An overview of the core model induction

What's this talk about?

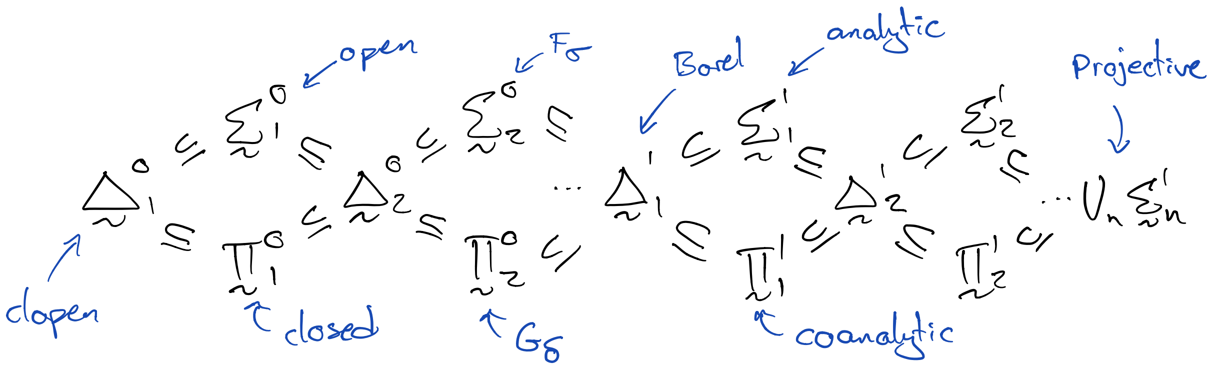
The purpose of this informal talk is to give a bird's eye view of the core model induction (at least up to $AD^{L(\mathbb{R})}$, maybe a bit more).

I will not delve into technicalities here — this is what Martin, Trevor and Nam are doing. Instead, I will try to stitch these talks together.

What is CMI trying to do?

CMI is a tool for proving that many sets of reals are determined. Note that this is a <u>direct implication</u> and not simply a consistency result.

We do this by working progressively, showing that larger and larger point-classes are determined:



Traversing through the projectives

We get a long way in ZFC alone:

<u>Theorem (Martin '75).</u> $\mathrm{Det}(\underset{\sim}{\Delta}^1_1)$. $\dashv$

From here we leave ZFC:

<u>Theorem (Harrington–Martin, '70 & '78).</u> $\mathrm{Det}(\underset{\sim}{\Pi}^1_1)$ is equivalent to $x^\#$ exists for all $x \in \mathbb{R}$. $\dashv$

Note that $x^\#$ can be seen as postulating the existence of a <u>mouse</u>. This parallel between mice and determinacy is the key to CMI.

How do we then show that $x^\#$ exists? This is where <u>covering</u> enters the picture:

<u>Theorem (Jensen '75).</u> TFAE for $x \in \mathbb{R}$:

- $x^\#$ exists
- For every $X \subseteq On$ there's a $Y \in L[x]$ such that $X \subseteq Y$ and $|Y| \leq \aleph_1 \cdot |X|$. $\dashv$

  Jensen covering

Jensen covering for $L[x]$ intuitively says that $L[x]$ is "close to V". We then have the following way to show $\mathrm{Det}(\boldsymbol{\Pi}^1_1)$:

$$\begin{aligned} \mathrm{Det}(\boldsymbol{\Pi}^1_1) &\iff \forall x \in \mathbb{R} : x^\# \text{ exists} \\ &\iff \forall x \in \mathbb{R} : \text{Jensen cov. fails for } L[x]. \end{aligned}$$

This general pattern will continue:

$$\boxed{\text{Determinacy}} \Leftarrow \boxed{\text{Mice}} \Leftarrow \boxed{\text{Covering failure}}$$

Let's see how this manifests itself in the projective case.

For the projective steps we have the following key theorem, analogous to the above:

<u>Theorem (Martin-Müller-Neeman-Steel-Woodin).</u> For all $n \geq 1$, $\mathrm{Det}(\underset{\sim}{\Pi}^1_{n+1}) \Leftrightarrow \forall x \in \mathbb{R}: M_n^\#$ exists and is ω_1-iterable. $\dashv$

A sudden jump in consistency strength, but again we have an equivalence between determinacy and mice.

Now, for the covering part of the triad.

If $\neg x^\#$ then $K = L[x]$.

<u>This</u> is where the core model K comes in. Here the K is a robust and canonical L-like model, in the sense that

1) K is an iterable mouse
2) K cannot be changed by set forcing;
3) K satisfies weak covering: $\mathrm{cof}^V(\kappa) = \mathrm{Card}^V(\kappa)$ for all K-cardinals, reflecting that K contains enough sets to only have a small error $(\leq \mathrm{cof}^K(\kappa) - \mathrm{Card}^V(\kappa))$ when computing cofinalities. This means that "K is close to V".

Sometimes K exists:

<u>Theorem (Jensen-Steel '13).</u> Assume there is no inner model with a Woodin cardinal. Then K exists. $\dashv$

Sometimes it doesn't:

<u>Theorem (Woodin).</u> If there exists a Woodin cardinal then K doesn't exist. $\dashv$

This looks like we won't get anywhere, but note that Woodin's theorem only mentions <u>true</u> Woodins and not Woodins in inner models.

Indeed, we have the following:

<u>Theorem (K-existence dichotomy).</u> Assume $M_\omega^\#$ doesn't exist and that $M_n^\#$ exists for all $x \in \mathbb{R}$. Then either

1) $K(x)$ exists; or
2) $M_{n+1}^\#(x)$ exists. $\dashv$

So, in particular, if we can show that $x^\#$ exists for all $x \in \mathbb{R}$ and that $K(x)$ does <u>not</u> exist, then we get determinacy of all projective sets!

This seems to exhaust our hierarchy, so we need another one, to be able to keep getting more and more determined sets.

Moving to $L(\mathbb{R})$

The first observation is that

$$\begin{aligned}\mathcal{P}(\mathbb{R}) \cap J_1(\mathbb{R}) &= \{A \subseteq \mathbb{R} \mid A \text{ is definable with parameters from } J_0(\mathbb{R}) = V_{\omega+1}\} \\ &= \bigcup_{n<\omega} \underset{\sim}{\Sigma}^1_n,\end{aligned}$$

which is the projective sets! We then have our next hierarchy: $J_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ for $\alpha \in On$.

Now, again we'd like to convert determinacy of these pointclasses into existence (and iterability) of mice, as with the projective sets.

First of all, we don't have to show that <u>all</u> $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ are determined:

what Trevor proved

<u>Theorem (Kechris-Woodin).</u> If every scaled-co-scaled $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is determined then every $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is as well. $\dashv$

This means that we only have to worry about α such that a new scaled-co-scaled set appears in $J_{\alpha+1}(\mathbb{R})$. We call such α critical.

We seem to need an auxiliary "coarse mouse witness condition":

W_α^*: Let $\mathcal{U} \subseteq \mathbb{R}$ be scaled-co-scaled with the two scales being elements of $J_\alpha(\mathbb{R})$. Then $\forall k<\omega \forall x \in \mathbb{R} \exists (N, \Sigma)$:

1) $x \in N$, N is a coarse $(k, \mathcal{U})$-Woodin mouse, witnessed by Σ; and
2) $\Sigma \restriction HC \in J_\alpha(\mathbb{R})$.

The key point is that the notion of "coarse $(k,\mathcal{U})$-Woodin mouse" has been chosen specifically such that:

Proposition. $W_\alpha^* \Rightarrow J_\alpha(\mathbb{R}) \models AD.$ $\dashv$

Now, we can move from W_α^* to actual mice using the following.

special case – more generally this is done in $V[G]$.
sometimes called dichotomy

Theorem (Mouse witness equivalence). Let α be critical and assume that W_β^* holds for all $\beta \leq \alpha$. Then there's a "hybrid mouse operator" $\mathcal{N}$ on $\mathbb{R}$ such that

maybe equivalence?

$$L(\mathbb{R}) \models W_{\alpha+1}^* \Longleftarrow \forall n \in \omega \, \forall x \in \mathbb{R} : M_n^{\mathcal{N}}(x) \text{ exists} \quad \dashv$$

Think of $M_n^{\mathcal{N}}(x)$ as having n Woodins and then "closing off with $\mathcal{N}$" rather than with the usual $\#$.

Here's an analogue to the K-existence dichotomy:

Theorem (K^F-existence dichotomy). Let $\mathcal{N}$ be a "hybrid mouse operator" on V and let $x \in V$. Then either

1) $K^F(x)$ exists; or
2) $M_1^F(x)$ exists. $\dashv$

Here "K^F" behaves almost like K; in particular it satisfies weak covering.

So again, we show that $K^F(x)$ can't exist, yielding $M_1^F(x)$ and then also $L(\mathbb{R}) \models W_\alpha^*$ for all α, which then gives $AD^{L(\mathbb{R})}$. Hooray!

The witness equivalence.

This is where we use the scale analysis.

The scale analysis allows us to split the critical α into cases, and construct a mouse operator $\mathcal{N}$ satisfying the theorem in each case.

The two main cases are called

① the inadmissible case
② the end-of-gap / admissible case.

In ① we construct a mouse operator that behaves like a normal mouse (a "pure" mouse operator), and in ② we construct a hybrid mouse operator, which has both an extender sequence and a strategy sequence.

The strategy sequence comes from <u>another</u> mouse, so we first have to find such one!

② can also be done in Trevor's way, without using mice at all. This stays true for $\Omega > 0$.

Beyond $L(\mathbb{R})$.

We need a new hierarchy yet again. First let's define an ordering on $\mathcal{P}(\mathbb{R})$:

<u>Def.</u> The Wadge order on $\mathcal{P}(\mathbb{R})$ is defined as $A \leq_W B$ iff A is a cts pre-image of B. $\dashv$

Intuition: Cts functions are uniquely determined by what they do on $\mathbb{Q}$, so they can be coded as reals. So $A \leq_W B$ is saying that from B and a real, we can compute A. $\dashv$

Lemma (Wadge) For all $A, B \subseteq \mathbb{R}$, either $A \leq_W B$ or $B \leq_W \neg A$.

Thm (Martin). Under $ZF + AD + DC_{\mathbb{R}}$, $\leq_W$ is well-founded. $\dashv$

So, modulo complements, $\leq_W$ is a well-order of $\mathcal{P}(\mathbb{R})$ under $AD + DC_{\mathbb{R}}$.

Def. The Solovay sequence $(\theta_\alpha \mid \alpha \leq \Omega)$ is given as

$$\theta_0 := \sup\{\alpha \mid \text{there's an OD surjection } f : \mathbb{R} \twoheadrightarrow \alpha\}$$

$$\theta_{\beta+1} := \sup\{\alpha \mid \text{there's an } OD(A) \text{ surjection } f : \mathbb{R} \twoheadrightarrow \alpha, \text{ for any } A \subseteq \mathbb{R} \text{ with } w(A) < \theta_\beta\}$$

$$\theta_\lambda := \sup_{\alpha < \lambda} \theta_\alpha. \qquad \dashv$$

Note that $\Theta = \theta_\Omega$.

<u>Fact.</u> $L(\mathbb{R}) \models AD \Rightarrow L(\mathbb{R}) \models \Omega = 0$. $\dashv$

<u>Thm (Woodin).</u> ish Assuming there's no inner model with a Woodin limit of Woodins, the Solovay sequences of all AD^+ $(+DC_{\mathbb{R}}?)$ models containing all the reals line up. $\dashv$

This theorem shows that we have a new hierarchy: traversing the Solovay sequence.

<u>First step:</u> find a model of $AD^+ + \Omega > 0$, containing all of $\mathbb{R}$. By Woodin's result, any one such will do.

We first "close off under $\Omega = 0$". Define

$$\Gamma_0 := \{ A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \models AD^+ + \Omega = 0 \}.$$

Then one can show that

internal core model induction

$$L(\Gamma_0,\mathbb{R})\models AD^+ + \Omega=0$$

this is the pure mouse aspect.

as well, making it the "maximal" model of $AD^+ + \Omega=0$.

we need MSC here

$\mathcal{H}^+ := Lp(HOD|\theta_0)$

We the perform a hod analysis of $HOD^{L(\Gamma_0,\mathbb{R})}$ and show that it's iterable; let Σ "$\subseteq$" $\mathbb{R}$ be its strategy.

From this we can show that $\Sigma\notin L(\Gamma_0,\mathbb{R})$ and an internal CMI shows that $L(\Sigma,\mathbb{R})\models AD^+$, meaning

$$L(\Gamma_0,\mathbb{R})\models AD^+ + \Omega>0,$$

which is the model we where looking for

For $\Omega>1$ we do basically the same thing, except that we move from pure mice to <u>only</u> dealing with hod mice.

For limit Ω we take "the limit of the previous hods". We have to show that this limit has a nice strategy — doing this for Ω singular limit will show $AD_{\mathbb{R}}$ + "Θ regular".