Determinacy – seminar

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Basic descriptive set theory

Before we move to infinite games, we introduce some notions from descriptive set theory. This field of set theory studies *definable* subsets of reals.

The Baire space

An easier way to work with reals is *not* to work with $\mathbb{R}$, but to work with the so-called **Baire space** ω^ω of functions $\omega \to \omega$. A convenient way to view this space is as a *tree* with ω many branches at each node and height ω. The topology on ω^ω is generated by the **basic open sets** $N_s := \{x \in \omega^\omega \mid s \subseteq x\}$, where $s \in \omega^{<\omega} := \bigcup_{n<\omega} \omega^n$.

The reason why it's okay to work in the Baire space instead of $\mathbb{R}$ is justified by the fact that the Baire space is homeomorphic to the irrationals $\mathbb{R} - \mathbb{Q}$. Since the rationals is a Lebesgue null-set, we're fine with working in these spaces.

Borel and projective hierachies

Definability in the context of descriptive set theory is defined via a mixture of topology and measure theory. We define the **Borel hierachy** as follows, where $\alpha < \omega_1$. A set $A \subseteq \omega^\omega$ is..

- $\mathbf{\Sigma}^0_1$ if it's open in the standard topology;
- $\mathbf{\Pi}^0_\alpha$ if $\omega^\omega - A$ is $\mathbf{\Sigma}^0_\alpha$;
- $\mathbf{\Sigma}^0_\alpha$ if there's an ω-sequence of sets $A_0, A_1, A_2, \dots$ such that A_i is $\mathbf{\Pi}^0_{\alpha_i}$ for $\alpha_i < \alpha$ and $A = \bigcup_{i<\omega} A_i$;
- $\mathbf{\Delta}^0_\alpha$ if it's both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$.

Then the **Borel sets** is the sets $\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$. Alternatively, one could equivalently define $\mathbb{B}$ as the σ-algebra generated by the open sets of reals. We can extend this hierachy to the **projective hierachy**, defined as follows, where $n < \omega$. A set $A \subseteq \omega^\omega$ is..

- $\mathbf{\Sigma}^1_1$ if there exists a surjective continuous function $f : \omega^\omega \to A$;
- $\mathbf{\Pi}^1_n$ if $\omega^\omega - A$ is $\mathbf{\Sigma}^1_n$;
- $\mathbf{\Sigma}^1_{n+1}$ if there is a $\mathbf{\Pi}^1_n$ set $B \subseteq A \times \omega^\omega$ such that A is the first projection of B;
- $\mathbf{\Delta}^1_n$ if it's both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

It's a deep theorem that $\mathbb{B} = \mathbf{\Delta}^1_1$, so the projective hierachy is really a continuation of the Borel hierachy.

Example 1.3.

(i) $\mathbf{\Pi}^0_1$ is exactly the closed sets;

(ii) $\mathbb{Q}$ is $\mathbf{\Sigma}^0_2$;

(iii) $\mathbb{R} - \mathbb{Q}$ is $\mathbf{\Pi}^0_2$;

(iv) To construct a $\mathbf{\Sigma}^0_\alpha$ set for $\alpha > 4$ one **has** to use axiom of choice, as whether or not every set is $\mathbf{\Sigma}^0_4$ is independent of ZF.

Infinite game theory

Basic game theory

Given a subset $A \subseteq \omega^\omega$ we can associate a **game** $G(A)$, defined as follows. We imagine two players I and II each playing natural numbers $n \in \omega$:

$$\begin{array}{lcccccccc} \text{I} & x_0 & & x_2 & & x_4 & & \cdots & \\ \text{II} & & x_1 & & x_3 & & x_5 & & \cdots \end{array}$$

Then the sequence $x := (x_i)_{i<\omega}$ is an element of ω^ω and we say that player I **wins** if $x \in A$ – otherwise player II wins. A **strategy** for player I in $G(A)$ is a function $\sigma : \bigcup_{n<\omega} \omega^{2n} \to \omega$ and a strategy for player II is a function $\tau : \bigcup_{n<\omega} \omega^{2n+1} \to \omega$. We think of such strategies as supplying the given player with his moves. Say, if σ is a strategy for player I, then a generic play goes like

$$\begin{array}{lcccccccc} \text{I} & \sigma(\langle\rangle) & & \sigma(\langle\sigma(\langle\rangle), y_0\rangle) & & \sigma(\langle\sigma(\langle\sigma(\langle\rangle), y_1\rangle), y_1\rangle & & \cdots & \\ \text{II} & & y_0 & & y_1 & & y_2 & & \cdots \end{array}$$

We denote this play as $\sigma * y \in \omega^\omega$. A strategy σ is **winning** in $G(A)$ if $\sigma * y \in A$ for every $y \in \omega^\omega$. That is, no matter what player II plays, player I will always win. Strategies for player II are defined analogously. A game $G(A)$ is **determined** if one of the players has a winning strategy.

We say that a game $G(A)$ is $\mathbf{\Gamma}$ if A is $\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a pointclass – this could for instance be open sets $\mathbf{\Sigma}^0_1$, Borel sets $\mathbf{\Delta}^1_1$ or *analytic sets* $\mathbf{\Sigma}^1_1$.

Determinacy

Why do we care whether or not games are determined? Let's introduce an axiom, called the **axiom of determinacy**, also written AD, saying that $G(A)$ is determined for every $A \subseteq \omega^\omega$. It turns out that AD implies that reals are well-behaved:

Theorem 2.3. *Assuming AD, every set of reals is Lebesgue measurable and satisfies the Continuum Hypothesis (i.e. that for every $A \subseteq \mathbb{R}$, either A is countable or $|A| = |\mathbb{R}|$).* ■

This is just a few of the consequences of AD. But AD turns out to be a bit *too* nice: it's inconsistent with AC.

Proposition 2.4 (ZF). *Assuming AC, there exists a non-determined set $A \subseteq \mathbb{R}$.*

Proof. (sketch) We'll construct a set which isn't Lebesgue measurable, so that Theorem 2.3 implies that AD is false. We'll show the construction and leave out the argument showing that it isn't measurable.

Consider the quotient group $\mathbb{R}/\mathbb{Q}$. Use the axiom of choice to ensure that a subset $X \subseteq [0,1]$ exists with exactly one element from each equivalence class $[x] \in \mathbb{R}/\mathbb{Q}$. Such a set is called a *Vitali set*. ■

But what if we restrict ourselves to a smaller class of sets? Could we find such a class in which all sets in the class are determined, but it doesn't contradict AC? Let's start with the simplest sets, according to our hierarchies: the open sets.

Theorem 2.5 (Gale-Stewart '53). *Every open game $G(A)$ is determined.*

Proof. Assume that player I has no winning strategy in $G(A)$. Say that a partial play $\langle x_0, \dots, x_{2n} \rangle \in \omega^{<\omega}$ is **not losing for player II** if player I has no winning strategy from that point on. Note that by definition $\langle\rangle$ is not losing for player II.

Assume now that player II is not losing at $p := \langle x_0, \dots, x_{2n} \rangle$. We claim that there exists some x_{2n+1} such that for every x_{2n+2}, $p^\frown\langle x_{2n+1}, x_{2n+2} \rangle$ is not losing for player II. Indeed, assuming it wasn't the case we would have that no matter what player II played, player I would have a play such that player I would have a winning strategy at that point. But this means exactly that player I had a winning strategy at p, so p would be losing for player II, $\lightning$. Define the strategy τ for player II as these "non-losing" moves.

We now claim that τ is a winning strategy for player II. Fix some $x \in \omega^\omega$. Then $(x * \tau) \upharpoonright 2k+1$ is not losing for player II for every $k < \omega$. Assume for a contradiction that $x * \tau \in A$. Then since A is open, find an open neighbourhood $N_q \subseteq A$ of $x * \tau$ satisfying that $\text{len}(q)$ is odd. But now $q = (x * \tau) \upharpoonright 2k+1$ for some $k < \omega$, so q is both losing and not losing for player II, $\lightning$. Hence $x * \tau \notin A$

and τ is winning for player II. ■

Over 20 years later, Martin improved this result greatly, to *all* Borel sets.

Theorem 2.6 (Martin '75). *Every Borel game $G(A)$ is determined.* ■

The proof of this subliminal result was almost 10 pages long, and improved upon in '82 to a purely inductive argument, shortening it to about 5 pages. The proof features an ingenious idea of providing a general sufficient condition for games to be determined, called *unraveling*. Details can be read in my project.

What about the next step, analytic sets? This turns out to be independent of ZFC. But under strong assumptions, it turns out that we can prove it anyway. Recall from my last talk that a **measurable cardinal** exists iff there exists an elementary embedding $j : V \to M$ for some universe M in which ZFC holds.

Theorem 2.7 (Martin '70). *If there exists a measurable cardinal then every analytic game $G(A)$ is determined.* ■

So.. what about the rest of the projective hierarchy? This turns out to be unprovable even assuming a measurable. However, there is a strictly stronger notion of a measurable cardinal called a **Woodin cardinal**[1], which helps us.

Theorem 2.8 (Martin, Steel '85). *If there exists infinitely many Woodin cardinals, then every projective game $G(A)$ is determined.* ■

Okay, so we can subsume our entire hierarchy assuming these Woodin cardinals exist. But, can we do more? Recall the definition of a constructible set from my last talk, which is a set that can be described by a formula. Every projective set can (by definition) be described by a formula. Is every constructible set determined?

Theorem 2.9 (Woodin '85). *If there exists infinitely many Woodin cardinals with a measurable above them, then every constructible game $G(A)$ is determined.* ■

[1]The precise definition is as follows. κ is Woodin if for every function $f : \kappa \to \kappa$ there's a cardinal $\lambda < \kappa$, a transitive inner model M and an elementary embedding $j : V \to M$ such that f restricts to a function $f \upharpoonright \lambda : \lambda \to \lambda$, $\mathrm{crit}(j) = \lambda$ and $V_{j(f)(\lambda)} \subseteq M$.