Dan S. Nielsen: "Game-theoretic Ramsey-like cardinals"

Ramsey-like cardinals were introduced in Gitman's 2011 paper of the same name, and are roughly speaking cardinals $\kappa$ which can be characterised as critical points of elementary embeddings between $ZFC^-$-models of size $\kappa$. We can then vary the consistency strength by requiring more closure of the domain model and requiring more properties of the embedding. These cardinals lie between the weakly compacts and the measurables. Of course, Ramsey cardinals are of this type, as Mitchell showed in his 1979 paper "Ramsey cardinals and constructibility".

Holy and Schlicht introduced a new hierarchy of Ramsey-like cardinals in their recent 2018 paper, called the $\alpha$-Ramsey cardinals, and in my forthcoming paper with Philip Welch in JSL, we analyse this hierarchy. We connect the $\alpha$-Ramseys to previously introduced Ramsey-like cardinals and large cardinals in general, as well as investigating which properties they satisfy. I will here be talking about a couple of those results, related to long games.

We start with a handful of definitions.

__Definition__ For a cardinal $\kappa$, a __weak $\kappa$-model__ is a set $M$ of size $\kappa$ satisfying that $\kappa+1 \subseteq M$ and $(M,\in) \models ZFC^-$.

Definition. A set $\mu$ is an $\mathcal{M}$-measure on $\kappa$ if $(\mathcal{M}, \in, \mu) \models \ulcorner \mu$ is a $\kappa$-complete ultrafilter on $\kappa \urcorner$.

Definition. Let $\mathcal{M}$ be a weak $\kappa$-model and $\mu$ an $\mathcal{M}$-measure. Then $\mu$ is

- weakly amenable if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with $\mathcal{M}$-cardinality $\kappa$;
- countably complete if $\bigcap \vec{X} \neq \emptyset$ for every $\omega$-sequence $\vec{X} \in {}^{\omega}\mu$;
- $\mathcal{M}$-normal if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^{\kappa}\mu : \triangle \vec{X} \in \mu$;
- normal if $\triangle \vec{X}$ is stationary in $\kappa$ for every $\kappa$-sequence $\vec{X} \in {}^{\kappa}\mu$;
- good if it has a wellfounded ultrapower.

Note that normal $\mathcal{M}$-measures are both $\mathcal{M}$-normal and countably complete.

Definition (Holy, Schlicht, N.). Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ an ordinal. Then define the perfect information two-player game as follows.

$$\boxed{\mathcal{G}_\gamma(\kappa)} \qquad \begin{array}{lccccccc} \mathrm{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & \mathcal{M}_\gamma & \\ \mathrm{II} & & \mu_0 & & \mu_1 & \cdots & & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha$ is a weak $\kappa$-model for all $\alpha \leq \gamma$, $\mu_\alpha$ is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$ and $\mu_\gamma$ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure. We require that the $\mathcal{M}_\alpha$'s and $\mu_\alpha$'s are $\subseteq$-increasing and for $\alpha \leq \gamma$ a limit ordinal that $\mathcal{M}_\alpha = \bigcup_{\xi < \alpha} \mathcal{M}_\xi$ and $\mu_\alpha = \bigcup_{\xi < \alpha} \mu_\xi$. Player II wins iff she can continue playing throughout all $(\gamma+1)$-many rounds.

Definition. Let $\kappa=\kappa^{<\kappa}$ be a cardinal and $\gamma\leq\kappa$ an ordinal. Then $\kappa$ is $\gamma$-Ramsey if player I doesn't have a winning strategy in $\mathcal{G}_\gamma(\kappa)$, and strategic $\gamma$-Ramsey if player II does have a winning strategy in $\mathcal{G}_\gamma(\kappa)$.

The reason why this still qualifies as being a Ramsey-like cardinal is the following.

Theorem (Holy-Schlicht). For regular cardinals $\lambda$, a cardinal $\kappa$ is $\lambda$-Ramsey iff for arbitrarily large $\theta>\kappa$ and every $A\subseteq\kappa$ there is a weak $\kappa$-model $\mathcal{M}\prec H_\theta$ with $\mathcal{M}^{<\lambda}\subseteq\mathcal{M}$ and $A\in\mathcal{M}$ with an $\mathcal{M}$-normal weakly amenable good $\mathcal{M}$-measure $\mu$ on $\kappa$.

The $\alpha$-Ramsey cardinals can't reach the measurables:

Fact (Holy-Schlicht). Every measurable cardinal $\kappa$ is ~~an~~ $\kappa$-Ramsey limit of $\kappa$-Ramseys.

The strategic ones do, however:

Theorem (Welch). If $\kappa$ is strategic $\omega_1$-Ramsey then either $0^\P$ exists or $\kappa$ is measurable in $K$.

Proof. Assume $\neg 0^\P$ and let $\tau$ be a winning strategy for II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$ where $g\subseteq Col(\omega_1,\kappa^+)$ is $V$-generic. Since $Col(\omega_1,\kappa^+)$ is $\omega$-closed, $V$ and $V[g]$ have the same countable sequences of $V$, so $\tau$ is still a strategy for II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$ as long as player I only plays models in $V$.

Let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle \in V[g]$ be a sequence of regular $K$-cardinals cofinal in $\kappa^+$, let player I play $\mathcal{M}_\alpha := K|\kappa_\alpha$ in $\mathcal{G}_{\omega_1}(\kappa)$ and let player II follow $\tau$. This results in a countably complete weakly amenable $K$-measure $\mu_{\omega_1}$, so the "beaver argument" (i.e. $K$-correctness of $\mu_{\omega_1}$) shows that $\mu_{\omega_1} \in K$, making $\kappa$ measurable in $K$. □

This also shows that, together with the previous fact, $\mathcal{G}_\alpha(\kappa)$ is <u>not</u> determined in general, for any uncountable $\alpha$. So, how about the countable ones? We will show that $\mathcal{G}_\omega(\kappa)$ <u>is</u> determined, and to show that we need to introduce games with <u>hidden</u> information. This is games of the form

$$\begin{array}{lllll} \text{I} & x_0[\hat{x}_0] & & x_1[\hat{x}_1] & \cdots \\ \text{II} & & y_0[\hat{y}_0] & & y_1[\hat{y}_1] \quad \cdots, \end{array}$$

where the $\hat{x}_\alpha$'s and the $\hat{y}_\alpha$'s are <u>hidden</u>, meaning that strategies don't include the opposing player's hidden information. We will use the following simple fact.

**Proposition.** Let $\mathcal{G}$ be a closed $\omega$-length game in which only player I has hidden information (i.e. that $\hat{y}_k = \emptyset$). Then $\mathcal{G}$ is determined.

**Proof.** This is exactly the same argument as the one showing that every closed game is determined: we assume that player II has no winning strategy in $\mathcal{G}$ and let player I play the "non-losing" strategy. The important thing to note here is that, from the point of view of player I, the game is a game with perfect information. $\square$

**Theorem (N., Welch).** Every $\omega$-Ramsey cardinal is strategic $\omega$-Ramsey.

**Proof.** Let $\kappa$ be $\omega$-Ramsey and define the following auxilliary game $\tilde{\mathcal{G}}(\kappa)$ of length $\omega$, in which player II has hidden information. Fix a regular $\theta > \kappa$.

| | | | | | |
|---|---|---|---|---|---|
| I | $\mathcal{M}_0$ | | $\mathcal{M}_1$ | | $\cdots$ |
| II | | $\mathcal{T}^{(0)}[\hat{\mathcal{T}}^{(0)}]$ | | $\mathcal{T}^{(1)}[\hat{\mathcal{T}}^{(1)}]$ | $\cdots$ |

Here $\mathcal{M}_n \prec H_\theta$ is a weak $\kappa$-model, $\mathcal{T}^{(n)}$ is a tree of height $n+1$ and $\hat{\mathcal{T}}^{(n)}$ is a pruned subtree of $\mathcal{T}^{(n)}$ of the same height. Writing $\mathcal{T}^{(n)}_k$ for the $k$'th level of $\mathcal{T}^{(n)}$, we firstly require $\mathcal{T}^{(n)}_0 = \{\emptyset\}$ and for $k<n$ that $\mathcal{T}^{(n)}_{k+1}$ is the set of all elementary embeddings $j : (\mathcal{M}_k, \in) \to (\mathcal{N}_j, \in)$ with $\mathcal{N}_j \in H_{\kappa^+}$ and $\mu_j = \mu$ for some fixed normal $\mathcal{M}_k$-measure $\mu$ on $\kappa$, where $\mu_j$ is the measure

induced by $j$. For $j, i \in \mathcal{J}^{(n)}$ we put $j \leq_{\mathcal{J}^{(n)}} i$ iff $j \subseteq i$ and $\mathcal{N}_j \prec \mathcal{N}_i$. We also require that the $\mathcal{M}_n$'s are $\subseteq$-increasing and that $\mathcal{J}^{(n)}$ and $\hat{\mathcal{J}}^{(n)}$ are initial segments of $\mathcal{J}^{(n+1)}$ and $\hat{\mathcal{J}}^{(n+1)}$, respectively. Player II wins iff she can play throughout all $\omega$-many rounds.

We note a few facts about this game. Firstly, since it's an open game of length $\omega$ in which only player II has hidden information, it is determined by the previous proposition. Secondly, if $\vec{\mathcal{M}} * \vec{\mathcal{J}}$ is winning for player II in $\bar{\mathcal{G}}(\kappa)$ then $\hat{\mathcal{J}} := \bigcup_{n<\omega} \hat{\mathcal{J}}^{(n)}$ is a pruned tree of height $\omega$, so we can choose a cofinal branch $\vec{j} \in [\hat{\mathcal{J}}]$. Since we required that $\mathcal{N}_j \prec \mathcal{N}_i$ whenever $j \leq_{\hat{\mathcal{J}}} i$ we get that

$$j := \bigcup \vec{j} : (\mathcal{M}, \in) \to (\mathcal{N}, \in)$$

is an elementary embedding, where $\mathcal{M} := \bigcup \vec{\mathcal{M}}$ and $\mathcal{N} := \bigcup \vec{\mathcal{N}}$. This means that if player II has a winning strategy in $\bar{\mathcal{G}}(\kappa)$ then she also has one in $\mathcal{G}_\omega(\kappa)$, by simply letting $\mu_n$ be the derived measure of any $j \in \hat{\mathcal{J}}^{(n)}_{n+1}$.

The above facts therefore mean that it suffices to show that player I does not have a winning strategy in $\bar{\mathcal{G}}(\kappa)$, so let $\sigma$ be a strategy for player I in $\bar{\mathcal{G}}(\kappa)$ and assume wlog that whenever $\mathcal{M}_n$ is a move following $\sigma$ then $\mathcal{M}_0, \dots, \mathcal{M}_{n-1} \in \mathcal{M}_n$ — we'll show that $\sigma$ is not a winning strategy.

Define an associated strategy $\tilde{\sigma}$ for player I in $\mathcal{G}_\omega(\kappa)$ as

$$\tilde{\sigma}(\langle M_k, \mu_k \mid k<n\rangle) := \sigma(\langle M_k, \mathcal{J}^{(k)} \mid k<n\rangle),$$

where $\mathcal{J}^{(k)}_{m+1}$ consists of all elementary embeddings $j : (M_m, \in) \to (N_j, \in)$ with $N_j \in H_{\kappa^+}$ and $\mu_j = \mu_m$ for every $m<k$. In other words, we simply replace all the $\mu_k$'s by the $\mathcal{J}^{(k)}$'s as generated by $\mu_0, \dots, \mu_{k-1}$ and apply $\sigma$.

As player I doesn't have a winning strategy in $\mathcal{G}_\omega(\kappa)$ there exists a play $\tilde{\sigma} * \vec{\mu}$ for which player II wins. Let $M$ be the $\omega$-th model in this play, $N$ be the transitive collapse of $\mathrm{Ult}(M, \mu_\omega)$ and set

$$j := \pi \circ j_{\mu_\omega} : (M, \in) \to (N, \in),$$

where $\pi : (\mathrm{Ult}(M, \mu_\omega), \in_{\mu_\omega}) \to (N, \in)$ is the transitive collapse.

Now define $\mathcal{J}^{(n)}$ as in the definition of $\tilde{\sigma}$. We now have that $j_n := j \restriction M_n : M_n \to j(M_n)$ is elementary with derived measure $\mu_n$. Note that $M_n \in \operatorname{dom} j = M_\omega$ by the above choice of $\sigma$, so since $N \in H_{\kappa^+}$ we also get that $j(M_n) \in H_{\kappa^+}$, so that $j_n \in \mathcal{J}^{(n)}_{n+1}$ for all $n<\omega$.

Furthermore, since $M_n \prec H_\theta$ for every $n<\omega$ we also get that $M_n \prec M_{n+1}$ and therefore also $j(M_n) \prec j(M_{n+1})$ by elementarity of $j$. ~~A~~ ~~quick diagram chase shows that we also get~~

This means that $j_n \leq_{\mathcal{T}} j_{n+1}$ for all $n<\omega$.

But then $\mathcal{T}^{(n)}$ does contain a pruned subtree of the same height which extends the previous ones, namely the tree whose $(k+1)$'st level only has $j_k$ as an element, so that the $\mathcal{T}^{(n)}$'s are indeed valid moves for player II in $\tilde{\mathcal{G}}(\kappa)$. The only thing that could potentially go wrong at this point is if player I in $\tilde{\mathcal{G}}$ decides to react to player II's moves with something different from the $M_n$'s (i.e. the models played in $\sigma * \vec{\mu}$).

But this cannot happen, since firstly player I's strategy can only depend upon the $\mathcal{T}^{(n)}$'s by definition of hidden information, and secondly as $\tilde{\mathcal{G}}(\kappa)$ is defined so that to every sequence $\langle \mu_0, \dots, \mu_n \rangle$ there's a unique tree $\mathcal{T}^{(n)}$ corresponding to these measures, in the same manner as described above, so the definition of $\tilde{\sigma}$ ensures that player I in $\tilde{\mathcal{G}}(\kappa)$ reacts to the $\mathcal{T}^{(n)}$'s precisely as player I in $\mathcal{G}_\omega(\kappa)$ reacts to the $\mu_n$'s. As player II can continue like this throughout all $\omega$-many rounds, player II wins, which by the above means that $\kappa$ is strategic $\omega$-Ramsey. $\square$