Model Theory
An introduction

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- Awesome theorems

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So.. what *is* model theory?

- Model theory is about *models* and *theories*
- Classical model theory = algebra + logic
- Syntax and semantics

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Structures

Definition

Let $\mathcal{L}$ be a language. Then an *$\mathcal{L}$-structure* $\mathfrak{M}$ is a set M along with

- an n-ary relation $R_i^{\mathfrak{M}} \subseteq M^n$ for each n-ary relation symbol $R_i \in \mathcal{L}$,
- an n-ary function $f_j^{\mathfrak{M}} : M^n \to M$ for each n-ary function symbol $f_j \in \mathcal{L}$
- a distinguished element $c_k^{\mathfrak{M}} \in M$ for each constant symbol $c_k \in \mathcal{L}$.

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Completeness (first kind)

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A *dense linear ordering* $(M, <)$ is an ordered set on which the ordering $<$ is

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- transitive
- total
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Completeness (first kind)

Let $\mathcal{L}_{strict} := \{<\}$ be the language of strict orderings.

Proposition

Let $\mathfrak{M}$, $\mathfrak{N}$ be two DLO's without endpoints and let σ be an $\mathcal{L}_{strict}$-sentence. Then

$$\mathfrak{M} \models \sigma \Leftrightarrow \mathfrak{N} \models \sigma.$$

Proposition

Let $\mathfrak{M}$ be a DLO without endpoints. There exists no $\mathcal{L}_{strict}$-sentence σ such that $\mathfrak{M} \models \sigma$ iff the ordering on $\mathfrak{M}$ is complete.

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Let $\mathcal{L}$ be a language.

Definition

A theory $\mathcal{T}$ is *complete* if for every $\mathcal{L}$-sentence σ, either $\mathcal{T} \models \sigma$ or $\mathcal{T} \models \neg\sigma$.

Theorem (Vaught's test)

Let $\mathcal{T}$ be a satisfiable $\mathcal{L}$-theory with no finite models and every model of cardinality κ is isomorphic, for some cardinal $\kappa \geq |\mathcal{L}|$. Then $\mathcal{T}$ is complete.

Corollary

Vec_∞ is complete in $\mathcal{L}_{vec}$, the language of vector spaces.

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Completeness (third kind)

Let $\mathcal{L}$ be a language and $\mathcal{T}$ an $\mathcal{L}$-theory.

Theorem (Gödels Completeness Theorem)

Let σ be an $\mathcal{L}$-sentence. Then

$$\mathcal{T} \models \sigma \Leftrightarrow \mathcal{T} \vdash \sigma.$$

Theorem (Compactness Theorem)

$\mathcal{T}$ is satisfiable iff every finite $\Delta \subseteq \mathcal{T}$ is satisfiable.

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If $\mathcal{T} \models \sigma$ then $\Delta \models \sigma$ for some finite $\Delta \subseteq \mathcal{T}$.

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Let σ be a sentence in the language of rings. Then $\mathrm{ACF}_0 \models \sigma$ if $\mathrm{ACF}_p \models \sigma$ for all primes p. In particular $\mathbb{C} \models \sigma$.

Theorem (Ax)

Every injective polynomial $p : \mathbb{C}^n \to \mathbb{C}^n$ is surjective.

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