Playing games until the end of time

MINGLE '18

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Let's play a stupid game

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I pick a natural number n_0

Let's play a stupid game

I pick a natural number n_0

You pick another natural number n_1

Let's play a stupid game

I pick a natural number n_0

You pick another natural number n_1

You win if $n_1 > n_0$

Let's model the stupid game

I
II

Let's model the stupid game

I	n_0
II	

Let's model the stupid game

I	n_0	
II		n_1

Let's model the stupid game

$$\begin{array}{lcc} \mathrm{I} & \underbrace{n_0 & \\ \mathrm{II} & & n_1}_{\text{1 round}} \end{array}$$

Let's model the stupid game

$$\begin{array}{lll} \text{I} & n_0 & \\ \text{II} & & n_1 \end{array}$$

$$\underbrace{\hspace{4em}}_{\text{1 round}}$$

You always have a winning strategy

Let's model the stupid game

$$\begin{array}{lll} \mathrm{I} & n_0 & \\ \mathrm{II} & & n_1 \end{array}$$

$$\underbrace{\hspace{4em}}_{\text{1 round}}$$

You always have a winning strategy

For every n_0 there's an n_1 such that $n_1 > n_0$

Let's model the stupid game

$$\begin{array}{lll} \mathrm{I} & n_0 & \\ \mathrm{II} & & n_1 \end{array}$$

$$\underbrace{\quad n_0 \quad n_1 \quad}_{1\ \text{round}}$$

You always have a winning strategy

$$\forall n_0 \exists n_1 : n_1 > n_0$$

Let's model the generalised stupid game

$$\begin{array}{lcccccccc} \text{I} & n_0 & & n_2 & & \cdots & & n_{2k} & \\ \text{II} & & n_1 & & n_3 & & \cdots & & n_{2k+1} \end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{k+1 rounds}}$$

You always have a winning strategy

$$\forall n_0 \exists n_1 \forall n_2 \exists n_3 \cdots \forall n_{2k} \exists n_{2k+1} : n_{2k+1} > n_{2k}$$

Why are we playing stupid games?

First of all, games aren't stupid. Calm down.

Why are we playing silly games?

We saw that player II has a winning strategy iff

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Analogously, player I has a winning strategy iff
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We say that finite games are determined

Chess is determined

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(if we remove draws)

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(say White wins if it's a draw)

Infinite chess?

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Same rules, same pieces, infinite board

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Still determined?

Infinite chess!

Note that if white loses then they've lost at a finite stage

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White can start out by playing a non-losing move

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Keep playing non-losing moves

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but they're playing non-losing moves, contradiction!

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So infinite chess is determined

I know what you're thinking

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Yes, there are non-determined games out there

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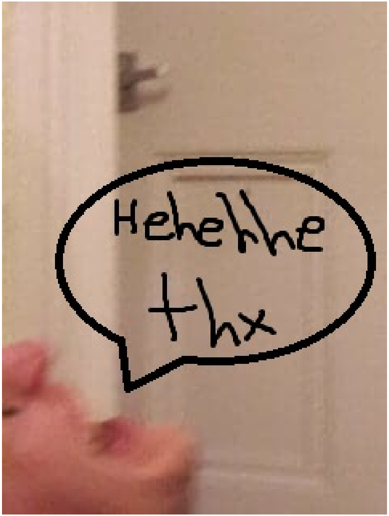
Let's do that another time


Hehehhe
thx