Q6)

A. Suppose y is an n-dimensional vector.

$y^T P y = y^T A^T A y$ as P = $A^T A$

$y^T P y = (Ay)^T A y$

Now, Ay is an m-dimensional vector and $(Ay)^T A y$ is simply the magnitude of that vector. So $y^T P y \geq 0$

Similarly, for Q, z will be an m-dimensional vector and $z^T Q z$ will be equal to the magnitude of Az.

If y is an eigenvector for P , Py = λy,

$y^T P y = \lambda \|y\|_2$. Since, both $y^T P y$ and $\|y\|_2$ are positive, λ must also be positive. (The same reasoning applies for the eigenvalues of Q.)

B. We know, Pu = λu u ∈ $R^{n \times 1}$

$A^T A u = \lambda u$

Now multiply both sides by A.

$(AA^T)Au = \lambda Au$

Also, Q = (AA^T)

Qv = λv where v = Au is the eigenvector of Q. v ∈ $R^{m \times 1}$

If, Qv = μv

Also, Q = AA^T

$AA^T v = \mu v$

Multiply both sides by $A^{T.}$

$(A^T A)A^T v = \mu A^T v$

Since, $A^T A$ = P, we can write above equation as,

$P(A^T v) = \mu(A^T v)$

We can see that $A^T v$ is the eigen vector of P with eigen value as μ.

Size of u = $R^{n \times 1}$. Hence, the elements in u are **n**.

Size of v = $R^{m \times 1}$. Hence, the elements in v are **m**.

C. We know, Q v_i = μ v_i .

Also, $u_i = \dfrac{A^T v_i}{\|A^T v_i\|_2}$. ….(1)

Q = AA^T

$A(A^T v_i) = \mu v_i$. ….(2)

Substituting value of ui from eq 1 in eq 2.

$A u_i \|A^T v_i\|_2 = \mu v_i$

$A u_i = \dfrac{\mu}{\|A^T v_i\|_2} v_i$ ….(3)

Let $\frac{\mu}{||A^T v_i||_2} = \lambda_i$ ….(4)

Using eq 3 and 4,

A u_i = λ_i v_i .

Hence, proved.

D. Consider the matrix $V^T V$. Since V is the matrix formed by column vectors u_i , the (i, j) th entry of $V^T V$ is given by $u_i^T u_j$ which is 0 whenever i $\neq$ j and 1 whenever i = j. So $V^T V$ is an identity matrix.

From above part we have, A u_i = λ_i v_i .

U Γ = [λ 1 v1 | λ 2 v2 | λ 3 v3 |...| λ m vm]

=[Au1 | Au2 | Au3 |...| Aum]

=A[u1 |u2 |u3 |...|um]

=AV

U Γ = AV

Multiply both sides by V^T .

U ΓV^T = A.

Hence, proved.