Games and Ramsey-like cardinals

Set Theory Today conference, Vienna

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Let's play a game $\mathcal{G}_\alpha^\theta(\kappa)$

I

II

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I	$\mathcal{M}_0$
II	

1. $\mathcal{M}_\eta \prec H_\theta$ is a κ-sized model of ZFC^- containing $\kappa+1$

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3. The $\mathcal{M}_\eta$'s and μ_η's are $\subseteq$-increasing

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4. We take unions at limit rounds
5. The game lasts for $\alpha{+}1$ rounds
6. Player II wins iff they can continue playing all rounds
7. IMPORTANT REMARK: The $\mathcal{M}_\eta$'s are **not** necessarily transitive!

Yet another large cardinal notion?!

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For $\alpha \leq \kappa$, a cardinal κ is **strategic α-Ramsey** if, for all regular $\theta > \kappa$, player II has a strategy in $\mathcal{G}_\alpha^\theta(\kappa)$ which is always winning.

Yet another large cardinal notion?!

Definition

For $\alpha \leq \kappa$, a cardinal κ is **weakly strategic α-Ramsey** if,
for all regular $\theta > \kappa$, player II has a strategy in $\mathcal{G}^\theta_\alpha(\kappa)$ which is always winning the first α rounds.

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- κ is strategic 0-Ramsey $\Leftrightarrow$ κ is weakly compact
- κ is strategic 1-Ramsey $\Rightarrow$ κ is ineffable ($\Rightarrow$ κ is strategic 0-Ramsey)

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A cardinal κ is weakly strategic ω-Ramsey iff κ is completely ineffable.

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The proof uses the previous theorem as the base case. In the "$\Leftarrow$" direction, care must be taken at successor stages to ensure that the measures cohere.

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A cardinal κ is **remarkable** if to every regular $\lambda > \kappa$ there is a regular $\nu > \lambda$ and a transitive M closed under λ-sequences such that, in a generic extension, there's an elementary embedding $\pi : H_\nu^V \to M$ with $\text{crit}\,\pi = \kappa$ and $\pi(\kappa) > \lambda$.

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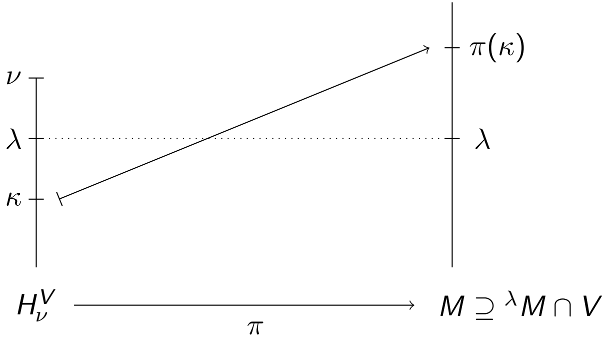

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Proposition (Schindler '00)

Remarkable cardinals are downwards absolute to L.

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2. (Schindler '04) semi-proper forcing cannot change the theory of $L(\mathbb{R})$

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1. (Schindler '00) proper forcing cannot change the theory of $L(\mathbb{R})$
2. (Schindler '04) semi-proper forcing cannot change the theory of $L(\mathbb{R})$
3. (Cheng-Schindler '15) 3^{rd} order number theory + Harrington's Principle ("there is a real x such that every x-admissible ordinal is an L-cardinal")

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Theorem (Schindler-N.)

Every remarkable cardinal is strategic ω-Ramsey, and if κ is strategic ω-Ramsey then either κ is remarkable in L or

$$L_\kappa \models \text{There is a proper class of strategic } \omega\text{-Ramseys.}$$

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Consequently, strategic ω-Ramseys are equiconsistent with remarkables.

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The proof goes via the notion of a so-called *virtually measurable cardinal.*

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Proof: In $\mathcal{G}_\kappa^\theta(\kappa)$ simply play the measure on κ. $\square$

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Every strategic ω_1-Ramsey cardinal κ is measurable in the core model K below the sharp of a strong cardinal, $0^\P$.

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Proof sketch.

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Proof sketch. Jump to $V^{\mathrm{Col}(\omega_1,\kappa^{+K})}$, let $\eta_\alpha \to \kappa^{+K}$ and let player I in $\mathcal{G}^\theta_{\omega_1}(\kappa)^V$ play models

$$\mathcal{M}_\alpha := \qquad K|\eta_\alpha$$

Player II follows their winning strategy.

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The final measure μ_{ω_1} is then a K-measure and we can wlog assume that μ_{ω_1} is countably complete and weakly amenable.

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Then, since we're below $0^\P$, inner model theory magic ("the beaver argument") implies that $\mu_{\omega_1} \in K$ and we're done. $\square$

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Strategic ω_1-Ramseys are therefore equiconsistent with measurables.

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The proof is similar, but different inner model theory magic is used.

So what happens between ω and ω_1?

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Theorem (Mitchell '79, Dodd '82)

A cardinal κ is Ramsey iff every $A \subseteq \kappa$ can be put into a κ-sized model $M \models \mathsf{ZFC}^-$ containing $\kappa+1$ such that there exists a weakly amenable countably complete M-measure on κ.

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Then $\mu_\omega \subseteq \mu_{\omega+1}$ implies that, wlog,

$$\mathcal{M}_{\omega+1} \models \mu_\omega \text{ is countably complete and weakly amenable,}$$

making μ_ω countably complete and weakly amenable by elementarity.

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$$(\mathcal{M}_{\omega+1}, \in, \mu_{\omega+1}) \models \mu_{\omega+1} \text{ is countably complete.}$$

Then $\mu_\omega \subseteq \mu_{\omega+1}$ implies that, wlog,

$$\mathcal{M}_{\omega+1} \models \mu_\omega \text{ is countably complete and weakly amenable,}$$

making μ_ω countably complete and weakly amenable by elementarity.

So κ is Ramsey. But then $\mathcal{M}_\omega \models \kappa$ is Ramsey and so $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega) \models \kappa$ is Ramsey, since $\mathcal{M}_\omega$ has the same subsets as the ultrapower by weak amenability of μ_ω.

So what happens between ω and ω_1?

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Łoś and elementarity then makes κ a limit of Ramseys. □

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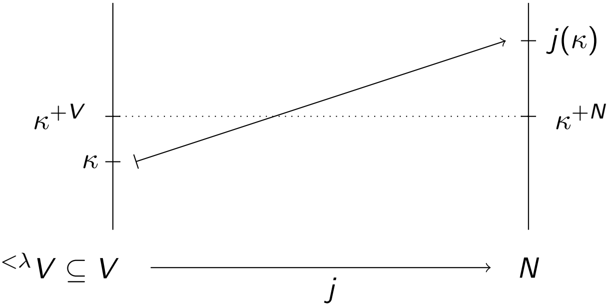

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The "$\Leftarrow$" direction is open – getting wellfoundedness of $\mathrm{Ult}(\mathcal{M}_\omega, \mu_\omega)$ seems hard. We've shown it in the case where $N \subseteq V$.

What's not known?

Question 1

Are the strategic α-Ramseys equivalent to some kind of "generic embedding property" when α is countably infinite, as in the uncountable regular case?

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Are the strategic α-Ramseys equivalent to some kind of "generic embedding property" when α is countably infinite, as in the uncountable regular case?

Question 2

Do the strategic α-Ramseys form a strict hierarchy for α countably infinite? More specifically, does

$$\mathsf{ZFC} + \exists\text{strategic } (\alpha{+}1)\text{-Ramsey} \vdash \mathrm{Con}(\exists\text{strategic } \alpha\text{-Ramsey})?$$

An overview

Measurable ∼ Strategic ω_1-Ramsey

Strategic α-Ramsey

Ramsey

Fully iterable

$V = L$

α-iterable

2-iterable

Remarkable ∼ Strategic ω-Ramsey

1-iterable

Completely ineffable = Weakly strategic ω-Ramsey

Strategic n-Ramsey

Weakly compact = Strategic 0-Ramsey

Thank you for your attention

Slides and preprint available at
`https://dsnielsen.com`