

# Virtual Large Cardinals

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Dan Saattrup Nielsen

University of Bristol

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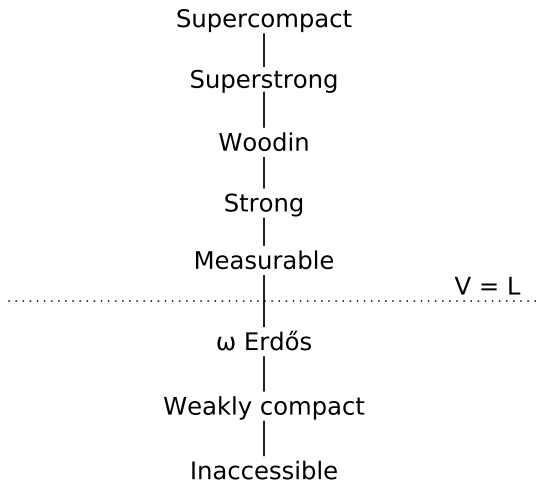
$$\pi: (\mathcal{M}, \in) \rightarrow (\mathcal{N}, \in)$$

## Examples

- $\kappa$  is a **measurable cardinal** if  $\mathcal{M} = H_{\kappa^+}$ .
- $\kappa$  is a  **$\theta$ -strong cardinal** if  $\mathcal{M} = H_\theta$ ,  $H_\theta \subseteq \mathcal{N}$  and  $\pi(\kappa) > \theta$ .

Recall that  $x \in H_\theta$  iff  $|\text{trcl}(x)| < \theta$ . This hierarchy is often more convenient than the  $V_\alpha$ 's since  $H_\theta \models \text{ZFC}^-$  if  $\theta$  is regular.

# The hierarchy of large cardinals



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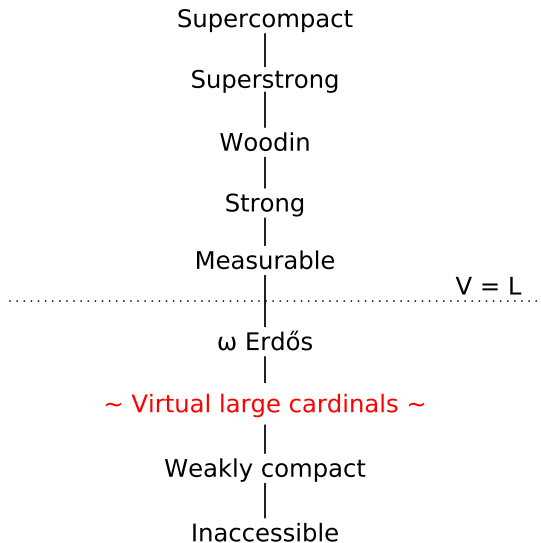
### "Definition"

Let  $\Phi$  be a large cardinal concept defined via elementary embeddings between *sets*, like the definitions on the previous slide.

Then  $\kappa$  is **virtually**  $\Phi$  if the same definition holds but where we only require the embeddings exist in a generic extension and that  $\mathcal{N} \subseteq V$ .



## A virtual addition to the hierarchy



Let us attach a **pre-** to our large cardinals if we do not require anything about where the critical point is sent:

### Example

$\kappa$  is **prestrong** if for every regular  $\theta > \kappa$  there is an elementary embedding  $\pi: (H_\theta, \in) \rightarrow (\mathcal{N}, \in)$  with  $\text{crit } \pi = \kappa$  and  $H_\theta \subseteq \mathcal{N}$ .

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This is really not an interesting concept in the real world:

### Proposition (folklore)

For regular cardinals  $\kappa < \theta$ :

- $\kappa$  is  $\theta$ -prestrong iff it is  $\theta$ -strong
- $\kappa$  is  $\theta$ -presupercompact iff it is  $\theta$ -supercompact

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It is interesting in the virtual world, however:

### Theorem (N.)

For regular cardinals  $\kappa < \theta$ ,  $\kappa$  is virtually  $\theta$ -prestrong iff either

- $\kappa$  is virtually  $\theta$ -strong, or
- $\kappa$  is virtually  $(\theta, \omega)$ -superstrong

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# Characterising a phenomenon

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Note that  $\omega$ -superstrong cardinals are inconsistent with ZFC!

### Definition

Let  $\kappa < \theta$  be regular and let  $A$  be a class. Then  $\kappa$  is **virtually**  $(\theta, A)$ -prestrong if there exists a generic elementary embedding

$$\pi: (H_\theta^V, \in, A \cap H_\theta^V) \rightarrow (\mathcal{N}, \in, B)$$

such that  $\text{crit } \pi = \kappa$ ,  $H_\theta^V \subseteq \mathcal{N}$ ,  $\mathcal{N} \subseteq V$  and  $A \cap H_\theta^V = B \cap H_\theta^V$ .

Further, if  $\pi(\kappa) > \theta$  then  $\kappa$  is **virtually**  $(\theta, A)$ -strong.



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Remember that we are looking for a large cardinal notion which is inconsistent in the real world.

## Definition

$\delta$  is **virtually berkeley** if for every transitive set  $\mathcal{M}$  there exists a generic elementary embedding  $\pi: \mathcal{M} \rightarrow \mathcal{M}$  with  $\text{crit } \pi < \delta$ .

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The real world versions of these are of course inconsistent with ZFC, but are currently being investigated in a choiceless context.

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Say that On is **virtually (pre)woodin** if for every class  $A$  there exists a virtually  $A$ -(pre)strong cardinal  $\kappa$ .

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## Definition

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## Theorem (N.)

The following are equivalent:

- On is virtually prewoodin iff it is virtually woodin
- There are no virtually berkeley cardinals

## Question

Is the existence of a virtually berkeley cardinal equivalent to the statement that, for every class  $A$ , every virtually  $A$ -prestrong cardinal is virtually  $A$ -strong?

## Question

Are virtually  $\omega$ -superstrongs equivalent to virtually berkeleys?

Wilson (2018) has shown that this is true in  $L$ .

Thank you for your attention.

