Virtual Large Cardinals

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Examples

- κ is a **measurable cardinal** if $\mathcal{M} = H_{\kappa^+}$.
- κ is a θ-**strong cardinal** if $\mathcal{M} = H_\theta$, $H_\theta \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

Recall that $x \in H_\theta$ iff $|\mathsf{trcl}(x)| < \theta$. This hierarchy is often more convenient than the V_α's since $H_\theta \models \mathsf{ZFC}^-$ if θ is regular.

The hierarchy of large cardinals

Supercompact

Superstrong

Woodin

Strong

Measurable

V = L

ω Erdős

Weakly compact

Inaccessible

What is a *virtual* large cardinal?

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"Definition"

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"Definition"

Let Φ be a large cardinal concept defined via elementary embeddings between *sets*, like the definitions on the previous slide.

Then κ is **virtually Φ** if the same definition holds but where we only require the embeddings exist in a generic extension and that $\mathcal{N} \subseteq V$.

A virtual addition to the hierarchy

Supercompact

Superstrong

Woodin

Strong

Measurable

V = L

ω Erdős

~ Virtual large cardinals ~

Weakly compact

Inaccessible

Attaching an adjective

Let us attach a **pre-** to our large cardinals if we do not require anything about where the critical point is sent:

Example

κ is **prestrong** if for every regular $\theta > \kappa$ there is an elementary embedding $\pi\colon (H_\theta, \in) \to (\mathcal{N}, \in)$ with $\mathsf{crit}\,\pi = \kappa$ and $H_\theta \subseteq \mathcal{N}$.

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This is really not an interesting concept in the real world:

Proposition (folklore)

For regular cardinals $\kappa < \theta$:

- κ is θ-prestrong iff it is θ-strong
- κ is θ-presupercompact iff it is θ-supercompact

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It *is* interesting in the virtual world, however:

Theorem (N.)

For regular cardinals $\kappa < \theta$, κ is virtually θ-prestrong iff either

- κ is virtually θ-strong, or
- κ is virtually (θ, ω)-superstrong

Characterising a phenomenon

Corollary (N.)

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- Virtually prestrongs are equivalent to virtually strongs
- There are no virtually ω-superstrongs.

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- There are no virtually ω-superstrongs.

Note that ω-superstrong cardinals are inconsistent with ZFC!

Adding parameters

Definition

Let $\kappa < \theta$ be regular and let A be a class. Then κ is **virtually (θ, A)-prestrong** if there exists a generic elementary embedding

$$\pi \colon (H_\theta^V, \in, A \cap H_\theta^V) \to (\mathcal{N}, \in, B)$$

such that $\operatorname{crit} \pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$ and $A \cap H_\theta^V = B \cap H_\theta^V$.

Further, if $\pi(\kappa) > \theta$ then κ is **virtually (θ, A)-strong**.

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Further, if $\pi(\kappa) > \theta$ then κ is **virtually (θ, A)-strong**.

Can we find some virtual large cardinal characterising exactly when the virtually A-prestrongs are equivalent to virtually A-strongs?

Remember that we are looking for a large cardinal notion which is inconsistent in the real world.

Definition

δ is **virtually berkeley** if for every transitive set $\mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\mathsf{crit}\,\pi < \delta$.

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The real world versions of these are of course inconsistent with ZFC, but are currently being investigated in a choiceless context.

Berkeley cardinals

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Definition

Say that On is **virtually (pre)woodin** if for every class A there exists a virtually A-(pre)strong cardinal κ.

Theorem (N.)

The following are equivalent:

- On is virtually prewoodin iff it is virtually woodin
- There are no virtually berkeley cardinals

Open questions

Question

Is the existence of a virtually berkeley cardinal equivalent to the statement that, for every class A, every virtually A-prestrong cardinal is virtually A-strong?

Question

Are virtually ω-superstrongs equivalent to virtually berkeleys?

Wilson (2018) has shown that this is true in L.

Thank you for your attention.

ω-erdős = Virtually berkeley = Virtually club berkeley = Virtually ω-superstrong = Virtually totally reinhardt

Virtually rank-into-rank

Virtually vopěnka = Virtually woodin

Virtually C(n)-extendible

Virtually extendible

Remarkable = Virtually measurable = Virtually strong = Virtually supercompact