Virtual large cardinals

European Set Theory Conference, Vienna

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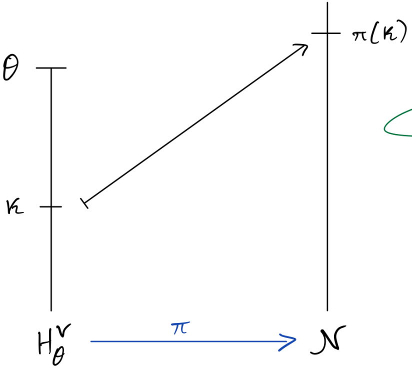
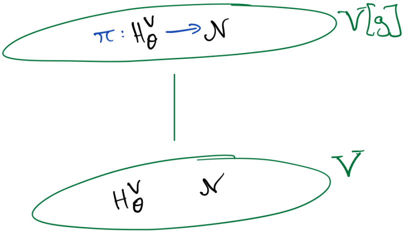
July 2019

What are they?

Rough definition

A large cardinal κ defined via *set-sized* elementary embeddings is **virtual** if the elementary embeddings exist in a generic extension.

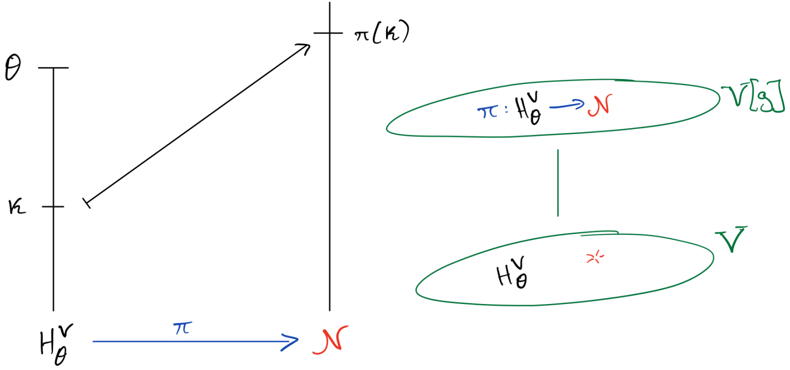




What are they?

Rough definition

A large cardinal κ defined via *set-sized* elementary embeddings is **generic** if the elementary embeddings and the target model exist in a generic extension.



Why should we care?

Theorem (Schindler '00)

A virtually strong cardinal is equiconsistent with $\mathrm{Th}(L(\mathbb{R}))$ being unchangeable by proper forcing.

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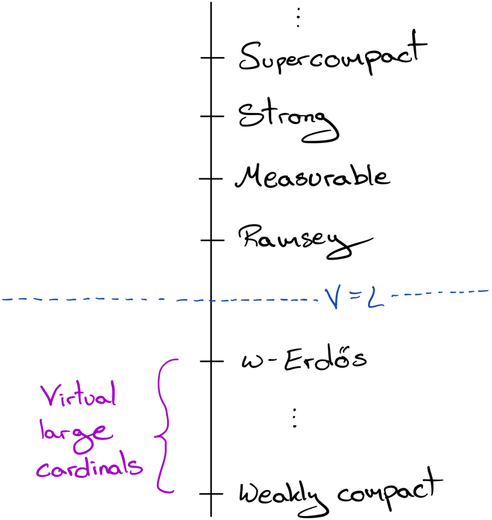
A virtually Shelah cardinal is equiconsistent with every universally Baire set of reals having the perfect set property.

Theorem (Wilson '19)

A virtually Vopěnka cardinal is equiconsistent with $\Theta = \omega_2$ and Σ^1_2 being the class of all ω_1-Suslin sets.

Where are they?



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Virtually strongs are equivalent to virtually supercompacts.

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Game interlude

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II

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8. This game is called $\mathcal{G}^\theta_\gamma(\kappa)$. If we restrict I to only add $<|\gamma|$ sets at a time then we call the game $\mathcal{C}^\theta_\gamma(\kappa)$.

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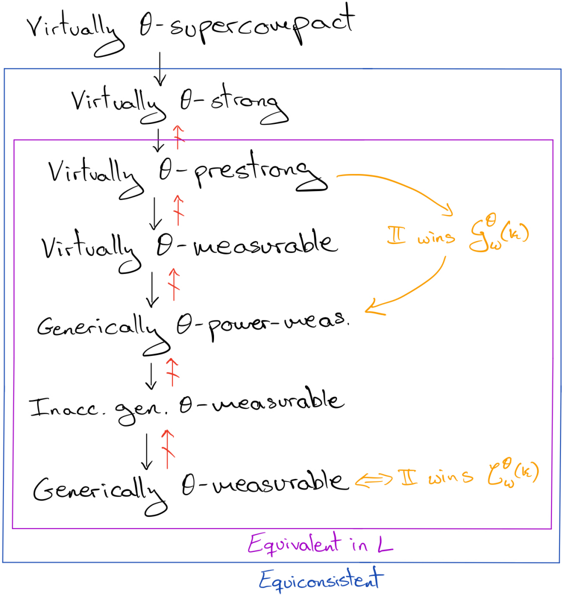
How do they behave level by level?



Future work

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6. Indestructibility properties?