SteepestDescent-Example-Solutions

Question 1

$$f(x,y) = \frac{1}{2}(cx^2 + y^2)$$

1. Show that the exact steepest descent is given by the equations:

$$x = (1 - ac)x, \qquad y = (1 - a)y,$$

where

$$a = \frac{c^2 + m^2}{c^3 + m^2}, \qquad m = \frac{y}{x}$$

Proof:

$$\frac{\partial f(x,y)}{\partial x} = cx, \quad \frac{\partial f(x,y)}{\partial y} = y, \quad \nabla f(x,y) = \begin{pmatrix} cx \\ y \end{pmatrix}, \quad r = \begin{pmatrix} -cx \\ -y \end{pmatrix},$$

$$\phi(t) = f(x - tcx, y - ty) = \frac{1}{2}(cx^2(1 - tc)^2 + y^2(1 - t)^2)$$

$$\phi'(t) = -c^2x^2(1 - tc) - y^2(1 - t) = t(c^3x^2 + y^2) - (c^2x^2 + y^2)$$

$$\phi'(t) = 0 \quad \Rightarrow \quad t = \frac{c^2x^2 + y^2}{c^3x^2 + y^2} = \frac{c^2 + m^2}{c^3 + m^2}, \quad \text{where } m = \frac{y}{x}$$

Therefore, starting with the initial guess x_0, y_0, the kth iteration of the exact steepest descent is given by:

$$x_k = (1 - a_k c)x_k, \qquad y_k = (1 - a_k)y_k,$$

where

$$a_k = \frac{c^2 + m_k^2}{c^3 + m_k^2}, \qquad m_k = \frac{y_k}{x_k}$$

2. Show that if we start with $x = 1, y = c$, then at the k'th iteration $a_k = 2/(1 + c)$ and

$$x_k = \left(\frac{1 - c}{1 + c}\right)^k, \qquad y_k = (-1)^k c \left(\frac{1 - c}{1 + c}\right)^k$$

Proof: Hint: use induction on k.

3. Use the results of the previous question to estimate the number of iterations required to obtain $|x| \leq 10^{-6}, |y| \leq 10^{-6}$, for $c = 0.001$.

Answer: Notice that for $c < 1$ we have $|x_k| > |y_k|$, so that it is enough to compute the number of iterations that achieves $x_k < 10^{-6}$. We have to solve the equation for the unknown k:

$$x_k = \left(\frac{1 - c}{1 + c}\right)^k \leq 10^{-6} \quad \Rightarrow k \log \frac{1 - c}{1 + c} \leq -6 \log 10 \quad \Rightarrow k \geq \frac{6 \log 10}{\log \frac{1+c}{1-c}} = 6907.7 \quad \Rightarrow k = 6908$$

4. Write a program that implements a variant of the exact steepest descent algorithm in which x is updated by the rule:

$$x = x + \beta a r$$

where a is as given by the exact steepest descent algorithm and $\beta > 0$.

Write the program for minimizing $f(x, y)$ with $c = 0.001$, and output the number of iterations needed in order to obtain $|x| \leq 10^{-6}, |y| \leq 10^{-6}$. Assume that the program always starts with $x = 1, y = c$. Choose $\beta = 1$ and verify the results of the previous problem. How many iterations are needed for the following values of β: $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1$?

Answer:

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
iterations	2272	1843	1227	892	504	565	435	357	246	6908	6908

5. Update your program for the previous question to compute the number of iterations when in each iteration β is chosen at random from the interval $[0, 1]$.

Answer: usually around 1000 iterations.

6. Implement the $\epsilon - step$ steepest descent algorithm for the same function. What value of ϵ is needed in order to get accuracy as above? How many iterations are needed?

Answer:

ϵ	0.1	0.5	0.9	1.5	1.99	1.999	1.9995
iterations	138149	27625	15344	9204	6936	6905	13815

These experiments give $\epsilon = 1.999$ as the best in our case.