

Virtual Large Cardinals

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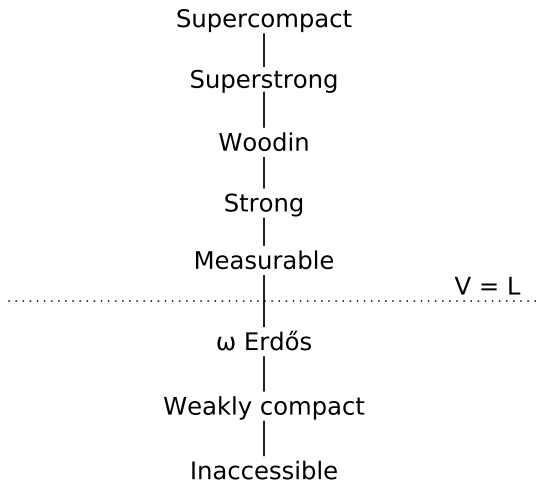
$$\pi: (\mathcal{M}, \in) \rightarrow (\mathcal{N}, \in)$$

Examples

- κ is a **measurable cardinal** if $\mathcal{M} = H_{\kappa^+}$.
- κ is a **θ -strong cardinal** if $\mathcal{M} = H_\theta$, $H_\theta \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

Recall that $x \in H_\theta$ iff $|\text{trcl}(x)| < \theta$. This hierarchy is often more convenient than the V_α 's since $H_\theta \models \text{ZFC}^-$ if θ is regular.

The hierarchy of large cardinals



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Let Φ be a large cardinal concept defined via elementary embeddings between *sets*, like the definitions on the previous slide.

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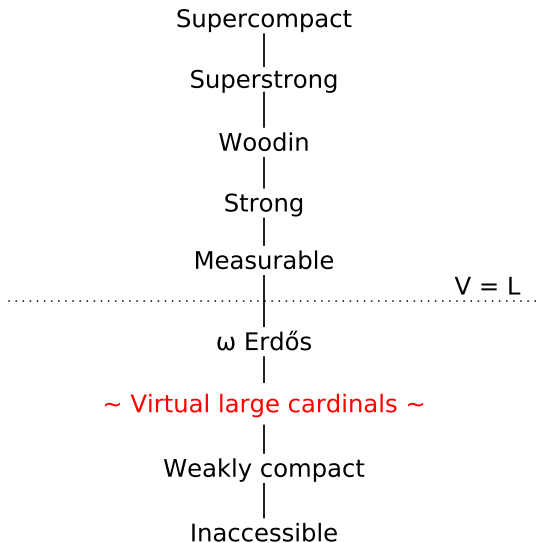
We basically just require that the embeddings exist in a *generic extension* rather than in V :

"Definition"

Let Φ be a large cardinal concept defined via elementary embeddings between *sets*, like the definitions on the previous slide.

Then κ is **virtually** Φ if the same definition holds but where we only require the embeddings exist in a generic extension and that $\mathcal{N} \subseteq V$.

A virtual addition to the hierarchy



Let us attach a **pre-** to our large cardinals if we do not require anything about where the critical point is sent:

Example

κ is **prestrong** if for every regular $\theta > \kappa$ there is an elementary embedding $\pi: (H_\theta, \in) \rightarrow (\mathcal{N}, \in)$ with $\text{crit } \pi = \kappa$ and $H_\theta \subseteq \mathcal{N}$.

Attaching an adjective

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This is really not an interesting concept in the real world:

Proposition (folklore)

For regular cardinals $\kappa < \theta$:

- κ is θ -prestrong iff it is θ -strong
- κ is θ -presupercompact iff it is θ -supercompact

Attaching an adjective

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It is interesting in the virtual world, however:

Theorem (N.)

For regular cardinals $\kappa < \theta$, κ is virtually θ -prestrong iff either

- κ is virtually θ -strong, or
- κ is virtually (θ, ω) -superstrong

Corollary (N.)

Virtually θ -prestrongs are equiconsistent with virtually θ -strongs.

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The following are equivalent:

- Virtually prestrongs are equivalent to virtually strongs
- There are no virtually ω -superstrongs.

Characterising a phenomenon

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Note that ω -superstrong cardinals are inconsistent with ZFC!

Definition

Let $\kappa < \theta$ be regular and let A be a class. Then κ is **virtually** (θ, A) -prestrong if there exists a generic elementary embedding

$$\pi: (H_\theta^V, \in, A \cap H_\theta^V) \rightarrow (\mathcal{N}, \in, B)$$

such that $\text{crit } \pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$ and $A \cap H_\theta^V = B \cap H_\theta^V$.

Further, if $\pi(\kappa) > \theta$ then κ is **virtually** (θ, A) -strong.

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Can we find some virtual large cardinal characterising exactly when the virtually A -prestrongs are equivalent to virtually A -strongs?

Remember that we are looking for a large cardinal notion which is inconsistent in the real world.

Definition

δ is **virtually proto-berkeley** if for every transitive set \mathcal{M} there exists a generic elementary embedding $\pi: \mathcal{M} \rightarrow \mathcal{M}$ with $\text{crit } \pi < \delta$. Further, δ is **virtually berkeley** if the critical point can be chosen to be arbitrarily large below δ .

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The real world versions of these are of course inconsistent with ZFC, but are currently being investigated in a choiceless context.

Berkeley cardinals

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Say that On is **virtually (pre)woodin** if for every class A there exists a virtually A -(pre)strong cardinal κ .

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Theorem (Dimopoulos-Gitman-N.)

The following are equivalent:

- On is virtually prewoodin iff it is virtually woodin
- There are no virtually berkeley cardinals

Definition

Generic Vopěnka's Principle (gVP) holds if for every class \mathcal{C} consisting of set-sized structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ and a generic elementary embedding $\pi: \mathcal{M} \rightarrow \mathcal{N}$.

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Theorem (Dimopoulos-Gitman-N.)

On is virtually prewoodin iff gVP holds.

Outline of argument cont.

Definition (GBC)

Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \text{On} \rangle$ is **natural** if there exists a class function $f: \text{On} \rightarrow \text{On}$ such that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for every $\alpha < \beta$, and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that

$$\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$$

for every α .

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Definition (GBC)

Define the **virtually vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that for any $\alpha < \beta$ and any generic elementary $\pi: \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta$, $\text{crit } \pi \in X$.

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This filter is normal, and proper iff gVP holds.

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Key Lemma (N.)

Assume gVP and that there are no virtually berkeley cardinals.
Then the virtually vopěnka filter on \mathbf{On} contains every class club.

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Assume gVP and that there are no virtually berkeley cardinals.
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Theorem (N.)

If there are no virtually berkeley cardinals then gVP implies that \mathbf{On} is virtually woodin.

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This, together with the equivalence of gVP and On being virtually woodin, gives us one direction of our theorem.

For the other direction we have the following:

Theorem (N.)

If there exists a virtually berkeley cardinal then gVP holds and On is not mahlo.

Outline of argument cont.

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Theorem (N.)

If there exists a virtually berkeley cardinal then gVP holds and On is not mahlo.

Virtually woodins are mahlo, and we (finally!) have our theorem:

Theorem (Dimopoulos-Gitman-N.)

The following are equivalent:

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A result that occurred along the way:

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This improves a result by Gitman and Hamkins (2019).

Question

Is the existence of a virtually berkeley cardinal equivalent to the statement that, for every class A , every virtually A -prestrong cardinal is virtually A -strong?

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Question

Are virtually ω -superstrongs equivalent to virtually berkeleys?

Wilson (2018) has shown that this is true in L .

Thank you for your attention.

ω -erős = Virtually berkeley = Virtually club berkeley =
Virtually ω -superstrong = Virtually totally reinhardt

Virtually rank-into-rank

Virtually vopěnka = Virtually woodin

Virtually $C(n)$ -extendible

Virtually extendible

Remarkable = Virtually measurable =
Virtually strong = Virtually supercompact