Truth

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What is truth?

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- “There is a person in this room”
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- *Absolute* and *relative* truth

Mathematical truth

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    - Checking...

Intermezzo: Gödel encoding

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logical symbol	natural number
(	0
)	1
$\wedge$	2
$\vee$	3
$\neg$	4
$\rightarrow$	5
$\forall$	6
$\exists$	7
$\in$	8
$=$	9
variables v_i	$10 + i$

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- Write $\ulcorner - \urcorner$ as the corresponding number
- For a formula $\varphi = s_0 \cdots s_n$, set $\ulcorner \varphi \urcorner = p_0^{\ulcorner s_0 \urcorner} \cdots p_n^{\ulcorner s_n \urcorner}$

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  - Or rather, not definable *within* mathematics itself
- What about if we step *outside* of mathematics?

Tarski's definition of truth

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Definition (Tarski, 1935)

For a *set* M, a formula $\varphi(v_0, \ldots, v_n)$ and $x_0, \ldots, x_n \in M$, we can define the truth relation $M \models \varphi[x_0, \ldots, x_n]$ as follows:

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- If φ is $\psi \wedge \chi$ then $M \models \varphi[\vec{x}]$ iff $M \models \psi[\vec{x}]$ and $M \models \chi[\vec{x}]$;

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Note that this is definable *inside* mathematics!

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For M any collection, we have to go *outside* mathematics

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Note: we cannot prove that $M \models$ ZFC, for *any* set M.

Some fun with truth

For $x \in M$, let's "close $\{x\}$ under truth"

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Fact

For any formula $\varphi(v_0, \ldots, v_n)$ there exists a function $f_\varphi : M^n \to M$ such that for every $\vec{x} \in M^n$, either

$$M \models \varphi[f_\varphi(\vec{x}), \vec{x}] \quad \text{or} \quad M \models \neg\exists v : \varphi[v, \vec{x}]$$

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$$\begin{aligned} F_0 &:= \{x\} \\ F_{n+1} &:= \{f_\varphi(\vec{y}) \mid y_i \in F_n \wedge \varphi \text{ is a formula}\} \\ \mathcal{H}_x &:= \bigcup_{n\in\mathbb{N}} F_n \end{aligned}$$

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The Skolem paradox (1922)

Let M be a "sufficiently big" set, such that $\mathbb{R} \in M$ and $M \models$ "$\mathbb{R}$ is uncountable". By the above fact, $\mathcal{H}_{\mathbb{R}}$ satisfies this as well.

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Is this really a paradox?

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Is this really a paradox? No! Truth is *relative*

Our journey in truth

What is truth?

Math truth?

Tarski's truth

Strange truth

Recap

Our journey in truth

- Absolute mathematical truth is not definable inside mathematics

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- Absolute mathematical truth is not definable inside mathematics
- Relative mathematical truth *can* be defined inside mathematics

Our journey in truth

- Absolute mathematical truth is not definable inside mathematics
- Relative mathematical truth *can* be defined inside mathematics
- Truth can really mess with your mind

The end of the journey

Thank you!