_0$-partial plays.

Given the above tree strategy, $\mathcal{T}^\eta_{\alpha+1}$ clearly satisfies the cofinally many responds property and the positivity property, simply by construction. For the unique pre-history, let $Y \in \mathcal{T}^\eta_{\alpha+1}$ and assume it has two distinct immediate $\mathcal{T}^\eta$-predecessors $Z_0, Z_1 \in \mathcal{T}^\eta_\alpha$. But then $Y \subseteq Z_0 \cap Z_1$ and $Y$ is $I^{\tau^\eta_\alpha}_{Z_0}$-positive by the positivity assumption, contradicting that $Z_0$ and $Z_1$ are almost disjoint by the almost disjointness property. Given the unique pre-history we then also get the almost disjointness property.

*Claim* 1.11. $\mathcal{T}^\eta \restriction \alpha + 2$ satisfies the hopeless ideal coherence property.

Proof of claim. Let $Y \in \mathcal{T}^\eta_{\alpha+1}$ – we have to show that

$$I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y). \tag{1}$$

It's clear that $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \subseteq I^{\tau^\eta_{\alpha+1}}_{Y}$, so let $Z \in I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y)$ and assume for a contradiction that $Z$ is $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle}$-positive. Letting $\vec{\alpha}_\xi * \vec{Y}_\xi$ be a play of $\mathcal{G}(\kappa) \restriction \eta$ consistent with $\tau^\eta_{\alpha+1}$ such that $Z$ is in the final measure, the definition of $\tau^\eta_{\alpha+1}$ yields that $Y_\alpha \in \mathcal{T}^\eta_{\alpha+1}$. As $Z \in I^{\tau^\eta_{\alpha+1}}_{Y}$ we have to assume that $Y \neq Y_\alpha$, so that the almost disjointness property implies that

$$|Y \cap Y_\alpha| < \kappa, \tag{2}$$

By the choice of $\vec{\alpha}_\xi * \vec{Y}_\xi$ there's some $\delta \in (\alpha, \lambda)$ such that $|Y_\delta - Z| < \kappa$, i.e. that $Y_\delta$ is a subset of $Z$ modulo a bounded set, since the $Y_\alpha$'s generate the final measure of the play. But then $Y_\delta \subseteq Y_\alpha$ by the rules of $\mathcal{G}(\kappa) \restriction \eta$, and also that $|Y_\delta - Y| < \kappa$ since $Z \subseteq Y$. But this means that $Y \cap Y_\alpha$ is $I^{\tau^\eta_{\alpha+1}}_{Y}$-positive since $Y_\delta$ is, contradicting (2). This shows (1). $\dashv$

This finishes the construction of $\mathcal{T}^\eta_{\alpha+1}$. For limit levels $\delta < \lambda$ we define $\tau^\eta_\delta$ as simply applying $\tau_0$ to the branches of $\mathcal{T}^\eta \restriction \delta$ – showing that the inductive assumptions hold at $\mathcal{T}^\eta_\delta$ is analogous to the above arguments, so we're now done with the construction of $\mathcal{T}^\eta$. Let $\tau^\eta := \bigcup_{\alpha<\lambda} \tau^\eta_\alpha \upharpoonright {}^{<\alpha} H_{\kappa^+}$ and define[10] $\mathcal{I}^\eta := I^{\tau^\eta}_{\langle\rangle}$.

Now note that $\mathcal{I}^{\eta+1} \subseteq \mathcal{I}^\eta$ and $\mathcal{T}^\eta \subseteq \mathcal{T}^{\eta+1}$ for every $\eta < \kappa^+$ – set $\mathcal{I} := \bigcap_{\eta<\kappa^+} \mathcal{I}^\eta$ and $\mathcal{T} := \bigcup_{\eta<\kappa^+} \mathcal{T}^\eta$. We showed that all hopeless ideals are $\kappa$-complete, normal and $(\kappa,\kappa)$-distributive, so this holds in particular for the $\mathcal{I}^\eta$'s and thus also for $\mathcal{I}$.

We claim that $\mathcal{T}$ is dense in $\mathscr{P}(\kappa)/\mathcal{I}$.[11] Let $X$ be an $\mathcal{I}$-positive set, making it $\mathcal{I}^\eta$-positive for some $\eta < \kappa^+$, meaning that there's a play $\vec{\alpha}_\gamma * \tau^\eta$ of $\mathcal{G}(\kappa) \upharpoonright \eta$ such that $X$ is in the final measure, which means that $|Y_\delta - X| < \kappa$ for some large $\delta < \lambda$ and in particular that $Y_\delta - X \in \mathcal{I}$. But $Y_\delta \in \mathcal{T}^\eta \subseteq \mathcal{T}$ by definition of $\tau^\eta$, which shows that $\mathcal{T}$ is dense.

It remains to show that $\mathcal{T}$ is $<\lambda$-closed. If $\lambda = \omega$ then this is trivial, so assume that $\lambda \geq \omega_1$. Let $\beta < \lambda$ and let $\langle Z_\alpha \mid \alpha < \beta\rangle$ be a $\subseteq$-decreasing sequence of elements $Z_\alpha \in \mathcal{T}$. We can fix some $\eta < \kappa^+$ such that $Z_\alpha \in \mathcal{T}^\eta$ for every $\alpha < \beta$ by regularity of $\kappa^+$, and since the $Z_\alpha$'s are $\subseteq$-decreasing they must also be $\leq_{\mathcal{T}^\eta}$-increasing by the hopeless ideal coherence for $\mathcal{T}^\eta$[12].

Let $\tilde{Z} \in \mathcal{T}^\eta$ be player II's $\tau^\eta$-response to the unique partial play of $\mathcal{G}(\kappa) \upharpoonright \eta$ corresponding to the branch containing the $Z_\alpha$'s, and pick $Z \in \mathcal{T}^\eta$ such that $\left|Z - \tilde{Z}\right| < \kappa$ and $Z \geq_{\mathcal{T}^\eta} Z_\alpha$ for all $\alpha < \beta$, again by the density claim and the hopeless ideal coherence. Then $Z$ witnesses $<\lambda$-closure of $\mathcal{T}$.[13] ■

With a bit more work we can from this result then derive the following equivalences.

**Corollary 1.12** (N.). *Let $\kappa$ be a regular cardinal and $\lambda \in [\omega_1, \kappa^+]$ be regular. Then the following are equivalent:*

(i) *$\kappa$ is $<\lambda$-closed faintly power-measurable;*
(ii) *$\kappa$ is $<\lambda$-closed ideally power-measurable;*
(iii) *$\kappa$ is $(\kappa,\kappa)$-distributive $<\lambda$-closed faintly measurable;*
(iv) *$\kappa$ is $(\kappa,\kappa)$-distributive $<\lambda$-closed ideally measurable;*
(v) *Player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$.*

[10] Note that the tree strategy property above ensures that the strategies *do* line up, so that $\tau^\eta$ is a well-defined strategy as well.
[11] This means that given any $\mathcal{I}$-positive set $X$ there's a $Y \in \mathcal{T}$ such that $Y - X \in \mathcal{I}$.
[12] This is the only place in which we're using hopeless ideal coherence.
[13] We're using that $\lambda$ is regular to get $Z$.

Proof. $(v) \Rightarrow (iv)$ is Theorem 1.9 above[14] and $(iv) \Rightarrow (iii) + (ii)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial, so we show $(i) \Rightarrow (v)$.

Assume $\kappa$ is $<\lambda$-closed faintly power-measurable, so there's a $<\lambda$-closed forcing $\mathbb{P}$ and a $V$-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there exists a transitive class $N$ and a $\kappa$-powerset preserving elementary embedding $\pi\colon V \to N$. Write $\mu$ for the induced weakly amenable $V$-normal $V$-measure on $\kappa$. Now, back in $V$, define a strategy $\sigma$ for player II in $G_\lambda(\kappa)$ as follows.

Whenever player I plays some model $M_\alpha$ then we let player II respond with a filter $\mu_\alpha$ such that, for some $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash \ulcorner \check{\mu}_\alpha = \dot{\mu} \cap \check{M}_\alpha \urcorner$ – such a filter exists because $\mu$ is weakly amenable. We require the $p_\alpha$'s to be decreasing, which is possible by $<\lambda$-closure. Now, all the $\mu_\alpha$'s are clearly $M_\alpha$-normal $M_\alpha$-measures on $\kappa$, which makes $\sigma$ a winning strategy. ■

Note that the above results all relied on $\lambda$ being uncountable to achieve wellfoundedness of the generic ultrapower. If we simply ignore this wellfoundedness aspect then we get the following similar equivalence in the $\lambda = \omega$ case, which then also includes completely ineffable cardinals.

**Corollary 1.13** (N.). *Let $\kappa$ be a regular cardinal. Then the following are equivalent:*[15]

*(i) There exists a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a weakly amenable $V$-normal $V$-measure on $\kappa$;*

*(ii) There exists a $(\kappa,\kappa)$-distributive forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a $V$-normal $V$-measure on $\kappa$;*

*(iii) $\kappa$ carries a normal $(\kappa,\kappa)$-distributive ideal;*

*(iv) Player II has a winning strategy in $\mathcal{G}^-_\omega(\kappa)$;*

*(v) $\kappa$ is completely ineffable.*

Proof. $(iv) \Leftrightarrow (v)$ was shown in Theorem ??, and $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. $(i) \Rightarrow (iv)$ is as $(i) \Rightarrow (v)$ in Corollary 1.12, and $(iv) \Rightarrow (iii)$ is Theo-

[14] Here wellfoundedness of the generic ultrapower is automatic since $\lambda$ has uncountable cofinality.

[15] Points $(i)$ and $(ii)$ look a lot like the definition of faintly power-measurable and $(\kappa,\kappa)$-distributive ideally measurable, but here we're not requiring the ultrapowers to be well-founded, so that would be stretching the definition of being measurable.

rem 1.9. ■

As an immediate consequence we then get another ideal-absoluteness result.

**Corollary 1.14.** *"$(\kappa, \kappa)$-distributive $<\lambda$-closed" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■

We get the following similar results for the $\mathcal{C}$-games[16].

**Theorem 1.15** (N.). *Let $\kappa$ and $\lambda \leq \kappa^+$ be regular infinite cardinals such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. If player II has a winning strategy in $\mathcal{C}^-_\lambda(\kappa)$ then $\kappa$ carries a $\lambda$-complete ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $Add(\lambda, 1)$.*

Proof. If $\lambda = \kappa^+$ then we're done by Theorem 1.9, since $\mathcal{G}_{\kappa^+}(\kappa)$ is equivalent to $\mathcal{C}_{\kappa^+}(\kappa)$, so assume that $\lambda \leq \kappa$. We follow the proof of Theorem 1.9 closely. Set $\mathbb{P} := \mathrm{Col}(\lambda, 2^\kappa)$. Fix a wellordering $<_{\kappa^+}$ of $H_{\kappa^+}$ and a $\mathbb{P}$-name $\pi$ for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \lambda \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \lambda$;
- $\kappa{+}1 \subseteq \mathcal{N}_\gamma$ and $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for every $\gamma < \lambda$;
- If $\delta < \lambda$ is a limit ordinal then $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$, $\mathcal{N}_\delta \prec H_{\kappa^+}$ and $\mathcal{N}_\delta \models \mathsf{ZFC}^-$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for all $\gamma < \beta < \lambda$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\lambda} \mathcal{N}_\gamma$.

Define the auxilliary game $\mathcal{G}(\kappa)$ as in the proof of Theorem 1.9 but where player I plays ordinals $\alpha_\eta < \lambda$ and where we use the above $\mathcal{N}_\gamma$'s. Here we only need $<\lambda$-closure of $\mathbb{P}$ to get an equivalence between $\mathcal{G}(\kappa)$ and $\mathcal{C}^-_\lambda(\kappa)$, since $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for all $\gamma < \lambda$.

To every limit ordinal $\eta < \lambda$ we define the restricted auxilliary game $\mathcal{G}(\kappa) \upharpoonright \eta$ as in the proof of Theorem 1.9, and to every winning strategy $\tau$ in $\mathcal{G}(\kappa) \upharpoonright \eta$ and

[16] Theorem 1.15 is the reason for naming the $\mathcal{C}$-games "Cohen games".

partial play $p$ of $\mathcal{G}(\kappa)\restriction\eta$ consistent with $\tau$ define the associated **hopeless ideal**[17]

$$I_p^\tau\restriction\eta := \{X\subseteq\kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction\eta,$$
$$X \text{ is } \textit{not} \text{ in the final measure}\}.$$

As in the proof of Claim 1.10 we get that every hopeless ideal is $\lambda$-complete.

Now, if $\kappa$ is measurable then we trivially get the conclusion,[18] so assume $\kappa$ isn't measurable. Then $\mathrm{sat}(\kappa)\geq\lambda$ since $2^{<\theta}<\kappa$ for every $\theta<\lambda$,[19] so that we can continue exactly as in the proof of Theorem 1.9 to construct ($\lambda$-sized) trees $\mathcal{T}^\eta$ and winning strategies $\tau^\eta$ for all limit ordinals $\eta<\lambda$ such that, setting $\mathcal{I}:=\bigcap_{\eta<\lambda} I_{\langle\rangle}^{\tau^\eta}$ and $\mathcal{T}:=\bigcup_{\eta<\lambda}\mathcal{T}^\eta$, $\mathcal{T}$ is a dense $<\lambda$-closed subset of $\mathscr{P}(\kappa)/\mathcal{I}$ of size $\lambda$, so that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $\mathrm{Add}(\lambda,1)$. ■

**Corollary 1.16** (N.). *Let $\kappa$ and $\lambda\in[\omega_1,\kappa^+]$ be regular such that $2^{<\theta}<\kappa$ for every $\theta<\lambda$. Then the following are equivalent:*

(i) *$\kappa$ is $<\lambda$-closed faintly measurable;*
(ii) *$\kappa$ is $<\lambda$-closed ideally measurable;*
(iii) *$\kappa$ is $<\lambda$-closed $\lambda$-sized faintly measurable;*
(iv) *$\kappa$ is $<\lambda$-closed $\lambda$-sized ideally measurable;*
(v) *Player II has a winning strategy in $\mathcal{C}_\lambda(\kappa)$.*

Proof. $(iv)\Rightarrow(iii)+(ii)$, $(ii)\Rightarrow(i)$ and $(iii)\Rightarrow(i)$ all trivial, and $(i)\Rightarrow(v)$ is like $(i)\Rightarrow(v)$ in Corollary 1.12, and $(v)\Rightarrow(iv)$ is Theorem 1.15. ■

Again, if we ignore wellfoundedness then we get the same equivalence in the $\lambda=\omega$ case:

**Corollary 1.17** (N.). *Let $\kappa$ be regular infinite. Then:*

(i) *Player II has a winning strategy in $\mathcal{C}_\omega^-(\kappa)$; and*
(ii) *$\kappa$ carries an ideal $I$ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to Add$(\omega,1)$.*

[17] This terminology is due to Matt Foreman.
[18] Take $\mathcal{I}(\mathrm{Add}(\lambda,1),\check{\mu})$ for $\mu$ the measure on $\kappa$.
[19] See Proposition 16.4 in [?].

PROOF. Player II has a winning strategy in $\mathcal{C}^-_\omega(\kappa)$ as we're simply measuring finitely many sets without any demand for wellfoundedness, showing $(i)$. Since $2^{<n} < \kappa$ for all $n < \omega$ as $\kappa$ is infinite, Theorem 1.15 then implies $(ii)$. ■

**COROLLARY 1.18.** *"<$\lambda$-closed $\lambda$-sized" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■