AlgTop & HomAlg talk

Brainstorm.

- CW complexes - what are they and why are they nice?
- Exactness
- Brouwer's fixpoint theorem
- Fundamental group
- $\mathbb{R}^2 \not\simeq \mathbb{R}^2 - *$.
- 5-lemma, diagram chasing

Sketch.

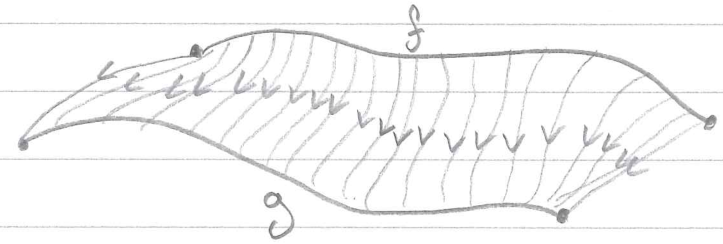
- AlgTop is interested in spaces up to homotopy equivalence
- $\mathbb{R} \simeq 0$ via $\lambda x.0$ and $\lambda y.0$, using $\lambda x \lambda t. xt$.
- How do we determine $X \not\simeq Y$? This is where algebra comes in.
- Homology groups H_n
  - $f : X \to Y \Rightarrow H_n f : H_n X \to H_n Y$
  - $f \simeq g \Rightarrow H_n f = H_n g$
  - $X \simeq Y \Rightarrow H_n X \cong H_n Y$.
  - $H_1 S^1 \cong \mathbb{Z}$, $H_1 D^2 = 0$, so $S^1 \not\simeq D^2$.
- Brouwer's fixpoint theorem
- HomAlg is used to define H_n
- HomAlg is the study of modules and chain complexes - very diagrammatic.

- Exactness
- 5-lemma sketch

Presentation:

Algebraic topology concerns itself with topological spaces up to homotopy equivalence as opposed to homeomorphism in normal topology.

Homotopy equivalence is based on the notion of homotopy, which is a "continuous deformation of functions" $f, g: X \to Y$:



Formally it's a continuous function $h: X \times [0,1] \to Y$ with $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Now write $f \simeq g$ if they are homotopic. Two spaces X and Y are homotopy equivalent if there are functions $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ with $fg \simeq id_Y$ and $gf \simeq id_X$.

To determine if two spaces are homotopy equivalent, we can write up an explicit homotopy inverse between them. Consider e.g. $\mathbb{R}$ and $\{0\}$. Define $f: \mathbb{R} \to \{0\}$ as the constant function and $g : \{0\} \to \mathbb{R}$ as $g(0) := 0$. Then $fg(0) = 0$, so $fg = id_{\{0\}}$. Conversely $gf(x) = g(0) = 0$, so we need to show that $0 \simeq id_{\mathbb{R}}$. Define $h : \mathbb{R} \times [0,1] \to \mathbb{R}$ as

$$h(x,t) := xt.$$

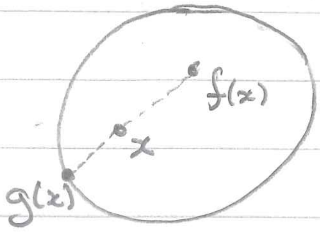
Then $h : 0 \simeq id_{\mathbb{R}}$ and it's shown.

What about showing that $X \not\simeq Y$? This is where algebra enters the picture. In the algebraic topology course, we will define groups $H_n(X)$, $n \geq 0$, associated to a space X, with the following properties:

- If $f : X \to Y$ is a continuous map between spaces then we have an induced map $f_* : H_n X \to H_n Y$;
- If $X \simeq Y$ then $H_n X \cong H_n Y$;
- $id_* = id$ and $(gf)_* = g_* f_*$.

One can for instance show that $H_1 S^1 \cong \mathbb{Z}$ and $H_1 D^2 = 0$, so $S^1 \not\simeq D^2$. An application of these groups is Brouwer's fixpoint theorem, which says that any continuous map $D^2 \to D^2$ has a fixpoint.

Assume for a contradiction that $f : D^2 \to D^2$ doesn't have a fixpoint. Then we can define a function $g: D^2 \to S^1$ as follows:



One can show that this map is continuous, and we see that $gi = id_{S^1}$, where $i: S^1 \hookrightarrow D^2$ is the inclusion. How is this a contradiction? Let's try to apply H_1 to the following diagram:

$$\begin{tikzcd} & D^2 \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ S^1 \arrow[ur, "i"] \arrow[rr, "id"'] & \, & S^1 \end{tikzcd}$$

As $H_1 S^1 \cong \mathbb{Z}$, $H_1 D^2 = 0$ and $id_* = id$, we get the diagram

$$\begin{tikzcd} & 0 \arrow[dr, "g_*"] \arrow[d, phantom, "="] & \\ \mathbb{Z} \arrow[ur, "i_*"] \arrow[rr, "id"'] & \, & \mathbb{Z} \end{tikzcd}$$

Then $id_{\mathbb{Z}} = 0$, a contradiction. This shows how powerful these H_n groups are, turning a geometric problem into an algebraic one.

The methods used in AlgTop is geometric in the beginning, but as more tools are developed, such as the H_n's, it becomes almost purely algebraic. These algebraic notions are not your usual kind of group- and ring theory arguments, but a particular kind of algebra called homological algebra, which is taught in the HomAlg course.

Homological algebra is used to define the H_n's, and more generally homological algebra is the structural study of modules, which is a natural generalization of both a vector space and an abelian group. You have addition and scalar multiplication from a ring.

A key notion in homological algebra is something called exactness. A sequence of abelian groups $\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots$ is exact if $\ker d_n = \operatorname{im} d_{n+1}$ for all n. To illustrate some use of exactness, we'll show a part of the so-called 5-lemma. Say we have a commutative diagram with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

Then the 5-lemma says that if every f_i

is an isomorphism for $i \neq 3$, then f_3 is an isomorphism as well.

<u>Injective:</u> Assume $x \in \ker(f_3)$.

$$
\begin{tikzcd}
a_1 \arrow[r, mapsto, "11"] \arrow[d, mapsto, "10"'] & a_2 \arrow[r, mapsto, "6"] \arrow[d, mapsto, "7\ \text{iso}"] & x \arrow[r, mapsto, "3"] \arrow[d, mapsto, "1"] & a_4 \overset{5}{=} 0 \arrow[d, mapsto, "4\ \text{iso}"] \\
b_1 \arrow[r, mapsto, "9"'] & b_2 \arrow[r, mapsto, "8"'] & 0 \arrow[r, mapsto, "2"'] & 0
\end{tikzcd}
\qquad \therefore x = 0
$$

<u>Surjective:</u> Let $y \in B_3$.

$$
\begin{tikzcd}
a_3 \arrow[r, mapsto, "6"] \arrow[d, mapsto, "7"'] & a_4 \arrow[r, mapsto, "5"] \arrow[d, mapsto, "2\ \text{iso}"] & 0 \arrow[d, mapsto, "4\ \text{iso}"] \\
b_3 \arrow[r, mapsto, "8"] & b_4 \arrow[r, mapsto, "3"'] & 0 \\
y \arrow[ur, mapsto, "1"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
a_2 \arrow[r, mapsto, "12\ \text{inj}"] \arrow[d, mapsto, "\text{iso}\ 11"'] & a_3' \arrow[d, mapsto, "13"] & \\
b_2 \arrow[r, mapsto, "10"'] & y - b_3 \arrow[r, mapsto, "9"'] & 0
\end{tikzcd}
\qquad \therefore f_3(a_3' + a_3) = y
$$

And that's it!