Jónsson cardinals - BLC '17

Question. Does every structure have a proper elementary substructure of the same size?

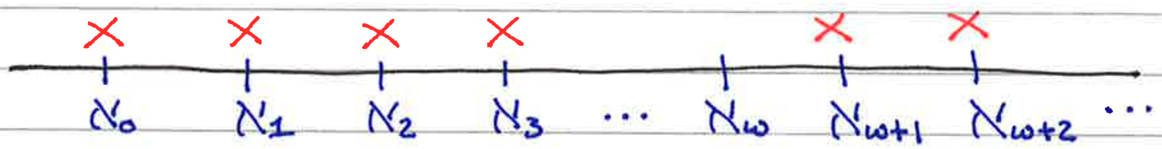
Nope! We can find countable groups with no countable subgroups. ($\mathbb{Z}_{p^\infty} := \bigcup_n \mathbb{Z}/p^n\mathbb{Z}$ with p prime)

Question. How about uncountable structures?

Nope! Using AC we can build uncountable counter-examples. (Due to Shelah ('80). It's a (simple) group of order $\aleph_1$)

Definition. A Jónsson cardinal is a cardinal κ such that every structure (in a countable language) of size κ has a proper elementary substructure of the same size.

But do they even exist? We saw before that $\aleph_0$ is not Jónsson and not every uncountable cardinal is Jónsson. But we know more than that.



The theorems used are the following:

- (Folklore) $\aleph_0$ is not Jónsson;
- (Tryba-Woodin '84) If λ is regular then λ^+ is not Jónsson
- (Rowbottom-Devlin '73) The least Jónsson cardinal is either weakly inaccessible or singular of countable cofinality.

It is still not known if $\aleph_\omega$ can be Jónsson, or if there can exist a successor Jónsson.

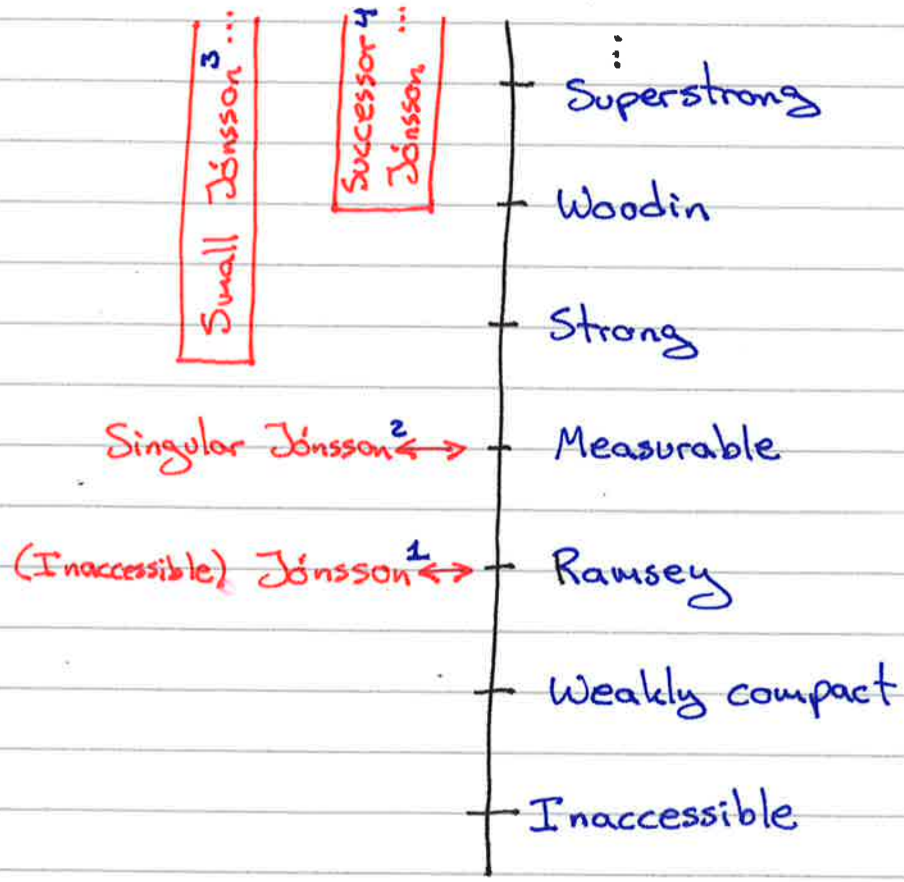
But why do we care about these cardinals?

This is mainly because several different characterisations have been found:

- Existence of elementary substructures
- Existence of embeddings into the universe
- Combinatorial colouring principles

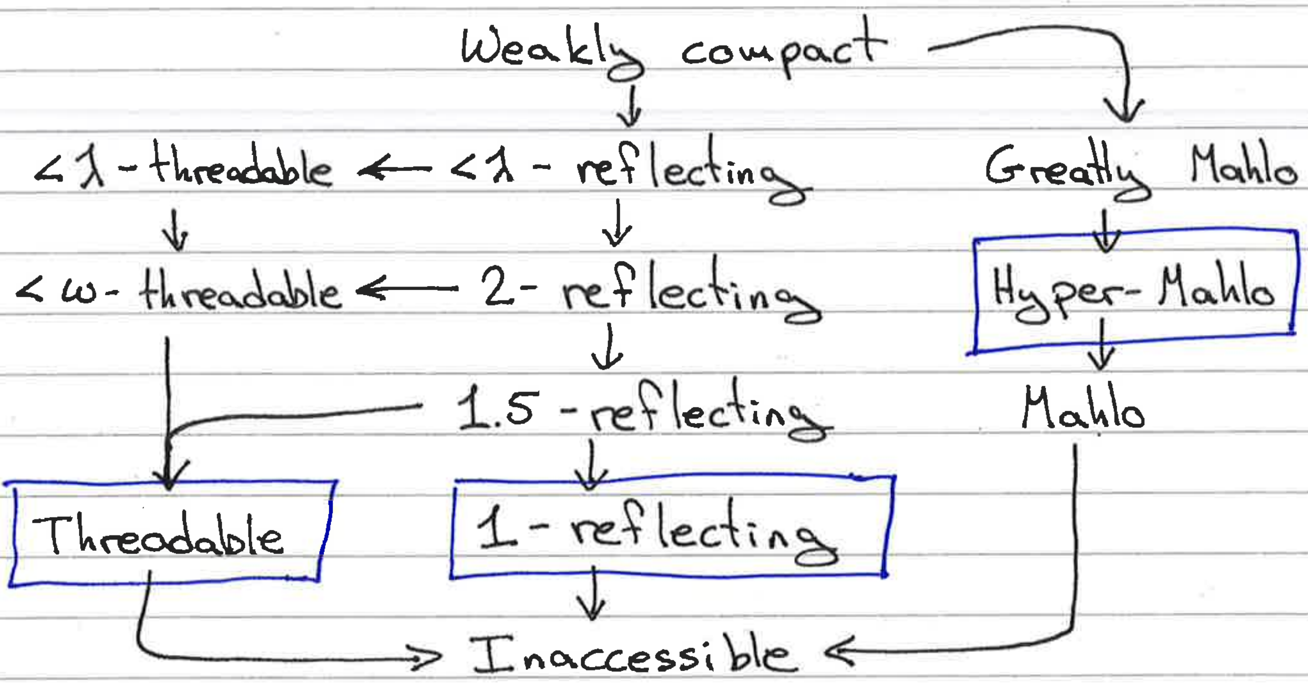
This makes the concept quite natural and they appear naturally "in nature".

We now know that Jónsson cardinals are "large cardinals", so that we can't prove the existence of them in ZFC. But how do they compare with the other known large cardinals?



1. Due to Mitchell ('[illegible]).
2. Due to Mitchell ('79) for the lower bound and Přikrý ('70) for the upper.
3. Due to Koepke ('88).
4. Due to Mitchell, Schimmerling and Steel ('97).

How about direct implication strength? Say, what does an inaccessible Jónsson directly imply? Let's zoom in on the bottom part of the large cardinal hierarchy, between inaccessibles and weakly compacts[5].



Let's introduce a few notions.

Definition. Let α be an ordinal. Then a subset $C \subseteq \alpha$ is *closed* if it contains all its limit points, and *unbounded* if for every $\gamma < \alpha$ we can find $\beta \in C$ with $\beta > \gamma$. We abbreviate closed unbounded to *club*. Further, say a subset $S \subseteq \alpha$ is *stationary* if it intersects every club.

Stationary sets are always unbounded, as tail segments of α are club. But stationary sets are not necessarily closed. It's another "largeness" concept.

5. The interactions between reflection and threadability is due to Hayut and Lambie-Hanson ('16).

We can now introduce two of the three hierarchies below weakly compact cardinals: the Mahlo-type cardinals and the reflection-type cardinals.

Definition. A cardinal κ is a Mahlo cardinal if the subset

$$\{\lambda < \kappa \mid \lambda \text{ is inaccessible}\}$$

is stationary.

Definition. A cardinal κ is a 1-reflecting cardinal if it's inaccessible and given any stationary subset $S \subseteq \kappa$ there's an $\alpha < \kappa$ satisfying that $S \cap \alpha$ is stationary (in α). We call such an α a reflection point.

The last hierarchy, the threadable-type cardinals, is a bit different.

Definition. A cardinal κ is threadable if it's inaccessible and every time we have a sequence $\vec{C} := \langle C_\alpha \mid \alpha < \kappa \rangle$ satisfying that

1) $C_\alpha \subseteq \alpha$ is club;
2) If $\gamma \in C_\alpha$ is a limit point then $C_\gamma = C_\alpha \cap \gamma$;

then there exists a club $C \subseteq \kappa$ such that for every $\alpha, \beta \in C$ with $\alpha \leq \beta$ it holds that $C_\alpha = C_\beta \cap \alpha$. We call such a C a thread through the sequence $\vec{C}$.

These are all the starting points for the three hierarchies, and the remaining notions are of the same idea, but simply strengthened.

Now, let's return to the Jónssons. We have the following three theorems concerning their direct implication strength.

- (Tryba-Woodin '84) Inaccessible Jónssons are 1-reflecting.
- (Shelah '98) Inaccessible Jónssons are hyper-Mahlo.
- (Rinot '14) Inaccessible Jónssons are threadable.

It's still open whether every inaccessible Jónsson is weakly compact. There's a distinction between truth and proof here though. It's been shown that[6]

ZFC + large cardinals[7]

can't prove the existence of an inaccessible Jónsson which isn't weakly compact, so the question is whether it can then prove the converse — the conjecture is no.[8] Either answer would reveal a great deal about the nature of these large cardinals though.

6. This is due to Apter ('09).
7. More precisely, ZFC + "$\mathcal{C}$ is the class of supercompacts".
8. Welch ('98).

As a last point, I would like to mention a result regarding something called sharps.

Definition. For any set X we say that $X^\#$ exists if there exists an elementary embedding

$$j : L(X) \longrightarrow L(X).$$

If an inaccessible Jónsson is weakly compact then $A^\#$ exists for every subset $A \subseteq \kappa$, so we can see this as another approximation to weak compactness. So is this true for any inaccessible Jónsson?

Theorem (Welch '98). If there's no inner model with a strong cardinal then every subset of a regular Jónsson cardinal has a sharp.

I recently improved the assumption to instead assuming that there's no inner model with a Woodin cardinal. This was quickly trumped though, as a recent result of Schindler and Steel ('14), paired with Rinot's ('14) result that regular Jónsson cardinals are threadable, shows that the conclusion of the above theorem is always the case! This then yields yet another property of Jónsson cardinals that approximate weak compactness.

— the end —