IIT Gandhinagar

MATHEMATICS NUMERICAL METHOD PROJECT

Three-body problem

PROJECT OVERVIEW

OUTLINE OF TOPICS

Introduction
Basics of Gravitation
The Three-body System
Modelling
Future Scope

INTRODUCTION

The Three-Body Problem is a special case of the n-body problem. Unlike the two-body problem, there is no closed solution to this problem. We require numerical methods to solve equations that describe the system.

Basics of Gravitation

Newton's law of gravitation	$\overline{F} = \frac{G m_1 m_2}{r^2} \hat{r}$
Second law of motion	$\overline{F} = m.\overline{a}$
Position of centre of mass	$\overline{r}_{com} = \frac{\sum_{i=1}^{n} m_i \overline{r}_i}{\sum_{i=1}^{n} m_i}$
Velocity of centre of mass	$\overline{v}_{com} = \frac{\sum_{i=1}^{n} m_i \overline{v}_i}{\sum_{i=1}^{n} m_i}$

Three Body Problem

OVERVIEW

THREE-BODY PROBLEM

What is the three-body problem?
What's so special about this problem?
How is it different from the one and two body problems?
Why is it chaotic and unpredictable?

WHAT IS THE THREE-BODY PROBLEM?

- Three masses moving under mutual gravitational force.
- All other forces are neglected.
- Their motion is calculate based on their initial positions and velocities of the masses.
- System is driven using Newton's laws of motion and gravitation.

WHAT'S SO SPECIAL ABOUT THIS PROBLEM?

- Newton's laws successfully provided solutions to the two-body problem. However, those equations couldn't give a solution for the three-body problem.

HOW IS IT DIFFERENT FROM THE ONE AND TWO-BODY PROBLEM?

- The three masses have their positions defined by 3 vectors in space. This accounts for 18 degrees of freedom in total.
- Using Newton's laws we get

$$m_1\ddot{\vec{r}}_1 = \frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12} + \frac{Gm_1m_3}{r_{13}^3}\vec{r}_{13}$$

$$m_2\ddot{\vec{r}}_2 = -\frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12} + \frac{Gm_2m_3}{r_{23}^3}\vec{r}_{23}$$

$$m_3\ddot{\vec{r}}_3 = -\frac{Gm_1m_3}{r_{13}^3}\vec{r}_{13} - \frac{Gm_2m_3}{r_{23}^3}\vec{r}_{23}$$

- Relations between the position vectors are

$$\vec{r}_{12} = \vec{r1} - \vec{r2}$$

$$\vec{r}_{23} = \vec{r2} - \vec{r3}$$

$$\vec{r}_{13} = \vec{r1} - \vec{r3}$$

WHY IS IT CHAOTIC AND UNPREDICTABLE

TIME

For a two-body system, we could find a generalized expression of the trajectory that does not involve time. However, adding the third mass makes the system chaotic.

CHAOTIC SYSTEM

This is a deterministic system that is difficult to predict. No general solution can be calculated as it is very sensitive to initial conditions.

PHYSICS BACKGROUND

NEWTON'S LAW OF GRAVITATION

$$\overline{F} = \frac{G m_1 m_2}{r^2} \hat{r}$$

EQUATION OF MOTION

First-order differential equation:

$$m.\frac{d\overline{v}}{dt} = \frac{G m_1 m_2}{r^3}.\overline{r}$$

Second-order differential equation:

$$m.\frac{d^2\overline{r}}{dt^2} = \frac{G m_1 m_2}{r^3}.\overline{r}$$

CENTRE OF MASS

$$\overline{r}_{com} = \frac{\sum_{i=1}^{n} m_i \overline{r}_i}{\sum_{i=1}^{n} m_i} \qquad \overline{v}_{com} = \frac{\sum_{i=1}^{n} m_i \overline{v}_i}{\sum_{i=1}^{n} m_i}$$

Modeling our Three-body system

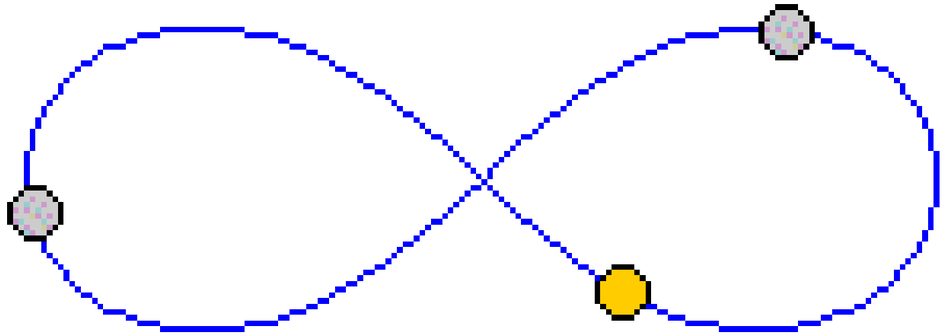
2 - D

OUR MODEL

- The three particles lie and move along a 2-D plane.
- They are under the influence of mutual graviation.
- There are no external forces acting upon the system
- The initial position and velocities of the masses forms the initial state of the sytem.

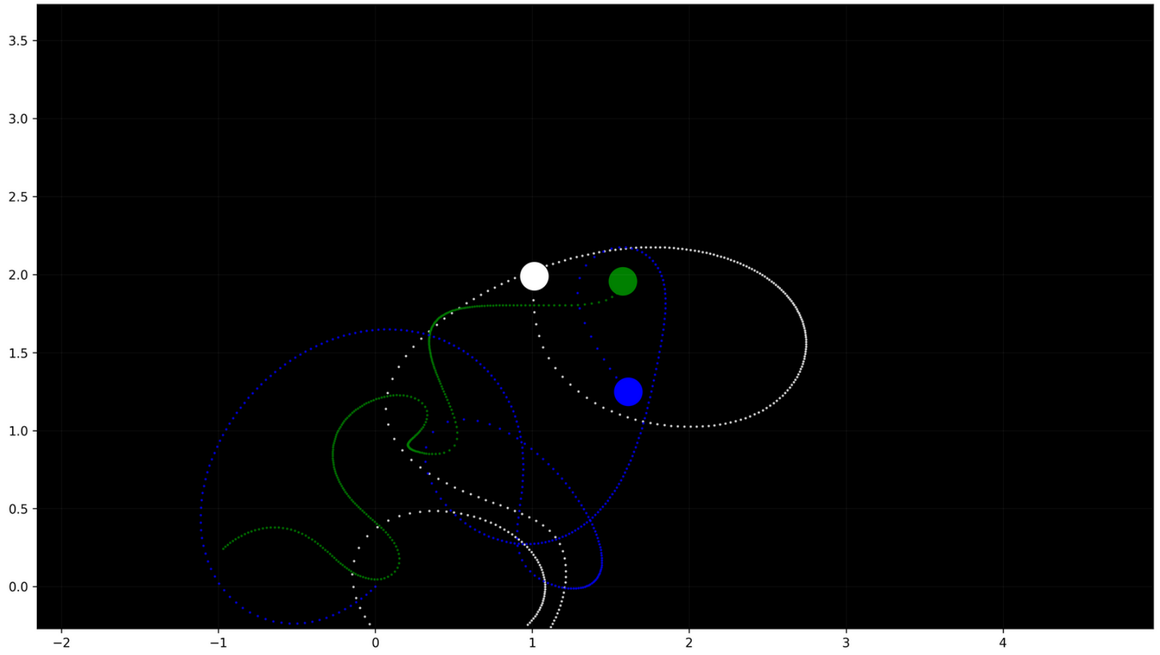
A Special Case: Figure 8

- All bodies have equal mass
- The system has remarkable symmetry
- The solution is a periodic trajectory
- The particles always form an equilateral triangle.



A special case

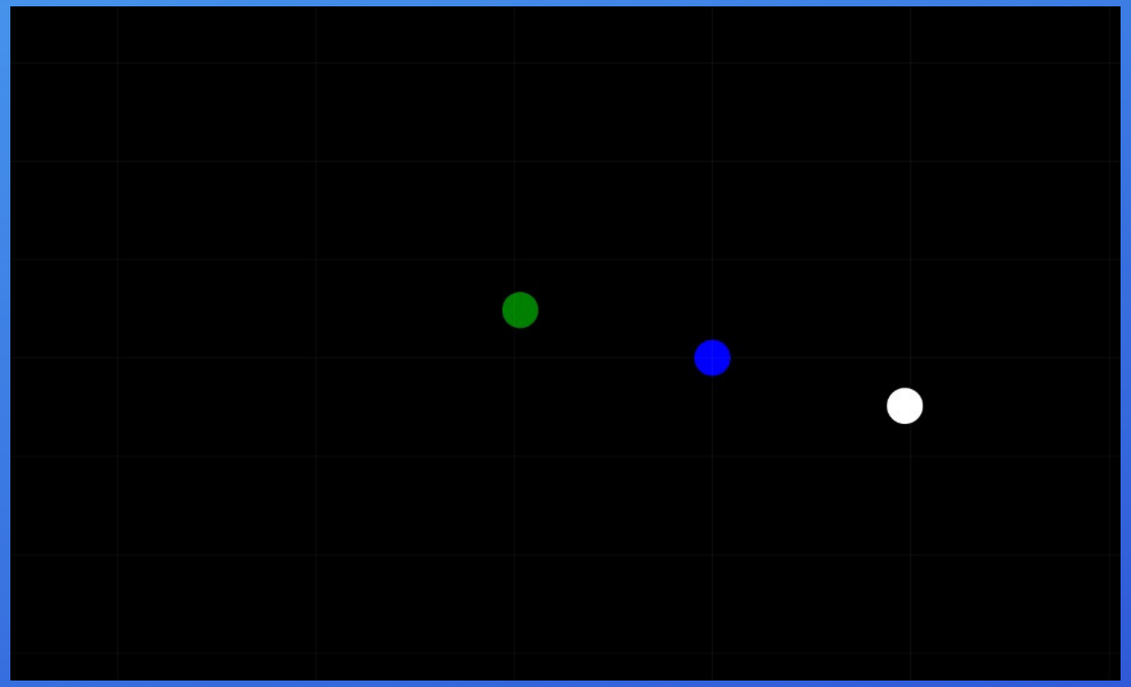
- The masses of the bodies are distinct.
- This system cannot be solved analytically.
- The future states are sensitive to the initial states.



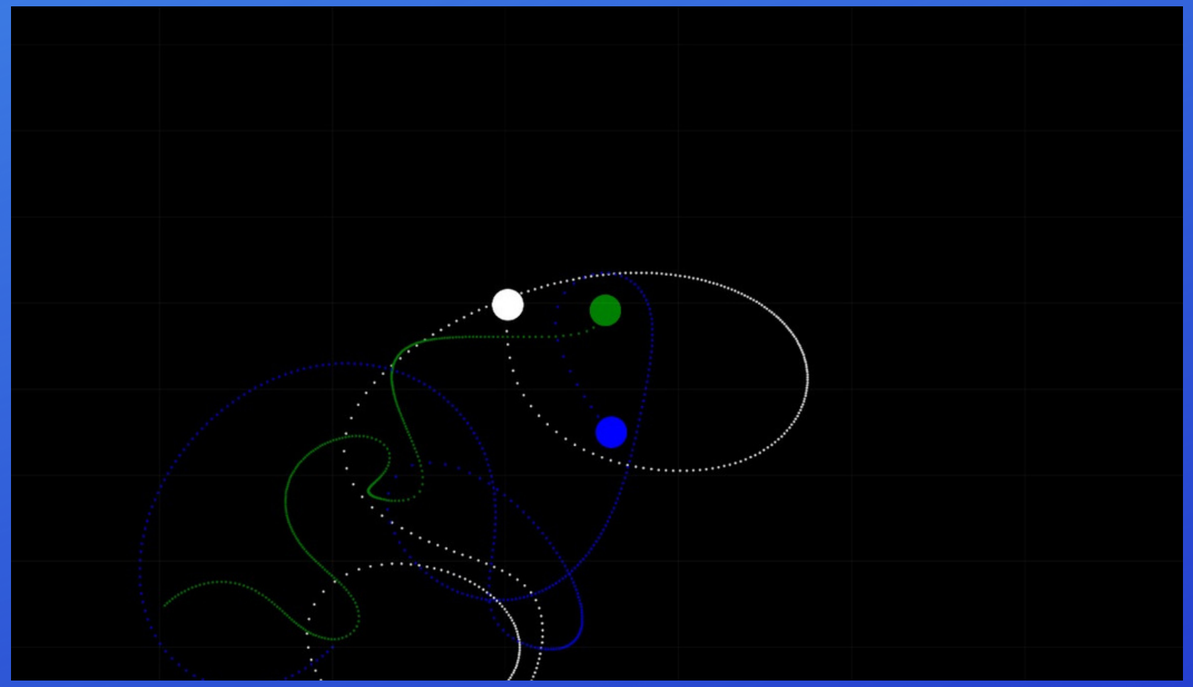
The general case

FOR A 2-D PLANE

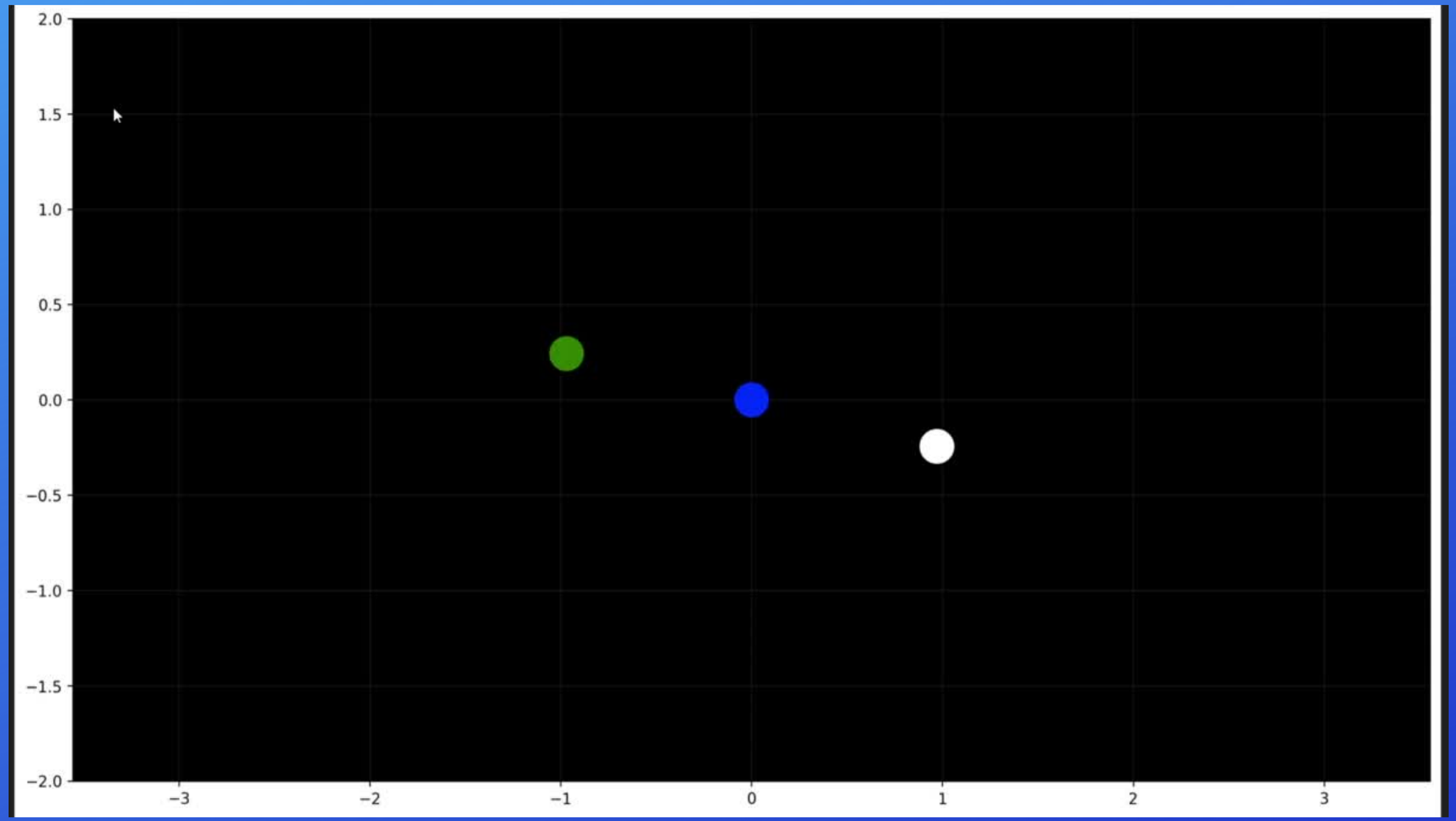
INITIAL STATE



FUTURE STATE



2-D SIMULATION

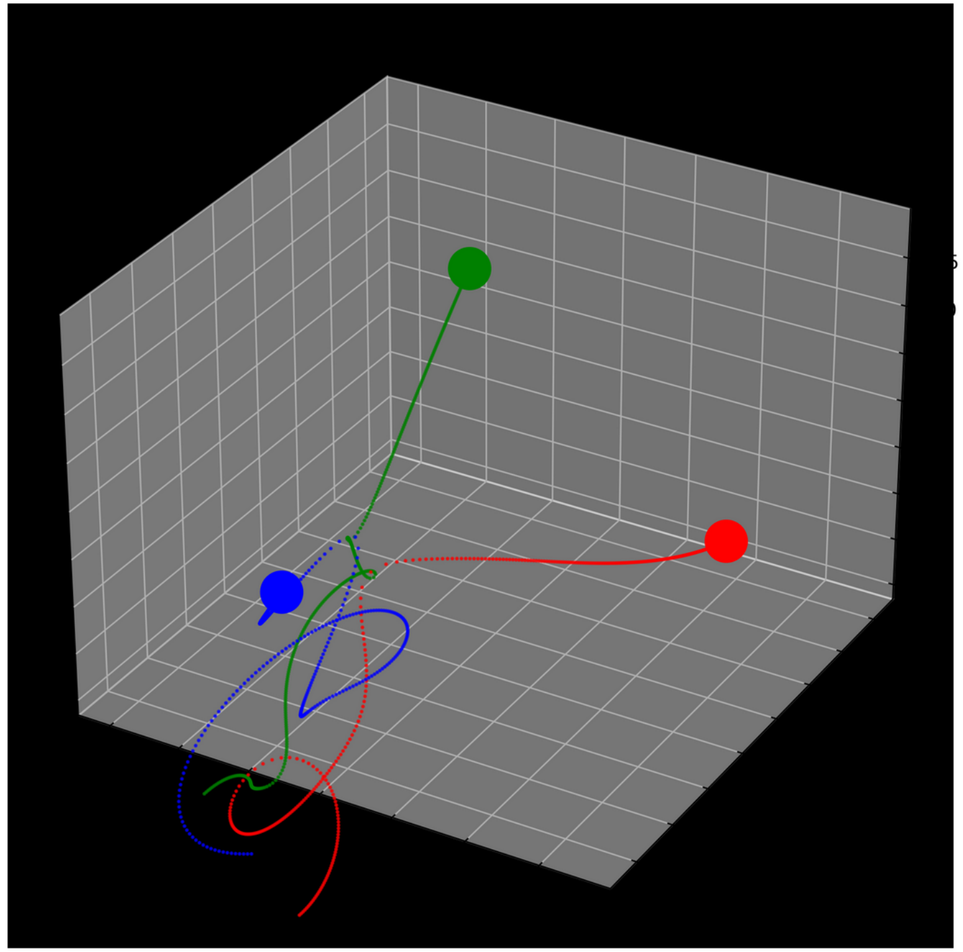


3- D

OUR MODEL

- The three particles lie and move in a 3D space.
- They are under the influence of mutual graviation.
- There are no external forces acting upon the system
- The initial position and velocities of the masses forms the initial state of the sytem.

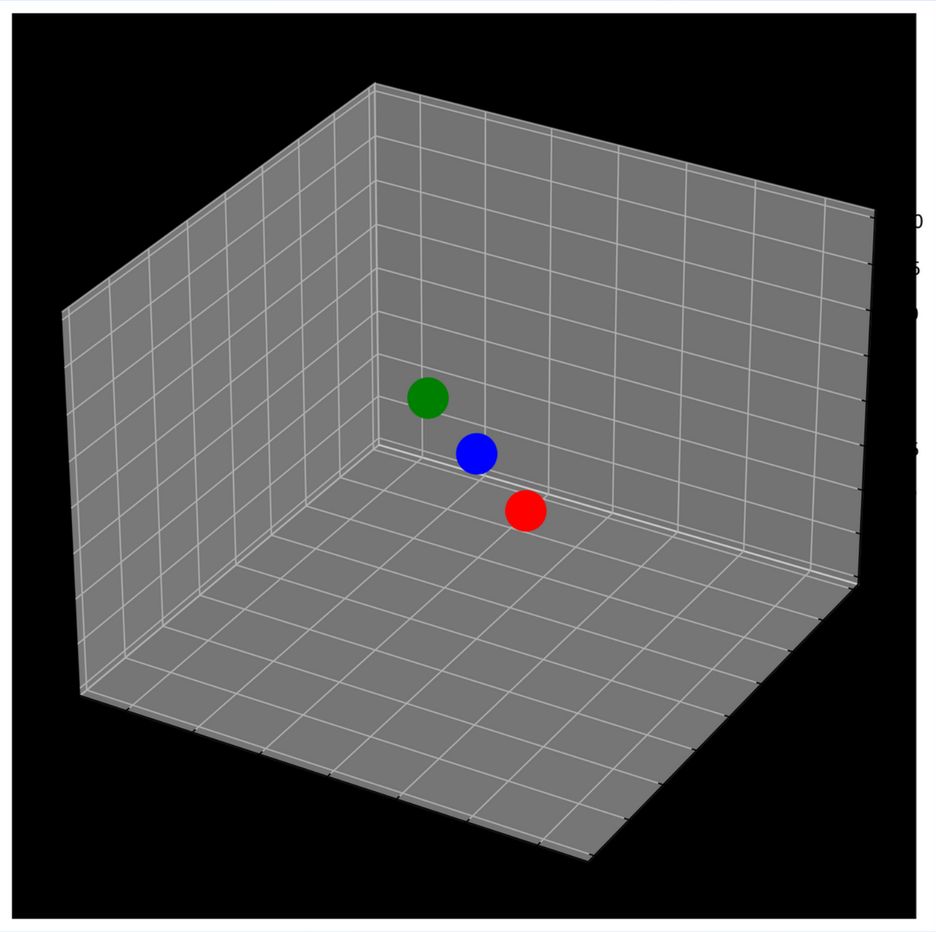
- This is the most chaotic version of the Three Body System.
- This system cannot be solved analytically and one must resort to numerical analysis
- Solved by the fourth order Runge-Kutta Method
- The future states are sensitive to the initial states.



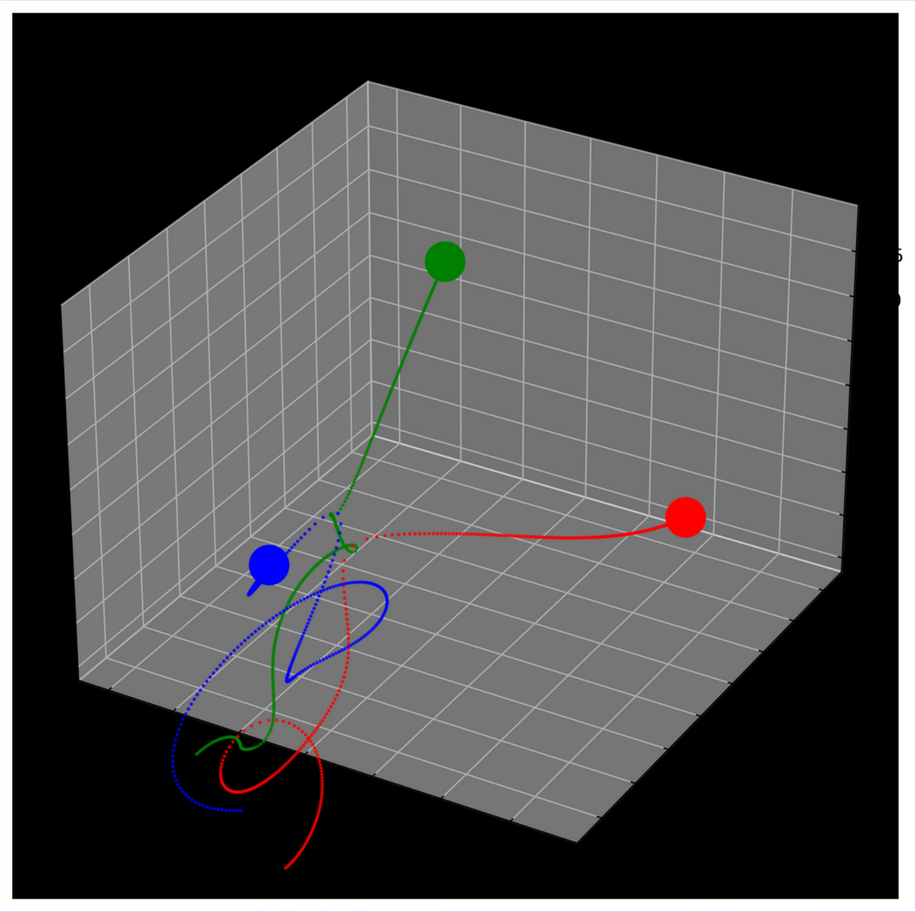
The 3D case

FOR A 3-D SPACE

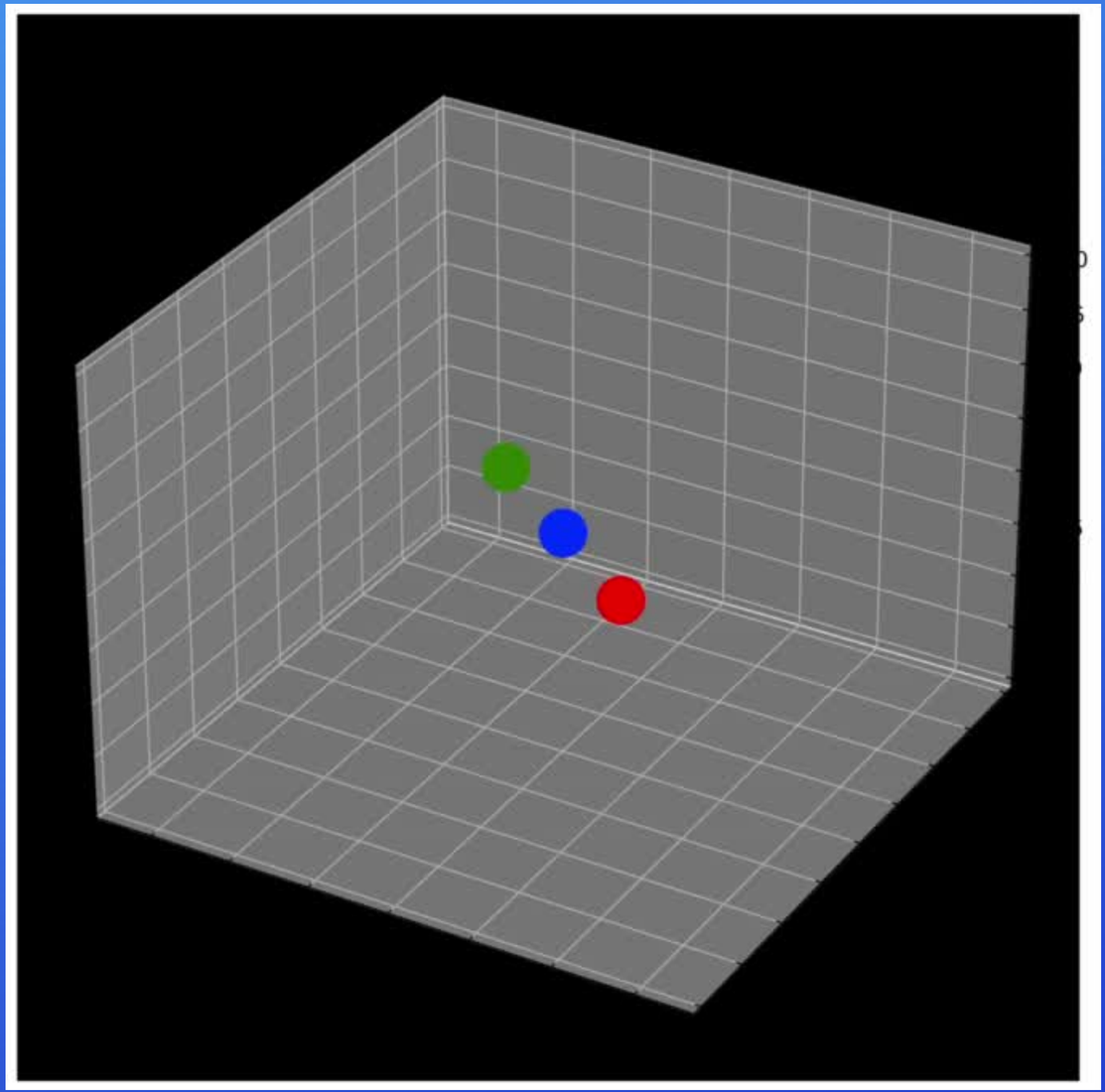
INITIAL STATE



FUTURE STATE



3-D SIMULATION



LINK TO CODE: https://github.com/saatvikrao/Three-Body-Problem-MA202/invitations

Runge-Kutta fourth order method

RUNGE-KUTTA FOURTH-ORDER METHOD

NUMERICAL METHOD

Solves first order Linear Ordinary Differential Equations

PRINCIPLE IDEA

Move from yi to yi+1 by multiplying some estimated slope by a timestep

TYPES OF EQUATIONS IT SOLVES

$dy / dx = f(x, y)$

INPUTS REQUIRED

TWO INPUTS ARE REQURIED

1. An ordinary differential equation that defines dy/dx in the form of a function of x and y.
2. Functional value of y(x) for given initial value of x.

SLOPES CALCULATED

The four slopes calculated at each iteration is given on the right side.

FINAL VALUE

The final value of the next iteration y(i+1) is calculated using the formula given on the bottom.

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

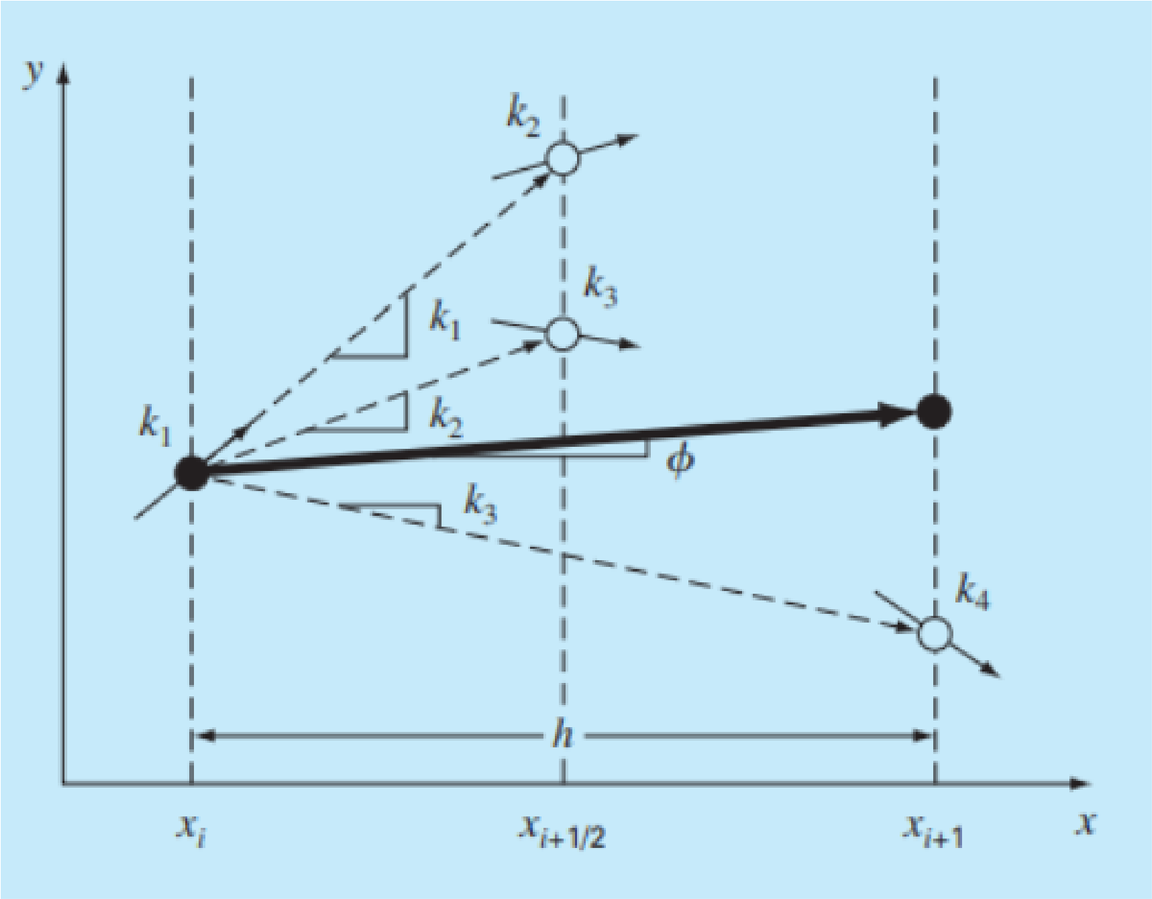
$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

FIGURE

Graph showing the points calculated in the previous slide



FUTURE SCOPE

We have modeled our system in a 2-D plane. However, the code and methodology can be further extended to cater to a three-body system in a 3-dimensional coordinate system