Virtual large cardinals

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University of Bristol
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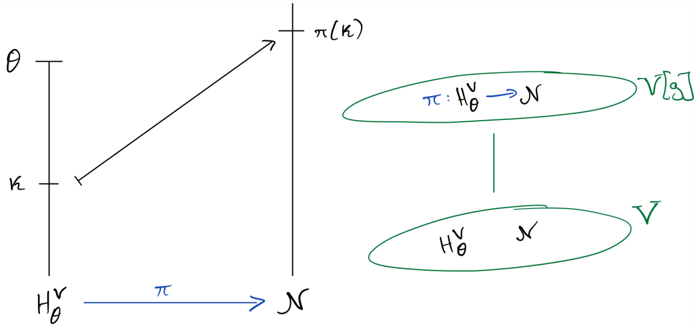
What are they?

Rough definition

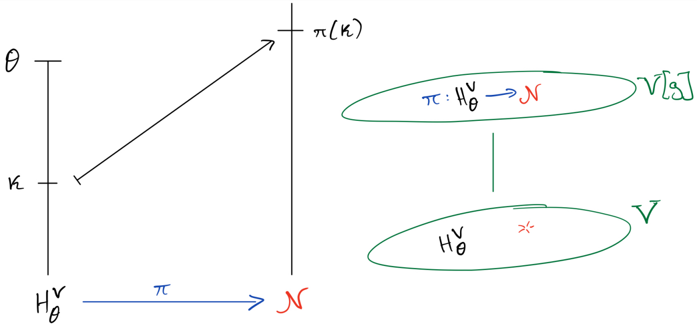
A large cardinal κ defined via *set-sized* elementary embeddings is **virtual** if the elementary embeddings exist in a generic extension.



What are they?

Rough definition

A large cardinal κ defined via *set-sized* elementary embeddings is **generic** if the elementary embeddings and the target model exist in a generic extension.



Why should we care?

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A virtually strong cardinal is equiconsistent with $\mathrm{Th}(L(\mathbb{R}))$ being unchangeable by proper forcing.

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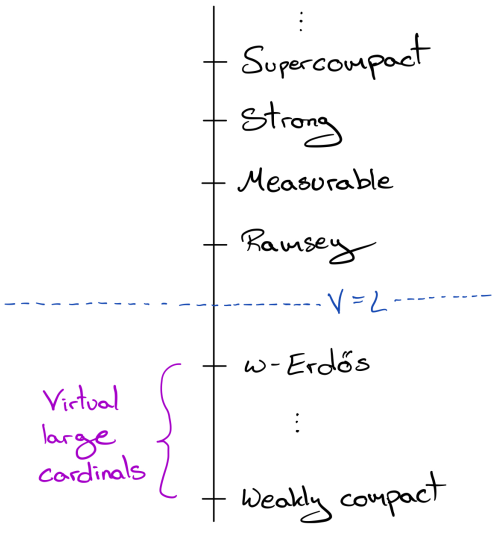
Theorem (Wilson '19)

A virtually Vopěnka cardinal is equiconsistent with $\Theta = \omega_2$ and $\boldsymbol{\Sigma}^1_2$ being the class of all ω_1-Suslin sets.

Where are they?



How do they behave?

Theorem (Gitman)

Virtually strongs are equivalent to virtually supercompacts.

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Theorem (N.)

Virtually measurables are equiconsistent with virtually strongs.

Game interlude

I

II

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4. We take unions at limit rounds

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7. IMPORTANT REMARK: The $\mathcal{M}_\alpha$'s are **not** necessarily transitive!
8. This game is called $\mathcal{G}_\gamma^\theta(\kappa)$. If we restrict I to only add $<|\gamma|$ sets at a time then we call the game $\mathcal{C}_\gamma^\theta(\kappa)$.

How do they behave level by level?

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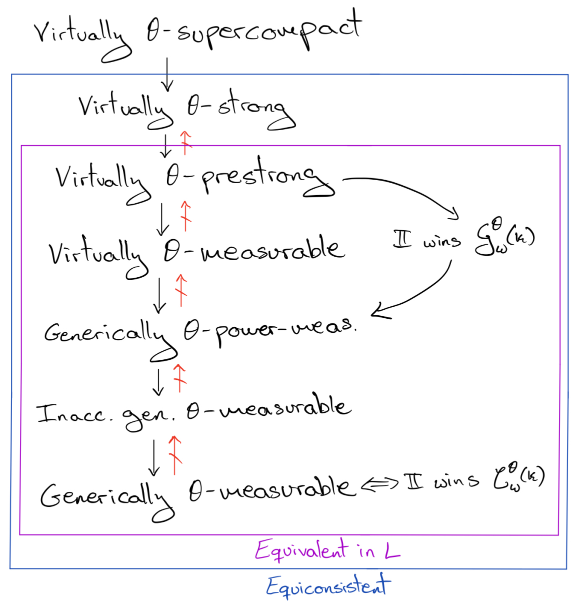

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