An introduction to Model Theory
(work in progress)

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Chapter 1

Preliminaries

We will denote the language by L, or more precisely L will denote the set of non-logical symbols: symbols of constant, function symbols and relation symbols. The relation symbols will be denoted by capital letters (usually $P, Q, R \ldots$), the function symbols will be denoted usually by $f, g, h \ldots$, while the constants will be denoted by $c, d \ldots$.

The L-terms are build recursively, starting from constants and variables, by applying the following rule: if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is also a term. To denote terms we will use parenthesis $f(t_1, \ldots, t_n)$ although they are not necessary. We assume a countable set of variables Var usually denoted by, $x, y, z, x_1, \ldots, x_n, \ldots$.

The firs-order formulas are built starting with atomic formulas (formulas of type $t_1 = t_2$ or $Rt_1, \ldots, t_n$) using the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and the quantifiers $\forall, \exists$. We will use the fact that each formula is equivalent to one using only $\neg, \wedge, \exists$ (i.e., where $\vee, \rightarrow, \leftrightarrow, \forall$ do not occur) to simplify proofs by induction.

Usually we will use the same letter $M, N, \ldots$ to indicate the structure and its domain. The cardinal of a structure M is the cardinal of its domain and is denoted by $|M|$. A tuple $\overline{a}$ of M is a finite sequence $a_1, \ldots, a_n$ of elements of the domain of M and this will be denoted informally by putting $\overline{a} \in M$. The length of the tuple $a_1, \ldots, a_n$ is n.

If t is a term, $t(\overline{x})$ indicates that the variables of t are among those of $\overline{x}$, i.e., a subset of $\{x_1, \ldots, x_n\}$ where $\overline{x}$ denotes $x_1, \ldots, x_n$. If M is a structure and $\overline{a}$ is a tuple of elements of M of the same length as $\overline{x}$, $t^M(\overline{a})$ (or also $t(\overline{a})$ if there is no possible confusion) will denote the interpretation of t in M when assigning a_i to x_i. When we use this notation we are assuming that $\overline{x}$ and $\overline{a}$ have the same length and ordered in some manner, although not explicitly stated.

The interpretation of a term $t^M(\overline{a})$ is defined by recursion below. Observe that if $t = f(t_1, \ldots, t_n)$ has its variables among $\overline{x}$ the same happens with each term t_i. So we will always assume that if $t = f(t_1, \ldots, t_n) = t(\overline{x})$ then also $t_i = t_i(\overline{x})$.

Definition 1.0.1. Given a term $t(\overline{x})$ a structure M and a tuple $\overline{a} \in M$, we define the interpretation $t^M(\overline{a})$ as follows:

- If $t(\overline{x})$ is a constant c we put $t^M = c^M$,
- If $t(\overline{x})$ is a variable x_i we put $t^M = a_i$,
- If $t(\overline{x})$ is $f(t_1, \dots, t_n)$, we put $t^M(\overline{a}) = f^M(t_1^M(\overline{a}), \dots, t_n^M(\overline{a}))$.

For a formula φ, we write $\varphi(\overline{x})$ to indicate that $\overline{x}$ is a tuple of variables and the free variables φ are among those of $\overline{x}$. If $\overline{a}$ is a tuple of M we write $M \models \varphi(\overline{a})$ to indicate that that the formula φ holds in M when assigning the variables in the tuple $\overline{x}$ to the elements of the tuple $\overline{a}$ in an ordered manner (sending x_i by a_i for each and i). Again, we are assuming that $\overline{x}$ and $\overline{a}$ have the same length.

Below we define by recursion the satisfaction relation $M \models \varphi(\overline{a})$. In order to simplify, we assume here the formulas are built using the connectives $\neg, \wedge$ and the existential quantifier $\exists$. Observe that if $\exists y \psi$ has its free variables among $\overline{x}$ then ψ has its free variables among $y, \overline{x}$.

Definition 1.0.2. Given a formula $\varphi(\overline{x})$, a structure M and a tuple $\overline{a} \in M$, we say that φ holds in M with the assignment $x_i \mapsto a_i$, in symbols $M \models \varphi(\overline{a})$, as follows:

- If $\varphi(\overline{x})$ is $t_1 = t_2$ we put $M \models \varphi(\overline{a})$ iff $t_1^M(\overline{a}) = t_2^M(\overline{a})$.
- If $\varphi(\overline{x})$ is $R(t_1, \dots, t_n)$ we put $M \models \varphi(\overline{a})$ iff $(t_1^M(\overline{a}), \dots, t_n^M(\overline{a})) \in R^M$.
- If $\varphi(\overline{x})$ is $\neg\psi$ we put $M \models \neg\psi(\overline{a})$ iff $M \not\models \psi(\overline{a})$.
- If $\varphi(\overline{x})$ is $\psi_1 \wedge \psi_2$ we put $M \models \varphi(\overline{a})$ iff $M \models \psi_1(\overline{a})$ and $M \models \psi_2(\overline{a})$.
- If $\varphi(\overline{x})$ is $\exists y \psi(y, \overline{x})$ we put $M \models \varphi(\overline{a})$ iff $M \models \psi(b, \overline{a})$ for some $b \in M$.

Given a formula $\varphi(y_1, \dots, y_m)$ and terms $t_1(\overline{x}), \dots, t_m(\overline{x})$ we will denote by $\varphi(t_1, \dots, t_m)$ the result of substituting y_i by t_i simultaneously in a lexicographic variant of φ. A lexicographic variant of a formula is obtained by replacing bound variables by other new variables. It is known that replacing bound variables in φ by variables not occurring in any term t_i (in fact it is enough to replace only those that occur in some t_i) the substitution x_i/t_i turns out to be free. The substitution lemma becomes, with this notation

$$M \models \varphi(t_1, \dots, t_m)(\overline{a}) \text{ iff } M \models \varphi(t_1^M(\overline{a}), \dots, t_m^M(\overline{a})) \tag{1.0.1}$$

Very often these terms t_i will be constants c_i. In this case, the substitution lemma becomes:

$$M \models \varphi(c_1, \dots, c_m) \text{ iff } M \models \varphi(c_1^M, \dots, c_m^M) \tag{1.0.2}$$

To end, a little bit more notation:

Notation 1.0.3. When working with particular examples we will freely write assignments inside formulas. For instance, let L denote the language that contains only a single binary relation $<$ and M be the structure $(\mathbb{Z}, <^{\mathbb{Z}})$, where we denote here also by $<^{\mathbb{Z}}$ the strict order relation between integers. Let the formula $\varphi(x, y) = \exists z(x < z \land z < y)$. We will write $(\mathbb{Z}, <^{\mathbb{Z}}) \models \exists z(1 < z \land z < 3)$ to denote $M \models \varphi(1, 3)$. This is very useful notation in order to 'understand the formula'. However one must know that $\exists z(1 < z \land z < 3)$ is not a formula, since 1 and 3 are not constants of the language.

Likewise, if $\varphi(\overline{x})$ is $\exists y \psi(y, \overline{x})$ we put $M \models \exists y \psi(y, \overline{a})$ to denote $M \models \varphi(\overline{a})$. Again, here $\exists y \psi(y, \overline{a})$ is not a formula.

remark 1.0.4. The notation $\varphi(\ldots, \ )$ has many (maybe too many) uses and one has to be careful with the notation. If we write $\varphi(x_1, \ldots, x_n)$ it usually denotes that the free variables of φ is a subset of $\{x_1, \ldots, x_n\}$. Of course, this is because we assume that $x_1, \ldots, x_n$ are variables. If $y_1, \ldots, y_n$ is a new tuple of variables $\varphi(y_1, \ldots, y_n)$ denotes the same formula after the substitution $x_i \to y_i$. If $c_1, \ldots, c_n$ is a tuple of constants $\varphi(c_1, \ldots, c_n)$ denotes the result after the substitution $x_i \to c_i$. If $a_1, \ldots, a_n$ is a tuple of elements in some L-structure M, $\varphi(a_1, \ldots, a_n)$ is not a formula (so may no be written!), but we write $M \models \varphi(a_1, \ldots, a_n)$ to denote the satisfaction of φ in M under the assignment $x_i \to a_i$.

Infinitely many free variables can occur in an infinite set of formulas. In this situation one needs to assign infinitely many variables, say all variables to simplify. If Var denotes the set of all variables, an assignment into M is a map $s : Var \to M$. For a given formula $\varphi(\overline{x})$, we denote $M \models \varphi(s)$ when $M \models \varphi(\overline{a})$, where $a_i = s(x_i)$.

Definition 1.0.5. Let Σ be a set of formulas and φ a formula.

- The set Σ is called ***satisfiable*** iff there is some structure M and some assignment $s : Var \to M$ such that $M \models \varphi(s)$ for each formula $\varphi \in \Sigma$. This fact is denoted by $M \models \Sigma(s)$.
- the formula φ is a ***logical consequence*** of Σ (or Σ ***entails*** φ), in symbols: $\Sigma \models \varphi$, if $M \models \varphi(s)$ for each M and s such that $M \models \Sigma(s)$.

Usually, the formulas will be closed (without free variables) and called sentences. The cardinal of L, denoted by $|L|$, is the number of symbols of L (constant, function and relation). Unless otherwise stated, we work with a countable set of variables. In this situation there are $|L| + \aleph_0$ formulas.

1.1 The compactness Theorem

Our starting point are the theorems of compactness (Theorem 1.1.1), stated without proof.

Theorem 1.1.1 (compactness theorem, first version)**.** *Given a set of formulas Σ and a formula φ then:*

$$\Sigma \models \varphi \quad \textit{iff there is a finite subset } \Sigma_0 \textit{ of } \Sigma \textit{ such that } \Sigma_0 \models \varphi$$

Equivalently, the compactness theorem can be stated as follows:

Theorem 1.1.2 (compactness theorem, second version)**.** *Every set of formulas finitely satisfiable is satisfiable.*

Corollary 1.1.3. *Let Σ be a set of sentences. If every finite subset of Σ has a model then Σ has a model.*

Exercise 1.1.4.

1. Prove each versions of the compactness theorem using the other.
2. prove that from corollary 1.1.3 one can recover the compactness theorem 1.1.1.

Exercise 1.1.5. A class $\mathcal{K}$ of L-structures is called axiomatizable (also called Δ-elementary class or EC_Δ class) if it is of the form $Mod(T)$ for some L-theory T. In case T is finite we call $\mathcal{K}$ finitely axiomatizable (also called an elementary class). Use compactness to show:

1. The class of torsion groups is not axiomatizable.
2. The class of connected graphs is not axiomatizable.

Exercise 1.1.6. Let M be an L-structures. A subset $A \subseteq M^n$ is called definable if there is an L-formula $\varphi(\overline{x})$ such that $A = \{\overline{a} \in M^n \mid M \models \varphi(\overline{a})\}$. Here we implicitly assume that the length of $\overline{x}$ is n. Use compactness to show:

1. The torsion part of a group is not definable (more precisely: there is no formula $\varphi(x)$ such that in any group G φ defines its torsion part).
2. The connected component of a graph is not definable (more precisely: there is no formula $\varphi(x, y)$ such that in any graph G $M \models \varphi(a, b)$ iff a and b are in the same connected component).
3. The set of all algebraic (over the prime field) elements in a given field are not definable (more precisely: there is no formula $\varphi(x)$ such that in any field K, φ defines the algebraic elements of K).

1.2 Löwenheim-SkolemTheorems

Definition 1.2.1 (embeddings and substructures). An ***embedding*** is an injective map $h : M \to N$ satisfying the following properties:

1. $h(c^M) = c^N$ for every constant symbol c.

2. $h(f^M(\overline{a})) = f^N(h\overline{a})$ for every function symbol f and every tuple $\overline{a}$ of elements of M of the same length than the arity of f.

3. $\overline{a} \in R^M$ iff $h\overline{a} \in R^N$ for every tuple $\overline{a}$ of M and every relation symbol R.

Here $\overline{a}$ denotes $a_1, \ldots, a_n$ and $h\overline{a}$ denotes $ha_1, \ldots, ha_n$. When the identity is an embedding of M into N it is said that M is a ***substructure*** of N and we will denote it by $M \subseteq N$. An ***Isomorphism*** is a surjective embedding, i.e., a bijection preserving all the symbols of the language.

If M is a substructure of N, then obviously the domain of M is a subset of the domain of N. Moreover, the following holds:

1. $c^M = c^N$ for every constant symbol c.

2. $f^M(\overline{a}) = f^N(\overline{a})$ for every function symbol f and every tuple $\overline{a}$ of elements of M of the same length than the arity of f.

3. $\overline{a} \in R^M$ iff $\overline{a} \in R^N$ for every tuple $\overline{a}$ of M and every relation symbol R.

Observe that a substructure of N is determined by its domain, since the interpretation of the symbols of function and relation is just the restriction of the interpretation of the symbols in N. The domains D of the substructures of N is a subsets of N 'closed under application of functions'. That is, satisfying the following:

1. $c^N \in D$ for every constant symbol c.

2. $f^N(\overline{a}) \in D$ for every tuple $\overline{a}$ of D and functional symbol f[1].

Although not every subset A of N is the domain of a substructure, there is a smallest substructure of N conntaining A, denoted by $\langle A\rangle_M$. The domain of $\langle A\rangle_M$ is

$$\left\{t^N(\overline{a}) \mid t = t(\overline{x}) \text{ is a term and } \overline{a} \text{ is a tuple of elements of } A\right\} \tag{1.2.1}$$

Exercise 1.2.2. Prove that (1.2.1) is the smallest subset of N containing A and closed under application of functions.

We observe that if $h : M \to N$ is an embedding, $h(M)$ is the domain of a substructure of N, which we will denote also by $h(M)$, and h is an isomorphism from M to $h(M)$. Conversely, every isomorphism of M onto a substructure of N is an embedding. In other words, an embedding of M into N is an isomorphism of M onto a substructure of N. Hence the structures that may be embedded in N are the isomorphic copies of substructures of N.

Proposition 1.2.3. *If $h : M \to N$ is an embedding then there is an extension M' of M (M is a substructure of M') and an isomorphism $h' : M' \to N$ extending h.*

[1] although not stated it should be clear that the arity of f is the same as the length of $\overline{a}$

Proof. Let A be a set containing M and h' an extension of h which is a bijection from A to N. Now we define a structure M' with domain A by putting, for each tuple $\overline{a} \in A$:

$$\begin{aligned} \overline{a} \in R^{M'} &\Leftrightarrow h'\overline{a} \in R^N \\ f^{M'}(\overline{a}) &= h'^{-1} f^N(h'\overline{a}) \end{aligned} \tag{1.2.2}$$

By definition, h' is an isomorphism extending h. It remains to prove that M is a substructure of M'. For a tuple $\overline{a} \in M$, and a relation symbol we have: $\overline{a} \in R^{M'}$ iff $h\overline{a} = h'\overline{a} \in R^N$ iff $\overline{a} \in R^M$, since h is an embedding. For a function symbol f we have: $f^{M'}(\overline{a}) = h'^{-1} f^N(h'\overline{a}) = h'^{-1} f^N(h\overline{a}) = h'^{-1} h f^N(\overline{a}) = f^N(\overline{a})$. □

Exercise 1.2.4 (homomorphism's theorem). Let h be an embedding $h : M \to N$. Prove that:

1. For each term $t(\overline{x})$ of the language and each $\overline{a} \in M$ it holds that $ht^M(\overline{a}) = t^N(h\overline{a})$. [2]

2. h preserves all the formulas without quantifiers. That is, if $\varphi(\overline{x})$ is a formula without quantifiers, then

$$M \models \varphi(\overline{a}) \text{ iff } N \models \varphi(h\overline{a}) \text{ for all } \overline{a} \in M \tag{1.2.3}$$

3. If h is an isomorphism then it preserves all formulas, i.e., (1.2.3) holds for every formula φ.

Exercise 1.2.5. Let h be an embedding from M to N. Prove that:

1. If $\varphi(\overline{x})$ is an existential formula (it is of the form $\exists\overline{y}\psi$ with ψ quantifier-free), $\overline{a} \in M$ and $M \models \varphi(\overline{a})$ then $N \models \varphi(h\overline{a})$.

2. If $\varphi(\overline{x})$ is a universal formula (it is of the form $\forall\overline{y}\psi$ with ψ quantifier-free), $\overline{a} \in M$ and $N \models \varphi(h\overline{a})$ then $M \models \varphi(\overline{a})$.

Exercise 1.2.6. Prove the following are equivalent:

1. h is an embedding

2. h preserves all quantifier-free formulas

3. h preserves all atomic formulas

4. h preserves the following formulas: $x = y$, $x = c$ for each constant c, $Rx_1, \ldots, x_n$ for each relation symbol R, $fx_1, \ldots, x_n = y$ for each function symbol f.

Definition 1.2.7. An ***elementary embedding*** is an embedding preserving all formulas, that is, satisfying (1.2.3) for every formula φ. In the case where the identity is elementary we will say that M is an ***elementary substructure*** of N and we will denote it by $M \preccurlyeq N$.

[2] In fact this is true for any homomorphism h

Point *3.* in 1.2.4 says that any isomorphism is an elementary embedding.

Definition 1.2.8. Two structures M, N are ***elementarily equivalent***, $M \equiv N$ in symbols, if they satisfy the same sentences, that is, have the same theory.

If there is an elementary embedding of M into N then M and N are elementarily equivalent. However not every embedding between elementarily equivalent structures is elementary: $(2\mathbb{Z}, <)$ is a substructure of $(\mathbb{Z}, <)$ that is isomorphic to it, however the formula $\exists x(0 < x < 2)$ holds in $(\mathbb{Z}, <)$ but not in $(2\mathbb{Z}, <)$.

We observe that an elementary embedding of M into N is an isomorphism of M onto an elementary substructure of N.

Exercise 1.2.9. Prove that if $h : M \to N$ is an elementary embedding then there is an elementary extension M' of M (M is an elementary substructure of M') and an isomorphism $h : M' \to N$ that extends h.

Theorem 1.2.10 (Tarski-Vauth's test)**.** *Let N be a structure and A a subset of N. Then A is the domain of an elementary substructure of N iff for every formula $\varphi(x, \overline{y})$ and every tuple $\overline{a}$ of A such that $N \models \exists x\varphi(x, \overline{a})$ there exists an element b of A such that $N \models \varphi(b, \overline{a})$.*

Proof. The implication from left to right is easy and we leave it to the reader. In order to see the converse we assume that M satisfies the condition of the statement. By applying the condition to the formula $\exists x(x = f(\overline{y}))$ we begin by seeing that A is the domain of a substructure M of N. Next we see, by induction on $\varphi(\overline{x})$, that for every tuple $\overline{a}$ of M that $M \models \varphi(\overline{a})$ iff $N \models \varphi(\overline{a})$. For atomic formulas it is done in exercise 1.2.4. For the conjunction and the negation is easy to check. The only delicate point is when $\varphi(\overline{x}) = \exists y\psi(y, \overline{x})$. If $M \models \exists y\psi(y, \overline{a})$ then $M \models \psi(b, \overline{a})$ for a certain $b \in M$ and therefore $N \models \psi(b, \overline{a})$ by hypothesis of induction. Conversely, if $N \models \exists y\psi(y, \overline{a})$, by hypothesis, there is a b in M such that $N \models \psi(b, \overline{a})$. But then, by hypothesis of induction again, $M \models \psi(b, \overline{a})$ and therefore $M \models \exists y\psi(y, \overline{a})$. $\square$

Exercise 1.2.11.
1. Let M be a substructure of N satisfying the following property: given a finite tuple $\overline{a} \in M$ and $b \in N$ there is an automorphism h of N with $h(b) \in M$ and $h(\overline{a}) = \overline{a}$. Show that then $M \preceq N$. Hint: use Theorem 1.2.10.
2. Apply 1 this to show that given $M \subseteq N$ two dense linear orderings without endpoints $M \preceq N$.
3. Let R be any ring. Let $R[X]$ denote the polynomial ring in the set of variables X. Prove that if X is an infinite set of variables and $X \subseteq Y$ then $R[X] \preceq R[Y]$ in the ring language $\{+, -, ., 0, 1\}$. Hint: Use 1.

The following result is an improvement of the downward Löwenheim-Skolem theorem.

Theorem 1.2.12 (Löwenheim-Skolem-Tarski)**.** *If A is a subset of N, then there is an elementary substructure of N containing A of cardinal at most $|A| + |L| + \aleph_0$.*

Proof. For each formula $\varphi(y,\overline{x})$ we consider the following choice function $F_{\varphi(y,\overline{x})}:N^n \to N$ (where n is the length of $\overline{x}$).

$$F_{\varphi(y,\overline{x})}(\overline{a}) = \begin{cases} b, & \text{if it exists an element } b \text{ of } N \text{ such that } N \models \varphi(b,\overline{a}); \\ c, & \text{otherwise.} \end{cases},$$

where c is a fixed element of N. If M contains A and is closed under all the maps $F_{\varphi(y,\overline{x})}$, by Tarski-Vaught's test 1.2.10 M is an elementary substructure of N. Let M be the closure of A under all the maps $F_{\varphi(y,\overline{x})}$. This M is built in a countable number of steps as follows. We start with $A_0 = A$ and take $A_{m+1} = A_m \cup \bigcup_{\varphi(y,\overline{x})\in L} F_{\varphi(y,\overline{x})}(A_m^n)$. Of course at any step $|A_n| \leq |A|+|L|+\aleph_0$ and therefore if we put $M = \bigcup_n A_n$, the cardinal of M is also bounded by $|A|+|L|+\aleph_0$. □

As a corollary we obtain the usually called downwards Löwenheim-Skolem theorem. Although it is also true for formulas in general (appropriately formulated) we state it for sentences.

Theorem 1.2.13 (downwards Löwenheim-Skolem Theorem)**.** *If a set of sentences has models, it also has models of cardinality at most $|L|+\aleph_0$.*

Proof. Let Σ be a set of sentences and N be a models of Σ. By Theorem 1.2.12 (with $A = \emptyset$) there is some elementary substructure M of N os size at most $|L|+\aleph_0$. Since $M \preceq N$, M and N must satisfy the same sentences, so M is the desired model of Σ. □

A simple consequence of downwards Löwenheim-Skolem theorem using compactness, is the upwards Löwenheim-Skolem theorem. It says that a theory having infinite models (or arbitrarily large finite models) also has models in any cardinal bigger or equal to $|L|+\aleph_0$.

Theorem 1.2.14 (upwards Löwenheim-Skolem Theorem)**.** *Let Σ be a set of sentences such that for each n, Σ has a model of size at least n. Then, for each cardinal $\lambda \geq |L|+\aleph_0$, Σ has a model of cardinal λ.*

Proof. We take a set C of new constants of cardinal λ and we consider the theory:

$$\Psi := T \cup \{c \neq d \mid c,d \text{ are two different constants of } C\}.$$

Every finite subset Ψ_0 of Ψ contains finitely many constants, say n. By hypothesis Σ has a model M of size at least n. Hence Ψ_0 is satisfied in a suitable expansion of M interpreting the new constant as different elements. Because of compactness, Ψ has a model. By downwards Löwenheim-Skolem Theorem 1.2.13 there is a model M of Ψ of cardinal at most $\lambda+|L|+\aleph_0 = \lambda$. But as the λ different constant have to be interpreted as different elements it follows that M has cardinal exactly λ. The reduct of M to L is a models of Σ of cardinal λ. □

As a consequence, if a theory has only finite models, there is a finite bound on the cardinality of its models. The upwards Löwenheim-Skolem theorem says us that the set of infinite cardinals where a countable theory has models is either the empty set or all infinite cardinals. If the language is uncountable we know that either the set of infinite cardinals where a theory has models is either the empty set or contains $\{\lambda \mid \lambda \text{ is a cardinal and } |L| + \aleph_0 \leq \lambda\}$. However it does no says anything about the cardinals in the interval $\{\lambda \mid \lambda \text{ is a cardinal and } \aleph_0 \leq \lambda < |L|\}$. The following examples show that we cannot say much more with absolute generality.

Example 1.2.15.

1. Let λ be an uncountable cardinal and C a set of κ constants. The set $\{c \neq d \mid c, d \text{ are two different constants of } C\}$ has no models of size less than $|L| = \kappa$.
2. Consider the language L which contains, for each subset S of $\mathbb{N}$ a unary predicate P_S. Let the L-structure N whose domain is $\mathbb{N}$ and P_S is interpreted as S. Obviously $\mathrm{Th}(N)$ has a countable model although $|L| = 2^{\aleph_0}$.

In the following exercise we will see that for finite cardinals the situation is radically different.

Exercise 1.2.16. The finite spectrum of a sentence φ is the set of finite cardinals where the sentence has models: $\{n \in \omega \mid \varphi \text{ has a model of cardinal } n\}$. The finite spectrum can be quite complicated. Construct formulas such that its finite spectrum is the following one: $n\mathbb{N} = \{nm \in \mathbb{N} \mid m \in \mathbb{N}\}$, where n a fixed natural number n. The set of all square numbers: $\left\{m^2 \in \mathbb{N} \mid m \in \mathbb{N}\right\}$. The set of all prime numbers. The set of all non prime numbers. The set of all powers of some prime number. the set of all powers of a fixed prime number p.

1.3 Complete theories: the test of Łoš-Vaught

A theory will be a set of sentences. Many textbooks of mathematical logic define a theory as a set of sentences closed under logical consequence. Every set of sentences Σ determines a unique set closed under logical consequence having the same models: the set of their logical consequences

$$\Sigma^{\models} := \{\varphi \mid \varphi \text{ is a sentence and } \Sigma \models \varphi\}.$$

Since we are interested on its models this does not make any difference.

Exercise 1.3.1. For a class $\mathcal{K}$ of L-structures, denote

$$\mathrm{Th}(\mathcal{K}) = \{\varphi \mid \varphi \text{ is a sentence and } M \models \varphi \text{ for each } M \in \mathcal{K}\}.$$

1. Shows that $\Sigma^{\models} = \mathrm{Th}(\mathrm{Mod}(\Sigma))$.
2. Let Σ_1, Σ_2 be sets of sentences. Shows that the following are equivalent:
   (a) $\Sigma_1^{\models} = \Sigma_2^{\models}$.
   (b) Σ_1 and Σ_2 have the same models.

Definition 1.3.2. A theory T is said to be ***complete*** if for every sentence φ we have that $T \models \varphi$ or $T \models \neg\varphi$.

The basic example of complete theory is the theory of a structure: $\operatorname{Th} M = \{\varphi \mid \varphi \text{ is a sentence and } M \models \varphi\}$. As a matter of fact, apart from the inconsistent theory (that consists of all sentences) there are not more: it is easy to verify that every consistent and complete theory is the theory of any of its models.

Exercise 1.3.3. 1. $M \equiv N$ iff N is a model of $\operatorname{Th}(M)$.

2. A theory is complete iff all its models are elementarily equivalent.

Definition 1.3.4. Let λ be an infinite cardinal. A theory T is called ***categorical in λ or λ-categorical*** if any two models of T of cardinal λ are isomorphic.

The λ-categoricity of T says exactly that, except of isomorphism, T has at most one model. If T has infinite models and $\lambda \geq |L| + \aleph_0$, by upwards Löwenheim-Skolem there has to be exactly one model up to isomorphism.

By the homomorphism theorem (exercise 1.2.4) if M is isomorphic to N then M and N are elementarily equivalent. Given any infinite structure M, by the upwards Löwenheim-Skolem theorem $\operatorname{Th} M$ has models N of arbitrarily large size, hence not isomorphic to M. In other words, if M is infinite there are structures elementarily equivalent but not isomorphic to M. The next proposition shows that the case where M is finite is completely different.

Proposition 1.3.5. *1. If M and L are finite, there is a sentence φ_M that characterizes M up to isomorphism, that is, for any N, N is a model of φ_M iff N it is isomorphic to M.*

2. If M is finite and $N \equiv M$ then M and N are isomorphic.

Proof. 1. Let $M = \{a_1, \dots, a_n\}$. Consider the sentence $\varphi_M := \exists x_1, \dots, x_n \psi_M$, where

$$\psi_M := \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_i \wedge \forall y \left(\bigvee_{i=1}^{n} y = x_i \right) \wedge \bigwedge_{s \in L} \psi_s \right),$$

and the formula ψ_s describes the interpretation of the symbol s in M as follows. If s is a constant c and $c^M = a_i$ then ψ_c is the formula $c = x_i$. If s is an m-ari function symbol f,

$$\psi_f := \bigwedge_{1 \leq i_1, \dots, i_m \leq n} f(x_{i_1}, \dots, x_{i_m}) = x_{g(i_1, \dots, i_m)},$$

where $f^M(a_{i_1}, \dots, a_{i_m}) = a_{g(i_1, \dots, i_m)}$. When s is an m-ari relation symbol R

$$\psi_R := \bigwedge_{(a_{i_1}, \dots, a_{i_m}) \in R^M} R x_{i_1}, \dots, x_{i_m} \wedge \bigwedge_{(a_{i_1}, \dots, a_{i_m}) \in M^m \setminus R^M} \neg\, R x_{i_1}, \dots, x_{i_m}.$$

Obviously $N \models \psi_s(b_1, \dots, b_n)$ iff the map $a_i \mapsto b_i$ preserves the symbol s. Thus $N \models \psi_M(b_1, \dots, b_n)$ iff $a_i \mapsto b_i$ is an isomorphism and $N \models \varphi_M$ iff $N \cong M$.

2. By *1*, the case when L is finite is obvious. Assume now $M \equiv N$ and L arbitrary. Obviously N is finite and of the same cardinalilty as M. Towards a contradiction, assume $M \not\cong N$. For each bijection g from M to M there is a symbol s_g not preserved by g. Since there are finitely many bijections from M to N the Language $L' := \{s_g \mid g : M \to N \text{ bijective}\}$ is finite and $M \restriction L' \not\cong N \restriction L'$ whereas $M \restriction L' \equiv N \restriction L'$, a contradiction by *1*. □

Observe that, by the previous proposition, if a complete theory only has finite models, as a matter of fact it has only one model (up to isomorphism). A quite important question is to know when a given theory (without finite models) is complete. A first simple case is when it is categorical in some cardinal.

Theorem 1.3.6 (Łoš-Vaught's test)**.** *A theory without finite models and categorical in some cardinal greater or equal to $|L| + \aleph_0$ is complete.*

Proof. Suppose that T is λ-categorical, with $\lambda \geq |L| + \aleph_0$ and without finite models. We will see that any two models M and N of T are elementarily equivalent. Consider the theories of M and N respectively. Being M and N infinite we can apply upwards Löwenheim-Skolem and we obtain M' and N' elementarily equivalent to M and N respectively of cardinal λ. Then M' and N' are two models of T of cardinal λ, therefore isomorphic. Finally M' and N' are elementarily equivalent, and therefore also M and N. □

For instance, the theory of dense linear orders without endpoints is complete, because is $\aleph_0$-categorical and has no finite models. It is easily seen that this theory is not categorical in any other cardinality.

If λ is an infinite cardinal number, $I(\lambda, T)$ denotes the number of models of T of cardinal λ up to isomorphism. Each model of cardinality λ has an isomorphic copy with domain λ. Hence $I(\lambda, T)$ is at most the number of structures with domain λ. For each constant symbol we have λ choices, and for each function symbol and relation symbol we have 2^λ choices. This shows that there are at most $(2^\lambda)^{|L|} = 2^{\lambda+|L|}$ structures with domain λ. This shows that for $\lambda \geq |L|+\aleph_0$ one gets $I(\lambda, T) \leq 2^\lambda$. If T has infinite models and $\lambda \geq |L|+\aleph_0$ the Löwenheim-Skolem theorem 1.2.14 tells us that $I(\lambda, T) \geq 1$.

Exercise 1.3.7. Prove that:

1. The theory of an infinite set is categorical in any cardinal, i.e., $I(\lambda, T) = 1$ for each infinite cardinal λ. This is the theory, in the empty language axiomatized by $\{\varphi_{\geq n} \mid n \in \mathbb{N}\}$, where $\varphi_{\geq n}$ is the following formula: $\exists x_1 \cdots \exists x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right)$.

2. The theory of an equivalence relation with infinitely many classes all infinite. The language contains a single binary relation. This theory is $\aleph_0$-categorical but it is not categorical in any other cardinal. It is not difficult to see that $I(\aleph_n, T) = \aleph_0$ for $0 < n < \omega$ and $I(\aleph_\alpha, T) = 2^{|\alpha|}$ for $\alpha \geq \omega$ (Hint: each model is characterized by the function $\alpha+1 \to \omega \cup \{\aleph_\beta \mid \beta \leq \alpha\}$ which computes, for each $\beta \leq \alpha$ the number of classes of size $\aleph_\beta$.

3. If k is a finite field, the theory of the infinite k-vector spaces is categorical in any infinite cardinal.

4. If k is an infinite field, the theory of nontrivial k-vector spaces is categorical only in cardinality $> |k|$. Observe that in this case $|L| = |k|$. Show that $I(\lambda, T) = 0$ for $\aleph_0 \leq \lambda < |k|$, $I(|k|, T) = |k|$ and $I(\lambda, T) = 1$ for $\lambda > |k|$.

5. The theory of all algebraically closed fields of a fixed characteristic is categorical in any non countable cardinal, however it is not $\aleph_0$-categorical. This uses the criterion of Steinitz for isomorphism of algebraically closed fields.

6. The theory T_{DLO} of the dense linear orders (more precisely: dense linear orders without endpoints) i $L = \{<\}$. T_{DLO} is axiomatized by $\forall x \forall y \exists z (x < y \to x < y < z)$, $\forall x \exists y \exists z (y < x < z)$ plus an axiom expressing that $<$ is linear order. Then T_{DLO} is $\aleph_0$-categorical (we will prove it later using Back and Forth) but it is not categorical in another infinite cardinal. In fact $I(\aleph_\alpha, T) = \aleph_\alpha$ for any $\alpha \geq 1$.

By Theorem 1.3.6, all these are examples of complete theories.

Exercise 1.3.8. Prove that:

1. If T is finitely axiomatizable (there is a finite set of sentences with the same models as T) then there exists a finite $T_0 \subseteq T$ such that T_0 has the same models as T.

2. Deduce that T_{DLO} is, among the of theories of the previous exercise, the only one which is finitely axiomatizable.

3. Show tat the class of fields of characteristic zero is not finitely axiomatizable.

M. Morley proved, in the year 1967, that if a countable theory (a theory in a countable language) is categorical in some non countable cardinal the it is categorical in all non countable cardinals. This result is known as Morley's Theorem. This is not true if we include $\aleph_0$ as the examples above show. We have seen examples of totally categorical (in countable and uncountable cardinals), categorical theories only in countable cardinal, categorical theories in all the non countable cardinals but not in $\aleph_0$, and complete theories that are not categorical in any cardinal. For countable T, the function $I(\cdot, T)$ is increasing for $\lambda \geq \aleph_1$ although this is a very deep result of Shelah. In fact Shelah proves the following conjecture of Morley: for countable T and not uncountably categorical, the function $I(\cdot, T)$ is increasing for $\lambda \geq \aleph_0$.

1.4 Elementary chains

Definition 1.4.1 (chain, elementary chain). A ***chain of structures*** is a collection $\langle M_i \mid i \in I \rangle$, where $(I, \leq)$ is totally orderly and such that $M_i \subseteq M_j$

whenever $i \leq j$. When M_i is an elementary substructure of M_j for every $i \leq j$ we will say that it is an ***elementary chain***.

When we have a chain of structures, the structure ***union of the chain***, denoted by $\bigcup_{i\in I} M_i$, is constructed as follows. Its domain is the union of the domains and the symbols are interpreted also taking the union of their interpretations. For instance, if R is a relation symbol $R^{\bigcup_{i\in\alpha} M_i} = \bigcup_{i\in\alpha} R^{M_i}$. For the function symbols we do the same with the graph of the function. Equivalently, if f is an n-ari function symbol and $\overline{a}$ is an n-tuple of elements of $\bigcup_{i\in I} M_i$, we take some M_j where all the elements of the tuple belong. Then we put $f^{\bigcup_{i\in I} M_i}(\overline{a}) := f^{M_j}(\overline{a})$. Being a chain, this does not depend on the chosen M_j. We treat the symbols of constant like symbols of function of arity 0. It is easy to verify that each M_i is a substructure of the union.

Exercise 1.4.2. Sow that $\forall\exists$-sentences (sentences of the form $\forall\overline{x}\exists\overline{y}\psi$ with ψ quantifier-free) are preserved under unions of chains. More precisely, if $\langle M_i \mid i \in I\rangle$ is a chain of structures and a $\forall\exists$-sentence φ holds in any M_i then φ also holds in the union of the chain $\bigcup_{i\in I} M_i$. Use 1.2.4.

The followings exercise aims to show that we can do the same construction in a wider situation: working wit embeddings and a non-necessarily ordered set.

Exercise 1.4.3. A directed system of embeddings consists is a collection of structures $\langle M_i \mid i \in I\rangle$, where $(I, \leq)$ is a directed (for any $i, j \in I$ there is some $k \in I$ with $i \leq k, j \leq k$) partially ordered set, together with a collection of embeddings $f_{i,j} : M_i \to M_j$ for any pair $i, j \in I$ with $i \leq j$, satisfying $f_{j,k} \circ f_{i,j} = f_{i,k}$. Define the direct limit $\lim_i M_i$ of the system and embeddings $h_i : M_i \to \lim_i M_i$ and show that $h_j \circ f_{i,j} = h_i$.

Lemma 1.4.4 (chain's lemma)**.** *If the chain $\langle M_i \mid i \in I\rangle$ is elementary, then each M_i is an elementary substructure of the union:*

$$M_i \preccurlyeq \bigcup_{j\in I} M_j.$$

Proof. We prove, by induction on the formula $\varphi(\overline{x})$, that for each i and each tuple $\overline{a}$ of M_i the following holds

$$M_i \models \varphi(\overline{a}) \text{ iff } \bigcup_{i\in I} M_i \models \varphi(\overline{a})$$

For the atomic formulas we know it because M_i is a substructure of the union. The cases of the negation and the conjunction are trivial. Let now $\varphi(\overline{a}) = \exists x\psi(x, \overline{a})$. If $M_i \models \exists x\psi(x, \overline{a})$ there is some $b \in M_i$ with $M_i \models \psi(b, \overline{a})$. By hypothesis of induction, $\bigcup_{i\in I} M_i \models \psi(b, \overline{a})$ and thus $\bigcup_{i\in I} M_i \models \exists x\psi(x, \overline{a})$. Conversely, if the formula $\exists x\psi(x, \overline{a})$ holds in $\bigcup_{i\in I} M_i$ we have that $\bigcup_{i\in I} M_i \models \psi(b, \overline{a})$ for a certain $b \in M_j$ with $j \geq i$. By hypothesis of induction $M_j \models \psi(b, \overline{a})$. Hence $M_j \models \exists x\psi(x, \overline{a})$ and, as the chain is elementary $M_i \models \exists x\psi(x, \overline{a})$. □

The chains of structures are usually indexes in an ordinal α and in the limits are take unions (that is, $M_i = \bigcup_{j<i} M_j$ for every $i < \alpha$ limit). In this case the chain is is said to be ***continuous***.

Exercise 1.4.5. A continuous chain of structures (indexed in an ordinal) is elementary if for each i, M_i is an elementary substructure of M_{i+1}.

Exercise 1.4.6. This is a continuation of exercise 1.4.3. Show that the chain lemma holds for a direct limit of structures. More precisely, If $(M_i \mid i \in I)$ and $(f_{i,j}) \mid i \leq j \in I$ for a directed system of elementary embeddings (i.e., each $f_{i,j}$ is elementary) then the maps defined in $h_i : M_i \to \lim_i M_i$ defined in 1.4.3 al also elementary.

Exercise 1.4.7. A theory is called model complete iff every embedding between models of T is elementary. Prove that T is model complete if given M, N models of T and an embedding $f : M \to N$ there is some elementary exetension M^* of M and an embeddings $g : N \to M^*$ such that $\mathrm{Id} = g \circ f$. *Hint: given* $f : M \to N$ *build elementary chains* $\langle M_n \mid n \in N \rangle$, $\langle M_n \mid n \in N \rangle$ *and embeddings* $f_n : M_n \to N_n$ $g_n : N_n \to M_{n+1}$ *with* $g_n \circ f_n = \mathrm{Id}$ $f_{n+1} \circ g_n = \mathrm{Id}$. *Start with* $M_0 = M$, $N_0 = N$ *and* $f_0 = f$. *Apply the hypothesis to obtain* $M_1 = M^*$ *and* $g_0 = g$. *Apply again the hypothesis to* $g_0 : N_0 \to M_1$ *to obtain* N_1 *and* f_1 *and so on.*

1.5 Diagrams

We often work with two languages: L and an extension L' of it (that is, all symbols of L are in L'), which is denoted by $L \subseteq L'$. In this case any L'-structure M' becomes an L-structure M naturally forgetting the interpretation of the symbols of $L' - L$. In this case M is called the ***restriction of*** M' ***to*** L and denoted by $M = M' \restriction L$. One can also say that M' is an ***expansion of*** M ***to*** L.

Now we will consider expansions of a language L by constants. If M is a L-structure, we will expand L by adding a new constant for each element of M. For simplicity we will use the same symbol to denote the constant and the element of M that designates. We will denote this expansion by $L(M)$. We will denote by M_M the natural expansion of M to $L(M)$, where each new constant a is interpreted as the element a of M. If N is another L-structure and h a map from M to N the expansion of N to $L(M)$ that interprets a as ha will be denoted by $(N, ha)_{a \in M}$.

Every sentence of $L(M)$ is obtained by replacement of the variables $\overline{x}$ of an L-formula $\varphi(\overline{x})$ by constants $\overline{a} \in M$. This sentence will be denoted by $\varphi(\overline{a})$. With this notation, the Substitution Lemma (1.0.2) becomes:

$$M_M \models \varphi(\overline{a}) \text{ iff } M \models \varphi(\overline{a}). \tag{1.5.1}$$

In the same way

$$(N, ha)_{a \in M} \models \varphi(\overline{a}) \text{ iff } N \models \varphi(h\overline{a}). \tag{1.5.2}$$

remark 1.5.1. in (1.5.1) the expression $\varphi(\overline{a})$ denotes a formula in $M_M \models \varphi(\overline{a})$ while the same expression $\varphi(\overline{a})$ does not denote a formula in $M \models \varphi(\overline{a})$. The same applies to (1.5.2). The equivalence (1.5.1) says that the notational confusion remarked in 1.0.4 is harmless here.

The ***diagram of*** M is the set of $L(M)$-formulas without quantifiers holding in M_M:

$$Diag(M) = \{\varphi \in L(M) \mid \varphi \text{ is without quantifiers } M_M \models \varphi\}\,.$$

The ***elementary diagram of*** M is the theory of M_M:

$$\Delta(M) = \{\varphi \in L(M) \mid M_M \models \varphi\} = \mathrm{Th}(M_M).$$

Next proposition says that every model of the (elementary) diagram of M consists of a structure N plus an (elementary) embedding of M in N.

Proposition 1.5.2. *Let M, N be two L-structures and h a map from M to N.*

1. *The map h is an embedding (of M into N) iff $(N, ha)_{a\in M}$ is a model of $Diag(M)$.*

2. *The map h is an elementary embedding (of M into N) iff $(N, ha)_{a\in M}$ it is a model of $\Delta(M)$.*

Proof. We begin by proving *1*. Suppose that h is an embedding. If $\varphi(\overline{a}) \in Diag(M)$ then $M_M \models \varphi(\overline{a})$ and thus $M \models \varphi(\overline{a})$. By 1.2.4 $N \models \varphi(h\overline{a})$ and therefore $(N, ha)_{a\in M} \models \varphi(\overline{a})$. Hence $(N, ha)_{a\in M}$ is a model of $Diag(M)$. Conversely, if $(N, ha)_{a\in M}$ is a model of $Diag(M)$, h is injective because $a \neq b \in Diag(M)$ for each couple of elements a, b different of M. Let R is a relation symbol and $\overline{a}$ a tuple of M. If $\overline{a} \in R^M$ then $R(\overline{a}) \in Diag(M)$. Therefore $(N, ha)_{a\in M} \models R(\overline{a})$ and thus $h\overline{a} \in R^N$. If $\overline{a} \notin R^M$ then $\neg R(\overline{a}) \in Diag(M)$. Therefore $(N, ha)_{a\in M} \models \neg R(\overline{a})$ and $h\overline{a} \notin R^N$. If f is a function symbol, $\overline{a}$ is a tuple of M and $b = f^M(\overline{a})$ we have that the formula $b = f(\overline{a})$ belongs to the diagram of M, $(N, ha)_{a\in M} \models b = f(\overline{a})$ and therefore $h(b) = f^N(h\overline{a})$.

To prove *2*, we suppose that h is an elementary embedding. Then if $\varphi(\overline{a}) \in \Delta(M)$ it follows that $M_M \models \varphi(\overline{a})$, $M \models \varphi(\overline{a})$, $N \models \varphi(h\overline{a})$ and therefore $(N, ha)_{a\in M} \models \varphi(\overline{a})$. Hence $(N, ha)_{a\in M}$ is a model of $\Delta(M)$. Conversely $(N, ha)_{a\in M}$ is model of $\Delta(M)$, $\varphi(\overline{x}) \in L$ and $\overline{a}$ is a tuple of M, $M \models \varphi(\overline{a})$ iff $\varphi(\overline{a}) \in \Delta(M)$ iff $(N, ha)_{a\in M} \models \varphi(\overline{a})$ iff $N \models \varphi(h\overline{a})$. □

Exercise 1.5.3. The diagram of a structure could be defined with a smaller set of formulas. Let us denote $diag(M)$ the following set of formulas:

- $R(\overline{a})$ for each relation symbol R and each tuple $\overline{a} \in R^M$.
- $\neg R(\overline{a})$ for each relation symbol R and each tuple $\overline{a} \notin R^M$.
- $f(\overline{a}) = b$ for each function symbol f, each tuple $\overline{a} \in M$ and each $b \in M$ such that $f^M(\overline{a}) = b$.
- $a \neq b$ for each pair of different elements a, b of M.

Sow that $Diag(M)$ and $diag(M)$ have the same models.

We next see an application of the diagrams.

Proposition 1.5.4. *Two structure M_1, M_2 are elementarily equivalent iff there is a structure N and elementary embeddings of each M_i in N.*

Proof. Suppose that M_1 and M_2 are elementarily equivalent. By proposition 1.5.2 everything is reduced to find a model of $\Delta(M_1) \cup \Delta(M_2)$. We can suppose that the domains of M_1 and M_2 are disjoint (otherwise we take an isomorphic copy of M_1). By compactness and the fact that $\Delta(M_i)$ is closed under conjunction we have to find a model of $\varphi(\overline{a}) \wedge \psi(\overline{b})$, where $\varphi(\overline{a})$ is from the elementary diagram of M_1 and $\psi(\overline{b})$ is from the elementary diagram of M_2. Here we assume $\overline{a} \in M_1$, $\overline{b} \in M_2$ and $\varphi(\overline{x}), \psi(\overline{y})$ are L-formulas.

Since $M_2 \models \psi(\overline{b})$ also $M_2 \models \exists \overline{x} \psi(\overline{x})$. As M_1 is elementarily equivalent to M_2 we have that $M_1 \models \exists \overline{x} \psi(\overline{x})$. Therefore $M_1 \models \psi(\overline{c})$ for a certain tuple $\overline{c}$ of elements of M_1. Then the expansion M' of ${M_1}_{M_1}$ interpreting $\overline{b}$ by $\overline{c}$ satisfies $\psi(\overline{b})$ and therefore is a model of $\varphi(\overline{a}) \wedge \psi(\overline{b})$. □

remark 1.5.5. In proposition 1.5.4 we can improve the conclusion by stipulating that N is an elementary extension of M_1, i.e. f_1 is the identity. This is done using lemma 1.2.3. But we cannot get, in general, a common elementary extension of M_1 and M_2. For instance, if $M_1 \subseteq M_2$ the existence of a common elementary extension implies $M_1 \preceq M_2$. Next exercises deal with this problem.

Exercise 1.5.6. Let M_1, M_2 be structures and let A denote the intersection $M_1 \cap M_2$ of their domains.

1. Show that if M_1 and M_2 have a common extension (i.e., there is N with $M_i \subseteq N$) then A is the domain of a common substructure of M_1 and M_2.
2. (to be checked) Show that M_1 and M_2 have a common extension iff
$$Diag((M_1)_A) \cup Diag((M_2)_A) \cup \{m_1 \neq m_2 \mid m_1 \in M_1 \setminus A,\ m_2 \in M_2 \setminus A\}$$
is consistent.
3. (to be checked) Show that if each pair of finitely generated substructures N_1 of M_1 and N_2 of M_2 have a common extension then also M_1 and M_2 have a common extension. Q: can we do it with ultraproducts????

Exercise 1.5.7. Let M_1, M_2 be structures and let A denote the intersection $M_1 \cap M_2$ of their domains.

1. Show that if M_1 and M_2 have a common elementary extension (i.e., there is N with $M_i \preceq N$) then A is an algebraically closed subset of M_1 and M_2.
2. (to be checked) Show that M_1 and M_2 have a common elementary extension iff
$$\Delta((M_1)_A) \cup \Delta((M_2)_A) \cup \{m_1 \neq m_2 \mid m_1 \in M_1 \setminus A,\ m_2 \in M_2 \setminus A\}$$
is consistent.

3. (to be checked) Show that M_1 and M_2 have a common elementary extension iff $(M_1)_A \equiv (M_2)_A$ and A is algebraically closed in each M_i. Hint: use 1 and 2.

Exercise 1.5.8. Prove one can generalize proposition 1.5.4: any collection of pairwise elementarily equivalent models can be embedded elementarily in a common model.

Corollary 1.5.9. *Every structure elementarily equivalent to a finite structure is isomorphic to it.*

Proof. Let M_1, M_2 be elementary equivalent structures and suppose M_1 is finite. Let N and $f_i : M_i \to N$ as in proposition 1.5.4. Since the embedding f_1 is elementary N is finite with the same cardinality as M_1. This implies f_1 is an isomorphism. Again, since the embedding f_2 is elementary M_2 is finite with the same cardinality as N. This implies f_2 is an isomorphism. Now $f_2^{-1} \circ f_1$ is an isomorphism from M_1 to M_2. □

A theory T is said to be ***closed under substructures*** if each substructure of each model of T is a model of T. A universal formula is a formula of the form $\forall \overline{x} \varphi$, where φ is a quantifier-free formula. The universal formulas are also called Π_1-formulas or sometimes $\forall$-formulas. One can also define existential formulas (also Σ_1-formulas or $\exists$-formulas) as formulas of the form $\exists \overline{x} \varphi$, where φ is a quantifier-free formula. In fact there is a whole hierarchy: Π_{n+1}-formulas are of the form $\forall \overline{x} \varphi$, where φ is a $\Sigma_n$formula and Σ_{n+1}-formulas are of the form $\exists \overline{x} \varphi$, where φ is a $\Pi_n$formula. A Π_n-formula is of the form $\forall \overline{x_1} \exists \overline{x_2} \cdots Q \overline{x_n} \varphi$ where φ is quantifier-free. Also the Σ_n-formulas have n blocs of alternating quantifiers in this case starting by al existential block.

Let's see an application of diagrams:

Proposition 1.5.10. *A theory is closed under substructures iff it can be axiomatized by a set of universal sentences.*

Proof. In order to prove $\Leftarrow$, it suffices to check the following: If M is a substructure of N, M satisfies each universal sentence true in N. If $N \models \forall \overline{x} \varphi(\overline{x})$ for some quantifier-free formula $\varphi(\overline{x})$, then for every tuple of elements $\overline{a} \in N$, $N \models \varphi(\overline{a})$. In particular $N \models \varphi(\overline{a})$ for each tuple $\overline{a}$ in M. As φ is quantifier-free, $M \models \varphi(\overline{a})$ for each tuple $\overline{a}$ in M and thus $M \models \forall \overline{x} \varphi(\overline{x})$.

Conversely, assume T is closed under substructures. We are going to prove that $T_\forall = \{\varphi \mid \varphi \text{ is a universal sentence and } T \models \varphi\}$ is a set of axioms for T, i.e., $\mathrm{Mod}(T) = \mathrm{Mod}(T_\forall)$. We must see $\mathrm{Mod}(T_\forall) \subseteq \mathrm{Mod}(T)$, the other inclusion being trivial. Let M be a model of $T_\forall$. It suffices to prove that $T \cup Diag(M)$ has a model. This will provide us, by proposition 1.5.2, with a model N of T and an embedding $h : M \to N$. Since T is closed under substructures $h(M)$ is a model of T hence also M (as M is isomorphic to $h(M)$. By compactness it is enough to prove that $T \cup \{\varphi(\overline{a})\}$ has a model for any formula $\varphi(\overline{a}) \in Diag(M)$. If not, for some formula $\varphi(\overline{a}) \in Diag(M)$ the theory $T \cup \{\varphi(\overline{a})\}$ has no models, where $\overline{a}$ is a tuple of constants $\overline{a} \in M$ and $\varphi(\overline{x})$ is a quantifier-free formula of L. This implies $T \models \neg\varphi(\overline{a})$ and, since the constants $\overline{a}$ do not belong to L, $T \models \forall\overline{x}\neg\varphi(\overline{x})$. This means that the formula $\forall\overline{x}\neg\varphi(\overline{x})$ belongs to $T_\forall$. But this is a contradiction, since the M is a model of $\forall\overline{x}\neg\varphi(\overline{x})$ and $M \models \varphi(\overline{a})$. $\square$

Corollary 1.5.11. *A sentence is preserved under substructures iff it is equivalent to an existential sentence.*

Proof. If φ is preserved under substructures, applying proposition 1.5.10, the theory $\{\varphi\}$ has a set Σ of universal axioms, that is, $\mathrm{Mod}(\Sigma) = \mathrm{Mod}(\varphi)$. By compactness there are $\psi_1, \ldots, \psi_n \in \Sigma$ such that $\{\psi_1, \ldots, \psi_n\} \models \varphi$. Then φ is equivalent to $\psi_1 \wedge \ldots \wedge \psi_n$. But a conjunction of existential formulas is equivalent to an existential formula, by replacing the bound variables is necessary. $\square$

Given an embedding $f : M \to N$ we say that f is ***existentially closed*** if the embedding preserve all existential formulas. More precisely, if (1.2.3) holds for all existential formulas. In fact is suffices that for any existential formula $\varphi(\overline{x})$ and any tuple $\overline{a} \in M$, if $N \models \varphi(h\overline{a})$ then $M \models \varphi(\overline{a})$. This is equivalent to preserve universal formulas in the sense of (1.2.3) (and equivalent also to: the universal formulas transfer from M to N). We will denote it by $f : M \to_1 N$ to suggest f preserves formulas with one block of quantifiers. When M is a substructure of N one says M is existentially closed in N when the identity $\mathrm{Id}_M : M \to N$ is an existially closed embedding. This will be denoted by $M \preceq_1 N$.

Lemma 1.5.12. *Let $M \subseteq N$. Then M is existentially closed in N iff there is an extension M' of N with $M \preceq M'$.*

Proof. Assume M is existentially closed in N. By proposition 1.5.2, a model of $\Delta(M) \cup Diag(N)$ consists of an L-structure M^* and embeddings $f : M \to M^*$ and $g : N \to M^*$ with f elementary. Observe that g extends f, as we are assuming $L(M) \subseteq L(N)$, i.e., we use the same constants to denote an individual of M in $\Delta(M)$ and in $Diag(N)$. By proposition 1.2.3 there is an extension M' of N and an isomorphism $h : M' \to M^*$ extending g. Then $M \preceq M'$, since $\mathrm{Id}_M = h^{-1} \circ f : M \to M'$ is the composition of two elementary embeddings. This shows it is enough then to prove that $\Delta(M) \cup Diag(N)$ has a model, or equivalently, by compactness, that $\Delta(M) \cup \{\psi\}$ has a model for each formula $\psi \in Diag(N)$. Separating the constants from M than those of $N \setminus M$, ψ is of the form $\varphi(\overline{a}, \overline{b})$ for some quantifier-free L-formula $\varphi(\overline{x}, \overline{y})$ and tuples $\overline{a} \in M$, $\overline{b} \in N \setminus M$ with $N \models \varphi(\overline{a}, \overline{b})$. Since M is existentially closed in N and $N \models \exists\overline{y}\varphi(\overline{a}, \overline{y})$ we get $M \models \exists\overline{y}\varphi(\overline{a}, \overline{y})$. Let $\overline{c}$ a tuple of elements in M such that $M \models \varphi(\overline{a}, \overline{c})$. This implies that M'', the expansion of M_M to $L(M\overline{b})$ by interpreting the tuple of constants $\overline{b}$ as the elements $\overline{c}$ is a model of $\psi = \varphi(\overline{a}, \overline{b})$. Hence M'' is a model of $\Delta(M) \cup \{\psi\}$. $\square$

Exercise 1.5.13. Prove that an embedding f from M to N is existentially closed iff there are embeddings $g : M \to M'$ and $h : N \to M'$, g elementary and $g = h \circ f$.

A theory T is ***closed under unions of chains*** if any union of models of T is also a model of T.

Theorem 1.5.14 (Chang-Łos-Suzsko)**.** *A theory T is closed under unions of chains iff T can be axiomatized by $\forall\exists$-sentences.*

Proof. We let the reader check that any $\forall\exists$-sentence is preserved under unions of chains. This gives the easy part of the proof. To prove the converse, assume T is preserved under unions of chains. Wa are going to see that $\mathrm{Mod}(T) = \mathrm{Mod}(T_{\forall\exists})$, where $T_{\forall\exists}$ denotes the set of $\forall\exists$-consequences of T.

Claim Assume M is a model of $T_{\forall\exists}$. Then there is some N model of T such that $M \preceq_1 N$.

Proof. Denote by $\Delta_1(M)$ the $\forall$-theory of M_M, i.e., the set of all $\forall$-sentences in $L(M)$ true in M_M. We first show that $\Delta_1(M) \cup T$ is consistent. By compactness it suffices to show $\forall\overline{x}\psi(\overline{x}, \overline{a}) \cup T$ is consistent, where $M_M \models \forall\overline{x}\psi(\overline{x}, \overline{a})$ and $\psi(\overline{x}, \overline{y})$ is quantifier-free. Otherwise $T \models \neg\forall\overline{x}\psi(\overline{x}, \overline{a})$ and thus $T \models \forall\overline{y}\exists\overline{x}\neg\psi(\overline{x}, \overline{y})$. This implies $\forall\overline{y}\exists\overline{x}\neg\psi(\overline{x}, \overline{y}) \in T_{\forall\exists}$ and thus $\forall\overline{y}\exists\overline{x}\neg\psi(\overline{x}, \overline{y})$ is true in M, a contradiction to $M_M \models \forall\overline{x}\psi(\overline{x}, \overline{a})$. This ends the proof of the claim.

Let M be a models of $T_{\forall\exists}$, we must show M is also a model of T. We will build a chain of structures $(M_n \mid n \in \omega)$ starting with $M_0 = M$. By the claim, there is $M_1 \in \mathrm{Mod}(T)$ extending M_0 with $M_0 \preceq_1 M_1$. By Lemma 1.5.12 there is some extension M_2 of M_1 with $M_0 \preceq M_2$. As M_2 is also a model of $T_{\forall\exists}$ by the claim again we obtain $M_3 \in \mathrm{Mod}(T)$ extending M_2 with $M_2 \preceq_1 M_3$. Again, by Lemma 1.5.12 there is some extension M_4 of M_3 with $M_2 \preceq M_4$. Repeating this construction ω times obtain a chain $(M_n \mid n \in \omega)$ in such a way that $M_{2i+1} \in \mathrm{Mod}(T)$ and $M_0 \preceq M_2 \preceq M_4 \preceq \cdots M_{2i} \preceq M_{2i+2} \cdots$. As T is closed under unions of chains $\bigcup_{i\in\omega} M_{2i+1}$ is a model of T. As $(M_{2i} \mid n \in \omega)$ is an elementary chain $M \preceq \bigcup_{i\in\omega} M_{2i} = \bigcup_{i\in\omega} M_{2i+1}$ thus M is a model of T. □

Example 1.5.15. Very often, nice mathematica theories are axiomatized with $\forall\exists$-axioms: groups, rings, fields, algebraically closed fields, DLO...

Corollary 1.5.16. *A sentence is preserved under unions of chains iff it is equivalent to a $\forall\exists$-sentence.*

Proof. The same proof as corollary 1.5.11 using theorem 1.5.14 and the fact that a conjunction of $\forall\exists$-formulas is equivalent to a $\forall\exists$-formula. □

Exercise 1.5.17. In this exercise we will prove another preservation theorem: a characterization of theories preserved under extension. By this we mean that any extension of any model of the theory is also a model of the theory. As usual, $T_\exists$ denotes the set of all existential sentences which are logical consequences of T. For an structure M, $\mathrm{Th}_\forall(M)$ denotes all the universal sentences true in M.

1. Prove that any existential sentence is preserved under extensions.

2. If M is a model of $T_\exists$ then $\mathrm{Th}_\forall(M) \cup T$ is consistent.

3. If N is a model of $\mathrm{Th}_{\forall}(M)$ then there is some N' extending N such that $M \equiv N'$.

4. Deduce from *2.* and *3.* that $\mathrm{Mod}(T_{\exists})$ is the class of all structures elementarily equivalent to some extension of a model of T.

5. Deduce from *1.* and *4.* that T is preserved under extensions iff T is axiomatized by a set of existential sentences.

6. Deduce from *5.* that a sentence is preserved under extensions iff it is equivalent to an existential sentences.

Exercise 1.5.18. Prove the following:

1. $Mod(T_{\forall}) = \{M \mid M \subseteq N \text{ for some } M \in \mathrm{Mod}(T)\} =$
   $= \{M \mid f : M \to N \text{ for some } M \in \ \mathrm{mod}\ (T)\}$

2. $Mod(T_{\forall\exists}) = \{M \mid M \preceq_1 N \text{ for some } M \in \mathrm{Mod}(T)\} =$
   $= \{M \mid f : M \to_1 N \text{ for some } M \in \mathrm{Mod}(T)\}$

Exercise 1.5.19. Prove the converse of exercise 1.4.7. Use diagrams and exercise 1.2.9.

Exercise 1.5.20. Let M be a given L-structure and T be a theory in the language L. Show that if for any finitely generated substructure M' of M there is an embedding of M' into a model of T then there is an embedding of M into a model of T. Hint: use diagrams and compactness.

Exercise 1.5.21. An Ordered Abelian Group is an Abelian Group together a linear order $\leq$ compatible with the group structure, i.e., satisfying $\forall x \forall y \forall z (x \leq y \to x + y \leq y + z)$. We assume the natural language for these structures: $L = \{+, -, 0, \leq\}$.

1. Show that an Ordered Abelian Group is torsion free.

2. Show that an Abelian Group can be ordered (in a compatible way) if any finitely generated subgroup can be ordered. Hint: Use that Ordered Abelian Group have universal axioms and exercise 1.5.20.

3. Prove that any torsion-free Abelian group can be turned into an Ordered Abelian Group . Hint: use that any finitely generatedAbelian group is a finite product (the direct sum of finitely many) cyclic groups.

Exercise 1.5.22. Let $\mathcal{K}$ be a class of structures in a given language closed under isomorphism. Show that:

1. $\mathcal{K}$ is axiomatizable by universal sentences (i.e., $\mathcal{K} = Mod(\Sigma)$ for some set Σ of universal sentences) iff $\mathcal{K}$ is closed under substructures and ultraproducts.

2. $\mathcal{K}$ is axiomatizable by existential sentences (i.e., $\mathcal{K} = Mod(\Sigma)$ for some set Σ of existential sentences) iff $\mathcal{K}$ is closed under ultraproducts and the complement is closed under substructures and ultrapowers.

Chapter 2

Types and Saturation

2.1 Types

Here we will consider again expansions of the language by adding new constants. Let M be a structure and A be a subset of the domain of M. We will denote by $L(A)$ the set of formulas of expansion of L consisting in adding a new constant for each element of A. There is no harm on taking the same elements of A as new constants. Then M expands in a natural way to an $L(A)$-structure denoted by M_A. Such an A is usually called a set of ***parameters***. The formulas of $L(A)$ are called formulas with parameters in A.

Until now we have considered languages with a countable amount of variables. Here we will broaden the language by adding an arbitrary amount of new variables. This set of variables will be considered as a sequence $\langle x_i \mid i \in I \rangle$ indexed in an a given set I, very often a cardinal number. This sequence $\langle x_i \mid i \in I \rangle$ will be denoted by $\overline{x}$ and we will call it a tuple of variables.

The cardinality of this language including the variables $\overline{x}$ is $|L| + |\overline{x}|$ instead of $|L| + \aleph_0$. Here $\overline{x}$ denotes the number of variables. Moreover, if A is a set of parameters of a model M, the number of formulas in this languages with parameters from A is $|L| + |\overline{x}| + |A|$.

We will denote by $L_{\overline{x}}(A)$ the set of formulas with parameters in A and free variables in $\overline{x}$. If $\varphi \in L_{\overline{x}}(A)$ we put $\varphi = \varphi(\overline{x})$ to emphasize that the formula φ has its free variables among those of $\overline{x}$. Obviously φ uses only a finite number of those variables. If we have a set $\pi \subseteq L_{\overline{x}}(A)$ obviously the free variables of each formula of π are among those of the tuple $\overline{x}$, and we will write $\pi = \pi(\overline{x})$ to emphasize it.

Definition 2.1.1. Let $\pi \subseteq L_{\overline{x}}(A)$, where A is a set of parameters from M. A ***realization*** of π in M is a tuple $\overline{a} = \langle a_i \mid i \in I \rangle$ of elements of M satisfying all the formulas of π, denoted by $M_A \models \pi(\overline{a})$. This means that for each formula $\varphi(\overline{x}) \in \pi$ it holds $M_A \models \varphi(\overline{a})$. A set π which has a realization in M is called ***realized in*** M, otherwise is called ***omitted in*** M. A set π is called ***finitely satisfiable in*** M if each finite subset of π has a realization in M. A ***type of***

M ***over*** A is a subset of $L_{\overline{x}}(A)$ finitely satisfiable in M_A.

When we say π is a type over A we assume there is some set of variables $\overline{x}$ and a set of parameters A in some model M and $\pi \subseteq L_{\overline{x}}(A)$.

When $\overline{x}$ has length n a type (of M over A) with this free variables is called an n-types (of M over A). When we speak about n-types, a fixed set of n variable is presupposed. In this sense $L_n(A)$ is used to indicate $L_{\overline{x}}(A)$ for some tuple $\overline{x}$ of n variables. As a matter of fact the name of the variables is irrelevant, what really counts is the number of variables in $\overline{x}$.

Example 2.1.2. We first provide examples of types without parameters.

1. Let now $L = \{+, 0\}$ the language of groups(one of the possible languages). Let $\pi(x) = \{\exists y(x = ny) \mid n > 0\}$. This is a 1-type over $\emptyset$ of $(\mathbb{Z}, +, 0)$ non realized. It is realized in $(\mathbb{Q}, +, 0)$.

2. Let now $L = \{+, 0\}$ the language of groups (one of the possible languages). Let $\pi(x) = \{nx \neq 0 \mid n > 0\}$. This is a 1-type over $\emptyset$ realized in $(\mathbb{Q}, +)$. It is also a non-realized a 1-type of $(\mu, \cdot)$. Here $(\mu, \cdot)$ denotes the set of complex roots of unity with multiplication,i.e. $\mu = \left\{e^{\pi i r} \mid r \in \mathbb{Q}\right\}$.

3. In the language of graphs (containing a single binary relation R) let $\pi(x, y)$ denote the following set of formulas:

$$\{x \neq y, \neg R(x, y)\} \cup \left\{\neg\exists z_1 \cdots \exists z_n \big(R(x, z_1) \wedge R(z_1, z_2) \wedge \cdots \wedge R(z_n, y)\big) \mid n \geq 1\right\}$$

   In any non-connected graph π is a realized 2-type over the empty set. In the graph with domain $\mathbb{Z}$ and $R(x, y)$ is interpreted as $|x - y| = 1$, π is a non-realized type. In fact π is omitted in the graph M iff M is connected.

4. In the language of order, let $\overline{x} = (x_n \mid n \in \mathbb{N})$ and let $\pi(\overline{x})$ be the following set of formulas:

$$\{x_0 > x_1, x_1 > x_2, x_2 > x_3, \ldots\}.$$

   In any infinite linearly ordered set π is a type. It is omitted in M iff M is well-ordered.

Now we provide examples of types with parameters:

1. Let $M = (\mathbb{Q}, \leq)$ be the rationals with the usual order. The following is a 1-type of M over $\mathbb{Q}$:

$$\left\{x > r \mid r \in \mathbb{Q}, r < \sqrt{2}\right\} \cup \left\{x < r \mid r \in \mathbb{Q}, r > \sqrt{2}\right\}$$

   This type is realized in $(\mathbb{R}, \leq)$ (an elementary extension of $(\mathbb{Q}, \leq)$ as we will see later) and omitted (not realized) in $(\mathbb{Q}, \leq)$.

If N is an elementary extension of M ($M \preccurlyeq N$but in symbols), any subset A of M is also a subset of N and we can also speak about types of N over A. As a matter of fact there is no difference between type of M and types of N over

A provided A is a subset of M. Any realization in M of an type over A is also a realization in N of the same type because the extension is elementary. Hence every type of M over A is a type of N over A. Conversely if $\pi(\overline{x})$ is a type of N over A, for each finite subset $\pi_0(\overline{x})$ of π, as $N_A \models \exists\overline{x}\,(\bigwedge \pi_0(\overline{x}))$ and $M \preccurlyeq N$ we have that $M_A \models \exists\overline{x}\,(\bigwedge \pi_0(\overline{x}))$ and therefore π is also finitely satisfiable in M. When considering elementary extensions we may omit the superscript.

Proposition 2.1.3. *Let $A \subseteq M$ and $\pi \subseteq L_{\overline{x}}(A)$. The following are equivalent:*

1. *π is a type of M.*
2. *$\pi(\overline{x}) \cup \mathrm{Th}\, M_A$ is consistent.*
3. *$\pi(\overline{x}) \cup \Delta(M)$ is consistent.*
4. *There is an elementary extension of M realizing π.*

Proof. *1⇒2.* Each finite subset of $\pi(\overline{x}) \cup \mathrm{Th}\, M_A$ is a subset of $\pi_0(\overline{x}) \cup \mathrm{Th}\, M_A$ for some finite subset $\pi_0(\overline{x})$ of $\pi(\overline{x})$, thus satisfiable in M. By compactness $\pi(\overline{x}) \cup \mathrm{Th}\, M_A$ is satisfiable.

2⇒3. By compactness we have to show that $\pi(\overline{x}) \cup \varphi(\overline{a}, \overline{b})$ is satisfiable, where $\varphi(\overline{a}, \overline{b}) \in \Delta(M)$ and $\overline{a}$ denotes the parameters of φ contained in A and $\overline{b}$ denotes the parameters not contained in A. As $\overline{b}$ has nothing in common with π, the satisfiability of $\pi(\overline{x}) \cup \varphi(\overline{a}, \overline{b})$ is equivalent to that of $\pi(\overline{x}) \cup \exists\overline{y}\varphi(\overline{a}, \overline{y})$. Since $\exists\overline{y}\varphi(\overline{a}, \overline{y}) \in \mathrm{Th}\, M_A$ the consistency of this set is ensured by hypothesis.

3⇒4. Let $(N, ha)_{a\in M}$ be a model of $\Delta(M)$ and let $\overline{b}$ be a tuple of N such that $(N, ha)_{a\in M} \models \pi(\overline{b})$. If $\overline{a}$ is an enumeration of M, we can write $\pi(\overline{x}) = \pi(\overline{x}, \overline{a})$. By lemma 1.5.2 and exercise 1.2.9 there is some $M' \succeq M$ and $h' \supseteq h$ such that h' is an isomorphism from M' to N. Now $(N, ha)_{a\in M} \models \pi(\overline{b}, \overline{a})$ implies $N \models \pi(\overline{b}, h\overline{a})$ whence, since h'^{-1} is an isomorphism $M' \models \pi(h'^{-1}\overline{b}, \overline{a})$. Thus $h'^{-1}\overline{b}$ is a realization of $\pi(\overline{x})$ in M'.

4⇒1. π is a type of some elementary extension of M hence a type of M. □

Definition 2.1.4. A type $\pi(\overline{x})$ of M over A is said to be ***complete*** if for each formula $\varphi \in L_{\overline{x}}(A)$ we have that either $\varphi \in \pi$ or $\neg\varphi \in \pi$.

Facts 2.1.5. *Let $p(\overline{x})$ be a complete type of M over A. Then*

1. *If $\pi \subseteq L_{\overline{x}}(A)$ and $\pi \cup p$ is consistent, then $\pi \subseteq p$.*
2. *$\mathrm{Th}(M_A) \subseteq p$.*
3. *p is closed under consequence. That is, if $\varphi \in L_{\overline{x}}(A)$ and $p \models \varphi$ then $\varphi \in p$.*
4. *For all formulas $\varphi_1, \ldots, \varphi_n \in L_{\overline{x}}(A)$, $\varphi_1 \wedge \cdots \wedge \varphi_n \in p$ iff each $\varphi_i \in p$. In particular p is closed under conjunction.*
5. *For all formulas $\varphi_1, \ldots, \varphi_n \in L_{\overline{x}}(A)$, $\varphi_1 \vee \cdots \vee \varphi_n \in p$ iff $\varphi_i \in p$ for some $i = 1 \ldots n$.*

6. *If $\varphi(\overline{y},\overline{z}) \in p$ then $\exists\overline{z}\varphi(\overline{y},\overline{z}) \in p$. Here $\overline{y}$ and $\overline{z}$ are assumed to be subtuples of $\overline{x}$.*

Proof. 1. Otherwise there is some formula $\varphi \in \pi \setminus p$. Then $\neg\varphi \in p$, a contradiction to the consistency of $\pi \cup p$.

2. By proposition 2.1.3 and *1.*

3. If $p \models \varphi$, then $p \cup \varphi$ is consistent. By *1.* $\varphi \in p$.

4. It follows by *2.* and $\{\varphi_1, \dots, \varphi_n\} \models \varphi_1 \wedge \dots \wedge \varphi_n$ and $\varphi_1 \wedge \dots \wedge \varphi_n \models \varphi_i$.

5. If for some i $\varphi_i \in p$ then $p \models \varphi_1 \vee \dots \vee \varphi_n$. Conversely, if $\varphi_i \notin p$ for $i = 1 \dots n$ then $\neg\varphi_i \in p$ for $i = 1 \dots n$ and thus $p \models \neg(\varphi_1 \vee \dots \vee \varphi_n)$. Hence $\neg(\varphi_1 \vee \dots \vee \varphi_n) \in p$ and thus $\varphi_1 \vee \dots \vee \varphi_n \notin p$.

6. Let $\overline{a}$ be a realization of $p(\overline{x})$ is some elementary extension N of M. Then $N \models p(\overline{a})$ in particular $N \models \varphi(\overline{b},\overline{c})$ and $N \models \exists\overline{z}\varphi(\overline{b},\overline{z})$, where $\overline{b},\overline{c}$ are the subtuples of $\overline{a}$ corresponding to the subtuples $\overline{y}$ and $\overline{z}$ of $\overline{x}$ respectively. This implies $p \cup \exists\overline{z}\varphi(\overline{y},\overline{z})$ is consistent, as $\overline{a}$ realizes all these formulas in N. By *2.* it follows $\exists\overline{z}\varphi(\overline{y},\overline{z}) \in p$. □

Definition 2.1.6. Let $\overline{a}$ is a tuple of elements of M and A a subset of M. We fix a tuple $\overline{x}$ of variables of the same length as $\overline{a}$. The ***type of $\overline{a}$ over A in M***, denoted by $\mathrm{tp}^M(\overline{a}/A)$, is the following set of formulas:

$$\{\varphi(\overline{x}) \in L_{\overline{x}}(A) \mid M_A \models \varphi(\overline{a})\}.$$

Observe that $\mathrm{tp}^M(\overline{a}/A)$ is a complete type (of M over A) with free variables $\overline{x}$. Of course, although not explicit in the notation $\mathrm{tp}^M(\overline{a}/A)$, this depends on the choice of $\overline{x}$, but is 'unique up to replacement of variables'. Moreover, if $M \preceq N$ and $\overline{a} \in M$ then $\mathrm{tp}^M(\overline{a}/A) = \mathrm{tp}^N(\overline{a}/A)$. Hence when playing with elementary extensions, we can omit the superscript.

Facts 2.1.7. 1. *Any type can be completed: given a type $\pi \subseteq L_{\overline{x}}(A)$ there is some complete type p (of M over A) with $\pi \subseteq p$.*

2. *If $p(\overline{x})$ is a complete type of M over A and $\overline{a}$ is a tuple is M, then:*

$$\overline{a} \text{ realizes } p \quad \text{iff } p = \mathrm{tp}^M(\overline{a}/A).$$

Proof. 1. Let, by proposition 2.1.3, $\overline{a}$ be a realization of π in an elementary extension N of M. Then $\mathrm{tp}^N(\overline{a}/A)$ is a complete type of N, hence of M, extending π. □

The set of all complete types of M over A with free variables $\overline{x}$ is denoted by $S^M_{\overline{x}}(A)$. Observe that if N is an elementary extension of M then $S^M_{\overline{x}}(A) = S^N_{\overline{x}}(A)$. Moreover,

$$\left|S^M_{\overline{x}}(A)\right| \le 2^{|L|+|A|+|\overline{x}|+\aleph_0}.$$

This is an immediate consequence of the following inequality:

$$|L_{\overline{x}}(A)| \le |L| + |A| + |\overline{x}| + \aleph_0.$$

If $\overline{x}$ and $\overline{y}$ are tuples of the same cardinality, a bijection from $\overline{x}$ to $\overline{y}$ extends to a bijection between $L_{\overline{x}}(A)$ and $L_{\overline{y}}(A)$ and thus also to $S^M_{\overline{x}}(A)$ and $S^M_{\overline{y}}(A)$. These spaces of complete types will be topologized later. This bijection turns out to be a homeomorphism between the spaces $S^M_{\overline{x}}(A)$ and $S^M_{\overline{y}}(A)$. This shows that what really matters is the number of free variables. When $\overline{x}$ has n variables and we do not want to put any emphasis in the name of the variables it is usually denoted by $S^M_n(A)$. One could also use the notation $S^M_\kappa(A)$ to denote the set of complete types over A with κ free variables for any infinite cardinal κ.

Obviously always

$$|A| \leq \left|S^M_n(A)\right|,$$

since the types $\mathrm{tp}^M(a/A)$ for $a \in A$ provides a collection of $|A|$ types: for $a, b \in A$, with $a \neq b$, $\mathrm{tp}^M(a/A) \neq \mathrm{tp}^M(b/A)$ since the first one contains the formula $x = a$ while the second contains $x \neq a$.

Usually $|A| \geq |L| + \aleph_0$. In this case the above inequalities becomes more compact:

$$|A| \leq \left|S^M_n(A)\right| \leq 2^{|A|}.$$

The following examples show that these extreme values occur in practice.

Example 2.1.8. 1. Let T the theory of an infinite set. Let M be infinite and $A \subseteq M$ also infinite. Let us see that $\left|S^M_n(A)\right| = |A|$. Apart from the types $\mathrm{tp}^M(a/A)$ for $a \in M$ there is only one extra type: $\mathrm{tp}^N(b/A)$ for some elementary extension N of M and $b \in N \setminus A$. Since any permutation of N is an automorphism, all elements of $N \setminus A$ have the same type over A: give $a, b \in N \setminus A$ there is a permutation fixing A pointwise and sending a to b this implies that a and b must have the same type over A.

2. Let T_{RAND} denote the theory of the random graph. This theory is in the language of a single binary relation R and expressing that (M, R^M) is a graph (i.e.,R is irreflexive and symmetric) with the following property: given two finite and disjoint subsets A, B of M, there is $c \in M$ such that $(c, a) \in R^M$ for each $a \in A$ and $(c, b) \notin R^M$ for each $b \in B$. Let us see that $(V_\omega, \overline{\in})$ is a model of T_{RAND}, where V_ω denotes the ω step in the construction of the Von-Neumann Universe and $\overline{\in}$ denotes the simetrized of the membership relation: $a \overline{\in} b$ iff $a \in b$ or $b \in a$. Given finite disjoint subset X, Y of V_ω there are infinitely many finite subset Z of V_ω with $X \subseteq Z$ and $Z \cap Y = \emptyset$. Choosing one of those Z such that $Z \notin \cup Y$ (which exists because $\cup Y$ is finite) we are done, since then $x \in Z$ for each $x \in X$ and $y \notin Z$, $Z \notin y$ for each $y \in Y$. Since Z is finite, $Z \in V_\omega$. Let's check now that given any model M of T_{RAND} and any infinite subset $A \subseteq M$. $\left|S^M_1(A)\right| = 2^{|A|}$. Given any subset $B \subseteq A$, consider the partial type $\pi_B(x) = \{xRb \mid b \in B\} \cup \{\neg xRa \mid a \in A \setminus B\}$. This is a type of M because M is a model of T_{RAND}. Let $p_B(x) \in S^M_1(A)$ be a completion of $\pi_B(x)$ for each B. If B, B' are two different subsets of A then some formula $x = a$ belongs to one of them and the negation $\neg x = a$ belongs to the other (choose a is the symmetric difference). This implies $p_B \neq p_{B'}$ and thus we have an injection from $P(A)$ to $S^M_1(A)$ showing that $\left|S^M_1(A)\right| = 2^{|A|}$.

Exercise 2.1.9. Let M be any model. Let $\overline{m} = (m_i \mid i \in I)$ be an injective enumeration of M (that is, the map from I to M is bijective). Let $p(\overline{x}) = \mathrm{tp}^M(\overline{m}/\emptyset)$, where $\overline{x} = (x_i \mid i \in I)$ is a tuple of variable of the same length as $\overline{m}$. Show that a tuple $\overline{a}$ is a realization of $p(\overline{x})$ in N iff the map $M \to N$ given by $m_i \mapsto a_i$ is an elementary embedding. In general enumerations need not be injective, surjectivity is enough. Show that all works also in this case.

Exercise 2.1.10. Show that a type of M over A is complete iff it is a maximal (by inclusion) in the set of all types of M over A with the same free variables. More precisely $p \in S^M_{\overline{x}}(A)$ iff it is maximal among subset of $L_{\overline{x}}(A)$ consistent with $\mathrm{Th}(M_A)$.

Exercise 2.1.11. Let M be a structure.

1. Let $A \subseteq M$ be a set of parameters and let $\overline{a} = (a_i \mid i \in I)$ be an enumeration of A. Let $\overline{y} = (y_i \mid i \in I)$ be a set of variables of the same length as $\overline{a}$. Let $q(\overline{y}) = \mathrm{tp}^M(\overline{a}/\emptyset)$. Given $p(\overline{x}, \overline{a}) \in S^M_{\overline{x}}(A)$ let $p(\overline{x}, \overline{y})$ the result of replacing each parameter a_i by y_i in all formulas.

   (a) Show that the map $S^M_{\overline{x}}(A) \to S^M_{\overline{x},\overline{y}}$ given by this replacement $p(\overline{x}, \overline{a}) \mapsto p(\overline{x}, \overline{y})$ is a bijection from $S^M_{\overline{x}}(A)$ onto $[q(\overline{y})] = \{p \in S^M_{\overline{x},\overline{y}} \mid q \subseteq p\}$.

   (b) Let $\overline{b}$ denote another realization of $q(\overline{x})$ in some structure N, and denote $B = \{b_i \mid i \in I\}$. Show that the map $S^M_{\overline{x}}(A) \to S^N_{\overline{x}}(B)$ given by $p(\overline{x}, \overline{a}) \mapsto p(\overline{x}, \overline{b})$ is a bijection.

2. Let Now N be a structure realizing any $q \in S^M_{\overline{y}}$. Next Lemma 2.1.12 show that such N exists. For each $q \in S^M_{\overline{y}}$, let $\overline{a}$ be a realization of $q(\overline{y})$ in N and denote $A_q = \{a_i \mid i \in I\}$. Show that

$$\left|S^M_{\overline{x},\overline{y}}\right| = \sum_{q \in S^M_{\overline{y}}} \left|S^N_{\overline{x}}(A_q)\right|.$$

Now we will see that we can realize any set of types of M in a suitable elementary extension of M. Any type over a set $A \subseteq M$ is also a type over M. Same wise, any type $\pi(\overline{y})$ for a certain subtuple $\overline{y}$ of $\overline{x}$ can be considered an type with variables in $\overline{x}$. For this reason we can consider all types with the same tuple of free variables.

Lemma 2.1.12. *Let M be an L-structure and P a set of types of M over M with free variables in $\overline{x}$. Then there is an elementary extension of M of cardinal at most $|M| + |L| + |P| + |\overline{x}| + \aleph_0$ realizing all the type of P.*

Proof. For each type π of P take $\overline{c_\pi}$ an I-tuple of new and different (between them and from those denoting the elements of M) constants. Denote by $\pi(\overline{c_\pi})$ the result of substituting the free variable of π for the constants of $\overline{c_\pi}$. We consider now the set of formulas

$$\Sigma = \Delta(M) \cup \bigcup_{\pi \in P} \pi(\overline{c_\pi}).$$

As the types are finitely satisfiable in M_M, every finite part of Σ is satisfied in a suitable expansion of M_M. By compactness and downwards Löwenheim-Skolem, Σ has a model of cardinal at most $|M| + |L| + |P| + |\overline{x}| + \aleph_0$. This model will be of the form $(N, ha, \overline{c_\pi}^N)_{a \in M,\ \pi \in P}$ where $h : M \to N$ denotes the interpretation of the constants of M. By proposition 1.5.2 h is an elementary embedding from M into N. Using exercise 1.2.9 there exists $M' \succcurlyeq M$ and an isomorphism $h' : M' \to N$ that extends h. We finally verify, for every $\pi \in P$, that $h'^{-1}(\overline{c_\pi}^N)$ realizes π in M'. If $\varphi(\overline{x}, \overline{a}) \in \pi$ then, as $(N, ha, \overline{c_\pi}^N)_{a \in M,\ \pi \in P}$ is a model of Σ, $(N, ha)_{a \in M} \models \varphi(\overline{c_\pi}^N, \overline{a})$, that is, $N \models \varphi(\overline{c_\pi}^N, h\overline{a})$. As h' is an isomorphism extending h we have $M' \models \varphi(h'^{-1}(\overline{c_\pi}^N), \overline{a})$. □

2.2 Saturation

Definition 2.2.1. Let M a L-structure and κ an infinite cardinal. The structure M is said to be ***κ-saturated*** if for each subset $A \subseteq M$ of cardinal less than κ, M realizes any 1-type of M over A.

Since one can extend every type to a complete type, it is enough to check the realization of complete 1-types.

Facts 2.2.2. *1. Every finite structure is κ-saturated for any cardinal κ.*

2. If M is infinite and κ-saturated then $|M| \geq \kappa$.

3. If M is κ-saturated and N is the restriction of M to a smaller language then N is κ-saturated.

4. If M is κ-saturated and $A \subseteq M$ with $|A| < \kappa$ then M_A is κ-saturated.

Proof. *1.* Since any type of M is realized in an elementary extension it is realized in the same M.

2. Otherwise the type $\{x \neq a \mid a \in M\}$ is over a set of less than κ parameters and it is not realized in M.

3. Obvious.

4. Any type of M_A over B is a type of M over AB. □

By facts *3.* and *4.* above, when $|A| < \kappa$ M is κ-saturated iff M_A is κ-saturated.

By fact 2 above the maximum degree of saturation that an infinite structure M can have is $|M|$-saturation. This makes a sense in the following definition:

Definition 2.2.3. A model M is called ***saturated*** if it is $|M|$-saturated.

Theorem 2.2.4. *Let M be a structure and κ be an infinite cardinal. Then there is a κ^+-saturated elementary extension of M of cardinal at most $2^{|L|}\,|M|^\kappa$.*

Proof. We may assume the structure M infinite. Then M has $|M|^\kappa$ subsets A of cardinal at most κ. Moreover, for each one of these subsets A we have that $\left|S_1^M(A)\right| \leq 2^{|L|+\kappa}$. By lemma 2.1.12 there is an elementary extension M_1 of M

of cardinal at most $2^{|L|}\,|M|^{\kappa}$ realizing all 1-types of M over a set of at most κ parameters from M. Applying again lemma 2.1.12 to M_1 we obtain M_2 an elementary extension of M_1 of cardinality at most $2^{|L|}\left(2^{|L|}\,|M|^{\kappa}\right)^{\kappa} = 2^{|L|}\,|M|^{\kappa}$ that realizes all 1-types of M_1 over a set of at most κ parameters from M_1. Iterating this procedure we construct a continuous chain $(M_\alpha \mid \alpha < \kappa^+)$ with $M_0 = M$ and where $M_{\alpha+1}$ is an elementary extension of M_α of cardinal at most $2^{|L|}\,|M|^{\kappa}$ that realizes all the types of M_α over a set of at most κ parameters from M_α. Using exercise 1.4.5 the chain is elementary and, by the lemma of the chain 1.4.4, M is an elementary substructure of the union of the chain N. By regularity of κ^+ N is κ^+-saturated and its cardinality is at most $2^{|L|}\,|M|^{\kappa}$. □

Usually $\kappa \geq |L|$. Then the cardinal of the extension provided by Theorem 2.2.4 is limited by $|M|^{\kappa}$. If moreover $\kappa \geq |M|$ the upper bound of the cardinal of the extension becomes 2^{κ}. Hence if $2^{\kappa} = \kappa^+$ any theory in a language with $|L| \leq \kappa$ has saturated models of size κ^+.

remark 2.2.5. If $2^{\kappa} = \kappa^+ \geq |L|$ then any theory has a saturated model of cardinality κ^+.

However if $2^{\kappa} > \kappa^+ \geq |L|$ in some theories there are not saturated models of cardinality κ^+. For instance, in T_{RAND}. If M is a model of T_{RAND} of cardinality κ^+, pick a subset A of M of cardinality κ. Observe the two different complete types cannot have a common realization: if $a \in M$ realizes p and $p \in \left|S_1^M(A)\right|$ then $p = \mathrm{tp}^M(a/A)$. Since $\left|S_1^M(A)\right| = 2^{\kappa}$ and $|M| < 2^{\kappa}$, M cannot realize all types in $S_1^M(A)$.

Exercise 2.2.6. Prove that every structure M has an ω-saturated elementary extension of cardinality at most $2^{|L|+\aleph_0}\,|M|$.

Exercise 2.2.7. Assume that $\kappa > |L| + \aleph_0$ and $\kappa^{\mu} = \kappa$ for each $\mu < \kappa$ (in particular if κ is an strongly inaccessible cardinal). Prove that every structure M with $|M| \leq \kappa$ has a saturated elementary extension of cardinality κ.

2.3 Partial elementary maps

In this section we will consider partial maps from M to N. A partial map $f : M \to N$ is an map that goes from a subset of M to N. Observe that an injective partial map from M to N is a bijection between subset of M and N respectively.

Definition 2.3.1. A partial map $f : M \to N$ is ***elementary*** if for every finite tuple $\overline{a}$ of the domain of f and every $\varphi(\overline{x}) \in L$ we have

$$M \models \varphi(\overline{a}) \text{ iff } N \models \varphi(f\overline{a}).$$

Equivalently, for every (finite) tuple $\overline{a}$ of the domain of f we have that $\overline{a}$ and $f\overline{a}$ have the same type over the empty set (in symbols: $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^N(f\overline{a})$). If $\overline{a}$ is an enumeration of the domain of f, it is also equivalent to $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^N(f\overline{a})$.

Observe that every partial elementary map is injective and its inverse is also elementary. If there is a partial elementary map between M and N then M and N have to be elementarily equivalent. In this sense it is useful to think that when M and N elementarily equivalent the empty map (between M and N) is elementary.

The following is easy but important: it will be used several times later.

Example 2.3.2.
1. Any elementary embedding $h : M \to N$ is a partial elementary map.
2. Any subset of a partial elementary map is also a partial elementary map.
3. Let $M = (\mathbb{Z}, s)$, where s denotes the successor function. The following partial maps $\{(n, m)\}$, $\{(n_1, n_1 + m), \ldots, (n_k, n_k + m)\}$ are elementary because they are subsets of an automorphism of M. But $\{(1, 8), (3, 5)\}$ is not a partial elementary map: $M \models 3 = s(s(1))$ but $M \not\models 5 = s(s(8))$. In other words, $y = ss(x) \in p(x, y) = \mathrm{tp}^M((1, 3))$ while $y = ss(x) \notin p(x, y) = \mathrm{tp}^M((8, 5))$. In this example, the partial elementary maps are the partial maps that preserve the 'distance'(the distance between n and m is number of times you must apply s to n to obtain m, possibly a negative number).

Exercise 2.3.3. Let $A \subseteq M$, let $\overline{a}, \overline{b}$ be tuples (of the same length) of M and let $\overline{c}$ be an enumeration of A. Show that the following are equivalent:

1. $\mathrm{tp}^M(\overline{a}/A) = \mathrm{tp}^M(\overline{b}/A)$.
2. $\mathrm{tp}^M(\overline{a}\overline{c}/\emptyset) = \mathrm{tp}^M(\overline{b}\overline{c}/\emptyset)$.
3. $id_A \cup (\overline{a}, \overline{b})$ is an elementary map.

Exercise 2.3.4. Prove that $(N, ha)_{a\in A}$ is a model of $Th(M_A)$ iff the partial map (with domain A) $h : M \to N$ is elementary. Hence, a model of $Th(M_A)$ consist on a structure N together with an elementary partial map $h : M \to N$ with domain A.

Definition 2.3.5. Suppose that f is a partial elementary map from M to N and π is a type of M over $\operatorname{dom} f$. The conjugate of π by f, denoted by π^f, is defined as

$$\{\varphi(\overline{x}, f\overline{a}) \mid \varphi(\overline{x}, \overline{a}) \in \pi\}.$$

Obviously, if π is over $A \subseteq \mathrm{dom}(f)$ then π^f is over $f(A) \subseteq \mathrm{Im}(f)$.

Lemma 2.3.6. *Let f a partial elementary map from M to N.*

1. *If π is a type of M over $\operatorname{dom} f$ then π^f is a type of N over $\operatorname{Im} f$.*
2. *If $\overline{a}$ is a tuple of $\operatorname{dom} f$ and $A \subseteq \operatorname{dom} f$ then $f\overline{a}$ realizes $\mathrm{tp}^f(\overline{a}/A)$. Equivalently $\mathrm{tp}(f\overline{a}/f(A)) = \mathrm{tp}^f(\overline{a}/A)$.*
3. *Let $\overline{a}$, $\overline{b}$ be tuples of the same length of M and N respectively. If $\overline{b}$ realizes $\mathrm{tp}^f(\overline{a}/\operatorname{dom} f)$ then $f \cup \{(\overline{a}, \overline{b})\}$ is elementary.*

Proof. 1. Observe that π^f is a set of formulas with parameters in $\operatorname{Im} f$ with the same variables as π. A finite subset of π^f is of the form

$$\{\varphi_1(\overline{x}, f\overline{a}), \dots, \varphi_n(\overline{x}, f\overline{a})\},$$

where $\varphi_i(\overline{x}, \overline{a}) \in \pi$ for each i. As π is a type of M, $M \models \exists \overline{x} \bigwedge_{i=1\dots n} \varphi_i(\overline{x}, \overline{a})$. Since f is elementary $N \models \exists \overline{x} \bigwedge_{i=1\dots n} \varphi_i(\overline{x}, f\overline{a})$ and therefore π^f is finitely satisfiable in N.

2. A formula in $\operatorname{tp}^f(\overline{a}/A)$ is of the form $\varphi(\overline{x}, f\overline{b})$ for some $\varphi(\overline{x}, \overline{b}) \in \operatorname{tp}(\overline{a}/A)$, where $\varphi(\overline{x}, \overline{y})$ an L-formula and $\overline{b}$ is a tuple in $\operatorname{dom}(f)$. If $\varphi(\overline{x}, \overline{b}) \in \operatorname{tp}(\overline{a}/A)$ then $M \models \varphi(\overline{a}, \overline{b})$ and, as f is elementary, $N \models \varphi(f\overline{a}, f\overline{b})$.

3. We have to show that for every tuple $\overline{c}$ of the domain of f and every formula $\varphi(\overline{x}, \overline{y}) \in L$, $M \models \varphi(\overline{a}, \overline{c})$ iff $N \models \varphi(\overline{b}, f\overline{c})$. In fact one of these two implications suffice. If $M \models \varphi(\overline{a}, \overline{c})$ then $\varphi(\overline{x}, \overline{c}) \in \operatorname{tp}(\overline{a}/\operatorname{dom} f)$ and thus $\varphi(\overline{x}, f\overline{c}) \in \operatorname{tp}^f(\overline{a}/\operatorname{dom} f)$. Since $\overline{b}$ realizes $\operatorname{tp}^f(\overline{a}/\operatorname{dom} f)$ we get $N \models \varphi(\overline{b}, f\overline{c})$. □

Observe that in point *3.* of Lemma 2.3.6, by point *2.*, the converse also holds. Obviously the conjugate by an elementary map of a complete type is also a complete type. Therefore, if $A \subseteq \operatorname{dom}(f)$, conjugation by f is a bijection between $S^M_{\overline{x}}(A)$ and $S^N_{\overline{x}}(f(A))$. In fact this bijection will be a homeomorphism of topological spaces.

Exercise 2.3.7. Here we will see that partial elementary maps can always be amalgamated is the following sense. Assume that for each $i \in I$, $f_i : M_0 \to M_i$ are partial elementary maps. Show that there is N and partial elementary maps $g_i : M_i \to N$ such that $g_i \circ f_i$ makes sense and $g_i \circ f_i = g_j \circ f_j$. Let $\overline{a}$ be an enumeration of $\bigcap_i \operatorname{dom}(f_i)$ and $\overline{x}$ a tuple of variable of the same length. Let $p(\overline{x})$ denote $\operatorname{tp}^{M_0}(\overline{a})$. For each i let $\overline{b_i}$ be an enumeration of $\operatorname{dom}(f_i) \setminus \bigcap_i \operatorname{dom}(f_i)$ and $\overline{y}_i$ a tuple of variables of the same length. Let $q_i(\overline{x}, \overline{y}_i)$ denote $\operatorname{tp}^{M_0}(\overline{a}, \overline{b}_i)$. Show that $\bigcup_i q_i(\overline{x}, \overline{y}_i)$ is consistent. Here we assume all variables $\overline{y_i}$ are distinct. Show that, if a $\overline{c}, (\overline{c_i} \mid i \in I)$ is realization of $\bigcup_i q_i(\overline{x}, \overline{y}_i)$ in N the maps f_i which send $g_i(\overline{a}), g_i(\overline{b}_i)$ to $\overline{c}, \overline{c_i}$ provide a solution.

2.4 More about saturation

The following shows that a complete theory has at most one saturated model in each cardinality.

Theorem 2.4.1. *Two saturated models of the same cardinality and elementarily equivalent are isomorphic.*

Proof. Suppose that M, N are elementarily equivalent and saturated models, both of cardinal λ. Let $(a_i \mid i \in \lambda)$ and $(b_i \mid i \in \lambda)$ be enumerations of M and N respectively. We will construct a chain $(f_i \mid i \in \lambda)$ of partial elementary maps with $a_i \in \operatorname{dom} f_{i+1}$ and $b_i \in \operatorname{Im} f_{i+1}$ and $|f_i| < \kappa$. The union of the chain will be the searched isomorphism. We start with $f_0 = \emptyset$ the empty map and in the limits we take unions. In order to construct f_{i+1} we take, because of saturation, a realization c of $\mathrm{tp}^{f_i}(a_i/\operatorname{dom} f_i)$. By lemma 2.3.6 $f' = f \cup \{(a_i, c)\}$ is elementary. Now we take a realization d of $\mathrm{tp}^{f'^{-1}}(b_i/\operatorname{Im}(f'))$. By 2.3.6 again, $f_{i+1} = f_i \cup \{(a_i, c), (d, b_i)\}$ is elementary. Obviously $|f_{i+1}| \leq |f_i| + 2 < \kappa$. $\square$

This proof in fact shows that if M and N are saturated and of the same cardinality, every partial elementary map of cardinal smaller than $|M| = |N|$ from M to N extends to an isomorphism between M and N. In particular:

Corollary 2.4.2. *If M is saturated then every partial elementary map from M to M of cardinal less than $|M|$ extends to an automorphism of M.*

The set of automorphisms of M which are the identity on A (A a subset of M) is usually denoted by $\mathrm{Aut}(M/A)$ and is a group by composition. There is a natural action of $\mathrm{Aut}(M/A)$ on M (or on M^I for any I) by putting $g \cdot \overline{a} := g(\overline{a})$. Assume now that $|A| < |M|$ and let $\overline{a}, \overline{b}$ be tuples in M of (the same) length $< |M|$. By the homomorphism theorem 1.2.4, if $\overline{a}$ and $\overline{b}$ lie in the same orbit of the action of $\mathrm{Aut}(M/A)$ obviously $\mathrm{tp}(\overline{a}/A) = \mathrm{tp}(\overline{b}/A)$. Conversely, if $\overline{a}, \overline{b}$ have the same type over A, then $id_A \cup (\overline{a}, \overline{b})$ is an elementary map of size $< |M|$. By corollary 2.4.2 there is an automorphism f of M extending $id_A \cup (\overline{a}, \overline{b})$. Hence $\overline{a}$ and $\overline{b}$ lie in the same orbit of the action of $\mathrm{Aut}(M/A)$. Thus for saturated models M and small A, A-orbits correspond to A-types. This holds in fact for any ***strongly homogeneous*** M (by definition: each elementary map of size $< |M|$ extends to an automorphism of M). Corollary 2.4.2 says that saturated models are strongly homogeneous.

Definition 2.4.3. A structure M is ***κ-homogeneous*** if given a partial elementary map $f : M \to M$ of cardinal $< \kappa$ and an element a of M there is an extension of f to a partial elementary map from M to M with a in the domain. This is equivalent to the existence of an element b in M such that $f \cup \{(a, b)\}$ is elementary.

The following exercise is a characterization of κ-saturation in terms of extensions of elementary maps.

Exercise 2.4.4.

1. Prove that M is κ-saturated iff given a partial elementary map $f : N \to M$ of cardinal $< \kappa$ and an element a of N there is an extension of f to a partial elementary map from N to M with a in the domain. Equivalently, given a partial elementary map $f : N \to M$ of cardinal $< \kappa$ and an element $a \in N$ there exists an element b in M such that $f \cup \{(a, b)\}$ is elementary from N to M.
2. Conclude that every κ-saturated model is κ-homogeneous.

The following shows that we can improve the extensibility of elementary maps in saturated models:

Proposition 2.4.5. *Let M be κ-saturated and let $f : N \to M$ be a partial elementary map with $|f| < \kappa$. Let B be a subset of N of cardinality at most κ. Then There is a partial elementary map $g : N \to M$, extending f with $B \subseteq \operatorname{dom}(g)$.*

Proof. Let $(b_i \mid i \in \kappa)$ be an enumeration of B. We construct a chain $(f_i \mid i \in \lambda)$ of partial elementary maps with $b_i \in \operatorname{dom} f_{i+1}$ and $|f_i| < \kappa$. The union of the chain will be the desired elementary map. We start with $f_0 = f$ and in the limits we take unions. In order to construct f_{i+1} we take, because of saturation, a realization c of $\operatorname{tp}^{f_i}(b_i/\operatorname{dom} f_i)$. By lemma 2.3.6 $f_{i+1} = f \cup \{(a_i, c)\}$ is elementary. Obviously $|f_{i+1}| \leq |f_i| + 1 < \kappa$. □

Corollary 2.4.6. *Let M, N be elementarily equivalent structures, where N is κ-saturated and M is of cardinal at most κ. Then there is an elementary embedding from M to N.*

Proof. Apply proposition 2.4.5 to the empty map with $B = N$. □

It is usual to say that M is ***κ-universal*** when for every N of cardinal less than κ and elementarily equivalent to M there is elementary embedding of N into M. With this terminology, proposition 2.4.7 can be stated as follows: every κ-saturated model is κ^+-universal.

The next result shows that we could have replaced 1-types by κ-types in the definition of κ-saturation.

Proposition 2.4.7. *Let M be κ-saturated. Then M realizes every type of M over any sets of parameters of cardinal less than κ with at most κ variables.*

Proof. We assume the type is complete. Let $p \in S^M_{\overline{x}}(A)$, where $|A| < \kappa$ and $|\overline{x}| \leq \kappa$. Let $\overline{b} \in N$ be a realization of $p(\overline{x})$ in an elementary extension N of M. Applying proposition 2.4.5 to the elementary map $\mathrm{Id}_A : N \to M$ and $B = \{b_i \mid i \in I\}$ we get some elementary g with $\mathrm{Id}_A \subseteq g$ and $\{b_i \mid i \in I\} \subseteq \operatorname{dom}(g)$. Now, by lemma 2.3.6, $g\overline{b}$ realizes $p^g = p$ in M. □

Exercise 2.4.8.

1. Assume M and N are κ-saturated. Let $A \subseteq M$ and $B \subseteq N$ such that $|A| + |B| \leq \kappa$ and $f : M \to N$ a partial elementary map with $|f| < \kappa$. Show that there is a partial elementary map $g : M \to N$ extending f with $A \subseteq \operatorname{dom}(f)$ and $B \subseteq \operatorname{Im}(g)$. Hint: pick enumerations $(a_\alpha \mid \alpha \in \kappa)$ and $(b_\alpha \mid \alpha \in \kappa)$ of $(a_\alpha \mid \alpha \in \kappa)$ A and B respectively and build a chain of extensions $(f_\alpha \mid \alpha \in \kappa)$ of f with $a_\alpha \in \operatorname{dom}(f_{\alpha+1})$ and $b_\alpha \in \operatorname{Im}(f_{\alpha+1})$ and proceed as in the proof of Theorem 2.4.1.

2. obtain Theorem 2.4.1 from 1.

3. obtain Corollary 2.4.2 from 1.

Exercise 2.4.9. M is said to be ***strongly κ-homogeneous*** if any partial elementary map $f : M \to M$ with $|f| < \kappa$ can be extended to an automorphism of M.

1. Assume M is κ-homogeneous. Let $A, B \subseteq M$ such that $|A| + |B| \leq \kappa$ and $f : M \to M$ a partial elementary map with $|f| < \kappa$. Show that there is a partial elementary map $g : M \to M$ extending f with $A \subseteq \operatorname{dom}(f)$ and $B \subseteq \operatorname{Im}(g)$. Hint: pick enumerations $(a_\alpha \mid \alpha \in \kappa)$ and $(b_\alpha \mid \alpha \in \kappa)$ of $(a_\alpha \mid \alpha \in \kappa)$ A and B respectively and build a chain of extensions

$(f_\alpha \mid \alpha \in \kappa)$ of f with $a_\alpha \in \mathrm{dom}(f_{\alpha+1})$ and $b_\alpha \in \mathrm{Im}(f_{\alpha+1})$ and proceed as in the proof of Theorem 2.4.1.

2. Show that strongly κ-homogeneous implies κ-homogeneous. The converse is not true in general. However:

3. Show that $|M|$-homogeneous implies $|M|$-strongly homogeneous. Hint: use 1.

Exercise 2.4.10. 1. Let $\overline{a} = (a_i \mid i \in I)$ be an enumeration of M. Let $\pi(\overline{x}) = \mathrm{atp}^M(\overline{a})$ be the atomic type of $\overline{a}$, where $\overline{x} = (x_i \mid i \in I)$ is a tuple of variables. Show that a tuple $\overline{b} = (b_i \mid i \in I)$ in a structure N realizes $\pi(\overline{x})$ iff the map from M to N given by $a_i \to b_i$ is an embedding.

2. Let N be an $|M|$-saturated L-structure. Show that if for each finitely generated substructure M' of M there is an embedding of M' into N then there is an embedding of M into N.

We know that a κ-saturated models is κ-homogeneous and κ^+-universal. In the next theorem we will see that the converse also holds in a strong form: It is enough $(|L| + \aleph_0)^+$-universality plus κ-homogeneity.

Theorem 2.4.11. *Let M be a model and $\kappa \geq |L| + \aleph_0$. The following are equivalent:*

1. *M is κ-saturated.*
2. *M is κ-homogeneous and $(|L| + \aleph_0)^+$-universal.*
3. *M is κ-homogeneous and realizes all types in $S_n^M(\emptyset)$ for each finite n.*

Proof. *1.* implies *2.* is in exercise 2.4.4 and corollary 2.4.6.

2. implies *3.* Let $p \in S_n^M(\emptyset)$. Let, by Löwenheim-Skolem, $M' \preceq M$ with $|M'| \leq |L| + \aleph_0$ and let $\overline{a}$ be a realization of $p(\overline{x})$ in an elementary extension N of M' with $|N| \leq |L| + \aleph_0$. By $(|L| + \aleph_0)^+$-universality there is an elementary embedding $f : N \to M$. Now, by Lemma 2.3.6, $f\overline{a}$ realizes $p^f(\overline{x}) = p(\overline{x})$ in M.

3. implies *1.* Assume M is κ-homogeneous and realizes all types (of M) in finitely many variables (over the empty set). We first show that M realizes all types in at most κ variables over empty. This will be done by induction on λ, the number of variables. Let $p(\overline{x}) \in S_\lambda^M(\emptyset)$, where $\overline{x} = (x_\alpha \mid \alpha \in \lambda)$ and $\lambda \leq \kappa$. We will build, by induction on α, a sequence $(a_\alpha \mid \alpha \in \lambda)$ of elements of M in such a way that $\overline{a_{\leq\alpha}} = (a_\beta \mid \beta \leq \alpha)$ realizes the restriction $p_{\leq\alpha}$ of p to $(x_\beta \mid \beta \leq \alpha)$. This will suffice. Assume we have $\overline{a_{<\alpha}} = (a_\beta \mid \beta < \alpha)$ realizing $p_{<\alpha}$, the restriction of p to $(x_\beta \mid \beta < \alpha)$. We must find a_α in such a way $\overline{a_{\leq\alpha}}$ realizes $p_{\leq\alpha}$. By inductive hypothesis let $\overline{b_{\leq\alpha}} = (b_\beta \mid \beta \leq \alpha)$ be a realization of $p_{\leq\alpha}$. Since $\overline{a_{<\alpha}}$ and $\overline{b_{<\alpha}}$ have the same type over empty, namely $p_{<\alpha}$, the map sending a_β to b_β for $\beta < \alpha$ is elementary. By homogeneity there is some a_α such that adding (a_α, b_α) to this map remains elementary. This implies that $\mathrm{tp}^M(\overline{a_{\leq\alpha}}/\emptyset) = \mathrm{tp}^M(\overline{b_{\leq\alpha}}/\emptyset) = p_{\leq\alpha}$.

Let A be a subset of M of size less than κ and let $p(x) \in S_1^M(A)$. Let b be a realization of p in an elementary extension N of M. Let $\overline{a} = (a_i \mid i \in I)$ be an enumeration of A and let $q(\overline{y}, x) = \mathrm{tp}^N(\overline{a}, b)$. As q is a type (of N hence also of M) over empty with less than κ variables it has a realization $(\overline{c}, d)$ in M. Since $\mathrm{tp}^N(\overline{a}, b) = q(\overline{y}, x) = \mathrm{tp}^M(\overline{c}, d)$ the map (from N to M) sending $\overline{a}, b$ to $\overline{c}, d$ is elementary and thus the map (from N to M and also from M to M) sending $\overline{c}$ to $\overline{a}$ is also elementary. By homogeneity there is $e \in M$ such that the map (from M to M) sending $\overline{c}, d$ to $\overline{a}, e$ is elementary. Now the map $g : N \to M$ sending $\overline{a}, b$ to $\overline{a}, e$ is elementary. Since b realizes p, by 2.3.6, e realizes $p^g = p$. $\square$

Exercise 2.4.12. Show that 1. and 2. above are equivalent. Show that if moreover $\kappa > |L| + \aleph_0$ then 1. and 2. are also equivalent to 3.

1. M realizes all κ-types of M over empty.
2. For any $N \equiv M$ and any $A \subseteq N$ with $|A| \leq \kappa$ there is a partial elementary map $f : N \to M$ with $A \subseteq \mathrm{dom}(f)$.
3. M is κ^+-universal.

Exercise 2.4.13.
1. Let M be κ-homogeneous $N \equiv M$ and assume each finitary type (with finitely many variables) over empty realized in N is also realized in M.
   (a) Show that each type in at most k variables over empty realized in N is also realized in M. Hint: by induction on λ, the number of variables, see the proof of *3.* implies *1.* in Theorem 2.4.11.
   (b) If $|N| \leq \kappa$, show there is an elementary embedding from N to M. This shows M is k^+-universal in the class
   $$\{N \mid N \equiv M \text{ and } N \text{ omits any finitary type over empty omitted by } M\}.$$
   Hint: use (a).
   (c) Let $f : N \to M$ is a partial elementary map with $|f| < \kappa$ and $b \in N$. Show one can extend f to an elementary map $g : N \to M$ with $b \in \mathrm{dom}(g)$. Hint: use (a).
   (d) Let $f : N \to M$ is a partial elementary map with $|f| < \kappa$ and $B \subseteq N$ with $|B| \leq \kappa$. Show one can extend f to an elementary map $g : N \to M$ with $B \subseteq \mathrm{dom}(g)$. Hint: either iterate (c). with a good enumeration of B or use (a). and exercise 2.4.9.
2. Assume M and N are κ-homogeneous elementarily equivalent and realize the same finitary types over empty.
   (a) Le $f : N \to M$ is a partial elementary map with $|f| < \kappa$ and $B \subseteq N$, $A \subseteq M$ with $|A| + |B| \leq \kappa$. Show one can extend f to an elementary map $g : N \to M$ with $B \subseteq \mathrm{dom}(g)$ and $A \subseteq \mathrm{Im}(g)$. Hint: iterate 1.(c) with good enumerations of A and B.

(b) Show that if moreover $|M| = |N| = \kappa$ then M and N are isomorphic. This generalizes Theorem 2.4.1. Hint apply 2.(a).

Exercise 2.4.14. A theory T is called λ-stable if for any model M of T and any $A \subseteq M$ with $|A| = \lambda$ then $\left|S_1^M(A)\right| = \lambda$. Show that if $\lambda \geq |L| + \aleph_0$ is regular and T is λ-stable theory with infinite models then T has a saturated model of cardinality λ. Hint: build a continuous chain of length λ of models of cardinality λ each one realizing all 1-types over the predecessor (if it has). It also true for non-regular λ, but the proof is much more difficult.

Exercise 2.4.15. Suppose that N is λ-saturated (λ^+-universal suffice, using exercise) $|M| \leq \lambda$ and each existential sentence holding in M also holds in N. Show that there is an embedding from M to N. Hint: consider an enumeration $\overline{a}$ of M and consider $\mathrm{atp}^M(\overline{a}/\emptyset)$, the atomic type of $\overline{a}$ over $\emptyset$.

Exercise 2.4.16. Let k be an algebraically closed field. Show that

1. k is λ-saturated iff the transcendence degree of k over the prime field is at least λ.

2. Any uncountable algebraically closed field is saturated.

Chapter 3

Omitting Types

3.1 A little bit of topology

Definition 3.1.1. A topological space is a pair (X, τ), where X is a set and $\tau \subseteq P(X)$ is a family of subsets of X satisfying the following properties:

- For every $\mu \subseteq \tau$, $\cup\mu \in \tau$.
- For every finite $\mu \subseteq \tau$, $\cap\mu \in \tau$.
- $\emptyset, X \in \tau$

The elements of X are called points and the subsets of X sets. The sets which belong to μ will be called ***open*** sets and their complements ***closed*** sets. The definition says that the collection of open sets is closed under taking unions, finite intersections and contain the empty set and the whole space. Thus the collection of closed sets is closed under taking intersections and finite unions. The empty set and the whole space are also closed. The subsets of X which are both open and closed are called ***clopen*** sets. For instance, $\emptyset$ and X are clopen. In order to define a topology in X it is quite common to provide a nice subcollection of open sets, instead of providing the whole collection of all open sets. A ***base*** of (X, τ) is a subcollection $\mu \subseteq \tau$ such that one gets all open sets as unions of sets in the base. The elements of μ are called basic open sets. More precisely, each open set is a union of basic open sets i.e., for every $V \in \tau$ there is $\mu' \subseteq \mu$ such that $V = \cup\mu'$. It is easily checked (and let as an exercise for the reader) that a base μ for a topology in X satisfies the following two properties:

- $X = \cup\mu$.
- For every $V_1, V_2 \in \mu$ and for every $x \in V_1 \cap V_2$ there is $V_3 \in \mu$ such that $x \in V_3 \subseteq V_1 \cap V_2$.

Conversely, given a set X and $\mu \subseteq P(X)$ satisfying the two properties above the collection $\tau = \{\cup\mu' \mid \mu' \subseteq \mu\}$ is a topology on X. This is again left as an exercise.

Definition 3.1.2. Given a subset A of X, the ***interior*** of A, denoted by $\overset{\circ}{A}$, is the biggest open set contained in A:

$$\overset{\circ}{A} = \cup \{V \mid V \text{ is open and } V \subseteq A\} .$$

The closure of A is the smallest closed set containing A:

$$\overline{A} = \cap \{C \mid C \text{ is closed and } A \subseteq C\} .$$

As taking complements turns open sets to closed sets and reverse the inclusions, it is straightforward to check that

$$\left(\overset{\circ}{A}\right)^C = \overline{A^C}.$$

Here we use A^C to denote $X \setminus A$, the complement of A in X.

A ***neighborhood*** of a point x is any set containing an open set containing x. We let as an exercise to show the following: $x \in \overset{\circ}{A}$ iff there is a neighborhood of x contained in A. Also: $x \in \overline{A}$ iff every neighborhood of x intersects (has nonempty intersection with) A.

Definition 3.1.3. A subset of X is called ***dense*** if $\overline{A} = X$.

Equivalently, a dense set A is a subset of X which has nonempty intersection with any nonempty open set. Moreover, A is dense iff A^C has empty interior (in other words: $\emptyset$ is he only open set contained in the complement of A).

Definition 3.1.4. A topological space (X, τ) is said to be ***Haussdorff*** iff given two different points $x, y \in X$ there are neighborhoods $V_x \ni x$ and $V_y \ni y$ of x and y respectively such that $V_x \cap V_y = \emptyset$.

Definition 3.1.5. A subset A of X is called ***compact*** (in (X, τ)) iff for every family of open sets $\{V_i \mid i \in I\}$ such that $A \subseteq \bigcup_{i \in I} V_i$ there is a finite $I_0 \subseteq I$ such that $A \subseteq \bigcup_{i \in I} C_i$. In words: every open covering of A has a finite subcovering. When X is compact we say that the topological space (X, μ) is compact.

Taking complements, one gets the following characterization: A is compact iff for every family of closed sets $\{C_i \mid i \in I\}$ such that $A \cap \bigcap_{i \in I} C_i = \emptyset$ there is a finite $I_0 \subseteq I$ such that $A \cap \bigcap_{i \in I} C_i = \emptyset$. Observe that the space X is compact whenever given any family of closed sets with empty intersection one can find a finite subfamily with empty intersection. This is usually stated the following way: given a collection of closed sets, if any finite subcollection has nonempty intersection then also the whole family has nonempty intersection.

Compact sets behave as finite sets in some sense. For instance, let us as see that in a Haussdorff space we can extend the 'separation' of points to compact sets: given two disjoint compact sets K_1, K_2 there are disjoint open sets V_1, V_2 with $K_i \subseteq V_i$. Let us start with a point x and a compact set K. For each $y \in K$ let V_y, W_y be open neighborhoods of x and y respectively with $V_y \cap W_y = \emptyset$.

Since $(W_y \mid y \in K)$ is an open covering of K, by compactness, there is some finite $F \subseteq K$ such that $(W_y \mid y \in F)$ also covers K. Now it is easily checked that $\bigcap_{y \in F} V_y$ and $\bigcup_{y \in F} W_y$ are the desired neighborhood of x and K respectively. This implies, in particular that K is closed. Repeating the argument again we can separate disjoint compact sets.

There is a close relation between 'compact' and 'closed' subset of a topological space:

remark 3.1.6.

1. Every closed set of a compact space is compact.
2. Every compact subset of a Haussdorff space is closed.

Hence, for compact Haussdorff spaces, 'closed subset' and 'compact subset' coincide.

Proof. *1.* Let C be a closed subset of X. If $(V_i \mid i \in I)$ is an open covering of C, adding C^C one gets an open covering of X. A finite subcovering of X provides a finite subcovering of C.

2. By the separation argument above. □

Definition 3.1.7. A topological space is ***locally compact*** if any point has a compact neighborhood.

Obviously any compact space is locally compact: the whole space is a compact neighborhood.

remark 3.1.8. In a Haussdorff locally compact space each point has a base of compact neighborhoods: for each point x and each neighborhood V of x there is some compact neighborhood K of x contained in V.

Proof. Let K be a compact neighborhood of x an open V neighborhood of x. Let W an open neighborhood of x contained in K. We will find a compact neighborhood of x contained in $W \cap V$. By remark 3.1.6 $K_1 = K \setminus (V \cap W)$ is compact. By separation of x and K_1 let V_1, V_2 disjoint open sets with $x \in V_1$ and $K_1 \subseteq V_2$. Now $x \in V_1 \cap W \subseteq V_2^C \cap K$ and thus $V_2^C \cap K$ is a compact neighborhood of x contained in $V \cap W$. □

Next result, the Baire category theorem also holds for complete metric spaces with a slight modification on the proof.

Theorem 3.1.9 (Baire Category theorem)**.** *In a locally compact Haussdorf space, the intersection of countably many open dense sets is dense.*

Proof. Let X be locally compact and Haussdorf. Let $(V_n \mid n \in \mathbb{N})$ be a collection of dense open sets. In order to show that $\bigcap_{n \in \mathbb{N}} V_i$ is dense we must show that $V \cap \bigcap_{i \in \mathbb{N}} V_i \neq \emptyset$ for any nonempty open V. Since V_1 is dense $V_1 \cap V \neq \emptyset$ and thus, by remark 3.1.8, we can find a compact set K_1 with nonempty interior with $K_1 \subseteq V \cap V_1$. Since V_2 is dense and K_1 has nonempty interior, $V_2 \cap K_1$ has also nonepty interior and thus, by remark 3.1.8 again, we can find a compact set K_2 with nonempty interior with $K_2 \subseteq K_1 \cap V_1$. Observe that $K_2 \subseteq V \cap\ V_1 \cap V_2$. Repeating this procedure we build a decreasing chain of compact sets $(K_n \mid n \in$

$\mathbb{N}$) with $K_n \subseteq V \cap V_1 \cap \cdots \cap V_n$. Since K_1 is compact and each K_i is closed, $\bigcap K_{n\in\mathbb{N}}$ is nonempty, hence $V \cap \bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$. $\square$

3.2 The spaces of complete types of a theoty

In this section we assume a theory T is given, and $\overline{x}$ will denote a tuple of variables, maybe of infinite length.

Definition 3.2.1. A type of T is a set of formulas $\pi(\overline{x}) \subseteq L_{\overline{x}}$ such that $T \cup \pi(\overline{x})$ is consistent.

Hence, a subset of $L_{\overline{x}}$ is a type of T iff has a realization in a model of T. Equivalently, by the compactness theorem, iff every finite subset of $\pi(\overline{x})$ has a realization in a model of T. As in section 2.1 we say that a type is complete iff for every formula $\varphi \in L_{\overline{x}}$ the type contains exactly one of $\varphi, \neg\varphi$. The set of all complete types of T with free variables on $\overline{x}$ is denoted by $S^T_{\overline{x}}$.

The notion of 'type of T' and 'type of M' should not be confused. First, in a type of T we do not allow parameters, whereas we allow parameter sets $A \subseteq M$ for the second notion. Moreover, compactness allows us to conclude that 'finitely realizable in T' (more precisely: in a model of T) is the same as 'realizable in T'. Obviously 'finitely realizable in M' is not the same as 'realizable in M'. However both notions are intimately related and can be seen each one as a particular case of the other if T is complete.

remark 3.2.2.
1. Let M be a model, A a set of parameters $A \subseteq M$ and $\pi(\overline{x}) \subseteq L_{\overline{x}}(A)$. Then $\pi(\overline{x})$ is a type of M over A iff it is a type of the theory $\mathrm{Th}\, M_A$. This is an consequence of 2.1.3. In particular:
$$S^M_{\overline{x}}(A) = S^{\mathrm{Th}(M_A)}_{\overline{x}}.$$
2. Let T be a theory. Any type $\pi(\overline{x})$ of T is a type of some model of T, since $\pi(\overline{x})$ is realized in some model of T. If M is a model of T and $\pi(\overline{x})$ is type of M, $\pi(\overline{x})$ is also s type of T because the consistency of $\mathrm{Th}(M) \cup \pi$ implies the consistency of $T \cup \pi$. Hence, π is a type of T iff π is a type of some model of T. For complete types, one gets:
$$S^T_{\overline{x}} = \bigcup_{M\in\mathrm{Mod}(T)} S^M_{\overline{x}}.$$
In the expression above it suffices to pick a model for each completion of T. When T is a complete theory and M a is model of T, then $\pi(\overline{x})$ is a type of M (over $\emptyset$) iff it is a type of T. In particular:
$$S^T_{\overline{x}} = S^M_{\overline{x}}.$$

The space $S^T_{\overline{x}}$ is topologized as follows. For every formula $\varphi \in L_{\overline{x}}$, consider
$$[\varphi] = \left\{p \in S^T_{\overline{x}} \mid \varphi \in p\right\}.$$

It is easily checked that $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$, hence $\{[\varphi] \mid \varphi \in L_{\overline{x}}\}$ forms a base of a topology. As $S^T_{\overline{x}} \setminus [\varphi] = [\neg\varphi]$, the sets of the base are clopen, hence is a totally disconnected space.

As closed sets are intersections of basic clopen sets, any closed set C is of the form $C = \bigcap_{\varphi\in\pi}[\varphi] = \left\{p \in S^T_{\overline{x}} \mid \pi \subseteq p\right\}$ for a certain set $\pi \subseteq L_{\overline{x}}$. Hence closed sets correspond to sets of formulas π. We will denote $\left\{p \in S^T_{\overline{x}} \mid \pi \subseteq p\right\}$ by $[\pi]$. One also verifies that $[\pi] = \emptyset$ iff $T \cup \pi$ is inconsistent iff $\pi(\overline{x})$ has no realization in a model of T. Hence nonempty closed sets correspond to types.

Proposition 3.2.3. *The space $S^T_{\overline{x}}$ is compact.*

Proof. Suppose we are given a family $([\pi_i], i \in I)$ of closed sets with empty intersection. Since $\emptyset = \bigcap_{i\in I}[\pi_i] = [\bigcup_{i\in I}\pi_i]$ we get that $\bigcup_{i\in I}\pi_i$ is inconsistent with T. By compactness there is a finite $I_0 \subseteq I$ such that $\bigcup_{i\in I_0}\pi_i \cup T$ is inconsistent. This means that $\bigcap_{i\in I_0}[\pi_i] = [\bigcup_{i\in I_0}\pi_i] = \emptyset$. □

Exercise 3.2.4. Show the following properties

1. $[\varphi \vee \psi] = [\varphi] \cup [\psi]$
2. $[\varphi] \subseteq [\psi]$ iff $T \models \varphi \to \psi$
3. $[\varphi] = [\psi]$ iff $T \models \varphi \leftrightarrow \psi$
4. $[\varphi] = S^T_{\overline{x}}$ iff $T \models \varphi$.
5. $[\varphi] = \emptyset$ iff $T \cup \varphi$ is inconsistent.

Exercise 3.2.5. Let T be a theory in a language L. Consider the following equivalence relation $\sim$ in $L_{\overline{x}}$: $\varphi \sim \psi$ iff $T \models \varphi \leftrightarrow \psi$.

1. Show that it is an equivalence relation and define operations such that $L_{\overline{x}}/\sim$ is a boolean algebra. It is called the Tarski-Lindembaum algebra. Let us denote it by $B^T_{\overline{x}}$.
2. Given $p \in S^T_{\overline{x}}$, show that $\psi \sim \varphi$ and $\varphi \in p$ then $\psi \in p$. Let us denote $p/\sim$ the quotient of p by $\sim$. Show that $p/\sim$ is an ultrafilter of $B^T_{\overline{x}}$.
3. given an ultrafilter U of $B^T_{\overline{x}}$, show that $\bigcup U \in S^T_{\overline{x}}$.
4. Show the maps $p \mapsto p/\sim$ and $U \mapsto \bigcup U$ are homeomorphisms, one inverse to the other, between $S^T_{\overline{x}}$ and the stone space of $B^T_{\overline{x}}$.

3.3 The omitting types theorem

By definition, a type of T is realized in a model of T. Here we will try to answer the following question: it is realized in every model? In other words: is there a model of T not relizing the type? We begin by naming things:

Definition 3.3.1. We say that a structure M ***omits*** a set of formulas $\pi(\overline{x})$ if M does not realize $\pi(\overline{x})$.

If $\pi(\overline{x})$ is inconsistent with T then obviously every model of T omits $\pi(\overline{x})$.

Definition 3.3.2. An n-type $\pi(\overline{x})$ of T is ***isolated*** iff there is a formula $\varphi(\overline{x})$, consistent with T, such that $T \models \varphi \to \psi$ for every $\psi \in \pi$. This is denoted by $T \models \varphi \to \pi$, and we say that φ isolates π in T.

Let us remark that the requirement that the formula isolating the type has (at most) the same free variables as the type is avoidable. If $T \models \varphi(\overline{x}, \overline{y}) \to \pi(\overline{x})$ then also $T \models \exists \overline{y} \varphi(\overline{x}, \overline{y}) \to \pi(\overline{x})$. This notion has a nice topological translation in the space S_n^T. The consistency of φ with T becomes $\emptyset \neq [\varphi]$. The fact that $T \models \varphi \to \pi$ translates to $[\varphi] \subseteq [\pi]$. In other words, an n-type π of T is isolated iff $[\pi]$ has nonempty interior in the space S_n^T. As we have seen above this characterization does not depend on the n chosen. Even more, one gets the same characterization in any space $S_{\overline{x}}^T$ (provided that I contains the variables of the type). Taking complements, one gets that π is non-isolated iff $[\pi]^C = S_{\overline{x}}^T \setminus [\pi]$ is dense. In other words, non-isolated types correspond (in the space $S_{\overline{x}}^T$) to complements of open dense sets[1].

Let's return to the question about if there is a model of T that omits a given type (or, more generally, a set of types). If the theory T is complete and the type π is isolated in T, it is realized in any model of T. For, if $\varphi(\overline{x})$ is consistent with T, by completeness $T \models \exists \overline{x} \varphi(\overline{x})$. Hence in any model of T there is a tuple satisfying φ. If φ isolates π, $T \models \varphi(\overline{x}) \to \pi(\overline{x})$, hence any tuple satisfying φ in a model of T realizes π. Hence, in any model od T we can find a realization of π. The following theorem provides a sufficient condition in order to omit a collection of types in a countable language.

Theorem 3.3.3 (Omitting types theorem)**.** *Let T be a consistent theory in a countable language L. Let P be a countable set of non-isolated types, each one in a finite number of variables. There is a model of T that omits all the types in P.*

Proof. We begin by fixing a countable set of variables $\overline{x} = (x_n \mid n \in \mathbb{N})$. We will work in the space $S_{\overline{x}}^T$ of complete types in those variables. For every $\pi \in P$ let us denote by $n(\pi)$ the (finite) number of free variables of π. Given a map $\sigma : \{1, \ldots, n(\pi)\} \to \mathbb{N}$ let p^σ denote the type obtained by replacing the variables of π according to σ. More precisely, replacing the i^{th} variable of π by $x_{\sigma(i)}$. Let us check that π^σ remains non-isolated (in case σ is injective is clear: we are only renaming variables). For, suppose that $\{y_1, \ldots, y_n\}$ are the variables of π and $\varphi(x_{\sigma(1)}, \ldots, x_{\sigma(n))}$ isolates π^σ. Then one easily checks that the formula

$$\varphi(y_1, \ldots, y_n) \wedge \bigwedge_{\substack{\sigma(i)=\sigma(j) \\ 1 \leq i < j \leq n}} y_i = y_j$$

isolates π in T.

[1]this are the so called closed nowhere dense sets

Now for every formula $\varphi(y, \overline{y}) \in L_{\overline{x}}$ let $TV_{\varphi(y,\overline{y})}$ denote the following open set

$$TV_{\varphi(y,\overline{y})} = \bigcup_{n\in\mathbb{N}} [\exists y\varphi(y, \overline{y}) \to \varphi(x_n, \overline{y})]$$

Claim 1. Each set $TV_{\varphi(y,\overline{y})}$ is dense.

Proof. We have to show that for every $\psi \in L_{\overline{x}}$ consistent with T, $[\psi]$ has nonempty intersection with $TV_{\varphi(y,\overline{y})} = [\neg\exists y\varphi(y, \overline{y})] \cup \bigcup_{n\in\mathbb{N}} [\varphi(x_n, \overline{y})]$. If $\psi \wedge \neg\exists y\varphi(y, \overline{y})$ is consistent with T, $[\psi] \cap [\neg\exists y\varphi(y, \overline{y})]$ is nonempty. Otherwise $T \models \psi \to \exists y\varphi(y, \overline{y})$ and thus the formula $\psi \wedge \varphi(x_n, \overline{y})$ is consistent with T, where x_n is a variable not occuring in $\psi \wedge \exists y\varphi(y, \overline{y})$. In this case $[\psi] \cap [\varphi(x_n, \overline{y})]$ is nonempty. This ends the proof of the claim.

Now consider the following set

$$\bigcap_{\varphi(y,\overline{y})\in L_{\overline{x}}} TV_{\varphi(y,\overline{y})} \cap \bigcap_{\pi\in P} \bigcap_{\sigma\in^{n(\pi)}\mathbb{N}} [\pi^\sigma]^C.$$

We already know that each $TV_{\varphi(y,\overline{y})}$ is dense. Each $[\pi^\sigma]^C$ is also dense because the type π^σ is non-isolated. Hence the set above is a countable intersection of open dense sets. By the Baire category theorem it contains a point $p = p(x_n \mid n \in \mathbb{N})$. Let $\overline{a} = (a_n \mid n \in \mathbb{N})$ be a realization of p in some model M of T. We are going to see that $\{a_n \mid n \in \mathbb{N}\}$ is the domain of a model of T that omits all the types of P.

Claim 2. The set $\{a_n \mid n \in \mathbb{N}\}$ is the domain of an elementary substructure of M.

Proof. We use the Tarski-Vaught test, Theorem 1.2.10. Suppose $M \models \exists y\varphi(y, a_{i_1}, \ldots, a_{i_m})$ for some formula $\varphi(y, x_{i_1}, \ldots, x_{i_m}) \in L_{\overline{x}}$. This implies $p \in [\exists y\varphi(y, x_{i_1}, \ldots, x_{i_m})]$. Since p also belongs to $TV_{\varphi(y,x_{i_1},\ldots,x_{i_m})}$ then $p \in [\exists y\varphi(y, x_{i_1}, \ldots, x_{i_m}) \to \varphi(x_n, x_{i_1}, \ldots, x_{i_m})]$ for some $n \in \mathbb{N}$. This implies $p \in [\varphi(x_n, x_{i_1}, \ldots, x_{i_m})]$, hence $M \models \varphi(a_n, a_{i_1}, \ldots, a_{i_m})$. This ends the proof of the claim.

Let us denote by N this substructure. Obviously N is a model of T. The proof ends with the following:

Claim 3. N omits all types in P.

Proof. We show that given any $\pi = \pi(y_1, \ldots, y_n) \in P$, and any $\sigma : \{1, \ldots, n\} \to \mathbb{N}$ the subtuple $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ of $\overline{a}$ does not realize π. Since $p \in [\pi^\sigma]^C = \bigcup_{\varphi\in\pi}[\neg\varphi^\sigma]$, we get $\neg\varphi^\sigma \in p$ for some $\varphi \in \pi$. But this implies $M \models \neg\varphi^\sigma(\overline{a})$. Equivalently $M \models \neg\varphi(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. Hence $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ does no realize π. □

Chapter 4

Countable models

In this chapter the language will be countable.

4.1 atomic models

We start with a remark about isolated types in a theory T. If a complete type $p \in S^T_{\overline{x}}$ is isolated by a consistent formula $\varphi \in L_{\overline{x}}$ then $\varphi \in p$. Otherwise $\neg\varphi \in p$ and thus $T \models \varphi \to \neg\varphi$, contradicting the consistency of φ with T.

Let's now make some more remarks about isolated types in the space $S^M_{\overline{x}}(A)$, where M is any model and $A \subseteq M$ a parameter set. Recall that $S^M_{\overline{x}}(A) = S^{\mathrm{Th}(M_A)}_{\overline{x}}$.

remark 4.1.1. 1. $\pi \subseteq L_{\overline{x}}(A)$ is isolated iff there is some $\varphi(\overline{x}) \in L_{\overline{x}}(A)$ with

$$M_A \models \exists\overline{x}\varphi(\overline{x}) \quad M_A \models \forall\overline{x}\,(\varphi(\overline{x}) \to \pi(\overline{x}))$$

2. If $\pi \subseteq L_{\overline{x}}(A)$ is isolated then it is realized in M_A.

Proof. 1. If $\varphi(\overline{x})$ is consistent with $\mathrm{Th}(M_A)$, since $\mathrm{Th}(M_A)$ is complete it follows $\mathrm{Th}(M_A) \models \exists\overline{x}\varphi(\overline{x})$. Hence $M_A \models \exists\overline{x}\varphi(\overline{x})$. We use $M_A \models \forall\overline{x}\,(\varphi(\overline{x}) \to \pi(\overline{x}))$ to denote $M_A \models \forall\overline{x}\,(\varphi(\overline{x}) \to \mu(\overline{x}))$ for any $\mu \in \pi$. Obviously $M_A \models \forall\overline{x}\,(\varphi(\overline{x}) \to \pi(\overline{x}))$ is equivalent to $\mathrm{Th}(M_A) \models \forall\overline{x}\,(\varphi(\overline{x}) \to \pi(\overline{x}))$.

2. By 1, any tuple of M realizing $\varphi(\overline{x})$ realizes $\pi(\overline{x})$. □

If $f : M \to N$ is a partial elementary map and $A \subseteq \mathrm{dom}(f)$, the map $p \mapsto p^f$ from $S^M_{\overline{x}}(A)$ to $S^N_{\overline{x}}(f(A))$ is a homeomorphism. We already know it is a bijection. But the image of the basic clopen $[\varphi]$ is the basic clopen $[\varphi^f]$. This means this map is open. By the same reason the inverse map, given by $p \mapsto p^{f^{-1}}$ is also open. In particular, $p \in S^M_{\overline{x}}(A)$ is isolated iff p^f is isolated in $S^N_{\overline{x}}(f(A))$. In fact this is true also for partial (not necessarily complete) types.

Definition 4.1.2. Let M be a structure. We say that M is ***atomic*** iff for every finite tuple $\overline{a} \in M$, $\mathrm{tp}^M(\overline{a})$ is isolated.

Lemma 4.1.3. *Let M be a structure and $A \subseteq M$ be a finite set of parameters. If M is atomic then also M_A is atomic.*

Proof. Let $\overline{a}$ be a finite tuple in M. Let $\overline{b}$ be an enumeration of A. Since M is atomic, the type $p(\overline{x}, \overline{y})$ of the tuple $\overline{a}\overline{b}$ is isolated. Let $\varphi(\overline{x}, \overline{y}) \in L_{\overline{x},\overline{y}}$ be a formula that isolates the type: $M \models \exists\overline{x}\exists\overline{y}\varphi(\overline{x}, \overline{y}) \wedge \forall\overline{x}\forall\overline{y}(\varphi(\overline{x}, \overline{y}) \to p(\overline{x}, \overline{y}))$. Obviously $M_A \models \forall\overline{x}(\varphi(\overline{x}, \overline{b}) \to p(\overline{x}, \overline{b}))$. Moreover $\varphi \in p$ implies $M \models \varphi(\overline{a}, \overline{b})$ and thus $M_A \models \exists\overline{x}\varphi(\overline{x}, \overline{b})$. This means that $\mathrm{tp}(\overline{a}/A) = p(\overline{x}, \overline{b})$ is isolated by $\varphi(\overline{x}, \overline{b})$. $\square$

Next result shows that a complete theory has at most a countable atomic model.

Proposition 4.1.4. *Suppose M and N are countable and atomic. Let $f : M \to N$ be a finite partial elementary map. then f extends to an isomorphism from M to N. In particular two elementarily equivalent atomic countable models are isomorphic.*

Proof. Let $(a_n \mid n \in \omega)$ and $(b_n \mid n \in \omega)$ be enumerations of $M \setminus \mathrm{dom}(f)$ and $N \setminus \mathrm{Im}(f)$ respectively. We are going t build a chain of elementary maps $(f_n \mid n \in \omega)$ from M to N with $f_0 = f$, $a_n \in \mathrm{dom}\, f_{n+1}$, $b_n \in \mathrm{Im}\, f_{n+1}$ and f_n finite. The union will be the isomorphism. Assume f_n is already constructed. By Lemma 4.1.3 M is atomic over $\mathrm{dom}(f_n)$, hence $p = \mathrm{tp}(a_n/\mathrm{dom}(f_n))$ is isolated. Thus p^{f_n} is also isolated and thus realized by some $b \in N$. Now $g = f \cup \{(a_n, b)\}$ is elementary. Again by Lemma 4.1.3 N is atomic over $\mathrm{Im}(g)$ and thus we can realize $tp^{g^{-1}}(b_n/\mathrm{Im}(g))$ by some $a \in M$. Finally we put $f_{n+1} = f_n \cup \{(a_n, b), (a, b_n)\}$. $\square$

Exercise 4.1.5. Show that an atomic model is ω-homogeneous.

Next exercise shows that countable atomic models are prime.

Exercise 4.1.6. Let M be countable and atomic. Show that M is prime, i.e. for any $N \equiv M$ there is an elementary embedding from M to N.

Exercise 4.1.7. Prove the converse of the previous exercise. More precisely: show that any prime models is atomic and countable.

4.2 The Ryll-Nardzewski's theorem

Theorem 4.2.1 (Engeler-Ryll-Nardzewski-Svenonious)**.** *Let T be a complete theory in a countable language without finite models. The following are equivalent:*

1. *T is ω-categorical.*
2. *for all n all points in S_n^T are isolated.*
3. *for all n S_n^T is finite.*

4. *Each model of T is atomic.*

Proof. *1* implies *2*. If some p in some S_n^T is non-isolated, by the omitting types Theorem 3.3.3 T has a countable model omitting p and also, by proposition 2.1.12, a countable model realizing p. Since p has no parameters, these two models cannot be isomorphic.

2 implies *3*. The collection of isolated points of S_n^T is an open covering of S_n^T, which must be finite by compactness.

3 implies *4*. Because S_n^T is Haussdorff all points in S_n^T are isolated. Since T is complete, for any model M of T, $S_n^T = S_n^M$ whence M is atomic.

4 implies *1*. All countable models of T are atomic hence isomorphic by proposition 4.1.4. □

Exercise 4.2.2. Let T be a complet theory without finite models. Prove that the following are equivalent:

1. T is ω-categorical.
2. T has a model which is both atomic and saturated.
3. T has a countable model which is both atomic and saturated.
4. T has a model which is both atomic and $\aleph_1$-universal.
5. T has a countable model which is both atomic and $\aleph_1$-universal.
6. T has a model realizing only finitely many n-types for each n.
7. T has a countable model realizing only finitely many n-types for each n.
8. All models of T are ω-saturated.
9. All countable models of T are ω-saturated.
10. All countable models of T are atomic.
11. All models of T realize only finitely many n-types for each n.
12. for all n B_n^T is finite.
13. For any model M of T and any finite $A \subseteq M$ $S_1^M(A)$ is finite.
14. For any model M of T any finite $A \subseteq M$ and any n, $S_n^M(A)$ is finite.

Exercise 4.2.3. Provide some more characterizations of ω-categoricity.

Exercise 4.2.4. Let T be a theory without finite models. Prove that T is ω-categorical iff all its models are partially isomorphic.

Exercise 4.2.5.
1. Show that a complete theory with infinitely many different constants (the language contains infinitely constants and the theory says that the are different) cannot be ω-categorical. Even more holds:

2. Show that any model of an ω-categorical theory is locally finite. That is: each finitely generated substructure is finite. In fact any model of the theory is uniformly locally finite: there is a function $d : \mathbb{N} \to \mathbb{N}$ (depending only on the theory, not the model) such that $|\langle A\rangle_M| \leq d(|A|)$ for any finite subset A of any model M of the theory.

4.3 countable atomic and countable saturated models of a complete theory

In the section we well characterize, in terms of the spaces S_n^T, when a given complete theory has a countable atomic model and when it has a countable saturated model.

We start with the existence of countable saturated models.

remark 4.3.1. Let T be a theory, M a model of T and $A \subseteq M$ a set of parameters.

1. If $\overline{y} = (y_i \mid i \in I)$ a tuple of new variables of the same cardinality as A, $\left|S_{\overline{x}}^M(A)\right| \leq \left|S_{\overline{xy}}^T\right|$.
2. If T is complete $\left|S_{\overline{x}}^T\right| \leq \left|S_{\overline{x}}^M(A)\right|$.

Proposition 4.3.2. *Let T be a complete theory with infinite models. The following are equivalent:*

1. *T has a countable saturated model.*
2. *For all n $\left|S_n^T\right| \leq \aleph_0$.*

Proof. *1* implies *2*. Let M be the countable saturated model of T. Since T is complete, by remark 3.2.2 $S_n^T = S_n^M$. Since M is saturated, any type in S_n^M is realized in M. But the number of possible realizations is countable whence S_n^M is at most countable.

2 implies *1*. For any model M of T and any finite $A \subseteq M$, by remark 4.3.1, $\left|S_1^M(A)\right| \leq \left|S_{|A|+1}^T\right| \leq \aleph_0$. We build a chain $(M_n \mid n \in \omega)$ of countable models of T as follows. Starting with any countable model, we pick, by 2.1.12 M_{n+1} an elementary extension of M_n realizing all 1-types of M_n over a finite set of parameters. By hypothesis there are only countably many such types and we can pick M_{n+1} to be countable. The union of the chain is a countable saturated model of T. □

The following exercise is a generalization of proposition 4.3.2

Exercise 4.3.3. Let T be a complete theory and λ a cardinal with $\lambda^{<\lambda} = \lambda$ (i.e., λ is regular and $2^\mu \leq \lambda$ for any $\mu < \lambda$). Show the following are equivalent:

1. T has a saturated model of cardinality λ.
2. $S_n^T \leq \lambda$ for each n.

3. $S_{\mu}^{T} \leq \lambda$ for each $\mu < \lambda$.

4. $S_1^M(A) \leq \lambda$ for each model M of T and each $A \subseteq M$ with $|A| < \lambda$.

Exercise 4.3.4. Let T be any theory (not necessarily complete).

1. Show the following are equivalent:

   (a) For all n $\left|S_n^T\right| \leq \aleph_0$.

   (b) T has at most countably many completions and each completion of T has a finite model or a countable saturated model.

2. If for all n $\left|S_n^T\right| \leq \aleph_0$, then the number of completions of T coincide with the number of saturated model of T of cardinality at most countable (up to isomorphism).

3. If T has at most countably many countable models and only one of them is saturate then T is complete.

Exercise 4.3.5. Show that if a complete theory has a countable ω_1-universal model then it also has a countable saturated model.

Corollary 4.3.6. *Let T be any theory with infinite models. If T has at most countably many countable models then T has a countable saturated model.*

Proof. Replacing T by any of its completions we may assume T complete. Since by 2.1.12 each type in S_n^T is realized in a countable model of T, S_n^T is at most countable. □

Now we go to the question about countable atomic models.

remark 4.3.7. If N is atomic and $N \preceq N$ then also M is atomic.

Proof. It follows from $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^N(\overline{a})$ for every tuple $\overline{a} \in M$. □

Hence, the existence of an atomic model of a theory implies the existence of a countable atomic model. Next proposition characterize the existence of countable atomic models.

Theorem 4.3.8. *Let T be a complete theory. Then T has an atomic model iff for all n, the set of isolated points is dense in S_n^T.*

Proof. Assume T has a prime model M, and let $\varphi(\overline{x}) \in L_n$ consistent with T. Since T is complete $T \models \exists \overline{x} \varphi(\overline{x})$ and thus there is a tuple $\overline{a}$ in M satisfying $\varphi(\overline{x})$. Since M is atomic, $p = \mathrm{tp}(\overline{a}/\emptyset)$ is isolated. This implies that p is an isolated point in $[\varphi]$.

Conversely, assume that the set of isolated points are dense in every space S_n^T. For each n, consider the following n-type:

$$\pi_n = \left\{\neg\varphi \mid \varphi \in L_n \text{ and } \varphi \text{ isolates a complete type } p \in S_n^T\right\}.$$

Now we show that each π_n is non-isolated in T. Otherwise, assume that for some n there is $\psi \in L_n$ consistent with T such that $T \models \psi \to \pi_n$. Let, by density of isolated points, $p \in [\psi]$ isolated in T. Let $\varphi \in L_n$ a formula isolating p in T. Since $\psi \in p$ it follows that $T \models \varphi \to \psi$. But $\neg\varphi \in \pi_n$ implies $T \models \psi \to \neg\varphi$ and thus $T \models \varphi \to \neg\varphi$, a contradiction with the consistency of φ with T. By the omitting types theorem there is a model M of T omitting all π_n. We end by checking that M is atomic. Given a tuple $\overline{a}$ of M, $\overline{a}$ does not realize π_n, where n is the length of $\overline{a}$. This implies $\overline{a}$ satisfies some formula $\varphi(\overline{x})$ isolating a type $p \in S_n^T$. Hence p is the type of $\overline{a}$, which is isolated. □

We end this section by showing that the existence of a countable saturated model implies the existence of an atomic one.

Theorem 4.3.9. *Let T be any theory without finite models. If T has a countable saturated model then T has an atomic model.*[1]

Proof. Replacing T by an appropriate completions we may assume T complete. Assume T has not atomic model. By theorem 4.3.8 there is some n such that in S_n^T the isolated point are not dense. Let $\varphi \in L_n$ such that $[\varphi]$ does not contain isolated points (i.e. $[\varphi]$ perfect). Let p, q be two different points in $[\varphi]$. Let $\varphi_o \in p$ with $\varphi_o \notin q$ and put $\varphi_1 = \neg\varphi_0$. Observe that $[\varphi]$ is a disjoint union $[\varphi] = [\varphi_0] \cup [\varphi_1]$ where each $[\varphi_i]$ is nonempty and perfect again. We can partition again $[\varphi_i] = [\varphi_{i0}] \cup [\varphi_{i1}]$ with $[\varphi_{ij}]$ nonempty and perfect. Iterating this process we build a binary tree of formulas $(\varphi_s \mid s \in^{<\omega} 2)$ indexed by finite binary sequences. By compactness of S_n^T, for each binary sequence of infinite length $s \in^{\omega} 2$ the intersection $C_s = \bigcap_{n\in\omega}[\varphi_{s\restriction n}]$ is nonempty. here $s \restriction n$ denotes sequence of the first n binary values of s. Since the sets C_s are pairwise disjoint we have that $\left|S_n^T\right| \geq 2^{\omega}$, thus, by proposition 4.3.2 T has not countable saturated model. □

In fact, the prove given above shows that a perfect set in a compact Haussdorff space has size at least 2^{ω}.

4.4 On the number of countable models

Theorem 4.4.1 (Vaught)**.** *There is no complete theory with exactly 2 countable models up to isomorphism.*

Proof. Assume T is a complete theory with exactly 2 countable models. We will end the proof exhibiting a third countable model of T, a contradiction. By corollary 4.3.6 T has a countable saturated model M_2 and by theorem 4.3.9 a prime model M_1. By Ryll-Nardzewski for some n S_n^T contains a non-isolated type p. Since M_2 realizes p and M_1 omits p (realizes only isolated types) they cannot be isomorphic. Let $\overline{a} \in M_2$ be a tuple realizing p. We now consider the

[1] we do not claim the prime model of T is elementarily embeddable in any model of T, only to those models of T elementarily equivalent to the prime model.

theory of M_A, where A denotes the set .... As $\left|S_m^{T(\overline{a})}\right| \leq \left|S_{m+|\overline{a}|}^{T}\right| \leq \aleph_0$, by proposition 4.3.2 and theorem 4.3.9 there is a prime model $(M_3)_{\overline{b}}$ of $T(\overline{a})$. As $\mathrm{tp}^{M_3}(\overline{b}) = p$, M_3 is not atomic, hence not isomorphic to M_1. Since $\aleph_0 \leq \left|S_n^T\right| \leq \left|S_n^{T(\overline{a})}\right|$, there should be some $q \in S_n^{T(\overline{a})}$ non-isolated. Obviously $(M_3)_{\overline{b}}$ does not realize q hence M_3 is not saturated and cannot be isomorphic to M_2. M_3 is our third countable model of T. $\square$

Exercise 4.4.2. Let L be a language containing countably many unary predicates, $L = (P_i \mid i \in \mathbb{N})$. For each each pair A, B of finite subsets of $\mathbb{N}$ consider the formula $\varphi_{A,B} := \exists x \left(\bigwedge_{i\in A} P_i(x) \wedge \bigwedge_{i\in B} \neg P_i(x)\right)$. Let T be the following theory $\{\varphi_{A,B} \mid A, B \text{ finite and disjoint subsets of } \mathbb{N}\}$. Show the following:

1. T is consistent and has no finite models.
2. In an ω-saturated model M of T the following holds: For each subset A of $\mathbb{N}$ the set $\bigcap_{i\in A} P_i^M \cap \bigcap_{i\in\mathbb{N}\setminus A} M \setminus P_i^M$ is infinite.
3. T is complete and eliminates quantifiers.
4. The converse of 2. holds.
5. T has no prime model nor countable saturated model nor countable ω_1-universal model.
6. T has $2^{\aleph_0}$ countable models.
7. for each n, S_n^T is perfect (i.e., has no isolated points. Hint: show that an n-type π is isolated then only finitely may predicates occur in π).
8. $I(T, \aleph_\alpha) = \begin{cases} 2^{\aleph_\alpha}, & \text{if } \aleph_\alpha \leq 2^{\aleph_0}; \\ |\alpha|^{2^{\aleph_0}}, & \text{if } \aleph_\alpha > 2^{\aleph_0}. \end{cases}$

Exercise 4.4.3. Modify previous exercise to obtain a theory with prime model but no countable saturated model (Hint: consider $T(A)$ for a countable set A).

Exercise 4.4.4. Let T be a complete theory in a countable language which is not ω-categorical. Let $\pi_1, \ldots, \pi_n$ be a finite collection of types of T each one in a finite number of variables. Show that T has a countable non-saturated model realizing all the types $\pi_1, \ldots, \pi_n$.

Exercise 4.4.5. Let L be a language containing countably many unary predicates $\{P_i \mid i \in \mathbb{N}\}$ and countably many constants $\left\{a_{i,j} \mid (i,j) \in \mathbb{N}^2\right\}$. Let T be the theory which says $a_{i,j} \neq a_{i,k}$ for any i and any pair $j < k$, $\forall x(\neg P_i(x) \vee \neg P_j(x))$ for any pair $i < j$ and $P_i(a_{i,j})$ for any pair i, j. Show the following:

1. In any ω-saturated model M of T the following sets are infinite: $M \setminus \bigcup_{i\in\mathbb{N}} P_i^M$ and $P_i^M \setminus \left\{a_{i,j}^M \mid j \in \mathbb{N}\right\}$ for each i.
2. T is complete and eliminates quantifiers.
3. The converse of 1. holds.

4. T has a prime model and a countable saturated model. Describe them.

5. T has $2^{\aleph_0}$ countable models.

6. What are the isolated types in S^T_n?

7. $I(\aleph_\alpha, T) = (2 + |\alpha|)^{\aleph_0}$.

Exercise 4.4.6. Proceed as in previous exercise with the following theory. The language contains countably many unary predicates: $L = \{P_s \mid s \in^{<\omega} 2\}$. T says $P_\emptyset = M$, P^M_s is nonempty and P^M_s is the disjoint union of P^M_{s0} and P^M_{s1}. You must show S^T_n is perfect.

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exemple 3,4,..models

4.5 restes

An ***atom*** of a boolean algebra B is a minimal element $a \neq 0$. Equivalently, a is an atom iff for every $b \in B$ either $a \leq b$ or $a \leq \neg b$. We are interested in the following boolean algebra, called the Tarski-Lindenbaum algebra. Consider L_n modulo the following equivalence relation: $\varphi \sim \psi$ iff $T \models \varphi \leftrightarrow \psi$. It is easily checked that $\sim$ is compatible with $\wedge, \vee$ and $\neg$. Hence $L_n/\sim$ is naturally a boolean algebra denoted by B^T_n. Observe that in B^T_n the following holds:

remark 4.5.1. Let φ, ψ be L_n-formulas and let $\overline{\varphi}, \overline{\psi}$ denote its classes in B^T_n.

1. $\overline{\varphi} \leq \overline{\psi}$ iff $T \models \varphi \to \psi$.

2. $\overline{\varphi} = 0$ iff φ is inconsistent with T.

3. The following are equivalent:

   (a) $\overline{\varphi}$ is an atom of B^T_n

   (b) φ consistent with T and for every L_n-formula ψ, either $T \models \varphi \to \psi$ or $T \models \varphi \to \neg\psi$

   (c) φ is consistent with T and isolates a complete type in S^T_n.

Proof. We prove *3.* The equivalence between (a) and (b) follows applying *1* and *2.* If (b) holds then φ is a formula consistent with T isolating $\{\psi \in L_n \mid T \models \varphi \to \psi\}$, a complete type in S^T_n. The converse is obvious. □

In order to make everything easier we will identify a formula with its class. This is harmless, since we are going the work only with models of T.

An atom φ of B^T_n isolated a complete type $p \in S^T_n$, i.e., $\{p\} = [\varphi]$. Conversely, if $p \in S^T_n$ is an isolated point, then $\{p\} = [\varphi]$ for some formula $\varphi \in L_n$. Since p is complete, φ is an atom. Hence, atoms of B^T_n are in correspondence with isolated points of S^T_n.

Let T be a theory (maybe not complete), M a model of T and $A \subseteq M$ a set of parameters. We will denote by $T(A)$ the theory of M_A. $T(A)$ is a complete theory in $L(A)$ extending T. In fact any complete theory in $L(A)$ extending T is of this kind. Recall (exercise 2.3.4) that $(N, ha)_{a\in A}$ is a model of $T(A)$ iff the partial map (with domain A) $h : M \to N$ is elementary. Hence, a model of $T(A)$ consist on a structure N together with an elementary partial map $h : M \to N$ with domain A. Observe that, by remark 3.2.2 (or 2.1.3), $S_n^{T(A)} = S_n^M(A)$. The atoms of $B_n^{T(A)}$ will be called simply atoms of $T(A)$.

Definition 4.5.2. Let M be a model and $A \subseteq M$ a set of parameters. We say that M is prime over A if every partial elementary map $f : M \to N$ with $A = \mathrm{dom}(f)$ extends to an elementary embedding from M to N. We say M ***is prime*** if M is prime over the empty set, i.e., there is an elementary embedding from M to any N elementarily equivalent to M.

Theorem 4.5.3. *Let M be a model and $A \subseteq M$ a countable set of parameters. Then M is prime over A iff M is atomic over A and countable. In particular M is prime iff M is atomic and countable.*

Proof. If M is prime over A, it is countable, since we can embed M into a countable model of $T(A)$. If M is not atomic over A, let $\overline{a} \in M$ such that $\mathrm{tp}(a/A)$ is non-isolated. By the omitting types theorem, let $(N, ha)_{a\in A}$ be a model of $T(A)$ omitting $\mathrm{tp}(a/A)$. This means that N omits $\mathrm{tp}^h(a/A)$. Since M is prime over A, h may be extended to an elementary embedding $f : M \to N$. But then $f(\overline{a})$ realizes $\mathrm{tp}^h(\overline{a}/A)$, a contradiction.

Assume now that M is countable and atomic over A and there is an elementary map $f : M \to N$ with domain A. Let $(a_n \mid n \geq 1)$ be an enumeration of $M \setminus A$. We are going to build a chain $(f_n \mid n \in \omega)$ of elementary maps from M to N with $\mathrm{dom}(f_n) = A \cup \{a_1, \dots, a_n\}$. Obviously $\bigcup_{n\in\omega} f_n$ will be an elementary embedding from M to N. We start with $f_0 = f$. Let us see how to build f_{n+1} from f_n. Since M is atomic over A, by lemma 4.1.3, M is atomic over $\mathrm{dom}(f_n)$. Hence $\mathrm{tp}(a_{n+1}/\mathrm{dom}(f_n))$ is isolated. Obviously $\mathrm{tp}^{f_n}(a_{n+1}/\mathrm{dom}(f_n))$ is also isolated and thus realized by b in N. Now we set $f_{n+1} = f_n \cup \{(a_{n+1}, b)\}$. $\square$

A Boolean algebra is atomic iff for every $0 \neq b \in B$ there is some atom a with $a \leq b$. Assume B_n^T is atomic. Let $\varphi \in L_n$ consistent with T. Let $\psi \in L_n$ an atom of B_n^T with $T \models \psi \to \varphi$. Let p the type isolated by ψ. Then $p \in [\varphi]$. This shows that the isolated points of S_n^T are dense. Conversely, assume the isolated points of S_n^T are dense. Let $\varphi \in L_n$ consistent with T and let $p \in [\varphi]$ an isolated point. let ψ be the formula that isolates p then ψ is an atom above φ. This shows that B_n^T is atomic. The same argument shows that a boolean algebra is atomic iff in its stone space the isolated points are dense.

The following theorem gives a criteria for existence of prime models of T.

Exercise 4.5.4. Prove that if $\varphi(x, \overline{y})$ is an atom of T (more precisely, $\overline{\varphi}$ is an atom in B_{n+1}^T, where n denotes the lenght of $\overline{y}$) then also $\exists x\varphi$ is an atom of T (more precisely, $\overline{\exists x\varphi}$ is an atom in B_n^T).

Chapter 5

Partial isomorphism

5.1 Partial isomorphism

In this chapter we will presuppose a fixed language L and when we speak about structures or models we mean L-structures.

Definition 5.1.1. A partial isomorphism f from M to N is a bijection between subsets of M and N respectively satisfying the following conditions:

1. For every predicate symbol R and every tuple $\overline{a}$ of the domain of f it holds that $\overline{a} \in R^M$ iff $f\overline{a} \in R^N$.
2. For every function symbol f, every tuple $\overline{a}$ and element a of the domain of f it holds that $f^M(\overline{a}) = a$ iff $f^N(f\overline{a}) = f(a)$
3. For every constant symbol c and every element a of the domain of f it holds that $c^M = a$ iff $c^M = f(a)$.

Equivalently, we could have described partial isomorphism as partial maps preserving, in the sense of 1.2.3, the formulas of type

$$x = y, \quad R(x_1, \ldots, x_n), \quad x = f(x_1, \ldots, x_n), \quad x = c.$$

Observe that the inverse of a partial isomorphism is also a partial isomorphism. For example, every part(subset) of an isomorphism between substructures of M and N is a partial isomorphism. It is not true, however, that every partial isomorphism from M to N extends to an isomorphism f between substructures of M and N (an embedding of $dom(f)$ to N). If we take $M = (\mathbb{Z}, s)$ the map $\{(1, 4), (3, 7)\}$ is a partial isomorphism from M to M which can not be extended by adding 2 in the domain. It is not possible, therefore, extend this map to an isomorphism between substructures of M.

Exercise 5.1.2. 1. If $f : M \to N$ is a partial isomorphism, $\mathrm{dom}(f)$ is a substructure of M iff $\mathrm{Im}(f)$ is a substructure of N.

2. If f is a partial isomorphism between substructures of M and N then f is an embedding from $\mathrm{dom}(f)$ to N.

Thus, when the domain of a partial isomorphism $f : M \to N$ is a substructure of M then f is an embedding of $\mathrm{dom}(f)$ in N. In particular, when the language is relational, a partial isomorphism is the same as an isomorphism between substructures.

Definition 5.1.3. A system of partial isomorphism from M to N is a nonempty collection I of partial isomorphisms from M to N satisfying the following properties:

Back For every $f \in I$ and $b \in N$ there is a $g \in I$ that extends f and such that b is in the image of g.

Forth For every $f \in I$ and $a \in M$ there is a $g \in I$ that extends f and such that a is in the domain of g.

When there is a system of partial isomorphism I from M to N it is said that M and N are partially isomorphic (via I) and this is denoted by $M \cong_p N$ ($I : M \cong_p N$ respectively).

Example 5.1.4.

1. Let M, N be two dense linear orders without endpoints. The collection of isomorphisms between finite substructures of M and N is a system of partial isomorphism.

2. If f is an isomorphism from M to N then $\{f\}$, is a system of partial isomorphism.

3. If $I : M \cong_p N$ is a system of partial isomorphisms then

$$\{f \mid f \text{ is a partial map and } f \subseteq g \text{ for some } g \in I\},$$

$$\{f \mid f \text{ is a finite partial map and } f \subseteq g \text{ for some } g \in I\}$$

are also systems of partial isomorphisms from M to N. Hence there is always a system of finite partial isomorphisms between partially isomorphic structures.

By the above examples two isomorphic structures are partially isomorphic. If the structures are countable the converse holds:

Proposition 5.1.5. *any two countable and partially isomorphic structures are isomorphic.*

Proof. Let $(a_i \mid\in \omega)$ and $(b_i \mid\in \omega)$ be enumerations of M and N and let I be a system of partial isomorphism from M to N. We will Construct a chain of partial isomorphisms $(f_i \mid i \in \omega)$ such that a_i is in the domain of f_{i+1} and b_i in the image of f_{i+1}. Then the union of the f_i is an isomorphism from M to N. Take as f_0 any element of I. Supposing f_i constructed. By forth there is $g \in I$ with $a \in \mathrm{dom}(g)$ such that extends f. By back, there is f_{i+1} that extends g and such that $b \in \mathrm{Im}(f_{i+1})$. $\square$

If we apply this proposition to the example 5.1.4 we obtain that there is a unique countable and dense linear order without endpoints.

Exercise 5.1.6.

1. Prove that any two atomless Boolean algebras are partially isomorphic via the set of isomorphisms between finite subalgebras.

2. The Random Graph. The language contains only a symbol of binary relation R. A graph is a symmetrical and antireflexive relation. The axioms say that given two finite sets A_1, A_2 of vertices (elements of the domain) there is another vertex related with all the vertices of A_1 and none of A_2. Prove that two models any of this theory are partially isomorphic via the set of isomorphisms between finite substructures.

3. The Random structure in a finite relational language. Let L be relational and finite. For any finite tuple of variables $\overline{x}$ let $\Delta(\overline{x}, y)$ denote the set of formulas of type $R(\overline{z})$ where $\overline{z}$ is a subtuple of $\overline{x}$ (any variable of $\overline{z}$ is in $\overline{x}$, we allow repetitions) and y occurs in $\overline{z}$. For any subset Σ of $\Delta(\overline{x}, y)$ let φ_Σ denote the formula

$$\forall x \left(\text{`}\overline{x} \text{ different'} \rightarrow \exists y \left( \text{`}y \notin \overline{x}\text{'} \wedge \bigwedge_{\psi \in \Sigma} \psi \wedge \bigwedge_{\psi \in \Delta(\overline{x}, y) \setminus \Sigma} \neg \psi \right) \right).$$

   Let $T_{RAND}(L)$ be the theory (of the random structure) axiomatized by all the sentences φ_Σ where Σ is any subset of some $\Delta(\overline{x}, y)$. Any two models M, N of the theory of the random structure are partially isomorphic via $I_f(M, N)$ the set of all finite partial isomorphisms from M to N.

4. Let L be the language containing a single binary relation $\leq$ and unary predicates $P_1, \ldots, P_n$. Let T the theory expressing that $\leq$ is a linear order without endpoints, the P_i's form a partition of the domain, and each P_i is dense in the domain (between any two points there is some point of P_i). Any two models M, N of this theory are partially isomorphic via $I_f(M, N)$ the set of all finite partial isomorphisms from M to N.

remark 5.1.7. If all the models of a countable theory are partially isomorphic, by 5.1.5 then the theory is ω-categorical and therefore complete. We will see later on that it also holds the converse one. All the examples of the former exercise are theories ω-categorical.

5.2 Karp's Theorem

The logic $L_{\infty\omega}$ is the extension of the logic of first order accepting arbitrarily great conjunctions and disjunctions. More precisely, the formulas are constructed from the atomic ones and closing under negation, arbitrary conjunction and quantifying a single variable. Unlike the logic of first order, the formulas of this logic always constitute a proper class. We will call also sentences (or closed formulas) the formulas without free variables in this language. We are going to see that the partial isomorphism characterizes the elementary equivalence in the logic $L_{\infty\omega}$.

Theorem 5.2.1 (Karp)**.** *Two structures are elementarily equivalents in the logic $L_{\infty\omega}$ (that is, they satisfy the same sentences of $L_{\infty\omega}$) iff they are partially isomorphic.*

Proof. Suppose that I is a system of partial isomorphism between M and N. We have to see that M and N are $L_{\infty,\omega}$-equivalent. We will make the proof in two steps.

Claim 1. Given any a term $t = t(\overline{x})$, any $f \in I$ and any finite tuple $\overline{a}$ of elements of $\mathrm{dom}(f)$, there is $g \in I$, $f \subseteq g$ such that $t^M(\overline{a}) \in \mathrm{dom}(g)$ and $g(t^M(\overline{a})) = t^N(g(\overline{a}))$.

Proof. By induction on t. The case where t is a variable is obvious. If $t = f(t_1, \ldots, t_n)$, by applying the hypothesis of induction n times we obtain $h \in I$, an extension of f, such that $\mathrm{dom}(h)$ contains $t_i^M(\overline{a})$ and $h(t_i^M(\overline{a})) = t_i^N(h(\overline{a}))$ for $i = 1 \ldots n$. Now we take $g \in I$ an extension of h with $t^M(\overline{a}) \in \mathrm{dom}(g)$. Now g is a partial isomorphism and its domain contains $t^M(\overline{a})$ and all $t_i^M(\overline{a})$. Since $t^M(\overline{a}) = f^M(t_1^M(\overline{a}), \ldots, t_n^M(\overline{a}))$ we have that

$$g(t^M(\overline{a})) = f^N(g(t_1^M(\overline{a})), \ldots, g(t_n^M(\overline{a}))) =$$

and as that $h(t_i^M(\overline{a})) = t_i^N(h(\overline{a}))$,

$$= f^N(t_1^N(g(\overline{a})), \ldots, t_n^N(g(\overline{a}))) = t^N(g(\overline{a})).$$

Now we will prove that all the maps of I preserve all the formulas $L_{\infty,\omega}$:

Claim 2. Given any formula $\varphi = \varphi(\overline{x})$ of $L_{\infty,\omega}$, any $f \in I$ and any tuple $\overline{a}$ of elements of $\mathrm{dom}(f)$

$$M \models \varphi(\overline{a}) \text{ iff } N \models \varphi(f\overline{a}).$$

Proof. By induction on the formula φ. If the formula is $t_1 = t_2$, by claim 1, we take $g \in I$ an extension of f such that $t_i^M(\overline{a}) \in \mathrm{dom}(g)$ and $g(t_i^M(\overline{a})) = t_i^N(g(\overline{a}))$ for $i = 1, 2$. Then, as g is injective $t_1^M(\overline{a}) = t_2^M(\overline{a})$ iff $t_1^N(g(\overline{a})) = t_2^M(g(\overline{a}))$. The case where the formula is $R(t_1, \ldots, t_n)$ is made analogously, applying the claim 1 n times in order to achieve an extension g of f with $t_i^M(\overline{a}) \in \mathrm{dom}\, g$ and $g(t_i^M(\overline{a})) = t_i^N(g(\overline{a}))$ for $i = 1 \ldots n$. Then g preserves $R(t_1, \ldots, t_n)$ since it is a partial isomorphism.

The cases in which the formula is the negation of another formula or the conjunction of a set of formulas come out without any problem applying the hypothesis of induction. Finally we treat the case in which the formula is $\exists x \varphi(x, \overline{x})$. If $M \models \exists x \varphi(x, \overline{a})$ then there is an element $a \in M$ such that $M \models \varphi(a, \overline{a})$. We take $g \in I$ an extension of f with $a \in \mathrm{dom}\, g$. By hypothesis of induction $N \models \varphi(g(a), g(\overline{a}))$ and therefore $N \models \exists x \varphi(x, g(\overline{a}))$. The converse is done analogously using Back instead of Forth.

Now we will prove the converse one. We suppose that M and N satisfy the same closed formulas of the logic $L_{\infty,\omega}$. We will prove that

$$I = \{f : M \to N \mid \ f \text{ is partial, finite and preserves all the formulas of } L_{\infty,\omega}\}$$

is a system of partial isomorphism. By hypothesis the empty map is in I. In order to prove Forth, let $f \in I$ and $a \in M$. Let $\overline{a}$ an enumeration of the domain of f. Everything is reduced to find $b \in M$ such that the tuples $a\overline{a}$ and $bf\overline{a}$ satisfy the same formulas. If there was not certain b we would have that for each $b \in N$ there is a formula $\varphi_b(x, \overline{x})$ of $L_{\infty,\omega}$ such that $M \models \varphi_b(a, \overline{a})$ but $N \models \neg\varphi_b(b, f\overline{a})$. This is an absurd, since then $M \models \exists x \bigwedge_{b \in N} \varphi_b(x, \overline{a})$ and $N \models \neg\exists x \bigwedge_{b \in N} \varphi_b(x, f\overline{a})$, contradicting the fact that f preserves all the formulas of $L_{\infty,\omega}$. Back is made analogously. □

Corollary 5.2.2. *Two partially isomorphic models are elementarily equivalent. Moreover, every partial isomorphism living on a system of partial isomorphism is an elementary map.*

Exercise 5.2.3. Let M, N be given partially isomorphic structures. Prove that:

1. If M is ω-saturated then N is also ω-saturated.
2. If M is ω-homogeneous then N is also ω-homogeneous.
3. M and N realize the same types in a finite number of variables over the empty set.

Exercise 5.2.4. Let M, N be given ω-homogeneous structures. Prove that M and N are partially isomorphic iff they realize the same types in a finite number of variables over the empty set.

5.3 Partial isomorphism and completeness

In this section we will provide proposition 5.3.2, a quite useful criterion to prove that a given theory is complete.

We begin by seeing that for ω-saturated models, 'being elementarily equivalents and being partially isomorphic is the same thing.

Proposition 5.3.1. *Two ω-saturated and elementarily equivalents models are partially isomorphic via the set of all finite partial elementary maps.*

Proof. Suppose that M, N are ω-saturated and elementarily equivalent. Let I denote set of all partial and finite elementary maps from M to N. I is non empty since then the empty map is in I. Everything is reduced to see that I has the Forth property(Back is done the same way). Let f be a finite partial elementary map and a an element of M. By lemma 2.3.6 everything is reduced to take, because of saturation, a realization of $\mathrm{tp}^f(a/\operatorname{dom} f)$. □

Corollary 5.3.2. *A theory T is complete iff all its ω-saturated models are partially isomorphic.*

Proof. The implication from left to right follows by applying the proposition 5.3.1. Conversely if M and N are models of T, by the proposition 2.2.4 we can take M' and N' elementary and ω-saturated extensions of M and N respectively. As $M' \cong_p N'$ then $M \equiv M' \equiv N' \equiv N$. □

Example 5.3.3. 1. The theory T of a total order with previous and next element. A $\mathbb{Z}$-chain of a model of T consists of an element and all its predecessors and all its successors. A $\mathbb{Z}$-chain is isomorphic to $\mathbb{Z}$. Let easy to see that in an ω-saturated model of T the $\mathbb{Z}$-chains are dense and without endpoints. Then, given two ω-saturated models of T the set of isomorphisms between a finite number of $\mathbb{Z}$-chains of M and of N is a set of partial isomorphism. Then this theory is complete and provides an axiomatization of $\operatorname{Th}\mathbb{Z}, <$.

2. The theory of a countable amount of independent relations. The language L consists of a countable amount of 1-ari predicates $\{P_i \mid i \in \omega\}$. The axioms consist of the formula $\exists x \left(\bigwedge_{i\in A} P_i(x) \bigwedge_{i\in B} \wedge\neg P_i(x)\right)$ for each couple A, B of finite and disjoint subset of ω.

Exercise 5.3.4. 1. Let M, N be given $(|L| + \aleph_0)^+$-saturated with $M \equiv N$. Prove that there is a system of partial isomorphisms constituted by isomorphism between substructures of M and N respectively (Hint: consider all the isomorphisms between elementary substructures of M and N respectively, of cardinal at most $|L| + \aleph_0$).

2. Prove that T is complete iff given M, N, two $(|L|+\aleph_0)^+$-saturated models of T, there is a system of partial isomorphism constituted by isomorphism between substructures of M and N respectively.

3. State and prove a generalization of point 1 replacing $|L| + \aleph_0$ by any given cardinal λ with $\lambda \geq |L| + \aleph_0$.

5.4 Elimination of quantifiers

In the literature there is some confusion of what exactly means 'Elimination of Quantifiers'. In a rather informal way it is said that a theory T 'Elimination of Quantifiers' (or has Quantifier Elimination, Q.E.) if any formula is equivalent, modulo T, to a quantifier-free formula. This is may be stated more precisely the following way:

Definition 5.4.1 (Q.E. first definition)**.** A theory T has ***Quantifier Elimination*** if for every formula φ, there is a formula ψ without quantifiers such that $T \models \varphi \leftrightarrow \psi$ [1].

Observe that we do not claim anything about the free variables of the formulas φ and ψ. A priori it may occur that ψ has more free variables that φ.

Sometimes Q.E. is stated asking that ψ has the same free variables as φ:

[1] for those who do not like free variables beside the symbol $\models$: this is equivalent to $T \models \forall\overline{x}(\varphi \leftrightarrow \psi)$, where $\overline{x}$ is the set of all free variables in the formula $\varphi \leftrightarrow \psi$

Definition 5.4.2 (Q.E. second definition)**.** A theory T has ***Quantifier Elimination*** if for every formula $\varphi = \varphi(\overline{x})$, where $\overline{x}$ is a nonempty tuple of variables, there is a formula $\psi = \psi(\overline{x})$ without quantifiers such that $T \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$ [2].

Observe that in this second definition we exclude the empty tuple $\overline{x}$. Of course any sentence φ has at most one free variable, hence by the second definition, it is equivalent to a quantifier-free formula $\psi(x)$ with (at most) one free variable x. But we cannot conclude that φ is equivalent to a quantifier-free sentence! For instance, if there are not constants in the language there are no quantifier-free sentences! But in fact this is the only obstruction as point *2* of next proposition shows.

Proposition 5.4.3.

1. *The above definitions of Q.E. are equivalent.*
2. *If T has Q.E. and the language has a constant, then any sentence is equivalent modulo T to a quantifier-free sentence.*

Proof. We start by proving that the first definition implies the second. Let $\varphi(x_1, \dots, x_n)$ be any formula with $x_1, \dots, x_n$ a non-empty (i.e. $n > 0$) tuple of variables. By hypothesis there is some quantifier-free formula $\psi(x_1, \dots, x_n, y_1, \dots, y_m)$ equivalent to φ mod T (more precisely $T \models \varphi \leftrightarrow \psi$). It is easily seen that then $\varphi(x_1, \dots, x_n)$ is equivalent mod T to the formula $\psi(x_1, \dots, x_n, x_1, \dots, x_1)$.

Conversely, it only remains to cover the case where φ is a sentence. But then we can apply the second definition to the formula $\varphi \wedge x = x$ (equivalent to φ mod T).

For the proof of *2*, let φ be any sentence and let $\psi(y_1, \dots, y_m)$ be a quantifier-free formula equivalent to φ modulo T. Then it is easily seen that φ is also equivalent mod T to $\psi(c, \dots, c)$, were c is any constant of the language. $\square$

The next exercise shows that in order to eliminates quantifiers it is enough to eliminate a single quantifier:

Exercise 5.4.4. Let T be a theory. Show that the following are equivalent:

1. T has Q.E.
2. Each existential formula with a single existential quantifier (i.e., of type $\exists x\psi$ with ψ quantifier-free) is equivalent, modulo T, to quantifier-free formula.
3. For every formula $\exists y\varphi(\overline{x}, y)$, where $\overline{x}$ is a nonempty tuple of variables, and $\varphi(\overline{x}, y)$ is quantifier-free, there is a formula $\psi(\overline{x})$ without quantifiers such that $T \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$.

Definition 5.4.5. If $\overline{a}$ is a tuple of elements of M, the ***atomic type of*** $\overline{a}$ (over the empty set), that we will denote by $\mathrm{atp}^M(\overline{a})$ (or $\mathrm{atp}^M(\overline{a}/\emptyset)$) is the set of atomic or negations of atomic formulas that $\overline{a}$ satisfies:

$$\mathrm{atp}(\overline{a}) = \{\varphi(\overline{x}) \in L \mid \varphi \text{ is atomic or negation of atomic and } M \models \varphi(\overline{a})\}.$$

[2] this is equivalent to $T \models \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$

The following criterion will be use later several times.

Proposition 5.4.6. *A theory T eliminates quantifiers iff given M, N models of T and any non empty finite tuples $\overline{a}, \overline{b}$ of M and N respectively such that* $\mathrm{atp}^M(\overline{a}) = \mathrm{atp}^N(\overline{b})$ *then also* $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^N(\overline{b})$*.*

Proof. The implication from left to right is easy. We concentrate in the converse one. Given $\varphi(\overline{x})$, where $\overline{x}$ is a non empty tuple, we consider

$$\Psi(\overline{x}) = \{\psi(\overline{x}) \mid \psi(\overline{x}) \text{ does not have quantifiers and } T \models \varphi(\overline{x}) \to \psi(\overline{x})\}\,.$$

By compactness everything is reduced to prove that $T \cup \Psi(\overline{x}) \models \varphi(\overline{x})$. Let M be a model of T and $\overline{a}$ be a tuple of M that realizes $\Psi(\overline{x})$. We have to show that $M \models \varphi(\overline{a})$.

We claim that $T \cup \mathrm{atp}^M(\overline{a}) \cup \{\varphi(\overline{x})\}$ is consistent (we assume that the type $\mathrm{atp}^M(\overline{a})$ is written using $\overline{x}$ as free variables). Otherwise we would have that $T \cup \{\varphi(\overline{x})\} \models \neg \bigwedge_{i=1}^n \psi_i(\overline{x})$ for certain $\psi_i(\overline{x}) \in \mathrm{atp}^M(\overline{a})$. This would imply that $\neg \bigwedge_{i=1}^n \psi_i(\overline{x}) \in \Psi(\overline{x})$ and therefore $M \models \neg \bigwedge_{i=1}^n \psi_i(\overline{a})$. This is absurd, since each $\psi_i(\overline{x})$ is satisfied by $\overline{a}$.

As $T \cup \mathrm{atp}^M(\overline{a}) \cup \{\varphi(\overline{x})\}$ is consistent, there is a model N of T and a tuple $\overline{b}$ of M that realizes $\mathrm{atp}^M(\overline{a}) \cup \{\varphi(\overline{x})\}$. Then $\overline{a}$ and $\overline{b}$ have the same atomic type (over empty) and by hypothesis they also have the same type (complete and over the empty set). Then $M \models \varphi(\overline{a})$, since $N \models \varphi(\overline{b})$. □

Proposition 5.4.6 has a useful straightforward generalization:

Exercise 5.4.7. Let $\overline{x}$ be a fixed nonempty finite tuple of variables and let Δ be a nonempty subset of $L_{\overline{x}}$. The Δ-type of a tuple $\overline{a}$ of a model M is defined by $\mathrm{tp}_\Delta(\overline{a}) = \{\varphi(\overline{x}) \mid \varphi \in \Delta \text{ and } M \models \varphi(\overline{a})\}$ Prove that they are equivalent:

1. For every given formula $\varphi(\overline{x})$ (where $\overline{x}$ is a non empty tuple of variables) there exists a boolean combination $\psi(\overline{x})$ of Δ-formulas such that $T \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$

2. Given a pair M, N of models of T and tuples $\overline{a}, \overline{b}$ of M and N respectively such that $\mathrm{tp}^M_\Delta(\overline{a}) = \mathrm{tp}^N_\Delta(\overline{b})$ then also $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^N(\overline{b})$.

The following exercise shows that Q.E. may be easily achieved by adding, for each formula, a new relation and its definition.

Exercise 5.4.8. Let T be a theory in the language L. Consider the following expansion of the language: for each L-formula $\varphi(\overline{x})$ add a new relation R_φ, whose arity is the length of $\overline{x}$. Consider the theory T' obtained by adding the axioms $\forall \overline{x}(\varphi(\overline{x}) \leftrightarrow R(\overline{x}))$ (for each L-formula $\varphi(\overline{x})$) to T. Prove that

1. Each model of T has a unique expansion to a model of T'.

2. If φ is an L-sentence and $T' \models \varphi$ then $T \models \varphi$.

3. for every $\varphi(\overline{x}) \in L'$ there is $\psi(\overline{x}) \in L$ such that $T' \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$

4. T' eliminates quantifiers.

This exercise shows that every theory eliminates quantifiers in an appropriate definitional expansion. A definitional expansion of T is a theory T' in an expansion L' of the language L satisfying *1*, *2* and *3*.

5.5 Partial isomorphism and Q.E.

Now we will give a modification of proposition 5.3.2 that is useful to prove elimination of quantifiers.

Let M be a structure. The substructure of M generated by a subset A of M is the smallest substructure of M containing A. We will denote it by $\langle A\rangle_M$. A substructure of M is called finitely generated if it is generated by a finite subset, i.e., it is of the form $\langle A\rangle_M$ for some finite subset A of M. Observe that if $\overline{a}$ is a (possibly infinite) tuple of elements of M, $\langle\overline{a}\rangle_M$ has as domain $\{t(\overline{a}) \mid t(\overline{x}) \text{ term of } L\}$.

Lemma 5.5.1. *Let M, N be structures and let $\overline{a}$ and $\overline{b}$ be (possibly infinite) tuples in M and N respectively. Then the following are equivalent:*

1. *$\overline{a}$ and $\overline{b}$ have the same atomic type.*

2. *there is an isomorphism from $\langle\overline{a}\rangle_M$ to $\langle\overline{b}\rangle_N$ sending $\overline{a}$ to $\overline{b}$.*

Proof. *1* implies *2*. Under the assumption $\mathrm{atp}^M(\overline{a}) = \mathrm{atp}^N(\overline{b})$ the map that sends $t(\overline{a}) \mapsto t(\overline{b})$ for every term t is well defined and an isomorphism between the substructures generated by $\overline{a}$ and $\overline{b}$. This fact may be easily checked by the reader.

Conversely, assume there is an isomorphism from $\langle\overline{a}\rangle_M$ to $\langle\overline{b}\rangle_N$ sending $\overline{a}$ to $\overline{b}$. Then, by the homomorphism's theorem (exercise **??**) $\overline{a}$ and $\overline{b}$ have the same atomic type. □

There are two quite interesting candidates to be a system of partial isomorphism. The first one is the set of all isomorphisms between finitely generated substructures of M and N respectively, denoted by $I_{fg}(M, N)$. The other one is $I_0(M, N) = \{(\overline{a}, \overline{b}) \mid \overline{a}, \overline{b} \text{ finite tuples of } M \text{ and } N \text{ respectively with the same atomic type}\}$.

Proposition 5.5.2. *Given a theory T the following are equivalent:*

1. *T is complete and eliminates quantifiers.*

2. *Any two ω-saturated models M, N of T are partially isomorphic via the set $I_0(M, N)$*

3. *Any two ω-saturated models M, N of T are partially isomorphic via the set $I_{fg}(M, N)$*

Proof. 1 ⇒ 2. Let M, N be ω-saturated models of T. By eliminates quantifiers any two nonempty tuples living in models of T have the same type iff they have the same atomic type. Therefore $I_0(M, N)$ coincides with the set of all finite elementary maps from M to N. By completeness M and N are elementarily equivalent. Now proposition 5.3.1 claims that $I_0(M, N)$ is a system of partial isomorphism.

2 ⇒ 3. Taking into account lemma 5.5.1, it is easy to check that if $I_0(M, N)$ is a system of partial isomorphism then so is $I_{fg}(M, N)$.

3 ⇒ 1. The completeness follows from corollary 5.3.2. For the elimination of quantifiers we apply the criterion of proposition 5.4.6. Let M, N be models of T and $\overline{a}, \overline{b}$ be nonempty tuples of M and N respectively. By proposition 2.2.4 we can suppose that M and N are ω-saturated. As $\mathrm{atp}^M(\overline{a}) = \mathrm{atp}^N(\overline{b})$, by lemma 5.5.1 it follows that there is an isomorphism $f : \langle\overline{a}\rangle_M \to \langle\overline{a}\rangle_N$ with $f(\overline{a}) = \overline{b}$. Then $f \in I_{fg}(M, N)$, which is a system of partial isomorphism. By corollary 5.2.2 f is a partial elementary map and therefore $\overline{a}$ and $\overline{b}$ have the same type over $\emptyset$. □

Example 5.5.3. Applying 5.5.2, in the examples 5.1.4, 5.1.6 and 5.3.3 we have seen that the following theories eliminate quantifiers:

1. The dense linear orders without endpoints in the language $L = \{\leq\}$.

2. The Random graph in the language $L = \{R\}$.

3. The random structure in a finite relational language.

4. The atomless Boolean algebras in the language $L = \{\wedge, \vee, 0, 1\}$.

5. The discrete orders without endpoints (the theory of $(\mathbb{Z}, \leq)$) in the language $L = \{\leq, s\}$, where s is the 'next' function. There is no Q.E. in the pure order language ($L = \{\leq\}$).

6. Infinite independent relations in the language $L = \{P_i \mid i \in \omega\}$.

Exercise 5.5.4. Prove the following variation of the proposition 5.5.2. Given a theory T the following are equivalent:

1. T eliminates quantifiers.

2. Any two ω-saturated models M, N of T either they are partially isomorphic via $I_{fg}(M, N)$ or $I_{fg}(M, N)$ is empty.

3. Any two ω-saturated models M, N of T either they are partially isomorphic via $I_0(M, N)$ or $I_0(M, N)$ is empty.

Exercise 5.5.5. Prove that T is complete and eliminates quantifiers iff given any two $(|L| + \aleph_0)^+$-saturated models M, N of T, the set of all isomorphism between substructures of M and N of cardinal at most $|L| + \aleph_0$ is a system of partial isomorphism. (Hint: for the left to right implication observe Q.E. implies that isomorphisms between substructures are elementary maps.)

Exercise 5.5.6. State and prove a generalization of exercise 5.5.5 replacing $|L|+\aleph_0$ by any given cardinal λ with $\lambda \geq |L|+\aleph_0$.

Exercise 5.5.7.
1. Prove that T eliminates quantifiers iff given any two models M, N of T with N $(|L|+\aleph_0)^+$-saturated, given M_1 a substructures of M of cardinal at most $|L|+\aleph_0$, given $a \in M$ and given an embedding f of M_1 into N we can extend f to an embedding of $M_1 \langle a \rangle$ into N. Here $M_1 \langle a \rangle$ denotes the substructure of M generated by M_1 and a. (Hint: observe that this is the forth property for the system provided in exercise 5.5.5).
2. Generalize the criteria of point 1 replacing $|L|+\aleph_0$ by any give cadina λ with $\lambda \geq |L|+\aleph_0$. (Hint: use 5.5.6 instead of 5.5.5).

Exercise 5.5.8. In the language that contains a 1-ari function symbol f alone, we consider the theory T that says that f is a permutation (bijective). Prove that T eliminates quantifiers. Give all its completions (hint: for each n describe the number of n-cycles). [To be worked, more hints]

Exercise 5.5.9. Let $L = \{P_i \mid i \in \omega\}$ be the language that contains countably many unary predicates. We will describe all complete theories is this language.