1 The Internal CMI

Introduction

1.1 Operators and hybrid mice

Define model operator, (hybrid) mouse operator, mouse reflection, condenses finely/well, determines itself on generic extensions, relativises well...

1.2 The hybrid core model dichotomy

Lemma 1.2.1. *Let θ be a regular uncountable cardinal or $\theta = \infty$ and let $\mathcal{N}$ be a tame hybrid mouse operator on H_θ which relativises well. Then $\mathcal{N}$ is countably iterable iff it's (θ, θ)-iterable, guided by $\mathcal{N}$. Furthermore, for every $x \in H_\theta$, if $M_1^{\mathcal{N}}(x)$ exists and is countably iterable, then it's also (θ, θ)-iterable, guided by $\mathcal{N}$.*

Define this

Change this to model operators; perhaps change parts of the proof and/or assumptions needed.

Proof. Fix $x \in H_\theta$. We first show that $\mathcal{N}(x)$ is (θ, θ)-iterable. Let $\mathcal{T} \in H_\theta$ be a normal tree of limit length on $\mathcal{N}(x)$. Let $\eta \gg \operatorname{rk}(\mathcal{T})$ and let

$$\mathcal{H} := \operatorname{cHull}^{H_\eta}(\{x, \mathcal{N}(x), \mathcal{T}\})$$

with uncollapse $\pi \colon \mathcal{H} \to H_\eta$. Set $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran} \pi$. Note that $\overline{\mathcal{N}(x)} = \mathcal{N}(\overline{x})$ since $\mathcal{N}$ relativises well. Now $\overline{\mathcal{T}}$ is a normal, countable iteration tree on $\mathcal{N}(\overline{x})$ and hence our iteration strategy yields a wellfounded cofinal branch $\overline{b} \in V$ for $\overline{\mathcal{T}}$. Note that $\overline{\mathcal{Q}} := \mathcal{Q}(\overline{b}, \overline{\mathcal{T}})$ exists, since if $\overline{b}$ drops then there's nothing to do, and otherwise we have that

$$\rho_1(\mathcal{M}_{\overline{b}}^{\overline{\mathcal{T}}}) = \rho_1(\mathcal{N}(\overline{x})) = \operatorname{rk} \overline{x} < \delta(\overline{\mathcal{T}}),$$

so $\delta(\overline{\mathcal{T}})$ is not definably Woodin over $\mathcal{M}_{\overline{b}}^{\overline{\mathcal{T}}}$.

Why is that?

Claim 1.2.1.1. $\overline{\mathcal{Q}} \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$

Proof of claim. If $\overline{\mathcal{Q}} = \mathcal{M}(\overline{\mathcal{T}})$ then the claim is trivial, so assume that $\mathcal{M}(\overline{\mathcal{T}}) \lhd \overline{\mathcal{Q}}$. Note that $\overline{\mathcal{Q}} \trianglelefteq M_b^{\overline{\mathcal{T}}}$ by definition of $\mathcal{Q}$-structures, and that $M_b^{\overline{\mathcal{T}}}$ satisfies (2) of the definition of relativises well, meaning that

Define this, cut-point and $\mathcal{M}_b^{\mathcal{T}}$

$$M_b^{\overline{\mathcal{T}}} \models \ulcorner \forall \eta \forall \zeta > \eta : \text{if } \eta \text{ is a cutpoint then } M_b^{\overline{\mathcal{T}}} | \zeta \not\models \varphi_{\mathcal{N}}[\bar{x}, p_{\mathcal{N}}] \urcorner. \tag{1}$$

This statement is Π^1_2 and $\overline{\mathcal{Q}}$ is Π^1_2-correct since it contains a Woodin cardinal, so that $\mathcal{Q}$ satisfies the statement as well. Since $\mathcal{N}$ is tame we get that $\delta(\overline{\mathcal{T}})$ is a cutpoint of $\overline{\mathcal{Q}}$, so that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ is *not* a proper initial segment of $\overline{\mathcal{Q}}$. Further, as we're assuming that both $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\mathcal{M}_b^{\overline{\mathcal{T}}}$ are (ω_1+1)-iterable above $\delta(\overline{\mathcal{T}})$ the same thing holds for $\overline{\mathcal{Q}} \trianglelefteq \mathcal{M}_b^{\overline{\mathcal{T}}}$, so that we can compare $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ with $\overline{\mathcal{Q}}$ (in V). Let

$$(\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})), \overline{\mathcal{Q}}) \rightsquigarrow (\mathcal{P}, \mathcal{R})$$

be the result of the coiteration. We claim that $\mathcal{R} \trianglelefteq \mathcal{P}$. Suppose $\mathcal{P} \lhd \mathcal{R}$. Then there is no drop in $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ and in fact $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P}$ since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ projects to $\delta(\overline{\mathcal{T}})$. Furthermore, as we established that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{N}(\overline{\mathcal{Q}}|\delta(\overline{\mathcal{T}}))$ isn't a proper initial segment of $\overline{\mathcal{Q}}$ it can't be a proper initial segment of $\mathcal{R}$ either, as the coiteration is above $\delta(\overline{\mathcal{T}})$. But we're assuming that $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) = \mathcal{P} \lhd \mathcal{R}$, a contradiction. So $\mathcal{R} \trianglelefteq \mathcal{P}$.

Since $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ and $\overline{\mathcal{Q}}$ agree up to $\delta(\overline{\mathcal{T}})$ and there is no drop $\overline{\mathcal{Q}} \rightsquigarrow \mathcal{R}$ we have that $\overline{\mathcal{Q}} = \mathcal{R}$. If $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}})) \rightsquigarrow \mathcal{P}$ doesn't move either we're done, so assume not. Let F be the first exit extender of $\mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ in the coiteration. We have $\operatorname{lh}(F) \leq o(\overline{\mathcal{Q}})$, $\overline{\mathcal{Q}} \trianglelefteq \mathcal{P}$ and $\operatorname{lh}(F)$ is a cardinal in $\mathcal{P}$.

Define this

As $\overline{\mathcal{Q}}$ is $\delta(\overline{\mathcal{T}})$-sound and projects to $\delta(\overline{\mathcal{T}})$ it follows that $J(\overline{\mathcal{Q}}|\operatorname{lh}(F))$ collapses $\operatorname{lh}(F)$, so it has to be the case that $\overline{\mathcal{Q}}|\operatorname{lh}(F) = \mathcal{P}$ and thus $o(\mathcal{P}) = \operatorname{lh}(F)$. But this means that $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ even though we assumed that $\mathcal{N}(\mathcal{M}(\mathcal{T})) \rightsquigarrow \mathcal{P}$ moved, a contradiction. $\dashv$

Now, in a sufficiently large collapsing extension extension of $\mathcal{H}$, $\bar{b}$ is the unique cofinal, wellfounded branch of $\overline{\mathcal{T}}$ such that $\mathcal{Q}(\bar{b}, \overline{\mathcal{T}}) \trianglelefteq \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$ exists. Hence, by the homogeneity of $\mathrm{Col}(\omega, \theta)$, $\bar{b} \in H$. By elementarity there is a unique cofinal, wellfounded branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$. This proves that M is (uniquely) On-iterable and virtually the same argument yields the iterability of M via successor-many stacks of normal trees.

To show that M is fully iterable, it remains to be seen that the unique iteration strategy (guided by $\mathcal{N}$) of M outlined above leads to wellfounded direct limits for stacks of normal trees on M of limit length. Let λ be a limit ordinal and $\vec{\mathcal{T}} = (\mathcal{T}_i \mid i < \lambda)$ a stack according to our iteration strategy. Suppose $\lim_{i<\lambda} \mathcal{M}_\infty^{\mathcal{T}_i}$ is illfounded.

Redefine $\eta \gg \mathrm{rk}(\vec{\mathcal{T}})$, $\mathcal{H} := \mathrm{cHull}^{H_\eta}(\{x, M, \vec{\mathcal{T}}\})$ and $\pi : \mathcal{H} \to H_\eta$ the uncollapse, again with $\overline{a} := \pi^{-1}(a)$ for every $a \in \operatorname{ran} \pi$. By elementarity we get that $\mathcal{H} \models \ulcorner \lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i}$ is illfounded$\urcorner$. But $\overline{\vec{\mathcal{T}}}$ is countable and according to the iteration strategy guided by $\mathcal{N}$, so that

$$V \models \ulcorner \lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i} \text{ is wellfounded.}\urcorner$$

Now note that $(\lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^{\mathcal{H}} = (\lim_{i<\overline{\lambda}} \mathcal{M}_\infty^{\overline{\mathcal{T}}_i})^V$ and wellfoundedness is absolute between $\mathcal{H}$ and V, a contradiction.

Now assume that $M_1^{\mathcal{N}}(x)$ exists for some $x \in H_\theta$, and that it's countably iterable. We then do exactly the same thing as with $\mathcal{N}(x)$ *except* that in the claim we replace (1) with

$$\overline{\mathcal{Q}} \models \forall \eta (\overline{\mathcal{Q}} | \eta \not\models \ulcorner \delta(\overline{\mathcal{T}}) \text{ is not Woodin} \urcorner),$$

so that if $\mathcal{P} \triangleleft \mathcal{R}$ then $\delta(\overline{\mathcal{T}})$ is still Woodin in $\mathcal{P} = \mathcal{N}(\mathcal{M}(\overline{\mathcal{T}}))$, contradicting the defining property of $M_1^{\mathcal{N}}(x)$ (and thus also of $\mathcal{R}$). The rest of the proof is a copy of the above. ■

Theorem 1.2.2 (Hybrid core model dichotomy)**.** *Let θ be a $\beth$-fixed point or $\theta = \infty$, and let F be a tame model operator on H_θ that condenses well. Let $x \in H_\theta$. Then either:*

I don't think tame is needed here, as we're only indexing extenders at F-initial segments. -Dan

(i) *The core model $K^F(x)|\theta$ exists and is (θ,θ)-iterable; or*

(ii) *$M_1^F(x)$ exists and is (θ,θ)-iterable.*

Proof. Assume first that $K^{c,F}(x)|\theta$ reaches a premouse which isn't F-small; let $\mathcal{N}_\xi$ be the first part of the construction witnessing this. Then $\mathfrak{C}(\mathcal{N}_\xi) = M_1^F(x)$, and by Lemma 1.2.1 it suffices to show that $M_1^F(x)$ is countably iterable.

Insert argument?

Show that $M_1^F(x)$ is countably iterable.

We can thus assume that $K^{c,F}(x)|\theta$ is F-small. Note that if $K^{c,F}(x)|\theta$ has a Woodin cardinal then because the model is F-closed we contradict F-smallness, so the model has no Woodin cardinals either, making it (θ,θ)-iterable.

Let $\kappa < \theta$ be any uncountable cardinal and let $\Omega := \beth_\kappa(\kappa)^+$. Note that $\Omega < \theta$ since we assumed that θ is a $\beth$-fixed point and $\kappa < \theta$. If Ω is a limit cardinal in $K^{c,F}(x)|\theta$ then let $\mathcal{S} := \mathrm{Lp}(K^{c,F}(x)|\Omega)$ and otherwise let $\mathcal{S} := K^{c,F}(x)|\Omega$. Then by Lemma 3.3 of [?] we get that $\mathcal{S}$ is countably iterable, with largest cardinal Ω in the "limit cardinal case".

This also means that Ω isn't Woodin in $L[\mathcal{S}]$, as it's trivial in the case where Ω is a successor cardinal of $K^{c,F}(x)|\theta$ by our case assumption, and in the "limit cardinal case" it also holds since

$$K^{c,F}(x)|\Omega^{+K^{c,F}(x)|\theta} \subseteq \mathcal{S}.$$

By [?] and [?] this means that we can build $K^F(x)|\kappa$, as the only places they use that there's no inner model with a Woodin are to guarantee that $K^{c,F}(x)|\Omega$ exists and has no Woodin cardinals, and in Lemma 4.27 of [?] in which they only require that Ω isn't Woodin in $L[\mathcal{S}]$.

As $\kappa < \theta$ was arbitrary we then get that $K^F(x)|\theta$ exists. Note that $K^F(x)|\theta$ has no Woodin cardinals either and is F-small, so that $\mathcal{Q}$-structures trivially exist, making it (θ,θ)-iterable. ■

1.3 The hybrid witness equivalence

Definition 1.3.1. asd

Define coarse (k, U, x)-Woodin pairs

$\circ$

Definition 1.3.2. Let F be a total condensing operator and let α be an ordinal. Then the **coarse mouse witness condition at** α **with** F, written $W^*_\alpha(F)$, states that given any scaled-co-scaled $U \subseteq \mathbb{R}$ whose associated sequences of prewellorderings are elements of $\mathrm{Lp}^F_\alpha(\mathbb{R})$, we have for every $k < \omega$ and $x \in \mathbb{R}$ a coarse (k, U, x)-Woodin pair (N, Σ) with $\Sigma \restriction \mathsf{HC} \in \mathrm{Lp}^F_\alpha(\mathbb{R})$. $\circ$ Check if this is a reasonable definition.

Theorem 1.3.3 (Hybrid witness equivalence)**.** *Let $\theta > 0$ be a cardinal, $g \subseteq \mathrm{Col}(\omega, {<}\theta)$ V-generic, $\mathbb{R}^g := \bigcup_{\alpha<\theta} \mathbb{R}^{V[g \restriction \alpha]}$, F a total radiant operator and α a critical ordinal of $\mathrm{Lp}^F(\mathbb{R}^g)$. Assume that $\mathrm{Lp}^F(\mathbb{R}^g) \models \mathsf{DC} + \ulcorner W^*_\beta(F)$ holds for all $\beta \leq \alpha\urcorner$. Then there is a hybrid mouse operator $\mathcal{N} \in V$ on $H_{\aleph_1^{V[g]}}$ such that*

$$\mathrm{Lp}^F(\mathbb{R}^g) \models W^*_{\alpha+1}(F) \quad \textit{iff} \quad V \models \ulcorner M^{\mathcal{N}}_n \textit{ is total on } H_{\aleph_1^{V[g]}} \textit{ for all } n < \omega\urcorner$$

Furthermore, if $\theta < \aleph_1^V$ then we only need to assume that F is total and condensing.

Be more explicit about what the given operator $\mathcal{N}$ looks like.

1.4 Determinacy in mice from DI

Proposition 1.4.1 (Folklore?)**.** *If ω_1 carries a saturated ideal then mouse reflection holds at ω_1.*

Proof. Let $\mathcal{N}$ be a mouse operator defined on HC and fix some $x \in H_{\omega_2}$; we want to show that $\mathcal{N}(x)$ is defined. Let $j : V \to M$ be the generic ultrapower with $\mathrm{crit}\, j = \omega_1^V$ and note that $j(\omega_1^V) = \omega_1^M = \omega_1^{V[g]} = \omega_2^V$ by saturation of the ideal. This means in particular that $\mathsf{HC} \prec H^M_{\omega_2}$. Since

$$\mathsf{HC} \models \ulcorner \mathcal{N}(y) \text{ exists for all sets } y\urcorner$$

we get that $H^M_{\omega_2}$ believes the same is true. But $H^V_{\omega_2} \subseteq H^M_{\omega_2}$ since $\operatorname{crit} j = \omega_1^V$, so that in particular $H^M_{\omega_2}$ believes that $x^\sharp$ exists. Since M is closed under ω-sequences in $V[g]$ by Proposition **??**, we get that $x^\sharp$ exists in $V[g]$ and hence also in V as set forcing can't add sharps. ■

Prove this or give a reference.

Proposition 1.4.2 (Folklore?)**.** *If ω_1 carries a precipitous ideal then HC is closed under sharps. If the ideal is furthermore saturated then H_{ω_2} is closed under sharps.*

Proof. Proposition 1.4.1 gives the latter statement if we show the former, so fix an $x \in \mathsf{HC}$ and let $j : V \to M$ be the generic ultrapower from a precipitous ideal on ω_1^V. Since $j(x) = x$ we get that $j : L[x] \to L[x]$ with $\operatorname{crit} j > \operatorname{rk} x$, implying that $x^\sharp$ exists in the generic extension. But set forcing can't add sharps so $x^\sharp$ exists in V as well. ■

Add argument or reference.

Definition 1.4.3. Let $j : V \to M$ be an elementary embedding in some $V[g]$ and let F be a model operator. Then F is j**-radiant** if it condenses well, determines itself on generic extensions and satisfies the **extension property**, which says that $F \subseteq j(F)$ and $j(F) \restriction \mathsf{HC}^{V[g]}$ is definable in $V[g]$. ○

Lemma 1.4.4 (DI)**.** *M_1^F is total on H_{ω_2} for any j-radiant model operator F on H_{ω_2}.*

Proof. We want to use the hybrid core model dichotomy 1.2.2, but the problem is that F is not total. We solve this by going to a smaller model; the model $W := L^F_{\omega_2^V}(\mathbb{R})$ will be a first attempt (note that $\mathbb{R} \in \operatorname{dom} F$ as we're assuming CH). To be able to apply the dichotomy in a model we need it to satisfy ZFC. The following claim is the first step towards this.

Claim 1.4.4.1. Given any real x, $L^F_{\omega_2}(x) \models \ulcorner \omega_1^V$ is inaccessible$\urcorner$.

Proof of claim. Letting $j : V \to M$ be the generic elementary embedding, note that j doesn't move x, so that

$$j \upharpoonright L^F_{\omega_2^V}(x) : L^F_{\omega_2^V}(x) \to L^{j(F)}_{\omega_2^M}(x).$$

Since F has the extension property, $L^{j(F)}_{\omega_2^M}(x)$ is just an end-extension of $L^F_{\omega_2^V}(x)$. In particular ω_1^V is still a cardinal in there, meaning that, for every $\alpha < \omega_1^V$,

$$L^{j(F)}_{\omega_1^M}(x) \models \ulcorner\text{there's a cardinal} > \alpha\urcorner.$$

By elementarity this makes ω_1^V a limit cardinal in $L^F_{\omega_2^V}(x)$ and by GCH in $L^F_{\omega_2^V}(x)$ it's inaccessible. $\dashv$

This claim is now transferred to M, and as $\mathbb{R}^V$ is a real from the point of view of M, we get that

$$L^{j(F)}_{\omega_2^M}(\mathbb{R}^V) \models \ulcorner\omega_1^M \text{ is inaccessible}\urcorner.$$

Noting that $\omega_1^M = \omega_2^V$ and again using the extension property of F, we get that $W \models \mathsf{ZF}$. We don't get choice in W as it doesn't contain a wellorder of the reals, so we we'll work with $W[h]$ instead, where $h \subseteq \mathrm{Col}(\omega_1, \mathbb{R})^W$ is W-generic. Since we're assuming CH we get that $g \in V$, making $W[h] \in V$ as well, $W[h]$ is still closed under F since F determines itself on generic extensions, and $W[h] \models \mathsf{ZFC}$.

We can now apply the hybrid core model dichotomy 1.2.2 inside $W[h]$ to conclude that, for every real x, either $K^F(x)^{W[h]}$ exists or $M_1^F(x)$ exists (note that (ω_1, ω_1)-iterability is absolute between $W[h]$ and V since $W[h]$ contains all the reals). Since mouse reflection holds at ω_1 by Proposition 1.4.1 if the latter conclusion held at all reals x then we would also get that M_1^F is total on H_{ω_2} and we'd be done. So assume $K := K^F(x)^{W[h]}$ exists.

Claim 1.4.4.2. $j(K) \in V$.

Proof of claim. This is where we'll be using homogeneity of our ideal. Firstly K is definable in $W[h]$ and thus also in W by homogeneity of $\mathrm{Col}(\omega_1, \mathbb{R})$, so that $j(K)$ is definable in $j(W)$. But $j(W)$ is definable in $V[g]$ as the unique $j(F)$-premouse over $\mathbb{R}$ of height ω_1, making $j(K)$ definable in $V[g]$ with $j(F) \restriction \mathsf{HC}$ as a parameter. But $j(F) \restriction \mathsf{HC}$ is definable in $V[g]$ since F satisfies the extension property, so homogeneity of our ideal implies that $j(F) \in V$ and hence $j(K) \in V$ as well. ⊣

This claim also implies that ω_1^V is inaccessible in K, as if it wasn't, say $\omega_1^V = \lambda^{+K}$, then $\omega_2^V = j(\omega_1^V) = j(\lambda)^{+j(K)} = \lambda^{+j(K)}$, so that ω_2^V isn't a cardinal in V, $\lightning$.

We then also get that $(\omega_1^V)^{+j(K)} < \omega_2^V$, since if they were equal then elementarity would imply that ω_1^V was a successor in K, $\lightning$.

Since $K|\omega_1^V = j(K)|\omega_1^V$, elementarity and the above implies that

$$j^2(K)|(\omega_1^V)^{+j^2(K)} = j(K)|(\omega_1^V)^{+j(K)},$$

which makes sense as $j(K) \in V$.

Let now E be the (ω_1^V, ω_2^V)-extender derived from $j \restriction j(K)$, and note that $E \restriction \alpha \in M$ for every $\alpha < \omega_2^V = \omega_1^M$ as M is closed under countable sequences in $V[g]$.

Claim 1.4.4.3. $E \restriction \alpha$ is on the $j(K)$-sequence for every $\alpha < \omega_2^V$.

Why is this sufficient?

Proof of claim. We need to show that

$$j(W) \models \ulcorner \langle \langle j(K), \mathrm{Ult}(j(K), E \restriction \alpha) \rangle, \alpha \rangle \text{ is } \mathsf{On}\text{-iterable} \urcorner.$$

What kind of reflection?

Assume not. Then by reflection we get, in $j(W)$, a countable $\overline{K}$ and an elementary $\sigma : \overline{K} \to \mathrm{Ult}(j(K), E \restriction \alpha)$ with $\sigma \restriction \alpha = \mathrm{id}$ and $\langle \langle j(K), \overline{K} \rangle, \alpha \rangle$ isn't ω_1-iterable.

Let $k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$ be the factor map with $k \restriction \alpha = \mathrm{id}$ and define $\psi := k \circ \sigma : \overline{K} \to j^2(K)$, so that $(k \circ \sigma) \restriction \alpha = \mathrm{id}$. We have both ψ and $\overline{K}$ in M, which is the generic ultrapower $\mathrm{Ult}(V, g)$, so we also get that $\psi = [\vec{\psi}_\xi]_g$, $\overline{K} = [\vec{K}_\xi]_g$ and $\alpha = [\vec{\alpha}_\xi]_g$. We need to show that

For g-almost every $\xi < \omega_1^V$ it holds that $W \models \ulcorner\langle\langle K, K_\xi\rangle, \alpha_\xi\rangle$ is ω_1-iterable$\urcorner$

By Łoś' Lemma we have that, in V and hence also in $V[g]$, there are embeddings $\psi_\xi : K_\xi \to j(K)$ with $\psi_\xi \restriction \alpha_\xi = \mathrm{id}$ for g-almost every $\xi < \omega_1^V$. As $j(W)$ is closed under countable sequences in $V[g]$ it sees that the K_ξ's are countable, so that an application of absoluteness of wellfoundedness shows that $j(W)$ also has elementary embeddings $\psi^*_\xi : K_\xi \to j(K)$ with $\psi^*_\xi \restriction \alpha_\xi$.

Include this argument perhaps.

But $j(K) = K^{j(F)}(x)^{j(W[h])}$, so $j(W[h])$ sees that $\langle\langle K, K_\xi\rangle, \alpha_\xi\rangle$ is ω_1-iterable, which is therefore also true in W since $W \cap \mathbb{R} \subseteq \mathbb{R}^{V[g]} = j(W[h]) \cap \mathbb{R}$. ⊣

Our desired contradiction is then showing that K has a Shelah cardinal, which is impossible. Let $f : \omega_1^V \to \omega_1^V$ be a function in $j(K)$ and pick some $\alpha \in (j(f)(\kappa), \omega_2^V)$. Letting

Insert argument?

$$k : \mathrm{Ult}(j(K), E \restriction \alpha) \to j^2(K)$$

be the factor map, we get that $\mathrm{crit}\, k \geq \alpha$ by coherence of extenders on the K-sequence and hence that $i_{E\restriction\alpha}(f)(\omega_1^V) < \alpha$ as well. This shows that ω_1^V is Shelah in $j(K)$ and hence K has a Shelah cardinal by elementarity, $\lightning$. ∎

Theorem 1.4.5 (DI). $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R}) \models$ *AD for all "nice"* Γ *and* Σ.

Specify niceness.

Proof.

Show that all the operators occuring in the $\mathrm{Lp}^{\Gamma,\Sigma}(\mathbb{R})$ induction are j-radiant.

∎