Taking the blue pill:
Virtual Set Theory

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Abstract

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Introduction

Gödel proved his Incompleteness Theorems in [Gödel, 1931], one of which showed that to every consistent sufficiently strong[1] theory there would be statements which the system can neither prove nor disprove; we say that such a theory is *incomplete* and say that the statement in question is *independent* of the theory. Of notable importance is ZFC, the established foundational theory of Mathematics. Mathematicians at the time were generally disinterested in his result, as they considered the statement he constructed in his proof to be "unnatural" and therefore have no real consequence to mathematical practice.

Proceeding Gödel's proof of the consistency of the *Continuum Hypothesis* in [Gödel, 1938], which was also the first problem that appeared on Hilbert's famous list of 23 problems in Mathematics published in year 1900, Gödel proposed a program in [Gödel, 1947], the goal of which was to "decide interesting mathematical propositions independent of ZFC in well-justified extensions of ZFC." His result had shown the "first half" for the Continuum Hypothesis, namely that ZFC cannot disprove it.

The second half of the proof that the Continuum Hypothesis is indeed independent of ZFC came about thirty years later, when Cohen used his newly developed notion of *forcing* in [Cohen, 1964] to prove the consistency of the *negation* of the Continuum Hypothesis, showing that there *are* natural statements which are independent of ZFC.

Today, many others have followed in Gödel's footsteps and have made great efforts to analyse the nature of these natural independent statements. This organically led to the development of *large cardinal axioms*, being axioms that extend ZFC in terms of consistency strength and seem to be the *canonical* such axioms, in that all natural theories found "in the wild" have been shown to be equiconsistent with a known large cardinal axiom.

A notable phenomenon is that for "natural" theories T and U, if T has smaller consistency strength than U then the Σ^0_ω consequences of T are also Σ^0_ω consequences of U – so by climbing this large cardinal hierarchy we in fact uncover more truths about the natural numbers. The reals *also* attain this monotone be-

[1]Being able to prove PA counts as "sufficiently strong".

haviour as long as one has moved sufficiently far up the hierarchy, namely past the existence of infinitely many so-called *Woodin cardinals*. This phenomenon also occurs for sets of reals.

Now, it has been found that most large cardinals having the strength of at least a measurable cardinal can be characterised in terms of *elementary embeddings*, enabling a uniform analysis of these cardinals. The large cardinals below the measurables have historically not had such uniform characterisations, but recently the notion of a *virtual large cardinal* was introduced in [Schindler, 2000a] and [Gitman and Schindler, 2018] that essentially *reflects* a lot of the behaviour of the larger large cardinals down to the lower realms. Here Cohen's method of forcing is in full force, as the definition of a virtual version of a large cardinal characterised by elementary embeddings is essentially stating that we can *force* such an embedding to exist, rather than postulating their existence in the universe.

This thesis is an extensive analysis of this virtual phenomenon. The thesis naturally splits into two parts, with the first part being an analysis of the virtuals in isolation and the second part being how these virtuals relate to commonly used set-theoretic objects. Chapter 2 covers the first part, and Chapters 3 and 4 the second.

In the first part we examine how the virtual large cardinals relate to each other, and highlights how they differ from their non-virtual counterparts. A crucial difference between the virtuals and the standard large cardinals is that we do not get a Kunen inconsistency for the virtuals. One consequence of this is that the property that $j(\kappa) > \theta$ always holds when κ is a θ-strong cardinal with $j\colon V \to \mathcal{M}$ being the associated elementary embedding, do not always hold in the virtual world. This leads us to define *prestrong* cardinals as the cardinals not having this property, and in Theorem 2.12 we characterise the virtual θ-prestrong cardinals into either virtually θ-strong cardinals or virtually (θ, ω)-superstrong cardinals. One consequence of this is that virtually measurable cardinals are *equiconsistent* with virtually strong cardinals, without being equivalent. Another consequence is Corollary 2.53 that "virtual Kunen inconsistencies", being the existence of virtual rank-into-rank cardinals, happen exactly when we can separate the virtually prestrong cardinals from the virtually strongs.

The virtual large cardinals also differ from the standard large cardinals by how they interact with structures closed under sequences. We first see this difference in Theorem 2.7, due to Ralf Schindler and Victoria Gitman, who showed that the virtually strongs are equivalent to the virtually supercompacts. This is expanded to virtually Woodin cardinals in Proposition 2.18, yielding a plethora of characterisations of these cardinals, as well as in Theorem 2.23, where we show that the Woodin cardinals and the Vopěnka cardinals are also equivalent in the virtual world. These two results are joint with Stamatis Dimopoulos and Victoria Gitman.

We next delve into a weak version of the Vopěnka principle, denoted WVP, which originates from category theory. Trevor Wilson has shown that WVP is equivalent to On being a Woodin cardinal, and we show that this equivalence *only* holds in the virtual world if we work with *pre*-Woodin cardinals, in analogy with the prestrongs mentioned above, as well as not explicitly requiring the target model to be well-founded. This result is joint with Victoria Gitman.

Since there are no Kunen inconsistencies in the virtual world, this allows us to study the virtual versions of the *Berkeley cardinals* in ZFC. We introduce these and show that in Theorem 2.43 that the virtual Vopěnka principle implies that On is Mahlo *exactly* when there are no virtually Berkeley cardinals, improving on a result by Victoria Gitman and Joel Hamkins. We furthermore show that the virtually Berkeley cardinals exist exactly when On is virtually pre-Woodin without being virtually Woodin, which parallels the result for the rank-into-rank cardinals mentioned above. This also hints at Berkeley cardinals being a natural large cardinal notion.

The virtual large cardinals all require the target model of the generic elementary embedding to be a subset of the ground model, and if we remove this condition then we get the *faint* large cardinals. We show in Corollary 2.50 that the virtuals and faints are consistently distinct notions. We provide further separations in Theorem 2.52, this theorem being joint with Victoria Gitman. However, in Theorem 2.46 we show that in L, $L[\mu]$ as well as in the core model K below a Woodin cardinal, the two notions are equivalent.

The first part ends with an analysis of indestructibility properties of the faints, which is joint with Philipp Schlicht. We work with a strengthening of the faintly supercompact cardinals in which the target model is closed under sequences in the generic extension and not just in the ground model. We show that these cardinals enjoy many indestructibility properties, including $<\kappa$-directed closed forcings and $\mathrm{Add}(\omega, \kappa)$. In an attempt to understand how strong these cardinals are, we show that no such cardinals can exist in neither L nor $L[\mu]$. Using the stationary tower we can show that a proper class of Woodin cardinals is an upper bound, but a recent unpublished result by Toshimichi Usuba shows that, surprisingly, virtually extendibles also provide an upper consistency bound for these cardinals.

The second part is split into two chapters, the first one exploring filters and games, with ideals being the focus of the second. We perform a thorough analysis of certain Ramsey-like cardinals introduced by Peter Holy and Philipp Schlicht, defined using *filter games*, in which player I plays set-sized structures $\mathcal{M}_\alpha$ and player II has to follow up with $\mathcal{M}_\alpha$-filters on the cardinal κ in question. They focused on the case in which player I doesn't have a winning strategy, where they showed that this results in a large cardinal notion characterised by elementary embeddings between set-sized structures.

Our main focus is when player II *does* have a winning strategy. When the games are of finite length we show in Theorems 3.18 and 3.20 that the resulting large cardinal notions form a strict hierarchy via the use of indescribability properties, and characterise in Theorem 3.29 the completely ineffable cardinals with these games.

As we move to infinite games this is when we reach the connection to the virtual large cardinals. Indeed, Theorem 3.30 shows that the faintly θ-measurable cardinals can be characterised in terms of a slight weaknening of the ω-length version of these games. Theorem 3.32 shows that this weakened game is equivalent to the original game in L, and is related to the above-mentioned separation- and core model results. These two theorems are joint with Ralf Schindler.

Taking one more step, to games of length $\omega+1$, our consistency strength suddenly dramatically increases to measurable cardinals, as shown in Theorem 3.48. For these countable length games we show that our resulting large cardinals can be characterised in terms of the *indiscernible games* introduced in [Sharpe and Welch, 2011]. We also include proofs due to Philip Welch and Ralf Schindler that the cardinals corresponding to the ω_1-length games are measurable in K below $0^\P$ and a Woodin cardinal, respectively.

The last case is when the games have length of uncountable cofinality, where we show that the resulting large cardinals are downward absolute to K below $0^\P$, the sharp of a strong cardinal. We also show how the cardinals relate to the *strongly-* and *super Ramseys* introduced by Victoria Gitman.

The following chapter asks the question of when these cardinals characterised by generic elementary embeddings can equivalently be characterised by the existence of ideals in the ground model. To organise our results we define a poset property to be *ideal-absolute* if this holds for forcings having that property. We show in Theorem 4.7 that distributivity properties are ideal-absolute and Theorem 4.9 and the subsequent Corollary 4.12 show that (κ,κ)-distributive $<\lambda$-closure is also ideal-absolute, for $\lambda \in [\omega_1, \kappa^+]$. This main result is an improvement of the proof of an unpublished result due to Matthew Foreman, Theorem 4.8.

Building on these results, we give in Corollary 4.13 another characterisation of the completely ineffables, in terms of ideals, and in Theorem 4.15 and Corollary 4.16 we show that $<\lambda$-closure is also ideal-absolute. This ties in with the above-mentioned weakening of the games, also showing that these games characterise the $<\lambda$-closed faintly- and ideally measurables.

We end with a final chapter containing a range of open questions, continuing on from our results in this thesis.

Notation

We will denote the class of ordinals by On. For X, Y sets we denote by ${}^X Y$ the set of all functions from X to Y. For an infinite cardinal κ, we let H_κ be the set of sets X such that the cardinality of the transitive closure of X is $<\kappa$. ZF^- will denote ZF with the Collection scheme but without the Power Set axiom, following the results of [Gitman et al., 2015]. We write GBC for Gödel-Bernays class theory with the Axiom of Choice, and GB for GBC without the Axiom of Choice. The symbol $\lightning$ will denote a contradiction and $\mathscr{P}(X)$ denotes the power set of X. We will sometimes denote elementary embeddings $\pi\colon (\mathcal{M}, \in) \to (\mathcal{N}, \in)$ by simply $\pi\colon \mathcal{M} \to \mathcal{N}$. Generally, $\alpha, \beta, \gamma, \zeta$ will denote ordinals and $\kappa, \lambda, \theta, \delta$ cardinals.

1 PRELIMINARIES

This chapter will give a rough overview of concepts that will be used in subsequent chapters. Large cardinal theory plays a prominent role to understand how the *virtual* large cardinals in Chapter 2 compares the other large cardinals. We will also routinely be working with the core model K, so the second section will give a high-level overview of what K is and which key properties it has. The last section in these preliminaries will cover some results related to working with elementary embeddings in different forcing extensions, which tend to not be covered in standard textbooks. In the interest of brewity we will references rather than proofs of most of these results.

1.1 LARGE CARDINALS

Since large cardinals came into existence in the beginning of the 20th century, a vast zoo of different types of such have appeared. The aim of this section is to act as a reference for the definitions of these as well as the relations between them.

Large cardinals are roughly split into two "sections": the small ones and the large ones. This distinction is a bit blurry and varies from set theorist to set theorist, but here the distinction will be made at the point where *global elementary embeddings* enter the picture, which starts at the measurable cardinals.

We will start from the bottom and only cover the large cardinals that we will be dealing with in this thesis.

1.1.1 Small large cardinals

The first large cardinal lies at the very bottom of the hierarchy: the inaccessibles.

DEFINITION 1.1. A cardinal κ is **regular** if $\operatorname{cof}\kappa = \kappa$; i.e. that there are no $\gamma < \kappa$ with a cofinal function $f\colon \gamma \to \kappa$. κ is a **strong limit** if $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$. If κ is both regular and a strong limit then we say that it is (strongly) **inaccessible**. ○

Every other large cardinal is either inaccessible or implies that there exists an inner model with an inaccessible cardinal. The following shows that inaccessible cardinals transcend ZFC.

Proposition 1.2 ([Kanamori, 2008] Proposition 1.2). *If κ is inaccessible then $(V_\kappa, \in) \models$ ZFC.* ■

Gödel's Second Incompleteness Theorem from [Gödel, 1931] then shows that ZFC can prove neither the existence of any inaccessible cardinals nor the mere consistency of inaccessible cardinals existing. This is the foundation of the large cardinal hierarchy. We say that a large cardinal is **stronger** than another large cardinal if the former proves the consistency of the latter, so that the same application of the Incompleteness Theorem shows that the weaker large cardinal theory can never prove the consistency of a stronger one.

Next, we move a handful of steps up the large hierarchy ladder and introduce the *weakly compact cardinals*. These have a multitude of different different equivalent definitions which we will not cover here, but instead define them in terms of a combinatorial colouring relation. We need a definition.

Definition 1.3. For any function $f\colon A \to B$, a subset $H \subseteq A$ is **homogeneous for** f if $f \restriction H$ is a constant function. ○

Think of f in the above Definition 1.3 as being a *colouring function*, that colours elements of A in colours from B. For $H \subseteq A$ to be homogeneous would then mean that everything in H has the same colour.

Definition 1.4. An uncountable cardinal κ is **weakly compact** if to every function $f\colon [\kappa]^2 \to \{0, 1\}$ there is a $H \subseteq [\kappa]^2$ of size κ which is homogeneous for f. ○

Again, thinking in terms of colourings, κ is weakly compact if whenever we colour pairs of ordinals below κ in two colours, then we can find a large (i.e. of size κ) set of such pairs all of the same colour.

The following result then shows that the weakly compact cardinals are indeed stronger than the inaccessibles.

Theorem 1.5 ([Jech, 2006] Lemma 9.9). *Every weakly compact cardinal is a limit of inaccessible cardinals.* ■

Moving a tiny step further, we introduce two strengthenings of the weakly compacts: the *ineffables* and the *completely ineffables*.

Definition 1.6. An uncountable cardinal κ is **ineffable** if to any function $f\colon [\kappa]^2 \to 2$ there exists a *stationary* $H \subseteq [\kappa]^2$ which is homogeneous for f. ○

Ineffable cardinals are weakly compact by definition, and the following theorem from [Friedman, 2001] shows that they are strictly stronger.

Theorem 1.7 (Friedman). *Ineffable cardinals are weakly compact limits of weakly compacts.* ■

A way of improving ineffability is to "close under homogeneity", in the sense that if H is homogeneous for $f\colon [\kappa]^2 \to 2$ and $g\colon [H]^2 \to 2$ is any function, then there is a subset of H which is homogeneous for g. To formalise this notion we use the concept of a *stationary class*.

Definition 1.8. For X any set, a collection $\mathcal{R} \subseteq \mathscr{P}(X)$ is a **stationary class** if

- $\mathcal{R} \neq \emptyset$;
- Every $A \in \mathcal{R}$ is a stationary subset of X;
- If $A \in \mathcal{R}$ and $B \supseteq A$ then $B \in \mathcal{R}$. ○

Definition 1.9. An uncountable cardinal κ is **completely ineffable** if there is a stationary class $\mathcal{R} \subseteq \mathscr{P}(\kappa)$ such that for every $A \in \mathcal{R}$ and $f\colon [A]^2 \to 2$ there exists a $H \in \mathcal{R}$ which is homogeneous for f. ○

As suspected, these completely ineffable cardinals are indeed strictly stronger than the ineffables, as the following theorem from [Abramson et al., 1977] shows.

Theorem 1.10 (Abramson et al). *Completely ineffable cardinals are ineffable limits of ineffable cardinals.* ■

Lastly, the *Ramsey cardinals* is a natural strengthening of the weakly compact cardinals.

Definition 1.11. An uncountable cardinal κ is **Ramsey** if to every function $f\colon [\kappa]^{<\omega} \to \{0,1\}$ there exists a subset $H \subseteq [\kappa]^{<\omega}$ such that, for every $n < \omega$, $H \cap [\kappa]^n$ is homogeneous for $f \restriction [\kappa]^n$. ○

See Figure 1.1 for an overview of these large cardinals.

1.1.2 Large large cardinals

Moving on to the higher reaches of the large cardinals, these are more uniformly defined and all involve the notion of an *elementary embedding*.

Definition 1.12. For two first-order structures $\mathcal{M}$ and $\mathcal{N}$ with underlying sets M and N, an **elementary embedding** $j\colon \mathcal{M} \to \mathcal{N}$ between them is a function $j\colon M \to N$ such that, for any first-order formula $\varphi(v_1, \dots, v_n)$ and sets $x_1, \dots, x_n \in \mathcal{M}$ it holds that $\mathcal{M} \models \varphi[x_1, \dots, x_n]$ iff $\mathcal{N} \models \varphi[j(x_1), \dots, j(x_n)]$. ○

As elementary embeddings in particular preserve equality, they are always injective. Identity embeddings are of course always elementary, so we say that an elementary embedding is **non-trivial** if it is not the identity. The following then shows that in most situations these non-trivial embeddings can be associated to a unique ordinal.

Proposition 1.13 ([Kanamori, 2008] Propostion 5.1). *If $j\colon (\mathcal{M}, \in) \to (\mathcal{N}, \in)$ is an elementary embedding such that $\mathcal{M}$ is transitive and either $\mathcal{N} \subseteq \mathcal{M}$ or $\mathcal{M} \models$ ZFC, then there exists an ordinal $\alpha < o(\mathcal{M})$ moved by j, i.e. that $j(\alpha) \neq \alpha$. We call the least such ordinal the **critical point** of j, and denote it by* $\operatorname{crit} j$. ■

As we will only be dealing with non-trivial elementary embeddings, we will always assume elementary embeddings to be non-trivial unless otherwise stated. Our first type of large large cardinal is the measurable cardinal, being the first cardinal witnessing an elementary embedding from the entire universe. We formalise this in Gödel-Bernays set theory with Choice, GBC, which is a class theory that is conservative over ZFC.[1]

[1]For more information about GBC, see [?].

Definition 1.14 (GBC). An uncountable cardinal κ is **measurable** if there exists a transitive class $\mathcal{M}$ and an elementary embedding $j\colon (V, \in) \to (\mathcal{M}, \in)$ with critical point κ. ○

The measurable cardinals were the first cardinals shown to "transcend" Gödel's constructible universe L.[2] This was proven by Dana Scott and has now become known as Scott's Theorem.[3]

Theorem 1.15 (Scott's Theorem, [Kanamori, 2008] Corollary 5.5). *L, Gödel's constructible universe, has no measurable cardinals.* ■

Given this result, it's not surprising that the measurables then exceed the strength of the previous large cardinals.

Proposition 1.16. *Measurable cardinals are completely ineffable limits of completely ineffable cardinals.*

Proof. (Sketch) If $j\colon V \to \mathcal{M}$ is a non-trivial elementary embedding then the **derived ultrafilter** $\mu \subseteq \mathscr{P}(\kappa)$ on $\kappa := \operatorname{crit} j$ is defined as $X \in \mu$ iff $\kappa \in j(X)$. Section 5 in [Kanamori, 2008] shows that it is indeed an ultrafilter and that its ultrapower $\operatorname{Ult}(V, \mu)$ is wellfounded. A reflection argument then shows that we can simply take $\mathcal{R} := \mu$. ■

Moving even further, we strengthen the definition of measurable cardinals to arrive at the *strong cardinals*.

Definition 1.17 (GBC). An uncountable cardinal κ is **strong** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$ satisfying that $H_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta\colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ. We say that κ is **θ-strong** if the property holds for a specific θ. ○

[2] For more information about L, see e.g. [Schindler, 2014]

[3] The measurables are not the weakest large cardinals with this property, however. For instance, the Ramsey cardinals enjoy this property too, but none of the others described in the previous subsection.

Proposition 1.18 ([Kanamori, 2008] 26.6). *Strong cardinals are measurable limits of measurable cardinals.* ■

One property of the strong cardinals that we will get back to in the next subsection and which will be important in Chapter 2 is the following.

Proposition 1.19 ([Kanamori, 2008] 26.7). *If $j\colon V \to \mathcal{M}_\theta$ witnesses that $\kappa := \operatorname{crit} j$ is a θ-strong cardinal then $j(\kappa) > \theta$.* ■

We can strengthen the strongs even more by requiring *sequence closure* rather than only containing an initial segment of the universe.

Definition 1.20 (GBC). An uncountable cardinal κ is **supercompact** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$ satisfying that ${}^{<\theta}\mathcal{M}_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta\colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ. ○

To get an intuition of why the sequence closure is a lot more powerful, note that bits of the elementary embedding itself are now elements of $\mathcal{M}_\theta$, so that $\mathcal{M}_\theta$ can now start reasoning about large cardinals, and j_θ being elementary, these facts will then be carried back into the universe. Here is an example of such an argument.

Proposition 1.21. *If κ is supercompact then*

$$V_\kappa \models \ulcorner \textit{There exists a proper class of strong cardinals} \urcorner. \tag{1}$$

Proof. (Sketch) By noting that the restrictions of the supercompact embedding is an element of the target model by supercompactness, κ is strong in the target model, so that a reflection argument shows (1). ■

Another way of strengthening the strong cardinals is by restricting the behaviour of what the elementary embedding can do on certain sets.

Definition 1.22. Let A be any set. An uncountable cardinal κ is A-**strong** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$ satisfying that $H_\theta \subseteq \mathcal{M}_\theta$, and an elementary $j_\theta\colon (V,\in) \to (\mathcal{M}_\theta,\in)$ with critical point κ, such that $A\cap H_\theta = j(A)\cap H_\theta$. ○

These A-strong cardinals are not used much in practice, but the following *Woodin cardinals* are immensely useful and can be seen as a "local" version of a proper class of A-strongs for every class A.

Definition 1.23. An uncountable cardinal δ is a **Woodin cardinal** if there to every subset $A\subseteq H_\delta$ exists $\kappa<\delta$ such that $(H_\delta,\in,A)\models\ulcorner\kappa$ is A-strong$\urcorner$. ○

Woodin cardinals can equivalently be defined in terms of functions instead of the A-strong cardinals.

Theorem 1.24 ([Kanamori, 2008] Theorem 26.14). *The following are equivalent for an uncountable cardinal κ.*

(i) *κ is a Woodin cardinal;*

(ii) *For any $f\colon\kappa\to\kappa$ there exists $\alpha<\kappa$ such that $f[\alpha]\subseteq\alpha$, a transitive $\mathcal{M}$ with $V_{j(f)(\alpha)}\subseteq\mathcal{M}$ and an elementary embedding $j\colon(V,\in)\to(\mathcal{M},\in)$ with $\operatorname{crit} j=\kappa$.*

Our last large cardinal in this section is ostensibly completely different from the others. It originates from category theory, and according to [ak, 2013] was originally proposed by Petr Vopěnka as a "bogus large cardinal property" which he believed was inconsistent with ZFC, but a proof of this never appeared.

Definition 1.25 (GBC). **Vopěnka's Principle** (VP) postulates that to any first-order language $\mathcal{L}$ and proper class $\mathcal{C}$ of $\mathcal{L}$-structures, there exist distinct $\mathcal{M},\mathcal{N}\in\mathcal{C}$ and an elementary embedding $j\colon\ \mathcal{M}\to\mathcal{N}$. ○

Definition 1.26. An uncountable cardinal δ is **Vopěnka** if $(V_\delta,\in;V_{\delta+1})\models\mathsf{VP}$. ○

Perlmutter showed in [Perlmutter, 2015] that the Woodin- and Vopěnka cardinals are closely connected, with Woodin cardinals relating to Vopěnka cardinals in the same way that strong cardinals relate to supercompacts.

Theorem 1.27 (Perlmutter). *Vopěnka cardinals are equivalent to cardinals that are "Woodin for supercompactness", meaning a cardinal δ such that to any subset $A \subseteq H_\delta$ there is a cardinal $\kappa < \delta$ such that $(H_\delta, \in, A) \models \ulcorner\kappa$ is A-supercompact$\urcorner$.*[4]
■

See Figure 1.1 for an overview of these large cardinals.

1.1.3 Inconsistent large cardinals

In these highest reaches of the large cardinal hierarchy we encounter large cardinals whose existence are inconsistent with ZFC. The reason why these are still interesting to us is because none of them have yet been proven inconsistent with ZF. The first such cardinal is the following.

Definition 1.28 (GBC). An uncountable cardinal κ is a **Reinhardt cardinal** if there exists an elementary embedding $j\colon (V, \in) \to (V, \in)$ with $\operatorname{crit} j = \kappa$. ∘

This was shown to be inconsistent in [Kunen, 1971].

Theorem 1.29 (Kunen inconsistency, GBC, [Kanamori, 2008] Theorem 23.12). *There are no Reinhardt cardinals. Even more, there is no non-trivial elementary $j\colon (V_{\lambda+2}, \in) \to (V_{\lambda+2}, \in)$ for any uncountable cardinal λ.*

The proof of Proposition 1.19, which stated that $j(\kappa) > \theta$ always holds for strong cardinals κ, relies heavily on the Kunen inconsistency. When we're going to deal with the *virtual* large cardinals in Chapter 2 we don't have such a Kunen inconsistency and we will show that in that case the property $j(\kappa) > \theta$ is a highly non-trivial assumption.

There is also the following strengthening of the Reinhardts, in analogy with the strong cardinals.

Definition 1.30. An uncountable cardinal κ is **super Reinhardt** if for all ordinals λ there exists an elementary embedding $j\colon (V, \in) \to (V, \in)$ with $\operatorname{crit} j = \kappa$ and $j(\kappa) > \lambda$. ∘

[4] Here κ is, in analogy with Definition 1.22, A-**supercompact** if there to every cardinal $\theta > \kappa$ exists a transitive class $\mathcal{M}_\theta$, closed under $<\theta$-sequences, and an elementary $j_\theta\colon (V, \in) \to (\mathcal{M}_\theta, \in)$ with critical point κ, such that $A \cap H_\theta = j(A) \cap H_\theta$.

We can improve this even further by defining a notion corresponding to Woodin cardinals. If we define κ to be A-**super Reinhardt** for a class A to be a super Reinhardt cardinal with $A = \bigcup_{\alpha \in \mathsf{On}} j(A \cap V_\alpha)$, in analogy with the A-strong cardinals, then we define the *totally Reinhardts* as follows.

Definition 1.31. An inaccessible cardinal κ is **totally Reinhardt** if for each $A \subseteq V_\kappa$ it holds that

$$(V_\kappa, \in; V_{\kappa+1}) \models \ulcorner\text{There exists an } A\text{-super Reinhardt cardinal}\urcorner. \qquad \circ$$

The last large cardinals that we will introduce are the Berkeley cardinals. These were introduced by Woodin at University of California, Berkeley around 1992. Similar to the Vopěnka cardinals, these were introduced as a large cardinal candidate that would "clearly" be inconsistent with ZF, but such as result has not yet been found.They trivially imply the Kunen inconsistency and are therefore at least inconsistent with ZFC, but that's as far as it currently goes.

We start with a preliminary definition.

Definition 1.32 (GB). An uncountable cardinal δ is a **proto-Berkeley cardinal** if to every transitive *set* $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists an elementary embedding $j\colon (\mathcal{M}, \in) \to (\mathcal{M}, \in)$ with $\operatorname{crit} j < \delta$. ○

Note that if κ is a proto-Berkeley cardinal then every $\lambda > \kappa$ is also proto-Berkeley, which makes it quite an uninteresting notion. But we can isolate the interesting cases, leading to the definition of a Berkeley cardinal. The following is Theorem 2.1.14 in [Cutolo, 2017].

Theorem 1.33 (Cutolo). *If δ_0 is the least proto-Berkeley cardinal then we can choose the critical point of the embedding to be arbitrarily large below δ_0.* ■

As this property is clearly not preserved upwards, this makes for a good candidate for the large cardinal notion.

Definition 1.34 (GB). A proto-Berkeley cardinal δ is **Berkeley** if we can choose the critical point of the embedding to be arbitrarily large below δ. If we furthermore

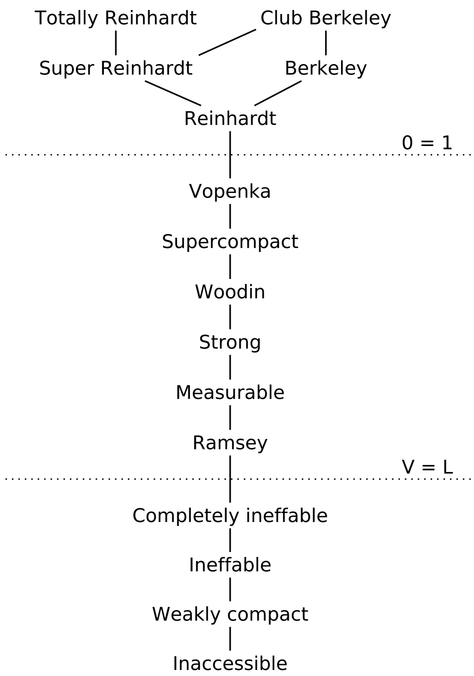

Figure 1.1: A subset of the large cardinal hierarchy, with lines indicating relative consistency implications.

can choose the critical point as an element of any club $C \subseteq \delta$ then we say that δ is **club Berkeley**. ○

In [Cutolo, 2017], they furthermore mention that, among the above-mentioned cardinals, the non-trivial relative consistency implications currently known are the following, being Theorem 2.2.1 and 2.2.2 in [Cutolo, 2017], respectively.

Theorem 1.35 (Cutolo). *Berkeley cardinals are consistency-wise strictly stronger than Reinhardt cardinals.* ■

Theorem 1.36 (Cutolo). *Club Berkeley cardinals are consistency-wise strictly stronger than super Reinhardt cardinals.* ■

See Figure 1.1 for an overview of these large cardinals.

1.2 Core model theory

Perhaps cut down on this section if part II is not happening?

As we will be utilising the *core model* at various points throughout this thesis, we give here an idea of what we mean by the *core model*. A convenient feature of core model theory is that most of the technical details regarding the construction is not needed for applications; it suffices to know only its abstract properties. That being said, we *will* provide a glimpse of the construction at the end of this section. To see the full construction we refer the interested reader to [Nielsen, 2016], [Zeman, 2001] and [Jensen and Steel, 2013a].

1.2.1 The core model K

The **core model**[5] K of a universe is the roughly speaking the subuniverse that strikes a balance between retaining the complexity of the universe while being as simple as possible. The problem is then making all of this precise. Some aspects of the definition is agreed upon by most researchers:

(i) We choose to define the *complexity* of a universe by its large cardinal structure. This is based on the empirical fact that large cardinals seem to capture the strength of every "naturally defined" hypothesis, and gives us a convenient yard stick. For instance, a universe containing a measurable cardinal is more complex than L, as Scott's Theorem 1.15 shows that L cannot contain any measurable cardinals (or any large cardinals stronger than measurables);

(ii) We further postulate that L is the simplest universe there is, and the simplicity of a universe should therefore be measured in terms of how much it resembles L. We will be more precise about what it means to "resemble L" below, but with this intuitive notion is should at least be clear that, say, L is simpler than $L[\mu]$.

Even though (i) captures what we mean by complexity, it leaves much to be desired. For instance, as the structure of the large cardinal hierarchy can only be verified empirically, we might end up in an unfortunate situation where we simply do not know whether a given universe is more complex than another one[6]. The famous

[5] K is short for *Kern*, meaning *core* in German.

[6] It might also be the case that the large cardinal hierarchy is not linear at all.

example of this is the current situation with the superstrong- and strongly compact cardinals, that we simply do not know which one is stronger[7]. Thus, given a universe whose strength corresponds to that of a strongly compact and another one at the level of superstrongs, we would not be able to say which one is more complex.

To remedy this unfortunate situation, we choose instead to define the complexity of a universe in terms of an intermediate property. A universe satisfying this property should then entail that it inherits the large cardinal structure of its surrounding universe. All the intermediate properties currently being used are all instances of a general phenomenon called *covering*. The intuitive idea is that every set in the universe can be "approximated" by a set in the subuniverse, and arose from a seminal theorem of Jensen, see [Schindler, 2014, Theorem 11.56], stating that $0^\sharp$ exists if and only if *strong covering* fails for L, defined as follows.

Definition 1.37 (Jensen). We say that **strong covering** holds for universes $\mathcal{U} \subseteq \mathcal{V}$ if to every $\alpha < o(\mathcal{V})$ and $X \in \mathscr{P}^{\mathcal{V}}(\alpha)$ there exists $A \in \mathcal{U}$ such that $X \subseteq A$ and $\mathrm{Card}^{\mathcal{V}}(X) = \mathrm{Card}^{\mathcal{V}}(A)$. ∘

We can then interpret Jensen's result as saying that, if the complexity of the surrounding universe $\mathcal{V}$ is below the strength of $0^\sharp$ then L is a good candidate for K. In a complex universe we would therefore be looking for the core model among subuniverses more complex than L, and it turns out that also requiring strong covering to hold in such models is too much to ask; the current definition of covering has thus been weakened to the following.

Definition 1.38. We say that **(weak) covering** holds for universes $\mathcal{U} \subseteq \mathcal{V}$ if $\mathrm{cof}^{\mathcal{V}}(\alpha^{+\mathcal{U}}) = \mathrm{Card}^{\mathcal{V}}(\alpha^{+\mathcal{U}})$ holds for any ordinal α with $\alpha^{+\mathcal{U}} \geq \aleph_2^{\mathcal{V}}$. ∘

This statement might seem very distant from the strong version, but one can think of weak covering as saying that $\mathcal{U}$ "knows" the true cofinality of its successor cardinals $\kappa \geq \aleph_2^{\mathcal{V}}$ within the error margin $\varepsilon := \kappa^{+\mathcal{U}} - \mathrm{Card}^{\mathcal{V}}(\kappa)$. More concretely, we could equivalently define weak covering as $\mathcal{U}$ containing all cofinal maps $f\colon \gamma \to \kappa$ in $\mathcal{V}$ for every $\gamma \in \mathrm{Card}^{\mathcal{V}}(\kappa)$, making it closer in spirit to the strong covering property.

[7] Although the general consensus is that the strongly compact cardinals should be equiconsistent with the supercompacts, making them stronger than the superstrongs.

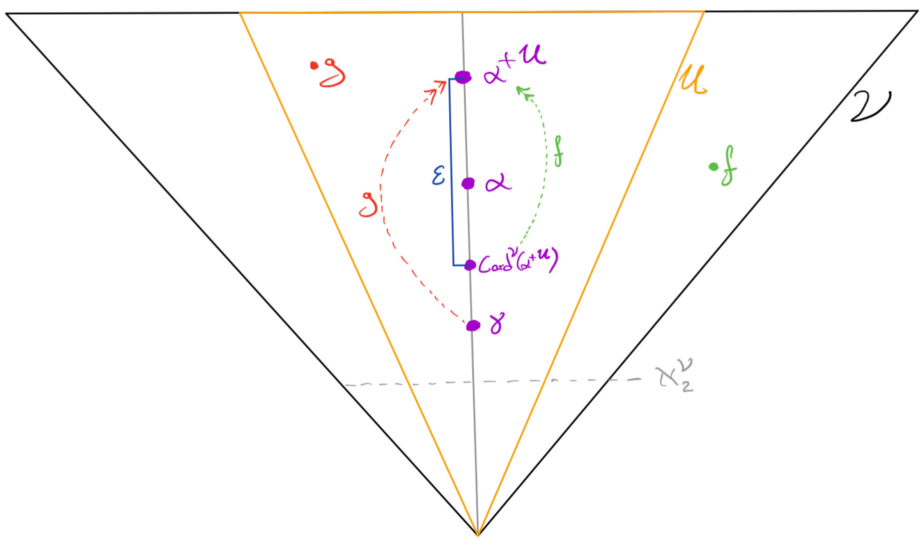

Figure 1.2: Weak covering property

When it comes to (ii) we have to define what we mean by "resembling L". Ultimately this boils down to the current working definition of a *mouse* and is still work in progress. If our universe is no more complex than the strength of a Woodin cardinal however, then we know what the correct definition of a mouse is, and hence also what "resembling L" would mean in this context. The definition of mice along with the assumption of covering then turns out to imply that the core model will indeed inherit the large cardinal strength of the universe[8].

To construct the core model one could then take a bottom-up approach, starting with L and then carefully include the complexity of the universe while remaining similar to L[9]. Alternatively, a top-down approach would be to define a structure which has *all* the complexity of the universe, and then showing that this structure indeed exhibits these L-like properties[10].

[8]To show this one first uses covering to show that K is *universal*, i.e. that it wins every coiteration. With universality at hand, a comparison argument with any $L[\vec{E}]$-model containing a large cardinal will then show that K will have an inner model with the large cardinal in question.

[9]This is the strategy undertaken by Steel and Sargsyan.

[10]Woodin is pursuing this path.

1.2.2 Constructing K

The standard construction of K takes the bottom-up approach. The first step towards this is the construction of K^c[11], which we build by recursion on the ordinals. We start with $K^c_0 := \emptyset$ and at every successor ordinal α we do one of two things:

(i) If there exists a "nice" extender indexed at α then we put it onto the extender sequence of $\mathfrak{C}(K^c_\alpha)$, where $\mathfrak{C}(X)$ is the transitive collapse of a certain hull of X[12];

(ii) Otherwise we let $K^c_\alpha := \mathcal{J}(\mathfrak{C}(K^c_{\alpha-1}))$, with $\mathcal{J}(x) := \operatorname{rud}(\operatorname{trcl}(x \cup \{x\}))$ being the usual operator we use to build L with Jensen's hierarchy.

In other words, we are essentially building L with extenders attached onto it in a canonical fashion. Taking cores at every step will ensure that the initial segments will be *sound*, which ultimately is what guarantees iterability of K^c. The fact that we put on all the relevant extenders from V is what will ensure the covering property of the model. It turns out that K^c isn't exactly what we want however, as it relies *too much* on the surrounding universe, in contrast with L whose construction procedure builds the exact same model in every universe. To attain this *canonicity* we are again taking certain "thick" hulls of K^c (again, think of it as removing the noise). The resulting construction *almost* gives us what we want and is dubbed *pseudo-K*. The problem with this is that the technicalities of the construction uses certain properties of a fixed cardinal Ω, so to build the true core model we "glue" these pseudo-K's together.

The takeaway here is that whenever we're working with an initial segment of K then that segment will be build using the recursive steps (i) and (ii) above, carefully including extenders from V.

1.2.3 Properties of K

In terms of applications of core model theory, the properties of K is usually what matters. We touched on the weak covering property above, but for completeness we state most of the properties usually employed when working with K, here. In [Jensen and Steel, 2013a] they isolate a set of properties of K which leads them to

[11] The "c" stands for *certified*, as the extenders we put on the sequence was historically called *certified extenders*.

[12] Think of $\mathfrak{C}(X)$ as "removing the noise of X".

define K as the structure satisfying the conjunction of these properties. These are as follows.[13]

(i) K is a transitive proper class premouse satisfying ZFC;

(ii) K is Σ_2-definable;

(iii) K has a Σ_2-definable iteration strategy Σ;

(iv) K is **generically absolute**, meaning that $K^V = K^{V[g]}$ and $\Sigma^{V[g]} \restriction V = \Sigma^V$ for any V-generic filter $g \subseteq \mathbb{P}$ for a set-sized forcing notion $\mathbb{P}$;

(v) K is **inductively defined**, meaning that $K|\omega_1^V$ is Σ_1-definable over $J_{\omega_1}(\mathbb{R})$;

(vi) K satisfies **weak covering** as in Definition 1.38.

On top of these properties, we will also employ the following property, which is proven in Lemmata 7.3.7–7.3.9 and 8.3.4 in [Zeman, 2001].

Check if Zeman is the author

Theorem 1.39 (Zeman). *Assume $0^{\P}$ does not exist. If μ is a countably complete weakly amenable K-measure then $\mu \in K$.* ■

1.3 Forcing lemmata

In this section we will cover a few results that we will need when working with elementary embeddings in different forcing extensions. These are the *lifting criterion*, which characterises when we can lift an elementary embedding to a forcing extension, *countable embedding absoluteness*, which allows us to "transfer" elementary embeddings from one forcing extension to another, and lastly a result that gives a sufficient condition for preserving sequence closure when moving to generic extensions.

1.3.1 Lifting criterion

When we are working with an elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$ between sets in the universe, we would sometimes like to lift such an embedding to a generic extension, meaning that given a forcing notion $\mathbb{P} \in \mathcal{M}$ and an $\mathcal{M}$-generic $g \subseteq \mathbb{P}$, we're interested in when we can lift π to an embedding

$$\pi^+\colon \mathcal{M}[g] \to \mathcal{N}[h],$$

[13]See REF for definitions of premice, iteration trees and iteration strategies.

where $h \subseteq \pi(\mathbb{P})$ is $\mathcal{N}$-generic. The **lifting criterion** shows exactly when this is possible. The following is Proposition 9.1 in [?].

PROPOSITION 1.40 (The Lifting Criterion). *Let $\pi\colon \mathcal{M} \to \mathcal{N}$ be an elementary embedding between weak κ-models. Fix a forcing notion $\mathbb{P} \in \mathcal{M}$, an $\mathcal{M}$-generic $g \subseteq \mathbb{P}$ and an $\mathcal{N}$-generic $h \subseteq \pi(\mathbb{P})$. Then the following are equivalent:*

- $\pi[g] \subseteq h$;
- *There exists an elementary $\pi^+\colon \mathcal{M}[g] \to \mathcal{N}[h]$ such that $\pi^+(g) = h$ and $\pi^+ \restriction \mathcal{M} = \pi$.*

PROOF. $(ii) \Rightarrow (i)$ is clear, so assume (i). Define $\pi^+\colon \mathcal{M}[g] \to \mathcal{N}[h]$ as $\pi^+(\dot\tau^g) := \pi(\tau)^h$. To see that π^+ is well-defined fix $\dot\sigma, \dot\tau \in \mathcal{M}^{\mathbb{P}}$ such that $\dot\sigma^g = \dot\tau^g$, and fix $p \in g$ such that $p \Vdash \dot\sigma = \dot\tau$. By elementarity $\pi(p) \Vdash \pi(\dot\sigma) = \pi(\dot\tau)$, so since $\pi(p) \in h$ by (i) we get that $\pi(\dot\sigma)^h = \pi(\dot\tau)^h$.

To show elementarity, note that for $x \in \mathcal{M}$ it holds that $\pi(\check x) = \check{\pi(x)}$, implying $\pi^+(x) = \pi^+(\check x^g) = \pi(\check x)^h = \pi(x)$. Further, letting $\dot g \in \mathcal{M}^{\mathbb{P}}$ be the standard $\mathbb{P}$-name for g, then $\pi(\dot g)$ is the standard $\pi(\mathbb{P})$-name for h and therefore $\pi^+(g) = h$. ■

1.3.2 Countable embedding absoluteness

A key folklore lemma which we will frequently need when dealing with elementary embeddings existing in generic extensions is the following.

LEMMA 1.41 (Countable Embedding Absoluteness). *Let $\mathcal{M}, \mathcal{N}$ be sets, $\mathcal{P}$ a transitive class with $\mathcal{M}, \mathcal{N} \in \mathcal{P}$, and let $\pi\colon \mathcal{M} \to \mathcal{N}$ be an elementary embedding. Assume that $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC} + \ulcorner\mathcal{M}$ is countable$\urcorner$ and fix any finite $X \subseteq \mathcal{M}$.*

Then $\mathcal{P}$ contains an elementary embedding $\pi^*\colon \mathcal{M} \to \mathcal{N}$ which agrees with π on X. If π has a critical point and if $\mathcal{M}$ is transitive then we can also assume that $\operatorname{crit}\pi = \operatorname{crit}\pi^*$.[14]

PROOF. Let $\{a_i \mid i < \omega\} \in \mathcal{P}$ be an enumeration of $\mathcal{M}$ and set $\mathcal{M} \restriction n := \{a_i \mid i < n\}$. Then, in $\mathcal{P}$, build the tree $\mathcal{T}$ of all partial isomorphisms between $\mathcal{M} \restriction n$ and $\mathcal{N}$ for $n < \omega$, ordered by extension. Then $\mathcal{T}$ is illfounded in V by assumption, so it's also illfounded in $\mathcal{P}$ since $\mathcal{P}$ is transitive and $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC}$. The branch then

[14]We are using transitivity of $\mathcal{M}$ to ensure that the *ordinal* $\operatorname{crit}\pi$ exists.

gives us the embedding π^*, and if $\operatorname{crit}\pi$ exists then we can ensure that it agrees with π on the critical point and finitely many values by adding these conditions to $\mathcal{T}$. ■

1.3.3 Preservation of sequence closure

The following lemma is from [Lücke and Schlicht, 2014] and gives a useful condition on when sequence closure is preserved when moving to generic extensions.

Lemma 1.42. *Let λ be an infinite cardinal, $\mathcal{M} \models ZF^-$ a transitive model, $\mathbb{P} \in \mathcal{M}$ a λ^+-cc forcing notion and $g \subseteq \mathbb{P}$ an $\mathcal{M}$-generic filter. Then $V \models {}^\lambda\mathcal{M} \subseteq \mathcal{M}$ implies that $V[g] \models {}^\lambda\mathcal{M} \subseteq \mathcal{M}$.*

Proof. Work in $V[g]$. Let $c := \langle c_\alpha \mid \alpha < \lambda\rangle$ be a λ-sequence such that $c_\alpha \in \mathcal{M}[g]$ for every $\alpha < \lambda$. Fix for every $\alpha < \lambda$ a $\mathbb{P}$-name $\dot{c}_\alpha$ such that $\dot{c}_\alpha^g = c_\alpha$. Also let $\dot{a}$ be a $\mathbb{P}$-name with $\dot{a}^g = \langle \dot{c}_\alpha \mid \alpha < \lambda\rangle$ and choose $p \in g$ such that $V \models \ulcorner p \Vdash \forall\alpha < \check{\lambda}\colon \dot{a}(\alpha) \in \mathcal{M}^{\mathbb{P}}\urcorner$.

Now, working in V, there is for each $\alpha < \lambda$ a maximal antichain A_α below p such that every $q \in A_\alpha$ decides $\dot{a}(\alpha)$; i.e., $q \Vdash \ulcorner \dot{a}(\alpha) = \check{x}\urcorner$ for some $x \in \mathcal{M}$. Define now

$$\sigma := \{((\alpha, x), q) \mid \alpha \in \lambda \wedge q \in A_\alpha \wedge q \Vdash \ulcorner \dot{a}(\alpha) = \check{x}\urcorner\}.$$

Then $p \Vdash \ulcorner \sigma = \dot{a}\urcorner$. Note that $|\sigma| \leq \lambda$, since $|A_\alpha| \leq \lambda$ for each $\alpha < \lambda$. Thus $\sigma \in \mathcal{M}$. Now it holds that

$$V[g] \models \ulcorner \langle \dot{c}_\alpha \mid \alpha < \lambda\rangle = \dot{a}^g = \sigma^g \in \mathcal{M}[g]\urcorner,$$

and we can compute $c = \langle c_\alpha \mid \alpha < \lambda\rangle = \langle \dot{c}_\alpha^g \mid \alpha < \lambda\rangle$ from $\langle \dot{c}_\alpha \mid \alpha < \lambda\rangle$ and g, so $c \in \mathcal{M}[g]$ by Replacement. ■

2 | Virtual large cardinals

In this chapter we investigate the properties of virtual versions of well-known large cardinals, including measurables, strongs, supercompacts, Woodins and Vopěnkas. This entails firstly analysing the relationships between them, and secondly looking at more general properties in terms of their behaviour in core models as well as their indestructibility. This virtual perspective also allows us to analyse virtualised versions of large cardinals that are otherwise inconsistent with ZFC, such as the Berkeley cardinals.

Before we start, we will briefly cover a few standard definitions and lemmata that we will be using freely throughout the chapter. When we're dealing with embeddings between set-sized structures, we will usually be interested in structures of the following form.

Definition 2.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa+1 \subseteq \mathcal{M}$ and $(\mathcal{M}, \in) \models \mathsf{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$, $\mathcal{M}$ is a **κ-model.**[1] $\circ$

Embeddings between these weak κ-models can equivalently be phrased in terms of ultrafilters, or *measures*. Recall that μ is an **$\mathcal{M}$-measure** if

$$(\mathcal{M}, \in, \mu) \models \ulcorner \mu \text{ is a } \kappa\text{-complete ultrafilter on } \kappa \urcorner.$$

Some common properties of such measures are the following.

Definition 2.2. For a weak κ-model $\mathcal{M}$, an $\mathcal{M}$-measure μ is...

- **weakly amenable** if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with $\mathrm{Card}^{\mathcal{M}}(x) = \kappa$;
- **countably complete** if $\bigcap \vec{X} \neq \emptyset$ for every ω-sequence $\vec{X} \in {}^{\omega}\mu$. $\circ$

[1] Note that our (weak) κ-models do not have to be transitive, in contrast to the models considered in [Gitman, 2011] and [Gitman and Welch, 2011]. Not requiring the models to be transitive was introduced in [Holy and Schlicht, 2018].

Weak amenability can equivalently be phrased in terms of a property concerning only the embedding.

Proposition 2.3 (Folklore). *Let $\mathcal{M}$ be a weak κ-model, μ an $\mathcal{M}$-measure and $j : \mathcal{M} \to \mathcal{N}$ the associated ultrapower embedding. Then μ is weakly amenable if and only if j is κ-**powerset preserving**, meaning that $\mathcal{M} \cap \mathscr{P}(\kappa) = \mathcal{N} \cap \mathscr{P}(\kappa)$.*
■

2.1 Strongs & supercompacts

We start out with measurables, strongs and supercompacts. Their (non-virtual) definitions can be found in Appendix 1.1.

Definition 2.4. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is...

- **faintly θ-measurable** if, in a forcing extension, there is a transitive class $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$;
- **faintly θ-strong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$;
- **faintly θ-supercompact** if it's faintly θ-measurable, ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

We further replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we attach a "**pre**" if we don't assume that $\pi(\kappa) > \theta$, and we will leave out θ when it holds for all regular $\theta > \kappa$. ○

As a quick example of this terminology, a *faintly prestrong cardinal* is a cardinal κ such that for all regular $\theta > \kappa$, κ is faintly θ-measurable with $H_\theta^V \subseteq \mathcal{N}$.

We note that even small cardinals can be faintly measurable: we may for instance have a precipitous ideal on ω_1; see [Jech, 2006, Theorem 22.33]. The "virtually" adverb implies that the cardinals are in fact large cardinals in the usual sense, as Proposition 2.5 below shows.

Proposition 2.5 (Virtualised folklore). *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is 1-iterable.*

Proof. Let κ be virtually θ-measurable, witnessed by a forcing $\mathbb{P}$, a transitive $\mathcal{N} \subseteq V$ and an elementary $\pi \colon H_\theta^V \to \mathcal{N}$ with $\pi \in V^{\mathbb{P}}$. If κ isn't a strong limit then we have a surjection $\pi(f) \colon \mathscr{P}(\alpha) \to \pi(\kappa)$ with $\operatorname{ran} \pi(f) = \operatorname{ran} f \subseteq \kappa$ for some $\alpha < \kappa$, $\lightning$. Note that we used $\mathcal{N} \subseteq V$ to ensure that $\mathscr{P}(\alpha)^V = \mathscr{P}(\alpha)^{\mathcal{N}}$. The same argument shows that κ is regular. By restricting the generic embedding and using that $\mathscr{P}(\kappa)^V = \mathscr{P}(\kappa)^N$ as $\mathcal{N} \subseteq V$ and $\mathscr{P}(\kappa)^V \subseteq \mathcal{N}$, we get that κ is 1-iterable. ■

Along with the above definition of faintly supercompactness we can also virtualise Magidor's characterisation of supercompact cardinals[2], which was one of the original characterisations of the remarkable cardinals in [Schindler, 2000a].

Definition 2.6. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-Magidor-supercompact** if there are $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\pi \colon H_{\bar\theta}^V \to H_\theta^V$ such that $\operatorname{crit} \pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. ∘

In the virtual world these two versions of supercompacts remain equivalent, but they also turn out to be equivalent to the virtually strongs:

Theorem 2.7 (Gitman-Schindler). *For an uncountable cardinal κ, the following are equivalent.*[3]

(i) κ is virtually strong;

(ii) κ is virtually supercompact;

(iii) κ is virtually Magidor-supercompact.

Proof. $(ii) \Rightarrow (i)$ is simply by definition.

$(i) \Rightarrow (iii)$: Fix $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi \colon H_{(2^{<\theta})^+}^V \to \mathcal{M}$ with[4] $\operatorname{crit} \pi = \kappa$, $\pi(\kappa) > \theta$, $H_{(2^{<\theta})^+}^V \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. Since $H_\theta^V, H_{\pi(\theta)}^{\mathcal{M}} \in \mathcal{M}$, Countable Embedding Absoluteness 1.41 implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H_\theta^V \to H_{\pi(\theta)}^{\mathcal{M}}$ with $\operatorname{crit} \pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H_\theta^V = H_\theta^{\mathcal{M}}$ as $\mathcal{M} \subseteq V$ and $H_\theta^V \subseteq \mathcal{M}$, elementarity

[2]See Appendix 1.1 for the non-virtual version of this characterisation.

[3]A cardinal satisfying any/all of these conditions is usually called **remarkable**.

[4]The domain of π is $H_{(2^{<\theta})^+}^V$ to ensure that $H_\theta^V \in \operatorname{dom} \pi$.

of π now implies that $H^V_{(2^{<\theta})^+}$ has ordinals $\bar\kappa < \bar\theta < \kappa$ and a generic elementary $\sigma\colon H^V_{\bar\theta} \to H^V_\theta$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa$. This shows (iii).

$(iii) \Rightarrow (ii)$: Fix $\theta > \kappa$ and $\delta := (2^{<\theta})^+$. By (iii) there exist ordinals $\bar\kappa < \bar\delta < \kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\delta} \to H^V_\delta$ with $\operatorname{crit}\pi = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$. We will argue that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, so that by elementarity κ is virtually θ-supercompact in H^V_δ and hence also in V by the choice of δ. Consider the restriction

$$\sigma := \pi \restriction H^V_{\bar\theta} : H^V_{\bar\theta} \to H^V_\theta.$$

Note that H^V_θ is closed under $<\bar\theta$-sequences (and more) in V. Now define

$$X := \bar\theta{+}1 \cup \{x \in H^V_\theta \mid \exists y \in H^V_{\bar\theta} \exists p \in \operatorname{Col}(\omega, H^V_{\bar\theta})\colon p \Vdash \dot\sigma(\check y) = \check x\} \in V.$$

Note that $|X| = \left|H^V_{\bar\theta}\right| = 2^{<\bar\theta}$ and that $\operatorname{ran}\sigma \subseteq X$. Now let $\overline{\mathcal{M}} \prec H^V_\theta$ be such that $X \subseteq \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{\mathcal{M}}$ of size $(2^{<\bar\theta})^{<\bar\theta} = 2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{\mathcal{M}}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}| = 2^{<\bar\theta} < \bar\delta$, making $\mathcal{M} \in H^V_{\bar\delta}$.

Countable Embedding Absoluteness 1.41 then implies that $H^V_{\bar\delta}$ has a generic elementary embedding $\sigma^*\colon H^V_{\bar\theta} \to \mathcal{M}$ with $\operatorname{crit}\sigma^* = \bar\kappa$, showing that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H^V_{\bar\delta}$, which is what we wanted to show. ■

Remark 2.8. The above proof in fact shows something stronger: if κ is virtually $(2^{<\theta})^+$-strong then it is virtually θ-supercompact, and if it's virtually $(2^{<\theta})^+$-Magidor-supercompact then it's virtually θ-supercompact. It's open whether they are equivalent level-by-level (see Question 5.4).

A key difference between the normal large cardinals and the virtual kind is that we don't have a virtual version of the Kunen inconsistency[5]: it's perfectly valid to have a generic elementary embedding $H^V_\theta \to H^V_\theta$ with θ much larger than the critical point.

[5]See Appendix 1.1 for the Kunen inconsistency.

Proposition 2.9 (Folklore). *If $0^\sharp$ exists then there are inaccessible cardinals $\kappa < \theta$ such that, in a generic extension of L, there is an elementary embedding $\pi\colon L_\theta \to L_\theta$. In other words, π witnesses a strong failure of the virtualised Kunen inconsistency.*

Proof. From $0^\sharp$ we get an elementary embedding $j\colon L \to L$. Let $C \subseteq \mathsf{On}$ be the proper class club of limit points of j above $\operatorname{crit} j$, which then contains an inaccessible cardinal θ as there are stationarily many such. Restrict j to $\pi := j \restriction L_\theta \to \mathcal{N}$ and note that $\mathcal{N} = L_\theta$ by condensation and because θ is a limit point of j. Let $\kappa := \operatorname{crit} \pi$. Now an application of Countable Embedding Absoluteness 1.41 shows that a generic extension of L contains an elementary embedding $\tilde{\pi} : L_\theta \to L_\theta$ with $\operatorname{crit} \tilde{\pi} = \kappa$. ∎

This becomes important when dealing with the "pre"-versions of the large cardinals. We next move to a virtualisation of the α-superstrong cardinals.

Definition 2.10. Let θ be a regular uncountable cardinal and and α an ordinal. Then a cardinal $\kappa < \theta$ is **faintly (θ, α)-superstrong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$[6]. We replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we say that κ is **faintly α-superstrong** if it's faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong.[7] ∘

As in the non-virtual case, the virtually superstrongs supercede the virtually strongs in consistency strength. Note that this then also implies that the superstrongs are stronger than the virtually supercompacts, which is *not* the case outside the virtual world.

Proposition 2.11 (N.). *If κ is faintly superstrong then H_κ has a proper class of virtually strong cardinals.*

Proof. Fix a regular $\theta > \kappa$ and a generic embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit} \pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) < \theta$. Then $\pi(\kappa)$ is a V-cardinal, so that $H_{\pi(\kappa)}^V$ thinks that κ is virtually strong. This implies that H_κ^V thinks there is a proper class

[6] Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.

[7] Note that the conventions stated here are different from the ones in Definition 2.4.

of virtually strong cardinals, using that $H^V_\kappa \prec H^V_{\pi(\kappa)}$. ■

The following theorem then shows that the only thing stopping prestrongness from being equivalent to strongness is the existence of "Kunen inconsistencies".

Theorem 2.12 (N.). *Let θ be an uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-prestrong iff either*

(i) κ is virtually θ-strong; or

(ii) κ is virtually (θ, ω)-superstrong.

Proof. ($\Leftarrow$) is trivial, so we show ($\Rightarrow$). Let κ be virtually θ-prestrong. Assume (i) fails, meaning that there's a generic extension $V^{\mathbb{P}}$ and an elementary embedding $\pi \in V^{\mathbb{P}}$ such that $\pi\colon H^V_\theta \to \mathcal{N}$ for some transitive $\mathcal{N}$ with $H^V_\theta \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) \leq \theta$. Assume $\pi^n(\kappa)$ is defined for all $n < \omega$ and define $\lambda := \sup_{n<\omega} \pi^n(\kappa)$. If $\lambda \leq \theta$ then κ is virtually (θ, ω)-superstrong by definition, so assume that there's some least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$.

This means that κ is virtually ν-strong for every regular $\nu \in (\kappa, \pi^n(\kappa))$, which is a Δ_0-statement in $\{H^V_{\nu^+}\}$ and hence downwards absolute to $H^V_{\pi^n(\kappa)}$. This means that κ is virtually strong in $H^V_{\pi^n(\kappa)}$ and also that $\pi^n(\kappa)$ is virtually strong in $H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$ by elementarity, and so in particular virtually θ-strong in $\mathcal{N}$. This means that there's some generic elementary embedding

$$\sigma\colon H^{\mathcal{N}}_\theta \to \mathcal{M}$$

with $H^{\mathcal{N}}_\theta \subseteq \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{N}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. We can now restrict σ to its critical point $\pi^n(\kappa)$ to get that

$$H^V_{\pi^n(\kappa)} = H^{\mathcal{N}}_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))},$$

using that $H^V_\theta = H^{\mathcal{N}}_\theta$ holds as π is a virtual embedding. Since κ is virtually strong in $H^V_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$. In particular, κ is virtually θ-strong in $\mathcal{M}$, and as $H^{\mathcal{M}}_\theta = H^{\mathcal{N}}_\theta = H^V_\theta$, this means that κ is virtually θ-strong in V, contradicting (i). ■

We then get the following consistency result.

Corollary 2.13 (N.). *For any uncountable regular θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

Proof. The above Proposition 2.11 and Theorem 2.12 show that virtually θ-prestrongs are equiconsistent with virtually θ-strongs. Now note that Countable Embedding Absoluteness 1.41 and condensation in L imply that every faintly θ-measurable cardinal is virtually θ-prestrong in L. ■

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_\theta^V \to H_\theta^V$ with $\operatorname{crit}\pi = \kappa$. We firstly note that the virtually ω-superstrongs coincide with the virtually rank-into-ranks.

Proposition 2.14 (N.). *A regular uncountable cardinal κ is virtually ω-superstrong iff it is virtually rank-into-rank.*

Proof. If κ is virtually ω-superstrong, witnessed by a generic embedding $\pi\colon H_\theta^V \to \mathcal{N}$, then $\lambda := \sup_{n<\omega} \pi^n(\kappa)$ is well-defined. By restricting π to $\pi \upharpoonright H_\lambda^V\colon H_\lambda^V \to H_\lambda^V$ we get a witness to κ being virtually λ-rank-into-rank.

Conversely, if κ is θ-rank-into-rank, witnessed by a generic embedding $\pi\colon H_\theta^V \to H_\theta^V$, then one readily checks that π also witnesses that κ is virtually ω-superstrong. ■

2.2 Woodins & Vopěnkas

In this section we will analyse the virtualisations of the Woodin and Vopěnka cardinals, which can be seen as "boldface" variants of strongs and supercompacts.

Definition 2.15. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, A)**-strong** for a set $A \subseteq H_\theta^V$ if there exists a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, A) \to (\mathcal{M}, \in, B)$$

with $\mathcal{M}$ transitive, such that $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta^V \subseteq \mathcal{M}$ and $B \cap H_\theta^V = A$. We say that κ is **faintly** (θ, A)**-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$ and say that κ is **faintly** (θ, A)**-extendible** if $\mathcal{M} = H_\mu^V$ for some V-cardinal μ. We will leave out θ if it holds for all regular $\theta > \kappa$. ∘

Definition 2.16. A cardinal δ is **faintly Woodin** if given any $A \subseteq H_\delta^V$ there exists a faintly $({<}\delta, A)$-strong cardinal $\kappa < \delta$. ∘

As with the previous definitions, for both of the above two definitions we substitute "faintly" for **virtually** when $\mathcal{M} \subseteq V$, and substitute "strong", "supercompact" and "Woodin" for **prestrong**, **presupercompact** and **pre-Woodin** when we don't require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with the real Woodin cardinals, virtually Woodin cardinals are Mahlo. This contrasts the virtually pre-Woodins since [Wilson, 2019a], together with Theorem 2.23 below, shows that they can be singular.

Proposition 2.17 (Virtualised folklore). *Virtually Woodin cardinals are Mahlo.*

Proof. Let δ be virtually Woodin. Note that δ is a limit of weakly compact cardinals by Proposition 2.5, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f\colon \alpha \to \delta$ with $f(0) > \alpha$ and $\alpha < \delta$, and note that f cannot have any closure points. Fix a virtually $({<}\delta, f)$-strong cardinal $\kappa < \delta$; we claim that κ is a closure point for f, which will yield our desired contradiction.

Let $\gamma < \kappa$ and choose a regular $\theta \in (f(\gamma), \delta)$. We then have a generic embedding $\pi\colon (H_\theta^V, \in, f \cap H_\theta^V) \to (\mathcal{N}, \in, f^+)$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and f^+ is a function such that $f^+ \cap H_\theta^V = f \cap H_\theta^V$. But then $f^+(\gamma) = f(\gamma) < \pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma) < \kappa$, making κ a closure point for f, $\lightning$. This shows that δ is inaccessible.

As for Mahloness, let $C \subseteq \delta$ be a club and $\kappa < \delta$ a virtually $({<}\delta, C)$-strong cardinal. Let $\theta \in (\min C, \delta)$ and let $\pi\colon H_\theta^V \to \mathcal{N}$ be the associated generic elementary embedding. Then for every $\gamma < \kappa$ there exists an element of C below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an

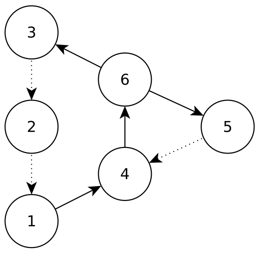

Figure 2.1: Proof strategy of Proposition 2.18, dotted lines are trivial implications.

element of C. As κ is regular, this shows that δ is Mahlo. ■

The well-known equivalence of the "function definition" and "A-strong" definition of Woodin cardinals[8] holds if we restrict ourselves to *virtually* Woodins, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Proposition 2.18 (Dimopoulos-Gitman-N.). *For an uncountable cardinal δ, the following are equivalent.*

(i) *δ is virtually Woodin.*

(ii) *for every $A \subseteq H_\delta^V$ there exists a virtually $({<}\delta, A)$-supercompact $\kappa < \delta$.*

(iii) *for every $A \subseteq H_\delta^V$ there exists a virtually $({<}\delta, A)$-extendible $\kappa < \delta$.*

(iv) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*

(v) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, ${}^{<\pi(f)(\kappa)}\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*

(vi) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\bar\theta < \kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ with $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\pi$.*

Proof. Firstly note that $(iii) \Rightarrow (ii) \Rightarrow (i)$ and $(v) \Rightarrow (iv)$ are simply by definition.

[8] See Appendix 1.1 for this characterisation of (non-virtual) Woodin cardinals.

$\boxed{(i) \Rightarrow (iv)}$ Assume δ is virtually Woodin, and fix a function $f\colon \delta \to \delta$. Let $\kappa < \delta$ be virtually $({<}\delta, f)$-strong and let $\theta := \sup_{\alpha\leq\kappa} f(\alpha)+1$. Then there's a generic elementary embedding $\pi\colon (H_\theta^V, \in, f\cap H_\theta^V) \to (\mathcal{M}, \in, f^+)$ where $f^+ \restriction \kappa = f \restriction \kappa$, $\mathcal{M} \subseteq V$ and $\pi(\kappa) > \theta$. We firstly want to show that κ is a closure point for f, so let $\alpha < \kappa$. Then

$$f(\alpha) = f^+(\alpha) = \pi(f)(\alpha) = \pi(f)(\pi(\alpha)) = \pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha < \kappa$. Now, if κ wasn't a closure point for f then, letting $\alpha < \kappa$ be the least such that $f(\alpha) \geq \kappa$,

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument wouldn't work if we had only assumed δ to be virtually pre-Woodin. Lastly, θ-strongness implies that $H_\theta^V \subseteq \mathcal{M}$, and $\mathcal{M} \subseteq V$ holds by assumption.

$\boxed{(iv) \Rightarrow (vi)}$ Assume (iv) holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$. By (iv) there's a $\kappa < \delta$ which is a closure point of g and there's a regular $\theta \in (\kappa, \delta)$ and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$. We want to find a regular $\bar\theta < \kappa$ and another elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f(\operatorname{crit}\sigma) = \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\sigma$.

Note that $\mathcal{M} \subseteq V$ and $H_\theta^V \subseteq \mathcal{M}$ implies that $H_\theta^V = H_\theta^{\mathcal{M}}$, so that both H_θ^V and $H_{\pi(\theta)}^{\mathcal{M}}$ are elements of $\mathcal{M}$ (we introduced g to ensure that $\pi(\theta)$ makes sense). An application of Countable Embedding Absoluteness 1.41 then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H_\theta^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$ and $\pi(f \restriction \kappa) \in \operatorname{ran}\pi^*$.

By elementarity of π, H_θ^V has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f \restriction \kappa \in \operatorname{ran}\sigma$ and $\bar\theta = f(\operatorname{crit}\sigma)$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta \to \delta$ be given. Define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$, so that by (vi) there exist regular $\bar\kappa < \bar\theta < \kappa < \theta$ such that κ is a closure point for g and there exists a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) = \bar\theta$ and $g \restriction \kappa \in \operatorname{ran}\pi$.

Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 2.7 we get a transitive $\mathcal{M} \in H^V_{g(\bar\kappa)}$ closed under $<f(\bar\kappa)$-sequences, and $H^V_{g(\bar\kappa)}$ has a generic elementary embedding $\sigma\colon H^V_{f(\bar\kappa)} \to \mathcal{M}$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa > f(\bar\kappa)$. In other words, $\bar\kappa$ is virtually $f(\bar\kappa)$-supercompact in $H^V_{\bar\theta}$. Elementarity of π then implies that κ is virtually $\pi(f)(\kappa)$-supercompact in H^V_θ, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all α such that $(H^V_\alpha, \in, A \cap H^V_\alpha) \prec (H^V_\delta, \in, A)$. Let $f\colon \delta \to \delta$ be given as $f(\alpha) = \langle \alpha_0, \alpha_1 \rangle$ with $\langle -, - \rangle$ being the Gödel pairing function, where α_0 is the first limit of elements of C above α and the α_1's are chosen such that $\{\alpha_1 \mid \alpha < \beta\}$ encodes $A \cap \beta$. This definition makes sense since δ is inaccessible by Proposition 2.5.

Let $\kappa < \delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa$, $\theta > \kappa$ and a generic elementary embedding $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$, and $f \restriction \kappa \in \operatorname{ran}\pi$. We claim that $\bar\kappa := \operatorname{crit}\pi$ is virtually $(<\delta, A)$-extendible. To see this, it suffices by the definition of C to show that

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \models \ulcorner\bar\kappa \text{ is virtually } (A \cap H_\kappa)\text{-extendible}\urcorner, \tag{1}$$

since $\kappa \in C$ because it is a closure point of f. Let $\beta := \min(C - \bar\kappa) < \bar\theta$ and note that β exists as $f(\bar\kappa) = \bar\theta$ so the definition of f says that $\bar\theta$ is a limit of elements of C above $\bar\kappa$. It then holds that $(H^V_{\bar\kappa}, \in, A \cap H^V_{\bar\kappa}) \prec (H^V_\beta, \in, A \cap H^V_\beta)$ as both $\bar\kappa$ and β are elements of C. Since f encodes A in the manner previously described and $\pi^{-1}(f) \restriction \bar\kappa = f \restriction \bar\kappa$, we get that $\pi(A \cap H^V_{\bar\kappa}) = A \cap H^V_\kappa$ and thus

$$(H^V_\kappa, \in, A \cap H^V_\kappa) \prec (H^V_{\pi(\beta)}, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H^V_\beta)$. Now, as $(H^V_\gamma, \in, A \cap H^V_\gamma)$ and $(H^V_{\pi(\gamma)}, \in, A^* \cap H^V_{\pi(\gamma)})$ are elements of $H^V_{\pi(\beta)}$ for every $\gamma < \kappa$, Countable Embedding Absoluteness 1.41 implies that $H^V_{\pi(\beta)}$ sees that $\bar\kappa$ is virtually $(<\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ■

Remark 2.19. The above proof shows that the $\mathcal{M} \subseteq V$ assumptions can be replaced by "sufficient" agreement between $\mathcal{M}$ and V: for (i)-(iii) this means that $H^{\mathcal{M}}_\theta =$

H_θ^V whenever $\mathcal{M}$ is the codomain of a virtual (θ, A)-strong/supercompact/extendible embedding, and in (iv)-(v) this means that $H^{\mathcal{M}}_{\pi(f)(\kappa)} = H^V_{\pi(f)(\kappa)}$. The same thing holds in the "lightface" setting of Theorem 2.7.

We will now step away from the Woodins for a little bit, and introduce the Vopěnkas. In anticipation of the next section we will work with the class-sized version here, but all the following results work equally well for inaccessible virtually Vopěnka cardinals[9].

Definition 2.20 (GBC). The **Generic Vopěnka Principle** (gVP) states that for any class C consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$. ∘

We will be using a standard variation of gVP involving the following *natural sequences*.

Definition 2.21 (GBC). Say that a class function $f\colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. ∘

Definition 2.22 (GBC). Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ is **natural** if there exists an indexing function $f\colon \mathsf{On} \to \mathsf{On}$ and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$ for every α. Denote this indexing function by $f^{\vec{\mathcal{M}}}$ and the unary relations as $R^{\vec{\mathcal{M}}}_\alpha$. ∘

The following Theorem 2.23 is then the main theorem of this section. Firstly it shows that inaccessible cardinals are virtually Vopěnka iff they are virtually pre-Woodin, but also that adding the "virtually" adverb doesn't do anything in this context, in contrast to Theorem 2.52.

Theorem 2.23 (GBC, Dimopoulos-Gitman-N.). *The following are equivalent:*

(i) gVP holds;

(ii) For any natural On-sequence $\vec{\mathcal{M}}$ there exists a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for some $\alpha < \beta$;

[9] Note however that we have to require inaccessibility here: see [Wilson, 2019a] for an analysis of the singular virtually Vopěnka cardinals.

(iii) *On is virtually pre-Woodin;*

(iv) *On is faintly pre-Woodin.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume On is faintly pre-Woodin and fix some On-sequence $\vec{\mathcal{M}} := \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ of structures in a common language. Let κ be $(<\mathsf{On}, \vec{\mathcal{M}})$-prestrong and fix some regular $\theta > \kappa$ satisfying that $\mathcal{M}_\alpha \in H_\theta^V$ for every $\alpha < \theta$, and fix a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, \vec{\mathcal{M}}) \to (\mathcal{N}, \in, \mathcal{M}^*)$$

with $H_\theta^V \subseteq \mathcal{N}$ and $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$. Set $\kappa := \operatorname{crit} \pi$.

We have that $\pi \restriction \mathcal{M}_\kappa\colon \mathcal{M}_\kappa \to \mathcal{M}^*_{\pi(\kappa)}$, but we need to reflect this embedding down below θ as we don't know whether $\mathcal{M}^*_{\pi(\kappa)}$ is on the $\vec{\mathcal{M}}$ sequence. Working in the generic extension, we have

$$\mathcal{N} \models \exists \bar\kappa < \pi(\kappa) \exists \dot\sigma \in V^{\mathrm{Col}(\omega, \mathcal{M}^*_{\bar\kappa})}\colon \ulcorner \dot\sigma\colon \mathcal{M}^*_{\bar\kappa} \to \mathcal{M}^*_{\pi(\kappa)} \text{ is elementary} \urcorner.$$

Here κ realises $\bar\kappa$ and $\pi \restriction \mathcal{M}_\kappa$ realises σ. Note that $\mathcal{M}^*_\kappa = \mathcal{M}_\kappa$ since we ensured that $\mathcal{M}_\kappa \in H_\theta^V$ and we are assuming that $\vec{\mathcal{M}} \cap H_\theta^V = \mathcal{M}^* \cap H_\theta^V$, so the domain of σ ($= \pi \restriction \mathcal{M}_\kappa$) *is* $\mathcal{M}^*_\kappa$ – also note that σ exists in a $\mathrm{Col}(\omega, \mathcal{M}_\kappa)$ extension of $\mathcal{N}$ by an application of Countable Embedding Absoluteness 1.41. Now elementarity of π implies that

$$H_\theta^V \models \exists \bar\kappa < \kappa \exists \dot\sigma \in V^{\mathrm{Col}(\omega, \mathcal{M}_{\bar\kappa})}\colon \ulcorner \dot\sigma\colon \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa \text{ is elementary} \urcorner,$$

which is upwards absolute to V, from which we can conclude that $\sigma\colon \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa$ witnesses that gVP holds.

$(ii) \Rightarrow (iii)$: Assume (ii) holds and assume that On is not virtually pre-Woodin, which means that there exists some class A such that there are no virtually A-prestrong cardinals. This allows us to define a function $f\colon \mathsf{On} \to \mathsf{On}$ as $f(\alpha)$ being the least regular $\eta > \alpha$ such that α is not virtually (η, A)-prestrong.

We also define $g\colon \mathsf{On} \to \mathsf{On}$ as taking α to the least strong limit cardinal above α which is a closure point for f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A \cap H^V_{g(\alpha)}$. (ii) supplies us with $\alpha < \beta$ and a generic elementary embedding[10]

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}).$$

Since $g(\alpha)$ is a closure point for f it holds that $f(\operatorname{crit}\pi) < g(\alpha)$, so fixing a regular $\theta \in (f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-prestrong, contradicting the definition of f. Hence On is virtually pre-Woodin. ■

2.2.1 Weak Vopěnka

We now move to a *weak* variant of gVP, introduced in a category-theoretic context in [Adámek and Rosický, 1994]. It starts with the following equivalent characterisation of gVP, which is the virtual analogue of the characterisation shown in [Adámek and Rosický, 1994].

Lemma 2.24 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < On \rangle$ satisfying that*[11]

(i) gVP

(ii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < On \rangle$ satisfying that
- *there is a generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

(iii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < On \rangle$ satisfying that
- *there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

Proof. Note that the only difference between (ii) and (iii) is that the homomorphism exists in V, making $(ii) \Rightarrow (iii)$ trivial.

$(iii) \Rightarrow (i)$: Assume that gVP fails, meaning by Theorem 2.23 that we have a natural On-sequence $\vec{\mathcal{M}}_\alpha$ such that, in every generic extension, there's no homomorphism between any two disctinct $\mathcal{M}_\alpha$'s. Define an On-sequence $\langle \mathcal{N}_\kappa \mid \kappa \in$

[10] Note that $V_{g(\alpha)} = H^V_{g(\alpha)}$ since $g(\alpha)$ is a strong limit cardinal.

[11] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [Adámek and Rosický, 1994].

Card⟩ as

$$\mathcal{N}_\kappa := \coprod_{\xi\leq\kappa}\mathcal{M}_\xi = \{(x,\xi) \mid \xi\leq\kappa\wedge\xi\in\text{Card}\wedge x\in\mathcal{M}_\xi\},$$ [12]

with a unary relation R^* given as $R^*(x,\xi)$ iff $\mathcal{M}_\xi \models R(x)$ and a binary relation $\sim^*$ given as $(x,\xi) \sim^* (x',\xi')$ iff $\xi = \xi'$. Whenever we have a homomorphism $f\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ we then get an induced homomorphism $\tilde{f}\colon \mathcal{M}_0 \to \mathcal{M}_\xi$, given as $\tilde{f}(x) := f(x,0)$, where $\xi \leq \kappa$ is given by preservation of $\sim^*$.

For any two cardinals $\kappa < \lambda$ we have a homomorphism $j_{\kappa\lambda}\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ in V, given as $j_{\kappa\lambda}(x,\xi) := (x,\xi)$. This embedding must also be the *unique* such embedding in all generic extensions, as otherwise we get a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s. Furthermore, there can't be any homomorphism $\mathcal{N}_\lambda \to \mathcal{N}_\kappa$ as that would also imply the existence of a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s.

$(i) \Rightarrow (ii)$: Assume that we have an On-sequence $\vec{\mathcal{M}}_\alpha$ as in the theorem, with generic homomorphisms $j_{\alpha\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ that are unique in all generic extensions for every $\alpha \leq \beta$, with no generic homomorphisms going the other way.

We first note that we can for every $\alpha \leq \beta$ choose the $j_{\alpha\beta}$ in a $\text{Col}(\omega,\mathcal{M}_\alpha)$-extension, by a proof similar to the proof of Lemma 1.41 and using the uniqueness of $j_{\alpha\beta}$. Next, fix a proper class $C \subseteq \mathsf{On}$ such that $\alpha \in C$ implies that

$$\sup_{\xi\in C\cap\alpha} |\mathcal{M}_\xi|^V < |\mathcal{M}_\alpha|^V .$$

and note that this implies that $V[g] \models |\mathcal{M}_\xi| < |\mathcal{M}_\alpha|$ for every V-generic $g \subseteq \text{Col}(\omega,\mathcal{M}_\xi)$. This means that for every $\alpha \in C$ we may choose some $\eta_\alpha \in \mathcal{M}_\alpha$ which is *not* in the range of any $j_{\xi\alpha}$ for $\xi < \alpha$. But now define first-order structures $\langle \mathcal{N}_\alpha \mid \alpha \in C\rangle$ as $\mathcal{N}_\alpha := (\mathcal{M}_\alpha, \eta_\alpha)$. Then, by our assumption on the $\mathcal{M}_\alpha$'s and construction of the $\mathcal{N}_\alpha$'s, there can be no generic homomorphism between any two distinct $\mathcal{N}_\alpha$, showing that gVP fails. ■

Note that the proof of the above lemma shows that we without loss of generality may assume that the generic homomorphism in (i) exists in V, which we record here:

Lemma 2.25 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle\mathcal{M}_\alpha \mid \alpha < On\rangle$ satisfying that*[13]

(i) there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;

(ii) there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.

■

The *weak* version of gVP is then simply "flipping the arrows around" in the above characterisation of gVP.

Definition 2.26 (GBC). **Generic Weak Vopěnka's Principle** (gWVP) states that there does *not* exist an On-sequence of first-order structures $\langle\mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ such that

- there is a generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\alpha \leq \beta$, which is unique in all generic extensions;
- there is *no* generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for any $\alpha < \beta$.

○

We start by showing that gWVP is indeed a weaker version of gVP.

Proposition 2.27. *gVP implies gWVP.*

Proof. Assume gVP holds and gWVP fails, and let $\langle\mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ be an On-sequence of first-order structures such that for every $\alpha \leq \beta$ there exists a generic homomorphism

$$j_{\beta\alpha}\colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$$

in some $V[g]$ which is unique in all generic extensions, with no generic homomorphisms going the other way. Here we may assume, as in the proof of Lemma 2.24, that $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\beta)$. We can then find a proper class $C \subseteq \mathsf{On}$ such that $|\mathcal{M}_\alpha|^V < |\mathcal{M}_\beta|^V$ for every $\alpha < \beta$ in C. By gVP there are then $\alpha < \beta$ in C and a

[13] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [Adámek and Rosický, 1994].

generic homomorphism

$$\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta .$$

in some $V[h]$, where again we may assume that $h \subseteq \mathrm{Col}(\omega, \mathcal{M}_\alpha)$. But then $\pi \circ j_{\beta\alpha} = \mathrm{id}$ by uniqueness of $j_{\beta\beta} = \mathrm{id}$, which means that $j_{\beta\alpha}$ is injective in $V[g \times h]$ and hence also in $V[g]$. But then $|\mathcal{M}_\beta|^{V[g]} \leq |\mathcal{M}_\alpha|^{V[g]}$, which implies that $|\mathcal{M}_\beta|^V \leq |\mathcal{M}_\alpha|^V$ by the $|\mathcal{M}_\beta|^{+V}$-cc of $\mathrm{Col}(\omega, \mathcal{M}_\beta)$, contradicting the definition of C. ■

Denoting the corresponding non-generic principle by WVP [Wilson, 2019b] showed the following.

Theorem 2.28 (Wilson). *WVP is equivalent to On being a Woodin cardinal.*

Given our 2.23 we may then suspect that in the virtual world these two are equivalent, which turns out to *almost* be the case. We will be roughly following the argument in [Wilson, 2019b], but we have to diverge from it at several points in which they're using the fact that they're working with class-sized elementary embeddings.

Indeed, in that paper they establish a correspondence between elementary embeddings and certain homomorphisms, a correspondence we won't achieve here. Proving that the elementary embeddings we *do* get are non-trivial seems to furthermore require extra assumptions on our structures. Let's begin.

Define for every strong limit cardinal λ and Σ_1-formula φ the relations

$$\begin{aligned} R^\varphi &:= \{x \in V \mid (V, \in) \models \varphi[x]\} \\ R^\varphi_\lambda &:= \{x \subseteq H^V_\lambda \mid \exists y \in R^\varphi \colon y \cap H^V_\lambda = x\} \end{aligned}$$

and given any class A define the structure

$$\mathscr{P}_{\lambda,A} := (H^V_{\lambda^+}, R^\varphi_\lambda, \{\lambda\}, A \cap H^V_\lambda)_{\varphi \in \Sigma_1}.$$

Say that a homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ is **trivial** if $h(x) \cap H^V_\eta = x \cap H^V_\eta$ for every $x \in H^V_{\lambda^+}$. Note that h can only be trivial if $\eta \leq \lambda$ since $h(\lambda) = \eta$.

Lemma 2.29 (GBC, Gitman-N.). *Let λ be a singular strong limit cardinal, η a strong limit cardinal and $A \subseteq V$ a class. If there exists a non-trivial generic homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ then there's a non-trivial generic elementary embedding*

$$\pi\colon (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$$

for some $\mathcal{M}$ such that, letting $\nu := \min\{\lambda, \eta\}$, it holds that $H^V_\nu = H^{\mathcal{M}}_\nu$, $A \cap H^V_\nu = B \cap H^V_\nu$ and $\operatorname{crit}\pi < \nu$.

Proof. Assume that we have a non-trivial homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ in a forcing extension $V[g]$, define in $V[g]$ the set

$$\mathcal{M}^* := \{\langle b, f\rangle \mid b \in [H_\nu]^{<\omega} \wedge f \in H^V_{\lambda^+} \wedge f\colon H^V_\lambda \to H^V_\lambda\},$$

and define the standard relations $\in^*$ and $=^*$ on $\mathcal{M}^*$ as

$$\begin{aligned}
\langle b_0, f_0\rangle \in^* \langle b_1, f_1\rangle \quad &\text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) \in f_1(y)\}) \\
\langle b_0, f_0\rangle =^* \langle b_1, f_1\rangle \quad &\text{iff} \quad b_0 b_1 \in h(\{xy \in [H^V_\lambda]^{<\omega} \mid f_0(x) = f_1(y)\})
\end{aligned}$$

Let $\mathcal{M} := \mathcal{M}^* / =^*$, and also call $\in^*$ the induced relation on $\mathcal{M}$, which is clearly well-defined. We then get a version of Łoś' Theorem, using that h preserves all Σ_1-relations and that $H^V_\lambda \models \mathsf{ZFC}^-$.

Claim 2.30. For every formula $\varphi(v_1, \ldots, v_n)$ and every $[b_1, f_1], \ldots, [b_n, f_n] \in \mathcal{M}$ the following are equivalent:

(i) $(\mathcal{M}, \in^*) \models \varphi[[b_1, f_1], \ldots, [b_n, f_n]]$;
(ii) $b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \ldots, f_n(a_n)]\})$.

Proof of claim. The proof is straightforward, using that h preserves Σ_1-relations. We prove this by induction on φ. If φ is $v_i \in v_j$ then we have that

$$\begin{aligned}
&(\mathcal{M}, \in^*) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]\\
\Leftrightarrow&\langle b_i, f_i\rangle \in^* \langle b_j, f_j\rangle\\
\Leftrightarrow&b_i b_j \in h(\{a_i a_j \in [H_\lambda^V]^{<\omega} \mid f_i(a_i) \in f_j(a_j)\})\\
\Leftrightarrow&b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid f_i(a_i) \in f_j(a_j)\})\\
\Leftrightarrow&b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).
\end{aligned}$$

The cases where φ is $\psi \wedge \chi$ or $\neg\psi$ is straightforward. If φ is $\exists x\psi$ then

$$\begin{aligned}
&(\mathcal{M}, \in^*) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]\\
\Leftrightarrow&\exists\langle b, f\rangle \in \mathcal{M}^* \colon (\mathcal{M}, \in) \models \psi[\langle b, f\rangle, \langle b_1, f_1\rangle, \dots, \langle b_n, f_n\rangle]\\
\Leftrightarrow&\exists\langle b, f\rangle \in \mathcal{M}^* \colon bb_1 \cdots b_n \in h(\{aa_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f(a), f_1(a_1), \dots, f_n(a_n)]\})\\
\Leftrightarrow&b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}),
\end{aligned}$$

finishing the proof. ⊣

Note that we haven't shown that $(\mathcal{M}, \in^*)$ is wellfounded, and indeed it might not be. However, the following claim will show that $(H_\nu^V, \in)$ is isomorphic to a rank-initial segment of $(\mathcal{M}^*, \in^*)$, giving wellfoundedness up to that point at least. Define the function $\chi \colon (H_\nu^V, \in) \to (\mathcal{M}^*, \in^*)$ as $\chi(a) := [\langle a\rangle, \mathrm{pr}]$, where $\mathrm{pr}(\langle x\rangle) := x$.

Claim 2.31. For every $[a, f] \in \mathcal{M}$ and $b \in H_\nu^V$,

$$[a, f] \in^* \chi(b) \quad \Leftrightarrow \quad \exists c \in H_\nu^V \colon [a, f] = \chi(c).$$

Proof of claim. We have that

$$\begin{aligned}
[a, f] \in^* \chi(b) = [\langle b\rangle, \mathrm{pr}] &\Leftrightarrow a\langle b\rangle \in h(\{x\langle y\rangle \mid f(x) \in y\})\\
&\Leftrightarrow a\langle b\rangle \in h(\{x\langle y\rangle \mid \exists z \in y \colon f(x) = z\})\\
&\Leftrightarrow \exists c \in b \colon a\langle c\rangle \in h(\{x\langle z\rangle \mid f(x) = z\})\\
&\Leftrightarrow \exists c \in b \colon [a, f] = [\langle c\rangle, \mathrm{pr}] = \chi(c),
\end{aligned}$$

yielding the wanted. $\dashv$

This claim implies that by taking the transitive collapse of $\operatorname{ran}\chi \subseteq \mathcal{M}$ we may assume that $H_\nu^V = H_\nu^{\mathcal{M}}$. Now define

$$B := \{[b, f] \in \mathcal{M} \mid b \in h(\{x \in H_\lambda^V \mid f(x) \in A\})\}.$$

and, in $V[g]$, let $\pi\colon (H_\lambda^V, \in, A \cap H_\lambda^V) \to (\mathcal{M}, \in, B)$ be given as $\pi(x) := [\langle\rangle, c_x]$.

Claim 2.32. π is elementary.

Proof of claim. For $x_1, \dots, x_n \in H_\lambda^V$ it holds that

$$\begin{aligned}(\mathcal{M}, \in^*, B) \models \varphi[\pi(x_1), \dots, \pi(x_n)] &\Leftrightarrow (\mathcal{M}, \in^*) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\ &\Leftrightarrow \langle\rangle \in h(\{\langle\rangle \mid \mathscr{P}_{\lambda,A} \models \varphi[x_1, \dots, x_n]\}) \\ &\Leftrightarrow (H_{\lambda^+}^V, \in, A \cap H_\lambda^V) \models \varphi[x_1, \dots, x_n]\end{aligned}$$

and we also get that, for every $x \in H_\lambda^V$,

$$x \in A \Leftrightarrow \langle\rangle \in h(\{a \in H_\lambda^V \mid x \in A\}) \Leftrightarrow \pi(x) \in B,$$

which shows elementarity. $\dashv$

We next need to show that $B \cap H_\nu^V = A \cap H_\nu^V$, so let $x \in H_\nu^V$. Note that $x = [\langle x\rangle, \mathrm{pr}]$ by Claim 2.31 and the observation proceeding it, which means that

$$x \in B \Leftrightarrow \langle x\rangle \in h(\{\langle y\rangle \in H_\lambda^V \mid y \in A\}) \Leftrightarrow x \in A.$$

The last thing we need to show is that $\operatorname{crit}\pi < \nu$. We start with an analogous result about h.

Claim 2.33. There exists some $b \in H_\nu^V$ such that $h(b) \neq b$.

Proof of claim. Assume the claim fails. We now have two cases.

Case 1: $\lambda \geq \eta$

By non-triviality of h there's an $x \in \mathscr{P}^V(H_\lambda^V)$ such that $h(x) \neq x \cap H_\eta^V$, which means that there exists an $a \in H_\eta^V$ such that $a \in h(x) \Leftrightarrow a \notin x$.

If $a \in x$ then $\{a\} = h(\{a\}) \subseteq h(x)$,[14] making $a \in h(x)$, $\lightning$, so assume instead that $a \in h(x)$. Since η is a strong limit cardinal we may fix a cardinal $\theta < \eta$ such that $a \in H_\theta^V$ and $H_\theta^V \in H_\eta^V$. We then have that[15]

$$\{a\} \subseteq h(x) \cap H_\theta^V = h(x) \cap h(H_\theta^V) = h(x \cap H_\theta^V) = x \cap H_\theta^V,$$

so that $a \in x$, $\lightning$.

Case 2: $\lambda < \eta$

In this case we are assuming that $h \restriction H_\lambda^V = \mathrm{id}$, but $h(\lambda) = \eta > \lambda$. Since λ is singular we can fix some $\gamma < \lambda$ and a cofinal function $f\colon \gamma \to \lambda$. Define the relation

$$R = \{(\alpha, \beta, \bar\alpha, \bar\beta, g) \mid \ulcorner g \text{ is a cofinal function } g\colon \alpha \to \beta \urcorner \land g(\bar\alpha) = \bar\beta\}.$$

Then $R(\gamma, \lambda, \alpha, f(\alpha), f)$ holds by assumption for every $\alpha < \gamma$, so that R holds for some $(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*)$ such that

$$\begin{aligned}(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*) \cap H_\eta^V &= (h(\gamma), h(\lambda), h(\alpha), h(f(\alpha)), h(f)) \\ &= (\gamma, \eta, \alpha, f(\alpha), h(f)),\end{aligned}$$

using our assumption that h fixes every $b \in H_\lambda^V$. Since γ, α and $f(\alpha)$ are transitive and bounded in H_λ^V it holds that $h(\gamma) = \gamma^*$, $h(\alpha) = \alpha^*$ and $h(f(\alpha)) = f(\alpha)^*$. Also, since $\mathrm{dom}(f^*) = \gamma = \mathrm{dom}(f)$ we must in fact have that $f^* = h(f)$. But this means that $h(f)\colon \gamma \to \eta$ is cofinal and $\mathrm{ran}(h(f)) \subseteq \lambda$, a contradiction! $\dashv$

To use the above Claim 2.33 to conclude anything about π we'll make use of the following standard lemma.

[14]Note that as h preserves Σ_1 formulas it also preserves singletons and boolean operations.
[15]Note that we're using $\lambda \geq \eta$ here to ensure that $H_\theta^V \in \mathrm{dom}\, h$.

Claim 2.34. For any $x \in H^V_\lambda$ it holds that $h(x) = \pi(x) \cap H^V_\eta$.

Proof of claim. For any $n < \omega$ and $\langle a_1, \dots, a_n\rangle \in [H^V_\eta]^n$ we have that

$$\begin{aligned}
&\langle a_1, \dots, a_n\rangle \in \pi(x)\\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle a_1, \dots, a_n\rangle \in \pi(x)\\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle[\langle a_1\rangle, \mathrm{pr}], \dots, [\langle a_n\rangle, \mathrm{pr}]\rangle \in [\langle\rangle, c_x]\\
\Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(\{\langle x_1, \dots, x_n\rangle \mid \mathscr{P}_{\lambda,A} \models \langle x_1, \dots, x_n\rangle \in x\})\\
\Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(x),
\end{aligned}$$

showing that $h(x) = \pi(x) \cap H^V_\eta$. ⊣

Now use Claim 2.33 to fix a $b \in H^V_\nu$ which is moved by h. Claim 2.34 then implies that

$$\pi(b) \cap H^V_\eta = h(b) \cap H^V_\eta = h(b) \neq b = b \cap H^V_\eta,$$

showing that $\pi(b) \neq b$ and hence $\operatorname{crit}\pi < \nu$. This finishes the proof of the lemma. ■

Theorem 2.35 (GBC, Gitman-N.). *gWVP holds iff On is W-virtually pre-Woodin.*

Proof. ($\Leftarrow$) is just observing that the virtualisation of the argument in [Wilson, 2019b] that WVP holds if On is Woodin works in the W-virtually pre-Woodin case, so we only give a brief sketch.

Assume On is W-virtually pre-Woodin and let $\vec{\mathcal{M}}$ be a counterexample to gWVP, so that we in V we have homomorphisms $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for all $\alpha \leq \beta$. Work in some generic extension $V[g]$, fix a W-virtually $\vec{\mathcal{M}}$-prestrong cardinal κ and let $\theta \gg \kappa$ be such that $\mathcal{M}_{\kappa+1} \in H^V_\theta$. Letting $\pi\colon (H^V_\theta, \in) \to (\mathcal{M}, \in^*)$ be the corresponding embedding we get that $\mathcal{M}_{\kappa+1} = \pi(\vec{\mathcal{M}})_{\kappa+1}$, so that

$$\pi \restriction \mathcal{M}_\kappa \colon (\mathcal{M}_\kappa, \in) \to (\pi(\mathcal{M}_\kappa), \in^*) = (\pi(\vec{\mathcal{M}})_{\pi(\kappa)}, \in^*).$$

But then the choice of θ and elementarity of π we get that $\mathcal{M}$ has a homomorphism

$$h\colon (\pi(\vec{\mathcal{M}})_{\pi(\kappa)}, \in^*) \to (\pi(\vec{\mathcal{M}})_{\kappa+1}, \in^*) = (\mathcal{M}_{\kappa+1}, \in),$$

making $h \circ (\pi \restriction \mathcal{M}_\kappa)\colon (\mathcal{M}_\kappa, \in) \to (\mathcal{M}_{\kappa+1}, \in)$ a counterexample to gWVP.

($\Rightarrow$): Assume that On is not W-virtually pre-Woodin. This means that there exists a class A such that there are no W-virtually A-prestrong cardinals. We can therefore assign to any cardinal κ the least cardinal $f(\kappa) > \kappa$ such that κ is not W-virtually $(f(\kappa), A)$-prestrong.

Also define a function $g\colon \mathsf{On} \to \mathrm{Card}$ as taking an ordinal α to the least singular strong limit cardinal above α closed under f. Then we're assuming that there's no non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

with $H^V_{g(\alpha)} \subseteq \mathcal{M}$ and $B \cap H^V_{g(\alpha)} = A \cap H^V_{g(\alpha)}$. Assume towards a contradiction that for some α, β there is a non-trivial generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$. Lemma 2.29 then gives us a non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that $H^V_\nu \subseteq \mathcal{M}$ with $\nu := \min\{g(\alpha), g(\beta)\}$ and $A \cap H^V_\nu = B \cap H^V_\nu$, a contradiction! Therefore every generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$ is trivial. Since there is a unique trivial homomorphism when $\alpha \geq \beta$ and no trivial homomorphism when $\alpha < \beta$ since $g(\alpha)$ is sent to $g(\beta)$, the sequence of structures

$$\langle \mathscr{P}_{g(\alpha),A} \mid \alpha \in \mathsf{On} \rangle$$

is a counterexample to gWVP, which is what we wanted to show. ■

2.3 Berkeleys

We next move to the higher realms of the virtual large cardinal hierarchy, and study cardinals whose non-virtual versions are inconsistent with ZFC.

In the virtual setting the virtually Berkeley cardinals, like all the other virtual large cardinals, are simply downwards absolute to L. It turns out that virtually Berkeley cardinals are natural objects, as the main theorem of this section, Theorem 2.43, shows that these large cardinals are precisely what separates virtually pre-Woodins from the virtually Woodins, as well as separating virtually Vopěnka cardinals from Mahlo cardinals.

Definition 2.36. Say that a cardinal δ is **virtually proto-Berkeley** if for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi < \delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ then δ is **virtually Berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C \subseteq \delta$ we say δ is **virtually club Berkeley**. ○

Virtually (proto-)Berkeley cardinals turn out to be equivalent to their "boldface" versions, the proof of which is a straightforward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [Cutolo, 2017].

Proposition 2.37 (Virtualised Cutolo). *If δ is virtually proto-Berkeley then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\pi < \delta$. If δ is virtually Berkeley then we can furthermore ensure that $\operatorname{crit}\pi$ is arbitrarily large below δ.*

Proof. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{\{\langle A, x\rangle \mid x \in \mathcal{M}\}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: a is the first element in the pairs belonging to the set of highest rank, and $\mathcal{M}$ is what remains if we remove the set with the highest rank. But this means that a generic elementary embedding $\pi\colon \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and a, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the wanted conclusion. ■

The following is a straightforward virtualisation of the usual definition of the Vopěnka filter (see e.g. [Kanamori, 2008]).

Definition 2.38 (GBC). Define the **virtually Vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon\ \mathcal{M}_\alpha \to \mathcal{M}_\beta$. ∘

Theorem 2.23 shows that $\emptyset$ is in the virtually Vopěnka filter iff gVP fails, in analogy with the non-virtual case. Normality also holds in the virtual context, as the following proof shows.

Lemma 2.39 (GBC, Virtualised folklore). *The virtually Vopěnka filter is a normal filter.*

Proof. Let F be the virtually Vopěnka filter. We first show that F is actually a filter. If $X \in F$ and $Y \supseteq X$ then $Y \in F$ simply by definition of F. If $X, Y \in F$, witnessed by natural sequences $\vec{\mathcal{M}}$ and $\vec{\mathcal{N}}$, then $X \cap Y \in F$ as well, witnessed by the natural sequence $\vec{\mathcal{P}}$ induced by the indexing function $f^{\vec{\mathcal{P}}} := \max(f^{\vec{\mathcal{M}}}, f^{\vec{\mathcal{N}}})$ and unary relations $R^{\vec{\mathcal{P}}}_\alpha := \operatorname{Code}(\langle R^{\vec{\mathcal{M}}}_\alpha, R^{\vec{\mathcal{N}}}_\alpha\rangle)$. Indeed, if $\pi\colon\ \mathcal{P}_\alpha \to \mathcal{P}_\beta$ is a generic elementary embedding with critical point μ then μ is also the critical point of both $\pi \restriction \mathcal{M}_\alpha\colon\ \mathcal{M}_\alpha \to \mathcal{M}_\beta$ and $\pi \restriction \mathcal{N}_\alpha\colon\ \mathcal{N}_\alpha \to \mathcal{N}_\beta$.

For normality, let $X \in F^+$ be F-positive, where we recall that this means that $X \cap C \neq \emptyset$ for every $C \in F$, and let $f\colon\ X \to \mathsf{On}$ be regressive. We want to show that f is constant on an F-positive set.

Assume this fails, meaning that there are natural sequences $\vec{\mathcal{M}}^\gamma$ for γ such that for any generic elementary $\pi\colon\ \mathcal{M}^\gamma_\alpha \to \mathcal{M}^\gamma_\beta$ satisfies that $f(\operatorname{crit}\pi) \neq \gamma$. Define a new natural sequence $\vec{\mathcal{N}}$ as induced by the indexing function $g\colon\ \mathsf{On} \to \mathsf{On}$ given as $g(\alpha) := \sup_{\gamma<\alpha} \operatorname{rk}\mathcal{M}^\gamma_\alpha + \omega$ and unary relations $R^{\vec{\mathcal{N}}}_\alpha$ given as

$$R^{\vec{\mathcal{N}}}_\alpha := \operatorname{Code}(\langle\langle \mathcal{M}^\gamma_\alpha \mid \gamma < \alpha\rangle, f \restriction \alpha\rangle).$$

Now since X is F-positive there exists a generic elementary embedding $\pi\colon\ \mathcal{N}_\alpha \to \mathcal{N}_\beta$ with $\operatorname{crit}\pi \in X$. As $f(\operatorname{crit}\pi) < \operatorname{crit}\pi$ we get that $\pi(f(\operatorname{crit}\pi)) = f(\operatorname{crit}\pi)$, so that we have a generic elementary embedding

$$\pi \restriction \mathcal{M}_\alpha^{f(\operatorname{crit}\pi)} \colon \mathcal{M}_\alpha^{f(\operatorname{crit}\pi)} \to \mathcal{M}_\beta^{f(\operatorname{crit}\pi)},$$

but this contradicts the definition of $\vec{\mathcal{M}}^{f(\operatorname{crit}\pi)}$! Thus F is normal. ∎

The reason why we are being careful in showing all these analogous properties for the virtual Vopěnka filter is that not all properties carry over. Indeed, note that uniformity of filters is non-trivial as we're working with proper classes[16], and we will see in Theorem 2.43 shows that uniformity of this filter is equivalent to there being no virtually Berkeley cardinals – the following lemma is the first implication.

Lemma 2.40 (GBC, N.). *Assume gVP and that there are no virtually Berkeley cardinals. Then the virtually Vopěnka filter F on* On *contains every class club C.*

Proof. The crucial extra property we get by assuming that there aren't any virtually Berkeleys is that F becomes uniform, i.e. contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin F$. Let M be a transitive set with $\delta \subseteq M$ and $\gamma < \delta$ a cardinal. As $(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function $f \colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\alpha+1, \delta+1)$$

and unary relations $R_\alpha := \langle M, \mathcal{N}_\alpha \rangle$. If $\pi \colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit}\pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$, then $\pi(R_\alpha) = R_\beta$ implies that $\pi \restriction \mathcal{M} \colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi \leq \delta$. We also get that $\operatorname{crit}\pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit}\pi} \colon \mathcal{N}_{\operatorname{crit}\pi} \to \mathcal{N}_{\pi(\operatorname{crit}\pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and hence $\operatorname{crit}\pi > \gamma$ as $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually Berkeley, a contradiction. Thus $\operatorname{crit}\pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$.

[16] This boils down to the fact that the class club filter is not provably normal in GBC, see [Gitman et al., 2019]

Note that the class $C_0 \subseteq \mathsf{On}$ of limit ordinals is in F, since it's the diagonal intersection of the tails $(\alpha+1, \mathsf{On})$. Now let $C \subseteq \mathsf{On}$ be a class club, and let $C = \{a_\alpha \mid \alpha < \mathsf{On}\}$ be its increasing enumeration. Then $C \supseteq C_0 \cap \triangle_{\alpha<\mathsf{On}}(a_\alpha, \mathsf{On})$, implying that $C \in F$. ■

Theorem 2.41 (GBC, N.). *If there are no virtually Berkeley cardinals then On is virtually pre-Woodin iff On is virtually Woodin.*

Proof. Assume On is virtually pre-Woodin, so gVP holds by Theorem 2.23 and we can let F be the virtually Vopěnka filter. The assumption that there aren't any virtually Berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails – we will follow the proof of Theorem 2.23.

Failure means that there is some class A and some class club C such that there are no virtually A-prestrong cardinals in C. Since there are no virtually Berkeley cardinals, Lemma 2.40 imples that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as sending α to the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-prestrong if $\alpha \in C$, and set $f(\alpha) := \alpha$ if $\alpha \notin C$. Also define $g\colon \mathsf{On} \to \mathsf{On}$ as $g(\alpha)$ being the least strong limit cardinal in C above α which is a closure point for f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := \mathrm{Code}(\langle A \cap H^V_{g(\alpha)}, \mathcal{N}_\alpha\rangle)$ and apply gVP to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi\restriction(H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)})\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}),$$

making $\operatorname{crit}\pi$ virtually $(g(\alpha), A)$-prestrong and thus $\operatorname{crit}\pi \notin C$. But as we also get the embedding $\pi\restriction\mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\operatorname{crit}\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \mathrm{Card} \mid (H^V_\kappa, \in, A \cap H^V_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-prestrong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it isn't virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, A \cap H_\theta^V) \to (M, \in, B)$$

with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq M$, $M \subseteq V$, $A \cap H_\theta^V = B \cap H_\theta^V$ and $\pi(\kappa) < \theta$.

Now $\pi(\kappa)$ is inaccessible, and $(H^V_{\pi(\kappa)}, \in, A \cap H^V_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$ believes that κ is virtually $(A \cap H^V_{\pi(\kappa)})$-strong as in the proof of Theorem 2.12, meaning that $(H^V_\kappa, \in, A \cap H^V_\kappa)$ believes that there is a proper class of virtually $(A \cap H^V_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually Woodin. ■

Theorem 2.42 (GBC, N.). *If there exists a virtually Berkeley cardinal δ then gVP holds and On is not Mahlo.*

Proof. If On was Mahlo then there would in particular exist an inaccessible cardinal $\kappa > \delta$, but then $H^V_\kappa \models \ulcorner$there exists a virtually berkeley cardinal$\urcorner$, contradicting the incompleteness theorem.

To show gVP we show that On is virtually pre-Woodin, which is equivalent by Theorem 2.23. Fix therefore a class A – we have to show that there exists a virtually A-prestrong cardinal. For every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta\colon (H^V_\theta, \in, A \cap H^V_\theta) \to (H^V_\theta, \in, A \cap H^V_\theta)$$

with $\operatorname{crit}\pi < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-prestrong, making On virtually pre-Woodin. ■

Theorem 2.43 (GBC, N.). *The following are equivalent:*

(i) *gVP implies that On is Mahlo;*

(ii) *On is virtually pre-Woodin iff On is virtually Woodin;*

(iii) *There are no virtually Berkeley cardinals.*

Proof. $(iii) \Rightarrow (ii)$ is Theorem 2.41, and the contraposed version of $(i) \Rightarrow (iii)$ is Theorem 2.42. For $(ii) \Rightarrow (i)$ note that gVP implies that On is virtually pre-Woodin by Theorem 2.23, which by (ii) means that it's virtually Woodin and the usual proof shows that virtually Woodins are Mahlo[17], showing (i). ■

This also immediately implies the following equiconsistency, as virtually Berkeley cardinals have strictly larger consistency strength than virtually Woodin cardinals.

Corollary 2.44 (N.). *The existence of an inaccessible virtually pre-Woodin cardinal is equiconsistent with the existence of an inaccessible virtually Woodin cardinal.* ■

2.4 Behaviour in core models

Most of the cardinals turn out to downwards absolute to most inner models, including L:

Proposition 2.45. *For any regular uncountable cardinal θ, faintly θ-measurable cardinals are downwards absolute to any transitive class $\mathcal{U} \subseteq V$ satisfying $\mathsf{ZF}^- + \mathsf{DC}$.*

Proof. Let κ be faintly θ-measurable, witnessed by a forcing poset $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there's a transitive $\mathcal{M}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a transitive class $\mathcal{U} \subseteq V$ which satisfies $\mathsf{ZF}^- + \mathsf{DC}$. Restricting the embedding to $\pi \restriction H_\theta^{\mathcal{U}}\colon H_\theta^{\mathcal{U}} \to \mathcal{N}$ we can now apply the Countable Absoluteness Lemma 1.41 to $\pi \restriction H_\theta^{\mathcal{U}}$ to get that there exists an embedding $\pi^*\colon H_\theta^{\mathcal{U}} \to \mathcal{N}^*$ in a generic extension of U, making κ faintly θ-measurable in $\mathcal{U}$. ■

[17] See e.g. Exercise 26.10 in [Kanamori, 2008].

Theorem 2.46 (N.). *Let θ be a regular uncountable cardinal.*

(i) $L \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-prestrongs*$\urcorner$.

(ii) *Assume that $L[\mu]$ exists. It then holds that*
$L[\mu] \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-measurables*$\urcorner$.

(iii) *Assume there is no inner model with a Woodin. It then holds that*
$K \models \ulcorner$*faintly θ-measurables are equivalent to virtually θ-measurables*$\urcorner$.

Proof. For (i) simply note that if $\pi\colon L_\theta \to \mathcal{N}$ is a generic elementary embedding with $\mathcal{N}$ transitive, then by condensation we have that $\mathcal{N} = L_\gamma$ for some $\gamma \geq \theta$, so that π also witnesses the virtual θ-prestrongness of $\operatorname{crit}\pi$.

(ii): Assume that $V = L[\mu]$ for notational simplicity and let κ be faintly θ-measurable, witnessed by a generic elementary embedding $\pi\colon L_\theta[\mu] \to \mathcal{N}$ existing in some generic extension $V[g]$. By condensation we get that $\mathcal{N} = L_\gamma[\overline{\mu}]$ for some $\gamma \geq \theta$ and $\overline{\mu} \in V[g]$, but we're not guaranteed that $\overline{\mu} \in V$ here. Let λ be the unique measurable cardinal of $V = L[\mu]$.

Note that $\bar\mu$ is a measure on $\pi(\lambda) \geq \lambda$. If $\pi(\lambda) = \lambda$ then $L[\mu] = L[\bar\mu]$ by [Kanamori, 2008, Theorem 20.10] and we trivially get that $\mathcal{N} \subseteq V$. Assume thus that $\pi(\lambda) > \lambda$, which implies that $L[\bar\mu]$ is an internal iterate of $L[\mu]$ by [Kanamori, 2008, Theorem 20.12]. In particular it then holds that $L[\bar\mu] \subseteq L[\mu]$, so again we get that $\mathcal{N} \subseteq V$.

(iii): Assume that $V = K = L[\mathcal{E}]$ and fix a faintly θ-measurable cardinal κ, witnessed by a generic embedding $\pi\colon L_\theta[\mathcal{E}] \to \mathcal{N} = L_\gamma[\overline{\mathcal{E}}]$ in some generic extension $V[g]$. Now coiterate $L[\mathcal{E}]$ with $L[\overline{\mathcal{E}}]$, and denote the last models by $\mathcal{P}$ and $\mathcal{Q}$. Since $K = K^{V[g]}$ and as K is universal we get that $\mathcal{Q} \trianglelefteq \mathcal{P}$. Then the $L[\overline{\mathcal{E}}]$-to-$\mathcal{Q}$ branch did not drop, giving us an iteration embedding $i\colon L[\overline{\mathcal{E}}] \to \mathcal{Q}$.

Note that $\operatorname{crit} i \geq \kappa$ as $\overline{\mathcal{E}}$ is simply the pointwise image of $\mathcal{E}$ under π, so nothing below κ is touched and is therefore not used in the comparison either. This means that $\operatorname{crit}(i \circ \pi) = \kappa$, so that $(i \circ \pi)\colon L_\theta[\mathcal{E}] \to \mathcal{Q}$ witnesses that κ is virtually θ-measurable, since $\mathcal{Q} \trianglelefteq \mathcal{P}$ implies that $\mathcal{Q} \subseteq K$. ∎

Note that the proofs of (ii) and (iii) above do not show that κ is virtually θ-prestrong, as it might still be the case that $\bar\mu \neq \mu$ or $\bar{\mathcal{E}} \neq \mathcal{E}$, so we cannot conclude that $L_\theta[\mu] \subseteq L_\theta[\bar\mu]$ or $L_\theta[\mathcal{E}] \subseteq L_\theta[\overline{\mathcal{E}}]$. It might still hold however; see Question 5.5.

2.5 Separation results

Having proving many positive results about the relations between the virtual large cardinals in the previous sections, this section is dedicated to the negatives. More precisely, we will aim to *separate* many of the defined notions (potentially under suitable large cardinal assumptions).

Our first separation result is that the virtuals form a level-by-level hierarchy.

Theorem 2.47 (N.). *Let $\alpha < \kappa$ and assume that κ is faintly $\kappa^{+\alpha+2}$-measurable. Then*

$$L_\kappa \models \ulcorner \textit{There's a proper class of } \lambda \textit{ which are virtually } \lambda^{+\alpha+1}\textit{-strong}\urcorner.$$

Proof. Write $\theta := \kappa^{+\alpha+1}$. Then by Theorem 2.12 we get that either κ is faintly θ^+-strong in L or otherwise, in particular, L_κ thinks that there's a proper class of remarkables. In the second case we also get that L_κ thinks that there's a proper class of λ such that λ is virtually $\lambda^{+\alpha+1}$-strong and we'd be done, so assume the first case. Then $L_\kappa \prec_2 L_{\theta^+}$, so define for each $\xi < \kappa$ the sentence ψ_ξ as

$$\psi_\xi :\equiv \exists\lambda < \xi \colon \ulcorner \lambda \text{ is virtually } \lambda^{+\alpha+1}\text{-strong}\urcorner.$$

Then ψ_ξ is $\Sigma_2(\{\alpha, \xi\})$ since being virtually β-strong is a $\Delta_2(\{\beta\})$-statement. As $L_{\theta^+} \models \psi_\xi$ for all $\xi < \kappa$ we also get that $L_\kappa \models \psi_\xi$ for all $\xi < \kappa$, which is what we wanted to show. ■

As we are only assuming κ to be *faintly* measurable in the above, this also shows that the faintly $\kappa^{+\alpha+1}$-measurable cardinals κ form a strict hierarchy whenever $\alpha < \kappa$.

A separation result in a similar vein is the following, showing that it is consistent to have an inaccessible faintly measurable cardinal which is not weakly compact.

Proposition 2.48 (N.). *Assuming κ is measurable, there's a generic extension of V in which κ is inaccessible and faintly measurable, but not weakly compact.*

Proof. Let $\mathbb{P}$ be the forcing notion that adds a κ-Suslin tree $\mathcal{T}$. By [Kunen, 1978] it then holds that $\mathbb{P} * \mathcal{T} \cong \mathrm{Add}(\kappa^+, 1)$, a $<\kappa^+$-closed forcing, which preserves the measurability of κ. Further, the $\mathbb{P}$ forcing is shown to preserve the inaccessibility of κ, making κ inaccesible and faintly measurable in $V[g]$. Lastly, it cannot be weakly compact in $V[g]$ because $\mathcal{T}$ is a κ-tree without a branch, by definition. ■

Next, we show that the virtuals are in fact different from the faints. This is trivial in general as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

A key ingredient is that virtually κ^+-cardinals are Π^2_1-indescribable, whose proof is identical to the standard proof in [Hanf and Scott, 1961] that measurable cardinals are Π^2_1-indescribable. It should be noted that we crucially need the "virtual" property for the proof to go through. Using this indescribability fact, the proof of the following theorem is precisely the same as Hamkins' Proposition 8.2 in [Holy and Schlicht, 2018].

Theorem 2.49 (Virtualised Hamkins). *Assuming κ is a κ^{++}-tall cardinal,*[18] *there's a forcing extension of V in which κ is inaccessible but not virtually κ^+-measurable, and becomes measurable in an $\mathrm{Add}(\kappa^+, 1)$-generic extension.* ■

This then gives us our first separation result.

Corollary 2.50 (N.). *Assuming κ is a κ^{++}-tall cardinal, it's consistent that κ is faintly measurable but not virtually κ^+-measurable.*

Proof. By the above Theorem 2.49 we may assume that κ is not virtually κ^+-measurable but that it's measurable in $V^{\mathbb{P}}$ for $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$, so that κ is κ-closed κ^+-sized faintly ∞-measurable. ■

For a slightly more fine-grained distinction let's define an intermediate large cardinal between the faintly and virtual.

[18] Recall that κ is **κ^{++}-tall** if there's an elementary embedding $j\colon V \to M$ with $\mathrm{crit}\, j = \kappa$, ${}^\kappa M \subseteq M$ and $j(\kappa) > \kappa^{++}$.

Definition 2.51. Let $\kappa < \theta$ be infinite regular cardinals. Say that κ is **faintly θ-power-Φ** for $\Phi \in \{\text{measurable}, \text{prestrong}, \text{strong}\}$ if it is faintly θ-Φ, witnessed by an embedding $\pi\colon H_\theta^V \to \mathcal{N}$, and $\mathscr{P}^V(\kappa) = \mathscr{P}^{\mathcal{N}}(\kappa)$. $\circ$

Note that the proof of Lemma 2.5 shows that faintly power-measurables are also 1-iterable and so in particular weakly compact. Our separation result is then the following.

Theorem 2.52 (Gitman-N.). *For $\Phi \in \{measurable, prestrong, strong\}$, if κ is virtually Φ then there exist forcing extensions $V[g]$ and $V[h]$ such that*

(i) In $V[g]$, κ is inaccessible and faintly Φ but not faintly power-Φ; and

(ii) In $V[h]$, κ is faintly power-Φ but not virtually Φ.

Proof. We start with (i). Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\lambda < \kappa$, and let $g \subseteq \mathbb{P}_\kappa$ be V-generic. Note that κ remains inaccessible in $V[g]$. Fix a regular $\theta > \kappa$ and let $\mathbb{Q}_\theta$ be a forcing witnessing that κ is virtually θ-measurable.

Since κ is *virtually* measurable we may without loss of generality assume that $\mathbb{Q}_\theta = \text{Col}(\omega, \theta)$ by applying Countable Embedding Absoluteness 1.41. Fixing a $V[g]$-generic $h \subseteq \mathbb{Q}_\theta$ we get a transitive $\mathcal{N} \subseteq V$ and in $V[h]$ an elementary embedding

$$\pi\colon H_\theta^V \to \mathcal{N}$$

with $\text{crit}\,\pi = \kappa$. Let's now work in $V[g][h] = V[h][g] = V[g \times h]$, in which we still have access to π. The lifting criterion[19] is trivial for $\mathbb{P}_\kappa$, so we get an $\mathcal{N}$-generic $\tilde{g} \subseteq \pi(\mathbb{P}_\kappa)$ and an elementary

$$\pi^+\colon H_\theta^{V[g]} \to \mathcal{N}[\tilde{g}]$$

with $\pi \subseteq \pi^+$. Note here that without loss of generality $\pi(\kappa)$ is countable as otherwise we replace $\mathcal{N}$ by a countable hull, so we can indeed construct such a $\tilde{g}$. By

[19] See Appendix 1.3 for the definition and characterisations of this criterion.

elementarity of π it holds that

$$\pi(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \prod_{\lambda\in[\kappa,\pi(\kappa))} \operatorname{Add}(\lambda, 1), \tag{1}$$

so that $\mathcal{N}[\tilde{g}] \not\subseteq V[g]$ as it in particular contains a new subset of κ. If $\Phi =$ measurable then we're done at this point. For $\Phi =$ prestrong we simply note that $g \in \mathcal{N}[\tilde{g}]$ by (1) so that $H_\theta^{V[g]} \subseteq \mathcal{N}[\tilde{g}]$ as well, and since π^+ lifts π it holds that $\pi^+(\kappa) = \pi(\kappa) > \theta$ in the $\Phi =$ strong case.

As for (ii), we simply change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$, which means that $\pi(\mathbb{P}_\kappa)$ doesn't add any subsets of κ and κ thus remains faintly power-Φ. By choosing $\theta > \kappa^+$ it *does* add a subset to κ^+ however, showing that κ is not virtually Φ. ■

Note however, that in contrast to the above separation result, Theorem 2.23 showed that the faintly-virtually distinction vanishes when we're dealing with Woodin cardinals.

Our next separation result is concerning the virtually prestrong and virtually strong cardinals.

Corollary 2.53 (N.). *There exists a virtually rank-into-rank cardinal iff there is an uncountable cardinal θ and a virtually θ-prestrong cardinal which is not virtually θ-strong.*

Proof. ($\Leftarrow$) is directly from the above Proposition 2.14 and Theorem 2.12.

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let (κ, θ) be the lexicographically least pair such that κ is virtually θ-rank-into-rank, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong then it would be Σ_2-reflecting, so that the statement that there exists a virtually rank-into-rank cardinal would reflect down to H_κ^V, contradicting the minimality of κ. ■

Figure 2.2 summarises the separation results along with the results from Section 2.4. Note that it *might* be the case that virtually θ-measurables are always virtually

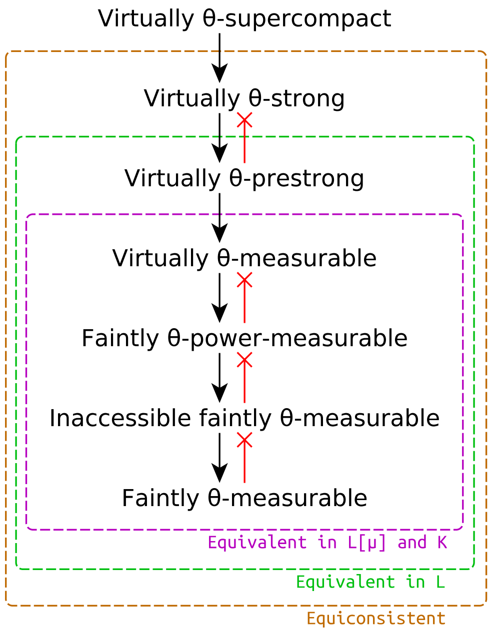

Figure 2.2: Direct implications between virtuals, where the red lines with crosses indicate that ZFC doesn't prove the reverse implication.

θ-prestrong (and hence also equivalent in $L[\mu]$ and K below a Woodin cardinal); see Question 5.5.

2.6 Indestructibility

It is well-known that supercompact cardinals κ can be made indestructible by all $<\kappa$-directed closed forcings by a suitable *Laver preparatory forcing*, which is the main theorem in the seminal paper [Laver, 1978]. A natural question, then, is whether similar results hold for the faintly and virtual versions. We noted in Proposition 2.5 that the virtuals are weakly compact, so the following theorem from [Jensen et al., 2009] shows that the consistency strength of indestructible virtual supercompacts is very large, potentially even in the realm of supercompacts themselves.

Theorem 2.54 (Schindler). *The consistency strength of a weakly compact cardinal κ which is indestructible by $<\kappa$-directed closed forcing is larger than the consis-*

tency strength of a proper class of strong cardinals and a proper class of Woodin cardinals.

This gets close to resolving the question about the indestructible virtuals, so what about the faintly supercompact cardinals? To make things a bit easier for ourselves, let us make the notion a bit stronger.

Definition 2.55. Fix uncountable cardinals $\kappa < \theta$. Then κ is **generically setwise θ-supercompact** if there exists a generic extension $V[g]$, a transitive $\mathcal{N} \in V[g]$ and a generic elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$, $\pi \in V[g]$, with $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and $V[g] \models {}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$. If it holds for all $\theta > \kappa$ then we say that κ is **generically setwise supercompact**. ∘

Note that the only difference between a generically setwise θ-supercompact cardinal and a virtually θ-supercompact cardinal is that the former is closed under sequences *in the generic extension*, where the latter is only closed under sequences in V; i.e., that $V \cap {}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$.

Ostensibly this seems to be an incredibly strong notion, as the target model now inherits a lot of structure from the generic extension. A first stab at an upper consistency bound could be to note that if there exists a proper class of Woodin cardinals then ω_1 is generically setwise supercompact. This can be shown using the countable stationary tower, see [Larson, 2004].

But, surprisingly, the following result from [Usuba, TBA] shows that they can exist in L.

Theorem 2.56 (Usuba). *If κ is virtually extendible then $\operatorname{Col}(\omega, {<}\kappa)$ forces that ω_1 is generically setwise supercompact.* ■

It turns out that this slightly stronger notion *does* have indestructiblity properties. We warm up by firstly showing that they are indestructible by small forcing.

Proposition 2.57 (N.-Schlicht). *Generic setwise supercompactness of κ is indestructible for forcing notions of size $< \kappa$.*

PROOF. Fix a forcing $\mathbb{P}$ of size $<\kappa$ and assume without loss of generality that $\mathbb{P} \in H_\kappa^V$, and fix also a cardinal $\theta > \kappa$. Using the setwise supercompactness of κ we may fix a forcing $\mathbb{Q}$ and a V-generic $h \subseteq \mathbb{Q}$ such that in $V[h]$ we have an elementary $\pi\colon M := H_\theta^V \to \mathcal{N}$ in $V[h]$ with $\mathcal{N}$ $<\theta$-closed.

Let $g \subseteq \mathbb{P}$ be $V[h]$-generic and work in $V[g \times h]$. By the lifting criterion we get a lift $\tilde{\pi}\colon M[g] \to \mathcal{N}[g]$ of π. If κ is a limit cardinal, then we may choose a cardinal $\lambda < \kappa$ such that $\mathbb{P} \in H_\lambda^V$. Since $\mathbb{P}$ has the λ^+-cc in V we get that $\pi(\mathbb{P}) = \mathbb{P}$ has the λ^+-cc in $\mathcal{N}$ and hence in $V[h]$ as well, making $\mathcal{N}[g]$ $<\theta$-closed by Lemma 1.42 and we're done.

If $\kappa = \nu^+$, then there are no cardinals between ν and $\pi(\kappa)$ in $\mathcal{N}$ and hence $|\theta| \leq \nu$. Thus it suffices to show that $\mathcal{N}[g]$ is ν-closed. Since $\pi(\mathbb{P}) = \mathbb{P}$ has size $\leq\nu$ in V, it has the $\nu^{+V[h]}$-cc in $V[h]$. Therefore $\mathcal{N}[g]$ is again $<\theta$-closed by Lemma 1.42. ■

Next, we show that these generic setwise supercompact cardinals κ *are* in fact indestructible for $<\kappa$-directed closed forcings, without having to do any preparation forcing at all.

THEOREM 2.58 (N.-Schlicht). *Generic setwise supercompactness of κ is indestructible for $<\kappa$-directed closed forcings.*

PROOF. Suppose that κ is generically setwise supercompact, $\mathbb{P}$ is a $<\kappa$-directed closed forcing and g is $\mathbb{P}$-generic over V. We'll show that κ is generically setwise supercompact in $V[g]$.

In V fix a regular $\theta > \kappa$ such that $\mathbb{P} \in H_\theta^V$, and let $\mathbb{Q}$ be the forcing given by the definition of setwise supercompactness. Let h be $\mathbb{Q}$-generic over $V[g]$. Let $\pi\colon H_\theta^V \to \mathcal{N}$ be as in the definition of generically setwise supercompactness, so that $\pi \in V[h] \subseteq V[g \times h]$. Work in $V[g \times h]$.

We may assume that $\theta = \theta^{<\theta}$ holds, as otherwise we just replace $\mathbb{Q}$ with $\mathbb{Q} * \mathrm{Col}(\theta, \theta^{<\theta})$ – we retain the $<\theta$-closure of $\mathcal{N}$ because $\mathrm{Col}(\theta, \theta^{<\theta})$ is $<\theta$-closed. We can further assume that $|\mathcal{N}| = \theta^{<\theta} = \theta$, as otherwise we can take a hull of $\mathcal{N}$ containing $\mathrm{ran}(\pi)$ and recursively close under $<\theta$-sequences, ending up with a $<\theta$-closed elementary substructure $\mathcal{H} \prec \mathcal{N}$ containing $\mathrm{ran}(\pi)$ – now replace $\mathcal{N}$ by the transitive collapse of $\mathcal{H}$.

Claim 2.59. There's a $\pi(\mathbb{P})$-generic filter $\tilde{g}$ over $\mathcal{N}$ that extends $\pi[g]$.

PROOF OF CLAIM. Since $\mathcal{N}$ is (in particular) $|\mathbb{P}|$-closed in $V[h]$ and $\mathbb{P}$ is trivially $|\mathbb{P}|^+$-cc, Lemma 1.42 implies that $\mathcal{N}$ is still $|\mathbb{P}|$-closed in $V[g\times h]$. As below, we thus still get that $\pi[g]\in\mathcal{N}$. Now work in $V[h]$, where we have full $<\theta$-closure of $\mathcal{N}$.

Since $\pi(\mathbb{P})$ is directed, there's a condition $q\leq\pi[g]$ in $\pi(\mathbb{P})$. Using the fact that $|\mathcal{N}|=\theta$ and $\pi(\mathbb{P})$ is $<\theta$-closed, we can construct a $\pi(\mathbb{P})$-generic filter $\tilde{g}$ over $\mathcal{N}$ with $q\in g$.[20] Then $\tilde{g}$ is as required. $\dashv$

Since we now have that $\pi[g]\subseteq\tilde{g}$ by the claim, the lifting criterion implies that we can lift π to $\tilde{\pi}\colon H_\theta^V[g]\to\mathcal{N}[\tilde{g}]$.

It thus remains to see that $\mathcal{N}[\tilde{g}]$ is $<\theta$-closed. To see this, take a sequence $\vec{x}=\langle x_i\mid i<\gamma\rangle$ with $\gamma<\theta$ and $x_i\in\mathcal{N}[\tilde{g}]$ and find names $\sigma_i\in\mathcal{N}$ with $\sigma_i^{\tilde{g}}=x_i$ for all $i<\gamma$. Since ${}^{<\theta}\mathcal{N}\subseteq\mathcal{N}$ we have that $\vec{\sigma}=\langle\sigma_i\mid i<\gamma\rangle\in\mathcal{N}$, and from $\vec{\sigma}$ we obtain a canonical name $\vec{\sigma}^\bullet\in\mathcal{N}$ with $\vec{\sigma}^{\bullet\tilde{g}}=\vec{x}\in\mathcal{N}[\tilde{g}]$. ■

Investigating further, we also show indestructibility for some forcings that do not fall into the above-mentioned categories.

PROPOSITION 2.60 (N.-Schlicht). *Generic setwise supercompactness of a regular cardinal κ is indestructible for* $\operatorname{Add}(\omega,\kappa)$. *If κ is a successor cardinal then it is also indestructible for* $\operatorname{Col}(\omega,<\kappa)$.

PROOF. Let g be $\operatorname{Add}(\omega,\kappa)$-generic over V. In V fix a regular $\theta>\kappa$ and let $\mathbb{Q}$ be the forcing given by the definition of generic setwise supercompactness. Let h be $\mathbb{Q}$-generic over $V[g]$ and work in $V[g\times h]$.

Let $\pi\colon H_\theta^V\to\mathcal{N}$ be as in the definition of generically setwise supercompactness. Moreover, let $\tilde{g}$ be $\operatorname{Add}(\omega,\pi(\kappa))$-generic over $V[g\times h]$. Since $\pi[g]=g$, the lifting criterion allows us to extend π to some $\tilde{\pi}\colon H_\theta^V[g]\to\mathcal{N}[g\times\tilde{g}]$. To show that $\mathcal{N}[g\times\tilde{g}]$ is $<\theta$-closed in $V[g\times h\times\tilde{g}]$, it suffices that $\operatorname{Add}(\omega,\pi(\kappa))$ has the ccc by Lemma 1.42.

[20] Namely, enumerate the dense subsets of $\pi(\mathbb{P})$ that are elements of $\mathcal{N}$ in order type θ and use the fact that the initial segments of the sequence, and of the corresponding sequence of conditions that we construct, are in $\mathcal{N}$.

For $\mathrm{Col}(\omega, {<}\kappa)$, we proceed similarly. Assume that $\kappa = \nu^+$. Take $\mathrm{Col}(\omega, {<}\kappa)$-, $\mathbb{Q}$- and $\mathrm{Col}(\omega, {<}\pi(\kappa))$-generic filters g, h and $\tilde{g}$. π and $\mathcal{N}$ are as above. Since $\nu < \kappa < \theta < \pi(\kappa)$ and there are no cardinals between ν and $\pi(\kappa)$ (in $\mathcal{N}$ and thus also in $V[h]$), ${<}\theta$-closure means ν-closure (in any model containing $V[h]$). By Lemma 1.42, it's thus sufficient to know that $\mathrm{Col}(\omega, {<}\pi(\kappa))$ has the ν^+-cc in $V[g \times h]$. This is because $\pi(\kappa) = \nu^{+\mathcal{N}} \leq \nu^{+V[g\times h]}$. ■

Usuba's Theorem 2.56 shows that the *consistency strength* of these generically setwise supercompact cardinals is small, but do they appear naturally anywhere? The following result shows that we cannot find any in neither L nor $L[\mu]$.

Proposition 2.61 (N.-Schlicht). *No cardinal κ is generically setwise supercompact in neither L nor $L[\mu]$ with μ being a normal ultrafilter.*

Proof. Assume first that $V = L$ and that κ is generically setwise supercompact. Let g be a generic filter and $\pi\colon L_\theta \to \mathcal{N}$ an embedding in $V[g]$ with $\pi \restriction L_{\kappa^{+L}} \in \mathcal{N}$. Then $\mathcal{N} = L_\alpha$ for some α by condensation and thus $\pi \restriction H_{\kappa^{+L}} \in L$. But this would induce a ${<}\kappa$-complete ultrafilter on κ, contradicting $V = L$.

Assume now that $V = L[\mu]$ and that κ is generically setwise supercompact, witnessed by a generic embedding $\pi\colon L_\theta[\mu] \to L_\alpha[\bar{\mu}]$. In particular this means that $\pi \restriction L_{\kappa^{+L[\mu]}}[\mu] \in L_\alpha[\bar{\mu}]$. If $\operatorname{crit}\mu < \kappa$ then $\mu = \bar{\mu}$ and $\mathscr{P}^{L[\mu]}(\kappa) = \mathscr{P}^{L[\bar{\mu}]}$, so that both $\pi(\kappa)$ and κ are now measurable cardinals in $L[\bar{\mu}]$, contradicting [Kanamori, 2008, Lemma 20.2]. So $\operatorname{crit}\mu \geq \kappa$.

If $\pi(\operatorname{crit}\mu) > \operatorname{crit}\mu$ then by [Kanamori, 2008, Theorem 20.12] we get that $L[\bar{\mu}]$ is an iterate of $L[\mu]$. But iteration embeddings preserve the subsets of their critical point, so again we have that $\mathscr{P}^{L[\mu]}(\kappa) = \mathscr{P}^{L[\bar{\mu}]}$ and we get the same contradiction as before.

Lastly, if $\operatorname{crit}\mu > \kappa$ and $\pi(\operatorname{crit}\mu) = \operatorname{crit}\mu$ then $\mu = \bar{\mu}$ by [Kanamori, 2008, Theorem 20.10], so we get a contradiction as in the $\operatorname{crit}\mu < \kappa$ ■

3 | Filters & Games

Moving away from the pure theory of the virtual large cardinals from Chapter 2, we now move to connections between these large cardinals and common set-theoretic objects of study. In this chapter those objects are filters and games, with the next chapter dealing with connections with ideals. This chapter covers the content in the paper [Nielsen and Welch, 2019], which started out as a further analysis of the results in [Holy and Schlicht, 2018] and somewhat surprisingly we ended up in the realm of virtual large cardinals. As is custom in mathematics, we will pretend that this was the goal all along.

We will in this section be dealing with many properties of $\mathcal{M}$-measures[1], so we start with a couple of definitions.

Definition 3.1. Let κ be a cardinal, $\mathcal{M}$ a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

- $\mathcal{M}$**-normal** if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \triangle \vec{X} \in \mu$;
- **genuine** if $|\triangle \vec{X}| = \kappa$ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- **normal** if $\triangle \vec{X}$ is stationary in κ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- 0-**good**, or simply **good**, if it has a well-founded ultrapower;
- α-**good** for $\alpha > 0$ if it is weakly amenable and has α-many well-founded iterates.

$\circ$

Note that a genuine $\mathcal{M}$-measure is $\mathcal{M}$-normal and countably complete, and a countably complete weakly amenable $\mathcal{M}$-measure is α-good for all ordinals α.

We will also be employing the following well-known result regarding set-sized embeddings.

[1] See the beginning of Chapter 2 for the definitions of weak κ-models $\mathcal{M}$ and their associated $\mathcal{M}$-measures.

Lemma 3.2 (Ancient Kunen Lemma). *Let κ be regular, $\mathcal{M},\mathcal{N}$ weak κ-models, $\theta\in(\kappa,o(\mathcal{M}))$ a regular $\mathcal{M}$-cardinal, and $\pi\colon\mathcal{M}\to\mathcal{N}$ an elementary embedding with $\operatorname{crit}\pi=\kappa$ and $H_\theta^{\mathcal{M}}\subseteq\mathcal{N}$. Then for every $X\in H_\theta^{\mathcal{M}}$ with $\operatorname{card}^{\mathcal{M}}(X)=\kappa$ it holds that $\pi\restriction X\in\mathcal{N}$.*

Proof. Let $f\colon\kappa\to X$, $f\in\mathcal{M}$, be a bijection and note that $\pi(x)=\pi(f)(f^{-1}(x))$ for all $x\in X$, so it suffices that $f,\pi(f)\in\mathcal{N}$, which is true since $f\in H_\theta^{\mathcal{M}}\subseteq\mathcal{N}$. ■

In [Holy and Schlicht, 2018] they provide the following characterisation of the normal measures.

Lemma 3.3 (Holy-Schlicht). *Let $\mathcal{M}$ be a weak κ-model and μ and $\mathcal{M}$-measure. Then μ is normal iff $\triangle\vec{X}$ is stationary for some enumeration $\vec{X}$ of μ.*

Proof. $(\Rightarrow)$ is trivial, so assume that $\vec{X}$ is an enumeration of μ such that $\triangle\vec{X}$ is stationary. Let $\vec{Y}\in{}^\kappa\mu$ be a κ sequence and define $g\colon\kappa\to\kappa$ such that $Y_\alpha=X_{g(\alpha)}$ for $\alpha<\kappa$. Letting $C_g\subseteq\kappa$ be the club of closure points of g we get that $\triangle\vec{X}\cap C_g\subseteq\triangle\vec{Y}\cap C_g$, making $\triangle\vec{Y}$ stationary. ■

The α-Ramsey cardinals in [Holy and Schlicht, 2018] are based upon the following game.

Definition 3.4 (Holy-Schlicht). For an uncountable cardinal $\kappa=\kappa^{<\kappa}$, a limit ordinal $\gamma\leq\kappa$ and a regular cardinal $\theta>\kappa$ define the game $wfG_\gamma^\theta(\kappa)$ of length γ as follows.

$$\begin{array}{lccccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \text{II} & & \mu_0 & & \mu_1 & & \mu_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha\prec H_\theta$ is a κ-model and μ_α is a filter for all $\alpha<\gamma$, such that μ_α is an $\mathcal{M}_\alpha$-measure, the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing and $\langle\mathcal{M}_\xi\mid\xi<\alpha\rangle,\langle\mu_\xi\mid\xi<\alpha\rangle\in\mathcal{M}_\alpha$ for every $\alpha<\gamma$. Letting $\mu:=\bigcup_{\alpha<\gamma}\mu_\alpha$ and $\mathcal{M}:=\bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$, player II wins iff μ is an $\mathcal{M}$-normal good $\mathcal{M}$-measure. ○

We will also be using the following fact from [Holy and Schlicht, 2018, Lemma 3.3], that the games $wfG^\theta_\gamma(\kappa)$ do not depend upon the values of θ:

Lemma 3.5 (Holy-Schlicht). *Let γ be a limit ordinal with* $\operatorname{cof}\gamma \neq \omega$. *Then $wfG^{\theta_0}_\gamma(\kappa)$ and $wfG^{\theta_1}_\gamma(\kappa)$ are equivalent for any regular $\theta_0, \theta_1 > \kappa$.* ■

We will be working with a variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I but more of player II. It will turn out that this change of game is innocuous, as Proposition 3.9 will show that they are equivalent.

Definition 3.6 (Holy-N.-Schlicht). Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following game $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ with $(\gamma{+}1)$-many rounds:

$$\begin{array}{cccccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & & \mathcal{M}_\gamma & \\ \mathrm{II} & & \mu_0 & & \mu_1 & & \cdots & & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leq \gamma$, μ_α is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$, μ_γ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure and the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore require that $\mathcal{M}_\alpha = \bigcup_{\xi<\alpha} \mathcal{M}_\xi$, $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$ and that μ_α is ζ-good. Player II wins iff they could continue to play throughout all $(\gamma{+}1)$-many rounds. ○

For convenience we will write $\mathcal{G}^\theta_\gamma(\kappa)$ for the game $\mathcal{G}^\theta_\gamma(\kappa, 0)$, and $\mathcal{G}_\gamma(\kappa)$ for $\mathcal{G}^\theta_\gamma(\kappa)$ whenever $\operatorname{cof}\gamma \neq \omega$, as again the existence of winning strategies in these games doesn't depend upon a specific θ. Note that we assume that $\kappa = \kappa^{<\kappa}$ is uncountable in the definition of the games that we're considering, so this is a standing assumption throughout this chapter, whenever any one of the above two games are considered.

Definition 3.7. Define the **Cohen game** $\mathcal{C}^\theta_\gamma(\kappa)$ as $\mathcal{G}^\theta_\gamma(\kappa)$ but where we require that $|\mathcal{M}_\alpha - H_\kappa| < \gamma$ for every $\alpha < \gamma$, i.e. that we only allow player I to add $<\gamma$ new elements to the models in each round, and where we only require $\mathcal{M}_\alpha \models \mathsf{ZFC}^-$ and $\mathcal{M}_\alpha \prec H_\theta$ for $\alpha \leq \gamma$ limit.[2]

Also define the **weak Cohen game** $\mathcal{C}^-_\gamma(\kappa)$ in analogy with $\mathcal{G}^-_\gamma(\kappa)$. ○

[2] $\mathcal{C}^\theta_\omega(\kappa)$ is similar to the $H(F, \lambda)$-games in [Donder and Levinski, 1989].

Proposition 3.8 (N.). *Assume $\gamma^{\aleph_0} = \gamma$ and let κ be regular. Then $\mathcal{C}^-_\gamma(\kappa)$ is equivalent to $\mathcal{C}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. In particular, if CH holds then $\mathcal{C}^-_{\omega_1}(\kappa)$ is equivalent to $\mathcal{C}^\theta_{\omega_1}(\kappa)$ for all regular $\theta > \kappa$.*

Proof. The assumption that $\gamma^{\aleph_0} = \gamma$ allows us to ensure that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ for all $\alpha < \gamma$. If player I has a winning strategy in $\mathcal{C}^\theta_\gamma(\kappa)$ for some regular $\theta > \kappa$ then they still win if we require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ (since they're only enlargening their models, making it even harder for player II to win), in which case the final measure μ_γ is countably complete and hence automatically has a wellfounded ultrapower.

If player II has a winning strategy in $\mathcal{C}^-_\gamma(\kappa)$ then they still win if player I plays $\mathcal{M}_\alpha$ such that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$, again ensuring that μ_γ has a wellfounded ultrapower. ■

Proposition 3.9 (Holy-N.-Schlicht). *$\mathcal{G}^\theta_\gamma(\kappa)$, $\mathcal{G}^\theta_\gamma(\kappa, 1)$ and $wfG^\theta_\gamma(\kappa)$ are all equivalent for all limit ordinals $\gamma \leq \kappa$, and $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$ whenever* $\operatorname{cof}\gamma > \omega$ *and $\zeta \in On$.*

Proof. We start by showing the latter statement, so assume that $\operatorname{cof}\gamma > \omega$. Consider now the auxilliary game, call it $\mathcal{G}$, which is exactly like $\mathcal{G}^\theta_\gamma(\kappa, 0)$, but where we also require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ and $\langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle, \langle \mu_\xi \mid \xi \leq \alpha\rangle \in \mathcal{M}_{\alpha+1}$ for every $\alpha < \gamma$.

Claim 3.10. $\mathcal{G}$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$.

Proof of claim. If player I has a winning strategy in $\mathcal{G}$ then they also have one in $\mathcal{G}^\theta_\gamma(\kappa)$, by doing exactly the same. Analogously, if player II has a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have one in $\mathcal{G}$. If player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ then we can construct a winning strategy σ' in $\mathcal{G}$, which is defined as follows. Fix some $\alpha \leq \gamma$ and, writing $\vec{\mathcal{M}}_\xi := \langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle$ and $\vec{\mu}_\xi := \langle \mu_\xi \mid \xi \leq \alpha\rangle$, we set

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) := \operatorname{Hull}^{H_\theta}(\sigma(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) \cup {}^\omega\mathcal{M}_\alpha \cup \{\vec{\mathcal{M}}_\xi, \vec{\mu}_\xi\}),$$

i.e. that we're simply throwing in the sequences into our models and making sure that we're still an elementary substructure of H_θ. This new strategy σ' is clearly winning. Assuming now that τ is a winning strategy for player II in $\mathcal{G}$, we define a winning strategy τ' for player II in $\mathcal{G}_\gamma^\theta(\kappa)$ by letting $\tau'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle)$ be the result of throwing in the appropriate sequences into the models $\mathcal{M}_\xi$, applying τ to get a measure, and intersecting that measure with $\mathcal{M}_\alpha$ to get an $\mathcal{M}_\alpha$-measure. $\dashv$

Now, letting $\mathcal{M}_\gamma$ be the final model of a play of $\mathcal{G}$, $\operatorname{cof}\gamma > \omega$ implies that any ω-sequence $\vec{X} \in \mathcal{M}_\gamma$ really is a sequence of elements from some $\mathcal{M}_\xi$ for $\xi < \gamma$, so that $\vec{X} \in \mathcal{M}_{\xi+1}$ by definition of $\mathcal{G}$, making $\mathcal{M}_\gamma$ closed under ω-sequences and thus also μ_γ countably complete. Since γ is a limit ordinal and the models contain the previous measures and models as elements, the proof of e.g. Theorem 5.6 in [Holy and Schlicht, 2018] shows that μ_γ is also weakly amenable, making it ζ-good for all ordinals ζ.

Now we deal with the first statement, so fix a limit ordinal γ. Firstly $\mathcal{G}_\gamma^\theta(\kappa)$ is equivalent to $\mathcal{G}_\gamma^\theta(\kappa, 1)$ as above, since both are equivalent to the auxilliary game $\mathcal{G}$ when γ is a limit ordinal. So it remains to show that $\mathcal{G}_\gamma^\theta(\kappa)$ is equivalent to $wfG_\gamma^\theta(\kappa)$. If player I has a winning strategy σ in $wfG_\gamma^\theta(\kappa)$ then define a winning strategy σ' for player I in $\mathcal{G}_\gamma^\theta(\kappa)$ as

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) := \sigma(\langle \mathcal{M}_0, \mu_0\rangle^\frown\langle \mathcal{M}_{\xi+1}, \mu_{\xi+1} \mid \xi + 1 \leq \alpha\rangle)$$

and for limit ordinals $\alpha \leq \gamma$ set $\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi < \alpha\rangle) := \bigcup_{\xi<\alpha} \mathcal{M}_\xi$; i.e. they simply follow the same strategy as in $wfG_\gamma^\theta(\kappa)$ but plugs in unions at limit stages. Likewise, if player II had a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ then they also have a winning strategy in $wfG_\gamma^\theta(\kappa)$, this time just by skipping the limit steps in $\mathcal{G}_\gamma^\theta(\kappa)$.

Now assume that player I has a winning strategy σ in $\mathcal{G}_\gamma^\theta(\kappa)$ and that player I *doesn't* have a winning strategy in $wfG_\gamma^\theta(\kappa)$. Then define a strategy σ' for player I in $wfG_\gamma^\theta(\kappa)$ as follows. Let $s = \langle \mathcal{M}_\alpha, \mu_\alpha \mid \alpha \leq \eta\rangle$ be a partial play of $wfG_\gamma^\theta(\kappa)$ and let s' be the modified version of s in which we have 'inserted' unions at limit steps, just as in the above paragraph. We can assume that every μ_α in s' is good and $\mathcal{M}_\alpha$-normal as otherwise player II has already lost and player I can play anything.

Now, we want to show that s' is a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$. All the models in s are κ-models, so in particular weak κ-models.

Claim 3.11. Every μ_α in s' is normal.

PROOF OF CLAIM. Assume without loss of generality that $\alpha = \eta$. Let player I play any legal response $\mathcal{M}$ to s in $wfG^\theta_\gamma(\kappa)$ (such a response always exists). If player II can't respond then player I has a winning strategy by simply following $s^\frown\langle\mathcal{M}\rangle$, $\lightning$, so player II *does* have a response μ to $s^\frown\mathcal{M}$. But now the rules of $wfG^\theta_\gamma(\kappa)$ ensures that $\mu_\eta \in \mathcal{M}$, so since

$$(\mathcal{M}, \in, \mu) \models \forall\vec{X} \in {}^\kappa\mu : \ulcorner\triangle\vec{X} \text{ is stationary in } \kappa\urcorner,$$

we then also get that $\mathcal{M} \models \ulcorner\triangle\mu_\eta$ is stationary in $\kappa\urcorner$ since $\mu_\eta \subseteq \mu$, so elementarity of $\mathcal{M}$ in H_θ implies that $\triangle\mu_\eta$ really *is* stationary in κ, making μ_η normal. $\dashv$

This makes s' a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$, so we may form the weak κ-model $\tilde{\mathcal{M}}_\eta := \sigma(s')$. Now let $\mathcal{M}_\eta \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_\eta \subseteq \mathcal{M}_\eta$ and $s \in \mathcal{M}_\eta$ and set $\sigma'(s) := \mathcal{M}_\eta$. This defines the strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$, which is winning since the winning condition for the two games is the same for γ a limit.[3]

Next, assume that player II has a winning strategy τ in $wfG^\theta_\gamma(\kappa)$. We recursively define a strategy $\tilde{\tau}$ for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ as follows. If $\tilde{\mathcal{M}}_0$ is the first move by player I in $\mathcal{G}^\theta_\gamma(\kappa)$, let $\mathcal{M}_0 \prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_0 \subseteq \mathcal{M}_0$, making $\mathcal{M}_0$ a valid move for player I in $wfG^\theta_\gamma(\kappa)$. Write $\mu_0 := \tau(\langle\mathcal{M}_0\rangle)$ and then set $\tilde{\tau}(\langle\tilde{\mathcal{M}}_0\rangle)$ to be $\tilde{\mu}_0 := \mu_0 \cap \tilde{\mathcal{M}}_0$, which again is normal by the same trick as above, making $\tilde{\mu}_0$ a legal move for player II in $\mathcal{G}^\theta_\gamma(\kappa)$. Successor stages $\alpha + 1$ in the construction are analogous, but we also make sure that $\langle\mathcal{M}_\xi \mid \xi < \alpha+1\rangle, \langle\mu_\xi \mid \xi < \alpha+1\rangle \in \mathcal{M}_{\alpha+1}$. At limit stages τ outputs unions, as is required by the rules of $\mathcal{G}^\theta_\gamma(\kappa)$. Since the union of all the μ_α's is good as τ is winning, $\tilde{\mu}_\gamma := \bigcup_{\alpha<\gamma}\tilde{\mu}_\alpha$ is good as well, making $\tilde{\tau}$ winning and we are done. ■

[3]More precisely, that σ is winning in $\mathcal{G}^\theta_\gamma(\kappa)$ means that there's a sequence $\langle f_n : \kappa \to \kappa \mid n < \omega\rangle$ with the f_n's all being elements of the last model $\tilde{\mathcal{M}}_\gamma$, witnessing the illfoundedness of the ultrapower. But then all these functions will also be elements of the union of the $\mathcal{M}_\alpha$'s, since we ensured that $\mathcal{M}_\alpha \supseteq \tilde{\mathcal{M}}_\alpha$ in the construction above, making the ultrapower of $\bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$ by $\bigcup_{\alpha<\gamma}\mu_\alpha$ illfounded as well.

We now arrive at the definitions of the cardinals we will be considering. They were in [Holy and Schlicht, 2018] only defined for γ being a cardinal, but given the above result we generalise it to all ordinals γ.

Definition 3.12. Let κ be a cardinal and $\gamma \leq \kappa$ an ordinal. Then κ is **γ-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. We furthermore say that κ is **strategic γ-Ramsey** if player II *does* have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$.

Define **(strategic) genuine γ-Ramseys** and **(strategic) normal γ-Ramseys** analogously, but where we require the last measure μ_γ to be genuine and normal, respectively. ○

Definition 3.13 (N.). A cardinal κ is **$<\gamma$-Ramsey** if it is α-Ramsey for every $\alpha < \gamma$, **almost fully Ramsey** if it is $<\kappa$-Ramsey and **fully Ramsey** if it is κ-Ramsey.

Further, say that κ is **coherent $<\gamma$-Ramsey** if it's strategic α-Ramsey for every $\alpha < \gamma$ and that there exists a choice of winning strategies τ_α in $\mathcal{G}_\alpha(\kappa)$ for player II satisfying that $\tau_\alpha \subseteq \tau_\beta$ whenever $\alpha < \beta$. In other words, there is a single strategy τ for player II in $\mathcal{G}_\gamma(\kappa)$ such that τ is a winning strategy for player II in $\mathcal{G}_\alpha(\kappa)$ for every $\alpha < \gamma$.[4] ○

This is not the original definition of (strategic) γ-Ramsey cardinals however, as this involved elementary embeddings between weak κ-models – but as the following theorem of [Holy and Schlicht, 2018] shows, the two definitions coincide whenever γ is a regular cardinal.

Theorem 3.14 (Holy-Schlicht). *For regular cardinals λ, a cardinal κ is λ-Ramsey iff for arbitrarily large $\theta > \kappa$ and every $A \subseteq \kappa$ there is a weak κ-model $\mathcal{M} \prec H_\theta$ with $\mathcal{M}^{<\lambda} \subseteq \mathcal{M}$ and $A \in \mathcal{M}$ with an $\mathcal{M}$-normal 1-good $\mathcal{M}$-measure μ on κ.* ■

[4] Note that, with this terminology, "coherent" is a stronger notion than "strategic". We could've called the cardinals *coherent strategic $<\gamma$-Ramseys*, but we opted for brevity instead.

3.1 The finite case

In this section we are going to consider properties of the n-Ramsey cardinals for finite n. Note in particular that the $\mathcal{G}_n^\theta(\kappa)$ games are determined, making the "strategic" adjective superfluous in this case. We further note that the θ's are also dispensible in this finite case:

Proposition 3.15 (N.). *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n \prec H_\theta$.*

Proof. $\Leftarrow$ is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays $\mathcal{M}_k$ in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \text{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of $\mathcal{M}_k$, and play $\mathcal{M}_k^*$ in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ-responses to the $\mathcal{M}_k^*$'s and let player II play the μ_k's in $\mathcal{G}_n(\kappa)$ as well.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\text{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω-sequence $\vec{f} := \langle f_k \mid k < \omega \rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \text{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\text{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \to H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \operatorname{ran} \pi$.

Now $\bar{A} = A$ for every $A \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the $\bar{f}_k$'s witness that $\text{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\text{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \text{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ's anymore we note that n-Ramseyness can now be described using a Π^1_{2n+2}-formula and normal n-Ramseyness using a Π^1_{2n+3}-formula.

We have the following characterisations, as proven in [Abramson et al., 1977].

Theorem 3.16 (Abramson et al.). *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then*

(i) κ is weakly compact if and only if it is 0-Ramsey;

(ii) κ is weakly ineffable if and only if it is genuine 0-Ramsey;

(iii) *κ is ineffable if and only if it is normal* 0*-Ramsey.*

PROOF. This is mostly just changing the terminology in [Abramson et al., 1977] to the current game-theoretic one, so we only show (i).

Theorem 1.1.3 in [Abramson et al., 1977] shows that κ is weakly compact if and only if every κ-sized collection of subsets of κ is measured by a $<\kappa$-complete measure, in the sense that every $<\kappa$-sequence (in V) of measure one sets has non-empty intersection.

For the $\Rightarrow$ direction we can let player II respond to any $\mathcal{M}_0$ by first getting the $<\kappa$-complete $\mathcal{M}_0$-measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \to \mathrm{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π, which is $\mathcal{M}_0$-normal and good. For $\Leftarrow$, if $X \subseteq \mathscr{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ-model $\mathcal{M}_0 \prec H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play $\mathcal{M}_0$ in $\mathcal{G}_0(\kappa)$ we get some $\mathcal{M}_0$-normal good $\mathcal{M}_0$-measure μ_0 on κ. Since $\mathcal{M}_0$ is closed under $<\kappa$-sequences we get that μ_0 is $<\kappa$-complete. ■

3.1.1 Indescribability

In this section we aim to prove that n-Ramseys are Π^1_{2n+1}-indescribable and that normal n-Ramseys are Π^1_{2n+2}-indescribable, which will also establish that the hierarchy of alternating n-Ramseys and normal n-Ramseys forms a strict hierarchy. Recall the following definition.

DEFINITION 3.17. A cardinal κ is **Π^1_n-indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ∘

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π^1_1-indescribable – see [Abramson et al., 1977].

THEOREM 3.18 (N.). *Every n-Ramsey κ is Π^1_{2n+1}-indescribable for $n < \omega$.*

PROOF. Let κ be n-Ramsey and assume that it is not Π^1_{2n+1}-indescribable, witnessed by a Π_{2n+1}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$

and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$-many quantifiers occuring in φ in $(n+1)$-many steps. We will here describe the first two steps with the remaining steps following the same pattern.

$\boxed{\textbf{First step.}}$ Write $\varphi(v) \equiv \forall v_1 \psi(v, v_1)$ for a Σ_{2n}-formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A^{(0)}_\alpha \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let $\mathcal{M}_0$ be a weak κ-model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good $\mathcal{M}_0$-normal $\mathcal{M}_0$-measure μ_0 on κ, using the 0-Ramseyness of κ. Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \mathrm{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. $\mathcal{M}_0$-normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, $\mathcal{M}_0$-normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}] \urcorner. \tag{1}$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0-formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

$\boxed{\textbf{Second step.}}$ Write $\psi(v, v_1) \equiv \exists v_2 \forall v_3 \chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$-formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3 \chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \tag{2}$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, v_3]. \tag{3}$$

Fix witnesses $A^{(1)}_\alpha \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S^{(0)}_\alpha := \{\xi < \kappa \mid A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S^{(0)}_\alpha \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and $\mathcal{M}_0$-normality of μ_0 then implies that $S^{(0)}_\alpha \in \mu_0$ is equivalent to

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ-model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an $\mathcal{M}_1$-normal $\mathcal{M}_1$-measure on κ, using the 1-Ramseyness of κ, so that $\mathcal{M}_1$-normality of μ_1 yields that $\triangle\vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle\vec{S}^{(0)}$ if and only if $A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A^{(0)}_\xi = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \mathrm{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A^{(1)}_\alpha$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha].$$

Now this, paired with the above observation regarding $\triangle\vec{S}^{(0)}$, means that for every $\alpha \in \triangle\vec{S}^{(0)} \cap \mathrm{Lim}$ we have that

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha],$$

so that $\mathcal{M}_1$-normality of μ_1 and Łoś' lemma implies that

$$\mathrm{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg\chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]\urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$-many steps, ending with a Δ_0-formula $\phi(v, v_1, \dots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\mathrm{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]\urcorner$. But now absoluteness of $\neg\phi$ means that $V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]$, contradicting (4).

■

Note that this is optimal, as n-Ramseyness can be described by a Π^1_{2n+2}-formula. As a corollary we then immediately get the following.

Corollary 3.19 (N.). *Every $<\omega$-Ramsey cardinal is Δ^2_0-indescribable.* ■

The second indescribability result concerns the normal n-Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π^1_2-indescribable – see [Abramson et al., 1977].

Theorem 3.20 (N.). *Every normal n-Ramsey κ is Π^1_{2n+2}-indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 3.18, as we would end up with a Π^1_1 statement in an ultrapower, and as Π^1_1 statements are not upwards absolute in general we would not be able to get our contradiction.

Proof. Let κ be normal n-Ramsey and assume that it is not Π^1_{2n+2}-indescribable, witnessed by a Π_{2n+2}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n-Ramsey to perform the same $n+1$ steps as in the proof of Theorem 3.18. This gives us a Σ_1-formula $\phi(v, v_1, \ldots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \cdots, \mathcal{A}^{(n)} \rangle$, $\langle B^{(0)}, \ldots, B^{(n-1)} \rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n \rangle$ of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \ldots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{1}$$

and, for μ_n-many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \ldots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, A^{(n)}_\alpha].$$

Now form $S^{(n)}_\alpha \in \mu_n$ as in the proof of Theorem 3.18. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 3.18 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for $k < n$), but we can now use normality[5] of μ_n to ensure that we *do* have that $\triangle\vec{S}^{(n)}$ is stationary in κ. This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \ldots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \quad (2)$$

Now note that since κ is inaccessible it is Σ^1_1-indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of [Abramson et al., 1977] shows that the set of reflection points of Σ^1_1-formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \ldots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n-Ramseyness can be described by a Π^1_{2n+3}-formula. In particular this then means that every $(n+1)$-Ramsey is a normal n-Ramsey stationary limit of normal n-Ramseys, and every normal n-Ramsey is an n-Ramsey stationary limit of n-Ramseys, making the hierarchy of alternating n-Ramseys and normal n-Ramseys a strict hierarchy.

3.1.2 Downwards absoluteness to L

Our absoluteness result below, Theorem 3.22, is inspired by arguments in [Abramson et al., 1977], and uses the following lemma from that paper.

Lemma 3.21 (Abramson et al). *There is a Π^1_1 formula $\varphi(A)$ such that, for any ordinal α, $(V_\alpha, V_{\alpha+1}) \models \varphi[A]$ iff α is a regular cardinal and A is a non-constructible subset of α.*[6] ■

Theorem 3.22 (N.). *Genuine- and normal n-Ramseys are downwards absolute to L, for every $n < \omega$.*

Proof. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ-model – we want to find a genuine $\mathcal{M}$-measure inside L. By

[5]Recall that this is stronger than just requiring it to be $\mathcal{M}_n$-normal – we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.
[6]This appears as Lemma 4.1.2 in [Abramson et al., 1977].

assumption we *can* find such a measure μ in V; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa \rangle \in L$ of $\mathscr{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

Claim 3.23. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

Proof of claim. Let $\vec{B}$ be the μ-**positive part** of $\vec{A}$, meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle\vec{B}$ has size κ, so we can pick $\delta \in \triangle\vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L. ⊣

Now let φ be the Π^1_1 formula given by Lemma 3.21. If we therefore assume that $T \notin L$ then $(V_\kappa, V_{\kappa+1}) \models \varphi[T]$, which by Π^1_1-indescribability of κ means that there exists some $\alpha < \kappa$ such that $(V_\alpha, V_{\alpha+1}) \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle\mu)^L = \triangle\mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V. The cases where κ is a genuine- or normal n-Ramsey cardinal is analogous. ■

Since $(n+1)$-Ramseys are normal n-Ramseys we then immediately get the following.

Corollary 3.24 (N.). *Every $(n+1)$-Ramsey is normal n-Ramsey in L, for every $n < \omega$. In particular, $<\omega$-Ramseys are downwards absolute to L.* ■

3.1.3 Complete ineffability

In this subsection we provide a characterisation of the *completely ineffable* cardinals[7] in terms of the α-Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$-Ramsey cardinals, namely the *coherent $<\omega$-Ramseys* as defined in 3.13. Note that a coherent $<\omega$-Ramsey is precisely a cardinal satisfying the ω-filter property, as defined in [Holy and Schlicht, 2018].

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is closely related to the proof of Theorem 3.20 (the indescribability of normal n-Ramseys), but the main difference

[7] See Appendix 1.1 for a definition of the completely ineffable cardinals.

is that we want everything to occur locally inside our weak κ-models. We'll need another lemma from [Abramson et al., 1977].

Lemma 3.25 (Abramson et al). *Let κ be inaccessible, $X \subseteq \kappa$ and φ a Σ^1_1-formula such that $(V_\kappa, \in, X) \models \varphi[X]$. Then*

$$\{\alpha < \kappa \mid (V_\alpha, \in, X \cap V_\alpha) \models \varphi[X \cap V_\alpha]\}$$

is a club. ■

Theorem 3.26 (N.). *Every coherent $<\omega$-Ramsey is a stationary limit of $<\omega$-Ramseys.*

Proof. Let κ be coherent $<\omega$-Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 \prec H_\theta$ be a weak κ-model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ-model $\mathcal{M} \prec H_\theta$ with an $\mathcal{M}$-normal $\mathcal{M}$-measure $\mu := \bigcup_{n<\omega} \mu_n$ on κ.

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } <\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by $\mathcal{M}$-normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k+1$ steps as in the proof of Theorem 3.20 with $\varphi(\xi)$ being $\ulcorner\xi \text{ is } k\text{-Ramsey}\urcorner$, so that we get a weak κ-model $\mathcal{M}_{k+1} \prec H_\theta$, an $\mathcal{M}_{k+1}$-normal $\mathcal{M}_{k+1}$-measure $\tilde{\mu}_{k+1}$ on κ, a Σ_1-formula $\varphi(v, v_1, v_2, \dots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \dots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \dots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \tag{2}$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg\varphi[\xi, A^{(0)}_\xi, B^{(0)} \cap V_\xi, A^{(1)}_\xi, B^{(1)} \cap V_\xi, \dots, A^{(k-1)}_\xi, B^{(k-1)} \cap V_\xi, A^{(k)}_\xi], \tag{3}$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \mathrm{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 3.18.

Since κ in particular is Σ^1_1-indescribable, Lemma 3.25 implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ-model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n+1)$-steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of $\mathcal{M}_{k+1}$. In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ, $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A^{(i)}_\xi \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa\rangle$ and note that since $\vec{S}$ is definable it is an element of $\mathcal{M}_{k+2}$ as well. Then $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $\triangle\vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \triangle\vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \triangle\vec{S}$ we see, as in the proof of Theorem 3.18, that

$$V_{\zeta+1} \models \varphi[\zeta, A^{(0)}_\zeta, B^{(0)} \cap V_\zeta, A^{(1)}_\zeta, B^{(1)} \cap V_\zeta, \dots, A^{(k)}_\zeta]$$

since $\triangle\vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} \prec H_\theta$ we have that $\mathcal{M}$ is correct about stationary subsets of κ, meaning that κ is a stationary limit of $<\omega$-Ramseys. ■

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

Definition 3.27. A collection $R \subseteq \mathscr{P}(\kappa)$ is a **stationary class** if

(i) $R \neq \emptyset$;
(ii) every $A \in R$ is stationary in κ;
(iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

○

Definition 3.28. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$ there is an $H \in R$ homogeneous for f. ○

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in [Abramson et al., 1977].

Theorem 3.29 (N.). *A cardinal κ is completely ineffable if and only if it is coherent $<\omega$-Ramsey.*

Proof. ($\Leftarrow$): Assume κ is coherent $<\omega$-Ramsey, witnessed by strategies $\langle\tau_n \mid n < \omega\rangle$. Let $f : [\kappa]^2 \to 2$ be arbitrary and form the sequence $\langle A^f_\alpha \mid \alpha < \kappa\rangle$ as

$$A^f_\alpha := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let $\mathcal{M}_f$ be a transitive weak κ-model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated $\mathcal{M}_f$-measure on κ given by τ_0.[8] 1-Ramseyness of κ ensures that μ_f is normal, meaning $\triangle\mu_f$ is stationary in κ. Define a new sequence $\vec{B}^f$ as the μ_f-positive part of $\vec{A}^f$.[9] Then $B^f_\alpha \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\triangle\vec{B}^f$ is stationary.

Let now $\mathcal{M}'_f$ be a new transitive weak κ-model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an $\mathcal{M}'_f$-measure $\mu'_f \supseteq \mu_f$ on κ. Then $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \in \mu_f\}$ and $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \notin \mu_f\}$ are both elements of $\mathcal{M}'_f$, so one of them is in μ'_f; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f.

Now let $g : [H_f]^2 \to 2$ be arbitrary and again form

$$A^g_\alpha := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ-model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$-measure $\mu_{f,g} \supseteq \mu'_f$ on κ. As before we then get a stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g. We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists\vec{f} : H_{\vec{f}} \subseteq A\},$$

where the $\vec{f}$'s range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \to 2$ and $f_{k+1} : [H_{f_k}] \to 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \to 2$, we can find $H \in R$ which is homogeneous

[8] Technically we would have to require that $\mathcal{M}_f \prec H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0, but note that we could simply get a measure on $\mathrm{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to $\mathcal{M}_f$. We will use this throughout the proof.

[9] The *μ-positive part* was defined in Claim 3.23.

for f. Indeed, if we let $\vec{f}$ be such that $H_{\vec{f}} \subseteq A$, which exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

($\Rightarrow$): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n-Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play $\mathcal{M}_0$ and enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A^0_\alpha \mid \alpha < \kappa\rangle$ such that $A^0_\xi \subseteq A^0_\zeta$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^0_\xi$. Let $<^\alpha_{\text{lex}}$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f, using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \restriction \alpha \mid \xi \in H_0 \wedge \xi > \alpha\rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least such η, which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C.

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \restriction \zeta = r_\gamma \restriction \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \restriction \zeta$ holds for every $\gamma \in H_0$, $\gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^0_\xi$ iff $\gamma \in A^0_\xi$. Define thus the $\mathcal{M}_0$-measure μ_0 on κ as

$$\begin{aligned} \mu_0(A^0_\xi) = 1 \quad &\text{iff} \quad (\forall \beta \in H_0 \cap C)(\beta > \xi \to \beta \in A^0_\xi) \\ &\text{iff} \quad (\exists \beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A^0_\xi), \end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa\rangle$ be the μ_0-positive part of $\vec{A}^0$, it is also simple to check that $H_0 \cap C \subseteq \triangle \vec{B}$, making μ_0 normal and hence also both $\mathcal{M}_0$-normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n-Ramsey and let $\langle\mathcal{M}_0,\mu_0,\dots,\mathcal{M}_n,\mu_n,\mathcal{M}_{n+1}\rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathscr{P}(\kappa)\cap\mathcal{M}_{n+1}$ as $\vec{A}^{n+1}=\langle A^{n+1}_\xi\mid\xi<\kappa\rangle$, again satisfying that $\xi\leq\zeta$ whenever $A^{n+1}_\xi\subseteq A^{n+1}_\zeta$, but also such that given any $\xi<\kappa$ there are $\zeta,\zeta'\in(\xi,\kappa)$ satisfying that $A^{n+1}_\zeta\in\mathscr{P}(\kappa)\cap\mathcal{M}_n$ and $A^{n+1}_{\zeta'}\in(\mathscr{P}(\kappa)\cap\mathcal{M}_{n+1})-\mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n\in R$ and C be club in κ such that $H_n\cap C\subseteq\triangle\mu_n$, which exist by our inductive assumption. For $\alpha<\kappa$ define $r_\alpha:\alpha\to 2$ as $r_\alpha(\xi)=1$ iff $\alpha\in A^{n+1}_\xi$, and define a colouring $f:[H_n]^2\to 2$ as

$$f(\{\alpha,\beta\}):=\begin{cases}0 & \text{if } r_{\min(\alpha,\beta)}<^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)}\restriction\min(\alpha,\beta)\\ 1 & \text{otherwise}\end{cases}$$

As $H_n\in R$ there is an $H_{n+1}\in R$ homogeneous for f. Just as before, define $g:\kappa\to\kappa$ as $g(\alpha)$ being the least $\eta\in[\alpha,\kappa)$ such that $r_\beta\restriction\alpha=r_\gamma\restriction\alpha$ for every $\beta,\gamma\in H_{n+1}$ with $\eta\leq\beta<\gamma$, and let D be the club of fixed points of g. As above we get that given any $\alpha,\gamma\in H_{n+1}\cap D$ with $\alpha\leq\gamma$ and $\xi<\alpha$, $\alpha\in A^{n+1}_\xi$ iff $\gamma\in A^{n+1}_\xi$. Define then the $\mathcal{M}_{n+1}$-measure μ_{n+1} on κ as

$$\begin{aligned}\mu_{n+1}(A^{n+1}_\xi)=1\quad&\text{iff}\quad(\forall\beta\in H_{n+1}\cap D\cap C)(\beta>\xi\to\beta\in A^{n+1}_\xi)\\ &\text{iff}\quad(\exists\beta\in H_{n+1}\cap D\cap C)(\beta>\xi\wedge\beta\in A^{n+1}_\xi).\end{aligned}$$

Then $H_{n+1}\cap D\cap C\subseteq\triangle\mu_{n+1}$, making μ_{n+1} normal, $\mathcal{M}_{n+1}$-normal and good, just as before. It remains to show that $\mu_n\subseteq\mu_{n+1}$. Let thus $A\in\mu_n$ be given, and say $A=A^{n+1}_\xi=A^n_\eta$, where $\vec{A}^n$ was the enumeration of $\mathscr{P}(\kappa)\cap\mathcal{M}_n$ used at the n'th stage. Then by definition of μ_n we get that for every $\beta\in H_n\cap C$ with $\beta>\eta$, $\beta\in A^n_\eta$. We need to show that

$$(\exists\beta\in H_{n+1}\cap D\cap C)(\beta>\xi\wedge\beta\in A^{n+1}_\xi)$$

holds. But here we can simply pick a $\beta>\max(\xi,\eta)$ with $\beta\in H_{n+1}\cap D\cap C\subseteq H_n\cap C$. This shows that $\mu_n\subseteq\mu_{n+1}$, making κ $(n+1)$-Ramsey and thus inductively also coherent $<\omega$-Ramsey. ■

3.2 The countable case

This section covers the (strategic) γ-Ramsey cardinals whenever γ has countable cofinality. This case is special because, as we cannot ensure that the final measure in $\mathcal{G}^\theta_\gamma(\kappa)$ is countably complete and so the existence of winning strategies *might* depend on θ, in contrast with the uncountable cofinality case.

3.2.1 [Strategic] ω-Ramsey cardinals

We now move to the strategic ω-Ramsey cardinals and their relationship to the (non-strategic) ω-Ramseys.

Theorem 3.30 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. Then κ is faintly θ-measurable iff player II has a winning strategy in $\mathcal{C}^\theta_\omega(\kappa)$.*

Proof. $(\Leftarrow)$: Fix a winning strategy σ for player II in $\mathcal{C}^\theta_\omega(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, H^V_\theta)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega \rangle$ of weak κ-models $\mathcal{M}_n \prec H^V_\theta$ such that $H^V_\theta \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^V_θ-normal H^V_θ-measure μ on κ.

We claim that $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \theta$ such that $g_n \in H^V_\theta$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f \colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n} \rangle$ is a partial play of $\mathcal{C}^\theta_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $C^\theta_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction.

Thus $\mathrm{Ult}(H_\theta^V, \mu)$ is wellfounded, so that the ultrapower embedding $\pi\colon H_\theta^V \to \mathrm{Ult}(H_\theta^V, \mu)$ witnesses that κ is faintly θ-measurable.

($\Rightarrow$) : Assume that κ is faintly θ-measurable. Let $\mathbb{P}$ be a forcing $\dot\mu$ a $\mathbb{P}$-name for an H_θ^V-normal H_θ^V-measure on κ and $\dot\pi$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}_\omega^\theta(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f_i^n \mid i < k\rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \check\mu \cap \mathcal{M}_n = \check\mu_n \cap \forall i < \check k\colon \dot\pi(\check f_i^n)(\check\kappa) = \check\alpha_i^n \urcorner,$$

with $\mu_n, \alpha_i^n \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f_{i_n}^{k_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check\alpha_{i_{n+1}}^{k_{n+1}} < \check\alpha_{i_n}^{k_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

We note that the above Theorem along with our results from Chapter 2 shows that winning the Cohen games doesn't guarantee weak compactness.

Corollary 3.31 (N.). *Let κ be inaccessible.*

(i) If player II wins $\mathcal{C}_\omega^\theta(\kappa)$ for all regular $\theta > \kappa$ then κ is not necessarily weakly compact;

(ii) If player II wins $\mathcal{C}_\kappa(\kappa)$ then κ is weakly compact.

Proof. The first claim is directly by Proposition 2.48 and Theorem 3.30, and the second claim is because the hypothesis implies that player II wins $\mathcal{G}_0(\kappa)$ so that inaccesibility of κ makes κ weakly compact – see e.g. [Gitman, 2011] for this characterisation of weak compactness. ■

Here's a near-analogous result of Theorem 3.30 for the $\mathcal{G}_\omega^\theta(\kappa)$ game.

Theorem 3.32 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$, and if player II has* *a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$ then κ is faintly θ-power-measurable. In particular, $\mathcal{G}^\theta_\omega(\kappa)^L \sim \mathcal{C}^\theta_\omega(\kappa)^L$.*

Proof. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi\colon H^V_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a name $\dot\mu$ and a $\mathbb{P}$-condition p such that

$p \Vdash \ulcorner\dot\mu$ is a weakly amenable $\check H_\theta$-normal $\check H_\theta$-measure with a wellfounded ultrapower$\urcorner$.

We now define a strategy σ for player II in $\mathcal{G}^\theta_\omega(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H^V_\theta$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n\colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot\mu \cap \check{\mathcal{M}}_n = \check\mu_n \cap \check\mu_n = \dot\mu \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma 3.2 we get that $\pi \restriction \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega}\mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n\colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega}\mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot\pi(\check g_{n+1})(\check\kappa) < \dot\pi(\check g_n)(\check\kappa)\urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check\pi_{n+1}(\check g_{n+1})(\check\kappa) < \check\pi_n(\check g_n)(\check\kappa)\urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy.

■

We get the following immediate corollary.

Corollary 3.33 (N.-Schindler). *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ■

Note also that the proof of Theorem 3.32 shows that whenever κ is strategic ω-Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H_ν^V-normal H_ν-measure on κ.

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω-Ramsey cardinals and ω-Ramsey cardinals lie, namely that strategic ω-Ramseys are equiconsistent with *remarkables* and ω-Ramseys are strictly below. Theorem 4.8 of [Gitman and Welch, 2011] showed that 2-iterables are limits of remarkables, and our Propositions 3.9 and 3.41 shows that ω-Ramseys are limits of 1-iterables, so that the strategic ω-Ramseys and the ω-Ramseys both lie strictly between the 2-iterables and 1-iterables. It was shown in [Holy and Schlicht, 2018] that ω-Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by [Schindler, 2000b], and [Gitman and Schindler, 2018] showed the following two equivalent formulations.

Definition 3.34. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

(i) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H_\lambda^V \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_\nu^V \to M$ with critical point κ and $\pi(\kappa) > \lambda$;

(ii) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^\lambda M \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H_\nu^V \to M$ with critical point κ and $\pi(\kappa) > \lambda$.

○

Theorem 3.35 (N.). *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models \ulcorner$there is a proper class of virtually measurables$\urcorner$. In particular, the two notions are equiconsistent.*

Remove this, but maybe move parts of the proof to the virtual large cardinals chapter

Proof. Virtually measurables are downwards absolute to L by Lemma 2.45, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H^V_\lambda \subseteq M$ and forcing poset $\mathbb{P}$ it holds that, in $V^{\mathbb{P}}$, there's no elementary embedding $\pi : H^V_\nu \to M$ with $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν-measurable to fix a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's an elementary $\pi : H^V_\nu \to M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H^V_\nu = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we're assuming that κ isn't remarkable. Then by restricting the generic embedding to H^V_κ we get that $H^V_\kappa \prec H^M_{\pi(\kappa)} = H^V_{\pi(\kappa)}$, using that $\pi(\kappa) < \nu$ and $H^V_\nu = H^M_\nu$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H^V_ν since $\pi(\kappa) < \nu$, and as $H^V_\nu \prec_1 V$ we get that $\pi(\kappa)$ *is* a cardinal. But then, again using that $H_{\pi(\kappa)} \prec_1 V$, κ is virtually measurable in $H^V_{\pi(\kappa)}$ since being virtually measurable is Π_2. This means that for every $\xi < \kappa$ it holds that

$$H^V_{\pi(\kappa)} \models \exists\alpha > \xi : \ulcorner\alpha \text{ is virtually measurable}\urcorner,$$

implying that $H^V_\kappa \models \ulcorner$There is a proper class of virtually measurables$\urcorner$. ■

Now Theorem 3.35 and Corollary 3.33 yield the following immediate corollary.

Corollary 3.36 (N.-Schindler). *Let κ be strategic ω-Ramsey. Then either κ is remarkable in L or otherwise*

$$L_\kappa \models \ulcorner\textit{there is a proper class of strategic } \omega\textit{-Ramseys}\urcorner.$$

In particular, the two notions are equiconsistent. ■

Now, using these results we show that the strategic ω-Ramseys have strictly stronger consistency strength than the ω-Ramseys.

Theorem 3.37 (N.). *Remarkable cardinals are strategic ω-Ramsey limits of ω-Ramsey cardinals.*

Proof. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ-sequences and a generic elementary embedding $\pi : H_\nu^V \to M$ for some $\nu > 2^\kappa$. We will show that κ is ω-Ramsey in M. Note that remarkables are clearly virtually measurable, and thus by Theorem 3.32 also strategic ω-Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}_\omega^\theta(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}_\omega^\theta(\kappa)^M$. Since M is closed under 2^κ-sequences it means that $\mathscr{P}(\mathscr{P}(\kappa)) \subseteq M$ and thus that M contains all possible filters on κ. We let player II follow τ, which produces a play $\sigma * \tau$ in which player II wins. But all player II's moves are in $\mathscr{P}(\mathscr{P}(\kappa))$ and hence in M, and as M is furthermore closed under ω-sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω-Ramsey in M.

This also implies that κ is a limit of ω-Ramseys in H_ν. But as κ is remarkable it holds that $H_\kappa \prec_2 V$, in analogy with the same property for strongs and supercompacts, and as being ω-Ramsey is a Π_2-notion this means that κ *is* a limit of ω-Ramseys. ■

This immediately yields the following corollary.

Corollary 3.38 (N.-Schindler). *If κ is a strategic ω-Ramsey cardinal then*

$$L_\kappa \models \ulcorner \textit{there is a proper class of } \omega\textit{-Ramseys} \urcorner. \qquad \dashv$$

3.2.2 (ω, α)-Ramsey cardinals

A natural generalisation of the γ-Ramsey definition is to require more iterability of the last measure. Of course, by Proposition 3.9 we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when $\operatorname{cof}\gamma > \omega$ so the next definition is only interesting whenever $\operatorname{cof}\gamma = \omega$.

DEFINITION 3.39 (N.). Let α, β be ordinals. Then a cardinal κ is (α, β)-**Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ for all regular $\theta > \kappa$.[10] ∘

DEFINITION 3.40 (Gitman). A cardinal κ is α-**iterable** if for every $A \subseteq \kappa$ there exists a *transitive* weak κ-model $\mathcal{M}$ with $A \in \mathcal{M}$ and an α-good $\mathcal{M}$-measure μ on $\mathcal{M}$. ∘

PROPOSITION 3.41. *If $\beta > 0$ then every (α, β)-Ramsey is a β-iterable stationary limit of β-iterables.*

PROOF. Let $(\mathcal{M}, \in, \mu)$ be a result of a play of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$ in which player II won. Then the transitive collapse of $(\mathcal{M}, \in, \mu)$ witnesses that κ is β-iterable, since μ is β-good by definition of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$.

That κ is β-iterable is reflected to some H_θ, so let now $(\mathcal{N}, \in, \nu)$ be a result of a play of $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ in which player II won. Then $\mathcal{N} \prec H_\theta$, so that κ is also β-iterable in $\mathcal{N}$. Since being β-iterable is witnessed by a subset of κ and $\beta > 0$ implies[11] that we get a κ-powerset preserving $j : \mathcal{N} \to \mathcal{P}$, $\mathcal{P}$ also thinks that κ is β-iterable, making κ a stationary limit of β-iterables by elementarity. ∎

We now move towards Theorem 3.45 which gives an upper consistency bound for the (ω, α)-Ramseys. We first recall a few definitions and a folklore lemma.

DEFINITION 3.42. For an infinite ordinal α, a cardinal κ is α-**Erdős** for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \to \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c; i.e. that $|c``[H]^n| \leq 1$ holds for every $n < \omega$. ∘

DEFINITION 3.43. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$.[12] ∘

[10] Note that an α-Ramsey cardinal is the same as an $(\alpha, 0)$-Ramsey cardinal.
[11] Recall that β-good for $\beta > 0$ in particular implies weak amenability.
[12] Note that this terminology is not at all related to remarkable *cardinals*.

Lemma 3.44 (Folklore). *Let κ be α-Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure $\mathcal{M}$ in a countable language $\mathcal{L}$ with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

Proof. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all $\mathcal{L}$-formulas and define $c : [C]^{<\omega} \to \kappa$ as follows. For an increasing sequence $\alpha_1 < \cdots < \alpha_{2n} \in C$ let

$$\begin{aligned} c(\{\alpha_1, \dots, \alpha_{2n}\}) := \text{the least } \lambda < \alpha_1 \text{ such that} & \\ \exists \delta_1 < \cdots \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \dots, \delta_k \rangle \wedge & \\ \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}] & \end{aligned}$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α-Erdős we get a homogeneous $I \in [C]^\alpha$ for c; i.e. that $|c``[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \dots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \dots, \alpha_{2n} \in I$, as otherwise there exists an $m < \omega$ and $\delta_1 < \cdots \delta_k$ such that for any $\alpha_1 < \ldots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}]. \tag{$\dagger$}$$

But then simply pick $\alpha_1 < \ldots \alpha_{2n} < \alpha'_1 < \cdots < \alpha'_{2n}$ so that both $\{\alpha_1, \dots, \alpha_{2n}\}$ and $\{\alpha'_1, \dots, \alpha'_{2n}\}$ witnesses (†); then either $\{\alpha_1, \dots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \dots, \alpha_n, \alpha'_{n+1}, \dots, \alpha'_{2n}\}$ also witnesses that (†) fails, $\lightning$. ■

Theorem 3.45 (N.). *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α-Erdős cardinal is a limit of (ω, α)-Ramsey cardinals.*

Proof. Let κ be α-Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 3.44 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I. We will show that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in [Holy and Schlicht, 2018] it suffices to find a weak ι-model $\mathcal{M} \prec H_\theta$ and an α-good $\mathcal{M}$-measure on ι. Define

$$\mathcal{M} := \text{Hull}^{H_\theta}(\iota \cup I) \prec H_\theta$$

and let $\pi : I \to I$ be the right-shift map. Since I is remarkable, I $(= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ with crit $j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \ldots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι-powerset preserving we get that $\mathcal{M} \prec H_\theta$ is a weak ι-model satisfying ZFC^- with a 1-good $\mathcal{M}$-measure μ_j on ι. Furthermore, as we can linearly iterate $\mathcal{M}$ simply by applying j we get an α-iteration of $\mathcal{M}$ since there are α-many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \ldots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\eta, \alpha)$. If we repeat this procedure for all regular $\theta > \kappa$ we get by the pidgeon hole principle that κ is a limit of (ω, α)-Ramsey cardinals. ■

As Theorem 4.5 in [Gitman and Schindler, 2018] shows that $(\alpha{+}1)$-iterable cardinals have α-Erdős cardinals below them for $\alpha \geq \omega$ additively closed, this shows that the (ω, α)-Ramseys form a strict hierarchy. Further, as α-Erdős cardinals are consistent with $V = L$ when $\alpha < \omega_1^L$ and ω_1-iterable cardinals aren't consistent with $V = L$, we also get that (ω, α)-Ramsey cardinals are consistent with $V = L$ if $\alpha < \omega_1^L$ and that they aren't if $\alpha = \omega_1$.

3.2.3 [Strategic] $(\omega{+}1)$-Ramsey cardinals

The next step is then to consider $(\omega{+}1)$-Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in [Mitchell, 1979] and in the proof of Lemma 1.3 in [Donder et al., 1981] – see also [Dodd, 1982] and [Gitman, 2011].

Theorem 3.46 (Dodd, Mitchell). *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model $\mathcal{M}$ such that there exists a weakly amenable countably complete $\mathcal{M}$-measure on κ.* ■

The following theorem then supplies us with a lower bound for the strength of the $(\omega+1)$-Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 3.57, but we include this Ramsey lower bound as well for completeness.

Theorem 3.47 (N.). *Every $(\omega+1)$-Ramsey cardinal is a Ramsey limit of Ramseys.*

Proof. Let κ be $(\omega+1)$-Ramsey and $A \subseteq \kappa$. Let σ be a strategy for player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A \in \mathcal{M}_0$ and $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Then σ isn't winning as κ is $(\omega+1)$-Ramsey, so we may fix a play $\sigma * \vec{\mu}_\alpha$ of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable $\mathcal{M}_\omega$-measure on κ, and by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 3.46.

Since κ is Ramsey, $\mathcal{M}_\omega \models \ulcorner\kappa$ is Ramsey$\urcorner$ as well. Letting $j : \mathcal{M}_\omega \to \mathcal{N}$ be the κ-powerset preservering embedding induced by μ_ω, we also get that $\mathcal{N} \models \ulcorner\kappa$ is Ramsey$\urcorner$ by κ-powerset preservation. This then implies that κ *is* a stationary limit of Ramsey cardinals inside $\mathcal{M}_\omega$, and thus also in V by elementarity. ■

As for the *consistency* strength of the strategic $(\omega+1)$-Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal – see e.g. p. 249 in [Kanamori and Magidor, 1978].

Theorem 3.48 (N.). *If κ is a strategic $(\omega+1)$-Ramsey cardinal then, in $V^{\mathrm{Col}(\omega,2^\kappa)}$, there's a transitive class N and an elementary embedding $j : V \to N$ with $\operatorname{crit} j = \kappa$. In particular, the existence of a strategic $(\omega+1)$-Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

Proof. Set $\mathbb{P} := \mathrm{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$. Let $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name of an ω-sequence $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n \prec H^V_{\kappa^+}$ and $\mathscr{P}(\kappa)^V \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, and let $\dot{\mu}$ be a $\mathbb{P}$-name for the ω-sequence of σ-responses to the $\mathcal{M}_n$'s in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)^V$.

Assume that there's a $\mathbb{P}$-condition p which forces the generic ultrapower $\mathrm{Ult}(V, \bigcup_n \dot\mu_n)$ to be illfounded, meaning that we can fix a $\mathbb{P}$-name $\dot f$ for an ω-sequence $\langle f_n \mid n < \omega\rangle$ such that

$$p \Vdash \dot X_n := \{\alpha < \kappa \mid \dot f_{n+1}(\alpha) < \dot f_n(\alpha)\} \in \bigcup_{n<\omega} \dot\mu_n.$$

Now, in V, we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} \prec H_\theta$ such that $\dot{\mathcal{M}}, \dot\mu, \dot f, H^V_{\kappa^+}, \sigma, p \in \mathcal{N}$. We can find an $\mathcal{N}$-generic $g \subseteq \mathbb{P}^{\mathcal{N}}$ in V with $p \in g$ since $\mathcal{N}$ is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}^g_n * \dot\mu^g_n$ is a play of $\mathcal{G}^{\kappa^+}_\omega(\kappa)^V$ which is according to σ, meaning that $\bigcup_{n<\omega} \dot\mu^g_n$ is normal and in particular countably complete (in V). Then $\bigcap_{n<\omega} \dot X^g_n \neq \emptyset$, but if $\alpha \in \bigcap_{n<\omega} \dot X^g_n$ then $\langle \dot f^g_n(\alpha) \mid n < \omega\rangle$ is a strictly decreasing ω-sequence of ordinals, $\lightning$. This means that $\mathrm{Ult}(V, \bigcup_n \mu_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in [Kellner and Shelah, 2011]. ■

The above Theorem 3.48 then answers Question 9.2 in [Holy and Schlicht, 2018] in the negative, asking if λ-Ramseys are strategic λ-Ramseys for uncountable cardinals λ, as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.

3.3 The general case

3.3.1 Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from [Gitman, 2011] and investigate further connections between these and the α-Ramsey cardinals. First, a definition.

Definition 3.49 (Gitman). A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ. If furthermore $\mathcal{M} \prec H_{\kappa^+}$ then we say that κ is **super Ramsey**. ∘

Note that since the model $\mathcal{M}$ in question is a *κ-model* it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$-sequences. [Gitman, 2011] shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M} \models \mathsf{ZFC}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ.

Now, a first connection between the α-Ramseys and the strongly- and super Ramseys is the result in [Holy and Schlicht, 2018] that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

Theorem 3.50 (N.-Welch). *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

Proof. Let κ be strongly Ramsey and let $\mathcal{M} \models \mathsf{ZFC}$ be a transitive κ-model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^{\mathcal{M}}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of $\mathcal{M}$ by weak amenability of μ. As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside $\mathcal{M}$, so $\mathcal{M}$ sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \to \mathcal{N}$ be the induced embedding, κ-powerset preservation of j_μ implies that μ is also a weakly amenable $\mathcal{N}$-normal $\mathcal{N}$-measure on κ. This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in $\mathcal{N}$, entailing that it is a stationary limit of almost fully Ramseys in $\mathcal{M}$. But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in [Holy and Schlicht, 2018], so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and $\mathcal{M}$ being transitive implies that κ really *is* a stationary limit of almost fully Ramseys. ■

3.3.2 Downwards absoluteness to K

Lastly, we consider the question of whether the α-Ramseys are downwards absolute to K, which turns out to at least be true in many cases. The below Theorem 3.52 then also answers Question 9.4 from [Holy and Schlicht, 2018] in the positive, asking whether α-Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of $0^{\P}$.

Definition 3.51. $0^{\P}$ is "the sharp for a strong cardinal", meaning the minimal sound active mouse $\mathcal{M}$ with $\mathcal{M} \,|\, \mathrm{crit}(\dot{F}^{\mathcal{M}}) \models \ulcorner$There exists a strong cardinal$\urcorner$, with $\dot{F}^{\mathcal{M}}$ being the top extender of $\mathcal{M}$. ○

Theorem 3.52 (N.-Welch). *Assume $0^{\P}$ does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ-Ramsey. Then $K \models \ulcorner\kappa$ is a λ-Ramsey cardinal$\urcorner$.*

Proof. Note first that $\kappa^{+K} = \kappa^+$ by [Schindler, 1997], since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ-models $\mathcal{M} \prec K|\kappa^+$. We can then define a strategy $\tilde{\sigma}$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde{\sigma}(\emptyset) := \mathrm{Hull}^{H_{\kappa^+}}(K|\kappa \cup \sigma(\emptyset))$. Assuming now that $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma\rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$, we have two cases. If $\tilde{\mu}_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle \mathcal{M}_\alpha \mid \alpha < \gamma\rangle$ be the corresponding models played in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde{\sigma}(\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma\rangle) := \mathrm{Hull}^{H_{\kappa^+}}(K|\kappa \cup \sigma(\langle \mathcal{M}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma\rangle)),$$

and otherwise let $\tilde{\sigma}$ play arbitrarily. As κ is λ-Ramsey (in V) there exists a play $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha \leq \lambda\rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ \prec K|\kappa^+$ so let $\mathcal{N}$ be the transitive collapse of $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+$. But if $j : \mathcal{N} \to K|\kappa^+$ is the uncollapse then $\mathrm{crit}\, j$ is both an $\mathcal{N}$-cardinal and also $> \kappa$ because we ensured that $K|\kappa \subseteq \mathcal{N}$. This means that $j = \mathrm{id}$ because κ is the largest $\mathcal{N}$-cardinal by elementarity in $K|\kappa^+$, so that $\tilde{\mathcal{M}}_\lambda \cap K|\kappa^+ = \mathcal{N}$ is a transitive elementary substructure of $K|\kappa^+$, making it an initial segment of K.

Now, since $\mu := \tilde{\mu}_\lambda$ is a countably complete weakly amenable $K|o(\mathcal{N})$-measure[13], the "beaver argument"[14] shows that $\mu \in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu \cap \mathcal{N} \in K$ whenever player I plays $\mathcal{N}$. Since $\mu = \tilde{\mu}_\lambda$ we also have that $\mu \cap \mathcal{M}_\alpha = \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$, so that σ

[13]Here we use that $\mathcal{N} \lhd K$.
[14]See Appendix 1.2 for details regarding the beaver argument.

will eventually play $\mathcal{N}$, making τ win against σ.[15] ■

Note that the only thing we used $\operatorname{cof}\lambda > \omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine- or normal α-Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

Corollary 3.53 (N.-Welch). *Assume $0^\P$ does not exist and let α be any limit ordinal. Then every genuine- and every normal α-Ramsey cardinal is downwards absolute to K. In particular, if α is a limit of limit ordinals then every $<\alpha$-Ramsey cardinal is downwards absolute to K as well.* ■

3.3.3 Indiscernible games

We now move to the strategic versions of the α-Ramsey hierarchy. The first thing we want to do is define *α-very Ramsey cardinals*, introduced in [Sharpe and Welch, 2011], and show the tight connection between these and the strategic α-Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 3.43.

Definition 3.54. A **good set of indiscernibles** for a structure $\mathcal{M}$ is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for $\mathcal{M}$ such that $\mathcal{M}|\iota \prec \mathcal{M}$ for any $\iota \in I$. ○

Definition 3.55 (Sharpe-Welch). Define the **indiscernible game** $G^I_\gamma(\kappa)$ in γ many rounds as follows

I	$\mathcal{M}_0$		$\mathcal{M}_1$		$\mathcal{M}_2$		$\cdots$	
II		I_0		I_1		I_2		$\cdots$

Here $\mathcal{M}_\alpha$ is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for $\mathcal{M}_\alpha$ and the I_α's are $\subseteq$-decreasing. Player II wins iff they can continue playing through all the rounds. ○

[15] Note that τ is not necessarily a winning strategy – all we know is that it is winning against this particular strategy σ.

Definition 3.56 (Sharpe-Welch). A cardinal κ is γ-**very Ramsey** if player II has a winning strategy in the game $G^I_\gamma(\kappa)$. ∘

The next couple of results concerns the connection between the strategic α-Ramseys and the α-very Ramseys. We start with the following.

Theorem 3.57 (N.). *Every $(\omega{+}1)$-Ramsey is an ω-very Ramsey stationary limit of ω-very Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey. We will describe a winning strategy for player II in the indiscernible game $G^I_\omega(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G^I_\omega(\kappa)$ then let player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ play

$$\mathcal{H}_0 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) \prec H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ which starts off with $\mathcal{H}_0$ and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}^{(\alpha)}_0 := \mathcal{M}_\alpha$ and $\mu^{(\alpha)}_0 := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{lcccccccc} \mathrm{I} & \mathcal{H}^{(0)}_0 & & \cdots & & \mathcal{H}^{(\omega)}_0 & & \mathcal{H}^{(\omega+1)}_0 & \\ \mathrm{II} & & \mu^{(0)}_0 & & \cdots & & \mu^{(\omega)}_0 & & \mu^{(\omega+1)}_0 \end{array}$$

By the choice of player I's strategy we get that $\mu^{(\omega)}_0$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ (it's even normal). Now Lemma 2.9 of [Sharpe and Welch, 2011] gives us a set of good indiscernibles $I_0 \in \mu^{(\omega)}_0$ for $\mathcal{M}_0$, as $\mathcal{M}_0 \in \mathcal{H}^{(\omega)}_0$ and $\mu^{(\omega)}_0$ is a countably complete weakly amenable $\mathcal{H}^{(\omega)}_0$-normal $\mathcal{H}^{(\omega)}_0$-measure on κ. Let player II play I_0 in $G^I_\omega(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G^I_\omega(\kappa)$.

$$\begin{array}{lccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 \\ \mathrm{II} & & I_0 & \end{array}$$

Since $\mu_0^{(\omega)} = \bigcup_n \mu_0^{(n)}$ we must have that $I_0 \in \mu_0^{(n_0)}$ for some $n_0 < \omega$. In the $(n_0{+}1)$'st round of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}_0^{(k)}, \mu_0^{(k)} \mid k \leq n_0 \rangle\}) \prec H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ thus looks like the following so far.

$$\begin{array}{llllllll} \mathrm{I} & \mathcal{H}_0^{(0)} & & \cdots & & \mathcal{H}_0^{(n_0)} & & \mathcal{H}_1 \\ \mathrm{II} & & \mu_0^{(0)} & & \cdots & & \mu_0^{(n_0)} & \end{array}$$

Now player II in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ is not losing at round n_0, so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I_1' \in \mu_1^{(n_1)}$ of good indiscernibles for $\mathcal{M}_1$, where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G_\omega^I(\kappa)$ play $I_1 := I_0 \cap I_1' \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G_\omega^I(\kappa)$, making κ ω-very Ramsey.

As for showing that κ is a stationary limit of ω-very Ramseys, let $\mathcal{M} \prec H_{\kappa^+}$ be a weak κ-model with a weakly amenable countably complete $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ, which exists by Theorem 3.47 as κ is $(\omega{+}1)$-Ramsey. Then by elementarity $\mathcal{M} \models \ulcorner\kappa$ is ω-very Ramsey$\urcorner$ and since κ being ω-very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ-ultrapower, meaning that κ is a stationary limit of ω-very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

Theorem 3.58 (N.). *For limit ordinals α, every coherent $<\omega\alpha$-Ramsey is $\omega\alpha$-very Ramsey.*

Proof. This is basically the same proof as the proof of Theorem 3.57. We do the "going-back" trick in ω-chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}_{\omega\alpha}^{\kappa^+}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$-Ramseyness. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω-chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 3.57. ∎

As for going from the α-very Ramseys to the strategic α-Ramseys we got the following.

Theorem 3.59 (N.). *For γ any ordinal, every coherent $<\gamma$-very Ramsey*[16] *is coherent $<\gamma$-Ramsey.*[17]

Proof. The reason why we work with $<\gamma$-Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ-very Ramsey and let τ be a winning strategy for player II in $G^I_\zeta(\kappa)$. Let $\mathcal{M}_\alpha \prec H_\theta$ be any move by player I in the α'th round of $\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in $\mathcal{M}_\alpha$ and form now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G^I_\zeta(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for $\mathcal{N}_\alpha$ such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in [Sharpe and Welch, 2011] we get a structure $\mathcal{P}_\alpha$ with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a $\mathcal{P}_\alpha$-measure $\tilde{\mu}_\alpha$ on κ, generated by I_α.[18] Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α's are generated by the I_α's, the μ_α's are $\subseteq$-increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$-Ramsey. ∎

The following result is then a direct corollary of Theorems 3.58 and 3.59.

[16] Here the coherency again just means that the winning strategies σ_α for player II in $G^I_\alpha(\kappa)$ are $\subseteq$-increasing.

[17] Here a "coherent $<\gamma$-very Ramsey cardinal" is defined from γ-very Ramseys in the same way as coherent $<\gamma$-Ramsey cardinals is defined from γ-Ramseys. When γ is a limit ordinal then coherent $<\gamma$-very Ramseys are precisely the same as γ-very Ramseys, so this is solely to "subtract one" when γ is a successor ordinal – i.e. a coherent $<(\gamma+1)$-very Ramsey cardinal is the same thing as a γ-very Ramsey cardinal.

[18] By *generated* here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α.

Corollary 3.60 (N.). *For limit ordinals α, κ is $\omega\alpha$-very Ramsey iff it is coherent $<\omega\alpha$-Ramsey. In particular, κ is λ-very Ramsey iff it is strategic λ-Ramsey for any λ with uncountable cofinality.* ■

We can now use this equivalence to transfer results from the α-very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in [Feng, 1990]. A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in [Sharpe and Welch, 2011]. We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in [Sharpe and Welch, 2011].

Theorem 3.61 (Sharpe-Welch). *A cardinal is completely Ramsey if and only if it is ω-very Ramsey.* ■

This, together with Theorem 3.57, immediately yields the following strengthening of Theorem 3.47.

Corollary 3.62 (N.). *Every $(\omega{+}1)$-Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* ■

The above Theorem 3.59 also yields the following consequence.

Corollary 3.63 (N.). *Every completely Ramsey cardinal is completely ineffable.*

Proof. From Theorem 3.61 we have that being completely Ramsey is equivalent to being ω-very Ramsey, so the above Theorem 3.59 then yields that a completely Ramsey cardinal is coherent $<\omega$-Ramsey, which we saw in Theorem 3.29 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 3.60 yields that strategic ω_1-Ramsey cardinals are ω_1-very Ramsey, and Theorem 3.50 in [Sharpe and Welch, 2011] states that ω_1-very Ramseys are measurable in the core model K, assuming $0^\P$ doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

THEOREM 3.64 (Welch). *Assuming $0^\P$ doesn't exist, every strategic ω_1-Ramsey cardinal is measurable in K.*

PROOF. Let κ be strategic ω_1-Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \mathrm{Col}(\omega_1, \kappa^+)$ is V-generic. Since $\mathrm{Col}(\omega_1, \kappa^+)$ is ω-closed, V and $V[g]$ have the same countable sequences of V, so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V.

Now let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of regular K-cardinals cofinal in κ^+, let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \mathrm{Hull}^{H_\theta}(K|\kappa_\alpha) \prec H_\theta$ and player II follow τ. This results in a countably complete weakly amenable K-measure μ_{ω_1}, which the "beaver argument"[19] then shows is actually an element of K, making κ measurable in K. ■

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1-Ramsey cardinal is also measurable in Steel's core model below a Woodin[20], a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from [Schindler, 2006a].[21]

THEOREM 3.65 (Schindler). *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ, and let $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.* ■

THEOREM 3.66 (Schindler). *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1-Ramsey cardinal is measurable in K.*

PROOF. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$

[19] See Appendix 1.2 for details regarding the beaver argument.
[20] See Appendix 1.2.
[21] That paper assumes the existence of a measurable as well, but by [Jensen and Steel, 2013b] we can omit that here.

fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ of weak κ-models $M_\alpha \prec H^V_\theta$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H^V_{\kappa^+} \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\mathrm{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Let $j : M_{\omega_1} \to \mathrm{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H^V_{\kappa^+} \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ with N transitive.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 3.65 applies, meaning that $\pi \restriction K : K \to K^N$ is an iteration map with critical point κ, making κ measurable in K. ■

4 | IDEAL ABSOLUTENESS

Historically, the idea of considering elementary embeddings existing only in generic extensions has been around for a while, but it all started as an analysis of *ideals*. *Precipitous ideals* were introduced in [Galvin et al., 1978] and further analysed in [Jech et al., 1980], being ideals that give rise to wellfounded generic ultrapowers.

In this chapter we will introduce the *ideally measurable cardinals*, essentially just switching perspective from the ideals themselves to the cardinals they are on. We then proceed to show how these cardinals relate to "pure" generic cardinals, being proper class versions of the faintly measurable cardinals that we have considered throughout Chapter 2. We start with a definition of the latter.

DEFINITION 4.1 (GBC). A cardinal κ is **generically measurable** if there is a generic extension $V[g]$, a transitive class $\mathcal{N} \subseteq V[g]$ and a generic elementary embedding $\pi\colon V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$. ○

Note that, trivially, every generically measurable cardinal is faintly measurable. The corresponding ideal version of this is then the following.

DEFINITION 4.2. A cardinal κ is **ideally measurable** if there exists an ideal $\mathcal{I}$ on θ such that the generic ultrapower $\mathrm{Ult}(V, \mathcal{I})$ is wellfounded in $V^{\mathbb{P}}$ for $\mathbb{P} := \mathscr{P}^V(\kappa)/\mathcal{I}$. ○

It should also be noted that [Claverie and Schindler, 2016] generalised the concept of ideally measurables to *ideally strong cardinals* by introducing the concept of *ideal extenders* to capture the strongness properties.

Throughout this chapter we will be interested in how properties of the *forcings* affect the large cardinal structure of a critical point of a generic embedding. We thus define the following.

Definition 4.3. Let θ be a regular uncountable cardinal, $\kappa < \theta$ a cardinal and $\Phi(\kappa)$ a poset property[1]. Then κ is $\Phi(\kappa)$ **faintly θ-measurable** if it is faintly θ-measurable, witnessed by a forcing poset satisfying $\Phi(\kappa)$. Similarly, κ is $\Phi(\kappa)$ **generically measurable** if it is generically measurable with the associated forcing satisfying $\Phi(\kappa)$. ○

Note that ω-distributive faintly θ-measurable cardinals are equivalent to ω-distributive generically measurable cardinals for all regular θ since wellfoundedness becomes automatic.

Definition 4.4. A poset property $\Phi(\kappa)$ is **ideal-absolute** if whenever κ satisfies that there's a $\Phi(\kappa)$ forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure μ on κ, then there's an ideal I on κ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(v)$. ○

Note that this is *almost* saying that $\Phi(\kappa)$ ideally measurables are equivalent to $\Phi(\kappa)$ generically measurables, but the only difference is that these definitions require wellfoundedness of the target model.

A typical ideal that we will be utilising is the following.

Definition 4.5. Let κ be a regular cardinal, $\mathbb{P}$ a poset and $\dot{\mu}$ a $\mathbb{P}$-name for a V-normal V-measure on κ. Then the **induced ideal** is

$$\mathcal{I}(\mathbb{P}, \dot{\mu}) := \{X \subseteq \kappa \mid ||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} = 0\},$$

where $\mathcal{B}(\mathbb{P})$ is the boolean completion of $\mathbb{P}$. ○

Note that if the generic measure μ is V-normal then $\mathcal{I}(\mathbb{P}, \dot{\mu})$ is also normal. This ideal will witness our first ideal-absoluteness result, which is a simple rephrasing of a folklore result.

Proposition 4.6 (Folklore). *"The κ^+-chain condition" is ideal-absolute.*

[1] Examples of these are having the κ-chain condition, being κ-closed, κ-distributive, κ-Knaster, κ-sized and so on. Formally speaking, $\Phi(\kappa)$ is a first-order formula $\varphi(\kappa, \mathbb{P})$ which is true iff $\mathbb{P}$ is a poset, κ is a cardinal and some first-order formula $\psi(\kappa, \mathbb{P})$ is true.

Proof. Assume $\mathbb{P}$ has the κ^+-chain condition such that there's a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot{\mu})$ – we will show that $\mathscr{P}(\kappa)/I$ has the κ^+-chain condition. Assume not and let $\langle X_\alpha \mid \alpha < \kappa^+\rangle$ be an antichain of $\mathscr{P}(\kappa)/I$, which by normality of I we may assume is pairwise almost disjoint. But this then makes $\langle ||\check{X}_\alpha \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid \alpha < \kappa^+\rangle$ an antichain of $\mathbb{P}$ of size κ^+, $\lightning$. ■

We next move to distributivity. This property is especially interesting in the context of our generic large cardinals, as an ideal I on some cardinal κ is ω-distributive precisely if it's precipitous[2], so that carrying an ω-distributive ideal coincides with our definition of *ideally measurable*.

Theorem 4.7 (N.). *"$<\lambda$-distributivity" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*

Proof. Assume that $\mathbb{P}$ is a $<\lambda$-distributive forcing such that there exists a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $\mathcal{I} := \mathcal{I}(\mathbb{P}, \dot{\mu})$ – we'll show that $\mathscr{P}(\kappa)/\mathcal{I}$ is $<\lambda$-distributive.

Let $\gamma < \lambda$ and let $\vec{\mathcal{A}}$ be a γ-sequence of maximum antichains $\mathcal{A}_\alpha \subseteq \mathscr{P}(\kappa)/\mathcal{I}$ such that $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ for $\alpha \leq \beta$. We have to show that there's a maximal antichain $\mathcal{A}$ which refines all the antichains in $\vec{\mathcal{A}}$.

Now define for every $\alpha < \gamma$ the sets

$$\mathcal{A}^*_\alpha := \{||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid X \in \mathcal{A}_\alpha\}.$$

Note that $\mathcal{A}^*_\alpha$ is an antichain in $\mathbb{P}$. They're also maximal, because if $p \in \mathbb{P}$ was incompatible with every condition in $\mathcal{A}^*_\alpha$ then, letting $X := \bigcap \mathcal{A}_\alpha$, we have that p is compatible with $||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$, so that $X \in \mathcal{I}^+$. But X is incompatible with everything in $\mathcal{A}_\alpha$, contradicting that $\mathcal{A}_\alpha$ is maximal.

By $<\lambda$-distributivity of $\mathbb{P}$ we get an antichain $\mathcal{A}^*$ which refines all the antichains in $\vec{\mathcal{A}}^*$. But note that for every $p \in \mathcal{A}^*$, if we define $s_p(\alpha)$ to be the unique $a \in \mathcal{A}_\alpha$ such that $p \leq a$, then it holds that $p \leq ||\Delta s_p \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$,[3] so that $\Delta b_p \in \mathcal{I}^+$. Now $\mathcal{A} := \{\Delta b_p \mid p \in \mathcal{A}^*\}$ gives us a maximal antichain consisting of limit points of branches of $\mathcal{T}$. ■

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[2]See [Jech et al., 1980] and [Foreman, 1983].

[3]Here we're using that $\lambda \leq \kappa^+$ to ensure that the diagonal intersection is in the measure.

In an unpublished paper, Foreman proved the following.

Theorem 4.8 (Foreman). *Let κ be a regular cardinal such that $2^\kappa = \kappa^+$, and let $\lambda \leq \kappa^+$ be an infinite successor cardinal. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal precipitous ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ has a dense $<\lambda$-closed subset of size κ^+.*

Here we improve that result by not relying on the CH-assumption, reaching the conclusion for all regular infinite λ and also showing (κ,κ)-distributivity of the ideal forcing. The argument follows the same overall structure as the original, with more technicalities to achieve the stronger result.

Theorem 4.9 (Foreman-N.). *Let κ be a regular cardinal and $\lambda \leq \kappa^+$ be regular infinite. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is (κ,κ)-distributive and has a dense $<\lambda$-closed subset of size κ^+.*

Proof. Set $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$ if $2^\kappa > \kappa^+$ and $\mathbb{P} := \{\emptyset\}$ otherwise. If κ is measurable then the dual ideal to the measure on κ satisfies all of the wanted properties, so assume that κ is not measurable. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \kappa^+ \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \kappa^+$;
- $\mathcal{N}_{\gamma+1} \prec H^V_{\kappa^+}$ is a κ-model for every $\gamma < \kappa^+$;
- $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$ for limit ordinals $\delta < \kappa^+$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for $\gamma < \beta < \kappa^+$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\kappa^+} \mathcal{N}_\gamma$.

Define now the auxilliary game $\mathcal{G}(\kappa)$ of length λ as follows.

I	α_0		α_1		$\cdots$
II		$p_0, \mathcal{M}_0, \mu_0, Y_0$		$p_1, \mathcal{M}_1, \mu_1, Y_1$	$\cdots$

Here $\langle\alpha_\gamma \mid \gamma < \lambda\rangle$ is an increasing continuous sequence of ordinals bounded in κ^+, $\vec{p}_\gamma$ is a decreasing sequence of $\mathbb{P}$-conditions satisfying that

$$p_\gamma \Vdash \ulcorner \check{\mathcal{M}}_\gamma = \pi(\check{\alpha}_\gamma) \wedge \check{\mu}_\gamma \text{ is a } \check{\mathcal{M}}_\gamma\text{-normal } \check{\mathcal{M}}_\gamma\text{-measure on } \check{\kappa}\urcorner$$

such that $Y_\gamma = \Delta_{\xi<\kappa} X_\xi^{\mu_\gamma}$, where $\vec{X}_\xi^{\mu_\gamma} \in H_{\kappa^+}^V$ is the $<_{\kappa^+}$-least enumeration of μ_γ.[4] We require that the μ_γ's are $\subseteq$-increasing, and player II wins iff she can continue playing throughout all λ rounds. Let $\mu_\lambda := \bigcup_{\xi<\lambda}\mu_\xi$ be the **final measure** of the play.

To every limit ordinal $\eta < \kappa^+$ define the **restricted auxilliary game** $\mathcal{G}(\kappa)\restriction\eta$ in which player I is only allowed to play ordinals $<\eta$. Note that a strategy τ for player II is winning in $\mathcal{G}(\kappa)$ if and only if it's winning in $\mathcal{G}(\kappa)\restriction\eta$ for all $\eta < \kappa^+$, simply because all sequences of ordinals played by player I are bounded in κ^+.

Note that μ_λ is precisely the tail measure on κ defined by the Y_γ's; i.e. that $X \in \mu_\lambda$ iff there exists a $\delta < \lambda$ such that $|Y_\delta - X| < \kappa$. From this it's simple to see that $\mathcal{G}(\kappa)$ is equivalent to $\mathcal{G}_\lambda^-(\kappa)$, so player II has a winning strategy τ_0 in $\mathcal{G}(\kappa)$.

For any winning strategy τ in $\mathcal{G}(\kappa)\restriction\eta$ and to every partial play p of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ, define the associated **hopeless ideal**[5]

$$I_p^\tau\restriction\eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction\eta,\\ X \text{ is } \textit{not} \text{ in the final measure}\}$$

Claim 4.10. Every hopeless ideal $I_p^\tau\restriction\eta$ is normal and (κ,κ)-distributive.

Proof of claim. For normality, if $\langle Z_\gamma \mid \gamma < \kappa\rangle$ is a sequence of elements of I_p^τ such that $Z := \nabla_\gamma Z_\gamma$ is I_p^τ-positive, then there exists a play of $\mathcal{G}(\kappa)\restriction\eta$ in which player II follows τ such that Z lies in the final measure. If we let player I play sufficiently large ordinals in $\mathcal{G}(\kappa)\restriction\eta$ we may assume that $\langle Z_\gamma \mid \gamma < \kappa\rangle$ is a subset and an element of the final model as well, meaning that one of the Z_γ's also lies in the final measure, $\lightning$.

We now show (κ,κ)-distributivity. Let $\mathcal{U} \subseteq \mathscr{P}(\kappa)/I_p^\tau$ be an unrooted tree of height κ such that every level $\mathcal{U}_\alpha$ is a maximal antichain of size $\leq \kappa$. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of

[4]We use that $\mathbb{P}$ is κ-closed to get the p_γ's as well as to ensure that $\mathcal{M}_\gamma, \mu_\gamma \in V$.
[5]This terminology is due to Matt Foreman.

branches of $\mathcal{U}$. Pick $X \in \mathcal{U}$ and let p be a play of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ with limit model $\mathcal{M}$ and limit measure μ, such that $X \in \mu$.

By letting player I in p play sufficiently large ordinals, we may assume that $\mathcal{U} \subseteq \mathcal{M}$, using that $|\mathcal{U}| \leq \kappa$, and also that $b_X := \mathcal{U} \cap \mu \in \mathcal{M}$. This means that $d_X := \Delta b_X \in \mathscr{P}(\kappa)/I_p^\tau$ is a limit point of the branch b_X through $\mathcal{U}$, so that $\mathcal{A} := \{d_X \mid X \in \mathcal{U}\}$ is a maximal antichain of limit points of branches of $\mathcal{U}$, making $\mathscr{P}(\kappa)/I_p^\tau$ (κ, κ)-distributive. $\dashv$

Fix some limit ordinal $\eta < \kappa^+$. We will recursively construct a tree $\mathcal{T}^\eta$ of height λ which consists of subsets $X \subseteq \kappa$, ordered by reverse inclusion. During the construction of the tree we will inductively maintain the following properties of $\mathcal{T}^\eta \restriction \alpha$ for $\alpha \leq \lambda$:

- Tree strategy: For every $\gamma < \alpha$ there is a winning strategy τ_γ^η for player II in $\mathcal{G}(\kappa) \restriction \eta$ such that for every $\beta < \gamma$, the β'th move by τ_γ^η is an element of $\mathcal{T}_\beta^\eta$ and τ_γ^η is consistent with τ_β^η for the first β-many rounds.
- Unique pre-history: Given any $\beta < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$ there's a unique partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ_β^η ending with Y – we define $I_Y^\tau := I_p^\tau$ for τ being any winning strategy for player II in $\mathcal{G}(\kappa) \restriction \eta$ satisfying that p is consistent with τ_β^η.
- Cofinally many responds: Let $\beta + 1 < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$, and set p to be the unique partial play of $\mathcal{G}(\kappa) \restriction \eta$ given by the unique pre-history of Y. Then the $\mathcal{T}^\eta$-successors of Y consists of player II's τ_β^η-responds to τ_β^η-partial plays extending p such that player I's last move in these partial plays are cofinal in η.[6]
- Positivity: If $\beta < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$ then Y is $I_X^{\tau_\gamma^\eta}$-positive for every $\gamma < \beta$ and every $X \in \mathcal{T}^\eta \restriction \gamma + 1$ with $X \leq_{\mathcal{T}^\eta} Y$.[7]
- Almost disjointness property: Every level $\mathcal{T}_\beta^\eta$ consists of pairwise almost disjoint sets.[8]

[6] The reason why we're dealing with the *restricted* auxilliary games is to achieve this property.
[7] This actually follows from the cofinally many responds, but we include it here for transparency.
[8] Two subsets $X, Y \subseteq \kappa$ are *almost disjoint* if $|X \cap Y| < \kappa$.

- Hopeless ideal coherence: $I_{\langle\rangle}^{\tau_\beta^\eta} \cap \mathscr{P}(Y) = I_Y^{\tau_\beta^\eta} \cap \mathscr{P}(Y)$ for every $\beta < \alpha$ and $Y \in \mathcal{T}_\beta^\eta$.

Note that what we're really aiming for is achieving the hopeless ideal coherence, since that enables us to ensure that if $X, Y \in \mathcal{T}^\eta$ and $X \subseteq Y$ then really $X \geq_{\mathcal{T}^\eta} Y$ – i.e. that we "catch" both X and Y in the same play of $\mathcal{G}(\kappa) \restriction \eta$. The rest of the properties are inductive properties we need to ensure this.

Set $\mathcal{T}_0^\eta := \{\kappa\}$. Assume that we've built $\mathcal{T}^\eta \restriction \alpha + 1$ satisfying the inductive assumptions[9] and let $Y \in \mathcal{T}_\alpha^\eta$ – we need to specify what the $\mathcal{T}^\eta$-successors of Y are. Since κ is weakly compact and not measurable it holds by Proposition 6.4 in [Kanamori, 2008] that $\text{sat}(I_Y^{\tau_\alpha^\eta}) \geq \kappa^+$, so we can fix a maximal antichain $\langle X_\gamma^Y \mid \gamma < \eta \rangle$ of $I_Y^{\tau_\alpha^\eta}$-positive sets. By κ-completeness of $I_Y^{\tau_\alpha^\eta}$ we can by Exercise 22.1 in [Jech, 2006] even ensure that all of the X_γ^Y's are pairwise disjoint.

To every $\gamma < \eta$ we fix a partial play p of even length of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ_α^η such that the last ordinal β_γ^Y in p played by player I is greater than or equal to γ and X_γ^Y has measure one with respect to the last measure in p. We then define the $\mathcal{T}^\eta$-successors of Y to be player II's τ_α^η-responses to the β_γ's (which are subsets of the X_γ^Y's modulo a bounded set and are therefore pairwise almost disjoint).

For limit stages $\delta < \lambda$ we apply τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ to get $\mathcal{T}_\delta^\eta$.

We now have to check that the inductive assumptions still hold; let's start with the tree strategy. Assume that we have a partial play p of length $2 \cdot \alpha + 1$ of $\mathcal{G}(\kappa) \restriction \eta$, i.e. the last move in p is by player II, consistent with τ_α^η; write ξ_p for player I's last move in p and Y_p for player II's response to ξ_p, which is also the last move in p. We can then pick a $\zeta < \eta$ such that $\beta_\zeta^{Y_p} > \xi_p$ by the cofinally many responds property and let $\tau_{\alpha+1}^\eta(p)$ be player II's τ_α^η-response to the partial play leading up to $\beta_\zeta^{Y_p}$. After this $(\alpha + 1)$'th round we just set $\tau_{\alpha+1}^\eta$ to follow τ_0. It's clear that $\tau_{\alpha+1}^\eta$ satisfies the required properties.

Before we move on to checking the remaining inductive assumptions, let's pause to get some intuition about the tree strategies. In the definition of $\tau_{\alpha+1}^\eta$ above, we took a partial play consistent with τ_α^η, applied τ_0 for a while, took note of player II's last τ_0-response and then included *only that* response in our new $\tau_{\alpha+1}^\eta$ partial play. This means that to every τ_α^η-partial play there's an ostensibly much longer

[9] In particular, we assume that τ_α^η is defined.

τ_0-partial play into which τ^η_α embeds; so we can look at the τ^η_α-partial plays as being "collapsed" τ_0-partial plays.

Given the above tree strategy, $\mathcal{T}^\eta_{\alpha+1}$ clearly satisfies the cofinally many responds property and the positivity property, simply by construction. For the unique pre-history, let $Y \in \mathcal{T}^\eta_{\alpha+1}$ and assume it has two distinct immediate $\mathcal{T}^\eta$-predecessors $Z_0, Z_1 \in \mathcal{T}^\eta_\alpha$. But then $Y \subseteq Z_0 \cap Z_1$ and Y is $I^{\tau^\eta_\alpha}_{Z_0}$-positive by the positivity assumption, contradicting that Z_0 and Z_1 are almost disjoint by the almost disjointness property. Given the unique pre-history we then also get the almost disjointness property.

Claim 4.11. $\mathcal{T}^\eta \restriction \alpha + 2$ satisfies the hopeless ideal coherence property.

Proof of claim. Let $Y \in \mathcal{T}^\eta_{\alpha+1}$ – we have to show that

$$I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y). \tag{1}$$

It's clear that $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle} \subseteq I^{\tau^\eta_{\alpha+1}}_{Y}$, so let $Z \in I^{\tau^\eta_{\alpha+1}}_{Y} \cap \mathscr{P}(Y)$ and assume for a contradiction that Z is $I^{\tau^\eta_{\alpha+1}}_{\langle\rangle}$-positive. Letting $\vec{\alpha}_\xi * \vec{Y}_\xi$ be a play of $\mathcal{G}(\kappa) \restriction \eta$ consistent with $\tau^\eta_{\alpha+1}$ such that Z is in the final measure, the definition of $\tau^\eta_{\alpha+1}$ yields that $Y_\alpha \in \mathcal{T}^\eta_{\alpha+1}$. As $Z \in I^{\tau^\eta_{\alpha+1}}_{Y}$ we have to assume that $Y \neq Y_\alpha$, so that the almost disjointness property implies that

$$|Y \cap Y_\alpha| < \kappa, \tag{2}$$

By the choice of $\vec{\alpha}_\xi * \vec{Y}_\xi$ there's some $\delta \in (\alpha, \lambda)$ such that $|Y_\delta - Z| < \kappa$, i.e. that Y_δ is a subset of Z modulo a bounded set, since the Y_α's generate the final measure of the play. But then $Y_\delta \subseteq Y_\alpha$ by the rules of $\mathcal{G}(\kappa) \restriction \eta$, and also that $|Y_\delta - Y| < \kappa$ since $Z \subseteq Y$. But this means that $Y \cap Y_\alpha$ is $I^{\tau^\eta_{\alpha+1}}_{Y}$-positive since Y_δ is, contradicting (2). This shows (1). $\dashv$

This finishes the construction of $\mathcal{T}^\eta_{\alpha+1}$. For limit levels $\delta < \lambda$ we define τ^η_δ as simply applying τ_0 to the branches of $\mathcal{T}^\eta \restriction \delta$ – showing that the inductive assumptions hold at $\mathcal{T}^\eta_\delta$ is analogous to the above arguments, so we're now done with the construction of $\mathcal{T}^\eta$. Let $\tau^\eta := \bigcup_{\alpha<\lambda} \tau^\eta_\alpha \restriction {}^{<\alpha}H_{\kappa^+}$ and define[10] $\mathcal{I}^\eta := I^{\tau^\eta}_{\langle\rangle}$.

Now note that $\mathcal{I}^{\eta+1} \subseteq \mathcal{I}^\eta$ and $\mathcal{T}^\eta \subseteq \mathcal{T}^{\eta+1}$ for every $\eta < \kappa^+$ – set $\mathcal{I} := \bigcap_{\eta<\kappa^+} \mathcal{I}^\eta$ and $\mathcal{T} := \bigcup_{\eta<\kappa^+} \mathcal{T}^\eta$. We showed that all hopeless ideals are κ-complete, normal and (κ,κ)-distributive, so this holds in particular for the $\mathcal{I}^\eta$'s and thus also for $\mathcal{I}$.

We claim that $\mathcal{T}$ is dense in $\mathscr{P}(\kappa)/\mathcal{I}$.[11] Let X be an $\mathcal{I}$-positive set, making it $\mathcal{I}^\eta$-positive for some $\eta < \kappa^+$, meaning that there's a play $\vec{\alpha}_\gamma * \tau^\eta$ of $\mathcal{G}(\kappa)\restriction\eta$ such that X is in the final measure, which means that $|Y_\delta - X| < \kappa$ for some large $\delta < \lambda$ and in particular that $Y_\delta - X \in \mathcal{I}$. But $Y_\delta \in \mathcal{T}^\eta \subseteq \mathcal{T}$ by definition of τ^η, which shows that $\mathcal{T}$ is dense.

It remains to show that $\mathcal{T}$ is $<\lambda$-closed. If $\lambda = \omega$ then this is trivial, so assume that $\lambda \geq \omega_1$. Let $\beta < \lambda$ and let $\langle Z_\alpha \mid \alpha < \beta\rangle$ be a $\subseteq$-decreasing sequence of elements $Z_\alpha \in \mathcal{T}$. We can fix some $\eta < \kappa^+$ such that $Z_\alpha \in \mathcal{T}^\eta$ for every $\alpha < \beta$ by regularity of κ^+, and since the Z_α's are $\subseteq$-decreasing they must also be $\leq_{\mathcal{T}^\eta}$-increasing by the hopeless ideal coherence for $\mathcal{T}^\eta$[12].

Let $\tilde{Z} \in \mathcal{T}^\eta$ be player II's τ^η-response to the unique partial play of $\mathcal{G}(\kappa)\restriction\eta$ corresponding to the branch containing the Z_α's, and pick $Z \in \mathcal{T}^\eta$ such that $\left|Z - \tilde{Z}\right| < \kappa$ and $Z \geq_{\mathcal{T}^\eta} Z_\alpha$ for all $\alpha < \beta$, again by the density claim and the hopeless ideal coherence. Then Z witnesses $<\lambda$-closure of $\mathcal{T}$.[13] ■

With a bit more work we can from this result then derive the following equivalences.

Corollary 4.12 (N.). *Let κ be a regular cardinal and $\lambda \in [\omega_1, \kappa^+]$ be regular. Then the following are equivalent:*

- *(i) κ is $<\lambda$-closed faintly power-measurable;*
- *(ii) κ is $<\lambda$-closed ideally power-measurable;*
- *(iii) κ is (κ,κ)-distributive $<\lambda$-closed faintly measurable;*
- *(iv) κ is (κ,κ)-distributive $<\lambda$-closed ideally measurable;*
- *(v) Player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$.*

[10] Note that the tree strategy property above ensures that the strategies *do* line up, so that τ^η is a well-defined strategy as well.

[11] This means that given any $\mathcal{I}$-positive set X there's a $Y \in \mathcal{T}$ such that $Y - X \in \mathcal{I}$.

[12] This is the only place in which we're using hopeless ideal coherence.

[13] We're using that λ is regular to get Z.

PROOF. $(v) \Rightarrow (iv)$ is Theorem 4.9 above[14] and $(iv) \Rightarrow (iii) + (ii)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial, so we show $(i) \Rightarrow (v)$.

Assume κ is $<\lambda$-closed faintly power-measurable, so there's a $<\lambda$-closed forcing $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there exists a transitive class N and a κ-powerset preserving elementary embedding $\pi : V \to N$. Write μ for the induced weakly amenable V-normal V-measure on κ. Now, back in V, define a strategy σ for player II in $G_\lambda(\kappa)$ as follows.

Whenever player I plays some model M_α then we let player II respond with a filter μ_α such that, for some $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash \ulcorner \check{\mu}_\alpha = \dot{\mu} \cap \check{M}_\alpha \urcorner$ – such a filter exists because μ is weakly amenable. We require the p_α's to be decreasing, which is possible by $<\lambda$-closure. Now, all the μ_α's are clearly M_α-normal M_α-measures on κ, which makes σ a winning strategy. ■

Note that the above results all relied on λ being uncountable to achieve wellfoundedness of the generic ultrapower. If we simply ignore this wellfoundedness aspect then we get the following similar equivalence in the $\lambda = \omega$ case, which then also includes completely ineffable cardinals.

COROLLARY 4.13 (N.). *Let κ be a regular cardinal. Then the following are equivalent:*[15]

- *(i) There exists a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a weakly amenable V-normal V-measure on κ;*
- *(ii) There exists a (κ, κ)-distributive forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure on κ;*
- *(iii) κ carries a normal (κ, κ)-distributive ideal;*
- *(iv) Player II has a winning strategy in $\mathcal{G}^-_\omega(\kappa)$;*
- *(v) κ is completely ineffable.*

PROOF. $(iv) \Leftrightarrow (v)$ was shown in Theorem 3.29, and $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. $(i) \Rightarrow (iv)$ is as $(i) \Rightarrow (v)$ in Corollary 4.12, and $(iv) \Rightarrow (iii)$ is Theo-

[14] Here wellfoundedness of the generic ultrapower is automatic since λ has uncountable cofinality.

[15] Points (i) and (ii) look a lot like the definition of faintly power-measurable and (κ, κ)-distributive ideally measurable, but here we're not requiring the ultrapowers to be well-founded, so that would be stretching the definition of being measurable.

rem 4.9. ■

As an immediate consequence we then get another ideal-absoluteness result.

Corollary 4.14. *"(κ,κ)-distributive $<\lambda$-closed" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■

We get the following similar results for the $\mathcal{C}$-games[16].

Theorem 4.15 (N.). *Let κ and $\lambda \leq \kappa^+$ be regular infinite cardinals such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. If player II has a winning strategy in $\mathcal{C}^-_\lambda(\kappa)$ then κ carries a λ-complete ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $Add(\lambda, 1)$.*

Proof. If $\lambda = \kappa^+$ then we're done by Theorem 4.9, since $\mathcal{G}_{\kappa^+}(\kappa)$ is equivalent to $\mathcal{C}_{\kappa^+}(\kappa)$, so assume that $\lambda \leq \kappa$. We follow the proof of Theorem 4.9 closely. Set $\mathbb{P} := \mathrm{Col}(\lambda, 2^\kappa)$. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \lambda\rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \lambda$;
- $\kappa{+}1 \subseteq \mathcal{N}_\gamma$ and $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for every $\gamma < \lambda$;
- If $\delta < \lambda$ is a limit ordinal then $\mathcal{N}_\delta = \bigcup_{\gamma<\delta}\mathcal{N}_\gamma$, $\mathcal{N}_\delta \prec H_{\kappa^+}$ and $\mathcal{N}_\delta \models \mathsf{ZFC}^-$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for all $\gamma < \beta < \lambda$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\lambda}\mathcal{N}_\gamma$.

Define the auxilliary game $\mathcal{G}(\kappa)$ as in the proof of Theorem 4.9 but where player I plays ordinals $\alpha_\eta < \lambda$ and where we use the above $\mathcal{N}_\gamma$'s. Here we only need $<\lambda$-closure of $\mathbb{P}$ to get an equivalence between $\mathcal{G}(\kappa)$ and $\mathcal{C}^-_\lambda(\kappa)$, since $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for all $\gamma < \lambda$.

To every limit ordinal $\eta < \lambda$ we define the restricted auxilliary game $\mathcal{G}(\kappa) \restriction \eta$ as in the proof of Theorem 4.9, and to every winning strategy τ in $\mathcal{G}(\kappa) \restriction \eta$ and

[16] Theorem 4.15 is the reason for naming the $\mathcal{C}$-games "Cohen games".

partial play p of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ define the associated **hopeless ideal**[17]

$$I_p^\tau\restriction\eta := \{X\subseteq\kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction\eta,$$
$$X \text{ is } \textit{not} \text{ in the final measure}\}.$$

As in the proof of Claim 4.10 we get that every hopeless ideal is λ-complete.

Now, if κ is measurable then we trivially get the conclusion,[18] so assume κ isn't measurable. Then $\text{sat}(\kappa)\geq\lambda$ since $2^{<\theta}<\kappa$ for every $\theta<\lambda$,[19] so that we can continue exactly as in the proof of Theorem 4.9 to construct (λ-sized) trees $\mathcal{T}^\eta$ and winning strategies τ^η for all limit ordinals $\eta<\lambda$ such that, setting $\mathcal{I} := \bigcap_{\eta<\lambda} I_{\langle\rangle}^{\tau^\eta}$ and $\mathcal{T} := \bigcup_{\eta<\lambda}\mathcal{T}^\eta$, $\mathcal{T}$ is a dense $<\lambda$-closed subset of $\mathscr{P}(\kappa)/\mathcal{I}$ of size λ, so that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $\text{Add}(\lambda,1)$. ■

Corollary 4.16 (N.). *Let κ and $\lambda\in[\omega_1,\kappa^+]$ be regular such that $2^{<\theta}<\kappa$ for every $\theta<\lambda$. Then the following are equivalent:*

(i) κ is $<\lambda$-closed faintly measurable;
(ii) κ is $<\lambda$-closed ideally measurable;
(iii) κ is $<\lambda$-closed λ-sized faintly measurable;
(iv) κ is $<\lambda$-closed λ-sized ideally measurable;
(v) Player II has a winning strategy in $\mathcal{C}_\lambda(\kappa)$.

Proof. $(iv)\Rightarrow(iii)+(ii)$, $(ii)\Rightarrow(i)$ and $(iii)\Rightarrow(i)$ all trivial, and $(i)\Rightarrow(v)$ is like $(i)\Rightarrow(v)$ in Corollary 4.12, and $(v)\Rightarrow(iv)$ is Theorem 4.15. ■

Again, if we ignore wellfoundedness then we get the same equivalence in the $\lambda=\omega$ case:

Corollary 4.17 (N.). *Let κ be regular infinite. Then:*

(i) Player II has a winning strategy in $\mathcal{C}_\omega^-(\kappa)$; and
(ii) κ carries an ideal I such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to $Add(\omega,1)$.

[17] This terminology is due to Matt Foreman.
[18] Take $\mathcal{I}(\text{Add}(\lambda,1),\check{\mu})$ for μ the measure on κ.
[19] See Proposition 16.4 in [Kanamori, 2008].

PROOF. Player II has a winning strategy in $\mathcal{C}^-_\omega(\kappa)$ as we're simply measuring finitely many sets without any demand for wellfoundedness, showing (i). Since $2^{<n} < \kappa$ for all $n < \omega$ as κ is infinite, Theorem 4.15 then implies (ii). ■

COROLLARY 4.18. *"$<\lambda$-closed λ-sized" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*
■

5 Further questions

Here we record many open questions related to the content of the preceeding chapters, broadly separated by topic.

5.1 Berkeleys

Question 1.7 in [Wilson, 2018] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually prestrong cardinal, and Lemmata 2.5 and 2.8 in the same paper also shows that being ω-Erdős is equivalent to being virtually club berkeley and that the least such is also the least virtually berkeley.[1]

Furthermore, they also showed that a non-Σ_2-reflecting virtually prestrong cardinal is equivalent to a virtually prestrong cardinal which isn't virtually strong. We can therefore reformulate their question to the following equivalent question.

Question 5.1 (Wilson). If there exists a virtually prestrong cardinal which is not virtually strong, is there then a virtually berkeley cardinal?

[Wilson, 2018] showed that their question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem 2.12 we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 5.2. If there exists a cardinal κ which is virtually (θ, ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually berkeley cardinal?

Theorem 2.43 from Chapter 2 at least gives a partially positive result, noting that the assumption by definition implies that On is virtually prewoodin but not virtually woodin.

[1] Note that this also shows that virtually club berkeley cardinals and virtually berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

Corollary 5.3 (N.). *If there exists a virtually A-prestrong cardinal for every class A and there are no virtually strong cardinals, then there exists a virtually berkeley cardinal.* ■

The assumption that there is a virtually A-prestrong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 2.43 shows that this is necessary, which might lead one to think that the question could have a negative answer.

5.2 Relations between virtuals

The analysis in Chapter 2 showed several implication and separation results between the virtual large cardinals. A few of these relations remain open, however.

Question 5.4. Are virtually θ-strong cardinals, virtually θ-supercompacts and virtually θ-Magidor-supercompacts all equivalent, for any uncountable regular cardinal θ?

Question 5.5. Let θ be an uncountable cardinal.

(i) Is every virtually θ-measurable cardinal also virtually θ-prestrong? What if we assume $V = L[\mu]$ or $V = K$, with K being the core model below a woodin cardinal?

(ii) Is every virtually θ-strong cardinal virtually θ-supercompact? Are they equiconsistent?

5.3 Indestructibility

Our original goal concerning indestructibility was to see what indestructibility properties the faintly supercompacts have, whether any analogy with the supercompact cardinals holds. This still remains open.

Question 5.6. Do faintly supercompact cardinals have indestructibility properties? For instance, if κ is faintly supercompact, does it remain supercompact after forcing with $\mathrm{Add}(\kappa, 1)$?

We proved several indestructibility properties of the ostensibly stronger notion of *generically setwise supercompacts*, and several questions then arise concerning the nature of these cardinals.

Question 5.7. What's the consistency strength of the generically setwise supercompact cardinals? The best upper bound is a virtually extendible, as given by Usuba's Theorem 2.56, and a lower bound is the trivial faintly supercompact one. What if we require the cardinal to be inaccessible?

Question 5.8. Is it consistent to have a faintly supercompact cardinal which isn't generically setwise supercompact?

Question 5.9. Assume there exists no inner model with a woodin cardinal. Can there then exist generically setwise supercompact cardinals in K?

5.4 Games and small embeddings

Our results in Chapter 3 provide answers to the following questions, which were posed in [Holy and Schlicht, 2018].

(i) If γ is an uncountable cardinal and the challenger does not have a winning strategy in the game $\mathcal{G}^\theta_\gamma(\kappa)$, does it follow that the judge has one?
(ii) If $\omega \leq \alpha \leq \kappa$, are α-Ramsey cardinals downwards absolute to the Dodd-Jensen core model?
(iii) Does 2-iterability imply ω-Ramseyness, or conversely?
(iv) Does κ having the strategic κ-filter property have the consistency strength of a measurable cardinal?

Here the "challenger" is player I and the "judge" is player II, so this is asking if every γ-Ramsey is strategic γ-Ramsey, when γ is an uncountable cardinal. Theorem 3.64 therefore gives a negative answer to (i) for all uncountable ordinals γ. Theorem 3.52 and Corollary 3.53 answer (ii) positively, for α-Ramseys with α having uncountable cofinality, and for $<\alpha$-Ramseys when α is a limit of limit ordinals. Note that (ii) in the $\alpha = \omega$ case was answered positively in [Holy and Schlicht, 2018].

As for (iii), it's mentioned in [Holy and Schlicht, 2018] that Gitman has showed that ω-Ramseys are not in general 2-iterable by showing that 2-iterables have strictly stronger consistency strength than the ω-Ramseys, which also follows from Theorem 3.37 and Theorem 4.8 in [Gitman and Welch, 2011]. Corollary 3.19 shows that ω-Ramsey cardinals are Δ^2_0-indescribable, and as 2-iterables are (at least) Π^1_3-definable it holds that any 2-iterable ω-Ramsey cardinal is a limit of 2-iterables, so that in general 2-iterables can't be ω-Ramsey either, answering (iii) in the negative. Lastly, Theorem 3.48 gives a positive answer to (iv).

We conjecture the following two questions to be true. The first is a direct analogue to Theorem 3.30, and the latter is a suspected analogy between the genuine n-Ramsey cardinals and the weakly ineffable cardinals.

Question 5.10. If κ is faintly θ-power-measurable, does player II then have a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$?

Question 5.11. Are genuine n-Ramsey cardinals limits of n-Ramsey cardinals? We conjecture this to be true, in analogy with the weakly ineffables being limits of weakly compacts. Since "weakly ineffable = Π^1_1-indescribability + subtlety", this might involve some notion of "n-iterated subtlety". The difference here is that n-Ramseys cannot be *equivalent* to Π^1_{2n+1}-indescribables for consistency reasons, so there is some work to be done.

We showed in Theorem 3.29, see also Corollary 4.13 that completely ineffable cardinals could be characterised in terms of player II having a winning strategy in $\mathcal{G}^-_\omega(\kappa)$. This lends itself to the following question.

Question 5.12. Are there higher analogues of ineffability which are equivalent to player II having a winning strategy in $\mathcal{G}^-_\alpha(\kappa)$ for $\alpha > \omega$?

5.5 Ideal absoluteness

One can ask of any poset property whether it is ideal-absolute, but we choose to only highlight one particular property here. We saw in Corollary 4.12 that $<\lambda$-closed faintly power-measurables "corresponds to" (κ, κ)-distributive $<\lambda$-closed forcings, and in Corollary 4.13 that completely ineffable cardinals "corresponds to" (κ, κ)-distributive forcings. In an attempt to find the forcing that corresponds to the faintly power-measurables, we arrive at the following question.

Question 5.13. For κ a regular cardinal, are the following equivalent?

(i) κ is faintly power-measurable;

(ii) κ is ideally power-measurable;

(iii) κ is (κ, κ)-distributive ω-distributive faintly measurable;

(iv) κ is (κ, κ)-distributive ω-distributive ideally measurable;

(v) Player II has a winning strategy in $\mathcal{G}_\omega(\kappa)$.

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