Determinacy of games

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November 25, 2016

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The plan

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- (Loose) introduction to games and determinacy

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- Consequences of determinacy

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- Which games *are* determined?

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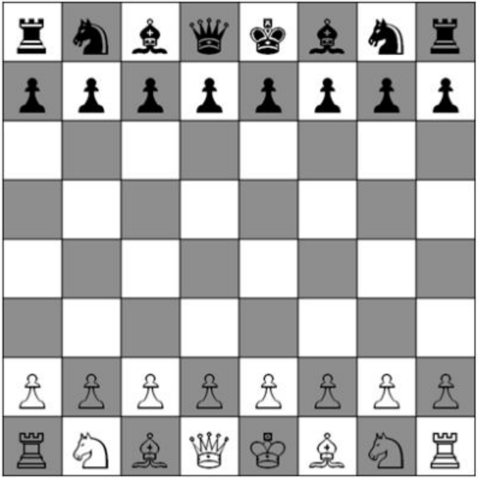
- Which games *are* determined?
- A bird's eye view

Games



Games

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Key properties:

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- 2 players

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Winning strategies

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Then player I has a winning strategy if

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Then player II has a winning strategy if

$$\forall x_0 \in \omega \exists x_1 \in \omega \cdots Q x_n \in \omega (\vec{x} \notin A)$$

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Say A is the set of winning moves for player I.

Then the game is determined if

$$\begin{aligned} &\neg\exists x_0 \in \omega \forall x_1 \in \omega \cdots Q x_n \in \omega (\vec{x} \in A) \\ \equiv &\forall x_0 \in \omega \exists x_1 \in \omega \cdots Q x_n \in \omega (\vec{x} \notin A) \end{aligned}$$

Switching focus

Key properties:

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We can identify the set of all such sequences $\vec{x}$ with the reals.

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The Davis Game

Let A be a set of reals, s_i finite $0-1$ sequences and $x_i \in \{0,1\}$. Then the Davis game $\mathcal{G}(A)$ is played as

$$\begin{array}{lcccccc} \mathrm{I} & s_0 & & s_1 & & \cdots & \\ \mathrm{II} & & x_0 & & x_1 & & \cdots \end{array}$$

Player I wins iff $s_0{}^\frown\langle x_0\rangle{}^\frown s_1{}^\frown\cdots \in A$.

Consequence of determinacy

The Continuum Hypothesis (CH)

Every infinite subset of the reals is either equinumerous with the integers or the reals.

Theorem (Davis 1964)

If $\mathcal{G}(A)$ is determined then CH holds for A. $\dashv$

Determinacy and choice

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Theorem (AC)

There is an undetermined game.

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There is an undetermined game.

(Proof on board)

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We also have the relativised versions $\Sigma^0_n(x)$ and $\Pi^0_n(x)$ for reals x, and say φ is $\boldsymbol{\Sigma}^0_n$ if φ is $\Sigma^0_n(x)$ for some real x.

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Theorem (Martin, Steel)

Large cardinals imply PD. $\dashv$

A bird's eye view

Note: $A \subset \mathbb{R}$ is projective iff A is definable in $\langle \mathcal{P}(\omega), \omega, \in, +, \cdot \rangle$

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Theorem (Woodin)

No matter what 'correct' axiom we choose, CH will turn out false.

Thank you!