Truth

Dan Saattrup Nielsen

What is truth?

Math truth!

T.1.22

Strange truth

Recap

Truth

Dan Saattrup Nielsen

February 26, 2016

Truth

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What is truth?

Math truth?

Tarski s trut

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Recar

A definition of truth?

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What is truth?

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Juange uut

- A definition of truth?
- "There is a person in this room"

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Recar

- A definition of truth?
- "There is a person in this room"
- "This dress is white and gold"

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What is truth?

Math truth?

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- A definition of truth?
- "There is a person in this room"
- "This dress is white and gold"
- "That guy is pretty"

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What is truth?

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Reca

- A definition of truth?
- "There is a person in this room"
- "This dress is white and gold"
- "That guy is pretty"
- Absolute and relative truth

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Absolute mathematical truth?

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- Absolute mathematical truth?
 - Checking...

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logical symbol	natural number
(0
)	1
\wedge	2
\vee	3
\neg	4
\rightarrow	5
\forall	6
3	7
\in	8
=	9
variables <i>v_i</i>	10 + i

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Recap

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■ Write ¬¬ as the corresponding number

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logical symbol	natural number
(0
)	1
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V	3
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∃	7
\in	8
=	9
variables <i>v_i</i>	10 + i

- Write 「—¬ as the corresponding number
- For a formula $\varphi = s_0 \cdots s_n$, set $\lceil \varphi \rceil = p_0^{\lceil s_0 \rceil} \cdots p_n^{\lceil s_n \rceil}$

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- Absolute mathematical truth?
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- Absolute mathematical truth?
 - Checking... No! (Proven by Tarski in 1936)

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- Absolute mathematical truth?
 - Checking... No! (Proven by Tarski in 1936)
 - Or rather, not definable within mathematics itself

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What is truth

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- Absolute mathematical truth?
 - Checking... No! (Proven by Tarski in 1936)
 - Or rather, not definable within mathematics itself
- What about if we step *outside* of mathematics?

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Definition (Tarski, 1935)

For a set M, a formula $\varphi(v_0, \ldots, v_n)$ and $x_0, \ldots, x_n \in M$, we can define the truth relation $M \models \varphi[x_0, \ldots, x_n]$ as follows:

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• If φ is $v_0 \in v_1$ then $M \models \varphi[\vec{x}]$ iff $x_0 \in x_1$;

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- If φ is $v_0 \in v_1$ then $M \models \varphi[\vec{x}]$ iff $x_0 \in x_1$;
- If φ is $v_0 = v_1$ then $M \models \varphi[\vec{x}]$ iff $x_0 = x_1$;

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- If φ is $\psi \wedge \chi$ then $M \models \varphi[\vec{x}]$ iff $M \models \psi[\vec{x}]$ and $M \models \chi[\vec{x}]$;

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Definition (Tarski, 1935)

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- If φ is $\neg \psi$ then $M \models \varphi[\vec{x}]$ iff $M \not\models \psi[\vec{x}]$;

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- If φ is $\exists w \psi$ then $M \models \varphi[\vec{x}]$ iff there exists $y \in M$ such that $M \models \psi[y, \vec{x}]$.

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Note that this is definable *inside* mathematics!

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For M any collection, we have to go *outside* mathematics

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Recap

■ Taking M to be the collection of all sets implies that $M \models \varphi$ means " φ is true"; i.e. absolute truth!

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■ Taking M to be the collection of all sets implies that $M \models \varphi$ means " φ is true"; i.e. absolute truth!

It turns out that this definition is "the right one":

(A variant of) Gödel's Completeness Theorem (1929)

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It turns out that this definition is "the right one":

(A variant of) Gödel's Completeness Theorem (1929)

Let φ be any formula. Then we can prove φ iff $M \models \varphi$ for all sets M satisfying ZFC.

Note: we cannot prove that $M \models \mathsf{ZFC}$, for any set M.

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For $x \in M$, let's "close $\{x\}$ under truth"

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Recap

For $x \in M$, let's "close $\{x\}$ under truth"

Fact

For any formula $\varphi(v_0,\ldots,v_n)$ there exists a function $f_{\varphi}:M^n\to M$ such that for every $\vec{x}\in M^n$, either

$$M \models \varphi[f_{\varphi}(\vec{x}), \vec{x}] \quad \text{or} \quad M \models \neg \exists v : \varphi[v, \vec{x}]$$

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Define now

$$F_0 := \{x\}$$

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Define now

$$F_0 := \{x\}$$

$$F_{n+1} := \{f_{\varphi}(\vec{y}) \mid y_i \in F_n \land \varphi \text{ is a formula}\}$$

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For $x \in M$, let's "close $\{x\}$ under truth"

Fact

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Define now

$$F_0 := \{x\}$$
 $F_{n+1} := \{f_{\varphi}(\vec{y}) \mid y_i \in F_n \land \varphi \text{ is a formula}\}$
 $\mathcal{H}_{\mathsf{x}} := \bigcup_{n \in \mathbb{N}} F_n$

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Facts about \mathcal{H}_x

 $\blacksquare \mathcal{H}_{\mathsf{x}} \models \varphi[\mathsf{x}] \text{ iff } M \models \varphi[\mathsf{x}];$

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Facts about \mathcal{H}_x

- $\mathcal{H}_x \models \varphi[x]$ iff $M \models \varphi[x]$;
- \blacksquare \mathcal{H}_{x} is countable.

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Facts about \mathcal{H}_x

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Is this strange?

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Facts about \mathcal{H}_{\times}

- $\mathcal{H}_x \models \varphi[x]$ iff $M \models \varphi[x]$;
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Is this strange?

The Skolem paradox (1922)

Let M be a "sufficiently big" set, such that $\mathbb{R} \in M$ and $M \models$ " \mathbb{R} is uncountable". By the above fact, $\mathcal{H}_{\mathbb{R}}$ satisfies this as well.

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Facts about \mathcal{H}_{x}

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Is this really a paradox?

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Let M be a "sufficiently big" set, such that $\mathbb{R} \in M$ and $M \models$ " \mathbb{R} is uncountable". By the above fact, $\mathcal{H}_{\mathbb{R}}$ satisfies this as well. But then $\mathcal{H}_{\mathbb{R}}$ is a *countable* set with an *uncountable* element!

Is this really a paradox? No! Truth is relative

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Recap

 Absolute mathematical truth is not definable inside mathematics

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- Absolute mathematical truth is not definable inside mathematics
- Relative mathematical truth can be defined inside mathematics

Truth

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Math truth?

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- Absolute mathematical truth is not definable inside mathematics
- Relative mathematical truth can be defined inside mathematics
- Truth can really mess with your mind

The end of the journey

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Thank you!