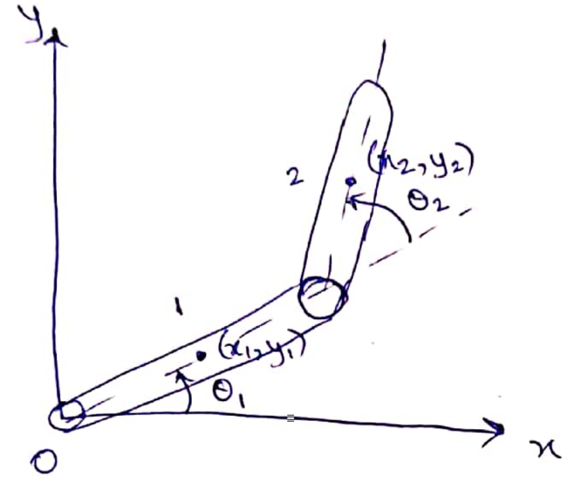
Lagrange's Equations :- $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = 0$

where $L = T - V \Rightarrow$ Lagrangian

T - kinetic energy

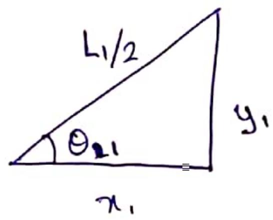
V - potential energy

Two - Link Planar Robot Manipulator :-



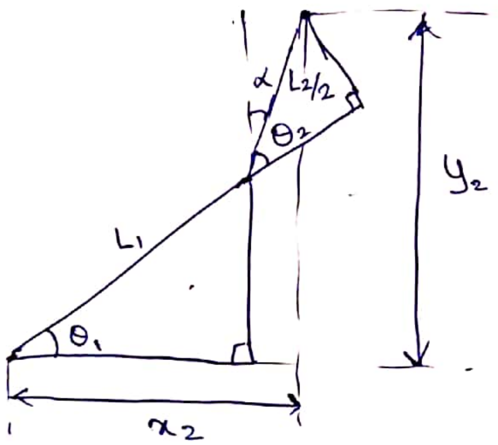
Masses - m_1, m_2

lengths - L_1, L_2



$x_1 = \dfrac{L_1}{2}\cos\theta_1$

$y_1 = \dfrac{L_1}{2}\sin\theta_1$



$\alpha = 90 - \theta_1 - \theta_2$

$x_2 = L_1\cos\theta_1 + \dfrac{L_2}{2}\sin\alpha$

$\Rightarrow x_2 = L_1\cos\theta_1 + \dfrac{L_2}{2}\cos(\theta_1+\theta_2)$

$y_2 = L_1\sin\theta_1 + \dfrac{L_2}{2}\cos\alpha$

$\Rightarrow y_2 = L_1\sin\theta_1 + \dfrac{L_2}{2}\sin(\theta_1+\theta_2)$

$V_1 = m_1 g y_1 = \dfrac{m_1 g L_1}{2}\sin\theta_1$

$V_2 = m_2 g y_2 = m_2 g\left(L_1\sin\theta_1 + \dfrac{L_2}{2}\sin(\theta_1+\theta_2)\right)$

$V = V_1 + V_2$

$\Rightarrow V = \dfrac{m_1 g L_1}{2}\sin\theta_1 + m_2 g\left(L_1\sin\theta_1 + \dfrac{L_2}{2}\sin(\theta_1+\theta_2)\right)$

$$T_1 = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} m_1 v_1^2 \quad , \quad T_2 = \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} m_2 v_2^2$$

$$\omega_1 = \dot{\theta}_1 \quad , \quad \omega_2 = \dot{\theta}_1 + \dot{\theta}_2 \qquad \{\dot{\theta} = d\theta/dt\}$$

$$I_1 = \frac{m_1 L_1^2}{12} \quad , \quad I_2 = \frac{m_2 L_2^2}{12}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow v^2 = \dot{x}^2 + \dot{y}^2$$

$$\therefore v_1^2 = \dot{x}_1^2 + \dot{y}_1^2$$

$$\Rightarrow v_1^2 = \frac{L_1^2 \dot{\theta}_1^2}{4}$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$

$$\dot{x}_2 = -L_1 \sin\theta_1 \cdot \dot{\theta}_1 - \frac{L_2}{2} \sin(\theta_1 + \theta_2) \cdot (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\dot{y}_2 = L_1 \cos\theta_1 \cdot \dot{\theta}_1 + \frac{L_2}{2} \cos(\theta_1 + \theta_2) \cdot (\dot{\theta}_1 + \dot{\theta}_2)$$

$$v_2^2 = L_1^2 \dot{\theta}_1^2 + \frac{L_2^2}{4} (\dot{\theta}_1 + \dot{\theta}_2)^2 + L_1 L_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2$$

$$\Rightarrow v_2^2 = \dot{\theta}_1^2 \left(L_1^2 + \frac{L_2^2}{4} + L_1 L_2 \cos\theta_2 \right) + \dot{\theta}_2 \left(\frac{L_2^2}{4} \right) + \dot{\theta}_1 \dot{\theta}_2 \left(\frac{L_2^2}{2} + L_1 L_2 \cos\theta_2 \right)$$

$$\therefore T_1 = \frac{1}{2} \frac{m_1 L_1^2}{12} \dot{\theta}_1^2 + \frac{1}{2} m_1 \frac{L_1^2}{4} \dot{\theta}_1^2$$

$$\Rightarrow T_1 = \frac{m_1 L_1^2 \dot{\theta}_1^2}{6}$$

$$T_2 = \frac{1}{2} \frac{m_2 L_2^2}{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 v_2^2$$

$$T_2 = \frac{1}{2} m_2 \left(v_2^2 + \frac{L_2^2}{12} \dot{\theta}_1^2 + \frac{L_2^2}{12} \dot{\theta}_2^2 + \frac{L_2^2}{6} \dot{\theta}_1 \dot{\theta}_2 \right)$$

$$\Rightarrow T_2 = \dot{\theta}_1^2\left(\frac{m_2L_1^2}{2} + \frac{m_2L_2^2}{6} + \frac{m_2L_1L_2}{2}\cos\theta_2\right) + \dot{\theta}_2^2\left(\frac{m_2L_2^2}{6}\right) + \dot{\theta}_1\dot{\theta}_2\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2\right)$$

③

$$\therefore T = T_1 + T_2$$

$$\Rightarrow T = \dot{\theta}_1^2\left(\frac{m_1L_1^2}{6} + \frac{m_2L_2^2}{6} + \frac{m_2L_1^2}{2} + \frac{m_2L_1L_2}{2}\cos\theta_2\right) + \dot{\theta}_2^2\left(\frac{m_2L_2^2}{6}\right) + \dot{\theta}_1\dot{\theta}_2\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2\right)$$

$$\therefore L = T - V$$

$$\Rightarrow L = \dot{\theta}_1^2\left(\frac{m_1L_1^2}{6} + \frac{m_2L_2^2}{6} + \frac{m_2L_1^2}{2} + \frac{m_2L_1L_2}{2}\cos\theta_2\right) + \dot{\theta}_2^2\left(\frac{m_2L_2^2}{6}\right) + \dot{\theta}_1\dot{\theta}_2\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2\right) - \frac{m_1gL_1}{2}\sin\theta_1 - m_2g\left(L_1\sin\theta_1 + \frac{L_2}{2}\sin(\theta_1+\theta_2)\right)$$

Lagrange's Equations are -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

~~Use chain rule -~~ $\cancel{\frac{dX}{dt} = \frac{\partial X}{\partial \theta_i}\cdot\frac{d\theta_i}{dt} + \frac{\partial X}{\partial \dot{\theta}_i}\cdot\frac{d\dot{\theta}_i}{dt}}$

$$\frac{\partial L}{\partial \dot\theta_1} = \dot\theta_1\left(\frac{m_1L_1^2}{3} + \frac{m_2L_2^2}{3} + m_2L_1^2 + m_2L_1L_2\cos\theta_2\right) + \dot\theta_2\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2\cos\theta_2}{2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = \left(\frac{m_1L_1^2}{3} + \frac{m_2L_2^2}{3} + m_2L_1^2 + m_2L_1L_2\cos\theta_2\right)\ddot\theta_1 - m_2L_1L_2\sin\theta_2\,\dot\theta_1\dot\theta_2$$

$$+ \ddot\theta_2\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2\right) - \frac{m_2L_1L_2}{2}\sin\theta_2 \cdot \dot\theta_2^2$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{m_1gL_1}{2}\cos\theta_1 - m_2g\left(L_1\cos\theta_1 + \frac{L_2}{2}\cos(\theta_1+\theta_2)\right)$$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\Rightarrow \ddot\theta_1\left(\frac{m_1L_1^2}{3} + \frac{m_2L_2^2}{3} + \overset{c_1}{m_2L_1^2} + m_2L_1L_2\cos\theta_2\right) + \ddot\theta_2\left(\overset{c_2}{\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2}\right)$$

$$-\overset{2c_3}{(m_2L_1L_2\sin\theta_2)}\,\dot\theta_1\dot\theta_2 - \overset{c_3}{\left(\frac{m_2L_1L_2}{2}\sin\theta_2\right)}\dot\theta_2^2 + \overset{c_4}{\left(\frac{m_1}{2} + m_2\right)}gL_1\cos\theta_1$$

$$+ \overset{c_5}{\frac{m_2gL_2}{2}}\cos(\theta_1+\theta_2) = 0$$

$\rightarrow$ (1)

$$\frac{\partial L}{\partial \dot\theta_2} = \dot\theta_2\left(\frac{m_2L_2^2}{3}\right) + \dot\theta_1\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = \ddot\theta_2\left(\frac{m_2L_2^2}{3}\right) + \ddot\theta_1\left(\frac{m_2L_2^2}{3} + \frac{m_2L_1L_2}{2}\cos\theta_2\right)$$

$$-\left(\frac{m_2L_1L_2}{2}\sin\theta_2\right)\dot\theta_1\dot\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -\left(\frac{m_2L_1L_2}{2}\sin\theta_2\right)\ddot\theta_1^2 - \left(\frac{m_2L_1L_2}{2}\sin\theta_2\right)\dot\theta_1\dot\theta_2$$

$$-\frac{m_2gL_2}{2}\cos(\theta_1+\theta_2)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\Rightarrow \ddot{\theta}_1\left(\frac{m_2 L_2^2}{3} + \overset{c_2}{\frac{m_2 L_1 L_2}{2}\cos\theta_2}\right) + \ddot{\theta}_2\overset{c_6}{\left(\frac{m_2 L_2^2}{3}\right)} + \overset{-c_3}{\left(\frac{m_2 L_1 L_2 \sin\theta_2}{2}\right)}\dot{\theta}_1^2$$

$$+ \overset{c_5}{\frac{m_2 g L_2}{2}\cos(\theta_1+\theta_2)} = 0 \quad \longrightarrow (2)$$

$$\therefore \text{Eq}^n\ (1) \Rightarrow c_1\ddot{\theta}_1 + c_2\ddot{\theta}_2 + 2c_3\dot{\theta}_1\dot{\theta}_2 + c_3\dot{\theta}_2^2 + c_4 + c_5 = 0$$

$$\text{Eq}^n\ (2) \Rightarrow c_2\ddot{\theta}_1 + c_6\ddot{\theta}_2 - c_3\dot{\theta}_1^2 + c_5 = 0$$

$$\text{from } (2),\quad \ddot{\theta}_2 = \frac{-c_2\ddot{\theta}_1 + c_3\dot{\theta}_1^2 - c_5}{c_6}$$

$$\text{from } (1),\quad \ddot{\theta}_1 = \frac{-c_2\ddot{\theta}_2 - 2c_3\dot{\theta}_1\dot{\theta}_2 - c_3\dot{\theta}_2^2 - c_4 - c_5}{c_1}$$

Substitute $\ddot{\theta}_2$ in eq^n (1),

$$c_1\ddot{\theta}_1 + c_2\left(\frac{-c_2\ddot{\theta}_1 + c_3\dot{\theta}_1^2 - c_5}{c_6}\right) + 2c_3\dot{\theta}_1\dot{\theta}_2 + c_3\dot{\theta}_2^2 + c_4 + c_5 = 0$$

$$(c_1c_6 - c_2^2)\ddot{\theta}_1 + c_2c_3\dot{\theta}_1^2 + 2c_3c_6\dot{\theta}_1\dot{\theta}_2 + c_3c_6\dot{\theta}_2^2 + c_4c_6 + c_5c_6 - c_2c_5 = 0$$

$$\therefore \ddot{\theta}_1 = \frac{-c_2c_3\dot{\theta}_1^2 - 2c_3c_6\dot{\theta}_1\dot{\theta}_2 - c_3c_6\dot{\theta}_2^2 + c_2c_5 - c_4c_6 - c_5c_6}{c_1c_6 - c_2^2} \quad \longrightarrow (3)$$

Substitute $\ddot{\theta}_1$ in eq^n 2,

$$C_2\left(\frac{-C_2\ddot{\theta}_2 - 2C_3\dot{\theta}_1\dot{\theta}_2 - C_3\dot{\theta}_2^2 - C_4 - C_5}{C_1}\right) + C_6\ddot{\theta}_2 - C_3\dot{\theta}_1^2 + C_5 = 0$$

$$(C_1C_6 - C_2^2)\ddot{\theta}_2 - C_1C_3\dot{\theta}_1^2 - 2C_2C_3\dot{\theta}_1\dot{\theta}_2 - C_2C_3\dot{\theta}_2^2 + C_1C_5 - C_2C_4 - C_2C_5 = 0$$

$$\Rightarrow \ddot{\theta}_2 = \frac{C_1C_3\dot{\theta}_1^2 + 2C_2C_3\dot{\theta}_1\dot{\theta}_2 + C_2C_3\dot{\theta}_2^2 + C_2C_4 + C_2C_5 - C_1C_5}{C_1C_6 - C_2^2} \longrightarrow 4$$

Let $\dot{\theta}_1 = \omega_1$

$\dot{\theta}_2 = \omega_2$

Energy Conservation :-

$T + V =$ constant

For animation,

$x_1' = l_1 \times \cos\theta_1$

$y_1' = l_1 \times \sin\theta_1$

$x_2' = l_1 \times \cos\theta_1 + l_2 \times \cos(\theta_1 + \theta_2)$

$y_2' = L_1 \times \sin\theta_1 + l_2 \times \sin(\theta_1 + \theta_2)$

Task 3 - To Follow a Certain Trajectory (& Task 2) ⑦
↳ Torque

Torque $\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \hat{\theta}_i}$

$$\therefore \tau_1 = c_1\ddot{\theta}_1 + c_2\ddot{\theta}_2 + 2c_3\dot{\theta}_1\dot{\theta}_2 + c_3\dot{\theta}_2^2 + c_4 + c_5$$
$$\tau_2 = c_2\ddot{\theta}_1 + c_6\ddot{\theta}_2 - c_3\dot{\theta}_1^2 + c_5$$

Thus, the eqⁿs are -

$$c_1\ddot{\theta}_1 + c_2\ddot{\theta}_2 + 2c_3\dot{\theta}_1\dot{\theta}_2 + c_3\dot{\theta}_2^2 + c_4 + c_5 - \tau_1 = 0 \longrightarrow (5)$$
$$c_2\ddot{\theta}_1 + c_6\ddot{\theta}_2 - c_3\dot{\theta}_1^2 + c_5 - \tau_2 = 0 \longrightarrow (6)$$

From (6), substitute $\ddot{\theta}_2$ in (5)

from (5), substitute $\ddot{\theta}_1$ in (6)

$$\ddot{\theta}_2 = \frac{-c_2\ddot{\theta}_1 + c_3\dot{\theta}_1^2 - c_5 + \tau_2}{c_6}$$

$$\ddot{\theta}_1 = \frac{-c_2\ddot{\theta}_2 - 2c_3\dot{\theta}_1\dot{\theta}_2 - c_3\dot{\theta}_2^2 - c_4 - c_5 + \tau_1}{c_1}$$

$$\therefore \ddot{\theta}_1 = \frac{-c_2c_3\dot{\theta}_1^2 - 2c_3c_6\dot{\theta}_1\dot{\theta}_2 - c_3c_6\dot{\theta}_2^2 + c_2c_5 - c_4c_6 - c_5c_6 + c_6\tau_1 - c_2\tau_2}{c_1c_6 - c_2^2}$$

$$\therefore \ddot{\theta}_2 = \frac{c_1c_3\dot{\theta}_1^2 + 2c_2c_3\dot{\theta}_1\dot{\theta}_2 + c_2c_3\dot{\theta}_2^2 + c_2c_4 + c_2c_5 - c_1c_5 - c_2\tau_1 + c_1\tau_2}{c_1c_6 - c_2^2}$$

Put $\dot{\theta}_1 = \omega_1$

$\dot{\theta}_2 = \omega_2$

Energy

1) Total Energy = KE + PE

2) Total Energy = $\int P\,dt = \int (\tau \cdot \omega)\,dt \quad \{= \int \tau\,d\theta\}$