Taking the blue pill:
Virtual Set Theory

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[Month] 2020

Abstract

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Acknowledgements

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Other chain conditions?

Introduction

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Notation

We will denote the class of ordinals by On. For X, Y sets we denote by ${}^{X}Y$ the set of all functions from X to Y. For an infinite cardinal κ, we let H_κ be the set of sets X such that the cardinality of the transitive closure of X is $<\kappa$. ZF^- will denote ZF with the Collection scheme but without the Power Set axiom, following the results of [Gitman et al., 2015]. The symbol $\lightning$ will denote a contradiction and $\mathscr{P}(X)$ denotes the power set of X. We will denote elementary embeddings $\pi\colon (\mathcal{M}, \in) \to (\mathcal{N}, \in)$ by simply $\pi\colon \mathcal{M} \to \mathcal{N}$.

Part I

A Tripod

1 | Virtual Large Cardinals

1.1 Getting started

A key folklore lemma which we will frequently need when dealing with elementary embeddings existing in generic extensions is the following.

Lemma 1.1.1 (Countable Embedding Absoluteness). *Let $\mathcal{M}, \mathcal{N}$ be sets, $\mathcal{P}$ a transitive class with $\mathcal{M}, \mathcal{N} \in \mathcal{P}$, and let $\pi\colon \mathcal{M} \to \mathcal{N}$ be an elementary embedding. Assume that*

$$\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC} + \ulcorner \mathcal{M} \text{ is countable} \urcorner$$

and fix any finite $X \subseteq \mathcal{M}$. Then $\mathcal{P}$ contains an elementary embedding $\pi^*\colon \mathcal{M} \to \mathcal{N}$ which agrees with π on X. If π has a critical point and if $\mathcal{M}$ and $\mathcal{N}$ are both transitive then we can also assume that $\operatorname{crit}\pi = \operatorname{crit}\pi^*$.[1]

Proof. Let $\{a_i \mid i < \omega\} \in \mathcal{P}$ be an enumeration of $\mathcal{M}$ and set $\mathcal{M}\restriction n := \{a_i \mid i < n\}$. Then, in $\mathcal{P}$, build the tree $\mathcal{T}$ of all partial isomorphisms between $\mathcal{M}\restriction n$ and $\mathcal{N}$ for $n < \omega$, ordered by extension. Then $\mathcal{T}$ is illfounded in V by assumption, so it's also illfounded in $\mathcal{P}$ since $\mathcal{P}$ is transitive and $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC}$. The branch then gives us the embedding π^*, and if $\operatorname{crit}\pi$ exists then we can ensure that it agrees with π on the critical point and finitely many values by adding these conditions to $\mathcal{T}$. ■

1.2 Strong and supercompact

We start out by defining virtual versions of a variety of large cardinal notions used in this section. We start out with measurables, strongs and supercompacts.

[1] We are using transitivity of $\mathcal{M}$ and $\mathcal{N}$ to ensure that the *ordinal* $\operatorname{crit}\pi$ exists.

Definition 1.2.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is...

Change the names of these?

- **faintly θ-measurable** if, in a forcing extension, there is a transitive class $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$;
- **faintly θ-strong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$;
- **faintly θ-supercompact** if it's faintly θ-measurable, ${}^{<\theta}\mathcal{N} \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

We further replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we attach a **"pre"** if we don't want to assume $\pi(\kappa) > \theta$, and when we don't mention θ we mean that it holds for all regular $\theta > \kappa$. For instance, a faintly prestrong cardinal is a cardinal κ such that for all regular $\theta > \kappa$, κ is faintly θ-measurable with $H_\theta^V \subseteq \mathcal{N}$. $\circ$

We note that even small cardinals can be faintly measurable: we may for instance have a precipitous ideal on ω_1; see [Jech, 2006, Theorem 22.33]. The "virtually" adverb implies that the cardinals are in fact large cardinals in the usual sense, as Proposition 1.2.2 below shows.

Recall from [Gitman and Schindler, 2018] that a cardinal κ is **1-iterable** if to every $A \subseteq \kappa$ there's a transitive $\mathcal{M} \models \mathsf{ZFC}^-$ with $\kappa, A \in \mathcal{M}$ and a weakly amenable $\mathcal{M}$-ultrafilter μ on κ with a wellfounded ultrapower.[2] 1-iterable cardinals are weakly compact limits of weakly compact cardinals; see [Gitman, 2011, Theorem 1.7].

Proposition 1.2.2 (Virtualised folklore). *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is 1-iterable.*

Proof. (Sketch) Let κ be virtually θ-measurable, witnessed by a forcing $\mathbb{P}$, a transitive $\mathcal{N} \subseteq V$ and an elementary $\pi\colon H_\theta^V \to \mathcal{N}$ with $\pi \in V^{\mathbb{P}}$. If κ isn't a strong limit then we have a surjection $\pi(f)\colon \mathscr{P}(\alpha) \to \pi(\kappa)$ with $\operatorname{ran}\pi(f) = \operatorname{ran} f \subseteq \kappa$ for some $\alpha < \kappa$, $\lightning$. Note that we used $\mathcal{N} \subseteq V$ to ensure that $\mathscr{P}(\alpha)^V = \mathscr{P}(\alpha)^{\mathcal{N}}$. The same argument shows that κ is regular. By restricting the generic embedding and using that $\mathscr{P}(\kappa)^V = \mathscr{P}(\kappa)^N$ as

[2] An $\mathcal{M}$-ultrafilter μ is **weakly amenable** if $\mu \cap X \in \mathcal{M}$ for every $X \in \mathcal{M}$ of $\mathcal{M}$-cardinality $\leq \kappa$.

$\mathcal{N} \subseteq V$ and $\mathscr{P}(\kappa)^V \subseteq \mathcal{N}$, we get that κ is 1-iterable. ■

Along with the above definition of faintly supercompactness we can also virtualise Magidor's characterisation of supercompact cardinals, which was one of the original characterisations of the remarkable cardinals in [Schindler, 2000a].

Definition 1.2.3. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-supercompact ala Magidor** if there are $\bar{\kappa} < \bar{\theta} < \kappa$ and a generic elementary $\pi \colon H^V_{\bar{\theta}} \to H^V_\theta$ such that $\operatorname{crit} \pi = \bar{\kappa}$ and $\pi(\bar{\kappa}) = \kappa$. ○

In the virtual world these two versions of supercompacts remain equivalent, but they also turn out to be equivalent to the virtually strongs:

Theorem 1.2.4 (Gitman-Schindler)**.** *For an uncountable cardinal κ, the following are equivalent.*[3]

(i) *κ is virtually strong;*

(ii) *κ is virtually supercompact;*

(iii) *κ is virtually supercompact ala Magidor.*

Proof. $(ii) \Rightarrow (i)$ is simply by definition.

$(i) \Rightarrow (iii)$: Fix $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi \colon H^V_{(2^{<\theta})^+} \to \mathcal{M}$ with[4] $\operatorname{crit} \pi = \kappa$, $\pi(\kappa) > \theta$, $H^V_{(2^{<\theta})^+} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. Since $H^V_\theta, H^{\mathcal{M}}_{\pi(\theta)} \in \mathcal{M}$, Countable Embedding Absoluteness 1.1.1 implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H^V_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ with $\operatorname{crit} \pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H^V_\theta = H^{\mathcal{M}}_\theta$ as $\mathcal{M} \subseteq V$ and $H^V_\theta \subseteq \mathcal{M}$, elementarity of π now implies that $H^V_{(2^{<\theta})^+}$ has ordinals $\bar{\kappa} < \bar{\theta} < \kappa$ and a generic elementary $\sigma \colon H^V_{\bar{\theta}} \to H^V_\theta$ with $\operatorname{crit} \sigma = \bar{\kappa}$ and $\sigma(\bar{\kappa}) = \kappa$. This shows (iii).

$(iii) \Rightarrow (ii)$: Fix $\theta > \kappa$ and $\delta := (2^{<\theta})^+$. By (iii) there exist ordinals $\bar{\kappa} < \bar{\delta} < \kappa$ and a generic elementary embedding $\pi \colon H^V_{\bar{\delta}} \to H^V_\delta$ with $\operatorname{crit} \pi = \bar{\kappa}$ and $\pi(\bar{\kappa}) = \kappa$. We will argue that $\bar{\kappa}$ is virtually $\bar{\theta}$-supercompact in $H^V_{\bar{\delta}}$,

[3] A cardinal satisfying any/all of these conditions is usually called **remarkable**.
[4] The domain of π is $H^V_{(2^{<\theta})^+}$ to ensure that $H^V_\theta \in \operatorname{dom} \pi$.

so that by elementarity κ is virtually θ-supercompact in H_δ^V and hence also in V by the choice of δ. Consider the restriction

$$\sigma := \pi \upharpoonright H_{\bar\theta}^V : H_{\bar\theta}^V \to H_\theta^V.$$

Note that H_θ^V is closed under $<\bar\theta$-sequences (and more) in V. Now define

$$X := \bar\theta{+}1 \cup \{x \in H_\theta^V \mid \exists y \in H_{\bar\theta}^V \exists p \in \mathrm{Col}(\omega, H_{\bar\theta}^V) \colon p \Vdash \dot\sigma(\check y) = \check x\} \in V.$$

Note that $|X| = \left|H_{\bar\theta}^V\right| = 2^{<\bar\theta}$ and that $\operatorname{ran}\sigma \subseteq X$. Now let $\overline{\mathcal{M}} \prec H_\theta^V$ be such that $X \subseteq \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{\mathcal{M}}$ of size $(2^{<\bar\theta})^{<\bar\theta} = 2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{\mathcal{M}}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}| = 2^{<\bar\theta} < \bar\delta$, making $\mathcal{M} \in H_{\bar\delta}^V$.

Countable Embedding Absoluteness 1.1.1 then implies that $H_{\bar\delta}^V$ has a generic elementary embedding $\sigma^* \colon H_{\bar\theta}^V \to \mathcal{M}$ with $\operatorname{crit}\sigma^* = \bar\kappa$, showing that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H_{\bar\delta}^V$, which is what we wanted to show. ■

Remark 1.2.5. The above proof shows that if κ is virtually $(2^{<\theta})^+$-strong then it's virtually θ-supercompact, and if it's virtually $(2^{<\theta})^+$-supercompact ala Magidor then it's virtually θ-supercompact. It's open whether they are equivalent level-by-level (see Question **??**).

A key difference between the normal large cardinals and the virtual kind is that we don't have a virtual version of the Kunen inconsistency: it's perfectly valid to have a generic elementary embedding $H_\theta^V \to H_\theta^V$ with θ much larger than the critical point. This becomes important when dealing with the "pre"-versions of the large cardinals. We start with a virtualisation of the α-superstrong cardinals.

Talk about what this is

Definition 1.2.6. Let θ be a regular uncountable cardinal and and α an ordinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ,α)**-superstrong** if it's faintly θ-measurable, $H_\theta^V \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$[5]. We replace "faintly" by **virtually**

[5]Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.

when $\mathcal{N} \subseteq V$, we say that κ is **faintly α-superstrong** if it's faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong.[6] $\circ$

Proposition 1.2.7 (N.). *If κ is faintly superstrong then H_κ has a proper class of virtually strong cardinals.*[7]

Proof. Fix a regular $\theta > \kappa$ and a generic embedding $\pi\colon H_\theta^V \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$ and $\pi(\kappa) < \theta$. Then $\pi(\kappa)$ is a V-cardinal, so that $H_{\pi(\kappa)}^V$ thinks that κ is virtually strong. This implies that H_κ^V thinks there is a proper class of virtually strong cardinals, using that $H_\kappa^V \prec H_{\pi(\kappa)}^V$. ■

The following theorem then shows that the only thing stopping prestrongness from being equivalent to strongness is the existence of "Kunen inconsistencies".

Theorem 1.2.8 (N.). *Let θ be an uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-prestrong iff either*

(i) κ is virtually θ-strong; or

(ii) κ is virtually (θ, ω)-superstrong.

Maybe replace this with virtually θ-rank-into-rank.

Proof. $(\Leftarrow)$ is trivial, so we show $(\Rightarrow)$. Let κ be virtually θ-prestrong. Assume (i) fails, meaning that there's a generic extension $V^{\mathbb{P}}$ and an elementary embedding $\pi \in V^{\mathbb{P}}$ such that $\pi\colon H_\theta^V \to \mathcal{N}$ for some transitive $\mathcal{N}$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) \leq \theta$. Assume $\pi^n(\kappa)$ is defined for all $n < \omega$ and define $\lambda := \sup_{n<\omega} \pi^n(\kappa)$. If $\lambda \leq \theta$ then κ is virtually (θ, ω)-superstrong by definition, so assume that there's some least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$.

This means that κ is virtually ν-strong for every regular $\nu \in (\kappa, \pi^n(\kappa))$, which is a Δ_0-statement in $\{H_{\nu^+}^V\}$ and hence downwards absolute to $H_{\pi^n(\kappa)}^V$. This means that κ is virtually strong in $H_{\pi^n(\kappa)}^V$ and also that $\pi^n(\kappa)$ is virtually strong in $H_{\pi^{n+1}(\kappa)}^{\mathcal{N}}$ by elementarity, and so in particular virtually θ-

[6] Note that the conventions stated here are different from the ones in Definition 1.2.1.

[7] Add more detail to this proof.

strong in $\mathcal{N}$. This means that there's some generic elementary embedding

$$\sigma\colon H_\theta^{\mathcal{N}} \to \mathcal{M}$$

with $H_\theta^{\mathcal{N}} \subseteq \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{N}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. We can now restrict σ to its critical point $\pi^n(\kappa)$ to get that

$$H^V_{\pi^n(\kappa)} = H^{\mathcal{N}}_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))},$$

using that $H_\theta^V = H_\theta^{\mathcal{N}}$ holds as π is a virtual embedding. Since κ is virtually strong in $H^V_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$. In particular, κ is virtually θ-strong in $\mathcal{M}$, and as $H_\theta^{\mathcal{M}} = H_\theta^{\mathcal{N}} = H_\theta^V$, this means that κ is virtually θ-strong in V, contradicting (i). ■

We then get the following consistency result.

Corollary 1.2.9 (N.). *For any uncountable regular θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

Add more details in the proof

Proof. The above Proposition 1.2.7 and Theorem 1.2.8 show that virtually θ-prestrongs are equiconsistent with virtually θ-strongs. Now note that Countable Embedding Absoluteness 1.1.1 and condensation in L imply that every faintly θ-measurable cardinal is virtually θ-prestrong in L. ■

reference

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_\theta^V \to H_\theta^V$ with $\operatorname{crit}\pi = \kappa$. We then have the following corollary.

Introduce large cardinals and their virtual versions in a separate chapter/appendix?

Corollary 1.2.10 (N.). *The following are equivalent:*

(i) *For every uncountable cardinal θ, every virtually θ-prestrong cardinal is virtually θ-strong;*

(ii) *There are no virtually rank-into-rank cardinals.*

Proof. ($\Leftarrow$): Note first that being virtually ω-superstrong is equivalent to being virtually rank-into-rank. Indeed, every virtually rank-into-rank cardinal is virtually ω-superstrong by definition, and if κ is virtually ω-superstrong and $\lambda := \sup_{n<\omega} \pi^n(\kappa)$ then $\pi \restriction H_\lambda^V : H_\lambda^V \to H_\lambda^V$ witnesses that κ is virtually λ-rank-into-rank. The above Theorem 1.2.8 then implies ($\Leftarrow$).

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal then there exists a $\theta > \kappa$ and a virtually θ-prestrong cardinal which is not virtually θ-strong. Let (κ, θ) be the lexicographically least pair such that κ is virtually θ-rank-into-rank, which trivially makes κ virtually θ-prestrong. If κ was also virtually θ-strong then it would be Σ_2-reflecting, so that the statement that there exists a virtually rank-into-rank cardinal would reflect down to H_κ^V, contradicting the minimality of κ. ■

As a final result of this section, we note that the "virtually" adverb *does* yield cardinals different from the faintly ones. This is trivial in general as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

For a slightly more fine-grained distinction let's prepend a **power-** adjective whenever the domain and codomain of the generic elementary embedding have the same subsets of κ. Note that the proof of Lemma 1.2.2 shows that faintly power-measurables are also 1-iterable.

Our separation result is then the following.

Theorem 1.2.11 (Gitman)**.** *For $\Phi \in \{measurable, prestrong, strong\}$, if κ is virtually Φ then there exist forcing extensions $V[g]$ and $V[h]$ such that*

(i) In $V[g]$, κ is inaccessible and faintly Φ but not faintly power-Φ; and

(ii) In $V[h]$, κ is faintly power-Φ but not virtually Φ.

Proof. We start with (i). Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\lambda < \kappa$, and let $g \subseteq \mathbb{P}_\kappa$ be V-generic. Note that κ remains inaccessible in $V[g]$. Fix a regular $\theta > \kappa$ and let $\mathbb{Q}_\theta$ be a forcing witnessing that κ is virtually θ-measurable.

Since κ is *virtually* measurable we may without loss of generality assume that $\mathbb{Q}_\theta = \mathrm{Col}(\omega, \theta)$ by applying Countable Embedding Absoluteness 1.1.1. Fixing a $V[g]$-generic $h \subseteq \mathbb{Q}_\theta$ we get a transitive $\mathcal{N} \subseteq V$ and in $V[h]$ an elementary embedding

Include this argument?

$$\pi \colon H_\theta^V \to \mathcal{N}$$

with $\mathrm{crit}\,\pi = \kappa$. Let's now work in $V[g][h] = V[h][g] = V[g \times h]$, in which we still have access to π. The lifting criterion is trivial for $\mathbb{P}_\kappa$, so we get an $\mathcal{N}$-generic $\tilde{g} \subseteq \pi(\mathbb{P}_\kappa)$ and an elementary

Define and give reference

$$\pi^+ \colon H_\theta^{V[g]} \to \mathcal{N}[\tilde{g}]$$

with $\pi \subseteq \pi^+$. Note here that without loss of generality $\pi(\kappa)$ is countable as otherwise we replace $\mathcal{N}$ by a countable hull, so we can indeed construct such a $\tilde{g}$. By elementarity of π it holds that

$$\pi(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \prod_{\lambda \in [\kappa, \pi(\kappa))} \mathrm{Add}(\lambda, 1), \tag{1}$$

so that $\mathcal{N}[\tilde{g}] \not\subseteq V$ as it in particular contains a new subset of κ. If $\Phi =$ measurable then we're done at this point. For $\Phi =$ prestrong we simply note that $g \in \mathcal{N}[\tilde{g}]$ by (1) so that $H_\theta^{V[g]} \subseteq \mathcal{N}[\tilde{g}]$ as well, and since π^+ lifts π it holds that $\pi^+(\kappa) = \pi(\kappa) > \theta$ in the $\Phi =$ strong case.

As for (ii), we simply change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$, which means that $\pi(\mathbb{P}_\kappa)$ doesn't add any subsets of κ and κ thus remains faintly power-Φ. By choosing $\theta > \kappa^+$ it *does* add a subset to κ^+ however, showing that κ is not virtually Φ. ■

In contrast to the above separation result, Theorem 1.3.9 will show that the faintly-virtually distinction vanishes when we're dealing with woodins.

1.3 Woodin and Vopěnka

In this section we will analyse the virtualisations of the woodin and vopěnka cardinals, which can be seen as "boldface" variants of strongs and supercompacts.

Definition 1.3.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly** (θ, A)**-strong** for a set $A \subseteq H_\theta^V$ if there exists a generic elementary embedding

$$\pi \colon (H_\theta^V, \in, A) \to (\mathcal{M}, \in, B)$$

with $\mathcal{M}$ transitive, such that $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta^V \subseteq \mathcal{M}$ and $B \cap H_\theta^V = A$. We say that κ is **faintly** (θ, A)**-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$ and say that κ is **faintly** (θ, A)**-extendible** if $\mathcal{M} = H_\mu^V$ for some V-cardinal μ. We will leave out θ if it holds for all regular $\theta > \kappa$. ∘

Definition 1.3.2. A cardinal δ is **faintly woodin** if given any $A \subseteq H_\delta^V$ there exists a faintly $({<}\delta, A)$-strong cardinal $\kappa < \delta$. ∘

As with the previous definitions, for both of the above two definitions we substitute "faintly" for **virtually** when $\mathcal{M} \subseteq V$, and substitute "strong", "supercompact" and "woodin" for **prestrong**, **presupercompact** and **prewoodin** when we don't require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with the real woodin cardinals, virtually woodin cardinals are mahlo. This contrasts the virtually prewoodins since [Wilson, 2019a], together with Theorem 1.3.9 below, shows that they can be singular.

Proposition 1.3.3 (Virtualised folklore)**.** *Virtually woodin cardinals are mahlo.*

Proof. Let δ be virtually woodin. Note that δ is a limit of weakly compact cardinals by Proposition 1.2.2, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f \colon \alpha \to \delta$ with $f(0) > \alpha$

and $\alpha < \delta$, and note that f cannot have any closure points. Fix a virtually $(<\delta, f)$-strong cardinal $\kappa < \delta$; we claim that κ is a closure point for f, which will yield our desired contradiction.

Let $\gamma < \kappa$ and choose a regular $\theta \in (f(\gamma), \delta)$. We then have a generic embedding $\pi\colon (H_\theta^V, \in, f \cap H_\theta^V) \to (\mathcal{N}, \in, f^+)$ with $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$ and f^+ is a function such that $f^+ \cap H_\theta^V = f \cap H_\theta^V$. But then $f^+(\gamma) = f(\gamma) < \pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma) < \kappa$, making κ a closure point for f, $\lightning$. This shows that δ is inaccessible.

As for mahloness, let $C \subseteq \delta$ be a club and $\kappa < \delta$ a virtually $(<\delta, C)$-strong cardinal. Let $\theta \in (\min C, \delta)$ and let $\pi\colon H_\theta^V \to \mathcal{N}$ be the associated generic elementary embedding. Then for every $\gamma < \kappa$ there exists an element of C below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an element of C. As κ is regular, this shows that δ is mahlo. ■

The well-known equivalence of the "function definition" and "A-strong" definition of woodin cardinals holds if we restrict ourselves to *virtually* woodins, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Proposition 1.3.4 (Dimopoulos-Gitman-N.). *For an uncountable cardinal δ, the following are equivalent.*

(i) *δ is virtually woodin.*
(ii) *for every $A \subseteq H_\delta^V$ there exists a virtually $(<\delta, A)$-supercompact $\kappa < \delta$.*
(iii) *for every $A \subseteq H_\delta^V$ there exists a virtually $(<\delta, A)$-extendible $\kappa < \delta$.*
(iv) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*
(v) *for every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, where κ is a closure point for f, and a generic elementary $\pi\colon H_\theta^V \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, ${}^{<\pi(f)(\kappa)}\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$.*

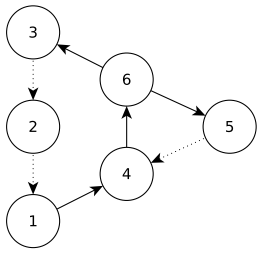


Figure 1.1: Proof strategy of Proposition 1.3.4, dotted lines are trivial implications.

(vi) *for every function* $f\colon \delta \to \delta$ *there are regular cardinals* $\bar\theta < \kappa < \theta < \delta$, *where* κ *is a closure point for* f, *and a generic elementary embedding* $\pi\colon H^V_{\bar\theta} \to H^V_\theta$ *with* $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$ *and* $f \restriction \kappa \in \operatorname{ran}\pi$.

Proof. Firstly note that $(iii) \Rightarrow (ii) \Rightarrow (i)$ and $(v) \Rightarrow (iv)$ are simply by definition.

$\boxed{(i) \Rightarrow (iv)}$ Assume δ is virtually woodin, and fix a function $f\colon \delta \to \delta$. Let $\kappa < \delta$ be virtually $({<}\delta, f)$-strong and let $\theta := \sup_{\alpha \leq \kappa} f(\alpha)+1$. Then there's a generic elementary embedding $\pi\colon (H^V_\theta, \in, f \cap H^V_\theta) \to (\mathcal{M}, \in, f^+)$ where $f^+ \restriction \kappa = f \restriction \kappa$, $\mathcal{M} \subseteq V$ and $\pi(\kappa) > \theta$. We firstly want to show that κ is a closure point for f, so let $\alpha < \kappa$. Then

$$f(\alpha) = f^+(\alpha) = \pi(f)(\alpha) = \pi(f)(\pi(\alpha)) = \pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha < \kappa$. Now, if κ wasn't a closure point for f then, letting $\alpha < \kappa$ be the least such that $f(\alpha) \geq \kappa$,

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument wouldn't work if we had only assumed δ to be virtually prewoodin. Lastly, θ-strongness implies that $H^V_\theta \subseteq \mathcal{M}$, and $\mathcal{M} \subseteq V$ holds by assumption.

$\boxed{(iv) \Rightarrow (vi)}$ Assume (iv) holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$. By (iv) there's a $\kappa < \delta$ which is a closure point of g and there's a regular $\theta \in (\kappa, \delta)$ and a generic elementary $\pi\colon H^V_\theta \to \mathcal{M}$

with $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\theta = \pi(f \restriction \kappa)(\kappa)$. We want to find a regular $\bar\theta < \kappa$ and another elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f(\operatorname{crit}\sigma) = \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\sigma$.

Note that $\mathcal{M} \subseteq V$ and $H_\theta^V \subseteq \mathcal{M}$ implies that $H_\theta^V = H_\theta^{\mathcal{M}}$, so that both H_θ^V and $H_{\pi(\theta)}^{\mathcal{M}}$ are elements of $\mathcal{M}$ (we introduced g to ensure that $\pi(\theta)$ makes sense). An application of Countable Embedding Absoluteness 1.1.1 then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H_\theta^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$ and $\pi(f \restriction \kappa) \in \operatorname{ran}\pi^*$.

By elementarity of π, H_θ^V has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta}^V \to H_\theta^V$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f \restriction \kappa \in \operatorname{ran}\sigma$ and $\bar\theta = f(\operatorname{crit}\sigma)$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta \to \delta$ be given. Define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<f(\alpha)})^+$, so that by (vi) there exist regular $\bar\kappa < \bar\theta < \kappa < \theta$ such that κ is a closure point for g and there exists a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) = \bar\theta$ and $g \restriction \kappa \in \operatorname{ran}\pi$.

Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 1.2.4 we get a transitive $\mathcal{M} \in H_{g(\bar\kappa)}^V$ closed under $<f(\bar\kappa)$-sequences, and $H_{g(\bar\kappa)}^V$ has a generic elementary embedding $\sigma\colon H_{f(\bar\kappa)}^V \to \mathcal{M}$ with $\operatorname{crit}\sigma = \bar\kappa$ and $\sigma(\bar\kappa) = \kappa > f(\bar\kappa)$. In other words, $\bar\kappa$ is virtually $f(\bar\kappa)$-supercompact in $H_{\bar\theta}^V$. Elementarity of π then implies that κ is virtually $\pi(f)(\kappa)$-supercompact in H_θ^V, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all α such that $(H_\alpha^V, \in, A \cap H_\alpha^V) \prec (H_\delta^V, \in, A)$. Let $f\colon \delta \to \delta$ be given as $f(\alpha) = \langle \alpha_0, \alpha_1 \rangle$ with $\langle -, - \rangle$ being the Gödel pairing function, where α_0 is the first limit of elements of C above α and the α_1's are chosen such that $\{\alpha_1 \mid \alpha < \beta\}$ encodes $A \cap \beta$. This definition makes sense since δ is inaccessible by Proposition 1.2.2.

Let $\kappa < \delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa$, $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_{\bar\theta}^V \to H_\theta^V$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) = \bar\theta$, and $f \restriction \kappa \in \operatorname{ran}\pi$. We claim that $\bar\kappa := \operatorname{crit}\pi$ is virtually $(<\delta, A)$-extendible. To see this, it suffices by the definition of C to show that

$$(H_\kappa^V, \in, A \cap H_\kappa^V) \models \ulcorner \bar\kappa \text{ is virtually } (A \cap H_\kappa)\text{-extendible}\urcorner, \tag{1}$$

since $\kappa \in C$ because it is a closure point of f. Let $\beta := \min(C - \bar\kappa) < \bar\theta$ and note that β exists as $f(\bar\kappa) = \bar\theta$ so the definition of f says that $\bar\theta$ is a limit of elements of C above $\bar\kappa$. It then holds that $(H_{\bar\kappa}^V, \in, A \cap H_{\bar\kappa}^V) \prec (H_\beta^V, \in, A \cap H_\beta^V)$ as both $\bar\kappa$ and β are elements of C. Since f encodes A in the manner previously described and $\pi^{-1}(f) \restriction \bar\kappa = f \restriction \bar\kappa$, we get that $\pi(A \cap H_{\bar\kappa}^V) = A \cap H_\kappa^V$ and thus

$$(H_\kappa^V, \in, A \cap H_\kappa^V) \prec (H_{\pi(\beta)}^V, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H_\beta^V)$. Now, as $(H_\gamma^V, \in, A \cap H_\gamma^V)$ and $(H_{\pi(\gamma)}^V, \in, A^* \cap H_{\pi(\gamma)}^V)$ are elements of $H_{\pi(\beta)}^V$ for every $\gamma < \kappa$, Countable Embedding Absoluteness 1.1.1 implies that $H_{\pi(\beta)}^V$ sees that $\bar\kappa$ is virtually $({<}\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ■

Remark 1.3.5. The above proof shows that the $\mathcal{M} \subseteq V$ assumptions can be replaced by "sufficient" agreement between $\mathcal{M}$ and V: for (i)-(iii) this means that $H_\theta^{\mathcal{M}} = H_\theta^V$ whenever $\mathcal{M}$ is the codomain of a virtual (θ, A)-strong/supercompact/extendible embedding, and in (iv)-(v) this means that $H_{\pi(f)(\kappa)}^{\mathcal{M}} = H_{\pi(f)(\kappa)}^V$. The same thing holds in the "lightface" setting of Theorem 1.2.4.

Make this the official definition? Perhaps even make Trevor's definition the official one, and call ours something different?

We will now step away from the woodins for a little bit, and introduce the vopěnkas. In anticipation of the next section we will work with the class-sized version here, but all the following results work equally well for inaccessible virtually vopěnka cardinals[8].

[8]Note however that we have to require inaccessibility here: see [Wilson, 2019a] for an analysis of the singular virtually vopěnka cardinals.

Definition 1.3.6 (GBC)**.** The **Generic Vopěnka Principle** (gVP) states that for any class C consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi \colon \mathcal{M} \to \mathcal{N}$. ∘

We will be using a standard variation of gVP involving the following *natural sequences.*

Definition 1.3.7 (GBC)**.** Say that a class function $f \colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. ∘

Definition 1.3.8 (GBC)**.** Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ is **natural** if there exists an indexing function $f \colon \mathsf{On} \to \mathsf{On}$ and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$ for every α. Denote this indexing function by $f^{\vec{\mathcal{M}}}$ and the unary relations as $R^{\vec{\mathcal{M}}}_\alpha$. ∘

The following Theorem 1.3.9 is then the main theorem of this section. Firstly it shows that inaccessible cardinals are virtually vopěnka iff they are virtually prewoodin, but also that adding the "virtually" adverb doesn't do anything in this context, in contrast to Theorem 1.2.11.

Theorem 1.3.9 (GBC, Dimopoulos-Gitman-N.)**.** *The following are equivalent:*

(i) *gVP holds;*
(ii) *For any natural On-sequence $\vec{\mathcal{M}}$ there exists a generic elementary embedding $\pi \colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for some $\alpha < \beta$;*
(iii) *On is virtually prewoodin;*
(iv) *On is faintly prewoodin.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume On is faintly prewoodin and fix some On-sequence $\vec{\mathcal{M}} := \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ of structures in a common language. Let κ be $({<}\mathsf{On}, \vec{\mathcal{M}})$-prestrong and fix some regular $\theta > \kappa$ satisfying that $\mathcal{M}_\alpha \in H^V_\theta$

for every $\alpha < \theta$, and fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, \vec{\mathcal{M}}) \to (\mathcal{N}, \in, \mathcal{M}^*)$$

with $H^V_\theta \subseteq \mathcal{N}$ and $\vec{\mathcal{M}} \cap H^V_\theta = \mathcal{M}^* \cap H^V_\theta$. Set $\kappa := \operatorname{crit}\pi$.

We have that $\pi \restriction \mathcal{M}_\kappa\colon\ \mathcal{M}_\kappa \to \mathcal{M}^*_{\pi(\kappa)}$, but we need to reflect this embedding down below θ as we don't know whether $\mathcal{M}^*_{\pi(\kappa)}$ is on the $\vec{\mathcal{M}}$ sequence. Working in the generic extension, we have

$$\mathcal{N} \models \exists\bar\kappa < \pi(\kappa)\exists\dot\sigma \in V^{\operatorname{Col}(\omega,\mathcal{M}^*_{\bar\kappa})}\colon \ulcorner\dot\sigma\colon\ \mathcal{M}^*_{\bar\kappa} \to \mathcal{M}^*_{\pi(\kappa)} \text{ is elementary}\urcorner.$$

Here κ realises $\bar\kappa$ and $\pi \restriction \mathcal{M}_\kappa$ realises σ. Note that $\mathcal{M}^*_\kappa = \mathcal{M}_\kappa$ since we ensured that $\mathcal{M}_\kappa \in H^V_\theta$ and we are assuming that $\vec{\mathcal{M}} \cap H^V_\theta = \mathcal{M}^* \cap H^V_\theta$, so the domain of σ ($= \pi \restriction \mathcal{M}_\kappa$) *is* $\mathcal{M}^*_\kappa$ — also note that σ exists in a $\operatorname{Col}(\omega, \mathcal{M}_\kappa)$ extension of $\mathcal{N}$ by an application of Countable Embedding Absoluteness 1.1.1. Now elementarity of π implies that

$$H^V_\theta \models \exists\bar\kappa < \kappa\exists\dot\sigma \in V^{\operatorname{Col}(\omega,\mathcal{M}_{\bar\kappa})}\colon \ulcorner\dot\sigma\colon\ \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa \text{ is elementary}\urcorner,$$

which is upwards absolute to V, from which we can conclude that $\sigma\colon\ \mathcal{M}_{\bar\kappa} \to \mathcal{M}_\kappa$ witnesses that gVP holds.

$(ii) \Rightarrow (iii)$: Assume (ii) holds and assume that On is not virtually prewoodin, which means that there exists some class A such that there are no virtually A-prestrong cardinals. This allows us to define a function $f\colon \mathsf{On} \to \mathsf{On}$ as $f(\alpha)$ being the least regular $\eta > \alpha$ such that α is not virtually (η, A)-prestrong.

We also define $g\colon \mathsf{On} \to \mathsf{On}$ as taking α to the least strong limit cardinal above α which is a closure point for f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A \cap H^V_{g(\alpha)}$. (ii) supplies us with $\alpha < \beta$ and a generic elementary embedding[9]

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}).$$

[9]Note that $V_{g(\alpha)} = H^V_{g(\alpha)}$ since $g(\alpha)$ is a strong limit cardinal.

Since $g(\alpha)$ is a closure point for f it holds that $f(\operatorname{crit}\pi) < g(\alpha)$, so fixing a regular $\theta \in (f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-prestrong, contradicting the definition of f. Hence On is virtually prewoodin. ■

1.4 Weak Vopěnka

Scrap this section?

We now move to a *weak* variant of gVP, introduced in a category-theoretic context in [Adámek and Rosický, 1994]. It starts with the following equivalent characterisation of gVP, which is the virtual analogue of the characterisation shown in [Adámek and Rosický, 1994].

Lemma 1.4.1 (GBC, Virtualised Adámek-Rosický). *gVP is equivalent to there not existing an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that*[10]

(i) gVP

(ii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that
- *there is a generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

(iii) There is not a natural On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ satisfying that
- *there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ in V for every $\alpha \leq \beta$, which is unique in all generic extensions;*
- *there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for any $\alpha < \beta$.*

Proof. Note that the only difference between *(ii)* and *(iii)* is that the homomorphism exists in V, making *(ii)* $\Rightarrow$ *(iii)* trivial.

(iii) $\Rightarrow$ *(i)*: Assume that gVP fails, meaning by Theorem 1.3.9 that we have a natural On-sequence $\vec{\mathcal{M}}_\alpha$ such that, in every generic extension, there's no homomorphism between any two disctinct $\mathcal{M}_\alpha$'s. Define an On-sequence

[10] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [Adámek and Rosický, 1994].

$\langle \mathcal{N}_\kappa \mid \kappa \in \text{Card} \rangle$ as

$$\mathcal{N}_\kappa := \coprod_{\xi \leq \kappa} \mathcal{M}_\xi = \{(x, \xi) \mid \xi \leq \kappa \wedge \xi \in \text{Card} \wedge x \in \mathcal{M}_\xi\},^{11}$$

with a unary relation R^* given as $R^*(x, \xi)$ iff $\mathcal{M}_\xi \models R(x)$ and a binary relation $\sim^*$ given as $(x, \xi) \sim^* (x', \xi')$ iff $\xi = \xi'$. Whenever we have a homomorphism $f\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ we then get an induced homomorphism $\tilde{f}\colon \mathcal{M}_0 \to \mathcal{M}_\xi$, given as $\tilde{f}(x) := f(x, 0)$, where $\xi \leq \kappa$ is given by preservation of $\sim^*$.

For any two cardinals $\kappa < \lambda$ we have a homomorphism $j_{\kappa\lambda}\colon \mathcal{N}_\kappa \to \mathcal{N}_\lambda$ in V, given as $j_{\kappa\lambda}(x, \xi) := (x, \xi)$. This embedding must also be the *unique* such embedding in all generic extensions, as otherwise we get a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s. Furthermore, there can't be any homomorphism $\mathcal{N}_\lambda \to \mathcal{N}_\kappa$ as that would also imply the existence of a generic homomorphism between two distinct $\mathcal{M}_\alpha$'s.

(i) $\Rightarrow$ *(ii)*: Assume that we have an On-sequence $\vec{\mathcal{M}}_\alpha$ as in the theorem, with generic homomorphisms $j_{\alpha\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ that are unique in all generic extensions for every $\alpha \leq \beta$, with no generic homomorphisms going the other way.

We first note that we can for every $\alpha \leq \beta$ choose the $j_{\alpha\beta}$ in a $\text{Col}(\omega, \mathcal{M}_\alpha)$-extension, by a proof similar to the proof of Lemma 1.1.1 and using the uniqueness of $j_{\alpha\beta}$. Next, fix a proper class $C \subseteq \mathsf{On}$ such that $\alpha \in C$ implies that

$$\sup_{\xi \in C \cap \alpha} |\mathcal{M}_\xi|^V < |\mathcal{M}_\alpha|^V .$$

and note that this implies that $V[g] \models |\mathcal{M}_\xi| < |\mathcal{M}_\alpha|$ for every V-generic $g \subseteq \text{Col}(\omega, \mathcal{M}_\xi)$. This means that for every $\alpha \in C$ we may choose some $\eta_\alpha \in \mathcal{M}_\alpha$ which is *not* in the range of any $j_{\xi\alpha}$ for $\xi < \alpha$. But now define first-order structures $\langle \mathcal{N}_\alpha \mid \alpha \in C \rangle$ as $\mathcal{N}_\alpha := (\mathcal{M}_\alpha, \eta_\alpha)$. Then, by our assumption on the $\mathcal{M}_\alpha$'s and construction of the $\mathcal{N}_\alpha$'s, there can be no generic homomorphism between any two distinct $\mathcal{N}_\alpha$, showing that gVP fails. ■

Note that the proof of the above lemma shows that we without loss of generality may assume that the generic homomorphism in (i) exists in V, which we record here:

Lemma 1.4.2 (GBC, Virtualised Adámek-Rosický)**.** *gVP is equivalent to there not existing an On-sequence of first-order structures* $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ *satisfying that*[12]

(i) there is a homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ *in* V *for every* $\alpha \leq \beta$*, which is unique in all generic extensions;*

(ii) there is no generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ *for any* $\alpha < \beta$*.*

■

The *weak* version of gVP is then simply "flipping the arrows around" in the above characterisation of gVP.

Definition 1.4.3 (GBC)**. Generic Weak Vopěnka's Principle** (gWVP) states that there does *not* exist an On-sequence of first-order structures $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ such that

- there is a generic homomorphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\alpha \leq \beta$, which is unique in all generic extensions;
- there is *no* generic homomorphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for any $\alpha < \beta$.

○

Denoting the corresponding non-generic principle by WVP [Wilson, 2019b] showed the following.

Theorem 1.4.4 (Wilson)**.** *WVP is equivalent to On being a Woodin cardinal.*

Given our 1.3.9 we may then suspect that in the virtual world these two are equivalent, which indeed turns out to be the case. We will be roughly following the argument in [Wilson, 2019b], but we have to diverge from it

[12] This is equivalent to saying that On, viewed as a category, can't be fully embedded into the category Gra of graphs, which is how it's stated in [Adámek and Rosický, 1994].

at several points in which they're using the fact that they're working with class-sized elementary embeddings.

Indeed, in that paper they establish a correspondence between elementary embeddings and certain homomorphisms, a correspondence we won't achieve here. Proving that the elementary embeddings we *do* get are non-trivial seems to furthermore require extra assumptions on our structures. Let's begin.

Define for every strong limit cardinal λ and Σ_1-formula φ the relations

$$\begin{aligned} R^\varphi &:= \{x \in V \mid (V, \in) \models \varphi[x]\} \\ R^\varphi_\lambda &:= \{x \subseteq H^V_\lambda \mid \exists y \in R^\varphi : y \cap H^V_\lambda = x\} \end{aligned}$$

and given any class A define the structure

$$\mathscr{P}_{\lambda,A} := (H^V_{\lambda^+}, R^\varphi_\lambda, \{\lambda\}, A \cap H^V_\lambda)_{\varphi\in\Sigma_1}.$$

Say that a homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ is **trivial** if $h(x)\cap H^V_\eta = x\cap H^V_\eta$ for every $x \in H^V_{\lambda^+}$. Note that h can only be trivial if $\eta \leq \lambda$ since $h(\lambda) = \eta$.

Lemma 1.4.5 (GBC, Gitman-N.)**.** *Let λ be a singular strong limit cardinal, η a strong limit cardinal and $A \subseteq V$ a class. If there exists a non-trivial generic homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ then there's a non-trivial generic elementary embedding*

$$\pi\colon (H^V_{\lambda^+}, \in, A \cap H^V_\lambda) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that, letting $\nu := \min\{\lambda, \eta\}$, it holds that $H^V_\nu \subseteq \mathcal{M}$, $A \cap H^V_\nu = B \cap H^V_\nu$ and $\operatorname{crit}\pi < \nu$.

Proof. Assume that we have a non-trivial homomorphism $h\colon \mathscr{P}_{\lambda,A} \to \mathscr{P}_{\eta,A}$ in a forcing extension $V[g]$, define in $V[g]$ the set

$$\mathcal{M}^* := \{\langle b, f\rangle \mid b \in [H_\nu]^{<\omega} \wedge f \in H^V_{\lambda^+} \wedge f\colon H^V_\lambda \to H^V_\lambda\},$$

and define the relation $\in^*$ on $\mathcal{M}^*$ as

$$\langle b_0, f_0\rangle \in^* \langle b_1, f_1\rangle \quad \text{iff} \quad b_0b_1 \in h(\{xy \in [H_\lambda^V]^{<\omega} \mid f_0(x) \in f_1(y)\}).$$

Claim 1.4.5.1. $\in^*$ is wellfounded.

Proof of claim. Assume not and let $\cdots \in^* \langle b_1, f_1\rangle \in^* \langle b_0, f_0\rangle$ be an $\in^*$-decreasing chain, which by definition means that, for every $n < \omega$,

$$b_{n+1}b_n \in h(\{xy \in [H_\lambda^V]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}). \tag{1}$$

Define the relation $R(v_0, v_1, v_2)$ on H_λ^V as

$$R(X, f, g) \text{ iff } X = \{xy \in [H_\lambda^V]^{<\omega} \mid f(x) \in g(y)\}.$$

This relation is equal to R_λ^φ for some φ, so h moves $\langle X, f, g\rangle \in R_\lambda^\varphi$ to

$$\langle h(X), h(f), h(g)\rangle \in R_\eta^\varphi,$$

meaning that

$$h(\{xy \in [H_\lambda^V]^{<\omega} \mid f_{n+1}(x) \in f_n(y)\}) = \{xy \in [H_\eta^V]^{<\omega} \mid f^*_{n+1}(x) \in f^*_n(y)\}$$

for some f^*_n such that $f^*_n \cap H_\eta^V = h(f_n)$ for all $n < \omega$. But now (1) implies that

$$b_{n+1}b_n \in \{xy \in [H_\eta^V]^{<\omega} \mid f^*_{n+1}(x) \in f^*_n(y)\}$$

and so $h(f_{n+1})(x) = f^*_{n+1}(x) \in f^*_n(y) = h(f_n)(y)$, giving an $\in$-decreasing sequence in $V[g]$ using transitivity of H_η^V, a contradiction!

Hence $\in^*$ is wellfounded. $\dashv$

$\mathcal{M}^*$ is a set, so $\in^*$ is trivially set-like. This means that we can take the transitive collapse $(\mathcal{M}, \in) \cong (\mathcal{M}^*, \in^*)$, and we note that $\mathcal{M} = \{[b, f] \mid \langle b, f\rangle \in \mathcal{M}^*\}$, where $[b, f] := \{[\bar{b}, \bar{f}] \mid \langle \bar{b}, \bar{f}\rangle \in^* \langle b, f\rangle\}$.

We now get a version of Łoś' Theorem whose proof is straight-forward, using that h preserves all Σ_1-relations and that $H^V_\lambda \models \mathsf{ZFC}^-$.

Claim 1.4.5.2. For every formula $\varphi(v_1, \dots, v_n)$ and every $[b_1, f_1], \dots, [b_n, f_n] \in \mathcal{M}$ the following are equivalent:

(i) $(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]]$;
(ii) $b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\})$.

Proof of claim. The proof is straightforward, using that h preserves Σ_1-relations. We prove this by induction on φ. If φ is $v_i \in v_j$ then we have that

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &[b_i, f_i] \in [b_j, f_j] \\
\Leftrightarrow &\langle b_i, f_i \rangle \in^* \langle b_j, f_j \rangle \\
\Leftrightarrow &b_i b_j \in h(\{a_i a_j \in [H^V_\lambda]^{<\omega} \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid f_i(a_i) \in f_j(a_j)\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\psi \wedge \chi$ then

$$\begin{aligned}
&(\mathcal{M}, \in) \models \varphi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &(\mathcal{M}, \in) \models \psi[[b_1, f_1], \dots, [b_n, f_n]] \wedge \chi[[b_1, f_1], \dots, [b_n, f_n]] \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \psi[f_1(a_1), \dots, f_n(a_n)]\}) \cap \\
&\qquad h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \chi[f_1(a_1), \dots, f_n(a_n)]\}) \\
\Leftrightarrow &b_1 \cdots b_n \in h(\{a_1 \cdots a_n \mid \mathscr{P}_{\lambda,A} \models \varphi[f_1(a_1), \dots, f_n(a_n)]\}).
\end{aligned}$$

If φ is $\neg\psi$ then

$$\begin{aligned}
&(\mathcal{M},\in)\models\varphi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&(\mathcal{M},\in)\models\neg\psi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&(\mathcal{M},\in)\not\models\psi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&b_1\cdots b_n\notin h(\{a_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\psi[f_1(a_1),\dots,f_n(a_n)]\})\\
\Leftrightarrow&b_1\cdots b_n\in h(\{a_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\varphi[f_1(a_1),\dots,f_n(a_n)]\}).
\end{aligned}$$

Finally, if φ is $\exists x\psi$ then

$$\begin{aligned}
&(\mathcal{M},\in)\models\varphi[[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&(\mathcal{M},\in)\models\exists x\psi[x,[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle\in\mathcal{M}^*\colon(\mathcal{M},\in)\models\psi[[b,f],[b_1,f_1],\dots,[b_n,f_n]]\\
\Leftrightarrow&\exists\langle b,f\rangle\in\mathcal{M}^*\colon bb_1\cdots b_n\in h(\{aa_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\psi[f(a),f_1(a_1),\dots,f_n(a_n)]\})\\
\Leftrightarrow&b_1\cdots b_n\in h(\{a_1\cdots a_n\mid\mathscr{P}_{\lambda,A}\models\varphi[f_1(a_1),\dots,f_n(a_n)]\}).
\end{aligned}$$

This finishes the proof. ⊣

Next up, we have the following standard lemma, which implies that $H_\eta^V\subseteq\mathcal{M}$:

Claim 1.4.5.3. For all $y\in H_\eta^V$ we have $y=[\langle y\rangle,\mathrm{pr}]$, where $\mathrm{pr}(\langle x\rangle):=x$.

Proof of claim. We prove this by $\in$-induction on $y\in H_\eta^V$, so suppose that $y'=[\langle y'\rangle,\mathrm{pr}]$ for every $y'\in y$, which implies that $y\subseteq\mathcal{M}$ by transitivity of $\mathcal{M}$. We then get that, for every $[b,f]\in\mathcal{M}$,

$$\begin{aligned}
[b,f]\in[\langle y\rangle,\mathrm{pr}]&\Leftrightarrow b\langle y\rangle\in h(\{a\langle x\rangle\mid f(a)\in\mathrm{pr}(\langle x\rangle)\})\\
&\Leftrightarrow\exists y'\in y\colon b\langle y'\rangle\in h(\{a\langle x\rangle\mid f(a)=x\})\\
&\Leftrightarrow\exists y'\in y\colon[b,f]=[\langle y'\rangle,\mathrm{pr}]=y'\\
&\Leftrightarrow[b,f]\in y,
\end{aligned}$$

showing that $y = [\langle y\rangle, \mathrm{pr}]$. $\dashv$

Now define

$$B := \{[b, f] \in \mathcal{M} \mid b \in h(\{x \in H_\lambda^V \mid f(x) \in A\})\}.$$

and, in $V[g]$, let $\pi\colon (H_\lambda^V, \in, A \cap H_\lambda^V) \to (\mathcal{M}, \in, B)$ be given as $\pi(x) := [\langle\rangle, c_x]$.

Claim 1.4.5.4. π is elementary.

Proof of claim. For $x_1, \dots, x_n \in H_\lambda^V$ it holds that

$$\begin{aligned}
&(\mathcal{M}, \in, B) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \varphi[\pi(x_1), \dots, \pi(x_n)] \\
\Leftrightarrow\ &\langle\rangle \in h(\{\langle\rangle \mid \mathscr{P}_{\lambda,A} \models \varphi[x_1, \dots, x_n]\}) \\
\Leftrightarrow\ &(H_{\lambda^+}^V, \in, A \cap H_\lambda^V) \models \varphi[x_1, \dots, x_n]
\end{aligned}$$

and we also get that, for every $x \in H_\lambda^V$,

$$x \in A \Leftrightarrow \langle\rangle \in h(\{a \in H_\lambda^V \mid x \in A\}) \Leftrightarrow \pi(x) \in B,$$

which shows elementarity. $\dashv$

We next need to show that $B \cap H_\nu^V = A \cap H_\nu^V$, so let $x \in H_\nu^V$. Note that $x = [\langle x\rangle, \mathrm{pr}]$ by Claim 1.4.5.3, which means that

$$x \in B \Leftrightarrow \langle x\rangle \in h(\{\langle y\rangle \in H_\lambda^V \mid y \in A\}) \Leftrightarrow x \in A.$$

The last thing we need to show is that $\operatorname{crit}\pi < \nu$. We start with an analogous result about h.

Claim 1.4.5.5. There exists some $b \in H_\nu^V$ such that $h(b) \neq b$.

Proof of claim. Assume the claim fails. We now have two cases.

$\boxed{\textbf{Case 1: } \lambda \geq \eta}$

By non-triviality of h there's an $x \in H^V_{\lambda^+}$ such that $h(x) \cap H^V_\eta \neq x \cap H^V_\eta$, which means that there exists an $a \in H^V_\eta$ such that $a \in h(x) \Leftrightarrow a \notin x$.

If $a \in x$ then $\{a\} = h(\{a\}) \subseteq h(x)$,[13] making $a \in h(x)$, $\lightning$, so assume instead that $a \in h(x)$. Since η is a strong limit cardinal we may fix a cardinal $\theta < \eta$ such that $a \in H^V_\theta$ and $H^V_\theta \in H^V_\eta$. We then have that[14]

$$\{a\} \subseteq h(x) \cap H^V_\theta = h(x) \cap h(H^V_\theta) = h(x \cap H^V_\theta) = x \cap H^V_\theta,$$

so that $a \in x$, $\lightning$.

$\boxed{\textbf{Case 2: } \lambda < \eta}$

In this case we are assuming that $h \restriction H^V_\lambda = \mathrm{id}$, but $h(\lambda) = \eta > \lambda$. Since λ is singular we can fix some $\gamma < \lambda$ and a cofinal function $f\colon \gamma \to \lambda$. Define the relation

$$R = \{(\alpha, \beta, \bar\alpha, \bar\beta, g) \mid \ulcorner g \text{ is a cofinal function } g\colon \alpha \to \beta \urcorner \wedge g(\bar\alpha) = \bar\beta\}.$$

Then $R(\gamma, \lambda, \alpha, f(\alpha), f)$ holds by assumption for every $\alpha < \gamma$, so that R holds for some $(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*)$ such that

$$\begin{aligned}(\gamma^*, \lambda^*, \alpha^*, f(\alpha)^*, f^*) \cap H^V_\eta &= (h(\gamma), h(\lambda), h(\alpha), h(f(\alpha)), h(f)) \\ &= (\gamma, \eta, \alpha, f(\alpha), h(f)),\end{aligned}$$

using our assumption that h fixes every $b \in H^V_\lambda$. Since γ, α and $f(\alpha)$ are transitive and bounded in H^V_λ it holds that $h(\gamma) = \gamma^*$, $h(\alpha) = \alpha^*$ and $h(f(\alpha)) = f(\alpha)^*$. Also, since $\mathrm{dom}(f^*) = \gamma = \mathrm{dom}(f)$ we must in fact have that $f^* = h(f)$. But this means that $h(f)\colon \gamma \to \eta$ is cofinal and $\mathrm{ran}(h(f)) \subseteq \lambda$, a contradiction! $\dashv$

[13] Note that as h preserves Σ_1 formulas it also preserves singletons and boolean operations.

[14] Note that we're using $\lambda \geq \eta$ here to ensure that $H^V_\theta \in \mathrm{dom}\, h$.

To use the above Claim 1.4.5.5 to conclude anything about π we'll make use of the following standard lemma.

Claim 1.4.5.6. For any $x \in H^V_\lambda$ it holds that $h(x) \cap H^V_\eta = \pi(x) \cap H^V_\eta$.

Proof of claim. For any $n < \omega$ and $\langle a_1, \dots, a_n\rangle \in [H^V_\eta]^n$ we have that

$$\begin{aligned}
&\langle a_1, \dots, a_n\rangle \in \pi(x)\\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle a_1, \dots, a_n\rangle \in \pi(x)\\
\Leftrightarrow\ &(\mathcal{M}, \in) \models \langle [\langle a_1\rangle, \mathrm{pr}], \dots, [\langle a_n\rangle, \mathrm{pr}]\rangle \in [\langle\rangle, c_x]\\
\Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(\{\langle x_1, \dots, x_n\rangle \mid \mathscr{P}_{\lambda,A} \models \langle x_1, \dots, x_n\rangle \in x\})\\
\Leftrightarrow\ &\langle a_1, \dots, a_n\rangle \in h(x),
\end{aligned}$$

showing that $h(x) \cap H^V_\eta = \pi(x) \cap H^V_\eta$. ⊣

Now use Claim 1.4.5.5 to fix a $b \in H^V_\nu$ which is moved by h. Claim 1.4.5.6 then implies that

$$\pi(b) \cap H^V_\eta = h(b) \cap H^V_\eta = h(b) \neq b = b \cap H^V_\eta,$$

showing that $\pi(b) \neq b$ and hence $\operatorname{crit}\pi < \nu$. This finishes the proof of the lemma. ■

Theorem 1.4.6 (GBC, Gitman-N.). *gVP is equivalent to gWVP.*

Proof. ($\Rightarrow$): Assume gVP holds and gWVP fails, and let $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ be an On-sequence of first-order structures such that for every $\alpha \leq \beta$ there exists a generic homomorphism

$$j_{\beta\alpha} \colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$$

in some $V[g]$ which is unique in all generic extensions, with no generic homomorphisms going the other way. Here we may assume, as in the proof of Lemma 1.4.1, that $g \subseteq \mathrm{Col}(\omega, \mathcal{M}_\beta)$. We can then find a proper class $C \subseteq \mathsf{On}$ such that $|\mathcal{M}_\alpha|^V < |\mathcal{M}_\beta|^V$ for every $\alpha < \beta$ in C. By gVP there are then $\alpha < \beta$ in C and a generic homomorphism

$$\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta\,.$$

in some $V[h]$, where again we may assume that $h \subseteq \mathrm{Col}(\omega, \mathcal{M}_\alpha)$. But then $\pi \circ j_{\beta\alpha} = \mathrm{id}$ by uniqueness of $j_{\beta\beta} = \mathrm{id}$, which means that $j_{\beta\alpha}$ is injective in $V[g \times h]$ and hence also in $V[g]$. But then $|\mathcal{M}_\beta|^{V[g]} \leq |\mathcal{M}_\alpha|^{V[g]}$, which implies that $|\mathcal{M}_\beta|^V \leq |\mathcal{M}_\alpha|^V$ by the $|\mathcal{M}_\beta|^{+V}$-cc of $\mathrm{Col}(\omega, \mathcal{M}_\beta)$, contradicting the definition of C.

($\Leftarrow$): Assume that gVP fails, which by Theorem 1.3.9 is equivalent to On not being faintly prewoodin. This means that there exists a class A such that there are no faintly A-prestrong cardinals. We can therefore assign to any cardinal κ the least cardinal $f(\kappa) > \kappa$ such that κ is not faintly $(f(\kappa), A)$-prestrong.

Also define a function $g\colon \mathsf{On} \to \mathrm{Card}$ as taking an ordinal α to the least singular strong limit cardinal above α closed under f. Then we're assuming that there's no non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

with $H^V_{g(\alpha)} \subseteq \mathcal{M}$ and $B \cap H^V_{g(\alpha)} = A \cap H^V_{g(\alpha)}$. Assume towards a contradiction that for some α, β there is a non-trivial generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$. Lemma 1.4.5 then gives us a non-trivial generic elementary embedding

$$\pi\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (\mathcal{M}, \in, B)$$

for some transitive $\mathcal{M}$ such that $H^V_\nu \subseteq \mathcal{M}$ with $\nu := \min\{g(\alpha), g(\beta)\}$ and $A \cap H^V_\nu = B \cap H^V_\nu$, a contradiction! Therefore every generic homomorphism $h\colon \mathscr{P}_{g(\alpha),A} \to \mathscr{P}_{g(\beta),A}$ is trivial. Since there is a unique trivial homomorphism when $\alpha \geq \beta$ and no trivial homomorphism when $\alpha < \beta$ since $g(\alpha)$ is sent to $g(\beta)$, the sequence of structures

$$\langle \mathscr{P}_{g(\alpha),A} \mid \alpha \in \mathsf{On} \rangle$$

is a counterexample to gWVP, which is what we wanted to show. ■

1.5 Berkeley

Berkeley cardinals was introduced by Woodin at University of California, Berkeley around 1992, and was introduced as a large cardinal candidate that would be inconsistent with ZF. They trivially imply the Kunen inconsistency and are therefore at least inconsistent with ZFC, but that's as far as it currently goes. In the virtual setting the virtually berkeley cardinals, like all the other virtual large cardinals, are simply downwards absolute to L.

It turns out that virtually berkeley cardinals are natural objects, as the main theorem of this section shows that these large cardinals are precisely what separates virtually prewoodins from the virtually woodins, as well as separating virtually vopěnka cardinals from mahlo cardinals.

Definition 1.5.1. Say that a cardinal δ is **virtually proto-berkeley** if for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi \colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi < \delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ then δ is **virtually berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C \subseteq \delta$ we say δ is **virtually club berkeley**. ∘

Virtually (proto-)berkeley cardinals turn out to be equivalent to their "boldface" versions, the proof of which is a straight-forward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [Cutolo, 2017].

Proposition 1.5.2 (Virtualised Cutolo)**.** *If δ is virtually proto-berkeley then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi \colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with* $\operatorname{crit}\pi < \delta$. *If δ is virtually berkeley then we can furthermore ensure that* $\operatorname{crit}\pi$ *is arbitrarily large below δ.*

PROOF. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{\{\langle A, x\rangle \mid x \in \mathcal{M}\}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: a is the first element in the pairs belonging to the set of highest rank, and $\mathcal{M}$ is what remains if we remove the set with the highest rank. But this means that a generic elementary embedding $\pi\colon \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and a, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the wanted conclusion. ■

The following is a straight-forward virtualisation of the usual definition of the vopěnka filter (see e.g. [Kanamori, 2008]).

Definition 1.5.3 (GBC). Define the **virtually vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$. ∘

Theorem 1.3.9 shows that this filter is proper iff gVP holds. The proof of Proposition 24.14 in [Kanamori, 2008] also shows that this filter is normal and is proper iff gVP holds. Note that uniformity of filters is non-trivial as we're working with proper classes[15]. Indeed, Theorem 1.5.7 shows that uniformity of this filter is equivalent to there being no virtually berkeley cardinals — the following lemma is the first implication.

Lemma 1.5.4 (GBC, N.). *Assume gVP and that there are no virtually berkeley cardinals. Then the virtually vopěnka filter F on On contains every class club C.*

PROOF. The crucial extra property we get by assuming that there aren't any virtually berkeleys is that F becomes uniform, i.e. contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin F$. Let M be a transitive set with $\delta \subseteq M$ and $\gamma < \delta$ a cardinal. As

[15]This boils down to the fact that the class club filter is not provably normal in GBC, see asd.

$(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function $f\colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\alpha + 1, \delta + 1)$$

and unary relations $R_\alpha := \langle M, \mathcal{N}_\alpha \rangle$. If $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit}\pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$, then $\pi(R_\alpha) = R_\beta$ implies that $\pi \restriction \mathcal{M}\colon \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit}\pi \leq \delta$. We also get that $\operatorname{crit}\pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit}\pi}\colon \mathcal{N}_{\operatorname{crit}\pi} \to \mathcal{N}_{\pi(\operatorname{crit}\pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and hence $\operatorname{crit}\pi > \gamma$ as $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually berkeley, a contradiction. Thus $\operatorname{crit}\pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$.

From here the proof of Lemma 8.11 in [Jech, 2006] shows us the wanted. ■

Theorem 1.5.5 (GBC, N.). *If there are no virtually berkeley cardinals then* On *is virtually prewoodin iff* On *is virtually woodin.*

Proof. Assume On is virtually prewoodin, so gVP holds by Theorem 1.3.9 and we can let F be the virtually vopěnka filter. The assumption that there aren't any virtually berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails — we will follow the proof of Theorem 1.3.9.

Failure means that there is some class A and some class club C such that there are no virtually A-prestrong cardinals in C. Since there are no virtually berkeley cardinals, Lemma 1.5.4 imples that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as sending α to the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-prestrong if $\alpha \in C$, and set $f(\alpha) := \alpha$ if $\alpha \notin C$. Also define $g\colon \mathsf{On} \to \mathsf{On}$ as $g(\alpha)$ being the least strong limit cardinal in C above α which is a closure point for f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := \text{Code}(\langle A \cap H^V_{g(\alpha)}, \mathcal{N}_\alpha\rangle)$ and apply gVP to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi \restriction (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)})\colon (H^V_{g(\alpha)}, \in, A \cap H^V_{g(\alpha)}) \to (H^V_{g(\beta)}, \in, A \cap H^V_{g(\beta)}),$$

making $\text{crit}\,\pi$ virtually $(g(\alpha), A)$-prestrong and thus $\text{crit}\,\pi \notin C$. But as we also get the embedding $\pi \restriction \mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\text{crit}\,\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \text{Card} \mid (H^V_\kappa, \in, A \cap H^V_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-prestrong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it isn't virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H^V_\theta, \in, A \cap H^V_\theta) \to (M, \in, B)$$

with $\text{crit}\,\pi = \kappa$, $H^V_\theta \subseteq M$, $M \subseteq V$, $A \cap H^V_\theta = B \cap H^V_\theta$ and $\pi(\kappa) < \theta$.

Now $\pi(\kappa)$ is inaccessible, and $(H^V_{\pi(\kappa)}, \in, A \cap H^V_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$ believes that κ is virtually $(A \cap H^V_{\pi(\kappa)})$-strong as in the proof of Theorem 1.2.8, meaning that $(H^V_\kappa, \in, A \cap H^V_\kappa)$ believes that there is a proper class of virtually $(A \cap H^V_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually woodin. ■

Theorem 1.5.6 (GBC, N.)**.** *If there exists a virtually berkeley cardinal δ then gVP holds and On is not mahlo.*

Proof. If On was Mahlo then there would in particular exist an inaccessible cardinal $\kappa > \delta$, but then $H_\kappa^V \models \ulcorner$there exists a virtually berkeley cardinal$\urcorner$, contradicting the incompleteness theorem.

To show gVP we show that On is virtually prewoodin, which is equivalent by Theorem 1.3.9. Fix therefore a class A — we have to show that there exists a virtually A-prestrong cardinal. For every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta \colon (H_\theta^V, \in, A \cap H_\theta^V) \to (H_\theta^V, \in, A \cap H_\theta^V)$$

with $\operatorname{crit} \pi < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-prestrong, making On virtually prewoodin. ■

Theorem 1.5.7 (GBC, N.)**.** *The following are equivalent:*

(i) gVP *implies that* On *is mahlo;*
(ii) On *is virtually prewoodin iff* On *is virtually woodin;*
(iii) There are no virtually berkeley cardinals.

Proof. *(iii)* $\Rightarrow$ *(ii)* is Theorem 1.5.5, and the contraposed version of *(i)* $\Rightarrow$ *(iii)* is Theorem 1.5.6. For *(ii)* $\Rightarrow$ *(i)* note that gVP implies that On is virtually prewoodin by Theorem 1.3.9, which by *(ii)* means that it's virtually woodin and the usual proof shows that virtually woodins are mahlo[16], showing *(i)*. ■

This also immediately implies the following equiconsistency, as virtually berkeley cardinals have strictly larger consistency strength than virtually woodin cardinals.

Corollary 1.5.8 (N.)**.** *The existence of an inaccessible virtually prewoodin cardinal is equiconsistent with the existence of an inaccessible virtually woodin cardinal.* ■

[16] See e.g. Exercise 26.10 in [Kanamori, 2008].

Question 1.7 in [Wilson, 2018] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually prestrong cardinal, and Lemmata 2.5 and 2.8 in the same paper also shows that being ω-Erdős is equivalent to being virtually club berkeley and that the least such is also the least virtually berkeley.[17]

Move the remaining bit of the chapter to the question chapter?

Furthermore, they also showed that a non-Σ_2-reflecting virtually prestrong cardinal is equivalent to a virtually prestrong cardinal which isn't virtually strong. We can therefore reformulate their question to the following equivalent question.

Question 1.5.9 (Wilson)**.** If there exists a virtually prestrong cardinal which is not virtually strong, is there then a virtually berkeley cardinal?

[Wilson, 2018] showed that their question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem 1.2.8 we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 1.5.10. If there exists a cardinal κ which is virtually (θ,ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually berkeley cardinal?

Our results above at least gives a partially positive result:

Corollary 1.5.11 (N.)**.** *If there exists a virtually A-prestrong cardinal for every class A and there are no virtually strong cardinals, then there exists a virtually berkeley cardinal.*

Proof. The assumption implies by definition that On is virtually prewoodin but not virtually woodin, so Theorem 1.5.7 supplies us with the desired. ■

[17] Note that this also shows that virtually club berkeley cardinals and virtually berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

The assumption that there is a virtually A-prestrong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 1.5.7 shows that this is necessary, which might lead one to think that the question could have a negative answer.

2 | Games and small embeddings

2.1 Setting the scene

In this section we will recall a handful of definitions concerning Ramsey-like cardinals, as well as define the α-Ramsey cardinals for arbitrary ordinals α. We start out with the models and measures that we are going to consider.

Definition 2.1.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa + 1 \subseteq \mathcal{M}$ and $(\mathcal{M}, \in) \models \mathsf{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$, $\mathcal{M}$ is a **κ-model**.[1] ○

Define this at some earlier point?

Recall that μ is an **$\mathcal{M}$-measure** if $(\mathcal{M}, \in, \mu) \models \ulcorner \mu$ is a κ-complete ultrafilter on $\kappa \urcorner$.

Definition 2.1.2. Let $\mathcal{M}$ be a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

- **weakly amenable** if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with $\mathcal{M}$-cardinality κ;
- **countably complete** if $\bigcap \vec{X} \neq \emptyset$ for every ω-sequence $\vec{X} \in {}^\omega\mu$;
- **$\mathcal{M}$-normal** if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa\mu : \triangle \vec{X} \in \mu$;
- **genuine** if $|\triangle \vec{X}| = \kappa$ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- **normal** if $\triangle \vec{X}$ is stationary in κ for every κ-sequence $\vec{X} \in {}^\kappa\mu$;
- **0-good**, or simply **good**, if it has a well-founded ultrapower;
- **α-good** for $\alpha > 0$ if it is weakly amenable and has α-many well-founded iterates.

○

Note that a genuine $\mathcal{M}$-measure is $\mathcal{M}$-normal and countably complete, and a countably complete weakly amenable $\mathcal{M}$-measure is α-good for all ordinals

[1] Note that our (weak) κ-models do not have to be transitive, in contrast to the models considered in [Gitman, 2011] and [Gitman and Welch, 2011]. Not requiring the models to be transitive was introduced in [Holy and Schlicht, 2018].

α. We'll use the fact shown in [Holy and Schlicht, 2018] that an $\mathcal{M}$-measure μ is normal iff $\triangle\vec{X}$ is stationary for some enumeration $\vec{X} = \langle X_\alpha \mid \alpha < \kappa\rangle$ of μ. We are also going to use the following alternative characterisation of weak amenability.

Proposition 2.1.3 (Folklore)**.** *Let $\mathcal{M}$ be a weak κ-model, μ an $\mathcal{M}$-measure and $j : \mathcal{M} \to \mathcal{N}$ the associated ultrapower embedding. Then μ is weakly amenable if and only if j is κ-**powerset preserving**, meaning that $\mathcal{M} \cap \mathscr{P}(\kappa) = \mathcal{N} \cap \mathscr{P}(\kappa)$.* ■

The α-Ramsey cardinals in [Holy and Schlicht, 2018] are based upon the following game.[2]

Definition 2.1.4 (Holy-Schlicht)**.** For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, a limit ordinal $\gamma \leq \kappa$ and a regular cardinal $\theta > \kappa$ define the game $wfG^\theta_\gamma(\kappa)$ of length γ as follows.

$$\begin{array}{lccccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots & \\ \mathrm{II} & & \mu_0 & & \mu_1 & & \mu_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a κ-model and μ_α is a filter for all $\alpha < \gamma$, such that μ_α is an $\mathcal{M}_\alpha$-measure, the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing and $\langle \mathcal{M}_\xi \mid \xi < \alpha\rangle, \langle \mu_\xi \mid \xi < \alpha\rangle \in \mathcal{M}_\alpha$ for every $\alpha < \gamma$. Letting $\mu := \bigcup_{\alpha<\gamma} \mu_\alpha$ and $\mathcal{M} := \bigcup_{\alpha<\gamma} \mathcal{M}_\alpha$, player II wins iff μ is an $\mathcal{M}$-normal good $\mathcal{M}$-measure. ○

Recall that two games G_1 and G_2 are **equivalent** if player I has a winning strategy in G_1 iff they have one in G_2, and player II has a winning strategy in G_1 iff they have one in G_2. [Holy and Schlicht, 2018] showed that the games $wfG^{\theta_0}_\gamma(\kappa)$ and $wfG^{\theta_1}_\gamma(\kappa)$ are equivalent for any γ with $\operatorname{cof}\gamma \neq \omega$ and any regular $\theta_0, \theta_1 > \kappa$. We will be working with a variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I but more of player II. It will turn out that this change of game is innocuous, as Proposition 2.1.8 will show that they are equivalent.

[2] Unless otherwise stated, every game considered will be a game with perfect information between two players I and II. For a formal framework modelling these games, see e.g. [Kanamori, 2008].

Definition 2.1.5 (Holy-N.-Schlicht)**.** Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following game $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ with $(\gamma{+}1)$-many rounds:

$$\begin{array}{lcccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & & \mathcal{M}_\gamma & \\ \text{II} & & \mu_0 & & \mu_1 & & \cdots & & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leq \gamma$, μ_α is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$, μ_γ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure and the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore require that $\mathcal{M}_\alpha = \bigcup_{\xi<\alpha} \mathcal{M}_\xi$, $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$ and that μ_α is ζ-good. Player II wins iff they could continue to play throughout all $(\gamma{+}1)$-many rounds. ∘

For convenience we will write $\mathcal{G}^\theta_\gamma(\kappa)$ for the game $\mathcal{G}^\theta_\gamma(\kappa, 0)$, and $\mathcal{G}_\gamma(\kappa)$ for $\mathcal{G}^\theta_\gamma(\kappa)$ whenever $\operatorname{cof}\gamma \neq \omega$, as again the existence of winning strategies in these games doesn't depend upon a specific θ. Note that we assume that $\kappa = \kappa^{<\kappa}$ is uncountable in the definition of the games that we're considering, so this is a standing assumption throughout the paper, whenever any one of the above two games are considered.

Definition 2.1.6. Define the **Cohen game** $\mathcal{C}^\theta_\gamma(\kappa)$ as $\mathcal{G}^\theta_\gamma(\kappa)$ but where we require that $|\mathcal{M}_\alpha - H_\kappa| < \gamma$ for every $\alpha < \gamma$, i.e. that we only allow player I to add $<\gamma$ new elements to the models in each round, and where we only require $\mathcal{M}_\alpha \models \mathsf{ZFC}^-$ and $\mathcal{M}_\alpha \prec H_\theta$ for $\alpha \leq \gamma$ limit.[3]

Also define the **weak Cohen game** $\mathcal{C}^-_\gamma(\kappa)$ in analogy with $\mathcal{G}^-_\gamma(\kappa)$. ∘

Proposition 2.1.7 (N.)**.** *Assume $\gamma^{\aleph_0} = \gamma$ and let κ be regular. Then $\mathcal{C}^-_\gamma(\kappa)$ is equivalent to $\mathcal{C}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. In particular, if CH holds then $\mathcal{C}^-_{\omega_1}(\kappa)$ is equivalent to $\mathcal{C}^\theta_{\omega_1}(\kappa)$ for all regular $\theta > \kappa$.*

Proof. The assumption that $\gamma^{\aleph_0} = \gamma$ allows us to ensure that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ for all $\alpha < \gamma$. If player I has a winning strategy in $\mathcal{C}^\theta_\gamma(\kappa)$ for some regular $\theta > \kappa$ then they still win if we require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$ (since they're only enlargening their models, making it even harder for player II

[3] $\mathcal{C}^\theta_\omega(\kappa)$ is similar to the $H(F, \lambda)$-games in [Donder and Levinski, 1989].

to win), in which case the final measure μ_γ is countably complete and hence automatically has a wellfounded ultrapower.

If player II has a winning strategy in $\mathcal{C}^-_\gamma(\kappa)$ then they still win if player I plays $\mathcal{M}_\alpha$ such that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_\gamma$, again ensuring that μ_γ has a wellfounded ultrapower. ■

Proposition 2.1.8 (Holy-N.-Schlicht). *$\mathcal{G}^\theta_\gamma(\kappa)$, $\mathcal{G}^\theta_\gamma(\kappa, 1)$ and $wfG^\theta_\gamma(\kappa)$ are all equivalent for all limit ordinals $\gamma \leq \kappa$, and $\mathcal{G}^\theta_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$ whenever* $\operatorname{cof}\gamma > \omega$ *and $\zeta \in \mathit{On}$.*

Proof. We start by showing the latter statement, so assume that $\operatorname{cof}\gamma > \omega$. Consider now the auxilliary game, call it $\mathcal{G}$, which is exactly like $\mathcal{G}^\theta_\gamma(\kappa, 0)$, but where we also require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ and $\langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle, \langle \mu_\xi \mid \xi \leq \alpha\rangle \in \mathcal{M}_{\alpha+1}$ for every $\alpha < \gamma$.

Claim 2.1.8.1. $\mathcal{G}$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa)$.

Proof of claim. If player I has a winning strategy in $\mathcal{G}$ then they also have one in $\mathcal{G}^\theta_\gamma(\kappa)$, by doing exactly the same. Analogously, if player II has a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have one in $\mathcal{G}$. If player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ then we can construct a winning strategy σ' in $\mathcal{G}$, which is defined as follows. Fix some $\alpha \leq \gamma$ and, writing $\vec{\mathcal{M}}_\xi := \langle \mathcal{M}_\xi \mid \xi \leq \alpha\rangle$ and $\vec{\mu}_\xi := \langle \mu_\xi \mid \xi \leq \alpha\rangle$, we set

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) := \operatorname{Hull}^{H_\theta}(\sigma(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle) \cup {}^\omega\mathcal{M}_\alpha \cup \{\vec{\mathcal{M}}_\xi, \vec{\mu}_\xi\}),$$

i.e. that we're simply throwing in the sequences into our models and making sure that we're still an elementary substructure of H_θ. This new strategy σ' is clearly winning. Assuming now that τ is a winning strategy for player II in $\mathcal{G}$, we define a winning strategy τ' for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ by letting $\tau'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha\rangle)$ be the result of throwing in the appropriate sequences into the models $\mathcal{M}_\xi$, applying τ to get a measure, and intersecting that measure with $\mathcal{M}_\alpha$ to get an $\mathcal{M}_\alpha$-measure. ⊣

Now, letting $\mathcal{M}_\gamma$ be the final model of a play of $\mathcal{G}$, $\operatorname{cof}\gamma > \omega$ implies that any ω-sequence $\vec{X} \in \mathcal{M}_\gamma$ really is a sequence of elements from some $\mathcal{M}_\xi$ for $\xi < \gamma$, so that $\vec{X} \in \mathcal{M}_{\xi+1}$ by definition of $\mathcal{G}$, making $\mathcal{M}_\gamma$ closed under ω-sequences and thus also μ_γ countably complete. Since γ is a limit ordinal and the models contain the previous measures and models as elements, the proof of e.g. Theorem 5.6 in [Holy and Schlicht, 2018] shows that μ_γ is also weakly amenable, making it ζ-good for all ordinals ζ.

Now we deal with the first statement, so fix a limit ordinal γ. Firstly $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $\mathcal{G}^\theta_\gamma(\kappa, 1)$ as above, since both are equivalent to the auxilliary game $\mathcal{G}$ when γ is a limit ordinal. So it remains to show that $\mathcal{G}^\theta_\gamma(\kappa)$ is equivalent to $wfG^\theta_\gamma(\kappa)$. If player I has a winning strategy σ in $wfG^\theta_\gamma(\kappa)$ then define a winning strategy σ' for player I in $\mathcal{G}^\theta_\gamma(\kappa)$ as

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle) := \sigma(\langle \mathcal{M}_0, \mu_0 \rangle^\frown \langle \mathcal{M}_{\xi+1}, \mu_{\xi+1} \mid \xi + 1 \leq \alpha \rangle)$$

and for limit ordinals $\alpha \leq \gamma$ set $\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi < \alpha \rangle) := \bigcup_{\xi<\alpha} \mathcal{M}_\xi$; i.e. they simply follow the same strategy as in $wfG^\theta_\gamma(\kappa)$ but plugs in unions at limit stages. Likewise, if player II had a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ then they also have a winning strategy in $wfG^\theta_\gamma(\kappa)$, this time just by skipping the limit steps in $\mathcal{G}^\theta_\gamma(\kappa)$.

Now assume that player I has a winning strategy σ in $\mathcal{G}^\theta_\gamma(\kappa)$ and that player I *doesn't* have a winning strategy in $wfG^\theta_\gamma(\kappa)$. Then define a strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$ as follows. Let $s = \langle \mathcal{M}_\alpha, \mu_\alpha \mid \alpha \leq \eta \rangle$ be a partial play of $wfG^\theta_\gamma(\kappa)$ and let s' be the modified version of s in which we have 'inserted' unions at limit steps, just as in the above paragraph. We can assume that every μ_α in s' is good and $\mathcal{M}_\alpha$-normal as otherwise player II has already lost and player I can play anything. Now, we want to show that s' is a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$. All the models in s are κ-models, so in particular weak κ-models.

Claim 2.1.8.2. Every μ_α in s' is normal.

Proof of claim. Assume without loss of generality that $\alpha = \eta$. Let player I play any legal response $\mathcal{M}$ to s in $wfG^\theta_\gamma(\kappa)$ (such a response always exists). If player II can't respond then player I has a winning strategy by simply following $s^\frown\langle\mathcal{M}\rangle$, $\lightning$, so player II *does* have a response μ to $s^\frown\mathcal{M}$. But now the rules of $wfG^\theta_\gamma(\kappa)$ ensures that $\mu_\eta\in\mathcal{M}$, so since

$$(\mathcal{M},\in,\mu)\models\forall\vec{X}\in{}^\kappa\mu:\ulcorner\triangle\vec{X}\text{ is stationary in }\kappa\urcorner,$$

we then also get that $\mathcal{M}\models\ulcorner\triangle\mu_\eta$ is stationary in $\kappa\urcorner$ since $\mu_\eta\subseteq\mu$, so elementarity of $\mathcal{M}$ in H_θ implies that $\triangle\mu_\eta$ really *is* stationary in κ, making μ_η normal. $\dashv$

This makes s' a valid partial play of $\mathcal{G}^\theta_\gamma(\kappa)$, so we may form the weak κ-model $\tilde{\mathcal{M}}_\eta:=\sigma(s')$. Now let $\mathcal{M}_\eta\prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_\eta\subseteq\mathcal{M}_\eta$ and $s\in\mathcal{M}_\eta$ and set $\sigma'(s):=\mathcal{M}_\eta$. This defines the strategy σ' for player I in $wfG^\theta_\gamma(\kappa)$, which is winning since the winning condition for the two games is the same for γ a limit.[4]

Next, assume that player II has a winning strategy τ in $wfG^\theta_\gamma(\kappa)$. We recursively define a strategy $\tilde\tau$ for player II in $\mathcal{G}^\theta_\gamma(\kappa)$ as follows. If $\tilde{\mathcal{M}}_0$ is the first move by player I in $\mathcal{G}^\theta_\gamma(\kappa)$, let $\mathcal{M}_0\prec H_\theta$ be a κ-model with $\tilde{\mathcal{M}}_0\subseteq\mathcal{M}_0$, making $\mathcal{M}_0$ a valid move for player I in $wfG^\theta_\gamma(\kappa)$. Write $\mu_0:=\tau(\langle\mathcal{M}_0\rangle)$ and then set $\tilde\tau(\langle\tilde{\mathcal{M}}_0\rangle)$ to be $\tilde\mu_0:=\mu_0\cap\tilde{\mathcal{M}}_0$, which again is normal by the same trick as above, making $\tilde\mu_0$ a legal move for player II in $\mathcal{G}^\theta_\gamma(\kappa)$. Successor stages $\alpha+1$ in the construction are analogous, but we also make sure that $\langle\mathcal{M}_\xi\mid\xi<\alpha+1\rangle,\langle\mu_\xi\mid\xi<\alpha+1\rangle\in\mathcal{M}_{\alpha+1}$. At limit stages τ outputs unions, as is required by the rules of $\mathcal{G}^\theta_\gamma(\kappa)$. Since the union of all the μ_α's is good as τ is winning, $\tilde\mu_\gamma:=\bigcup_{\alpha<\gamma}\tilde\mu_\alpha$ is good as well, making $\tilde\tau$ winning and we are done. ■

We now arrive at the definitions of the cardinals we will be considering. They were in [Holy and Schlicht, 2018] only defined for γ being a cardinal, but given the above result we generalise it to all ordinals γ.

[4]More precisely, that σ is winning in $\mathcal{G}^\theta_\gamma(\kappa)$ means that there's a sequence $\langle f_n:\kappa\to\kappa\mid n<\omega\rangle$ with the f_n's all being elements of the last model $\tilde{\mathcal{M}}_\gamma$, witnessing the illfoundedness of the ultrapower. But then all these functions will also be elements of the union of the $\mathcal{M}_\alpha$'s, since we ensured that $\mathcal{M}_\alpha\supseteq\tilde{\mathcal{M}}_\alpha$ in the construction above, making the ultrapower of $\bigcup_{\alpha<\gamma}\mathcal{M}_\alpha$ by $\bigcup_{\alpha<\gamma}\mu_\alpha$ illfounded as well.

Definition 2.1.9. Let κ be a cardinal and $\gamma \leq \kappa$ an ordinal. Then κ is **γ-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. We furthermore say that κ is **strategic γ-Ramsey** if player II *does* have a winning strategy in $\mathcal{G}^\theta_\gamma(\kappa)$ for all regular $\theta > \kappa$. Define **(strategic) genuine γ-Ramseys** and **(strategic) normal γ-Ramseys** analogously, but where we require the last measure μ_γ to be genuine and normal, respectively. ◦

Definition 2.1.10 (N.)**.** A cardinal κ is **$<\gamma$-Ramsey** if it is α-Ramsey for every $\alpha < \gamma$, **almost fully Ramsey** if it is $<\kappa$-Ramsey and **fully Ramsey** if it is κ-Ramsey. Further, say that κ is **coherent $<\gamma$-Ramsey** if it's strategic α-Ramsey for every $\alpha < \gamma$ and that there exists a choice of winning strategies τ_α in $\mathcal{G}_\alpha(\kappa)$ for player II satisfying that $\tau_\alpha \subseteq \tau_\beta$ whenever $\alpha < \beta$. In other words, there is a single strategy τ for player II in $\mathcal{G}_\gamma(\kappa)$ such that τ is a winning strategy for player II in $\mathcal{G}_\alpha(\kappa)$ for every $\alpha < \gamma$.[5] ◦

This is not the original definition of (strategic) γ-Ramsey cardinals however, as this involved elementary embeddings between weak κ-models – but as the following theorem of [Holy and Schlicht, 2018] shows, the two definitions coincide whenever γ is a regular cardinal.

Theorem 2.1.11 (Holy-Schlicht)**.** *For regular cardinals λ, a cardinal κ is λ-Ramsey iff for arbitrarily large $\theta > \kappa$ and every $A \subseteq \kappa$ there is a weak κ-model $\mathcal{M} \prec H_\theta$ with $\mathcal{M}^{<\lambda} \subseteq \mathcal{M}$ and $A \in \mathcal{M}$ with an $\mathcal{M}$-normal 1-good $\mathcal{M}$-measure μ on κ.* ■

2.2 The finite case

In this section we are going to consider properties of the n-Ramsey cardinals for finite n. Note in particular that the $\mathcal{G}^\theta_n(\kappa)$ games are determined, making the "strategic" adjective superfluous in this case. We further note that the θ's are also dispensible in this finite case:

[5] Note that, with this terminology, "coherent" is a stronger notion than "strategic". We could've called the cardinals *coherent strategic $<\gamma$-Ramseys*, but we opted for brevity instead.

Proposition 2.2.1 (N.)**.** *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n \prec H_\theta$.*

Proof. $\Leftarrow$ is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays $\mathcal{M}_k$ in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \mathrm{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of $\mathcal{M}_k$, and play $\mathcal{M}_k^*$ in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ-responses to the $\mathcal{M}_k^*$'s and let player II play the μ_k's in $\mathcal{G}_n(\kappa)$ as well.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\mathrm{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω-sequence $\vec{f} := \langle f_k \mid k < \omega\rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \mathrm{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\mathrm{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \to H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \mathrm{ran}\,\pi$.

Now $\bar{A} = A$ for every $A \in \mathscr{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the $\bar{f}_k$'s witness that $\mathrm{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\mathrm{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \mathrm{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ's anymore we note that n-Ramseyness can now be described using a Π^1_{2n+2}-formula and normal n-Ramseyness using a Π^1_{2n+3}-formula.

We already have the following characterisations, as proven in [Abramson et al., 1977].

Theorem 2.2.2 (Abramson et al.)**.** *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then*

(i) κ is weakly compact if and only if it is 0-Ramsey;

(ii) κ is weakly ineffable if and only if it is genuine 0-Ramsey;

(iii) κ is ineffable if and only if it is normal 0-Ramsey.

Proof. This is mostly a matter of changing terminology from [Abramson et al., 1977] to the current game-theoretic one, so we only show (i). Theorem 1.1.3 in

[Abramson et al., 1977] shows that κ is weakly compact if and only if every κ-sized collection of subsets of κ is measured by a $<\kappa$-complete measure, in the sense that every $<\kappa$-sequence (in V) of measure one sets has non-empty intersection.

For the $\Rightarrow$ direction we can let player II respond to any $\mathcal{M}_0$ by first getting the $<\kappa$-complete $\mathcal{M}_0$-measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \to \operatorname{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π, which is $\mathcal{M}_0$-normal and good. For $\Leftarrow$, if $X \subseteq \mathscr{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ-model $\mathcal{M}_0 \prec H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play $\mathcal{M}_0$ in $\mathcal{G}_0(\kappa)$ we get some $\mathcal{M}_0$-normal good $\mathcal{M}_0$-measure μ_0 on κ. Since $\mathcal{M}_0$ is closed under $<\kappa$-sequences we get that μ_0 is $<\kappa$-complete. ■

Indescribability

In this section we aim to prove that n-Ramseys are Π^1_{2n+1}-indescribable and that normal n-Ramseys are Π^1_{2n+2}-indescribable, which will also establish that the hierarchy of alternating n-Ramseys and normal n-Ramseys forms a strict hierarchy. Recall the following definition.

Definition 2.2.3. A cardinal κ is **Π^1_n-indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ○

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π^1_1-indescribable — see [Abramson et al., 1977].

Theorem 2.2.4 (N.). *Every n-Ramsey κ is Π^1_{2n+1}-indescribable for $n < \omega$.*

PROOF. Let κ be n-Ramsey and assume that it is not Π^1_{2n+1}-indescribable, witnessed by a Π_{2n+1}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$ and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$-many quantifiers occuring in φ in $(n+1)$-many steps. We will here describe the first two steps with the remaining steps following the same pattern.

$\boxed{\textbf{First step.}}$ Write $\varphi(v) \equiv \forall v_1 \psi(v, v_1)$ for a Σ_{2n}-formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A^{(0)}_\alpha \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let $\mathcal{M}_0$ be a weak κ-model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good $\mathcal{M}_0$-normal $\mathcal{M}_0$-measure μ_0 on κ, using the 0-Ramseyness of κ. Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \text{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. $\mathcal{M}_0$-normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, $\mathcal{M}_0$-normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\text{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}]\urcorner. \tag{1}$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0-formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

$\boxed{\textbf{Second step.}}$ Write $\psi(v, v_1) \equiv \exists v_2 \forall v_3 \chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$-formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3 \chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \tag{2}$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, v_3]. \tag{3}$$

Fix witnesses $A^{(1)}_\alpha \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S^{(0)}_\alpha := \{\xi < \kappa \mid A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S^{(0)}_\alpha \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and $\mathcal{M}_0$-normality of μ_0 then implies that $S^{(0)}_\alpha \in \mu_0$ is equivalent to

$$\mathrm{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ-model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an $\mathcal{M}_1$-normal $\mathcal{M}_1$-measure on κ, using the 1-Ramseyness of κ, so that $\mathcal{M}_1$-normality of μ_1 yields that $\triangle\vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle\vec{S}^{(0)}$ if and only if $A^{(0)}_\xi \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A^{(0)}_\xi = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \mathrm{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A^{(1)}_\alpha$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, A^{(0)}_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha].$$

Now this, paired with the above observation regarding $\triangle\vec{S}^{(0)}$, means that for every $\alpha \in \triangle\vec{S}^{(0)} \cap \mathrm{Lim}$ we have that

$$V_{\alpha+1} \models \neg\chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A^{(1)}_\alpha],$$

so that $\mathcal{M}_1$-normality of μ_1 and Łoś' lemma implies that

$$\mathrm{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg\chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]\urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$-many steps, ending with a Δ_0-formula $\phi(v, v_1, \dots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\mathrm{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]\urcorner$. But now absoluteness of $\neg\phi$ means that $V_{\kappa+1} \models \neg\phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]$, contradicting (4). ■

Note that this is optimal, as n-Ramseyness can be described by a Π^1_{2n+2}-formula. As a corollary we then immediately get the following.

Corollary 2.2.5 (N.). *Every $<\omega$-Ramsey cardinal is Δ^2_0-indescribable.* ■

The second indescribability result concerns the normal n-Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π^1_2-indescribable — see [Abramson et al., 1977].

Theorem 2.2.6 (N.). *Every normal n-Ramsey κ is Π^1_{2n+2}-indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 2.2.4, as we would end up with a Π^1_1 statement in an ultrapower, and as Π^1_1 statements are not upwards absolute in general we would not be able to get our contradiction.

Proof. Let κ be normal n-Ramsey and assume that it is not Π^1_{2n+2}-indescribable, witnessed by a Π_{2n+2}-formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n-Ramsey to perform the same $n+1$ steps as in the proof of Theorem 2.2.4. This gives us a Σ_1-formula $\phi(v, v_1, \dots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \cdots, \mathcal{A}^{(n)} \rangle$, $\langle B^{(0)}, \dots, B^{(n-1)} \rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n \rangle$ of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{1}$$

and, for μ_n-many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, A^{(n)}_\alpha].$$

Now form $S^{(n)}_\alpha \in \mu_n$ as in the proof of Theorem 2.2.4. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 2.2.4 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for

$k < n$), but we can now use normality[6] of μ_n to ensure that we *do* have that $\triangle\vec{S}^{(n)}$ is stationary in κ. This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg\phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \quad (2)$$

Now note that since κ is inaccessible it is Σ^1_1-indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of [Abramson et al., 1977] shows that the set of reflection points of Σ^1_1-formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \dots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n-Ramseyness can be described by a Π^1_{2n+3}-formula. In particular this then means that every $(n+1)$-Ramsey is a normal n-Ramsey stationary limit of normal n-Ramseys, and every normal n-Ramsey is an n-Ramsey stationary limit of n-Ramseys, making the hierarchy of alternating n-Ramseys and normal n-Ramseys a strict hierarchy.

Downwards absoluteness to L

The following proof is basically the proof of Theorem 4.1.1 in [Abramson et al., 1977].

Theorem 2.2.7 (N.)**.** *Genuine- and normal n-Ramseys are downwards absolute to L, for every $n < \omega$.*

Proof. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ-model — we want to find a genuine $\mathcal{M}$-measure inside L. By assumption we *can* find such a measure μ in V; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa\rangle \in L$ of $\mathscr{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

[6] Recall that this is stronger than just requiring it to be $\mathcal{M}_n$-normal — we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.

Claim 2.2.7.1. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

PROOF OF CLAIM. Let $\vec{B}$ be the μ-**positive part** of $\vec{A}$, meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle \vec{B}$ has size κ, so we can pick $\delta \in \triangle \vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L. $\dashv$

But now Lemma 4.1.2 in [Abramson et al., 1977] shows that there is a Π_1 formula $\varphi(v)$ such that, given any non-zero ordinal ζ, $V_{\zeta+1} \models \varphi[A]$ if and only if ζ is a regular cardinal and A is a non-constructible subset of ζ. If we therefore assume that $T \notin L$ then $V_{\kappa+1} \models \varphi[T]$, which by Π^1_1-indescribability of κ means that there exists some $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle\mu)^L = \triangle\mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V. The cases where κ is a genuine- or normal n-Ramsey cardinal is analogous. ■

Since $(n+1)$-Ramseys are normal n-Ramseys we then immediately get the following.

Corollary 2.2.8 (N.). *Every $(n+1)$-Ramsey is normal n-Ramsey in L, for every $n < \omega$. In particular, $<\omega$-Ramseys are downwards absolute to L.* ■

Complete ineffability

In this section we provide a characterisation of the *completely ineffable* cardinals in terms of the α-Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$-Ramsey cardinals, namely the *coherent $<\omega$-Ramseys* as defined in 2.1.10. Note that a coherent $<\omega$-Ramsey is precisely a cardinal satisfying the ω-filter property, as defined in [Holy and Schlicht, 2018].

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is very closely related to the proof of Theorem 2.2.6 (the indescribability of normal n-

Ramseys), but the main difference is that we want everything to occur locally inside our weak κ-models.

Theorem 2.2.9 (N.). *Every coherent $<\omega$-Ramsey is a stationary limit of $<\omega$-Ramseys.*

Proof. Let κ be coherent $<\omega$-Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 \prec H_\theta$ be a weak κ-model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ-model $\mathcal{M} \prec H_\theta$ with an $\mathcal{M}$-normal $\mathcal{M}$-measure $\mu := \bigcup_{n<\omega} \mu_n$ on κ.

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } {<}\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by $\mathcal{M}$-normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k+1$ steps as in the proof of Theorem 2.2.6 with $\varphi(\xi)$ being $\ulcorner\xi$ is k-Ramsey$\urcorner$, so that we get a weak κ-model $\mathcal{M}_{k+1} \prec H_\theta$, an $\mathcal{M}_{k+1}$-normal $\mathcal{M}_{k+1}$-measure $\tilde{\mu}_{k+1}$ on κ, a Σ_1-formula $\varphi(v, v_1, v_2, \dots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \dots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \dots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \tag{2}$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg\varphi[\xi, A^{(0)}_\xi, B^{(0)} \cap V_\xi, A^{(1)}_\xi, B^{(1)} \cap V_\xi, \dots, A^{(k-1)}_\xi, B^{(k-1)} \cap V_\xi, A^{(k)}_\xi], \tag{3}$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \mathrm{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 2.2.4.

Since κ in particular is Σ^1_1-indescribable, Lemma 3.4.3 of [Abramson et al., 1977] implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ-model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n+1)$-steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of $\mathcal{M}_{k+1}$. In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ, $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A^{(i)}_\xi \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa \rangle$ and note that since $\vec{S}$ is definable it is an element of $\mathcal{M}_{k+2}$ as well. Then $\mathcal{M}_{k+2}$-normality of $\tilde{\mu}_{k+2}$ ensures that $\triangle\vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \triangle\vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \triangle\vec{S}$ we see, as in the proof of Theorem 2.2.4, that

$$V_{\zeta+1} \models \varphi[\zeta, A^{(0)}_\zeta, B^{(0)} \cap V_\zeta, A^{(1)}_\zeta, B^{(1)} \cap V_\zeta, \ldots, A^{(k)}_\zeta]$$

since $\triangle\vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} \prec H_\theta$ we have that $\mathcal{M}$ is correct about stationary subsets of κ, meaning that κ is a stationary limit of $<\omega$-Ramseys. ■

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

Definition 2.2.10. A collection $R \subseteq \mathscr{P}(\kappa)$ is a **stationary class** if

(i) $R \neq \emptyset$;
(ii) every $A \in R$ is stationary in κ;
(iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

○

Definition 2.2.11. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \to 2$ there is an $H \in R$ homogeneous for f. ○

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in [Abramson et al., 1977].

Theorem 2.2.12 (N.)**.** *A cardinal κ is completely ineffable if and only if it is coherent $<\omega$-Ramsey.*

Proof. ($\Leftarrow$): Assume κ is coherent $<\omega$-Ramsey, witnessed by strategies $\langle \tau_n \mid n < \omega\rangle$. Let $f : [\kappa]^2 \to 2$ be arbitrary and form the sequence $\langle A^f_\alpha \mid \alpha < \kappa\rangle$ as

$$A^f_\alpha := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let $\mathcal{M}_f$ be a transitive weak κ-model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated $\mathcal{M}_f$-measure on κ given by τ_0.[7] 1-Ramseyness of κ ensures that μ_f is normal, meaning $\triangle\mu_f$ is stationary in κ. Define a new sequence $\vec{B}^f$ as the μ_f-positive part of $\vec{A}^f$.[8] Then $B^f_\alpha \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\triangle\vec{B}^f$ is stationary.

Let now $\mathcal{M}'_f$ be a new transitive weak κ-model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an $\mathcal{M}'_f$-measure $\mu'_f \supseteq \mu_f$ on κ. Then $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \in \mu_f\}$ and $\triangle\vec{B}^f \cap \{\xi < \kappa \mid A^f_\xi \notin \mu_f\}$ are both elements of $\mathcal{M}'_f$, so one of them is in μ'_f; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f.

Now let $g : [H_f]^2 \to 2$ be arbitrary and again form

$$A^g_\alpha := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ-model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$-measure $\mu_{f,g} \supseteq \mu'_f$ on κ. As before we then get a stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g. We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists \vec{f} : H_{\vec{f}} \subseteq A\},$$

where the $\vec{f}$'s range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \to 2$ and $f_{k+1} : [H_{f_k}] \to 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \to 2$, we can find $H \in R$ which is homogeneous for f. Indeed, if we let $\vec{f}$ be such that $H_{\vec{f}} \subseteq A$, which

[7] Technically we would have to require that $\mathcal{M}_f \prec H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0, but note that we could simply get a measure on $\mathrm{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to $\mathcal{M}_f$. We will use this throughout the proof.

[8] The *μ-positive part* was defined in Claim 2.2.7.1.

exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

($\Rightarrow$): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n-Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play $\mathcal{M}_0$ and enumerate $\mathscr{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A^0_\alpha \mid \alpha < \kappa \rangle$ such that $A^0_\xi \subseteq A^0_\zeta$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \to 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A^0_\xi$. Let $<^\alpha_{\text{lex}}$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \to 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha,\beta)} <^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)} \restriction \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f, using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \restriction \alpha \mid \xi \in H_0 \wedge \xi > \alpha \rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \restriction \alpha = r_\gamma \restriction \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \to \kappa$ as $g(\alpha)$ being the least such η, which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C.

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \restriction \zeta = r_\gamma \restriction \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \restriction \zeta$ holds for every $\gamma \in H_0$, $\gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha$, $\alpha \in A^0_\xi$ iff $\gamma \in A^0_\xi$. Define thus the $\mathcal{M}_0$-measure μ_0 on κ as

$$\begin{aligned} \mu_0(A^0_\xi) = 1 \quad & \text{iff} \quad (\forall \beta \in H_0 \cap C)(\beta > \xi \to \beta \in A^0_\xi) \\ & \text{iff} \quad (\exists \beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A^0_\xi), \end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa \rangle$ be the μ_0-positive part of $\vec{A}^0$, it is also simple to check that $H_0 \cap C \subseteq \triangle \vec{B}$, making μ_0 normal and hence also both $\mathcal{M}_0$-normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n-Ramsey and let $\langle\mathcal{M}_0,\mu_0,\dots,\mathcal{M}_n,\mu_n,\mathcal{M}_{n+1}\rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathscr{P}(\kappa)\cap\mathcal{M}_{n+1}$ as $\vec{A}^{n+1}=\langle A^{n+1}_\xi\mid\xi<\kappa\rangle$, again satisfying that $\xi\leq\zeta$ whenever $A^{n+1}_\xi\subseteq A^{n+1}_\zeta$, but also such that given any $\xi<\kappa$ there are $\zeta,\zeta'\in(\xi,\kappa)$ satisfying that $A^{n+1}_\zeta\in\mathscr{P}(\kappa)\cap\mathcal{M}_n$ and $A^{n+1}_{\zeta'}\in(\mathscr{P}(\kappa)\cap\mathcal{M}_{n+1})-\mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n\in R$ and C be club in κ such that $H_n\cap C\subseteq\triangle\mu_n$, which exist by our inductive assumption. For $\alpha<\kappa$ define $r_\alpha:\alpha\to 2$ as $r_\alpha(\xi)=1$ iff $\alpha\in A^{n+1}_\xi$, and define a colouring $f:[H_n]^2\to 2$ as

$$f(\{\alpha,\beta\}):=\begin{cases}0 & \text{if } r_{\min(\alpha,\beta)}<^{\min(\alpha,\beta)}_{\text{lex}} r_{\max(\alpha,\beta)}\restriction\min(\alpha,\beta)\\ 1 & \text{otherwise}\end{cases}$$

As $H_n\in R$ there is an $H_{n+1}\in R$ homogeneous for f. Just as before, define $g:\kappa\to\kappa$ as $g(\alpha)$ being the least $\eta\in[\alpha,\kappa)$ such that $r_\beta\restriction\alpha=r_\gamma\restriction\alpha$ for every $\beta,\gamma\in H_{n+1}$ with $\eta\leq\beta<\gamma$, and let D be the club of fixed points of g. As above we get that given any $\alpha,\gamma\in H_{n+1}\cap D$ with $\alpha\leq\gamma$ and $\xi<\alpha$, $\alpha\in A^{n+1}_\xi$ iff $\gamma\in A^{n+1}_\xi$. Define then the $\mathcal{M}_{n+1}$-measure μ_{n+1} on κ as

$$\begin{aligned}\mu_{n+1}(A^{n+1}_\xi)=1\quad&\text{iff}\quad(\forall\beta\in H_{n+1}\cap D\cap C)(\beta>\xi\to\beta\in A^{n+1}_\xi)\\ &\text{iff}\quad(\exists\beta\in H_{n+1}\cap D\cap C)(\beta>\xi\land\beta\in A^{n+1}_\xi).\end{aligned}$$

Then $H_{n+1}\cap D\cap C\subseteq\triangle\mu_{n+1}$, making μ_{n+1} normal, $\mathcal{M}_{n+1}$-normal and good, just as before. It remains to show that $\mu_n\subseteq\mu_{n+1}$. Let thus $A\in\mu_n$ be given, and say $A=A^{n+1}_\xi=A^n_\eta$, where $\vec{A}^n$ was the enumeration of $\mathscr{P}(\kappa)\cap\mathcal{M}_n$ used at the n'th stage. Then by definition of μ_n we get that for every $\beta\in H_n\cap C$ with $\beta>\eta$, $\beta\in A^n_\eta$. We need to show that

$$(\exists\beta\in H_{n+1}\cap D\cap C)(\beta>\xi\land\beta\in A^{n+1}_\xi)$$

holds. But here we can simply pick a $\beta>\max(\xi,\eta)$ with $\beta\in H_{n+1}\cap D\cap C\subseteq H_n\cap C$. This shows that $\mu_n\subseteq\mu_{n+1}$, making κ $(n{+}1)$-Ramsey and thus inductively also coherent $<\omega$-Ramsey. ∎

2.3 The countable case

This section covers the (strategic) γ-Ramsey cardinals whenever γ has countable cofinality. This case is special because, as mentioned in Section 2.1, we cannot ensure that the final measure is countably complete and so the existence of winning strategies in the $\mathcal{G}_\gamma^\theta(\kappa)$ *might* depend on θ, in contrast with the uncountable cofinality case; see e.g. Question **??**.

Fix ref

[Strategic] ω-Ramsey cardinals

Remove the virtual definitions and basic results if they're redundant

We now move to the strategic ω-Ramsey cardinals and their relationship to the (non-strategic) ω-Ramseys. For this we define a new addition to the family of *virtual cardinals* from [Gitman and Schindler, 2018], the *virtually measurable cardinals*.

Definition 2.3.1. A cardinal κ is **virtually measurable** if for every regular $\nu > \kappa$ there exists a transitive M and a forcing $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there exists an elementary embedding $j : H_\nu^V \to M$ with $\operatorname{crit} j = \kappa$. $\circ$

We'll need the following well-known lemmata; see Lemma 7.1 in [Holy and Schlicht, 2018] and Lemma 3.1 in [Gitman and Schindler, 2018] for their proofs.

Lemma 2.3.2 (Ancient Kunen Lemma)**.** *Let $M \models$ ZFC$^-$ and $j : M \to N$ an elementary embedding with critical point κ such that $\kappa + 1 \subseteq M \subseteq N$. Assume that $X \in M$ has M-cardinality κ. Then $j \restriction X \in N$.* ■

Lemma 2.3.3 (Absoluteness of embeddings on countable structures)**.** *Let M be a countable first-order structure and $j : M \to N$ an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of) ZFC such that M is countable in W and $N \in W$, then for any finite subset of M, W has some elementary embedding $j^* : M \to N$, which agrees with j on that subset. Moreover, if both M and N are transitive $\in$-structures and j has a critical point, we can also assume that* $\operatorname{crit}(j^*) = \operatorname{crit}(j)$. ■

Theorem 2.3.4 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. Then κ is generically θ-measurable iff player II has a winning strategy in $\mathcal{C}^\theta_\omega(\kappa)$.*

Proof. $(\Leftarrow)$: Fix a winning strategy σ for player II in $\mathcal{C}^\theta_\omega(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, H^V_\theta)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \prec H^V_\theta$ such that $H^V_\theta \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^V_θ-normal H^V_θ-measure μ on κ.

We claim that $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n \colon \kappa \to \theta$ such that $g_n \in H^V_\theta$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f \colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{C}^\theta_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $C^\theta_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so that the ultrapower embedding $\pi \colon H^V_\theta \to \mathrm{Ult}(H^V_\theta, \mu)$ witnesses that κ is generically θ-measurable.

$(\Rightarrow)$: Assume that κ is generically θ-measurable. Let $\mathbb{P}$ be a forcing $\dot\mu$ a $\mathbb{P}$-name for an H^V_θ-normal H^V_θ-measure on κ and $\dot\pi$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}^\theta_\omega(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f^n_i \mid i < k\rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \check\mu \cap \mathcal{M}_n = \check\mu_n \cap \forall i < \check k \colon \dot\pi(\check f^n_i)(\check\kappa) = \check\alpha^n_i \urcorner,$$

with $\mu_n, \alpha_i^n \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f_{i_n}^{k_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check{\alpha}_{i_{n+1}}^{k_{n+1}} < \check{\alpha}_{i_n}^{k_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

Here's a near-analogous result for the $\mathcal{G}_\omega^\theta(\kappa)$ game from [?], with a proof added for completeness.

Theorem 2.3.5 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$, and if player II has a winning strategy in $\mathcal{G}_\omega^\theta(\kappa)$ then κ is generically θ-power-measurable. In particular, $\mathcal{G}_\omega^\theta(\kappa)^L \sim \mathcal{C}_\omega^\theta(\kappa)^L$.*

Proof. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi\colon H_\theta^V \to \mathcal{M}$ with $\mathrm{crit}\,\pi = \kappa$. Fix a name $\dot{\mu}$ and a $\mathbb{P}$-condition p such that

$$p \Vdash \ulcorner \dot{\mu} \text{ is a weakly amenable } \check{H}_\theta\text{-normal } \check{H}_\theta\text{-measure with a wellfounded ultrapower} \urcorner.$$

We now define a strategy σ for player II in $\mathcal{G}_\omega^\theta(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H_\theta^V$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n\colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot{\mu} \cap \check{\mathcal{M}}_n = \check{\mu}_n \cap \check{\mu}_n = \dot{\mu} \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma **??** we get that $\pi \restriction \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence

$\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega} \mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot\pi(\check g_{n+1})(\check\kappa) < \dot\pi(\check g_n)(\check\kappa) \urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check\pi_{n+1}(\check g_{n+1})(\check\kappa) < \check\pi_n(\check g_n)(\check\kappa) \urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy. ■

Question 2.3.6. If κ is generically θ-power-measurable, does player II then have a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$?

We get the following immediate corollary.

Corollary 2.3.7 (N.-Schindler)**.** *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ■

Note also that the proof of Theorem **??** shows that whenever κ is strategic ω-Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H^V_ν-normal H_ν-measure on κ.

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω-Ramsey cardinals and ω-Ramsey cardinals lie, namely that strategic ω-Ramseys are equiconsistent with *remarkables* and ω-Ramseys are strictly below. Theorem 4.8 of [Gitman and Welch, 2011] showed that 2-iterables are limits of remarkables, and our Propositions 2.1.8 and 2.5.21 shows that ω-Ramseys are limits of 1-iterables, so that the strategic ω-Ramseys and the ω-Ramseys both lie strictly between the

2-iterables and 1-iterables. It was shown in [Holy and Schlicht, 2018] that ω-Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by [Schindler, 2000b], and [Gitman and Schindler, 2018] showed the following two equivalent formulations.

Definition 2.3.8. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

(i) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H^V_\lambda \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H^V_\nu \to M$ with critical point κ and $\pi(\kappa) > \lambda$;

(ii) For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^\lambda M \subseteq M$ and a forcing poset $\mathbb{P}$, such that in $V^{\mathbb{P}}$ there's an elementary embedding $\pi : H^V_\nu \to M$ with critical point κ and $\pi(\kappa) > \lambda$.

$\circ$

Theorem 2.3.9 (N.). *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models \ulcorner$there is a proper class of virtually measurables$\urcorner$. In particular, the two notions are equiconsistent.*

Proof. Virtually measurables are downwards absolute to L by Lemma 2.3.3, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H^V_\lambda \subseteq M$ and forcing poset $\mathbb{P}$ it holds that, in $V^{\mathbb{P}}$, there's no elementary embedding $\pi : H^V_\nu \to M$ with $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν-measurable to fix a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's an elementary $\pi : H^V_\nu \to M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H^V_\nu = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we're assuming that κ isn't remarkable. Then by restricting the generic embedding to H^V_κ we get that $H^V_\kappa \prec H^M_{\pi(\kappa)} = H^V_{\pi(\kappa)}$, using that $\pi(\kappa) < \nu$ and $H^V_\nu = H^M_\nu$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H^V_ν since $\pi(\kappa) < \nu$, and as $H^V_\nu \prec_1 V$ we get that $\pi(\kappa)$ *is* a cardinal. But then, again using that $H_{\pi(\kappa)} \prec_1 V$, κ is virtually measurable in $H^V_{\pi(\kappa)}$ since being virtually measurable is Π_2. This means that for every $\xi < \kappa$ it holds that

$$H^V_{\pi(\kappa)} \models \exists\alpha > \xi : \ulcorner\alpha \text{ is virtually measurable}\urcorner,$$

implying that $H^V_\kappa \models \ulcorner$There is a proper class of virtually measurables$\urcorner$. ■

Now Theorem 2.3.9 and Corollary 2.3.7 yield the following immediate corollary.

Corollary 2.3.10 (N.-Schindler). *Let κ be strategic ω-Ramsey. Then either κ is remarkable in L or otherwise*

$$L_\kappa \models \ulcorner\textit{there is a proper class of strategic } \omega\textit{-Ramseys}\urcorner.$$

In particular, the two notions are equiconsistent. ■

Now, using these results we show that the strategic ω-Ramseys have strictly stronger consistency strength than the ω-Ramseys.

Theorem 2.3.11 (N.). *Remarkable cardinals are strategic ω-Ramsey limits of ω-Ramsey cardinals.*

Proof. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ-sequences and a generic elementary embedding $\pi : H^V_\nu \to M$ for some $\nu > 2^\kappa$. We will show that κ is ω-Ramsey in M. Note that remarkables are clearly virtually measurable, and thus by Theorem **??** also strategic ω-Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}^\theta_\omega(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}^\theta_\omega(\kappa)^M$. Since M is closed under 2^κ-sequences it means that $\mathscr{P}(\mathscr{P}(\kappa)) \subseteq M$ and thus that M contains all possible filters on κ. We let player II follow τ, which produces a play $\sigma * \tau$ in which player II wins. But all player II's moves are in $\mathscr{P}(\mathscr{P}(\kappa))$ and hence in M, and as M is furthermore closed under ω-sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω-Ramsey in M.

This also implies that κ is a limit of ω-Ramseys in H_ν. But as κ is remarkable it holds that $H_\kappa \prec_2 V$, in analogy with the same property for strongs and supercompacts, and as being ω-Ramsey is a Π_2-notion this means that κ *is* a limit of ω-Ramseys. ■

This immediately yields the following corollary.

Corollary 2.3.12 (N.-Schindler)**.** *If κ is a strategic ω-Ramsey cardinal then*

$$L_\kappa \models \ulcorner \textit{there is a proper class of } \omega\textit{-Ramseys}\urcorner. \qquad \dashv$$

When we have a winning strategy

Theorem 2.3.13 (Schindler-N.)**.** *Let $\kappa < \theta$ be regular cardinals. Then κ is generically θ-measurable iff player II has a winning strategy in $\mathcal{C}^\theta_\omega(\kappa)$.*

PROOF. ($\Leftarrow$) : Fix a winning strategy σ for player II in $\mathcal{C}^\theta_\omega(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, H^V_\theta)$ be V-generic and in $V[g]$ fix an elementary chain $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \prec H^V_\theta$ such that $H^V_\theta \subseteq \bigcup_{n<\omega}\mathcal{M}_n$, using that θ is regular and has countable cofinality in $V[g]$. Player II follows σ, resulting in a H^V_θ-normal H^V_θ-measure μ on κ.

We claim that $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so assume not, witnessed by a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n \colon \kappa \to \theta$ such that $g_n \in H^V_\theta$ and

$$\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu.$$

Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f\colon \kappa \to \theta$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{C}^\theta_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of wellfoundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But

b now gives us a play of $C^\theta_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H^V_\theta, \mu)$ is wellfounded, so that the ultrapower embedding $\pi\colon H^V_\theta \to \mathrm{Ult}(H^V_\theta, \mu)$ witnesses that κ is generically θ-measurable.

($\Rightarrow$) : Assume that κ is generically θ-measurable. Let $\mathbb{P}$ be a forcing $\dot\mu$ a $\mathbb{P}$-name for an H^V_θ-normal H^V_θ-measure on κ and $\dot\pi$ a $\mathbb{P}$-name for the associated ultrapower embedding. Define a strategy for player II in $\mathcal{C}^\theta_\omega(\kappa)$ as follows: Whenever player I plays $\mathcal{M}_n$ then fix some $\mathbb{P}$-condition p_n such that, letting $\langle f^n_i \mid i < k\rangle$ enumerate all functions in $\mathcal{M}_n$ with domain κ,

$$p_n \Vdash \ulcorner \dot\mu \cap \mathcal{M}_n = \check\mu_n \cap \forall i < \check k\colon \dot\pi(\check f^n_i)(\check\kappa) = \check\alpha^n_i \urcorner,$$

with $\mu_n, \alpha^n_i \in V$. Note here that we can ensure $\mu_n \in V$ because it's finite. Also, ensure that the p_n's are $\leq$-decreasing. Assume now that $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ is illfounded, witnessed by functions $g_n \in {}^\kappa\mathcal{M}_\omega \cap \mathcal{M}_\omega$ for $n < \omega$. Then $g_n = f^{k_n}_{i_n}$ for some $k_n, i_n < \omega$, and hence $p_{k_{n+1}} \Vdash \ulcorner \check\alpha^{k_{n+1}}_{i_{n+1}} < \check\alpha^{k_n}_{i_n} \urcorner$ for every $n < \omega$, so in V we get an ω-sequence of strictly decreasing ordinals, $\lightning$. ■

Here's a near-analogous result for the $\mathcal{G}^\theta_\omega(\kappa)$ game from [?], with a proof added for completeness.

Theorem 2.3.14 (Schindler-N.). *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$, and if player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$ then κ is generically θ-power-measurable. In particular, $\mathcal{G}^\theta_\omega(\kappa)^L \sim \mathcal{C}^\theta_\omega(\kappa)^L$.*

Proof. The second statement is exactly like the ($\Leftarrow$) direction in the previous theorem, so we show the first statement. Assume κ is virtually θ-prestrong and fix a regular $\theta > \kappa$, a transitive $\mathcal{M} \in V$, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi\colon H^V_\theta \to \mathcal{M}$ with $\mathrm{crit}\,\pi = \kappa$. Fix a name $\dot\mu$ and a $\mathbb{P}$-condition p such that

$$p \Vdash \ulcorner \dot\mu \text{ is a weakly amenable } \check H_\theta\text{-normal } \check H_\theta\text{-measure with a wellfounded ultrapower} \urcorner.$$

We now define a strategy σ for player II in $\mathcal{G}^\theta_\omega(\kappa)$ as follows. Whenever player I plays a weak κ-model $\mathcal{M}_n \prec H^V_\theta$, player II fixes $p_n \in \mathbb{P}$, an $\mathcal{M}_n$-measure μ_n and a function $\pi_n \colon \mathcal{M}_n \to \pi(\mathcal{M}_n)$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \ulcorner \dot\mu \cap \check{\mathcal{M}}_n = \check\mu_n \cap \check\mu_n = \dot\mu \restriction \check{\mathcal{M}}_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma **??** we get that $\pi \restriction \mathcal{M}_n \in \mathcal{M} \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to $\mathcal{M}_n$ with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ has a wellfounded ultrapower. Assume it hasn't, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n \colon \kappa \to \mathcal{M}_\omega := \bigcup_{n<\omega} \mathcal{M}_n$ such that $g_n \in \mathcal{M}_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \tag{2}$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in \mathcal{M}_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot\pi(\check g_{n+1})(\check\kappa) < \dot\pi(\check g_n)(\check\kappa) \urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check\pi_{n+1}(\check g_{n+1})(\check\kappa) < \check\pi_n(\check g_n)(\check\kappa) \urcorner,$$

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω has a wellfounded ultrapower, making σ a winning strategy. ■

Question 2.3.15. If κ is generically θ-power-measurable, does player II then have a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$?

Separation results

Question 2.3.16. Is every generically θ-prestrong also generically θ-strong? In other words, do *(ii)* and *(iii)* in the above theorem imply that κ is generically θ-strong, or can we find an example of a generically θ-prestrong cardinal satisfying either *(ii)* or *(iii)* which isn't generically θ-strong?

[If $0^\sharp$ exists then letting (κ, λ) be the lexicographically least such that κ is virtually λ-rank-into-rank and virtually λ^+-prestrong in L, if κ waas virtually λ^+-strong in L then $L_\kappa \prec_2 L_{\lambda^+}$, so that L_κ has a $\bar\kappa$ which is $\bar\lambda$-rank-into-rank and $\bar\lambda^+$-prestrong, which is absolute to L, a contradiction. So θ-prestrong doesn't in general imply θ-strong.]

We first show that the virtuals form a level-by-level hierarchy.

Can we replace "prestrong" with "strong" below, as L_{θ^+} can see that κ is virtually θ-strong?

Theorem 2.3.17 (N.). *Let $\alpha < \kappa$ and assume that κ is generically $\kappa^{+\alpha+2}$-measurable. Then*

$$L_\kappa \models \ulcorner \textit{There's a proper class of } \lambda \textit{ which are virtually } \lambda^{+\alpha+1}\textit{-prestrong}\urcorner.$$

Proof. Write $\theta := \kappa^{+\alpha+1}$. Then by Theorem 1.2.8 we get that either κ is generically θ^+-strong in L or otherwise, in particular, L_κ thinks that there's a proper class of remarkables. In the second case we also get that L_κ thinks that there's a proper class of λ such that λ is virtually $\lambda^{+\alpha+1}$-prestrong and we'd be done, so assume the first case. Then $L_\kappa \prec_2 L_{\theta^+}$, so define for each $\xi < \kappa$ the sentence ψ_ξ as

$$\psi_\xi :\equiv \exists\lambda < \xi \colon \ulcorner \lambda \text{ is virtually } \lambda^{+\alpha+1}\text{-prestrong}\urcorner.$$

Then ψ_ξ is $\Sigma_2(\{\alpha, \xi\})$ since being virtually β-prestrong is a $\Delta_2(\{\beta\})$-statement. As $L_{\theta^+} \models \psi_\xi$ for all $\xi < \kappa$ we also get that $L_\kappa \models \psi_\xi$ for all $\xi < \kappa$, which is what we wanted to show. ■

This thus in particular shows that the generically $\kappa^{+\alpha+1}$-measurable cardinals κ form a strict hierarchy whenever $\alpha < \kappa$.

Proposition 2.3.18 (N.). *Assuming κ is measurable, there's a generic extension of V in which κ is inaccessible and κ-cc $<\kappa$-distributive generically ∞-measurable, but not weakly compact.*

Proof. By [?] we get that there are two generic extensions $V[g]$ and $V[g][h]$ such that κ is measurable in $V[g][h]$ and in $V[g]$ κ is inaccessible and there exists a κ-Suslin tree. But since forcing with a κ-Suslin tree is κ-cc and $<\kappa$-distributive we get that κ is immediately κ-cc $<\kappa$-distributive generically ∞-measurable in $V[g]$. ■

Corollary 2.3.19 (N.). *Let κ be inaccessible.*

(i) If player II wins $\mathcal{C}^\theta_\omega(\kappa)$ for all regular $\theta > \kappa$ then κ is not necessarily weakly compact;

(ii) If player II wins $\mathcal{C}_\kappa(\kappa)$ then κ is *weakly compact.*

Proof. The first claim is directly by Proposition 2.3.18 and Theorem 2.3.13, and the second claim is because the hypothesis implies that player II wins $\mathcal{G}_0(\kappa)$ so that inaccesibility of κ makes κ weakly compact — see e.g. [?] for this characterisation of weak compactness. ■

Question 2.3.20. How strong are cardinals κ such that player II wins $\mathcal{C}^\theta_\gamma(\kappa)$ for all $\theta > \kappa$, for various $\gamma < \kappa$? Can they be weakly compact? In the $\gamma = \omega$ case, can $\square(\kappa)$ hold?

We'll now show that the generic and ideal variants are all separated from the virtual ones. A key ingredient is that virtually critical cardinals are Π^2_1-indescribable, whose proof is identical to the standard proof in [Hanf and Scott, 1961] that measurable cardinals are Π^2_1-indescribable. It should be noted that we crucially need the "virtual" property for the proof to go through. Using this indescribability fact, the proof of the following theorem is precisely the same as Hamkins' Proposition 8.2 in [Holy and Schlicht, 2018].

Add the proof the of the following theorem.

Theorem 2.3.21 (Hamkins). *Assuming κ is a κ^{++}-tall cardinal,*[9] *there's a forcing extension of V in which κ is not virtually critical, but becomes measurable in an $Add(\kappa^+, 1)$-generic extension.* ■

This then gives us our separation result.

Corollary 2.3.22 (N.). *Assuming κ is a κ^{++}-tall cardinal, it's consistent that κ is $<\kappa$-closed κ-sized ideally measurable but not virtually critical.*

Proof. By the above Theorem 2.3.21 we may assume that κ is not virtually critical but that it's measurable in $V^{\mathbb{P}}$ for $\mathbb{P} := \text{Add}(\kappa^+, 1)$, so that κ is κ-closed κ^+-sized generically ∞-measurable. We will see in Theorem 2.5.11 that κ-closed κ^+-sized generically ∞-measurables are equivalent to κ-closed κ^+-sized *ideally* measurables, so that we achieve separation of these from the virtually criticals as well. ■

Question 2.3.23. Can we find a virtually ∞-measurable which isn't measurable?

As for the relationship between the generics and the ideals, [**?**] shows that, assuming the existence of a measurable cardinal, consistently we can get a generically ∞-measurable cardinal which isn't ideally measurable, separating the two. However, if $2^\kappa = \kappa^+$ then [Ferber and Gitik, 2010] shows that κ is ideally critical if and only if κ is generically critical.

Perhaps properly state these results, and maybe prove them?

2.4 What happens in core models?

Most of the cardinals turn out to downwards absolute to most inner models, including L:

[9] Recall that κ is κ^{++}**-tall** if there's an elementary embedding $j\colon V \to M$ with $\operatorname{crit} j = \kappa$, ${}^\kappa M \subseteq M$ and $j(\kappa) > \kappa^{++}$.

Proposition 2.4.1. *For any regular uncountable cardinal θ, generically θ-measurable cardinals are downwards absolute to any transitive class $\mathcal{U} \subseteq V$ satisfying $\mathsf{ZF}^- + \mathsf{DC}$.*

PROOF. Let κ be generically θ-measurable, witnessed by a forcing poset $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there's a transitive $\mathcal{M}$ and an elementary embedding $\pi\colon H^V_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$. Fix a transitive class $\mathcal{U} \subseteq V$ which satisfies $\mathsf{ZF}^- + \mathsf{DC}$. Restricting the embedding to $\pi \restriction H^{\mathcal{U}}_\theta\colon H^{\mathcal{U}}_\theta \to \mathcal{N}$ we can now apply the Countable Absoluteness Lemma 1.1.1 to $\pi \restriction H^{\mathcal{U}}_\theta$ to get that there exists an embedding $\pi^*\colon H^{\mathcal{U}}_\theta \to \mathcal{N}^*$ in a generic extension of U, making κ generically θ-measurable in $\mathcal{U}$. ■

Proposition 2.4.2 (N.). *Let θ be a regular uncountable cardinal.*

(i) $L \models \ulcorner$*generically θ-measurables are equivalent to virtually θ-prestrongs*$\urcorner$.

(ii) $L[\mu] \models \ulcorner$*generically θ-measurables are equivalent to virtually θ-measurable*$\urcorner$.[10]

PROOF. For (i) simply note that if $\pi\colon L_\theta \to \mathcal{N}$ is a generic elementary embedding with $\mathcal{N}$ transitive, then by condensation we have that $\mathcal{N} = L_\gamma$ for some $\gamma \geq \theta$, so that π also witnesses the virtual θ-prestrongness of $\operatorname{crit}\pi$.

(ii): Assume that $V = L[\mu]$ for notational simplicity and let κ be generically θ-measurable, witnessed by a generic elementary embedding $\pi\colon L_\theta[\mu] \to \mathcal{N}$ existing in some generic extension $V[g]$. By condensation we get that $\mathcal{N} = L_\gamma[\overline{\mu}]$ for some $\gamma \geq \theta$ and $\overline{\mu} \in V[g]$, but we're not guaranteed that $\overline{\mu} \in V$ here. Let λ be the unique measurable cardinal.

If $\kappa > \lambda$ then $\overline{\mu} = \mu$ as it simply isn't moved by π in that case, and $\mathcal{N} \subseteq V$. So assume that $\kappa \leq \lambda$ and compare $L_\theta[\mu]$ with $\mathcal{N}$ to a common iterate $L_\alpha[\hat{\mu}]$.

We don't necessarily get a common iterate, so we have to do initial segment cases — it still works out, however.

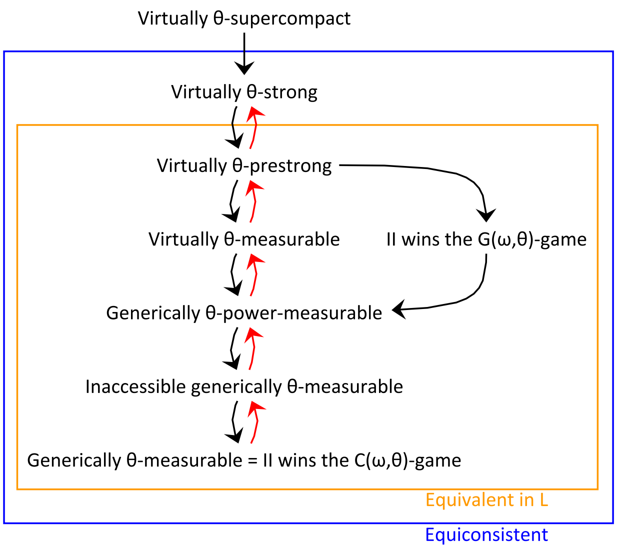
Let $i\colon \mathcal{N} \to L_\alpha[\hat{\mu}]$ be the iteration embedding, which is non-dropping. Note that $L_\alpha[\hat{\mu}] \in L[\mu]$ as it's an internal iterate of $L_\theta[\mu]$, and $\operatorname{crit}(i \circ \pi) = \kappa$ as $\kappa \leq \lambda$ holds by assumption, so $i\circ\pi$ witnesses that κ is virtually θ-measurable.

[10] Assuming that $L[\mu]$ exists, of course.

We might have that $\mu \neq \hat{\mu}$, however, so prestrongness of κ seems not to be guaranteed in this case. ■

Question 2.4.3. What happens in larger core models? It seems that in both $L[\mu]$ and K below $0^{\P}$ we get that generically θ-measurables are equivalent to virtually θ-measurables, but the measurable in $L[\mu]$ is virtually measurable and not virtually κ^{++}-strong. What happens to winning strategies in $\mathcal{G}^{\theta}_{\omega}(\kappa)$ then?

The following summarises the separation results from last chapter along with the above.



2.5 Ideal-absoluteness

Definition 2.5.1. A poset property[11] $\Phi(\kappa)$ is **ideal-absolute** if whenever κ satisfies that there's a $\Phi(\kappa)$ forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure μ on κ, then there's an ideal I on κ such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to a forcing satisfying $\Phi(v)$. ◦

Note that this is *almost* saying that $\Phi(\kappa)$ ideally measurables are equivalent to $\Phi(\kappa)$ generically ∞-measurables, but the only difference is that these definitions require well-foundedness of the above M.

Also note that ω-distributive generically θ_0-measurable cardinals are equivalent to ω-distributive generically θ_1-measurable cardinals for all regular $\theta_0, \theta_1 \in \infty \cup \{\infty\}$ since wellfoundedness becomes automatic, so in this case we will simply write "ω-distributive generically measurable".

Note that the ideally measurables aren't equiconsistent with the generically- and virtually measurables, since the ideally measurable cardinals are ideally ∞-measurable and are therefore equiconsistent with a measurable cardinal. Because of this proposition we will refrain from using the "ideally ∞-measurable" terminology and only use "ideally measurable" from now on.

Add proof?

We *do* get an equiconsistency at the critical level though, as Theorem 2.11 of [Ferber and Gitik, 2010] shows that if κ is generically critical then it's ideally critical in $L^{\mathrm{Col}(\omega,<\kappa)}$.

Definition 2.5.2. Let κ be a regular cardinal, $\mathbb{P}$ a poset and $\dot\mu$ a $\mathbb{P}$-name for a V-normal V-measure on κ. Then the **induced ideal** is

$$\mathcal{I}(\mathbb{P}, \dot\mu) := \{X \subseteq \kappa \mid \big|\big|\check X \in \dot\mu\big|\big|_{\mathcal{B}(\mathbb{P})} = 0\},$$

where $\mathcal{B}(\mathbb{P})$ is the boolean completion of $\mathbb{P}$. ◦

Note that if the generic measure μ is furthermore V-normal then $\mathcal{I}(\mathbb{P}, \dot\mu)$ is also normal.

[11] Examples of these are having the κ-chain condition, being κ-closed, κ-distributive, κ-Knaster, κ-sized and so on.

2.5.3 κ^+-chain condition

Other chain conditions?

Theorem 2.5.4 (Folklore)**.** *"The κ^+-chain condition" is ideal-absolute.*

PROOF. Assume $\mathbb{P}$ has the κ^+-chain condition such that there's a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot{\mu})$ — we will show that $\mathscr{P}(\kappa)/I$ has the κ^+-chain condition. Assume not and let $\langle X_\alpha \mid \alpha < \kappa^+\rangle$ be an antichain of $\mathscr{P}(\kappa)/I$, which by normality of I we may assume is pairwise almost disjoint. But this then makes $\langle ||\check{X}_\alpha \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid \alpha < \kappa^+\rangle$ an antichain of $\mathbb{P}$ of size κ^+, $\lightning$. ■

2.5.5 $<\lambda$-distributivity

Recall that an ideal I on some κ is ω-distributive if and only if it's precipitous[12], so that carrying an ω-distributive ideal coincides with our definition of *ideally measurable.*

Theorem 2.5.6 (N.)**.** *"$<\lambda$-distributivity" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.*

PROOF. Assume that $\mathbb{P}$ is a $<\lambda$-distributive forcing such that there exists a $\mathbb{P}$-name $\dot{\mu}$ for a V-normal V-measure on κ. Let $I := \mathcal{I}(\mathbb{P}, \dot{\mu})$ — we'll show that $\mathscr{P}(\kappa)/I$ is $<\lambda$-distributive. Let $\mathcal{T} \subseteq (\mathscr{P}(\kappa)/I)^{<\lambda}$ be an unrooted tree of height $<\lambda$ such that every level $\mathcal{T}_\alpha$ is a maximal antichain. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of branches of $\mathcal{T}$. Now define a corresponding tree $\mathcal{T}^* \subseteq \mathbb{P}^{<\lambda}$ as

Do this in terms of $\prec$-chains of antichains instead.

$$\mathcal{T}^*_\alpha := \{||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})} \mid X \in \mathcal{T}_\alpha\}.$$

Note that every level $\mathcal{T}^*_\alpha$ is an antichain in $\mathbb{P}$. They're also maximal, because if $p \in \mathbb{P}$ was incompatible with every condition in $\mathcal{T}^*_\alpha$ then, letting $X := \bigcap \mathcal{T}_\alpha$, we have that p is compatible with $||\check{X} \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$, so that $X \in I^+$. But X is incompatible with everything in $\mathcal{T}_\alpha$, contradicting that $\mathcal{T}_\alpha$ is maximal.

[12]See [?] and [Foreman, 1983].

By $<\lambda$-distributivity of $\mathbb{P}$ we get an antichain $\mathcal{A}^*$ consisting of limit points of branches of $\mathcal{T}^*$. But note that for every $p \in \mathcal{A}^*$ it holds that $p \leq ||\Delta b_p \in \dot{\mu}||_{\mathcal{B}(\mathbb{P})}$ with b_p being the branch of $\mathcal{T}^*$ with limit p,[13] so that $\Delta b_p \in I^+$. Now $\mathcal{A} := \{\Delta b_p \mid p \in \mathcal{A}^*\}$ gives us a maximal antichain consisting of limit points of branches of $\mathcal{T}$. ■

Question 2.5.7. Is "ω-distributive (κ,κ)-distributive" ideal-absolute? Does it correspond to generically power-measurables?

2.5.8 (κ,κ)-distributivity & $<\lambda$-closure

In this section we will prove a slightly stronger version of the following unpublished result by Foreman:

Theorem 2.5.9 (Foreman)**.** *Let κ be a regular cardinal such that $2^\kappa = \kappa^+$, and let $\lambda \leq \kappa^+$ be an infinite successor cardinal. If player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$ then κ carries a κ-complete normal precipitous ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ has a dense $<\lambda$-closed subset of size κ^+.*

Theorem 2.5.10 (Foreman-N.)**.** *Let κ be a regular cardinal and $\lambda \leq \kappa^+$ be regular infinite. If player II has a winning strategy in $\mathcal{G}^-_\lambda(\kappa)$ then κ carries a κ-complete normal ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is (κ,κ)-distributive and has a dense $<\lambda$-closed subset of size κ^+.*

Before we start the proof, let us note that the only difference between the two theorems is that we are requiring neither $2^\kappa = \kappa^+$ nor that λ is a successor cardinal. The proof strategy is similar to the original proof, but with some more technical details to ensure these strengthenings.

Proof. Set $\mathbb{P} := \mathrm{Add}(\kappa^+, 1)$ if $2^\kappa > \kappa^+$ and $\mathbb{P} := \{\emptyset\}$ otherwise. If κ is measurable then the dual ideal to the measure on κ satisfies all of the wanted properties, so assume that κ is not measurable. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \kappa^+\rangle \in V^{\mathbb{P}}$ such that

[13] Here we're using that all branches have length $<\kappa^+$, by choice of λ.

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \kappa^+$;
- $\mathcal{N}_{\gamma+1} \prec H^V_{\kappa^+}$ is a κ-model for every $\gamma < \kappa^+$;
- $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$ for limit ordinals $\delta < \kappa^+$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for $\gamma < \beta < \kappa^+$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\kappa^+} \mathcal{N}_\gamma$.

Define now the auxilliary game $\mathcal{G}(\kappa)$ of length λ as follows.

$$\begin{array}{llllll} \text{I} & \alpha_0 & & \alpha_1 & & \cdots \\ \text{II} & & p_0, \mathcal{M}_0, \mu_0, Y_0 & & p_1, \mathcal{M}_1, \mu_1, Y_1 & & \cdots \end{array}$$

Here $\langle \alpha_\gamma \mid \gamma < \lambda \rangle$ is an increasing continuous sequence of ordinals bounded in κ^+, $\vec{p}_\gamma$ is a decreasing sequence of $\mathbb{P}$-conditions satisfying that

$$p_\gamma \Vdash \ulcorner \check{\mathcal{M}}_\gamma = \pi(\check{\alpha}_\gamma) \wedge \check{\mu}_\gamma \text{ is a } \check{\mathcal{M}}_\gamma\text{-normal } \check{\mathcal{M}}_\gamma\text{-measure on } \check{\kappa} \urcorner$$

such that $Y_\gamma = \Delta_{\xi<\kappa} X^{\mu_\gamma}_\xi$, where $\vec{X}^{\mu_\gamma}_\xi \in H^V_{\kappa^+}$ is the $<_{\kappa^+}$-least enumeration of μ_γ.[14] We require that the μ_γ's are $\subseteq$-increasing, and player II wins iff she can continue playing throughout all λ rounds. Let $\mu_\lambda := \bigcup_{\xi<\lambda} \mu_\xi$ be the **final measure** of the play.

To every limit ordinal $\eta < \kappa^+$ define the **restricted auxilliary game** $\mathcal{G}(\kappa) \restriction \eta$ in which player I is only allowed to play ordinals $<\eta$. Note that a strategy τ for player II is winning in $\mathcal{G}(\kappa)$ if and only if it's winning in $\mathcal{G}(\kappa) \restriction \eta$ for all $\eta < \kappa^+$, simply because all sequences of ordinals played by player I are bounded in κ^+.

Note that μ_λ is precisely the tail measure on κ defined by the Y_γ's; i.e. that $X \in \mu_\lambda$ iff there exists a $\delta < \lambda$ such that $|Y_\delta - X| < \kappa$. From this it's simple to see that $\mathcal{G}(\kappa)$ is equivalent to $\mathcal{G}^-_\lambda(\kappa)$, so player II has a winning strategy τ_0 in $\mathcal{G}(\kappa)$.

[14] We use that $\mathbb{P}$ is κ-closed to get the p_γ's as well as to ensure that $\mathcal{M}_\gamma, \mu_\gamma \in V$.

For any winning strategy τ in $\mathcal{G}(\kappa)\restriction\eta$ and to every partial play p of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ, define the associated **hopeless ideal**[15]

$$I^\tau_p\restriction\eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa)\restriction\eta,\\ X \text{ is \textit{not} in the final measure}\}$$

Claim 2.5.10.1. Every hopeless ideal $I^\tau_p\restriction\eta$ is normal and (κ,κ)-distributive.

Proof of claim. For normality, if $\langle Z_\gamma \mid \gamma < \kappa\rangle$ is a sequence of elements of I^τ_p such that $Z := \nabla_\gamma Z_\gamma$ is I^τ_p-positive, then there exists a play of $\mathcal{G}(\kappa)\restriction\eta$ in which player II follows τ such that Z lies in the final measure. If we let player I play sufficiently large ordinals in $\mathcal{G}(\kappa)\restriction\eta$ we may assume that $\langle Z_\gamma \mid \gamma < \kappa\rangle$ is a subset and an element of the final model as well, meaning that one of the Z_γ's also lies in the final measure, $\lightning$.

We now show (κ,κ)-distributivity. Let $\mathcal{U} \subseteq \mathscr{P}(\kappa)/I^\tau_p$ be an unrooted tree of height κ such that every level $\mathcal{U}_\alpha$ is a maximal antichain of size $\leq\kappa$. We have to show that there's a maximal antichain $\mathcal{A}$ consisting of limit points of branches of $\mathcal{U}$. Pick $X \in \mathcal{U}$ and let p be a play of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ with limit model $\mathcal{M}$ and limit measure μ, such that $X \in \mu$.

By letting player I in p play sufficiently large ordinals, we may assume that $\mathcal{U} \subseteq \mathcal{M}$, using that $|\mathcal{U}| \leq \kappa$, and also that $b_X := \mathcal{U}\cap\mu \in \mathcal{M}$. This means that $d_X := \Delta b_X \in \mathscr{P}(\kappa)/I^\tau_p$ is a limit point of the branch b_X through $\mathcal{U}$, so that $\mathcal{A} := \{d_X \mid X \in \mathcal{U}\}$ is a maximal antichain of limit points of branches of $\mathcal{U}$, making $\mathscr{P}(\kappa)/I^\tau_p$ (κ,κ)-distributive. $\dashv$

Fix some limit ordinal $\eta < \kappa^+$. We will recursively construct a tree $\mathcal{T}^\eta$ of height λ which consists of subsets $X \subseteq \kappa$, ordered by reverse inclusion. During the construction of the tree we will inductively maintain the following properties of $\mathcal{T}^\eta\restriction\alpha$ for $\alpha \leq \lambda$:

[15] This terminology is due to Matt Foreman.

- Tree strategy: For every $\gamma < \alpha$ there is a winning strategy τ^η_γ for player II in $\mathcal{G}(\kappa)\restriction\eta$ such that for every $\beta < \gamma$, the β'th move by τ^η_γ is an element of $\mathcal{T}^\eta_\beta$ and τ^η_γ is consistent with τ^η_β for the first β-many rounds.
- Unique pre-history: Given any $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$ there's a unique partial play p of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ^η_β ending with Y — we define $I^\tau_Y := I^\tau_p$ for τ being any winning strategy for player II in $\mathcal{G}(\kappa)\restriction\eta$ satisfying that p is consistent with τ^η_β.
- Cofinally many responds: Let $\beta+1 < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$, and set p to be the unique partial play of $\mathcal{G}(\kappa)\restriction\eta$ given by the unique pre-history of Y. Then the $\mathcal{T}^\eta$-successors of Y consists of player II's τ^η_β-responds to τ^η_β-partial plays extending p such that player I's last move in these partial plays are cofinal in η.[16]
- Positivity: If $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$ then Y is $I^{\tau^\eta_\gamma}_X$-positive for every $\gamma < \beta$ and every $X \in \mathcal{T}^\eta\restriction\gamma+1$ with $X \leq_{\mathcal{T}^\eta} Y$.[17]
- Almost disjointness property: Every level $\mathcal{T}^\eta_\beta$ consists of pairwise almost disjoint sets.[18]
- Hopeless ideal coherence: $I^{\tau^\eta_\beta}_{\langle\rangle} \cap \mathscr{P}(Y) = I^{\tau^\eta_\beta}_Y \cap \mathscr{P}(Y)$ for every $\beta < \alpha$ and $Y \in \mathcal{T}^\eta_\beta$.

Note that what we're really aiming for is achieving the hopeless ideal coherence, since that enables us to ensure that if $X, Y \in \mathcal{T}^\eta$ and $X \subseteq Y$ then really $X \geq_{\mathcal{T}^\eta} Y$ — i.e. that we "catch" both X and Y in the same play of $\mathcal{G}(\kappa)\restriction\eta$. The rest of the properties are inductive properties we need to ensure this.

Set $\mathcal{T}^\eta_0 := \{\kappa\}$. Assume that we've built $\mathcal{T}^\eta\restriction\alpha+1$ satisfying the inductive assumptions[19] and let $Y \in \mathcal{T}^\eta_\alpha$ — we need to specify what the $\mathcal{T}^\eta$-successors of Y are. Since κ is weakly compact and not measurable it holds by Proposition 6.4 in [Kanamori, 2008] that $\mathrm{sat}(I^{\tau^\eta_\alpha}_Y) \geq \kappa^+$, so we can fix a

[16]The reason why we're dealing with the *restricted* auxilliary games is to achieve this property.
[17]This actually follows from the cofinally many responds, but we include it here for transparency.
[18]Two subsets $X, Y \subseteq \kappa$ are *almost disjoint* if $|X \cap Y| < \kappa$.
[19]In particular, we assume that τ^η_α is defined.

maximal antichain $\langle X^Y_\gamma \mid \gamma < \eta\rangle$ of $I^{\tau^\eta_\alpha}_Y$-positive sets. By κ-completeness of $I^{\tau^\eta_\alpha}_Y$ we can by Exercise 22.1 in [Jech, 2006] even ensure that all of the X^Y_γ's are pairwise disjoint.

To every $\gamma < \eta$ we fix a partial play p of even length of $\mathcal{G}(\kappa)\restriction\eta$ consistent with τ^η_α such that the last ordinal β^Y_γ in p played by player I is greater than or equal to γ and X^Y_γ has measure one with respect to the last measure in p. We then define the $\mathcal{T}^\eta$-successors of Y to be player II's τ^η_α-responses to the β_γ's (which are subsets of the X^Y_γ's modulo a bounded set and are therefore pairwise almost disjoint).

For limit stages $\delta < \lambda$ we apply τ_0 to the branches of $\mathcal{T}^\eta\restriction\delta$ to get $\mathcal{T}^\eta_\delta$.

We now have to check that the inductive assumptions still hold; let's start with the tree strategy. Assume that we have a partial play p of length $2\cdot\alpha+1$ of $\mathcal{G}(\kappa)\restriction\eta$, i.e. the last move in p is by player II, consistent with τ^η_α; write ξ_p for player I's last move in p and Y_p for player II's response to ξ_p, which is also the last move in p. We can then pick a $\zeta < \eta$ such that $\beta^{Y_p}_\zeta > \xi_p$ by the cofinally many responds property and let $\tau^\eta_{\alpha+1}(p)$ be player II's τ^η_α-response to the partial play leading up to $\beta^{Y_p}_\zeta$. After this $(\alpha+1)$'th round we just set $\tau^\eta_{\alpha+1}$ to follow τ_0. It's clear that $\tau^\eta_{\alpha+1}$ satisfies the required properties.

Before we move on to checking the remaining inductive assumptions, let's pause to get some intuition about the tree strategies. In the definition of $\tau^\eta_{\alpha+1}$ above, we took a partial play consistent with τ^η_α, applied τ_0 for a while, took note of player II's last τ_0-response and then included *only that* response in our new $\tau^\eta_{\alpha+1}$ partial play. This means that to every τ^η_α-partial play there's an ostensibly much longer τ_0-partial play into which τ^η_α embeds; so we can look at the τ^η_α-partial plays as being "collapsed" τ_0-partial plays.

Given the above tree strategy, $\mathcal{T}^\eta_{\alpha+1}$ clearly satisfies the cofinally many responds property and the positivity property, simply by construction. For the unique pre-history, let $Y \in \mathcal{T}^\eta_{\alpha+1}$ and assume it has two distinct immediate $\mathcal{T}^\eta$-predecessors $Z_0, Z_1 \in \mathcal{T}^\eta_\alpha$. But then $Y \subseteq Z_0\cap Z_1$ and Y is $I^{\tau^\eta_\alpha}_{Z_0}$-positive by the positivity assumption, contradicting that Z_0 and Z_1 are almost disjoint by the almost disjointness property. Given the unique pre-history we then also get the almost disjointness property.

Claim 2.5.10.2. $\mathcal{T}^\eta \upharpoonright \alpha+2$ satisfies the hopeless ideal coherence property.

Proof of claim. Let $Y \in \mathcal{T}^\eta_{\alpha+1}$ — we have to show that

$$I_{\langle\rangle}^{\tau^\eta_{\alpha+1}} \cap \mathscr{P}(Y) = I_Y^{\tau^\eta_{\alpha+1}} \cap \mathscr{P}(Y). \tag{1}$$

It's clear that $I_{\langle\rangle}^{\tau^\eta_{\alpha+1}} \subseteq I_Y^{\tau^\eta_{\alpha+1}}$, so let $Z \in I_Y^{\tau^\eta_{\alpha+1}} \cap \mathscr{P}(Y)$ and assume for a contradiction that Z is $I_{\langle\rangle}^{\tau^\eta_{\alpha+1}}$-positive. Letting $\vec{\alpha}_\xi * \vec{Y}_\xi$ be a play of $\mathcal{G}(\kappa)\upharpoonright\eta$ consistent with $\tau^\eta_{\alpha+1}$ such that Z is in the final measure, the definition of $\tau^\eta_{\alpha+1}$ yields that $Y_\alpha \in \mathcal{T}^\eta_{\alpha+1}$. As $Z \in I_Y^{\tau^\eta_{\alpha+1}}$ we have to assume that $Y \neq Y_\alpha$, so that the almost disjointness property implies that

$$|Y \cap Y_\alpha| < \kappa, \tag{2}$$

By the choice of $\vec{\alpha}_\xi * \vec{Y}_\xi$ there's some $\delta \in (\alpha, \lambda)$ such that $|Y_\delta - Z| < \kappa$, i.e. that Y_δ is a subset of Z modulo a bounded set, since the Y_α's generate the final measure of the play. But then $Y_\delta \subseteq Y_\alpha$ by the rules of $\mathcal{G}(\kappa)\upharpoonright\eta$, and also that $|Y_\delta - Y| < \kappa$ since $Z \subseteq Y$. But this means that $Y \cap Y_\alpha$ is $I_Y^{\tau^\eta_{\alpha+1}}$-positive since Y_δ is, contradicting (2). This shows (1). ⊣

This finishes the construction of $\mathcal{T}^\eta_{\alpha+1}$. For limit levels $\delta < \lambda$ we define τ^η_δ as simply applying τ_0 to the branches of $\mathcal{T}^\eta \upharpoonright \delta$ — showing that the inductive assumptions hold at $\mathcal{T}^\eta_\delta$ is analogous to the above arguments, so we're now done with the construction of $\mathcal{T}^\eta$. Let $\tau^\eta := \bigcup_{\alpha<\lambda} \tau^\eta_\alpha \upharpoonright {}^{<\alpha}H_{\kappa^+}$ and define[20] $\mathcal{I}^\eta := I_{\langle\rangle}^{\tau^\eta}$.

Now note that $\mathcal{I}^{\eta+1} \subseteq \mathcal{I}^\eta$ and $\mathcal{T}^\eta \subseteq \mathcal{T}^{\eta+1}$ for every $\eta < \kappa^+$ — set $\mathcal{I} := \bigcap_{\eta<\kappa^+} \mathcal{I}^\eta$ and $\mathcal{T} := \bigcup_{\eta<\kappa^+} \mathcal{T}^\eta$. We showed that all hopeless ideals are κ-complete, normal and (κ,κ)-distributive, so this holds in particular for the $\mathcal{I}^\eta$'s and thus also for $\mathcal{I}$.

We claim that $\mathcal{T}$ is dense in $\mathscr{P}(\kappa)/\mathcal{I}$.[21] Let X be an $\mathcal{I}$-positive set, making it $\mathcal{I}^\eta$-positive for some $\eta < \kappa^+$, meaning that there's a play $\vec{\alpha}_\gamma * \tau^\eta$ of $\mathcal{G}(\kappa)\upharpoonright\eta$ such that X is in the final measure, which means that $|Y_\delta - X| < \kappa$

[20] Note that the tree strategy property above ensures that the strategies *do* line up, so that τ^η is a well-defined strategy as well.

[21] This means that given any $\mathcal{I}$-positive set X there's a $Y \in \mathcal{T}$ such that $Y - X \in \mathcal{I}$.

for some large $\delta < \lambda$ and in particular that $Y_\delta - X \in \mathcal{I}$. But $Y_\delta \in \mathcal{T}^\eta \subseteq \mathcal{T}$ by definition of τ^η, which shows that $\mathcal{T}$ is dense.

It remains to show that $\mathcal{T}$ is $<\lambda$-closed. If $\lambda = \omega$ then this is trivial, so assume that $\lambda \geq \omega_1$. Let $\beta < \lambda$ and let $\langle Z_\alpha \mid \alpha < \beta \rangle$ be a $\subseteq$-decreasing sequence of elements $Z_\alpha \in \mathcal{T}$. We can fix some $\eta < \kappa^+$ such that $Z_\alpha \in \mathcal{T}^\eta$ for every $\alpha < \beta$ by regularity of κ^+, and since the Z_α's are $\subseteq$-decreasing they must also be $\leq_{\mathcal{T}^\eta}$-increasing by the hopeless ideal coherence for $\mathcal{T}^\eta$[22].

Let $\tilde{Z} \in \mathcal{T}^\eta$ be player II's τ^η-response to the unique partial play of $\mathcal{G}(\kappa) \restriction \eta$ corresponding to the branch containing the Z_α's, and pick $Z \in \mathcal{T}^\eta$ such that $\left|Z - \tilde{Z}\right| < \kappa$ and $Z \geq_{\mathcal{T}^\eta} Z_\alpha$ for all $\alpha < \beta$, again by the density claim and the hopeless ideal coherence. Then Z witnesses $<\lambda$-closure of $\mathcal{T}$.[23] ■

Theorem 2.5.11 (N.). *Let κ be a regular cardinal and $\lambda \in [\omega_1, \kappa^+]$ be regular. Then the following are equivalent:*

(i) κ is $<\lambda$-closed generically power-measurable;
(ii) κ is $<\lambda$-closed ideally power-measurable;
(iii) κ is (κ,κ)-distributive $<\lambda$-closed generically measurable;
(iv) κ is (κ,κ)-distributive $<\lambda$-closed ideally measurable;
(v) Player II has a winning strategy in $\mathcal{G}_\lambda(\kappa)$.

PROOF. $(v) \Rightarrow (iv)$ is Theorem 2.5.10 above[24] and $(iv) \Rightarrow (iii) + (ii)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial, so we show $(i) \Rightarrow (v)$.

Assume κ is $<\lambda$-closed generically power-measurable, so there's a $<\lambda$-closed forcing $\mathbb{P}$ and a V-generic $g \subseteq \mathbb{P}$ such that, in $V[g]$, there exists a transitive class N and a κ-powerset preserving elementary embedding $\pi\colon V \to N$. Write μ for the induced weakly amenable V-normal V-measure on κ. Now, back in V, define a strategy σ for player II in $G_\lambda(\kappa)$ as follows.

Whenever player I plays some model M_α then we let player II respond with a filter μ_α such that, for some $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash \ulcorner \check{\mu}_\alpha = \dot{\mu} \cap \check{M}_\alpha \urcorner$ — such

[22] This is the only place in which we're using hopeless ideal coherence.
[23] We're using that λ is regular to get Z.
[24] Here wellfoundedness of the generic ultrapower is automatic since λ has uncountable cofinality.

a filter exists because μ is weakly amenable. We require the p_α's to be decreasing, which is possible by $<\lambda$-closure. Now, all the μ_α's are clearly M_α-normal M_α-measures on κ, which makes σ a winning strategy. ■

Ignoring wellfoundedness we get the same equivalence in the $\lambda = \omega$ case.

Corollary 2.5.12 (N.). *Let κ be a regular cardinal. Then the following are equivalent:*[25]

(i) There exists a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a weakly amenable V-normal V-measure on κ;

(ii) There exists a (κ,κ)-distributive forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there's a V-normal V-measure on κ;

(iii) κ carries a normal (κ,κ)-distributive ideal;

(iv) Player II has a winning strategy in $\mathcal{G}_\omega^-(\kappa)$;

(v) κ is completely ineffable.

Proof. $(iv) \Leftrightarrow (v)$ was shown in [?], and $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are trivial. $(i) \Rightarrow (iv)$ is as $(i) \Rightarrow (v)$ in Theorem 2.5.11, and $(iv) \Rightarrow (iii)$ is Theorem 2.5.10. ■

Corollary 2.5.13. *"(κ,κ)-distributive $<\lambda$-closed" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ■

2.5.14 λ-density & $<\lambda$-closure

Can we get κ-complete below somehow? In this case, when $\lambda < \kappa$, κ cannot be inaccessible and cannot be a successor cardinal, by Kunen's "Saturated Ideals" paper.

Theorem 2.5.15 (N.). *Let κ and $\lambda \leq \kappa^+$ be regular infinite cardinals such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. If player II has a winning strategy in $\mathcal{C}_\lambda^-(\kappa)$*

[25] Points (i) and (ii) look a lot like what a definition of generically power-measurable and (κ,κ)-distributive ideally measurable *should* be, but here we're not requiring the ultrapowers to be well-founded, so that would be stretching the definition of being measurable.

then κ carries a λ-complete ideal $\mathcal{I}$ such that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $Add(\lambda, 1)$.

Proof. If $\lambda = \kappa^+$ then we're done by Theorem 2.5.10, since $\mathcal{G}_{\kappa^+}(\kappa)$ is equivalent to $\mathcal{C}_{\kappa^+}(\kappa)$, so assume that $\lambda \leq \kappa$. We follow the proof of Theorem 2.5.10 closely. Set $\mathbb{P} := \mathrm{Col}(\lambda, 2^\kappa)$. Fix a wellordering $<_{\kappa^+}$ of H_{κ^+} and a $\mathbb{P}$-name π for a sequence $\langle \mathcal{N}_\gamma \mid \gamma < \lambda \rangle \in V^{\mathbb{P}}$ such that

- $\mathcal{N}_\gamma \in V$ for every $\gamma < \lambda$;
- $\kappa+1 \subseteq \mathcal{N}_\gamma$ and $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for every $\gamma < \lambda$;
- If $\delta < \lambda$ is a limit ordinal then $\mathcal{N}_\delta = \bigcup_{\gamma<\delta} \mathcal{N}_\gamma$, $\mathcal{N}_\delta \prec H_{\kappa^+}$ and $\mathcal{N}_\delta \models \mathsf{ZFC}^-$;
- $\mathcal{N}_\gamma \cup \{\mathcal{N}_\gamma\} \subseteq \mathcal{N}_\beta$ for all $\gamma < \beta < \lambda$;
- $\mathscr{P}(\kappa)^V \subseteq \bigcup_{\gamma<\lambda} \mathcal{N}_\gamma$.

Define the auxilliary game $\mathcal{G}(\kappa)$ as in the proof of Theorem 2.5.10 but where player I plays ordinals $\alpha_\eta < \lambda$ and where we use the above $\mathcal{N}_\gamma$'s. Here we only need $<\lambda$-closure of $\mathbb{P}$ to get an equivalence between $\mathcal{G}(\kappa)$ and $\mathcal{C}^-_\lambda(\kappa)$, since $|\mathcal{N}_\gamma - H_\kappa|^V < \lambda$ for all $\gamma < \lambda$.

To every limit ordinal $\eta < \lambda$ we define the restricted auxilliary game $\mathcal{G}(\kappa) \restriction \eta$ as in the proof of Theorem 2.5.10, and to every winning strategy τ in $\mathcal{G}(\kappa) \restriction \eta$ and partial play p of $\mathcal{G}(\kappa) \restriction \eta$ consistent with τ define the associated **hopeless ideal**[26]

$$I^\tau_p \restriction \eta := \{X \subseteq \kappa \mid \text{For every play } \vec{\alpha}_\gamma * \tau \text{ extending } p \text{ in } \mathcal{G}(\kappa) \restriction \eta, \\ X \text{ is } \textit{not} \text{ in the final measure}\}.$$

As in the proof of Claim 2.5.10.1 we get that every hopeless ideal is λ-complete.

Now, if κ is measurable then we trivially get the conclusion,[27] so assume κ isn't measurable. Then $\mathrm{sat}(\kappa) \geq \lambda$ since $2^{<\theta} < \kappa$ for every $\theta < \lambda$,[28] so that we can continue exactly as in the proof of Theorem 2.5.10 to construct (λ-sized) trees $\mathcal{T}^\eta$ and winning strategies τ^η for all limit ordinals $\eta < \lambda$ such

[26] This terminology is due to Matt Foreman.
[27] Take $\mathcal{I}(\mathrm{Add}(\lambda, 1), \check{\mu})$ for μ the measure on κ.
[28] See Proposition 16.4 in [Kanamori, 2008].

that, setting $\mathcal{I} := \bigcap_{\eta<\lambda} I_{\langle\rangle}^{\tau^\eta}$ and $\mathcal{T} := \bigcup_{\eta<\lambda} \mathcal{T}^\eta$, $\mathcal{T}$ is a dense $<\lambda$-closed subset of $\mathscr{P}(\kappa)/\mathcal{I}$ of size λ, so that $\mathscr{P}(\kappa)/\mathcal{I}$ is forcing equivalent to $\mathrm{Add}(\lambda, 1)$. ∎

Corollary 2.5.16 (N.). *Let κ and $\lambda \in [\omega_1, \kappa^+]$ be regular such that $2^{<\theta} < \kappa$ for every $\theta < \lambda$. Then the following are equivalent:*

- *(i) κ is $<\lambda$-closed generically measurable;*
- *(ii) κ is $<\lambda$-closed ideally measurable;*
- *(iii) κ is $<\lambda$-closed λ-sized generically measurable;*
- *(iv) κ is $<\lambda$-closed λ-sized ideally measurable;*
- *(v) Player II has a winning strategy in $\mathcal{C}_\lambda(\kappa)$.*

Proof. *(iv)* ⇒ *(iii)* + *(ii)*, *(ii)* ⇒ *(i)* and *(iii)* ⇒ *(i)* all trivial, and *(i)* ⇒ *(v)* is like *(i)* ⇒ *(v)* in Theorem 2.5.11, and *(v)* ⇒ *(iv)* is Theorem 2.5.15. ∎

Again, if we ignore wellfoundedness then we get the same equivalence in the $\lambda = \omega$ case:

Corollary 2.5.17 (N.). *Let κ be regular infinite. Then:*

- *(i) Player II has a winning strategy in $\mathcal{C}_\omega^-(\kappa)$; and*
- *(ii) κ carries an ideal I such that $\mathscr{P}(\kappa)/I$ is forcing equivalent to $Add(\omega, 1)$.*

Proof. Player II has a winning strategy in $\mathcal{C}_\omega^-(\kappa)$ as we're simply measuring finitely many sets without any demand for wellfoundedness, showing *(i)*. Since $2^{<n} < \kappa$ for all $n < \omega$ as κ is infinite, Theorem 2.5.15 then implies *(ii)*. ∎

Corollary 2.5.18. *"$<\lambda$-closed λ-sized" is ideal-absolute for all regular $\lambda \in [\omega, \kappa^+]$.* ∎

(ω, α)-Ramsey cardinals

A natural generalisation of the γ-Ramsey definition is to require more iterability of the last measure. Of course, by Proposition 2.1.8 we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when $\operatorname{cof}\gamma > \omega$ so the next definition is only interesting whenever $\operatorname{cof}\gamma = \omega$.

Definition 2.5.19 (N.)**.** Let α, β be ordinals. Then a cardinal κ is **(α, β)-Ramsey** if player I does not have a winning strategy in $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ for all regular $\theta > \kappa$.[29] ◦

Definition 2.5.20 (Gitman)**.** A cardinal κ is **α-iterable** if for every $A \subseteq \kappa$ there exists a *transitive* weak κ-model $\mathcal{M}$ with $A \in \mathcal{M}$ and an α-good $\mathcal{M}$-measure μ on $\mathcal{M}$. ◦

Proposition 2.5.21. *If $\beta > 0$ then every (α, β)-Ramsey is a β-iterable stationary limit of β-iterables.*

Proof. Let $(\mathcal{M}, \in, \mu)$ be a result of a play of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$ in which player II won. Then the transitive collapse of $(\mathcal{M}, \in, \mu)$ witnesses that κ is β-iterable, since μ is β-good by definition of $\mathcal{G}^{\kappa^+}_\alpha(\kappa, \beta)$.

That κ is β-iterable is reflected to some H_θ, so let now $(\mathcal{N}, \in, \nu)$ be a result of a play of $\mathcal{G}^\theta_\alpha(\kappa, \beta)$ in which player II won. Then $\mathcal{N} \prec H_\theta$, so that κ is also β-iterable in $\mathcal{N}$. Since being β-iterable is witnessed by a subset of κ and $\beta > 0$ implies[30] that we get a κ-powerset preserving $j : \mathcal{N} \to \mathcal{P}$, $\mathcal{P}$ also thinks that κ is β-iterable, making κ a stationary limit of β-iterables by elementarity. ■

We now move towards Theorem 2.5.25 which gives an upper consistency bound for the (ω, α)-Ramseys. We first recall a few definitions and a folklore lemma.

[29] Note that an α-Ramsey cardinal is the same as an $(\alpha, 0)$-Ramsey cardinal.
[30] Recall that β-good for $\beta > 0$ in particular implies weak amenability.

Definition 2.5.22. For an infinite ordinal α, a cardinal κ is **α-Erdős** for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \to \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c; i.e. that $|c``[H]^n| \leq 1$ holds for every $n < \omega$. $\circ$

Definition 2.5.23. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$. $\circ$

Lemma 2.5.24 (Folklore). *Let κ be α-Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure $\mathcal{M}$ in a countable language $\mathcal{L}$ with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

Proof. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all $\mathcal{L}$-formulas and define $c : [C]^{<\omega} \to \kappa$ as follows. For an increasing sequence $\alpha_1 < \cdots < \alpha_{2n} \in C$ let

$$\begin{aligned} c(\{\alpha_1, \ldots, \alpha_{2n}\}) := \text{ the least } \lambda < \alpha_1 \text{ such that} \\ \exists \delta_1 < \cdots \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \ldots, \delta_k \rangle \wedge \\ \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}] \end{aligned}$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α-Erdős we get a homogeneous $I \in [C]^\alpha$ for c; i.e. that $|c``[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \ldots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \ldots, \alpha_{2n} \in I$, as otherwise there exists an $m < \omega$ and $\delta_1 < \cdots \delta_k$ such that for any $\alpha_1 < \ldots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \ldots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}]. \tag{\dagger}$$

But then simply pick $\alpha_1 < \ldots \alpha_{2n} < \alpha'_1 < \cdots < \alpha'_{2n}$ so that both $\{\alpha_1, \ldots, \alpha_{2n}\}$ and $\{\alpha'_1, \ldots, \alpha'_{2n}\}$ witnesses (†); then either $\{\alpha_1, \ldots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \ldots, \alpha_n, \alpha'_{n+1}, \ldots, \alpha'_{2n}\}$ also witnesses that (†) fails, $\lightning$. ■

Theorem 2.5.25 (N.). *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α-Erdős cardinal is a limit of (ω, α)-Ramsey cardinals.*

Proof. Let κ be α-Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 2.5.24 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I. We will show that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in [Holy and Schlicht, 2018] it suffices to find a weak ι-model $\mathcal{M} \prec H_\theta$ and an α-good $\mathcal{M}$-measure on ι. Define

$$\mathcal{M} := \mathrm{Hull}^{H_\theta}(\iota \cup I) \prec H_\theta$$

and let $\pi : I \to I$ be the right-shift map. Since I is remarkable, I $(= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \to \mathcal{M}$ with $\operatorname{crit} j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \dots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \dots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι-powerset preserving we get that $\mathcal{M} \prec H_\theta$ is a weak ι-model satisfying ZFC^- with a 1-good $\mathcal{M}$-measure μ_j on ι. Furthermore, as we can linearly iterate $\mathcal{M}$ simply by applying j we get an α-iteration of $\mathcal{M}$ since there are α-many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \dots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \dots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}^\theta_\omega(\eta, \alpha)$. If we repeat this procedure for all regular $\theta > \kappa$ we get by the pidgeon hole principle that κ is a limit of (ω, α)-Ramsey cardinals. ■

As Theorem 4.5 in [Gitman and Schindler, 2018] shows that $(\alpha{+}1)$-iterable cardinals have α-Erdős cardinals below them for $\alpha \geq \omega$ additively closed, this shows that the (ω, α)-Ramseys form a strict hierarchy. Further, as α-Erdős cardinals are consistent with $V = L$ when $\alpha < \omega_1^L$ and ω_1-iterable cardinals aren't consistent with $V = L$, we also get that (ω, α)-Ramsey cardinals are consistent with $V = L$ if $\alpha < \omega_1^L$ and that they aren't if $\alpha = \omega_1$.

[Strategic] $(\omega{+}1)$**-Ramsey cardinals**

The next step is then to consider $(\omega{+}1)$-Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in [Mitchell, 1979] and in the proof of Lemma 1.3 in [Donder et al., 1981] — see also [Dodd, 1982] and [Gitman, 2011].

Theorem 2.5.26 (Dodd, Mitchell). *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model $\mathcal{M}$ such that there exists a weakly amenable countably complete $\mathcal{M}$-measure on κ.* ■

The following theorem then supplies us with a lower bound for the strength of the $(\omega{+}1)$-Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 2.6.9, but we include this Ramsey lower bound as well for completeness.

Theorem 2.5.27 (N.). *Every $(\omega{+}1)$-Ramsey cardinal is a Ramsey limit of Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey and $A \subseteq \kappa$. Let σ be a strategy for player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A \in \mathcal{M}_0$ and $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Then σ isn't winning as κ is $(\omega{+}1)$-Ramsey, so we may fix a play $\sigma * \vec{\mu}_\alpha$ of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable $\mathcal{M}_\omega$-measure on κ, and by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 2.5.26.

Since κ is Ramsey, $\mathcal{M}_\omega \models \ulcorner\kappa$ is Ramsey$\urcorner$ as well. Letting $j : \mathcal{M}_\omega \to \mathcal{N}$ be the κ-powerset preservering embedding induced by μ_ω, we also get that $\mathcal{N} \models \ulcorner\kappa$ is Ramsey$\urcorner$ by κ-powerset preservation. This then implies that κ *is* a stationary limit of Ramsey cardinals inside $\mathcal{M}_\omega$, and thus also in V by elementarity. ■

As for the *consistency* strength of the strategic $(\omega{+}1)$-Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and

Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal — see e.g. p. 249 in [Kanamori and Magidor, 1978].

Theorem 2.5.28 (N.)**.** *If κ is a strategic $(\omega{+}1)$-Ramsey cardinal then, in $V^{\mathrm{Col}(\omega,2^\kappa)}$, there's a transitive class N and an elementary embedding $j : V \to N$ with $\operatorname{crit} j = \kappa$. In particular, the existence of a strategic $(\omega{+}1)$-Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

Proof. Set $\mathbb{P} := \mathrm{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$. Let $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name of an ω-sequence $\langle \mathcal{M}_n \mid n < \omega\rangle$ of weak κ-models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n \prec H^V_{\kappa^+}$ and $\mathscr{P}(\kappa)^V \subseteq \bigcup_{n<\omega} \mathcal{M}_n$, and let $\dot\mu$ be a $\mathbb{P}$-name for the ω-sequence of σ-responses to the $\mathcal{M}_n$'s in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)^V$.

Assume that there's a $\mathbb{P}$-condition p which forces the generic ultrapower $\mathrm{Ult}(V, \bigcup_n \dot\mu_n)$ to be illfounded, meaning that we can fix a $\mathbb{P}$-name $\dot f$ for an ω-sequence $\langle f_n \mid n < \omega\rangle$ such that

$$p \Vdash \dot X_n := \{\alpha < \kappa \mid \dot f_{n+1}(\alpha) < \dot f_n(\alpha)\} \in \bigcup_{n<\omega} \dot\mu_n.$$

Now, in V, we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} \prec H_\theta$ such that $\dot{\mathcal{M}}, \dot\mu, \dot f, H^V_{\kappa^+}, \sigma, p \in \mathcal{N}$. We can find an $\mathcal{N}$-generic $g \subseteq \mathbb{P}^{\mathcal{N}}$ in V with $p \in g$ since $\mathcal{N}$ is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}^g_n * \dot\mu^g_n$ is a play of $\mathcal{G}^{\kappa^+}_\omega(\kappa)^V$ which is according to σ, meaning that $\bigcup_{n<\omega} \dot\mu^g_n$ is normal and in particular countably complete (in V). Then $\bigcap_{n<\omega} \dot X^g_n \neq \emptyset$, but if $\alpha \in \bigcap_{n<\omega} \dot X^g_n$ then $\langle \dot f^g_n(\alpha) \mid n < \omega\rangle$ is a strictly decreasing ω-sequence of ordinals, $\lightning$. This means that $\mathrm{Ult}(V, \bigcup_n \mu_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in [Kellner and Shelah, 2011]. ■

The above Theorem 2.5.28 then answers Question 9.2 in [Holy and Schlicht, 2018] in the negative, asking if λ-Ramseys are strategic λ-Ramseys for uncountable cardinals λ, as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.

2.6 The general case

Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from [Gitman, 2011] and investigate further connections between these and the α-Ramsey cardinals. First, a definition.

Definition 2.6.1 (Gitman). A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ. If furthermore $\mathcal{M} \prec H_{\kappa^+}$ then we say that κ is **super Ramsey**. ◦

Note that since the model $\mathcal{M}$ in question is a *κ-model* it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$-sequences. [Gitman, 2011] shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ-model $\mathcal{M} \models \mathsf{ZFC}$ with a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ.

Now, a first connection between the α-Ramseys and the strongly- and super Ramseys is the result in [Holy and Schlicht, 2018] that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

Theorem 2.6.2 (N.-Welch). *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

Proof. Let κ be strongly Ramsey and let $\mathcal{M} \models \mathsf{ZFC}$ be a transitive κ-model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable $\mathcal{M}$-normal $\mathcal{M}$-measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^{\mathcal{M}}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of $\mathcal{M}$ by weak amenability of μ. As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside $\mathcal{M}$, so $\mathcal{M}$ sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \to \mathcal{N}$ be the induced embedding, κ-powerset preservation of j_μ implies that μ is also a weakly amenable $\mathcal{N}$-normal $\mathcal{N}$-measure on κ. This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in $\mathcal{N}$, entailing that it is a stationary limit of almost fully Ramseys in $\mathcal{M}$. But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in [Holy and Schlicht, 2018], so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and $\mathcal{M}$ being transitive implies that κ really *is* a stationary limit of almost fully Ramseys. ■

Downwards absoluteness to K

Lastly, we consider the question of whether the α-Ramseys are downwards absolute to K, which turns out to at least be true in many cases. The below Theorem 2.6.4 then also answers Question 9.4 from [Holy and Schlicht, 2018] in the positive, asking whether α-Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of $0^\P$.

Definition 2.6.3. $0^\P$ is "the sharp for a strong cardinal", meaning the minimal sound active mouse $\mathcal{M}$ with $\mathcal{M} \,l\, \text{crit}(\dot{F}^\mathcal{M}) \models \ulcorner$There exists a strong cardinal$\urcorner$, with $\dot{F}^\mathcal{M}$ being the top extender of $\mathcal{M}$. ∘

Theorem 2.6.4 (N.-Welch)**.** *Assume $0^\P$ does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ-Ramsey. Then $K \models \ulcorner\kappa$ is a λ-Ramsey cardinal$\urcorner$.*

Proof. Note first that $\kappa^{+K} = \kappa^+$ by [Schindler, 1997], since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ-models $\mathcal{M} \prec K l \kappa^+$. We can then define a strategy $\tilde\sigma$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde\sigma(\emptyset) := \text{Hull}^{H_{\kappa^+}}(K l \kappa \cup \sigma(\emptyset))$. Assuming now that $\langle \tilde{\mathcal{M}}_\alpha, \tilde\mu_\alpha \mid \alpha < \gamma\rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde\sigma$, we have two cases. If

$\tilde{\mu}_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle \mathcal{M}_\alpha \mid \alpha < \gamma \rangle$ be the corresponding models played in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde{\sigma}(\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma \rangle) := \mathrm{Hull}^{H_{\kappa^+}}(Kl\kappa \cup \sigma(\langle \mathcal{M}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma \rangle)),$$

and otherwise let $\tilde{\sigma}$ play arbitrarily. As κ is λ-Ramsey (in V) there exists a play $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha \leq \lambda \rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda \cap Kl\kappa^+ \prec Kl\kappa^+$ so let $\mathcal{N}$ be the transitive collapse of $\tilde{\mathcal{M}}_\lambda \cap Kl\kappa^+$. But if $j : \mathcal{N} \to Kl\kappa^+$ is the uncollapse then $\mathrm{crit}\, j$ is both an $\mathcal{N}$-cardinal and also $> \kappa$ because we ensured that $Kl\kappa \subseteq \mathcal{N}$. This means that $j = \mathrm{id}$ because κ is the largest $\mathcal{N}$-cardinal by elementarity in $Kl\kappa^+$, so that $\tilde{\mathcal{M}}_\lambda \cap Kl\kappa^+ = \mathcal{N}$ is a transitive elementary substructure of $Kl\kappa^+$, making it an initial segment of K.

Now, since $\mu := \tilde{\mu}_\lambda$ is a countably complete weakly amenable $Klo(\mathcal{N})$-measure[31], the "beaver argument"[32] shows that $\mu \in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu \cap \mathcal{N} \in K$ whenever player I plays $\mathcal{N}$. Since $\mu = \tilde{\mu}_\lambda$ we also have that $\mu \cap \mathcal{M}_\alpha = \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$, so that σ will eventually play $\mathcal{N}$, making τ win against σ.[33] ■

Note that the only thing we used $\mathrm{cof}\,\lambda > \omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine- or normal α-Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

Corollary 2.6.5 (N.-Welch)**.** *Assume $0^\P$ does not exist and let α be any limit ordinal. Then every genuine- and every normal α-Ramsey cardinal is downwards absolute to K. In particular, if α is a limit of limit ordinals then every $<\alpha$-Ramsey cardinal is downwards absolute to K as well.* ■

[31] Here we use that $\mathcal{N} \lhd K$.
[32] See Lemmata 7.3.7–7.3.9 and 8.3.4 in [Zeman, 2002] for this argument.
[33] Note that τ is not necessarily a winning strategy — all we know is that it is winning against this particular strategy σ.

Indiscernible games

We now move to the strategic versions of the α-Ramsey hierarchy. The first thing we want to do is define *α-very Ramsey cardinals*, introduced in [Sharpe and Welch, 2011], and show the tight connection between these and the strategic α-Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 2.5.23.

Definition 2.6.6. A **good set of indiscernibles** for a structure $\mathcal{M}$ is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for $\mathcal{M}$ such that $\mathcal{M} \restriction \iota \prec \mathcal{M}$ for any $\iota \in I$. ○

Definition 2.6.7 (Sharpe-Welch)**.** Define the **indiscernible game** $G^I_\gamma(\kappa)$ in γ many rounds as follows

$$\begin{array}{lccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & \cdots \\ \mathrm{II} & & I_0 & & I_1 & & I_2 & & \cdots \end{array}$$

Here $\mathcal{M}_\alpha$ is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for $\mathcal{M}_\alpha$ and the I_α's are $\subseteq$-decreasing. Player II wins iff they can continue playing through all the rounds. ○

Definition 2.6.8 (Sharpe-Welch)**.** A cardinal κ is **γ-very Ramsey** if player II has a winning strategy in the game $G^I_\gamma(\kappa)$. ○

The next couple of results concerns the connection between the strategic α-Ramseys and the α-very Ramseys. We start with the following.

Theorem 2.6.9 (N.)**.** *Every $(\omega{+}1)$-Ramsey is an ω-very Ramsey stationary limit of ω-very Ramseys.*

Proof. Let κ be $(\omega{+}1)$-Ramsey. We will describe a winning strategy for player II in the indiscernible game $G^I_\omega(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G^I_\omega(\kappa)$ then let player I in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ play

$$\mathcal{H}_0 := \mathrm{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) \prec H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ which starts off with $\mathcal{H}_0$ and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}_0^{(\alpha)} := \mathcal{M}_\alpha$ and $\mu_0^{(\alpha)} := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{lccccccc} \text{I} & \mathcal{H}_0^{(0)} & & \cdots & & \mathcal{H}_0^{(\omega)} & & \mathcal{H}_0^{(\omega+1)} & \\ \text{II} & & \mu_0^{(0)} & & \cdots & & \mu_0^{(\omega)} & & \mu_0^{(\omega+1)} \end{array}$$

By the choice of player I's strategy we get that $\mu_0^{(\omega)}$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ (it's even normal). Now Lemma 2.9 of [Sharpe and Welch, 2011] gives us a set of good indiscernibles $I_0 \in \mu_0^{(\omega)}$ for $\mathcal{M}_0$, as $\mathcal{M}_0 \in \mathcal{H}_0^{(\omega)}$ and $\mu_0^{(\omega)}$ is a countably complete weakly amenable $\mathcal{H}_0^{(\omega)}$-normal $\mathcal{H}_0^{(\omega)}$-measure on κ. Let player II play I_0 in $G^I_\omega(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G^I_\omega(\kappa)$.

$$\begin{array}{lccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 \\ \text{II} & & I_0 & \end{array}$$

Since $\mu_0^{(\omega)} = \bigcup_n \mu_0^{(n)}$ we must have that $I_0 \in \mu_0^{(n_0)}$ for some $n_0 < \omega$. In the $(n_0{+}1)$'st round of $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \operatorname{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}_0^{(k)}, \mu_0^{(k)} \mid k \leq n_0 \rangle\}) \prec H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ thus looks like the following so far.

$$\begin{array}{lcccccc} \text{I} & \mathcal{H}_0^{(0)} & & \cdots & & \mathcal{H}_0^{(n_0)} & & \mathcal{H}_1 \\ \text{II} & & \mu_0^{(0)} & & \cdots & & \mu_0^{(n_0)} & \end{array}$$

Now player II in $\mathcal{G}^{\kappa^+}_{\omega+1}(\kappa)$ is not losing at round n_0, so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I_1' \in \mu_1^{(n_1)}$ of good indiscernibles for $\mathcal{M}_1$, where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G_\omega^I(\kappa)$ play $I_1 := I_0 \cap I_1' \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G_\omega^I(\kappa)$, making κ ω-very Ramsey.

As for showing that κ is a stationary limit of ω-very Ramseys, let $\mathcal{M} \prec H_{\kappa^+}$ be a weak κ-model with a weakly amenable countably complete $\mathcal{M}$-normal $\mathcal{M}$-measure μ on κ, which exists by Theorem 2.5.27 as κ is $(\omega{+}1)$-Ramsey. Then by elementarity $\mathcal{M} \models \ulcorner\kappa \text{ is } \omega\text{-very Ramsey}\urcorner$ and since κ being ω-very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ-ultrapower, meaning that κ is a stationary limit of ω-very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

Theorem 2.6.10 (N.). *For limit ordinals α, every coherent $<\omega\alpha$-Ramsey is $\omega\alpha$-very Ramsey.*

Proof. This is basically the same proof as the proof of Theorem 2.6.9. We do the "going-back" trick in ω-chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}_{\omega\alpha}^{\kappa^+}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$-Ramseyness. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω-chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 2.6.9. ■

As for going from the α-very Ramseys to the strategic α-Ramseys we got the following.

Theorem 2.6.11 (N.). *For γ any ordinal, every coherent $<\gamma$-very Ramsey*[34] *is coherent $<\gamma$-Ramsey.*[35]

[34] Here the coherency again just means that the winning strategies σ_α for player II in $G_\alpha^I(\kappa)$ are $\subseteq$-increasing.

[35] Here a "coherent $<\gamma$-very Ramsey cardinal" is defined from γ-very Ramseys in the same way as coherent $<\gamma$-Ramsey cardinals is defined from γ-Ramseys. When γ is a limit ordinal then coherent $<\gamma$-very Ramseys are precisely the same as γ-very Ramseys, so this

PROOF. The reason why we work with $<\gamma$-Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ-very Ramsey and let τ be a winning strategy for player II in $G^I_\zeta(\kappa)$. Let $\mathcal{M}_\alpha \prec H_\theta$ be any move by player I in the α'th round of $\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in $\mathcal{M}_\alpha$ and form now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G^I_\zeta(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for $\mathcal{N}_\alpha$ such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in [Sharpe and Welch, 2011] we get a structure $\mathcal{P}_\alpha$ with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a $\mathcal{P}_\alpha$-measure $\tilde{\mu}_\alpha$ on κ, generated by I_α.[36] Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α's are generated by the I_α's, the μ_α's are $\subseteq$-increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$-Ramsey. ■

The following result is then a direct corollary of Theorems 2.6.10 and 2.6.11.

Corollary 2.6.12 (N.). *For limit ordinals α, κ is $\omega\alpha$-very Ramsey iff it is coherent $<\omega\alpha$-Ramsey. In particular, κ is λ-very Ramsey iff it is strategic λ-Ramsey for any λ with uncountable cofinality.* ■

We can now use this equivalence to transfer results from the α-very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in [Feng, 1990]. A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in [Sharpe and Welch, 2011]. We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in [Sharpe and Welch, 2011].

is solely to "subtract one" when γ is a successor ordinal — i.e. a coherent $<(\gamma+1)$-very Ramsey cardinal is the same thing as a γ-very Ramsey cardinal.

[36] By *generated* here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α.

Theorem 2.6.13 (Sharpe-Welch)**.** *A cardinal is completely Ramsey if and only if it is ω-very Ramsey.* ■

This, together with Theorem 2.6.9, immediately yields the following strengthening of Theorem 2.5.27.

Corollary 2.6.14 (N.)**.** *Every $(\omega{+}1)$-Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* ■

The above Theorem 2.6.11 also yields the following consequence.

Corollary 2.6.15 (N.)**.** *Every completely Ramsey cardinal is completely ineffable.*

Proof. From Theorem 2.6.13 we have that being completely Ramsey is equivalent to being ω-very Ramsey, so the above Theorem 2.6.11 then yields that a completely Ramsey cardinal is coherent $<\omega$-Ramsey, which we saw in Theorem 2.2.12 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 2.6.12 yields that strategic ω_1-Ramsey cardinals are ω_1-very Ramsey, and Theorem 3.50 in [Sharpe and Welch, 2011] states that ω_1-very Ramseys are measurable in the core model K, assuming $0^\P$ doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

Theorem 2.6.16 (Welch)**.** *Assuming $0^\P$ doesn't exist, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Let κ be strategic ω_1-Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \mathrm{Col}(\omega_1, \kappa^+)$ is V-generic. Since $\mathrm{Col}(\omega_1, \kappa^+)$ is ω-closed, V and $V[g]$ have the same countable sequences of V, so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V.

Now let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of regular K-cardinals cofinal in κ^+, let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \mathrm{Hull}^{H_\theta}(K l \kappa_\alpha) \prec H_\theta$ and player II follow τ. This results in a countably complete weakly amenable K-measure μ_{ω_1}, which the "beaver argument"[37] then shows is actually an element of K, making κ measurable in K. ■

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1-Ramsey cardinal is also measurable in Steel's core model below a Woodin, a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from [Schindler, 2006a].[38]

Theorem 2.6.17 (Schindler)**.** *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ, and let $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.* ■

Theorem 2.6.18 (Schindler)**.** *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ of weak κ-models $M_\alpha \prec H^V_\theta$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H^V_{\kappa^+} \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\mathrm{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Let $j : M_{\omega_1} \to \mathrm{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H^V_{\kappa^+} \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \mathrm{Ult}(V, \mu) \cong N$ with N transitive.

[37]See Lemmata 7.3.7–7.3.9 and 8.3.4 in [Zeman, 2002] for this argument.

[38]That paper assumes the existence of a measurable as well, but by [Jensen and Steel, 2013] we can omit that here.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 2.6.17 applies, meaning that $\pi \upharpoonright K : K \to K^N$ is an iteration map with critical point κ, making κ measurable in K. ■

3 Further questions

3.1 Section

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Part II

An Equiconsistency

4 The Internal CMI

Introduction

4.1 Operators and hybrid mice

Define model operator, (hybrid) mouse operator, mouse reflection, condenses finely/well, determines itself on generic extensions, relativises well...

4.2 The hybrid core model dichotomy

Lemma 4.2.1. *Let θ be a regular uncountable cardinal or $\theta = \infty$ and let $\mathcal{N}$ be a tame hybrid mouse operator on H_θ which relativises well. Then $\mathcal{N}$ is countably iterable iff it's (θ, θ)-iterable, guided by $\mathcal{N}$. Furthermore, for every $x \in H_\theta$, if $M_1^{\mathcal{N}}(x)$ exists and is countably iterable, then it's also (θ, θ)-iterable, guided by $\mathcal{N}$.*

Define this

Change this to model operators; perhaps change parts of the proof and/or assumptions needed.

Proof. Fix $x \in H_\theta$. We first show that $\mathcal{N}(x)$ is (θ, θ)-iterable. Let $\mathcal{T} \in H_\theta$ be a normal tree of limit length on $\mathcal{N}(x)$. Let $\eta \gg \operatorname{rk}(\mathcal{T})$ and let

$$\mathcal{H} := \operatorname{cHull}^{H_\eta}(\{x, \mathcal{N}(x), \mathcal{T}\})$$

with uncollapse $\pi \colon \mathcal{H} \to H_\eta$. Set $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran} \pi$. Note that $\overline{\mathcal{N}(x)} = \mathcal{N}(\overline{x})$ since $\mathcal{N}$ relativises well. Now $\overline{\mathcal{T}}$ is a normal, countable iteration tree on $\mathcal{N}(\overline{x})$ and hence our iteration strategy yields a wellfounded cofinal branch $\overline{b} \in V$ for $\overline{\mathcal{T}}$. Note that $\overline{\mathcal{Q}} := \mathcal{Q}(\overline{b}, \overline{\mathcal{T}})$ exists, since if $\overline{b}$ drops then there's nothing to do, and otherwise we have that

$$\rho_1(\mathcal{M}_{\overline{b}}^{\overline{\mathcal{T}}}) = \rho_1(\mathcal{N}(\overline{x})) = \operatorname{rk} \overline{x} < \delta(\overline{\mathcal{T}}),$$

so $\delta(\overline{\mathcal{T}})$ is not definably Woodin over $\mathcal{M}_{\overline{b}}^{\overline{\mathcal{T}}}$.

Why is that?

Claim 4.2.1.1. $\overline{\mathcal{Q}} \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$

Define this, cut-point and $\mathcal{M}_b^{\mathcal{T}}$

Proof of claim. If $\overline{\mathcal{Q}} = \mathcal{M}(\overline{\mathcal{T}})$ then the claim is trivial, so assume that $\mathcal{M}(\overline{\mathcal{T}}) \triangleleft \overline{\mathcal{Q}}$. Note that $\overline{\mathcal{Q}} \trianglelefteq M_b^{\overline{\mathcal{T}}}$ by definition of $\mathcal{Q}$-structures, and that $M_b^{\overline{\mathcal{T}}}$ satisfies (2) of the definition of relativises well, meaning that

$$M_b^{\overline{\mathcal{T}}} \models \ulcorner \forall \eta \forall \zeta > \eta : \text{if } \eta \text{ is a cutpoint then } M_b^{\overline{\mathcal{T}}} | \zeta \not\models \varphi_{\mathcal{N}}[\bar{x}, p_{\mathcal{N}}] \urcorner. \quad (1)$$

This statement is Π_2^1 and $\overline{\mathcal{Q}}$ is Π_2^1-correct since it contains a Woodin cardinal, so that $\mathcal{Q}$ satisfies the statement as well. Since $\mathcal{N}$ is tame we get that $\delta(\overline{\mathcal{T}})$ is a cutpoint of $\overline{\mathcal{Q}}$, so that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ is *not* a proper initial segment of $\overline{\mathcal{Q}}$. Further, as we're assuming that both $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\mathcal{M}_b^{\overline{\mathcal{T}}}$ are (ω_1+1)-iterable above $\delta(\overline{\mathcal{T}})$ the same thing holds for $\overline{\mathcal{Q}} \trianglelefteq \mathcal{M}_b^{\overline{\mathcal{T}}}$, so that we can compare $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ with $\overline{\mathcal{Q}}$ (in V). Let

$$(\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})), \overline{\mathcal{Q}}) \rightsquigarrow (\mathcal{P}, \mathcal{R})$$

be the result of the coiteration. We claim that $\mathcal{R} \trianglelefteq \mathcal{P}$. Suppose $\mathcal{P} \triangleleft \mathcal{R}$. Then there is no drop in $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ and in fact $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P}$ since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ projects to $\delta(\overline{\mathcal{T}})$. Furthermore, as we established that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ isn't a proper initial segment of $\overline{\mathcal{Q}}$ it can't be a proper initial segment of $\mathcal{R}$ either, as the coiteration is above $\delta(\overline{\mathcal{T}})$. But we're assuming that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P} \triangleleft \mathcal{R}$, a contradiction. So $\mathcal{R} \trianglelefteq \mathcal{P}$.

Since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\overline{\mathcal{Q}}$ agree up to $\delta(\overline{\mathcal{T}})$ and there is no drop $\overline{\mathcal{Q}} \rightsquigarrow \mathcal{R}$ we have that $\overline{\mathcal{Q}} = \mathcal{R}$. If $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ doesn't move either we're done, so assume not. Let F be the first exit extender of $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ in the coiteration. We have $\mathrm{lh}(F) \leq o(\overline{\mathcal{Q}})$, $\overline{\mathcal{Q}} \trianglelefteq \mathcal{P}$ and $\mathrm{lh}(F)$ is a cardinal in $\mathcal{P}$.

Define this

As $\overline{\mathcal{Q}}$ is $\delta(\overline{\mathcal{T}})$-sound and projects to $\delta(\overline{\mathcal{T}})$ it follows that $J(\overline{\mathcal{Q}}|\,\mathrm{lh}(F))$ collapses $\mathrm{lh}(F)$, so it has to be the case that $\overline{\mathcal{Q}}|\,\mathrm{lh}(F) = \mathcal{P}$ and thus $o(\mathcal{P}) = \mathrm{lh}(F)$. But this means that $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ even though we assumed that $\mathcal{N}(\mathcal{M}(\mathcal{T})) \rightsquigarrow \mathcal{P}$ moved, a contradiction. $\dashv$

Now, in a sufficiently large collapsing extension extension of $\mathcal{H}$, $\bar{b}$ is the unique cofinal, wellfounded branch of $\overline{\mathcal{T}}$ such that $\mathcal{Q}(\bar{b},\overline{\mathcal{T}}) \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ exists. Hence, by the homogeneity of $\mathrm{Col}(\omega,\theta)$, $\bar{b} \in H$. By elementarity there is a unique cofinal, wellfounded branch b of $\mathcal{T}$ such that $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$. This proves that M is (uniquely) On-iterable and virtually the same argument yields the iterability of M via successor-many stacks of normal trees.

To show that M is fully iterable, it remains to be seen that the unique iteration strategy (guided by $\mathcal{N}$) of M outlined above leads to wellfounded direct limits for stacks of normal trees on M of limit length. Let λ be a limit ordinal and $\vec{\mathcal{T}} = (\mathcal{T}_i \mid i < \lambda)$ a stack according to our iteration strategy. Suppose $\lim_{i<\lambda} \mathcal{M}_\infty^{\mathcal{T}_i}$ is illfounded.

Redefine $\eta \gg \mathrm{rk}(\vec{\mathcal{T}})$, $\mathcal{H} := \mathrm{cHull}^{H_\eta}(\{x, M, \vec{\mathcal{T}}\})$ and $\pi : \mathcal{H} \to H_\eta$ the uncollapse, again with $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran}\pi$. By elementarity we get that $\mathcal{H} \models \ulcorner \lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i}$ is illfounded$\urcorner$. But $\overline{\vec{\mathcal{T}}}$ is countable and according to the iteration strategy guided by $\mathcal{N}$, so that

$$V \models \ulcorner \lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i} \text{ is wellfounded.}\urcorner$$

Now note that $(\lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^{\mathcal{H}} = (\lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^V$ and wellfoundedness is absolute between $\mathcal{H}$ and V, a contradiction.

Now assume that $M_1^{\mathcal{N}}(x)$ exists for some $x \in H_\theta$, and that it's countably iterable. We then do exactly the same thing as with $\mathcal{N}(x)$ *except* that in the claim we replace (1) with

$$\overline{\mathcal{Q}} \models \forall\eta(\overline{\mathcal{Q}}|\eta \not\models \ulcorner\delta(\overline{\mathcal{T}}) \text{ is not Woodin}\urcorner),$$

so that if $\mathcal{P} \triangleleft \mathcal{R}$ then $\delta(\overline{\mathcal{T}})$ is still Woodin in $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$, contradicting the defining property of $M_1^{\mathcal{N}}(x)$ (and thus also of $\mathcal{R}$). The rest of the proof is a copy of the above. ■

Theorem 4.2.2 (Hybrid core model dichotomy). *Let θ be a $\beth$-fixed point or $\theta = \infty$, and let F be a tame model operator on H_θ that condenses well. Let $x \in H_\theta$. Then either:*

I don't think tame is needed here, as we're only indexing extenders at F-initial segments. -Dan *(i) The core model $K^F(x)|\theta$ exists and is (θ,θ)-iterable; or*

(ii) $M_1^F(x)$ exists and is (θ,θ)-iterable.

Insert argument?

PROOF. Assume first that $K^{c,F}(x)|\theta$ reaches a premouse which isn't F-small; let $\mathcal{N}_\xi$ be the first part of the construction witnessing this. Then $\mathfrak{C}(\mathcal{N}_\xi) = M_1^F(x)$, and by Lemma 4.2.1 it suffices to show that $M_1^F(x)$ is countably iterable.

Show that $M_1^F(x)$ is countably iterable.

We can thus assume that $K^{c,F}(x)|\theta$ is F-small. Note that if $K^{c,F}(x)|\theta$ has a Woodin cardinal then because the model is F-closed we contradict F-smallness, so the model has no Woodin cardinals either, making it (θ,θ)-iterable.

Let $\kappa < \theta$ be any uncountable cardinal and let $\Omega := \beth_\kappa(\kappa)^+$. Note that $\Omega < \theta$ since we assumed that θ is a $\beth$-fixed point and $\kappa < \theta$. If Ω is a limit cardinal in $K^{c,F}(x)|\theta$ then let $\mathcal{S} := \mathrm{Lp}(K^{c,F}(x)|\Omega)$ and otherwise let $\mathcal{S} := K^{c,F}(x)|\Omega$. Then by Lemma 3.3 of [Jensen et al., 2009] we get that $\mathcal{S}$ is countably iterable, with largest cardinal Ω in the "limit cardinal case".

This also means that Ω isn't Woodin in $L[\mathcal{S}]$, as it's trivial in the case where Ω is a successor cardinal of $K^{c,F}(x)|\theta$ by our case assumption, and in the "limit cardinal case" it also holds since

$$K^{c,F}(x)|\Omega^{+K^{c,F}(x)|\theta} \subseteq \mathcal{S}.$$

By [Fernandes, 2018] and [Jensen and Steel, 2013] this means that we can build $K^F(x)|\kappa$, as the only places they use that there's no inner model with a Woodin are to guarantee that $K^{c,F}(x)|\Omega$ exists and has no Woodin cardinals, and in Lemma 4.27 of [Jensen and Steel, 2013] in which they only require that Ω isn't Woodin in $L[\mathcal{S}]$.

As $\kappa < \theta$ was arbitrary we then get that $K^F(x)|\theta$ exists. Note that $K^F(x)|\theta$ has no Woodin cardinals either and is F-small, so that $\mathcal{Q}$-structures trivially exist, making it (θ,θ)-iterable. ■

4.3 The hybrid witness equivalence

Definition 4.3.1. asd

Define coarse (k, U, x)-Woodin pairs

$\circ$

Definition 4.3.2. Let F be a total condensing operator and let α be an ordinal. Then the **coarse mouse witness condition at** α **with** F, written $W^*_\alpha(F)$, states that given any scaled-co-scaled $U \subseteq \mathbb{R}$ whose associated sequences of prewellorderings are elements of $\mathrm{Lp}^F_\alpha(\mathbb{R})$, we have for every $k < \omega$ and $x \in \mathbb{R}$ a coarse (k, U, x)-Woodin pair (N, Σ) with $\Sigma \restriction \mathsf{HC} \in \mathrm{Lp}^F_\alpha(\mathbb{R})$. $\circ$

Check if this is a reasonable definition.

Theorem 4.3.3 (Hybrid witness equivalence)**.** *Let $\theta > 0$ be a cardinal, $g \subseteq \mathrm{Col}(\omega, {<}\theta)$ V-generic, $\mathbb{R}^g := \bigcup_{\alpha<\theta} \mathbb{R}^{V[g \restriction \alpha]}$, F a total radiant operator and α a critical ordinal of $\mathrm{Lp}^F(\mathbb{R}^g)$. Assume that $\mathrm{Lp}^F(\mathbb{R}^g) \models \mathsf{DC} + \ulcorner W^*_\beta(F)$ holds for all $\beta \leq \alpha\urcorner$. Then there is a hybrid mouse operator $\mathcal{N} \in V$ on $H_{\aleph_1^{V[g]}}$ such that*

$$\mathrm{Lp}^F(\mathbb{R}^g) \models W^*_{\alpha+1}(F) \quad \textit{iff} \quad V \models \ulcorner M_n^{\mathcal{N}} \textit{ is total on } H_{\aleph_1^{V[g]}} \textit{ for all } n < \omega\urcorner$$

Furthermore, if $\theta < \aleph_1^V$ then we only need to assume that F is total and condensing.

Be more explicit about what the given operator $\mathcal{N}$ looks like.

4.4 Determinacy in mice from DI

Proposition 4.4.1 (Folklore?)**.** *If ω_1 carries a saturated ideal then mouse reflection holds at ω_1.*

Proof. Let $\mathcal{N}$ be a mouse operator defined on HC and fix some $x \in H_{\omega_2}$; we want to show that $\mathcal{N}(x)$ is defined. Let $j : V \to M$ be the generic ultrapower with $\mathrm{crit}\, j = \omega_1^V$ and note that $j(\omega_1^V) = \omega_1^M = \omega_1^{V[g]} = \omega_2^V$ by saturation of the ideal. This means in particular that $\mathsf{HC} \prec H^M_{\omega_2}$. Since

$$\mathsf{HC} \models \ulcorner\mathcal{N}(y) \text{ exists for all sets } y\urcorner$$

we get that $H^M_{\omega_2}$ believes the same is true. But $H^V_{\omega_2} \subseteq H^M_{\omega_2}$ since $\operatorname{crit} j = \omega_1^V$, so that in particular $H^M_{\omega_2}$ believes that $x^\sharp$ exists. Since M is closed under ω-sequences in $V[g]$ by Proposition A.2.5, we get that $x^\sharp$ exists in $V[g]$ and hence also in V as set forcing can't add sharps. ■

Prove this or give a reference.

Proposition 4.4.2 (Folklore?). *If ω_1 carries a precipitous ideal then HC is closed under sharps. If the ideal is furthermore saturated then H_{ω_2} is closed under sharps.*

Proof. Proposition 4.4.1 gives the latter statement if we show the former, so fix an $x \in \mathsf{HC}$ and let $j : V \to M$ be the generic ultrapower from a precipitous ideal on ω_1^V. Since $j(x) = x$ we get that $j : L[x] \to L[x]$ with $\operatorname{crit} j > \operatorname{rk} x$, implying that $x^\sharp$ exists in the generic extension. But set forcing can't add sharps so $x^\sharp$ exists in V as well. ■

Add argument or reference.

Definition 4.4.3. Let $j : V \to M$ be an elementary embedding in some $V[g]$ and let F be a model operator. Then F is j**-radiant** if it condenses well, determines itself on generic extensions and satisfies the **extension property**, which says that $F \subseteq j(F)$ and $j(F) \restriction \mathsf{HC}^{V[g]}$ is definable in $V[g]$. ○

Lemma 4.4.4 (DI). *M_1^F is total on H_{ω_2} for any j-radiant model operator F on H_{ω_2}.*

Proof. We want to use the hybrid core model dichotomy 4.2.2, but the problem is that F is not total. We solve this by going to a smaller model; the model $W := L^F_{\omega_2^V}(\mathbb{R})$ will be a first attempt (note that $\mathbb{R} \in \operatorname{dom} F$ as we're assuming CH). To be able to apply the dichotomy in a model we need it to satisfy ZFC. The following claim is the first step towards this.

Claim 4.4.4.1. Given any real x, $L^F_{\omega_2}(x) \models \ulcorner\omega_1^V$ is inaccessible$\urcorner$.

Proof of claim. Letting $j : V \to M$ be the generic elementary embedding, note that j doesn't move x, so that

$$j \restriction L^F_{\omega_2^V}(x) : L^F_{\omega_2^V}(x) \to L^{j(F)}_{\omega_2^M}(x).$$

Since F has the extension property, $L^{j(F)}_{\omega_2^M}(x)$ is just an end-extension of $L^F_{\omega_2^V}(x)$. In particular ω_1^V is still a cardinal in there, meaning that, for every $\alpha < \omega_1^V$,

$$L^{j(F)}_{\omega_1^M}(x) \models \ulcorner\text{there's a cardinal} > \alpha\urcorner.$$

By elementarity this makes ω_1^V a limit cardinal in $L^F_{\omega_2^V}(x)$ and by GCH in $L^F_{\omega_2^V}(x)$ it's inaccessible. $\dashv$

This claim is now transferred to M, and as $\mathbb{R}^V$ is a real from the point of view of M, we get that

$$L^{j(F)}_{\omega_2^M}(\mathbb{R}^V) \models \ulcorner\omega_1^M \text{ is inaccessible}\urcorner.$$

Noting that $\omega_1^M = \omega_2^V$ and again using the extension property of F, we get that $W \models \mathsf{ZF}$. We don't get choice in W as it doesn't contain a wellorder of the reals, so we we'll work with $W[h]$ instead, where $h \subseteq \mathrm{Col}(\omega_1, \mathbb{R})^W$ is W-generic. Since we're assuming CH we get that $g \in V$, making $W[h] \in V$ as well, $W[h]$ is still closed under F since F determines itself on generic extensions, and $W[h] \models \mathsf{ZFC}$.

We can now apply the hybrid core model dichotomy 4.2.2 inside $W[h]$ to conclude that, for every real x, either $K^F(x)^{W[h]}$ exists or $M_1^F(x)$ exists (note that (ω_1, ω_1)-iterability is absolute between $W[h]$ and V since $W[h]$ contains all the reals). Since mouse reflection holds at ω_1 by Proposition 4.4.1 if the latter conclusion held at all reals x then we would also get that M_1^F is total on H_{ω_2} and we'd be done. So assume $K := K^F(x)^{W[h]}$ exists.

Claim 4.4.4.2. $j(K) \in V$.

Proof of claim. This is where we'll be using homogeneity of our ideal. Firstly K is definable in $W[h]$ and thus also in W by homogeneity of $\mathrm{Col}(\omega_1, \mathbb{R})$, so that $j(K)$ is definable in $j(W)$. But $j(W)$ is definable in $V[g]$ as the unique $j(F)$-premouse over $\mathbb{R}$ of height ω_1, making $j(K)$ definable in $V[g]$ with $j(F)\restriction\mathsf{HC}$ as a parameter. But $j(F)\restriction\mathsf{HC}$ is definable in $V[g]$ since F satisfies the extension property, so homogeneity of our ideal implies that $j(F) \in V$ and hence $j(K) \in V$ as well. $\dashv$

This claim also implies that ω_1^V is inaccessible in K, as if it wasn't, say $\omega_1^V = \lambda^{+K}$, then $\omega_2^V = j(\omega_1^V) = j(\lambda)^{+j(K)} = \lambda^{+j(K)}$, so that ω_2^V isn't a cardinal in V, $\lightning$.

We then also get that $(\omega_1^V)^{+j(K)} < \omega_2^V$, since if they were equal then elementarity would imply that ω_1^V was a successor in K, $\lightning$.

Since $K|\omega_1^V = j(K)|\omega_1^V$, elementarity and the above implies that

$$j^2(K)|(\omega_1^V)^{+j^2(K)} = j(K)|(\omega_1^V)^{+j(K)},$$

which makes sense as $j(K) \in V$.

Let now E be the (ω_1^V, ω_2^V)-extender derived from $j\restriction j(K)$, and note that $E\restriction\alpha \in M$ for every $\alpha < \omega_2^V = \omega_1^M$ as M is closed under countable sequences in $V[g]$.

Claim 4.4.4.3. $E\restriction\alpha$ is on the $j(K)$-sequence for every $\alpha < \omega_2^V$.

Why is this sufficient?

Proof of claim. We need to show that

$$j(W) \models \ulcorner\langle\langle j(K), \mathrm{Ult}(j(K), E\restriction\alpha)\rangle, \alpha\rangle \text{ is } \mathsf{On}\text{-iterable}\urcorner.$$

What kind of reflection?

Assume not. Then by reflection we get, in $j(W)$, a countable $\overline{K}$ and an elementary $\sigma : \overline{K} \to \mathrm{Ult}(j(K), E\restriction\alpha)$ with $\sigma\restriction\alpha = \mathrm{id}$ and $\langle\langle j(K), \overline{K}\rangle, \alpha\rangle$ isn't ω_1-iterable.

Let $k : \mathrm{Ult}(j(K), E\restriction\alpha) \to j^2(K)$ be the factor map with $k\restriction\alpha = \mathrm{id}$ and define $\psi := k\circ\sigma : \overline{K} \to j^2(K)$, so that $(k\circ\sigma)\restriction\alpha = \mathrm{id}$. We have both ψ and $\overline{K}$ in M, which is the generic ultrapower $\mathrm{Ult}(V, g)$, so we also get that $\psi = [\vec{\psi}_\xi]_g$, $\overline{K} = [\vec{K}_\xi]_g$ and $\alpha = [\vec{\alpha}_\xi]_g$. We need to show that

For g-almost every $\xi < \omega_1^V$ it holds that $W \models \ulcorner\langle\langle K, K_\xi\rangle, \alpha_\xi\rangle$ is ω_1-iterable$\urcorner$

By Łoś' Lemma we have that, in V and hence also in $V[g]$, there are embeddings $\psi_\xi : K_\xi \to j(K)$ with $\psi_\xi \restriction \alpha_\xi = \mathrm{id}$ for g-almost every $\xi < \omega_1^V$. As $j(W)$ is closed under countable sequences in $V[g]$ it sees that the K_ξ's are countable, so that an application of absoluteness of wellfoundedness shows that $j(W)$ also has elementary embeddings $\psi_\xi^* : K_\xi \to j(K)$ with $\psi_\xi^* \restriction \alpha_\xi$.

Include this argument perhaps.

But $j(K) = K^{j(F)}(x)^{j(W[h])}$, so $j(W[h])$ sees that $\langle\langle K, K_\xi\rangle, \alpha_\xi\rangle$ is ω_1-iterable, which is therefore also true in W since $W \cap \mathbb{R} \subseteq \mathbb{R}^{V[g]} = j(W[h]) \cap \mathbb{R}$. $\dashv$

Our desired contradiction is then showing that K has a Shelah cardinal, which is impossible. Let $f : \omega_1^V \to \omega_1^V$ be a function in $j(K)$ and pick some $\alpha \in (j(f)(\kappa), \omega_2^V)$. Letting

Insert argument?

$$k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$$

be the factor map, we get that $\operatorname{crit} k \geq \alpha$ by coherence of extenders on the K-sequence and hence that $i_{E\restriction\alpha}(f)(\omega_1^V) < \alpha$ as well. This shows that ω_1^V is Shelah in $j(K)$ and hence K has a Shelah cardinal by elementarity, $\lightning$. ■

Theorem 4.4.5 (DI). $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R}) \models$ *AD for all "nice" Γ and Σ.*

Specify niceness.

Proof.

Show that all the operators occuring in the $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R})$ induction are j-radiant.

■

5 | The External CMI

Introduction.

5.1 HOD mice

Provide overview of this section.

5.1.1 Iteration strategies

At some point we should mention that we adopt John's convention of hiding the degree of iteration trees, always taking the maximal possible degree. And that all of our trees are (stacks of) normal trees.

Definition 5.1.2. Let $\vec{\mathcal{T}}$ be a stack of normal trees. We write $\mathrm{lh}(\vec{\mathcal{T}})$ for the length of $\vec{\mathcal{T}}$ and $\mathcal{T}_\alpha$ for the αth tree in $\vec{\mathcal{T}}$, so that

$$\vec{\mathcal{T}} = (\mathcal{T}_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}})).$$

For $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$, $\gamma < \mathrm{lh}(\mathcal{T}_\alpha)$, $\eta < \mathrm{lh}(\mathcal{T}_\beta)$ we let $\mathcal{M}_\gamma^{\mathcal{T}_\alpha}$ be the model with index γ in the tree $\mathcal{T}_\alpha$ and write

$$\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} \colon \mathcal{M}_\gamma^{\mathcal{T}_\alpha} \to \mathcal{M}_\eta^{\mathcal{T}_\beta}$$

for the corresponding embedding, provided it exists.
We also write

$$\pi^{\vec{\mathcal{T}}}_{\alpha,\beta} \colon \mathcal{M}_0^{\mathcal{T}_\alpha} \to \mathcal{M}_0^{\mathcal{T}_\beta}.$$

If $\vec{\mathcal{T}}$ has a last model, i.e. if $\mathrm{lh}(\vec{\mathcal{T}}) = \xi + 1$ and $\mathcal{M}_\infty^{\mathcal{T}_\xi}$ exists, we let $\mathcal{M}_\infty^{\vec{\mathcal{T}}} := \mathcal{M}_\infty^{\mathcal{T}_\xi}$ and $\pi^{\vec{\mathcal{T}}} \colon \mathcal{M}_0^{\mathcal{T}_0} \to \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ be the associated embedding. ◦

Definition 5.1.3. Let Σ be an iteration strategy and $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$. We write $\Sigma_{\vec{\mathcal{T}},N}$ for the iteration strategy on N given by

$$\Sigma_{\vec{\mathcal{T}},N}(\vec{\mathcal{U}}) := \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}).$$

We call $\Sigma_{\vec{\mathcal{T}},N}$ the $(\vec{\mathcal{T}}, N)$**-tail strategy**of Σ. ◦

Definition 5.1.4. at the very end we should remove those definitions that we didn't need

Let Σ be an iteration strategy.

(i) Σ has the **Dodd-Jensen property** if for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$ and all $\pi\colon \mathcal{M}_\Sigma \to_{\Sigma_1} N$ we have $\pi^{\vec{\mathcal{T}}}(\alpha) \leq \pi(\alpha)$ for all $\alpha \in o(\mathcal{M}_\Sigma)$.

(ii) Σ has the **positional Dodd-Jensen property** if $\Sigma_{\vec{\mathcal{T}},N}$ has the Dodd-Jensen property for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iii) Σ is **weakly positional** if $\Sigma_{\vec{\mathcal{T}},N} = \Sigma_{\vec{\mathcal{U}},N}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(iv) Σ is **positional** if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly positional for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(v) Σ is **weakly commuting** if $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{U}}}$ for all $(\vec{\mathcal{T}}, N), (\vec{\mathcal{U}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vi) Σ is **commuting** if $\Sigma_{\vec{\mathcal{T}},N}$ is weakly commuting for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(vii) Σ is **weakly pullback consistent** if $\Sigma^{\vec{\mathcal{T}}} = \Sigma$ for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

(viii) Σ is **pullback consistent** if $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly pullback consistent for all $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$.

◦

Definition 5.1.5. If Σ is positional, $\Sigma_{\vec{\mathcal{T}},N}$ doesn't depend on $\vec{\mathcal{T}}$ and hence we simply write Σ_N for this tail strategy.

◦

Definition 5.1.6. An iteration strategy Σ has **branch condensation** (see Figure 5.1) if for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}_\Sigma$ such that

(i) $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ are according to Σ,

(ii) $\vec{\mathcal{U}}$ is a stack of successor length $\gamma + 1$ and $\vec{\mathcal{U}}$'s last component $\mathcal{U}_\gamma$is of limit length,

(iii) $\vec{\mathcal{T}}$ has a last model N such that $(\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)$,

$$\begin{tikzcd}
 & N \\
\mathcal{M}_\Sigma \arrow[ur, "\pi^{\vec{\mathcal{T}}}"] \arrow[dr, "\pi_c^{\vec{\mathcal{U}}}"'] & \\
 & \mathcal{M}_c^{\vec{\mathcal{U}}} \arrow[uu, "\pi"']
\end{tikzcd}$$

Figure 5.1: Branch condensation

(iv) there is some branch c such that $\pi_c^{\vec{\mathcal{U}}}$ exists and for some $\pi\colon \mathcal{M}_c^{\vec{\mathcal{U}}} \to_{\Sigma_1} N$ we have
$$\pi^{\vec{\mathcal{T}}} = \pi \circ \pi_c^{\vec{\mathcal{U}}}.$$

Then $c = \Sigma(\vec{\mathcal{U}})$. ○

Definition 5.1.7. Let $\mathcal{M}, \mathcal{N}$ be layered hybrid premice and $\mathcal{T}, \mathcal{U}$ be normal trees on $\mathcal{M}, \mathcal{N}$ respectively. $(\mathcal{M}, \mathcal{T})$ **is a hull of** $(\mathcal{N}, \mathcal{U})$ if there are

(i) an embedding, $\pi\colon \mathcal{M} \to_{\Sigma_1} \mathcal{N}$ and

(ii) an order-preserving map $\sigma\colon \operatorname{lh}(\mathcal{T}) \to \operatorname{lh}(\mathcal{U})$

such that

(i) $\alpha \leq_{\mathcal{T}} \beta \iff \sigma(\alpha) \leq_{\mathcal{U}} \sigma(\beta)$

(ii) $[\alpha, \beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}} = \emptyset \iff [\sigma(\alpha), \sigma(\beta)]_{\mathcal{U}} \cap \mathcal{D}^{\mathcal{U}} = \emptyset$,

(iii) $\pi_\alpha\colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_{\sigma(\alpha)}^{\mathcal{U}}$ and $\pi_\alpha(E_\alpha^{\mathcal{T}}) = E_{\sigma(\alpha)}^{\mathcal{U}}$,

(iv) for $\beta < \alpha$ we have $\pi_\alpha \restriction \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1 = \pi_\beta \restriction \operatorname{lh}(E_\beta^{\mathcal{T}}) + 1$,

(v) for $\alpha \leq_{\mathcal{T}} \beta$ with $[\alpha, \beta]_{\mathcal{T}} \cap \mathcal{D}^{\mathcal{T}}$ we have $\pi_\beta \circ \pi_{\alpha,\beta}^{\mathcal{T}} = \pi_{\sigma(\alpha),\sigma(\beta)}^{\mathcal{U}} \circ \pi_\alpha$,

(vi) if $\beta = \operatorname{pred}_{\mathcal{T}}(\alpha+1)$, then $\sigma(\beta) = \operatorname{pred}_{\mathcal{U}}(\sigma(\alpha+1))$ and $\pi_{\alpha+1}([a, f]_{E_\alpha^{\mathcal{T}}}) = [\pi_\alpha(a), \pi_\beta(f)]_{E_{\sigma(\alpha)}^{\mathcal{T}}}$ and

(vii) $0 \leq_{\mathcal{U}} \sigma(0)$, $[0, \sigma(0)] \cap \mathcal{D}^{\mathcal{U}} = \emptyset$ and $\pi_0 = \pi_{0,\sigma(0)}^{\mathcal{U}} \circ \pi$,

(See Figure 5.2) ○

Definition 5.1.8. Let $\mathcal{M}, \mathcal{N}$ be layered hybrid premice and $\vec{\mathcal{T}}, \vec{\mathcal{U}}$ be stacks of normal trees on $\mathcal{M}, \mathcal{N}$ respectively. $(\mathcal{M}, \vec{\mathcal{T}})$ **is a hull of** $(\mathcal{N}, \vec{\mathcal{U}})$ if there are

(i) an order presercing map $\sigma\colon \operatorname{lh}(\vec{\mathcal{T}}) \to \operatorname{lh}(\vec{\mathcal{U}})$,

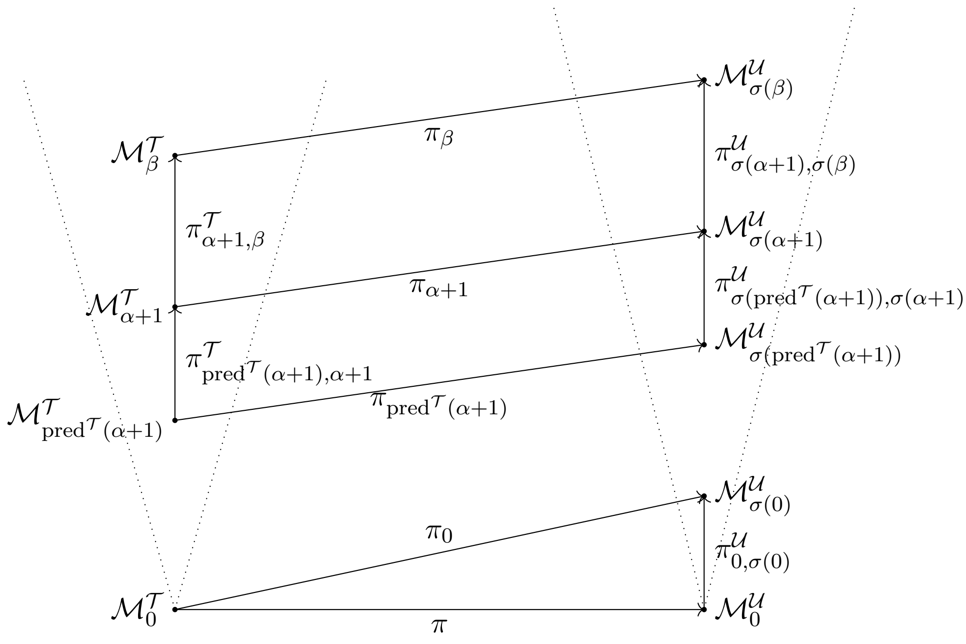

Figure 5.2: $\mathcal{T}$ is a hull of $\mathcal{U}$

(ii) a sequence $(\sigma_\alpha \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}))$ of order preserving maps $\sigma_\alpha\colon \mathrm{lh}(\mathcal{T}_\alpha) \to \mathrm{lh}(\mathcal{U}_{\sigma(\alpha)})$,

(iii) $(\pi_{\alpha,\beta} \mid \alpha < \mathrm{lh}(\vec{\mathcal{T}}) \wedge \beta < \mathrm{lh}(\mathcal{T}_\alpha))$ such that

  (a) $\pi_{0,0} = \pi^{\vec{\mathcal{U}}}_{0,\sigma(0)}$ (so that $\pi_{0,0} = \mathrm{id}$ if $\sigma(0) = 0$),

  (b) for $\alpha < \mathrm{lh}(\vec{\mathcal{T}})$

$$\pi_{\alpha,0}\colon \mathcal{M}^{\vec{\mathcal{T}}}_\alpha \to_{\Sigma_1} \mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}$$

  and $(\mathcal{M}^{\vec{\mathcal{T}}}_\alpha, \mathcal{T}_\alpha)$ is a $(\pi_{\alpha,0}, \sigma_0)$-hull of $(\mathcal{M}^{\vec{\mathcal{U}}}_{\sigma(\alpha)}, \mathcal{U}_{\sigma(\alpha)})$,

  (c) $\alpha < \beta < \mathrm{lh}(\vec{\mathcal{T}})$ and $\pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)}$ exists, then $\pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))}$ exists and

$$\pi_{\beta,\eta} \circ \pi^{\vec{\mathcal{T}}}_{(\alpha,\gamma),(\beta,\eta)} = \pi^{\vec{\mathcal{U}}}_{(\sigma(\alpha),\sigma_\alpha(\gamma)),(\sigma(\beta),\sigma_\beta(\eta))} \circ \pi_{\alpha,\gamma}.$$

(See Figure 5.3) $\circ$

Definition 5.1.9. Let $\mathcal{M}$ be a layered hybrid premouse and Σ be a (partial) iteration strategy for $\mathcal{M}$. Σ has **hull condensation** if the follwing holds

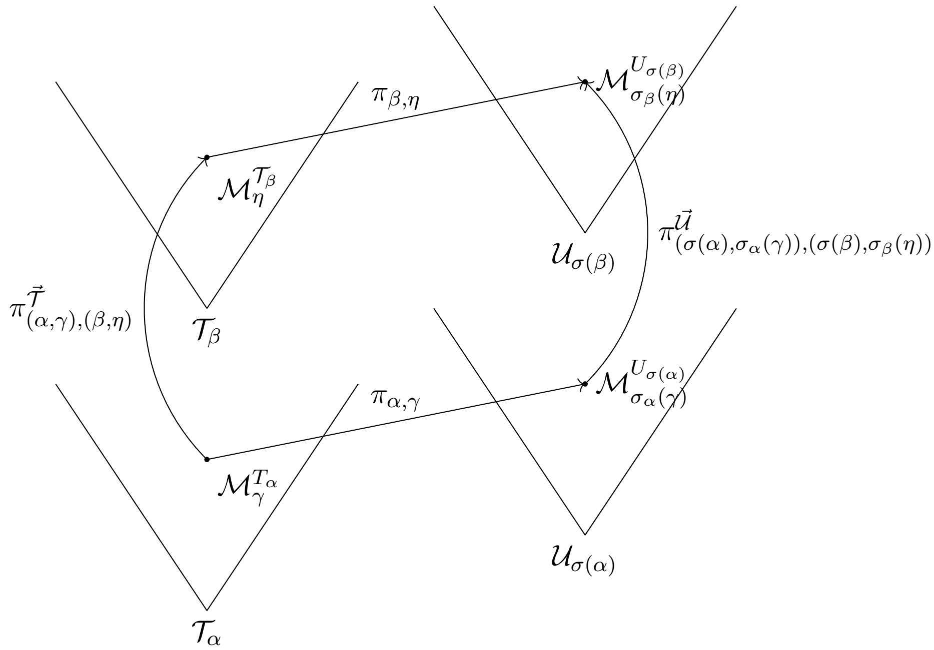

Figure 5.3: $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$

true for any two stacks $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ on $\mathcal{M}$.
If $\vec{\mathcal{U}}$ is according to Σ and $\vec{\mathcal{T}}$ is a hull of $\vec{\mathcal{U}}$, then $\vec{\mathcal{T}}$ is according to Σ. ◦

Lemma 5.1.10. *Let Σ be an iteration strategy. Then the following hold true.*

(i) If Σ has hull condensation then it is pullback consistent.

(ii) If Σ is positional and pullback consistent then it is commuting.

Proof. See [Sargsyan, 2015b, Proposition 2.36]. ∎

5.1.11 Layered Hybrid Mice

Define strategy mice as a particular kind of hybrid mice, hod mice/pairs and put in positional and commuting in the definition, state comparison. Introduce derived models of hod mice and how they relate to the Solovay hierarchy.
Define Σ-mouse

Definition 5.1.12. Let $\mathcal{M}$ be a transitive set (or structure). We let $o(\mathcal{M}) := \mathcal{M} \cap \mathsf{On}$ be the ordinal height of $\mathcal{M}$. ∘

Definition 5.1.13. Let $\mathcal{M}$ be a (hybrid) premouse and $\alpha \leq o(\mathcal{M})$. We let

(i) $\mathcal{M}||\alpha$ be the initial segment of $\mathcal{M}$ of height α including its top extender and

(ii) $\mathcal{M}|\alpha$ be the passive initial segment of $\mathcal{M}$ of height α, i.e. $\mathcal{M}||\alpha$ but without the top extender.

∘

Definition 5.1.14. Let $\mathcal{M}$ be a $\mathcal{J}$-structure [1] and $\alpha \leq o(\mathcal{M})$. We write $\mathcal{J}_\alpha^{\mathcal{M}}$ for the αth level of $\mathcal{M}$'s construction. ∘

Definition 5.1.15. A **potential layered hybrid premouse** (over X) is an acceptable $\mathcal{J}$-structure of the form $\mathcal{M} = (J_\alpha^{\vec{E},f}(X); \in, \vec{E}, B, f)$ X such that

(i) $\vec{E}$ is a fine extender sequence (over X),

(ii) f is a function with domain $Y \subseteq \alpha$ such that $f(\gamma)$, for each $\gamma \in Y$, is a shift of an amenable function that typically codes part of an iteration strategy for $\mathcal{M}$,

We will often write $\vec{E}^{\mathcal{M}}, f^{\mathcal{M}}, Y^{\mathcal{M}}$ for $\vec{E}, f, Y$ as above.
If all proper initial segments of $\mathcal{M}$ are sound, we say that $\mathcal{M}$ is a **layered hybrid premouse**. . ∘

In our case, assuming X is a self-well-ordered set, $Y^{\mathcal{M}}$ is determined by the **standard indexing scheme** (see [Sargsyan, 2015b, Definition 1.18]).
For more details, see [Sargsyan, 2015b].

Definition 5.1.16. Let Σ be a strategy for a layered hybrid premouse $\mathcal{M}$. For $\alpha \leq o(M)$ we let $\Sigma_{\mathcal{M}|\alpha}$ be the id-pullback iteration strategy on $\mathcal{M}|\alpha$ induced by Σ, i.e. a stack $\vec{\mathcal{T}}$ on $\mathcal{M}|\alpha$ is according to $\Sigma_{\mathcal{M}|\alpha}$ iff $\mathrm{id}\vec{\mathcal{T}}$ on $\mathcal{M}$, given by the copy construction via id (see [Steel, 2010, 4.1]), is according to Σ. ∘

[1]See [Zeman, 2011] for the basics on $\mathcal{J}$-structures, premice and their fine structure.

Definition 5.1.17. A **layered strategy premouse** $\mathcal{M}$ is a layered hybrid premouse such that

(i) $f^{\mathcal{M}}(\gamma)$ codes a partial iteration strategy $\Sigma^{\mathcal{M}}_{\gamma}$ for $\mathcal{M}|\gamma$ and
(ii) For $\gamma_0, \gamma_1 \in Y^{\mathcal{M}}$, if $\gamma_0 < \gamma_1$ then $(\Sigma^{\mathcal{M}}_{\gamma_1})_{\mathcal{M}|\gamma_0} \subseteq \Sigma^{\mathcal{M}}_{\gamma_0}$.

We also write $\Sigma^{\mathcal{M}}$ for the the strategy coded by $f^{\mathcal{M}}$. ◦

Definition 5.1.18. Let $\mathcal{M}$ be a layered strategy premouse and Σ be an iteration strategy for $\mathcal{M}$. $\mathcal{M}$ is a **Σ-premouse** if $\Sigma^{\mathcal{M}} \subseteq \Sigma$. ◦

Definition 5.1.19. Let Σ be an iteration strategy. We write $\mathcal{M}_\Sigma$ for the (layered hybrid) premouse $\mathcal{N}$ such that Σ is an iteration strategy for $\mathcal{N}$. We also let

$$I(\mathcal{M}_\Sigma, \Sigma) := \{(\vec{\mathcal{T}}, N) \mid \vec{\mathcal{T}} \text{ is a stack of normal trees on } \mathcal{M}_\Sigma \text{ according to } \Sigma, \\ \pi^{\vec{\mathcal{T}}} \text{ exist and } N \text{ is the last model of } \vec{\mathcal{T}}\}.$$

and

$$pI(\mathcal{M}_\Sigma, \Sigma) := \{N \mid \exists \vec{\mathcal{T}} \colon (\vec{\mathcal{T}}, N) \in I(\mathcal{M}_\Sigma, \Sigma)\}. \qquad \dashv$$

Definition 5.1.20. A Σ-premouse $\mathcal{M}$ is a **Σ-mouse** if there is a $\omega_1 + 1$-iteration strategy Λ such that all $\mathcal{N} \in pI(\mathcal{M}, \Lambda)$, $\mathcal{N}$ are themselves Σ-premice. ◦

Definition 5.1.21. Let a be a transitive self-well-ordered set and let Σ be an iteration strategy with hull-condensation such that $\mathcal{M}_\Sigma \in a$ and let Γ be a pointclass which is closed under Boolean operations and continuous images and preimages. Define the (Γ, Σ)-Lp stack over a recursively as follows:

(i) $\mathrm{Lp}^{\Gamma,\Sigma}_0(a) := a \cup \{a\}$,
(ii) $\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha+1}(a) := \bigcup\{\mathcal{M} \mid \mathcal{M}$ is a sound Σ-mouse over $\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha}(a)$ projecting to $o(\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha}(a))$ and having an iteration strategy in $\Gamma\}$,
(iii) $\mathrm{Lp}^{\Gamma,\Sigma}_{\lambda}(a) := \bigcup_{\alpha<\lambda} \mathrm{Lp}^{\Gamma,\Sigma}(a)$ for limit λ.

We also let $\mathrm{Lp}^{\Gamma,\Sigma}(a) := \mathrm{Lp}^{\Gamma,\Sigma}_1(a)$. ◦

5.1.22 HOD Mice

Definition 5.1.23. Suppose $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ is a layered strategic premouse. $\mathcal{P}$ is a HOD-**premouse** [2] provided the following hold:
Let $\lambda = \text{otp}(Y^{\mathcal{P}})$, $(\gamma_\beta \mid \beta < \lambda)$ be the strictly increasing enumeration of $Y^{\mathcal{P}}$ and let, for $\beta < \lambda$, $\mathcal{P}(\beta) := \mathcal{P}||\gamma_\beta$ and moreover $\mathcal{P}(\lambda) := \mathcal{P}$. Then there is a continuous, strictly increasing sequence $(\delta_\beta \mid \beta \leq \lambda)$ of $\mathcal{P}$-cardinals such that

(i) $B = \emptyset$,

(ii) $Y^{\mathcal{P}} \subseteq \delta_\lambda$,

(iii) $(\delta_\beta \mid \beta \leq \lambda)$ is sequence of Woodin cardinals and their limits in $\mathcal{P}$ and

(iv) for all $\beta \leq \lambda$

  (a) δ_β is a strong cutpoint of $\mathcal{P}$,

  (b) $\mathcal{P}(\beta) \models \ulcorner\mathsf{ZFC}\text{-Replacement}\urcorner$,

  (c) $\mathcal{P}(\beta) = \mathcal{O}^{\mathcal{P},\omega}_{\delta_\beta}$ [3],

  (d) if β is a limit then $\delta_\beta^{+\mathcal{P}} = \delta_\beta^{+\mathcal{P}(\beta)}$,

  (e) if $\beta < \lambda$ then $f(\gamma_\beta)$ codes a $(o(\mathcal{P}), o(\mathcal{P}))$-strategy, call it $\Sigma^{\mathcal{P}}_\beta$, for $\mathcal{P}(\beta)$ with hull condensation [4],

  (f) if $\alpha < \beta < \lambda$, then $(\Sigma^{\mathcal{P}}_\beta)_{\mathcal{P}(\alpha)} = \Sigma^{\mathcal{P}}_\alpha$,

  (g) if $\beta < \lambda$ and $\eta \in (\delta_\beta, \delta_{\beta+1})$ is a $\mathcal{P}$-successor cardinal, then $\mathcal{P}|\eta$ is a $\Sigma^{\mathcal{P}}_{\gamma_\beta}$-premouse over $\mathcal{P}(\beta)$ which is $(o(\mathcal{P}), o(\mathcal{P}))$-iterable for stacks above δ_β.

(v) $\forall n < \omega \colon \mathcal{P} \models \delta_\lambda^{+n}$ exists and $o(\mathcal{P}) = \sup_{n<\omega}(\delta_\lambda^{+n})^{\mathcal{P}}$.

define strong cut-point

confirm with Grigor that this is what he had in mind

See Figure 5.4.
We will often write $\delta^{\mathcal{P}}_\beta, \gamma^{\mathcal{P}}_\beta, \lambda^{\mathcal{P}}$ for $\delta_\beta, \gamma_\beta, \lambda$ as above and moreover let $\delta^{\mathcal{P}} := \delta_\lambda$. ◦

include an intutive description of HOD-mice

[2] These are in fact HOD-premice below $\ulcorner\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable$\urcorner$ in [Sargsyan, 2015b]. However, since all of our HOD-mice are of this form, we omit this.

[3] see [Sargsyan, 2015b, Definition 1.26]

[4] note that $\Sigma^{\mathcal{P}}_\beta \subseteq \mathcal{P}$ is an internal strategy, i.e. only defined on trees that are elements of $\mathcal{P}$

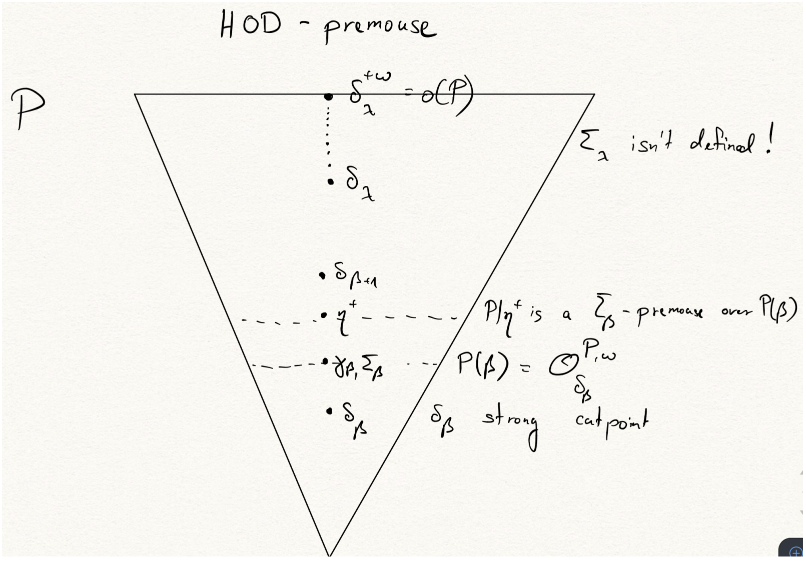

Figure 5.4: HOD-premouse

Definition 5.1.24. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse. We let

$$\mathcal{P}^- = \begin{cases} P|\gamma_{\lambda^{\mathcal{P}}-1} & \text{, if } \lambda^{\mathcal{P}} \text{ is a successor ordinal,} \\ \mathcal{P}|\delta^{\mathcal{P}} & \text{, otherwise.} \end{cases}$$

See Figure 5.5

add picture and figure out why we don't just let $\mathcal{P}^- = \mathcal{P}(\gamma^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1})$ in the successor case.

∘

Definition 5.1.25. Let $\mathcal{P}, \mathcal{Q}$ be HOD-premice. We write $\mathcal{P} \trianglelefteq_{\mathrm{HOD}} \mathcal{Q}$ if there is some $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We also write $\mathcal{P} \triangleleft_{\mathrm{HOD}} \mathcal{Q}$ if $\mathcal{P} \trianglelefteq_{\mathrm{HOD}} \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$.
In this case we say that $\mathcal{P}$ is a (proper) HOD-**initial segment** of $\mathcal{Q}$. ∘

Definition 5.1.26. Let $\mathcal{P} = (J^{\vec{E},f}(X); \in, \vec{E}, f, B)$ be a HOD-premouse and $\alpha \leq \lambda^{\mathcal{P}}$.

(i) If $\alpha < \lambda^{\mathcal{P}}$, we let $\Sigma^{\mathcal{P}}_\alpha$ be the internal iteration strategy of $\mathcal{P}(\alpha)$ coded by $f(\alpha)$ and

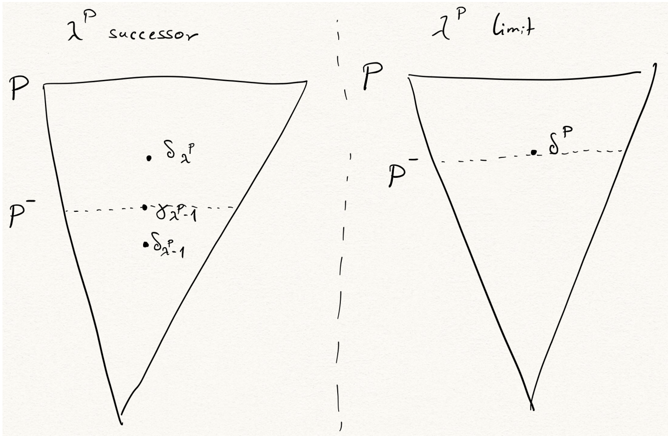

Figure 5.5: $\mathcal{P}^-$

(ii) $\Sigma^{\mathcal{P}}_{<\alpha} := \bigoplus_{\beta<\alpha} \Sigma^{\mathcal{P}}_{\beta}$.

We also let $\Sigma^{\mathcal{P}} := \Sigma^{\mathcal{P}}_{<\lambda^{\mathcal{P}}}$. ◦

Remark 5.1.27. By the agreement of the internal iteration strategies of HOD-premice (item 4f in subsection 5.1.23), $\Sigma^{\mathcal{P}}_{\alpha}$ already includes all of the information of $\Sigma^{\mathcal{P}}_{<\alpha}$ and can be identified with $\Sigma^{\mathcal{P}}_{<\alpha+1}$. reference broken

Definition 5.1.28. Let $\mathcal{P}$ be a HOD-premouse. Σ is a (κ, λ)**-iteration strategy** for $\mathcal{P}$ if it is a winning strategy for player II in the iteration game $\mathcal{G}(\mathcal{P}, \kappa, \lambda)$ and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, then $\mathcal{Q}$ is a HOD-premouse such that $\Sigma^{\mathcal{Q}} = \Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \cap \mathcal{Q}$. ◦ add reference

Remark 5.1.29. In particular, $\Sigma^{\mathcal{P}} = \Sigma \cap \mathcal{P}$, i.e. Σ extends the internal iteration strategy of $\mathcal{P}$.

Definition 5.1.30. $(\mathcal{P}, \Sigma)$ is a HOD**-pair** if

(i) $\mathcal{P}$ is a HOD-premouse and

(ii) Σ is a $(\omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{P}$ with hull condensation.

the definition of hod pair is different in both versions of Grigor‘s thesis. Verify that this is the intended one.

◦

5.1.31 HOD **Analysis**

gather all the information we need on HOD – this can be found in Grigor‘s thesis

Definition 5.1.32. Let $(P, \Sigma), (Q, \Lambda)$ be HOD-pairs. We let $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$, i.e. there is a $(\mathcal{P}, \Sigma)$-iterate $(\mathcal{T}, R)$ and a $(\mathcal{Q}, \Lambda)$-iterate $(\mathcal{U}, S)$ such that

$$\mathcal{R} \trianglelefteq_{\mathrm{HOD}} \mathcal{S} \text{ and } \Sigma_{\mathcal{R},\mathcal{T}} = \Lambda_{\mathcal{R},\mathcal{U}}.$$

We also let $(\mathcal{P}, \Sigma) <_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ iff $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ and $(\mathcal{Q}, \Lambda) \not\leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.

$\circ$

Definition 5.1.33. Let $(\mathcal{P}, \Sigma)$ be a HOD-pair such that Σ has branch condensation and is fullness preserving. We recursively define $\alpha(\mathcal{P}, \Sigma) := |(\mathcal{P}, \Sigma)|_{\leq_{\mathrm{DJ}}} \in \mathsf{On}$ via

$$\begin{aligned} |(\mathcal{P}, \Sigma)|_{\leq_{\mathrm{DJ}}} = \sup\{|(\mathcal{Q}, \Lambda)|_{\leq_{\mathrm{DJ}}} + 1 \mid & (\mathcal{Q}, \Lambda) \text{is a } \mathrm{HOD}\text{-pair such that} \\ & \Lambda \text{ has branch condensation} \\ & \text{and is fullness preserving } \} \end{aligned}$$

$\circ$

Remark 5.1.34. As in the case of ordinary premice, $\leq_{\mathrm{DJ}}$ (or rather $<_{\mathrm{DJ}}$) is a wellfounded relation. The interesting question is whether it‘s total.

Theorem 5.1.35 (Sargsyan)**.** *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ are* HOD*-pairs such that both Σ and Λ have branch condensation and are fullness preserving. Then $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ or $(\mathcal{Q}, \Lambda) \leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.*

Proof. [Sargsyan, 2015b, Theorem 5.10]. ■

Theorem 5.1.36 (Sargsyan)**.** *Assume* $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Suppose* $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ *are* HOD*-pairs such that both* Σ *and* Λ *have branch condensation and are* Γ*-fullness preserving for some pointclass* Γ *which is closed under continuous images and preimages. Suppose further that there is a good pointclass* Γ^* *such that* $\Gamma \cup \{\mathrm{Code}(\Sigma), \mathrm{Code}(\Lambda)\} \subseteq \underset{\sim}{\Delta}_{\Gamma^*}$. *Then* $(\mathcal{P}, \Sigma) \leq_{\mathrm{DJ}} (\mathcal{Q}, \Lambda)$ *or* $(\mathcal{Q}, \Lambda) \leq_{\mathrm{DJ}} (\mathcal{P}, \Sigma)$.

define Code Σ

Proof. [Sargsyan, 2015b, Theorem 2.33]. ■

Definition 5.1.37. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. We let

(i) $\mathcal{F}(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \mid \mathcal{Q} \in pB(\mathcal{P}, \Sigma)\}$ and

(ii) $\mathcal{F}^+(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \mid \mathcal{Q} \in pI(\mathcal{P}, \Sigma)\}$.

∘

Remark 5.1.38. By [Sargsyan, 2015b, Corollary 2.44] Σ is commuting, so that $\Sigma_{\mathcal{Q}}$ is indeed well-defined.

Definition 5.1.39. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$. We let $\mathcal{Q} \leq^{\mathcal{P},\Sigma} \mathcal{R}$ if

(i) $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and $R \in pI(\mathcal{Q}, \Sigma_Q)$ or

(ii) $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \leq_{\mathrm{DJ}} (\mathcal{R}, \Sigma_{\mathcal{R}})$.

∘

Lemma 5.1.40 (Sargsyan)**.** $\leq^{\mathcal{P},\Sigma}$ *is directed.*

Proof. [Sargsyan, 2015b, Lemma 4.17]. ■

Definition 5.1.41. Suppose Γ is a pointclass closed under Wadge reducibility and $(\mathcal{P}, \Sigma)$ is a HOD-pair such that Σ has branch condensation and is Γ-fullness preserving. Let $\mathcal{Q}, \mathcal{R} \in \mathrm{pI}(\mathcal{P}, \Sigma) \cup \mathrm{pB}(\mathcal{P}, \Sigma)$ be such that, for some

$\alpha \leq^{\mathcal{R}}$, $\mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. We let

$$\pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q} \to \mathcal{R}(\alpha)$$

be the iteration map given by $\Sigma_{\mathcal{Q}}$.

We let

(i) $\mathcal{M}_\infty(\mathcal{P}, \Sigma) = \operatorname{dirlim}(\mathcal{F}(\mathcal{P}, \Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q}, \mathcal{R} \in pB(\mathcal{P}, \Sigma) \wedge \exists \alpha \leq \lambda^{\mathcal{R}} \mathcal{R}(\alpha) \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}}))$ and

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma) = \operatorname{dirlim}(\mathcal{F}(\mathcal{P}, \Sigma), \pi^{\Sigma}_{\mathcal{Q},\mathcal{R}} \colon \mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \leq^{\Sigma}_{\mathcal{Q},\mathcal{R}} \mathcal{R})$.

For $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ we let

(i) $\pi^{\Sigma}_{\mathcal{Q},\infty} \colon \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$ and

(ii) $\sigma^{\Sigma}_{\mathcal{R},\infty} \colon \mathcal{R} \to \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$

be the direct limit maps. ◦

Definition 5.1.42. Let $(\mathcal{P}, \Sigma)$ be as above. We let

(i) $\delta_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$,

(ii) $\delta^+_\infty(\mathcal{P}, \Sigma)$ be the supremum of the Woodin cardinals of $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ and

(iii) $\lambda_\infty(\mathcal{P}, \Sigma) := \lambda^{\mathcal{M}^+_\infty(\mathcal{P},\Sigma)}$.

◦

Lemma 5.1.43 (Sargsyan)**.** *Let Γ be a pointclass closed under Wadge reducibility. Suppose $(\mathcal{P}, \Sigma)$ is a* HOD*-pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal and Σ has branch condensation and is Γ-fullness preserving. Then*

(i) $\delta_\infty(\mathcal{P}, \Sigma) = \delta^+_\infty(\mathcal{P}, \Sigma)$ *and*

(ii) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma) | \delta^+_\infty = \mathcal{M}_\infty(\mathcal{P}, \Sigma)$.

Proof. [Sargsyan, 2015b, Lemma 4.18]. ■

We will likely not need the entire theorem and should reduce it to the part that we need once we are done.

Theorem 5.1.44 (Sargsyan)**.** *Assume AD^+, let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Then the following holds:*

define ϕ

define Ω^Γ

(i) If $L(\Gamma, \mathbb{R}) \models \phi$ then for all $(\mathcal{P}, \Sigma) \in \Gamma$ such that $\alpha(\mathcal{P}, \Sigma) < \Omega^\Gamma$ we have, for all $\alpha \leq \alpha(\mathcal{P}, \Sigma)$,

(a) $\delta_\alpha^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)} = \theta_\alpha^\Gamma$ *and*

(b) $\mathcal{M}_\infty^+(\mathcal{P},\Sigma)|\theta_\alpha^\Gamma = (V^{\mathcal{H}}_{\theta_\alpha^\Gamma}; \in, \vec{E}^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)} \upharpoonright \theta_\alpha^\Gamma, \Lambda \upharpoonright \theta_\alpha^\Gamma)$,

where Λ *is the iteration strategy coded by* $f^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)}$.

define θ_α^Γ

(ii) *If* $L(\Gamma,\mathbb{R}) \models \psi$ *then for all* $\alpha \leq \Omega^\Gamma$

define ψ

(a) $\delta_\alpha^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)} = \theta_\alpha^\Gamma$ *and*

(b) $\mathcal{M}_\infty^+(\mathcal{P},\Sigma)|\theta_\alpha^\Gamma = (V^{\mathcal{H}}_{\theta_\alpha^\Gamma}; \in, \vec{E}^{\mathcal{M}_\infty^+(\mathcal{P},\Sigma)} \upharpoonright \theta_\alpha^\Gamma, \Lambda \upharpoonright \theta_\alpha^\Gamma)$.

(iii) *Suppose* $\Gamma^* \subseteq \mathcal{P}(\mathbb{R})$ *is such that* $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*,\mathbb{R}) \models \mathsf{AD}^+$ *and there is a* HOD*-pair* $(\mathcal{P},\Sigma) \in \Gamma^*$ *such that*

(a) Σ *has branch condensation and is* Γ*-fullness preserving,*

(b) $\lambda^{\mathcal{P}}$ *is a successor ordinal,* $\mathrm{Code}(\Sigma_{\mathcal{P}^-}) \in \Gamma$ *and* $L(\Gamma,\mathbb{R})$ *models that* $(\mathcal{P},\Sigma_{\mathcal{P}^-})$ *is a suitable pair such that* $\alpha(\mathcal{P}^-,\Sigma_{\mathcal{P}^-}) = \alpha$,

define suitable pair

(c) *there is a sequence* $(B_i \mid i < \omega) \subseteq \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *which guides* Σ *and*

define $\mathbb{B}(..)$ and what it means to be guided

(d) *for any* $B \in \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})^{L(\Gamma,\mathbb{R})}$ *there is some* $\mathcal{R} \in \mathrm{pI}(\mathcal{P},\Sigma)$ *such that* $\Sigma_{\mathcal{R}}$ *respects* B.

define respects B

Then $L(\Gamma,\mathbb{R}) \models \psi$ *and* $\mathcal{M}_\infty(\mathcal{P},\Sigma) = \mathcal{M}_\infty^+(\mathcal{P},\Sigma)$.

Proof. [Sargsyan, 2015b, Theorem 4.24]. ■

5.2 The tame case

Define

$$\Gamma_0 := \{A \subseteq \mathbb{R} \mid L(A,\mathbb{R}) \models \mathsf{AD} + \Omega = 0\}.$$

Lemma 5.2.1 (DI)**.** $\Gamma_0 = \mathrm{Lp}(\mathbb{R}) \cap \mathscr{P}(\mathbb{R})$.

Proof. ($\supseteq$) Let $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$ and let $A \subseteq \mathbb{R}$ be an element of $\mathcal{M}$. Since $\mathcal{M}$ projects to $\mathbb{R}$ and is sound, we get that A is OD_x for a real x, so that everything in $L(A,\mathbb{R})$ is also ordinal definable in a real as well. Since $\mathrm{Lp}(\mathbb{R}) \models \mathsf{AD}$ we then get that $\mathsf{AD} + \Omega = 0$ holds in $L(A,\mathbb{R})$, making $A \in \Gamma_0$.

Check this proof.

($\subseteq$) Let $A \in \Gamma_0$. Since we're assuming CH we get that $V[g] \models |\mathbb{R}| = \aleph_1^V = \aleph_0$, so fix a generic bijection $b : \omega \to \mathbb{R}^V$ in $V[g]$. Define $a_b \in \mathbb{R}$ as $n \in a_b$ iff $b(n) \in A$. As $L(A,\mathbb{R}) \models \mathsf{AD} + \theta_0 = \Theta$ it holds that A is $\mathrm{OD}_z^{L(A,\mathbb{R})}$

for $z \in \mathbb{R}$, so that

$$A = j(A) \cap \mathbb{R}^V \in \mathrm{OD}_{z,\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}.$$

In particular, as A and $\mathbb{R}^V$ are definable from b and a_b is definable from b, we get that $a_b \in \mathrm{OD}_b^{L(j(A),\mathbb{R}^{V[g]})}$. By MC we then get that there's some b-premouse $\mathcal{M} \in L(j(A), \mathbb{R}^{V[g]})$ projecting to b with $a_b \in \mathcal{M}$ and a Σ such that

$$L(j(A), \mathbb{R}^{V[g]}) \models \ulcorner \Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{M} \urcorner.$$

Why is it that we have to go through b in this fashion? Can't we just use MC and get $\mathcal{N}$ without going through $\mathcal{M}$? Is it because $L(j(A), \mathbb{R}^{V[g]})$ doesn't know that $\mathbb{R}^V$ is countable?

From this $\mathcal{M}$ we can then get an $\mathbb{R}^V$-premouse $\mathcal{N} \in L(j(A), \mathbb{R}^{V[g]})$ projecting to $\mathbb{R}^V$ with $A \in \mathcal{N}$ and

$$L(j(A), \mathbb{R}^{V[g]}) \models \ulcorner \Sigma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{N} \urcorner.$$

Now $\mathcal{N}$ is $\mathrm{OD}_{\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[g]})}$, and since we don't have divergent models of AD^+ it holds that, letting $\Theta^{j(A)} := \Theta^{L(j(A),\mathbb{R}^{V[g]})}$,

$$V[g] \models L(j(A), \mathbb{R}) = L(P_{\Theta^{j(A)}}(\mathbb{R})).$$

This means that $\mathcal{N} \in \mathrm{OD}_{\mathbb{R}^V}^{V[g]}$, so that homogeneity of I we get that $\mathcal{N} \in V$. It remains to show that $\mathcal{N} \trianglelefteq \mathrm{Lp}(\mathbb{R}^V)$, meaning that we need to show that $\mathcal{N}$ is countably $(\omega_1 + 1)$-iterable in V. But letting $\overline{\mathcal{N}} \to \mathcal{N}$ be a countable hull in V we get that $j(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$, so that elementarity of j implies that $\Sigma \upharpoonright V \in V$ is an $\omega_1^{V[g]} = \omega_2^V$-iteration strategy for $\overline{\mathcal{N}}$ and we're done. ■

Why's this?

Is this really this iterable?

Proposition 5.2.2 (DI). $\mathrm{cof}^V(\Theta^{\mathrm{Lp}(\mathbb{R})}) = \omega$.

Proof.

See Ketchersid's Thesis 3.17 or 7.4.2 in the CMI book. Perhaps we don't need it though, following Wilson's thesis.

■

Theorem 5.2.3. *Let Γ be an inductive-like pointclass. If $\mathcal{M}$ is a suitable quasi-iterable premouse, $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ is closed under recursive join and the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is both total on $\mathcal{M}$ and has the full factors property, then there's a unique Γ-fullness preserving (ω_1, ω_1)-strategy Φ for $\mathcal{M}$ such that, for every quasi-iterate $\mathcal{P}$ of $\mathcal{M}$,*

- *$\mathcal{P}$ is a non-dropping Φ-iterate of $\mathcal{M}$; and*
- *the Φ-iteration map $i : \mathcal{M} \to \mathcal{P}$ equals the $\mathcal{A}$-guided map $\pi^{\mathcal{A}}_{\mathcal{M},\mathcal{P}}$.*

Let $\Phi_{\mathcal{M}}$ be the unique strategy for $\mathcal{M}$ as in the above theorem. We now improve this to include branch condensation.

The 3d argument is quite similar to the proof of Theorem 7.19 in the outline.

Theorem 5.2.4. *Let Γ be an inductive-like pointclass and assume that $\mathbf{\Delta}_\Gamma$ is determined and that Γ-MC holds. Let $\mathcal{M}$ be an ω-suitable quasi-iterable premouse such that $\mathcal{D}(\mathcal{M}) \equiv \mathcal{M}_\Gamma$, let $\mathcal{A} \in [\mathrm{Env}(\Gamma)]^\omega$ be closed under recursive join, assume $\pi^{\mathcal{A}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and that it has the full factors property. Let $\Phi := \Phi_{\mathcal{M}}$. Then there's a $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ has $\mathcal{A}$-condensation, and hence also branch condensation, for every $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$.*

This is the companion of Γ, see Trevor's thesis. I'm not sure if we can find $\mathcal{M}$ like this, however.

Proof. Assume not and fix $A \in \mathrm{Env}(\Gamma)$ such that given any $(\mathcal{T}, \mathcal{P}) \in \mathrm{I}(\mathcal{M}, \Phi)$ there's a $(\mathcal{U}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Phi_{\mathcal{T},\mathcal{P}})$ such that $\Phi_{\mathcal{U},\mathcal{Q}}$ doesn't have A-condensation. Applying this inductively, we get a sequences $\langle \mathcal{Q}^0_n, \mathcal{R}^0_n, \mathcal{T}^0_n, \pi^0_n, \sigma^0_n, j^0_n \mid n < \omega\rangle$ such that

(i) $\mathcal{Q}^0_0 := \mathcal{M}$;

(ii) $\pi^0_n : \mathcal{Q}^0_n \to \mathcal{Q}^0_{n+1}$ is the iteration map through a tree of successor length, according to Φ;

$$
\begin{tikzcd}
\mathcal{Q}_0^0 \arrow[rr, "\pi_0^0"] \arrow[dr, "\sigma_0^0"'] \arrow[dd, "\tau_0^0"'] & & \mathcal{Q}_1^0 \arrow[rr, "\pi_1^0"] \arrow[dr, "\sigma_1^0"'] \arrow[dd, "\tau_1^0"'] & & \mathcal{Q}_2^0 \arrow[r, "\pi_2^0"] \arrow[dd, "\tau_2^0"'] & \cdots & \mathcal{Q}_\omega^0 \arrow[dd, "\tau_\omega^0"'] \\
& \mathcal{R}_0^0 \arrow[ur, "j_0^0"'] \arrow[dd, "k_0^0"] & & \mathcal{R}_1^0 \arrow[ur, "j_1^0"'] \arrow[dd, "k_1^0"] & & & \\
\mathcal{Q}_0^1 \arrow[rr, "\pi_0^1"] \arrow[dr, "\sigma_0^1"'] \arrow[dd, "\tau_0^1"'] & & \mathcal{Q}_1^1 \arrow[rr, "\pi_1^1"] \arrow[dr, "\sigma_1^1"'] \arrow[dd, "\tau_1^1"'] & & \mathcal{Q}_2^1 \arrow[r, "\pi_2^1"] \arrow[dd, "\tau_2^1"'] & \cdots & \mathcal{Q}_\omega^1 \arrow[dd, "\tau_\omega^1"'] \\
& \mathcal{R}_0^1 \arrow[ur, "j_0^1"'] \arrow[dd, "k_0^1"] & & \mathcal{R}_1^1 \arrow[ur, "j_1^1"'] \arrow[dd, "k_1^1"] & & & \\
\mathcal{Q}_0^2 \arrow[rr, "\pi_0^2"] \arrow[dr, "\sigma_0^2"'] \arrow[dd, "\tau_0^2"'] & & \mathcal{Q}_1^2 \arrow[rr, "\pi_1^2"] \arrow[dr, "\sigma_1^2"'] \arrow[dd, "\tau_1^2"'] & & \mathcal{Q}_2^2 \arrow[r, "\pi_2^2"] \arrow[dd, "\tau_2^2"'] & \cdots & \mathcal{Q}_\omega^2 \arrow[dd, "\tau_\omega^2"'] \\
& \mathcal{R}_0^2 \arrow[ur, "j_0^2"'] \arrow[dd, "k_0^2"] & & \mathcal{R}_1^2 \arrow[ur, "j_1^2"'] \arrow[dd, "k_1^2"] & & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
& \vdots & & \vdots & & & \\
\mathcal{Q}_0^\omega \arrow[rr, "\pi_0^\omega"] \arrow[dr, "\sigma_0^\omega"'] & & \mathcal{Q}_1^\omega \arrow[rr, "\pi_1^\omega"] \arrow[dr, "\sigma_1^\omega"'] & & \mathcal{Q}_2^\omega \arrow[r, "\pi_2^\omega"] & \cdots & \mathcal{Q}_\omega^\omega \\
& \mathcal{R}_0^\omega \arrow[ur, "j_0^\omega"'] & & \mathcal{R}_1^\omega \arrow[ur, "j_1^\omega"'] & & &
\end{tikzcd}
$$

Figure 5.6: The three-dimensional argument in Theorem 5.2.4

(iii) $\sigma_n^0 : \mathcal{Q}_n^0 \to \mathcal{R}_n^0$ an iteration map through a tree of limit length, according to Φ;

(iv) $j_n^0 : \mathcal{R}_n^0 \to \mathcal{Q}_{n+1}^0$ is elementary such that $\pi_n^0 = j_n^0 \circ \sigma_n^0$;

(v) $(j_n^0)^{-1}(\tau_{A,j_n^0(\kappa)}^{\mathcal{Q}_{n+1}^0}) \neq \tau_{A,\kappa}^{\mathcal{R}_n^0}$ for every $\mathcal{R}_n^0$-cardinal $\kappa \geq \delta_0^{\mathcal{R}_n^0}$.

Let $\mathcal{Q}_\omega^0$ be the direct limit of the $\mathcal{Q}_n^0$'s under the π_n^0 maps. Also let $\langle x_n \mid n < \omega \rangle$ enumerate the reals of $\mathcal{M}_\Gamma$ and pick $s \in [\mathsf{On}]^{<\omega}$ and a formula φ such that

$$\forall x \in \mathbb{R}(x \in A \Leftrightarrow \mathcal{M}_\Gamma \models \varphi[x, s]).$$

Our strategy now is now firstly to capture all the x_n's so that the derived models of the resulting structures become equal to $\mathcal{M}_\Gamma$. See Figure 5.6.

Perform a genericity iteration of $\mathcal{Q}_0^0$ above $\delta_0^{\mathcal{Q}_0^0}$ to $\mathcal{Q}_0^1$ to make x_0 generic over $\mathcal{Q}_0^1$ at $\delta_1^{\mathcal{Q}_0^1}$, while lifting the genericity iteration tree via the copy construction to the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and picking branches on the genericity iteration tree on $\mathcal{Q}_0^0$ by using $\Phi_{\mathcal{Q}_\omega^0}$ on the lifted tree on $\mathcal{Q}_\omega^0$. Let $\tau_0^0 : \mathcal{Q}_0^0 \to \mathcal{Q}_0^1$ be the genericity iteration map and $\mathcal{W}_0$ the last model of the lifted tree on $\mathcal{Q}_\omega^0$.

Now perform another genericity iteration of the last model of the lifted iteration tree on $\mathcal{R}_0^0$ above its δ_0 to $\mathcal{R}_0^1$ to make x_0 generic over $\mathcal{R}_0^1$ at $\delta_1^{\mathcal{R}_0^1}$, with branches being picked by lifting the iteration tree to $\mathcal{W}_0$ and using the branches according to $\Phi_{\mathcal{W}_0}$. Let $k_0^0 : \mathcal{R}_0^0 \to \mathcal{R}_0^1$ be the iteration embedding, $\sigma_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_0^1$ be the shift of σ_0^0 followed by latter genericity iteration, and $\mathcal{W}_1$ the last model of the lifted tree on $\mathcal{W}_0$.

Do a third genericity iteration of the last model of the lifted stack on $\mathcal{Q}_1^0$ above its δ_0 to $\mathcal{Q}_1^1$ to make x_0 generic at $\delta_1^{\mathcal{Q}_1^1}$, with branches being picked by lifting the tree to $\mathcal{W}_1$ and using branches picked by $\Phi_{\mathcal{W}_1}$. Let $\tau_1^0 : \mathcal{Q}_1^0 \to \mathcal{Q}_1^1$ be the iteration embedding, $j_0^1 : \mathcal{Q}_0^1 \to \mathcal{R}_1^1$ be the natural map, and $\pi_0^1 := j_0^1 \circ \sigma_0^1$.

Now continue this process to make x_0 generic over the $\mathcal{Q}_n^0$'s and $\mathcal{R}_n^0$'s, and let $\mathcal{Q}_\omega^1$ be the direct limit of the $\mathcal{Q}_n^1$'s under the π_n^1 maps. Then start at $\mathcal{Q}_0^1$ and repeat the same thing to make x_1 generic at the respective δ_2's and so on. Let $\mathcal{Q}_i^\omega$ be the direct limit of the $\mathcal{Q}_i^n$'s under the τ_i^n maps, $\mathcal{R}_i^\omega$

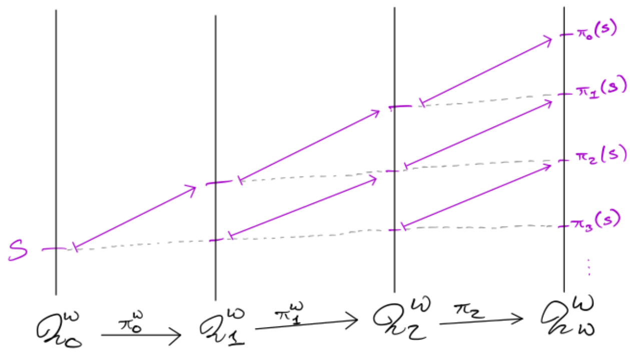

Figure 5.7: The argument in Claim 5.2.4.1.

the direct limit of the $\mathcal{R}_i^n$'s under the k_i^n maps and $\mathcal{Q}_\omega^n$ the direct limit of the $\mathcal{Q}_i^n$'s under the π_i^n maps.

By construction we get that the π_n^0's and τ_ω^n's are all by Φ and its tails, and that $\mathcal{Q}_\omega^\omega$ is wellfounded and Lp^Γ-full, so that the $\mathcal{Q}_n^\omega$'s and the $\mathcal{R}_n^\omega$'s are also wellfounded and Lp^Γ-full.

Claim 5.2.4.1. There exists some $k < \omega$ such that π_n^ω fixes s for every $n \geq k$.

Proof of claim. It suffices to show that $(\pi_n^\omega(\xi) \mid n < \omega)$ is eventually constant for all $\xi \in s$. Suppose this isn't the case. Fix $\xi \in s$ and a strictly increasing sequence $(i_n \mid n < \omega)$ such that $\pi_{i_n}^\omega(\xi) > \xi$ for all $n < \omega$. For $m < n < \omega$ we then have

$$\pi_{i_m,\infty}^\omega(\xi) = \pi_{i_n,\infty}^\omega \circ \pi_{i_m,i_n}^\omega(\xi) \geq \pi_{i_n,\infty}^\omega \circ \pi_{i_m}^\omega(\xi) > \pi_{i_n,\infty}^\omega(\xi),$$

so that $(\pi_{i_n}^\omega(\xi) \mid n < \omega)$ is a strictly decreasing sequence of ordinals in $\mathcal{Q}_\omega^\omega$ – contradicting its wellfoundedness. See Figure 5.7. $\dashv$

Let $k < \omega$ be as in the claim, and note that the j_n^ω's also fix s for $n \geq k$. Since $\mathcal{D}(\mathcal{R}_n^\omega) = \mathcal{M}_\Gamma$ for every $n < \omega$, the $\mathcal{Q}_n^\omega$'s and the $\mathcal{R}_n^\omega$'s have uniform definitions for the term relations for A when $n \geq k$, yielding that j_n^ω pulls back the term relation correctly whenever $n \geq k$, $\lightning$. ■

Theorem 5.2.5 (DI$^+$). $\mathrm{Lp}(\mathbb{R}) \models \ulcorner$*there's a fullness preserving hod pair below* $\omega_1\urcorner$.

Proof.

Show the above requirements in Wilson's theorem is satisfied? Double check the statement.

■

Theorem 5.2.6 (DI$^+$). *There is a model M containing all the reals such that $M \models AD^+ + \theta_0 < \Theta$.*

Proof.

Let $(\mathcal{M}, \Sigma)$ be a fullness preserving hod pair in $\mathrm{Lp}(\mathbb{R})$ given by the above theorem. Then $\Sigma \notin \mathrm{Lp}(\mathbb{R})$ by the proof of 7.4.3 in the CMI book, and in particular $\Sigma \notin \Gamma_0$. Then $M := L(\Sigma, \mathbb{R})$ is the wanted model.

■

5.3 The successor case

Definition 5.3.1. Let $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ be hod pairs below ω_1. We then say that $(\mathcal{Q}, \Lambda)$ **extends** $(\mathcal{P}, \Sigma)$, or is an **extension** of $(\mathcal{P}, \Sigma)$, if there exists some $\alpha < \lambda^{\mathcal{Q}}$ such that

(i) $\mathcal{Q}(\alpha) \in pI(\mathcal{P}, \Sigma)$; and

(ii) $\Sigma_{\mathcal{Q}(\alpha)} = \Lambda_{\mathcal{Q}(\alpha)}$.

We say that $(\mathcal{P}, \Sigma)$ **can be extended** if there exists an extension of $(\mathcal{P}, \Sigma)$.

∘

Theorem 5.3.2 (DI$^+$). *Every hod pair below ω_1 can be extended.*

Rough steps in the proof:

(i) Show that $M_1^{\sharp,\Sigma}$ exists

(ii) $\mathrm{Lp}^{\Sigma}(\mathbb{R}) \models \mathsf{AD}^+$ for some appropriate definition of $\mathrm{Lp}^{\Sigma}(\mathbb{R})$

(iii) The $\Omega > 0$ argument should show that there's an $A \notin \mathrm{Lp}^{\Sigma}(\mathbb{R})$ such that $L(A,\mathbb{R}) \models \mathsf{AD}^+$ and $\Sigma <_W A$

(iv) Show $L(A,\mathbb{R})$ then has the desired $(\mathcal{Q},\Lambda)$ (this step has already been done and can be black boxed)

5.4 The limit case

Theorem 5.4.1 (DI^+). *Assume there exists a sequence of hod pairs $(\mathcal{P}_\alpha, \Sigma_\alpha)$ below ω_1 with $(\mathcal{P}_{\alpha+1}, \Sigma_{\alpha+1})$ extending $(\mathcal{P}_\alpha, \Sigma_\alpha)$ for every α. Then either*

(i) There exists a hod pair $(\mathcal{H},\Lambda)$ below ω_1 such that $\lambda^{\mathcal{H}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$; or

(ii) There exists an $\mathcal{M}$ containing all the reals such that $\mathcal{M} \models \mathsf{AD}_{\mathbb{R}} +$ Θ is regular.

Rough steps in the proof:

(i) Do the easier countable cofinality case

(ii) Coiterate all the hod pairs to some $(\mathcal{P},\Sigma)$, which has $\lambda := \lambda^{\mathcal{P}} = \sup_\alpha \lambda^{\mathcal{P}_\alpha}$

(iii) If λ has non-measurable cofinality then $(\mathcal{P},\Sigma)$ is the hod pair that we're looking for, so assume this is not the case

(iv) Take the derived model $\mathcal{D}(\mathcal{P},\lambda)$, which then satisfies $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}+\Omega=\lambda$, where DC is because λ has uncountable cofinality

This is wrong, as we can't take this derived model. Instead we should form a directed system of all "nice" hod pairs having λ's below $\lambda^{\mathcal{P}}$ and take the Lp-closure of that, which should then be an initial segment of hod; call it $\mathcal{H}$.

(v) Show that $\mathcal{H}\,|\,\delta^{\mathcal{H}}$ is the union of M_∞^α for $\alpha < \lambda$, where M_∞^α is the hod limit of

$$\mathcal{F}_\alpha := \{(\mathcal{Q},\Psi) \mid \mathrm{Ult}(V,g) \models \ulcorner(\mathcal{Q},\Psi) \text{ is a hod pair and } \lambda^{\mathcal{Q}} = \alpha\urcorner\}.^5$$

Let Φ be the join of the strategies of the M_∞^α's and show that $\mathcal{H} = \mathrm{Lp}_\omega^\Phi(\mathcal{H}\,|\,\delta^{\mathcal{H}})$.

(vi) Show that $\mathcal{H} \models \ulcorner\delta^{\mathcal{H}}$ is singular$\urcorner$, since otherwise $\mathcal{D}(\mathcal{H},\delta^{\mathcal{H}}) \models \mathsf{AD}_{\mathbb{R}} +$ Θ is regular and we're done.

(vii) We want to construct a strategy Λ for $\mathcal{H}$ such that $(\mathcal{H}, \Lambda)$ is a hod pair below ω_1, as then this is the hod pair that we're looking for.

Definition 5.4.2. Let $(\mathcal{P}, \Sigma)$ be a hod pair. We let

(i) $\mathrm{I}(\mathcal{P}, \Sigma) := \{(\vec{\mathcal{T}}, \mathcal{Q}) \mid \vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ via Σ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists$\}$ be the collection of $(\mathcal{P}, \Sigma)$**-iterates** ,

(ii) $\mathrm{pI}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$,

(iii) $\mathrm{B}(\mathcal{P}, \Sigma) := \{(\mathcal{T}, \mathcal{M}) \mid \mathcal{M} \lhd_{\mathrm{HOD}} \mathcal{Q}$ and $(\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{I}(\mathcal{P}, \Sigma)\}$ be the collection of $(\mathcal{P}, \Sigma)$**-blowups** and

(iv) $\mathrm{pB}(\mathcal{P}, \Sigma) := \{\mathcal{Q} \mid (\vec{\mathcal{T}}, \mathcal{Q}) \in \mathrm{B}(\mathcal{P}, \Sigma)$ for some $\vec{\mathcal{T}}\}$.

$\circ$

Definition 5.4.3. Let $(\mathcal{P}, \Sigma)$ be a hod pair and Γ is a pointclass closed under Boolean operations and continuous images and preimages. Then Σ is Γ**-fullness preserving** if for all $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ and $\delta_\alpha^{\mathcal{Q}} < \eta$ which is a strong cutpoint of $\mathcal{Q}(\alpha + 1)$ we have

(i) $\mathcal{Q}|\eta^{+\mathcal{Q}(\alpha+1)} = \mathrm{Lp}^{\Gamma, \Sigma_{\mathcal{Q}(\alpha)}, \vec{\mathcal{T}}}(\mathcal{Q}|\eta)$ and

(ii) $\mathcal{Q}|\delta_\alpha^{+\mathcal{Q}} = \mathrm{Lp}^{\Gamma, \oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1)}, \vec{\mathcal{T}}}(\mathcal{Q}(\alpha))$.

Σ is **fullness preserving** iff it is $\mathcal{P}(\mathbb{R})$-fullness preserving.

Provide a motivation for this definition.

$\circ$

Lemma 5.4.4. *Let M, N be transitive models of ZFC$^-$ with largest cardinals δ^M, δ^N respectively. Let $\pi\colon M \to N$ be an elementary embedding, $\kappa := \mathrm{crit}(\pi)$ and let E be the long (κ, δ^N)-extender derived from π. Then $N = \mathrm{Ult}(M; E)$ and $\pi = \pi_E$ is the canonical ultrapower embedding.*

This will be useful in the proof of the A-condensing lemma.

Proof. We have the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\pi_E"'] & N \\ & \mathrm{Ult}(M; E) \arrow[u, "k"'] \end{tikzcd}$$

$$
\begin{tikzcd}
 & j(\mathcal{H}) \\
\mathcal{H} \arrow[ur, "j"] \arrow[r, "\pi"'] & \mathcal{R} \arrow[u, "\tau"']
\end{tikzcd}
$$

Figure 5.8: Full Factors Property

where k satisfies $k \restriction \delta^N = \mathrm{id}$. Let $\delta^{\mathrm{Ult}(M;E)}$ be the largest cardinal of $\mathrm{Ult}(M;E)$. By elementarity $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^N$, so that $\delta^{\mathrm{Ult}(M;E)} \leq \delta^N$. If $\delta^{\mathrm{Ult}(M;E)} < \delta^N$, then $k \restriction \delta^N = \mathrm{id}$ yields $k(\delta^{\mathrm{Ult}(M;E)}) = \delta^{\mathrm{Ult}(M;E)} < \delta^N$, which is absurd. Hence $\delta^{\mathrm{Ult}(M;E)} = \delta^N$ and $k \restriction (\delta^{\mathrm{Ult}(M;E)} + 1) = \mathrm{id}$. Since $\delta^{\mathrm{Ult}(M;E)}$ is the largest cardinal of $\mathrm{Ult}(M;E)$, it follows that k doesn't have a critical point. Therefore $k = \mathrm{id}$, $N = \mathrm{Ult}(M;E)$ and $\pi = \pi_E$. ■

Lemma 5.4.5. *$j \restriction \mathcal{H}$ has the **full factors property**[6], meaning that whenever $\mathcal{R}$*

R has to be countable in $V[g]$. How can we ensure that, as this only gives that it has size $\leq \aleph_1$? Do we have to resort to the (long) claim in Grigor's uB paper?

is a hod premouse and there are elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$, then $\mathcal{R}$ is $\Sigma^2_1(j(\Omega)^\tau)$-full.

Proof. Let $\Psi := j(\Omega)^\tau$ and assume the lemma fails, meaning that we have a hod mouse $\mathcal{R}$ and elementary embeddings $\pi : \mathcal{H} \to \mathcal{R}$ and $\tau : \mathcal{R} \to j(\mathcal{H})$ such that $j \restriction \mathcal{H} = \tau \circ \pi$ and $\mathcal{R} \neq \mathrm{Lp}^\Psi_\omega(\mathcal{R} | \delta^{\mathcal{R}})$, witnessed without loss of generality by an $\mathcal{M} \trianglelefteq \mathrm{Lp}^\Psi(\mathcal{R} | \delta^{\mathcal{R}})$ such that $\rho(\mathcal{M}) = \delta^{\mathcal{R}}$ and which is not an initial segment of $\mathcal{R}$.

$$
\begin{tikzcd}
(\mathcal{H}, \Omega) \arrow[r, "\pi"] \arrow[dr, "j \restriction \mathcal{H}"'] & (\mathcal{R}, \Psi) \arrow[d, "\tau"] \\
 & (j(\mathcal{H}), j(\Omega))
\end{tikzcd}
$$

[6] This terminology was introduced in [Wilson, 2012]; in [Sargsyan, 2015a] this was called *weak condensation*.

We can then fix some hod pair $(\mathcal{S}^*, \Lambda^*)$ such that $\tau``\mathcal{R}|\delta^{\mathcal{R}} \subseteq \operatorname{ran}(\pi^{\Lambda^*}_{\mathcal{S}^*,\infty})$, and furthermore let $\xi \leq \lambda^{\mathcal{S}^*}$ be least such that $\tau``\mathcal{R}|\delta^{\mathcal{R}} \subseteq \operatorname{ran}(\pi^{\Lambda^*}_{\mathcal{S}^*(\xi),\infty})$. Lastly let $(\mathcal{S}, \Lambda)$ be an extension of $(\mathcal{S}^*, \Lambda^*)$ such that $\lambda^{\mathcal{S}}$ is a limit ordinal.

Argue why $\mathcal{S}^*$ and $\mathcal{S}$ exist; we should be in the limit case to argue that $\mathcal{S}$ exists.

Let $\sigma : \mathcal{R}|\delta^{\mathcal{R}} \to \mathcal{S}|\delta^{\mathcal{S}}_{\gamma}$, where $\mathcal{S}^*(\xi)$ iterates to $\mathcal{S}(\gamma)$, be given by $\sigma(x) = y$ iff $\tau(x) = \pi^{\Lambda}_{\mathcal{S}(\gamma),\infty}(y)$.

$$\begin{tikzcd}
(\mathcal{R}|\delta^{\mathcal{R}}, \Psi) \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & (j(\mathcal{H})|\delta^{j(\mathcal{H})}, j(\Omega)) \\
& (\mathcal{S}|\delta^{\mathcal{S}}_{\gamma}, \bigoplus_{\beta<\gamma} \Lambda_{\mathcal{S}(\beta)}) \arrow[ur, "\pi^{\Lambda}_{\mathcal{S}(\gamma),\infty}"'] &
\end{tikzcd}$$

We can fix some hod pair $(\mathcal{S}', \Lambda')$ such that

$$L(\Lambda', \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner.$$

This should follow from generation of good pointclasses.

By coiterating $\mathcal{S}$ and $\mathcal{S}'$ we may assume without loss of generality that $\mathcal{S} = \mathcal{S}'$.

This requires us to work in an AD^+ model, so we better assume that somewhere.

Claim 5.4.5.1. There exists a hod pair $(\mathcal{Q}, \Phi)$ such that $\lambda^{\mathcal{Q}}$ is a limit ordinal and $L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse} \urcorner$.

Proof of claim.

This claim shouldn't be needed, as we should be able to take $\mathcal{Q}$ to be $\mathcal{S}$ in our case, using facts about the Γ-pointclasses. Also ensure that $\mathcal{Q} \supseteq \mathcal{H}$, which is possible as we're stretching by j.

$\dashv$

Fix $(\mathcal{Q}, \Phi)$ as in the claim and let $\mathcal{N}$ be some mouse such that $\mathcal{M} \lhd \mathcal{N}$ and $\mathcal{N}$ has ω many Woodins on top of $\mathcal{M}$.

Explain how this is done. In Grigor's paper he's using that "$j(\eta)$ is closed under hybrid $\mathcal{N}_\omega$-operators". In our measurable cofinality case there might be enough room to get this. Postpone until later, when we have an idea of how much operator closure we have at this point.

Then we get that

Why is $\Gamma(\mathcal{Q}, \Phi)$ in $\mathcal{D}(\mathcal{N})$?

$$\mathcal{D}(\mathcal{N}) \models \ulcorner L(\Gamma(\mathcal{Q}, \Phi), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner\urcorner.$$

In $V[g]$, I guess.

Now throw everything in sight into a countable hull, so that

$$\mathcal{D}(\overline{\mathcal{N}}) \models \ulcorner L(\Gamma(\overline{\mathcal{Q}}, \overline{\Phi}), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \overline{\Psi}\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner\urcorner.$$

I think that now $\overline{\mathcal{Q}}$ are taking the role of "$L[\mathcal{T}, \mathcal{H}]$", as Grigor's paper seems to indicate that $\mathcal{H} \subseteq \overline{\mathcal{Q}}$.

Now lift π to the ultrapower map π^+ given by the $(\delta^{\mathcal{H}}, \delta^{\mathcal{R}})$-extender over $\overline{\mathcal{Q}}$ derived from π, and let $\mathcal{R}^+$ be the ultrapower. Lift also σ, τ to corresponding σ^+, τ^+.

A it hand-wavy.

$$\begin{tikzcd}
(\overline{\mathcal{Q}}, \overline{\Phi}) \arrow[r, "\pi^+"] \arrow[dr, "\sigma^+"'] & (\mathcal{R}^+, \Phi^{**}) \arrow[d, "\tau^+"] \\
& (j(\overline{\mathcal{Q}}), \Phi^*)
\end{tikzcd}$$

Let now $\Phi^* := j(\overline{\Phi})$ and $\Phi^{**} := (\Phi^*)^{\tau^+}$, which is then a strategy for $\mathcal{R}^+$. Since $\overline{\Phi} = (\Phi^{**})^{\pi^+}$ we get that

Check this — might be by definition of pullback consistency, which is implied by hull condensation.

In Grigor's uB paper he uses a certain derived model C instead of $\mathcal{D}(\overline{\mathcal{N}})$, but I can't see how they're different from each other. Also, figure out why the following inclusion is true (it's probably folklore).

$$\mathcal{D}(\overline{\mathcal{N}}) \subseteq \mathcal{D}(\mathcal{R}^+, \Phi^{**}),$$

Note sure what's going on here.

implying that

$$L(\Gamma(\mathcal{R}^+, \Phi^{**}), \mathbb{R}) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R}\urcorner.$$

Why's that?

Because $\mathcal{R}^+$ is a Ψ-mouse over $\mathcal{R}|\delta^{\mathcal{R}}$, it follows that

$$\mathcal{D}(\mathcal{R}^+) \models \ulcorner \mathcal{M} \text{ is a } \Psi\text{-mouse which isn't an initial segment of } \mathcal{R} \urcorner,$$

which then implies that $\mathcal{M} \in \mathcal{R}^+$, so that $\mathcal{M} \trianglelefteq \mathcal{R}$, a contradiction. ■

I don't see how this last argument works.

Definition 5.4.6. For every $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ define $Q_X := \mathrm{cHull}^{j(\mathcal{H})}(X)$ and let

$$\tau_X \colon Q_X \to j(\mathcal{H})$$

be the uncollapse.

Say that $Y \in \mathcal{P}_{\omega_1}(j(\mathcal{H})|\delta^{j(\mathcal{H})})$ **extends** X if $X \cap j(\delta^{\mathcal{H}}) \subseteq Y$ and in that case let

(i) $\tau_{X,Y} := \tau_{X\cup Y}$,
(ii) $\Phi_{X,Y} := j(\Phi)^{\tau_{X,Y}}$,
(iii) $Q_{X,Y} := Q_{X\cup Y}$ and
(iv) $\pi_{X,Y} \colon Q_X \to Q_{X,Y}$ is the induced embedding given by

$$\pi_{X,Y}(x) = \tau_Y^{-1}(\tau_X(x)).$$

Furthermore define $T_X(A)$ for $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$ as

$$\begin{aligned} T_X(A) :&= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } Q_X \models \varphi[s, A]\} \\ &= \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_X}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\tau_X(s), \tau_X(A)]\} \end{aligned}$$

and let $T_{X,Y}(A)$ be given as

$$\begin{aligned} T_{X,Y}(A) := \{(\varphi, s) \mid \varphi \text{ is a formula, } s \in [\delta^{Q_{X,Y}}]^{<\omega} \text{ and } j(\mathcal{H}) \models \varphi[\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty}(s), \tau_X(A)], \\ \text{where } \alpha \text{ is least such that } s \in [\delta_\alpha^{Q_{X,Y}}]^{<\omega}\}. \end{aligned}$$

Here $\pi^{\Phi_{X,Y}}_{Q_{X,Y}(\alpha),\infty} \colon Q_{X,Y} \to j(\mathcal{H})|\nu_{X,Y}$

What is $\nu_{X,Y}$?

is given by

Missing! (It will be the iteration into an appropriate level of the directed system leading up to $j(\mathcal{H})$ followed by the direct limit embedding into some initial segment of $j(\mathcal{H})$)

$\circ$

Definition 5.4.7. Let $X \in \mathcal{P}_{\omega_1}(j(\mathcal{H}))$ and $A \in Q_X \cap \mathcal{P}(\delta^{Q_X})$. Then X is **A-condensing** if $\pi_{X,Y}(T_X(A)) = T_{X,Y}(A)$ for every Y extending X.
We say that X is **condensing** if X is A-condensing for all such A.

We want to show that $j``\mathcal{H}$ is condensing. We first show that it suffices to show that it's α-condensing for every $\alpha < \delta^{\mathcal{H}}$.

Lemma 5.4.8. *If $j``\mathcal{H}$ is $\alpha$-condensing for every $\alpha < \delta^{\mathcal{H}}$ then $j``\mathcal{H}$ is condensing.*

Proof.

Missing!

■

Theorem 5.4.9. *For every $\alpha < \delta^{\mathcal{H}}$ there exists an extension Y of $j``\mathcal{H}$ such that $j``\mathcal{H} \cup Y$ is α-condensing.*

Reduce this to $j``\mathcal{H}$ somehow?

Proof.

update notation

Set $X := j``\mathcal{H}$ and assume the theorem fails. Fix some $\alpha < \delta^{\mathcal{H}}$ such that X is not α-condensing. Fix some Y_0 extending X which witnesses this, meaning that $\pi^X_{Y_0}(T^X_\alpha) \neq T^{X,Y_0}_\alpha$. Since we're also assuming that $\pi^X_{Y_0}$ isn't α-condensing we can find Y_1 extending Y_0 such that $\pi^{Y_0}_{Y_1}(T^{Y_0}_\alpha) \neq T^{Y_0,Y_1}_\alpha$. Continue doing this, generating a sequence $\langle Y_n \mid n < \omega\rangle$ with Y_{n+1} extending Y_n and

$$\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) \neq T^{Y_n,Y_{n+1}}_\alpha \tag{1}$$

for all $n < \omega$. Let $\mathcal{P}_n := Q^X_{Y_n}$, $\pi_{m,n} := \pi^{Y_m}_{Y_n}$ and $\pi_n := \pi_{0,n}$. We want to show that such a sequence can't exist. Towards getting a contradiction we first need to make everything in sight countable, as that will allow us to reason using derived models (the problem is that $j(\mathcal{H})$ is too big, namely it has size $\aleph_1^{V[g]}$).

Using that $\delta^{j(\mathcal{H})}$ has uncountable cofinality we can find $\kappa < \delta^{j(\mathcal{H})}$ such that

$$\kappa = \mathrm{Hull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\operatorname{ran} \tau^X_{Y_n} \mid n < \omega\}) \cap \delta^{j(\mathcal{H})}.$$

Set $\mathcal{M} := \mathrm{cHull}^{j(\mathcal{H})}(\kappa \cup X \cup \{\operatorname{ran} \tau^X_{Y_n} \mid n < \omega\})$ and note that $\mathcal{M} = j(\mathcal{H})|\kappa^{+j(\mathcal{H})}$. Let $\pi : \mathcal{M} \to j(\mathcal{H})$ be the uncollapse and note that $\operatorname{crit} \pi = \kappa$ and that $\kappa = \delta^{\mathcal{M}}$. Define $\iota : \mathcal{H} \to \mathcal{M}$ as $\iota := \pi^{-1} \circ j$ and $\tau_n : \mathcal{P}_n \to \mathcal{M}$ as $\tau_n := \pi^{-1} \circ \tau^X_{Y_n}$. Note that $\mathcal{M}$ is countable in $V[g]$ and is hence an element of $\mathrm{Ult}(V, g)$.

Missing argument

Now define $\mathcal{H}^+$ as the hod limit of iterates of $\mathcal{H}$, so that $\mathcal{H}^+$ is a hod premouse with $\mathcal{H} \lhd_{\mathrm{hod}} \mathcal{H}^+$, $\mathcal{H}^+$ has a strategy Ψ extending Ω such that

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{H}^+, \Psi).$$

Provide more details.

We probably need $\mathcal{H}^+$ to be countable here, so we should probably apply the induced ideal and work in $V[g][h]$.

Also define $(\mathcal{P}^+_n, \Psi_n)$ as $P^+_n := \mathrm{Ult}(\mathcal{H}^+, E_{\pi_n})$, so that we also get that

$$(\{B \subseteq \mathbb{R} \mid w(B) < \kappa\})^{j(\mathcal{M})} \subseteq \mathcal{D}(\mathcal{P}^+_n, \Psi_n).$$

Missing argument. This might need that $\mathcal{H}^+, \Psi \restriction V \in V$, but we could probably also just work inside $\mathrm{Ult}(V, g)$, or the second ultrapower, all along.

Now $\mathcal{D}(\mathcal{P}^+_n, \Psi_n)$ has a definition of T^{X,Y_n}_α, so that $\pi^{Y_n}_{Y_{n+1}}(T^{Y_n}_\alpha) = T^{Y_n,Y_{n+1}}_{\pi_{n,n+1}(\alpha)}$. The three-dimensional argument then shows that α must be fixed by $\pi_{n,n+1}$ for some $n < \omega$, so that $X \cup Y_n$ *is* α-condensing, ↯. ■

What is meant by this?

Show this.

Define the strategy Λ for $\mathcal{H}$ and show that $(\mathcal{H}, \Lambda)$ is a hod pair.

6 | The Forcing Direction

Have a look at Trevor's thesis; he's doing something similar.

In this section we will prove the following unpublished theorem by Woodin.

Theorem 6.0.1 (Woodin)**.** *Assume* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular. Then there is a generic extension of* V *satisfying* DI^+*.*

Assume thus that $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Missing proof.

7 Further questions

7.1 Section

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

A | Preliminaries

A.1 Descriptive set theory

Convention A.1.1. We will be using the "logician's reals", meaning that $\mathbb{R} := {}^{\omega}\omega$ with the product topology, having the sets $\{x \in \mathbb{R} \mid x \supseteq s\}$ for $s \in {}^{<\omega}\omega$ as a clopen basis.

Definition A.1.2. Let $A, B \subseteq \mathbb{R}$. We say that A *is Wadge reducible to* B (in symbols $A \leq_W B$) iff there is some continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$A = f^{-1}[B] := \{a \in A \mid f(a) \in B\}.$$

We write $A <_W B$ iff $A \leq_W B$ and $B \not\leq_W A$. ○

Remark A.1.3. $\mathbb{R}$ and $\mathbb{R}^n$, for $1 \leq n \leq \omega$ are homeomorphic and we shall often identify them with one another.

Definition A.1.4. Subseteq of $\mathcal{R}$ are called *pointsets* . Subsets of $\mathcal{P}(\mathbb{R})$ are called *pointclasses* .

○

Definition A.1.5. Let Γ be a pointclass. We define

(i) $\exists^{\mathbb{R}}\Gamma := \{A \mid \exists B \in \Gamma \colon A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(x, y) \in B\}\}$,

(ii) $\forall^{\mathbb{R}}\Gamma := \{A \mid \exists B \in \Gamma \colon A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R}(x, y) \in B\}\}$.

○

Definition A.1.6. Let Γ be a pointclass. We define

(i) $\breve{\Gamma} := \{\mathbb{R} \setminus A \mid A \in \Gamma\}$,

(ii) $\Delta_\Gamma := \Gamma \cap \breve{\Gamma}$ and

(iii) $\underset{\sim}{\Gamma} := \exists^{\mathbb{R}}\Gamma$.

○

Lemma A.1.7 ([Wadge, 1972])**.** *Assume* $\mathsf{ZF}+\mathsf{AD}$ *and let* $A, B \subseteq \mathbb{R}$*. Then*

$$A \leq_W B \text{ or } B \leq_W \mathbb{R} \setminus A.$$

Lemma A.1.8 (Martin-Monk-Wadge)**.** *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*. Then* $\leq_W$ *is wellfounded.*[1]

Remark A.1.9. When considering $(\mathcal{P}(\mathbb{R}); \leq_W)$ in a $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ context, we will often tacitly identify $A \subseteq \mathbb{R}$ with its complement, making $<_W$ a wellorder.

Definition A.1.10 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$)**.** Let $A \subseteq \mathbb{R}$. Then the *Wadge rank of* A is defined recursively as $|A|_W := \sup\{|B|_W + 1 \mid B <_W A\}$. ∘

Definition A.1.11. Let X be a set. We write OD_X for the collection of all A for which there is some formula ϕ, ordinals $\alpha_0, \ldots, \alpha_k$ and $x_0, \ldots, x_l \in X$ with

$$A = \{a \mid \phi[a, \alpha_0, \ldots, \alpha_k, x_0, \ldots, x_l]\}. \quad \dashv$$

We write HOD_X for the collection of all A such that $\mathrm{trcl}(\{A\}) \subseteq \mathrm{OD}_X$. If $X = \emptyset$, we will often drop the subscript and simply write OD and HOD for $\mathrm{OD}_\emptyset$ and $\mathrm{HOD}_\emptyset$ respectively.

Definition A.1.12 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$)**.** For $B \subseteq \mathbb{R}$ let

$$\begin{aligned}\theta_B &:= \sup\{|A|_W \mid \exists x \in \mathbb{R}\colon A \in \mathrm{OD}_{\mathbb{R}\cup\{B\}}\} \\ &= \sup\{\alpha \in \mathsf{On} \mid \text{ there is a } \mathrm{OD}_{\mathbb{R}\cup\{B\}}\text{-surjection } f\colon \mathbb{R} \to \alpha\}.\end{aligned}$$

∘

Verify that these two values are in fact identical.

Definition A.1.13 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$)**.** Define the *Solovay sequence* $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ as follows:

(i) $\theta_0 := \theta_\emptyset$,

(ii) if there is some B such that $|B|_W = \theta_\alpha$ let $\theta_{\alpha+1} := \theta_B$.[2]

[1] See [Larson, 2018] for a proof.

[2] Since continuous functions are coded by reals, this is independent of the choice of B.

(iii) if α is a limit ordinal, we let $\theta_\alpha := \sup_{\beta<\alpha} \theta_\beta$.

Finally, Ω is the least ordinal such that $\theta_{\alpha+1} = \theta_\alpha$, and $\Theta := \theta_\Omega$. ∘

A.2 Ideals

Definition A.2.1. Let I be an ideal on a nonempty set Z. Let

(i) $I^+ := \mathcal{P}(Z) \setminus I$,

(ii) for $a, b \in I^+$ let $a \sim_I b$ iff $a \Delta b \in I$,

(iii) $\mathcal{P}(Z)/I := \mathcal{P}(Z)/{\sim_I}$ is the Boolean algebra with subset inclusion modulo $\sim_I$.

We call $\mathcal{P}(Z)/I$ the *associated forcing* to I. ∘

Definition A.2.2. If I is an ideal on a cardinal κ and $g \subseteq \mathcal{P}(\kappa)/I$ V-generic then g is a V-ultrafilter on κ in $V[g]$, so that we may take the *generic ultrapower* $\mathrm{Ult}(V, g)$. ∘

Proposition A.2.3. *Let I be an ideal on a cardinal κ. Then..*

(i) if I is κ-complete then so is any generic ultrafilter;

(ii) if I is normal then so is any generic ultrafilter. ⊣

define κ-complete and normal for an ideal. Add reference.

Definition A.2.4. Let λ be any cardinal. Then an ideal I on a cardinal κ is...

- *precipitous* if the generic ultrapower is wellfounded;
- *λ-saturated* if the associated forcing has the λ-chain condition;
- *λ-dense* if the associated forcing has a dense subset of size λ. ⊣

Note that λ-dense trivially implies λ^+-saturated. We'll need the following facts about ω_2-saturated ideals on ω_1:

Parts (ii)-(v) is Example 4.29 in Foreman's handbook chapter. Perhaps include the proof.

Proposition A.2.5. *Let I be an ω_2-saturated ideal on ω_1. Then I is precipitous, and letting $j\colon V \to M$ be the generic ultrapower map, it holds that*

(i) M is closed under ω-sequences in $V[g]$;

(ii) $j(\omega_1^V) = \omega_2^V = \omega_1^{V[g]}$;

(iii) $j(\omega_2^V) \in (\omega_2^V, \omega_3^V)$;

(iv) j *is continuous at* ω_2^V;

(v) $j(\omega_n^V) = \omega_n^V = \omega_{n-1}^{V[g]}$ *for all* $n \in [3, \omega]$. $\dashv$

As for the density, we in particular need the following fact:

Proposition A.2.6. *Let* I *be an* ω_1*-dense ideal on* ω_1*. Then the associated forcing is forcing equivalent to* $\mathrm{Col}(\omega, \omega_1)$*, so in particular it's homogeneous.*
■

PROOF. [Kanamori, 2008, Proposition 10.20]. ■

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