Pure Postgraduate Seminar

Inner Model Theory

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December 9, 2016

University of Bristol

What's inner model theory?

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- $\{\emptyset\}$
- $\emptyset$

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What's inner model theory?

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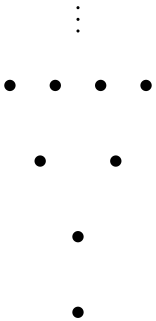

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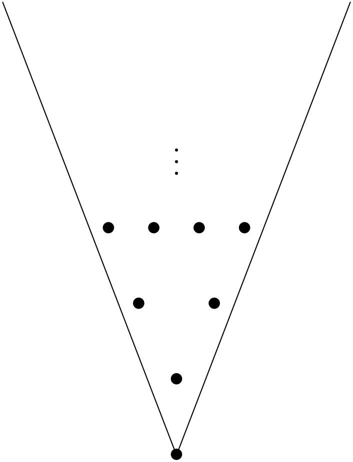

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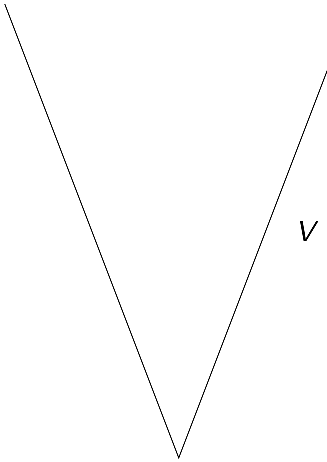

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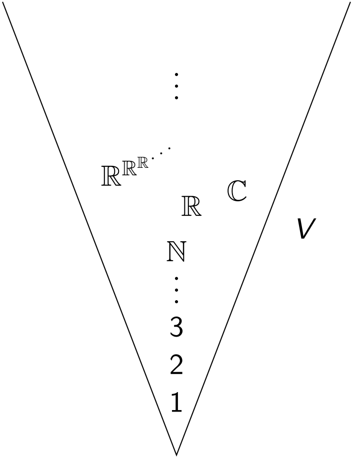

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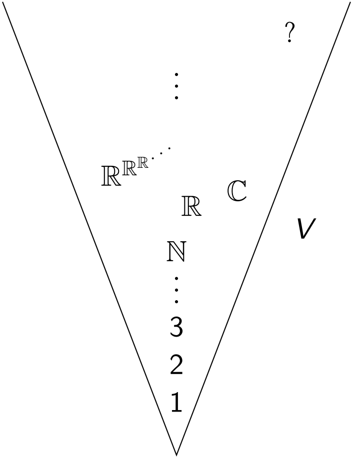

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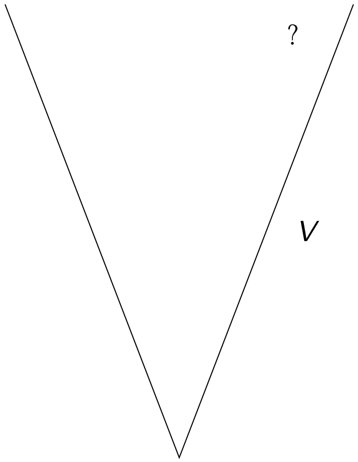

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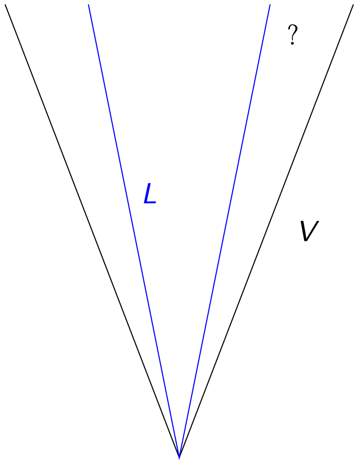

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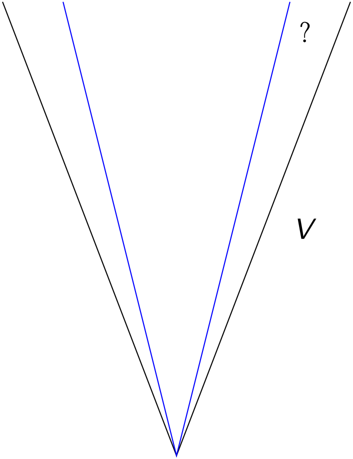

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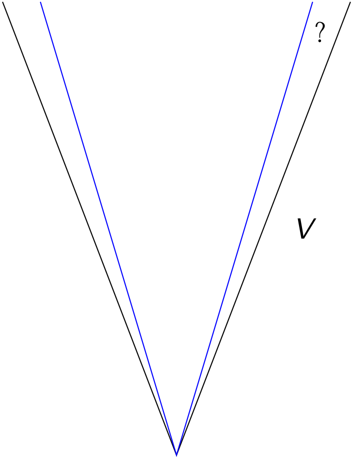

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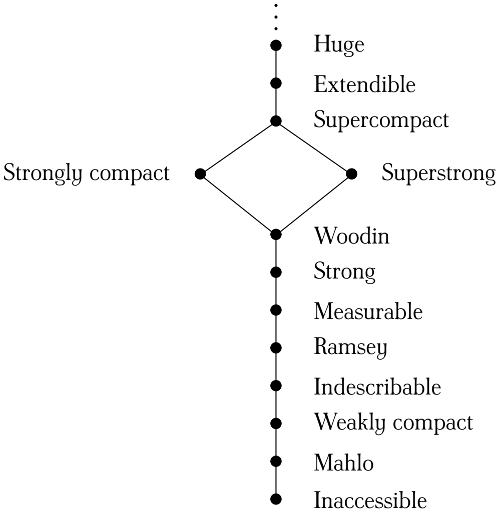

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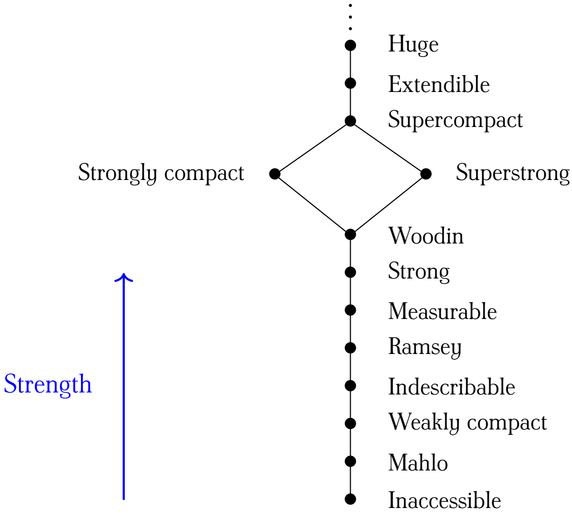

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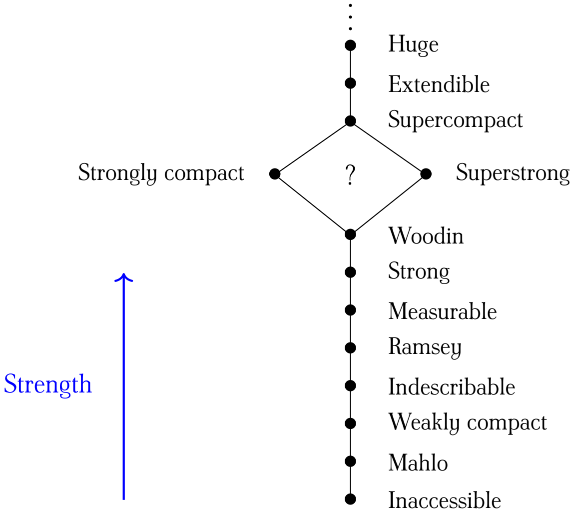

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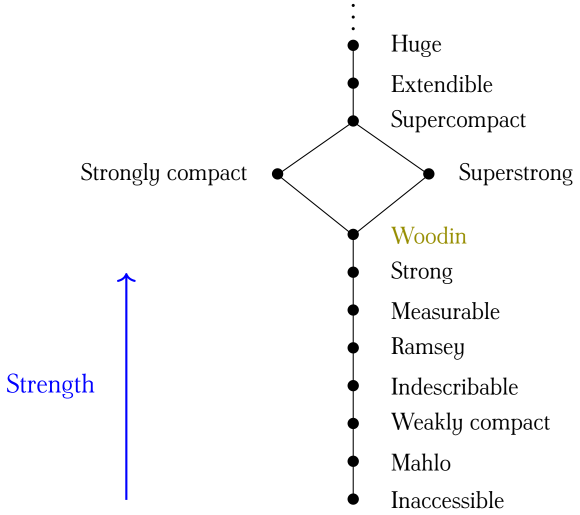

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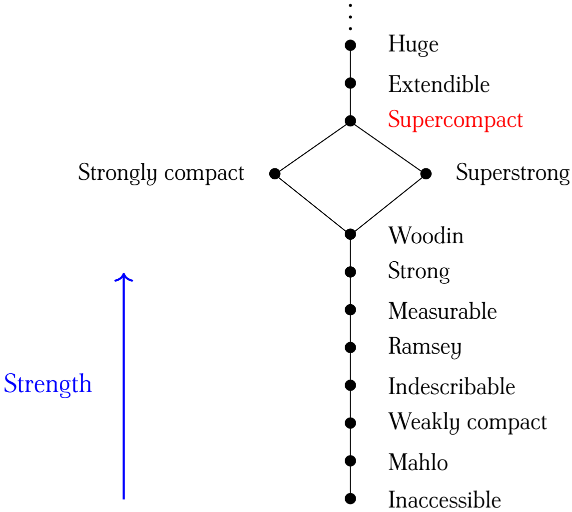

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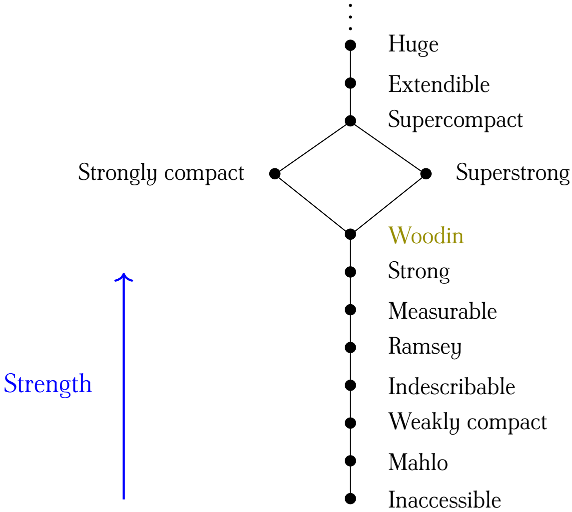

Elementary embeddings

Elementary embeddings

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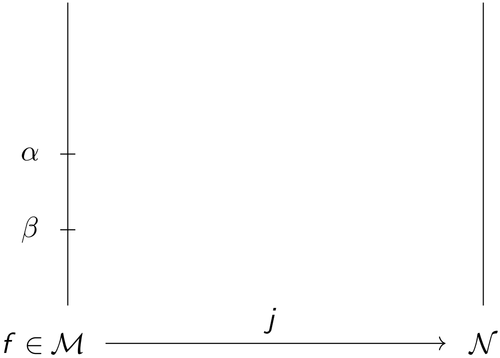
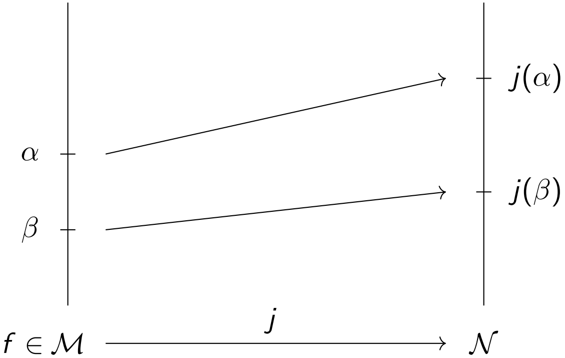
$$\mathcal{M} \xrightarrow{\quad j \quad} \mathcal{N}$$

Elementary embeddings



$\mathcal{M} \models$ "$f : \alpha \to \beta$"

Elementary embeddings



$\mathcal{M} \models$ "$f : \alpha \to \beta$"

$\mathcal{N} \models$ "$j(f) : j(\alpha) \to j(\beta)$"

Elementary embeddings

$$\mathcal{M} \xrightarrow{\quad j \quad} \mathcal{N}$$

$\mathcal{M} \models$“The dress is white and gold”

Elementary embeddings

$$\mathcal{M} \xrightarrow{\quad j \quad} \mathcal{N}$$

$\mathcal{M} \models$"The dress is white and gold"

$\mathcal{N} \models$"The dress is $\underbrace{j(\text{white})}_{\text{blue}}$ and $\underbrace{j(\text{gold})}_{\text{black}}$"

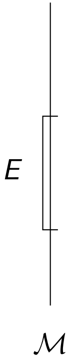
Elementary embeddings

$\mathcal{M}$

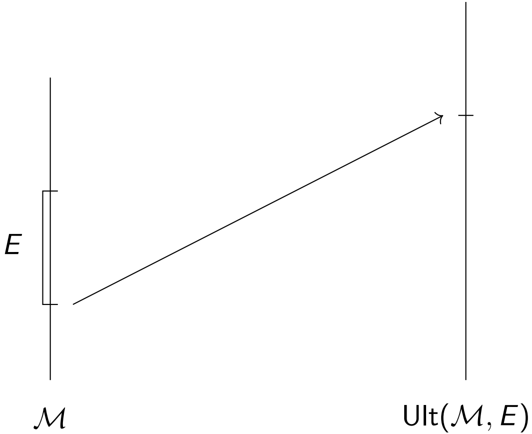
Elementary embeddings

$E \in \mathcal{M}$

Elementary embeddings



Elementary embeddings

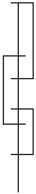


Premice

Definition

A (coarse) **premouse** is a structure of the form $\mathcal{M} = \langle L_\alpha^{\vec{E}}, \vec{E}, F \rangle$, where $\vec{E}$ is a *fine* extender sequence and every proper initial segment of $\mathcal{M}$ is sound.

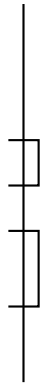
$=$



$\mathcal{M}$

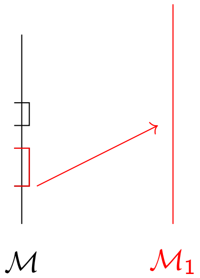
Linear iterations

Linear iterations

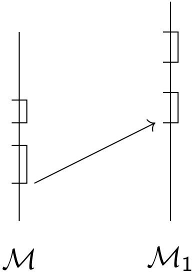


$\mathcal{M}$

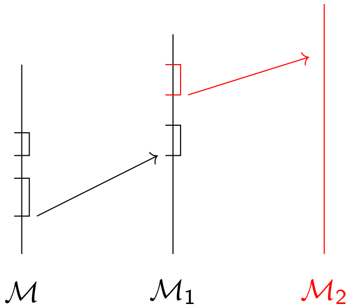
Linear iterations



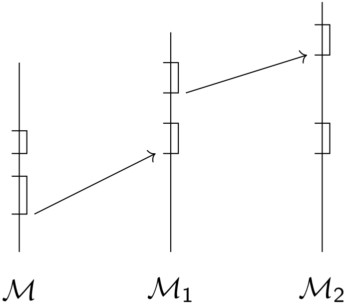
Linear iterations



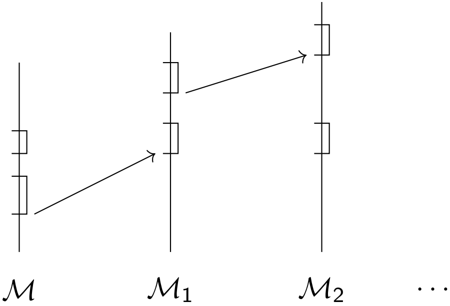
Linear iterations



Linear iterations



Linear iterations



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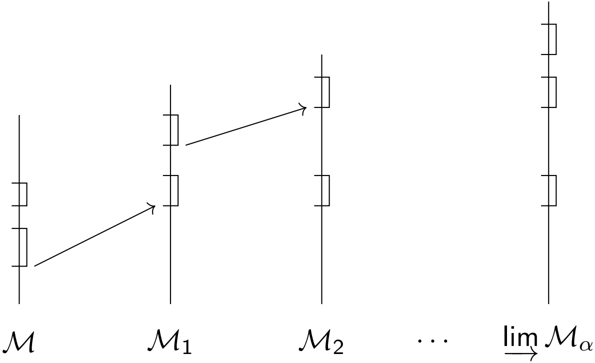
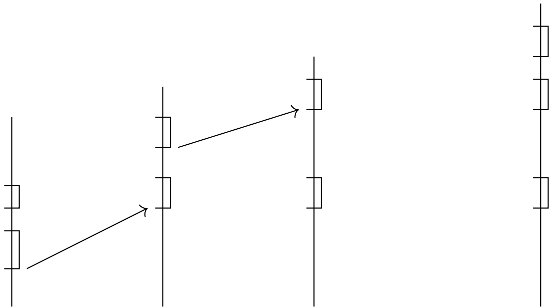


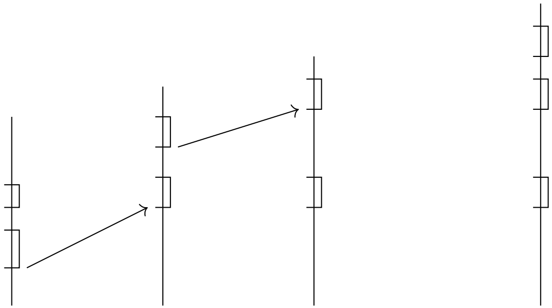
Linear iterations



$$\mathcal{M} \qquad \mathcal{M}_1 \qquad \mathcal{M}_2 \qquad \cdots \qquad \varinjlim \mathcal{M}_\alpha$$

$\mathcal{M}$ is **linearly iterable** if all these iterates are premice.

Linear iterations



$\mathcal{M} \qquad \mathcal{M}_1 \qquad \mathcal{M}_2 \qquad \cdots \qquad \varinjlim \mathcal{M}_\alpha$

$\mathcal{M}$ is **linearly iterable** if all these iterates are premice.

Note that the extenders here *don't overlap*.

The iteration game

The iteration game

$\mathcal{M}$

The iteration game

$$\begin{tikzcd}
 & \mathcal{M}_1 \\
\mathcal{M} \arrow[ur] &
\end{tikzcd}$$

The iteration game

$$\begin{array}{ccc} & \mathcal{M}_1 \ni \color{red}{E_1} \\ \nearrow & \\ \mathcal{M} & \end{array}$$

The iteration game

$$\begin{array}{ccc} \color{red}{\mathrm{Ult}(\mathcal{M}, E_1)} & & \mathcal{M}_1 \ni E_1 \\ & {\color{red}\nwarrow} \quad \nearrow & \\ & \mathcal{M} & \end{array}$$

The iteration game

$$\begin{tikzcd}
\mathcal{M}_2 & & \mathcal{M}_1 \\
& \mathcal{M} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The iteration game

$$\begin{tikzcd}[column sep=small]
E_2 \in \mathcal{M}_2 & & \mathcal{M}_1 \\
& \mathcal{M} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The iteration game

$$
\begin{tikzcd}
\mathcal{M}_3 & & & \\
& \mathcal{M}_2 \arrow[ul] & & \mathcal{M}_1 \\
& & \mathcal{M} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The iteration game

$$
\begin{tikzcd}[column sep=small]
E_3 \in \mathcal{M}_3 & & & \\
& \mathcal{M}_2 \arrow[ul] & & \mathcal{M}_1 \\
& & \mathcal{M} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The iteration game

$$\begin{tikzcd}
\mathcal{M}_3 & & & & \mathcal{M}_4 \\
& \mathcal{M}_2 \arrow[ul] & & \mathcal{M}_1 \arrow[ur] & \\
& & \mathcal{M} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

The iteration game

$$
\begin{tikzcd}[column sep=small]
\mathcal{M}_3 & & & & \mathcal{M}_4 \\
& \mathcal{M}_2 \arrow[ul] & & \mathcal{M}_1 \arrow[ur] & \\
& & \mathcal{M} \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

At limit steps player II picks a branch b through the tree

and take the direct limit along b.

Definition

An **iteration strategy** for a premouse $\mathcal{M}$ is a winning strategy for player II in the iteration game.

Mice

Definition

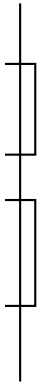
An **iteration strategy** for a premouse $\mathcal{M}$ is a winning strategy for player II in the iteration game.

Definition

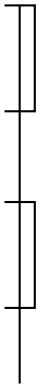
A **mouse** is a premouse for which an iteration strategy exists.

Comparison

Comparison

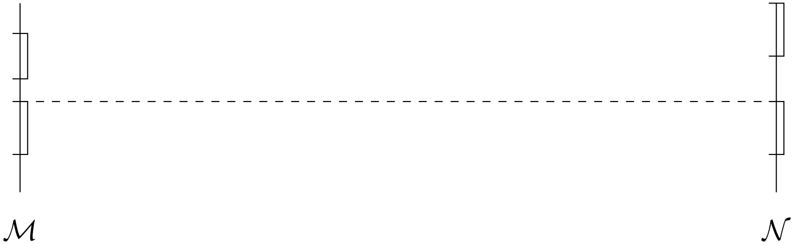


$\mathcal{M}$



$\mathcal{N}$

Comparison



Comparison

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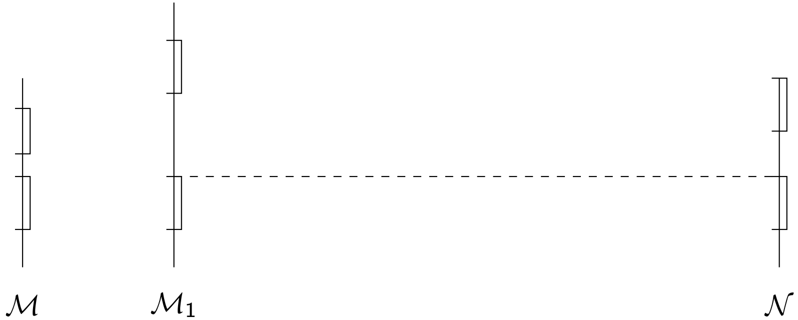

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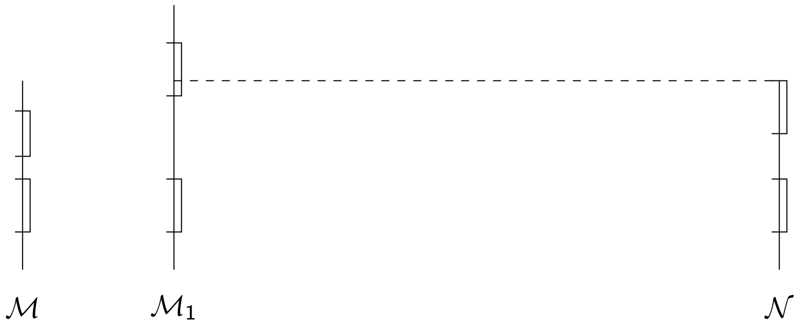

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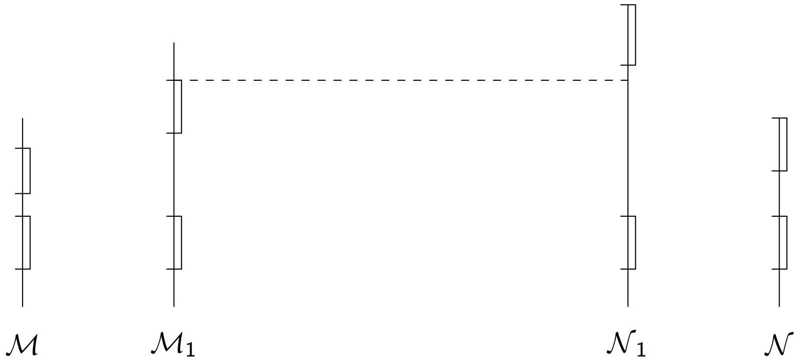

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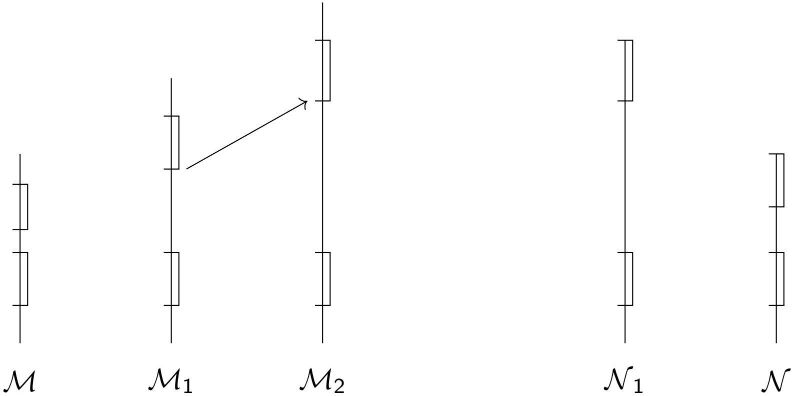

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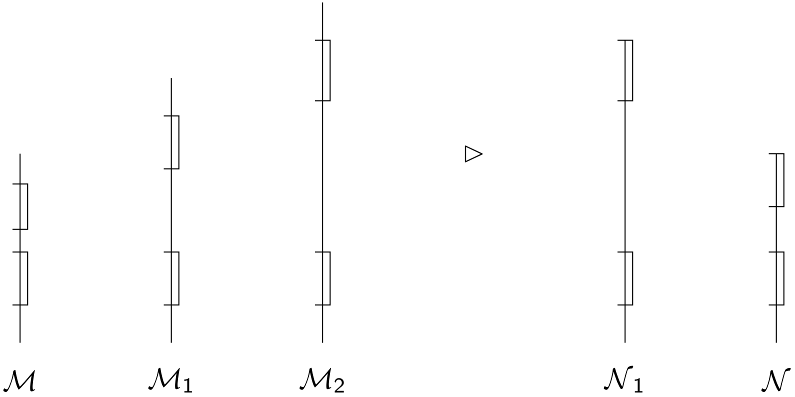
$\mathcal{M}$
$\mathcal{M}_1$
$\mathcal{M}_2$
$\triangleright$
$\mathcal{N}_1$
$\mathcal{N}$

Applications of comparison

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Condensation Theorem for L

If $j : \mathcal{H} \to L$ is elementary then $\mathcal{H} \lhd L$.

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Condensation Theorem for L

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Definition

The **projectum** of a mouse $\mathcal{M}$, written $\rho(\mathcal{M})$, is the least ordinal $\rho \leq \mathsf{On}^{\mathcal{M}}$ such that there exists a subset $A \subseteq \rho$ definable over $\mathcal{M}$ with parameters and satisfying that $A \notin \mathcal{M}$. The least such parameter is the **standard parameter**, denoted by $p(\mathcal{M})$.

Applications of comparison

Condensation Theorem for L

If $j : \mathcal{H} \to L$ is elementary then $\mathcal{H} \lhd L$.

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Condensation Theorem for mice

If $\mathcal{M}$ is a mouse and $j : \mathcal{H} \to \mathcal{M}$ is elementary with $\mathsf{crit}\, j \geq \rho(\mathcal{H})$ and $\mathcal{M} \models$ "$\rho(\mathcal{H})$ is a cardinal", then $\mathcal{H} \lhd \mathcal{M}$.

Applications of comparison

Definition

The **core** of a mouse $\mathcal{M}$, written $\mathfrak{C}(\mathcal{M})$, is a subset of $\mathcal{M}$ all of whose elements are definable from $\rho(\mathcal{M})$ and $p(\mathcal{M})$.

Applications of comparison

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If $\mathcal{M} \models \mathsf{ZF}^-$ or if $\mathcal{M}$ is sound then $\mathfrak{C}(\mathcal{M}) = \mathcal{M}$.

Applications of comparison

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The **core** of a mouse $\mathcal{M}$, written $\mathfrak{C}(\mathcal{M})$, is a subset of $\mathcal{M}$ all of whose elements are definable from $\rho(\mathcal{M})$ and $p(\mathcal{M})$.

If $\mathcal{M} \models \mathsf{ZF}^-$ or if $\mathcal{M}$ is sound then $\mathfrak{C}(\mathcal{M}) = \mathcal{M}$.

Theorem

$\mathfrak{C}(\mathcal{M})$ is sound whenever $\mathcal{M}$ is a mouse.

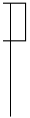
K^c constructions

K^c constructions

$$\mathcal{N}_0 := V_\omega$$

K^c constructions

All extenders used are *robust*.



$\mathcal{N}_1$

K^c constructions

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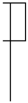
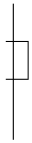


$\mathfrak{C}(\mathcal{N}_1)$

K^c constructions

All extenders used are *robust*.



$\mathcal{N}_2$

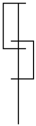
K^c constructions

All extenders used are *robust*.

$\mathfrak{C}(\mathcal{N}_2)$

K^c constructions

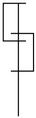
All extenders used are *robust*.



$\mathcal{N}_3$

K^c constructions

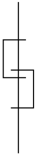
All extenders used are *robust*.



$\mathfrak{C}(\mathcal{N}_3)$

K^c constructions

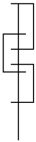
All extenders used are *robust*.



$\mathcal{N}_4$

K^c constructions

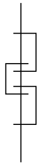
All extenders used are *robust*.



$\mathcal{N}_5$

K^c constructions

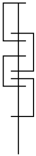
All extenders used are *robust*.



$\mathcal{N}_6$

K^c constructions

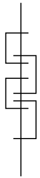
All extenders used are *robust*.



$\mathcal{N}_7$

K^c constructions

All extenders used are *robust*.

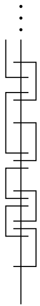


$\mathcal{N}_8$

K^c constructions

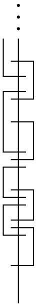
$\ldots$

K^c constructions



$\mathcal{N}_{\mathsf{On}}$

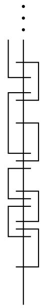
K^c constructions



$\mathcal{N}_{\mathsf{On}}$

That $\mathcal{N}_{\mathsf{On}}$ is a proper class is non-trivial.

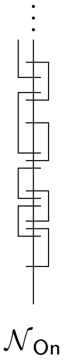
K^c constructions



That $\mathcal{N}_{\mathsf{On}}$ is a proper class is non-trivial.

We need every proper initial segment of $\mathcal{N}_{\mathsf{On}}$ to be sound.

K^c constructions



That $\mathcal{N}_{\mathsf{On}}$ is a proper class is non-trivial.

We need every proper initial segment of $\mathcal{N}_{\mathsf{On}}$ to be sound.

By the previous theorem we need to show that every $\mathfrak{C}(\mathcal{N}_\alpha)$ is iterable.

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Theorem (Steel-Mitchell-Jensen, 2003)

Assume there is no proper class model with a Woodin cardinal. Then $\mathfrak{C}(\mathcal{N}_\alpha)$ is iterable.

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- K is then constructed by "stitching together" these pseudo-K's.
- This K was built by Jensen and Steel in 2013.

The Main Theorem

Theorem (Jensen-Steel, 2013)

Assume that there is no proper class inner model with a Woodin cardinal. Then there are Σ_2 formulae $\psi_K(v)$ and $\psi_\Sigma(v)$ such that

(i) $K = \{v \mid \psi_K[v]\}$ is a transitive proper class mouse satisfying ZFC;

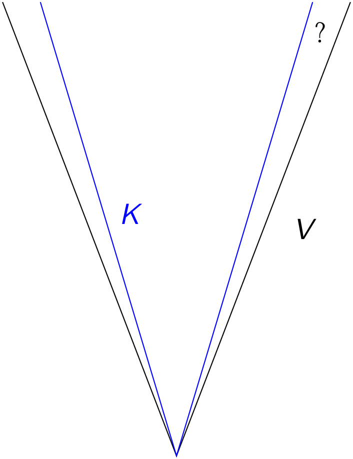
(ii) $\{v \mid \psi_\Sigma[v]\}$ is the unique iteration strategy for K acting on set-sized iteration trees;

(iii) (Generic absoluteness) $\psi_K^V = \psi_K^{V[g]}$ and $\psi_\Sigma^V = \psi_\Sigma^{V[g]} \cap V$ for any V-generic g over a set-sized poset;

(iv) (Inductive definition) $K|\omega_1^V$ is Σ_1-definable over $J_{\omega_1}(\mathbb{R})$;

(v) (Weak covering) For any $\lambda \geq \omega_2^V$ which is a successor K-cardinal, $\mathrm{cof}^V \lambda \geq |\lambda|^V$. Thus $\kappa^{+K} = \kappa^+$ whenever κ is a singular V-cardinal.

We're on the right track



Thank you!