Virtual Large Cardinals

British Logic Colloquium 2019, Oxford

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September 5, 2019

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Examples

- κ is a **measurable cardinal** if $\mathcal{M} = H_{\kappa^+}$.
- κ is a θ**-strong cardinal** if $\mathcal{M} = H_\theta$, $H_\theta \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

Recall that $x \in H_\theta$ iff $|\mathsf{trcl}(x)| < \theta$. This hierarchy is often more convenient than the V_α's since $H_\theta \models \mathsf{ZFC}^-$ if θ is regular.

The hierarchy of large cardinals

Supercompact

Superstrong

Woodin

Strong

Measurable

V = L

ω Erdős

Weakly compact

Inaccessible

What is a *virtual* large cardinal?

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"Definition"

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"Definition"

Let Φ be a large cardinal concept defined via elementary embeddings between *sets*, like the definitions on the previous slide.

Then κ is **virtually Φ** if the same definition holds but where we only require the embeddings exist in a generic extension and that $\mathcal{N} \subseteq V$.

A virtual addition to the hierarchy

Supercompact

Superstrong

Woodin

Strong

Measurable

V = L

ω Erdős

~ Virtual large cardinals ~

Weakly compact

Inaccessible

Attaching an adjective

Let us attach a **pre-** to our large cardinals if we do not require anything about where the critical point is sent:

Example

κ is **prestrong** if for every regular $\theta > \kappa$ there is an elementary embedding $\pi\colon (H_\theta, \in) \to (\mathcal{N}, \in)$ with $\operatorname{crit}\pi = \kappa$ and $H_\theta \subseteq \mathcal{N}$.

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This is really not an interesting concept in the real world:

Proposition (folklore)

For regular cardinals $\kappa < \theta$:

- κ is θ-prestrong iff it is θ-strong
- κ is θ-presupercompact iff it is θ-supercompact

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κ is **prestrong** if for every regular $\theta > \kappa$ there is an elementary embedding $\pi\colon (H_\theta, \in) \to (\mathcal{N}, \in)$ with $\operatorname{crit}\pi = \kappa$ and $H_\theta \subseteq \mathcal{N}$.

It *is* interesting in the virtual world, however:

Theorem (N.)

For regular cardinals $\kappa < \theta$, κ is virtually θ-prestrong iff either

- κ is virtually θ-strong, or
- κ is virtually (θ, ω)-superstrong

Characterising a phenomenon

Corollary (N.)

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- Virtually prestrongs are equivalent to virtually strongs
- There are no virtually ω-superstrongs.

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Note that ω-superstrong cardinals are inconsistent with ZFC!

Adding parameters

Definition

Let $\kappa < \theta$ be regular and let A be a class. Then κ is **virtually (θ, A)-prestrong** if there exists a generic elementary embedding

$$\pi\colon (H_\theta^V, \in, A \cap H_\theta^V) \to (\mathcal{N}, \in, B)$$

such that $\operatorname{crit}\pi = \kappa$, $H_\theta^V \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$ and $A \cap H_\theta^V = B \cap H_\theta^V$.

Further, if $\pi(\kappa) > \theta$ then κ is **virtually (θ, A)-strong**.

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Further, if $\pi(\kappa) > \theta$ then κ is **virtually (θ, A)-strong**.

Can we find some virtual large cardinal characterising exactly when the virtually A-prestrongs are equivalent to virtually A-strongs?

Remember that we are looking for a large cardinal notion which is inconsistent in the real world.

Berkeley cardinals

Definition

δ is **virtually proto-berkeley** if for every transitive set $\mathcal{M}$ there exists a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{M}$ with $\mathsf{crit}\,\pi < \delta$. Further, δ is **virtually berkeley** if the critical point can be chosen to be arbitrarily large below δ.

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The real world versions of these are of course inconsistent with ZFC, but are currently being investigated in a choiceless context.

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Definition

Say that On is **virtually (pre)woodin** if for every class A there exists a virtually A-(pre)strong cardinal κ.

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Definition

Say that On is **virtually (pre)woodin** if for every class A there exists a virtually A-(pre)strong cardinal κ.

Theorem (Dimopoulos-Gitman-N.)

The following are equivalent:

- On is virtually prewoodin iff it is virtually woodin
- There are no virtually berkeley cardinals

Definition

Generic Vopěnka's Principle (gVP) holds if for every class C consisting of set-sized structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in C$ and a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$.

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Theorem (Dimopoulos-Gitman-N.)

On is virtually prewoodin iff gVP holds.

Outline of argument cont.

Definition (GBC)

Say that an On-sequence $\langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On} \rangle$ is **natural** if there exists a class function $f\colon \mathsf{On} \to \mathsf{On}$ such that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for every $\alpha < \beta$, and unary relations $R_\alpha \subseteq V_{f(\alpha)}$ such that

$$\mathcal{M}_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$$

for every α.

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for every α.

Definition (GBC)

Define the **virtually vopěnka filter** F on On as $X \in F$ iff there's a natural On-sequence $\vec{\mathcal{M}}$ such that for any $\alpha < \beta$ and any generic elementary $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, $\mathsf{crit}\,\pi \in X$.

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This filter is normal, and proper iff gVP holds.

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Key Lemma (N.)

Assume gVP and that there are no virtually berkeley cardinals. Then the virtually vopěnka filter on On contains every class club.

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Key Lemma (N.)

Assume gVP and that there are no virtually berkeley cardinals. Then the virtually vopěnka filter on On contains every class club.

Theorem (N.)

If there are no virtually berkeley cardinals then gVP implies that On is virtually woodin.

Outline of argument cont.

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If there are no virtually berkeley cardinals then gVP implies that On is virtually woodin.

This, together with the equivalence of gVP and On being virtually woodin, gives us one direction of our theorem.

Outline of argument cont.

For the other direction we have the following:

Theorem (N.)

If there exists a virtually berkeley cardinal then gVP holds and On is not mahlo.

Outline of argument cont.

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Theorem (N.)

If there exists a virtually berkeley cardinal then gVP holds and On is not mahlo.

Virtually woodins are mahlo, and we (finally!) have our theorem:

Theorem (Dimopoulos-Gitman-N.)

The following are equivalent:

- On is virtually prewoodin iff it is virtually woodin
- There are no virtually berkeley cardinals

An extra consequence

A result that occured along the way:

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This improves a result by Gitman and Hamkins (2019).

Open questions

Question

Is the existence of a virtually berkeley cardinal equivalent to the statement that, for every class A, every virtually A-prestrong cardinal is virtually A-strong?

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Question

Are virtually ω-superstrongs equivalent to virtually berkeleys?

Wilson (2018) has shown that this is true in L.

Thank you for your attention.

ω-erdős = Virtually berkeley = Virtually club berkeley =
Virtually ω-superstrong = Virtually totally reinhardt

Virtually rank-into-rank

Virtually vopěnka = Virtually woodin

Virtually C(n)-extendible

Virtually extendible

Remarkable = Virtually measurable =
Virtually strong = Virtually supercompact