Virtual large cardinals

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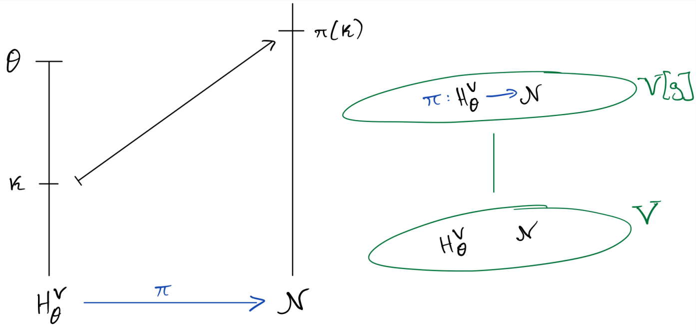
What are they?

Rough definition

A large cardinal κ defined via *set-sized* elementary embeddings is **virtual** if the elementary embeddings exist in a generic extension.



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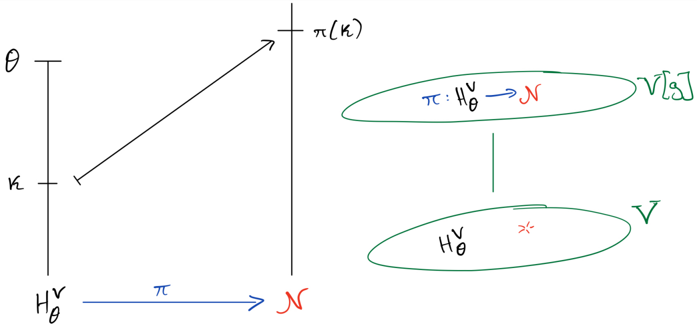
Rough definition

A large cardinal κ defined via *set-sized* elementary embeddings is **generic** if the elementary embeddings and the target model exist in a generic extension.



Why should we care?

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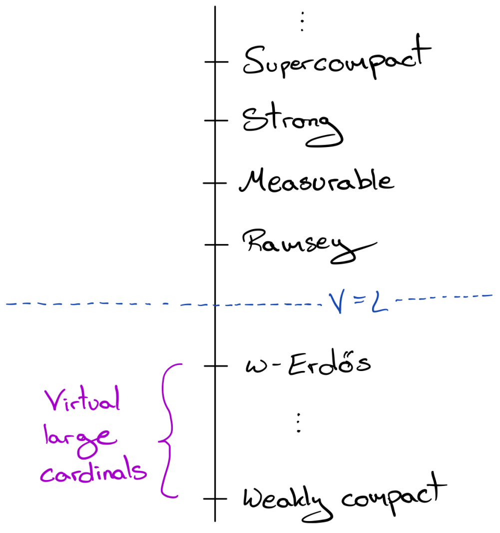
Theorem (Wilson '19)

A virtually Vopěnka cardinal is equiconsistent with $\Theta = \omega_2$ and $\boldsymbol{\Sigma}^1_2$ being the class of all ω_1-Suslin sets.

Where are they?



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II

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8. This game is called $\mathcal{G}^\theta_\gamma(\kappa)$. If we restrict I to only add $<|\gamma|$ sets at a time then we call the game $\mathcal{C}^\theta_\gamma(\kappa)$.

How do they behave level by level?

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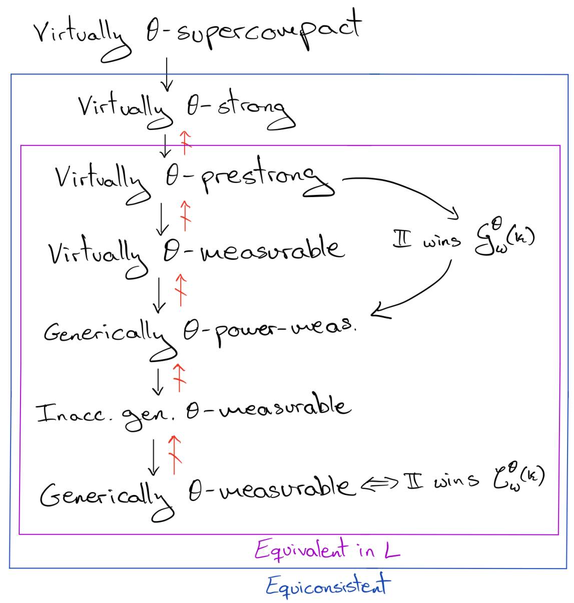

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4. Virtualising small embedding cardinals like Ramsey cardinals and below?
5. Indestructibility properties?

Thank you for your attention.