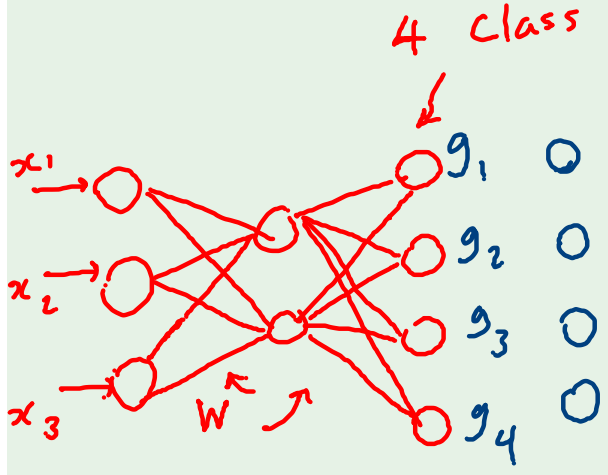
Applying SGD

All the weights $\mathbf{w} = \{w_{ij}^{(l)}\}$ determine $h(\mathbf{x})$

Applying SGD



$\frac{e^{g_1}}{\sum_{i=1}^{4} e^{g_i}} = h_1$ $= h_2$ $= h_3$ $= h_4$ $\rightarrow \sum_{j=1}^{4} h_j = 1$

All the weights $\mathbf{w} = \{w_{ij}^{(l)}\}$ determine $h(\mathbf{x})$

Error on example $(\mathbf{x}_n, y_n)$ is

$$\mathbf{e}\left(h(\mathbf{x}_n), y_n\right) = \mathbf{e}(\mathbf{w})$$

Reg. ← Sq. Err. ← Cross Ent: Multi-Class

$h(\mathbf{x}_n)$: predicted output; y_n: true output

$\min_{w} -\log h_3$ $\quad y_1 = 3$ $\quad y_2 = 1$ $\quad y_5 = 4$

$$\min_{w} \; -\log h_3^{(x_1)} - \log h_1^{(x_2)} \cdots - \log h_4^{(x_5)} - \log h_{y_n}(x_n)$$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{L(w)}$

Applying SGD

All the weights $\mathbf{w} = \{w_{ij}^{(l)}\}$ determine $h(\mathbf{x})$

Error on example $(\mathbf{x}_n, y_n)$ is

$$\mathbf{e}\left(h(\mathbf{x}_n), y_n\right) = \mathbf{e}(\mathbf{w})$$

To implement SGD, we need the gradient

$\nabla_{\mathbf{w}} \mathbf{e}(\mathbf{w})$:

Applying SGD

All the weights $\mathbf{w} = \{w_{ij}^{(l)}\}$ determine $h(\mathbf{x})$

Error on example $(\mathbf{x}_n, y_n)$ is

$$\mathbf{e}\left(h(\mathbf{x}_n), y_n\right) = \mathbf{e}(\mathbf{w})$$

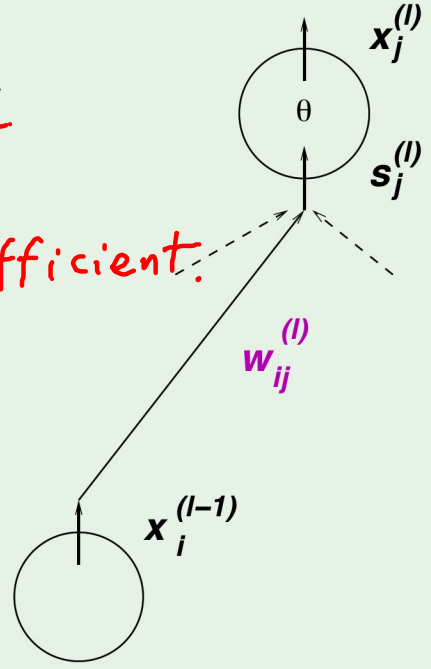
To implement SGD, we need the gradient

$$\nabla \mathbf{e}(\mathbf{w}): \quad \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, w_{ij}^{(l)}} \quad \text{for all } i, j, l$$

Computing $\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$

We can evaluate $\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$ one by one: analytically or numerically

not efficient.



$x_j^{(l)} = \theta(s_j^{(l)})$

Computing $\dfrac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$

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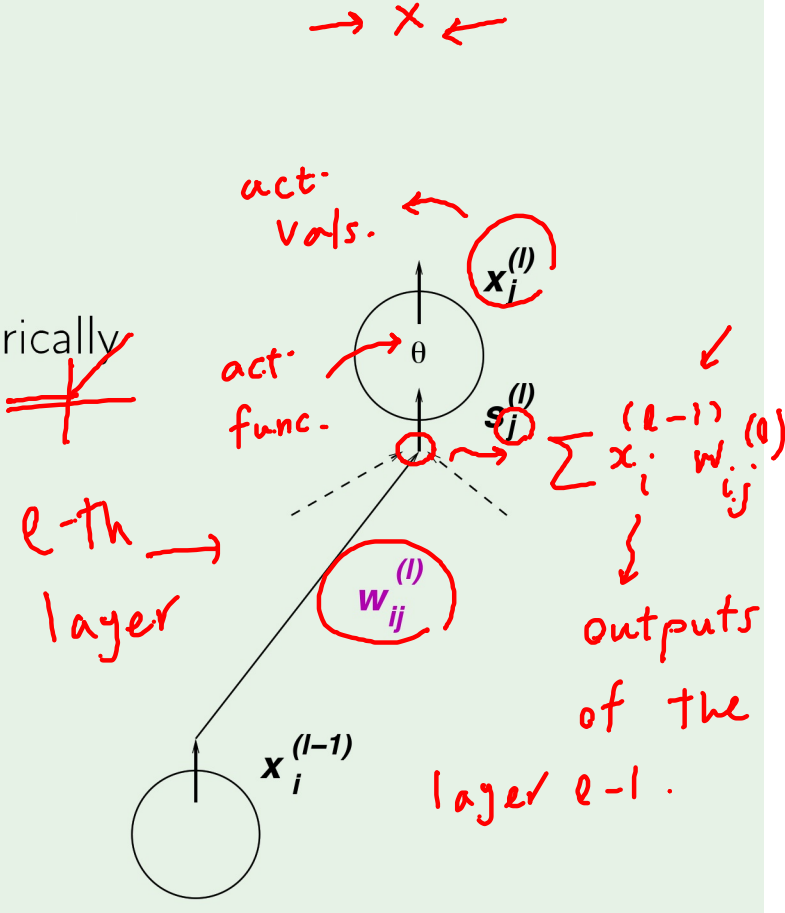
A trick for efficient computation:

$$\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}} = \frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ s_j^{(l)}} \times \frac{\partial\ s_j^{(l)}}{\partial\ w_{ij}^{(l)}}$$

$x_i^{(l-1)}$



l-th layer

outputs of the layer $l-1$.

Computing $\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$

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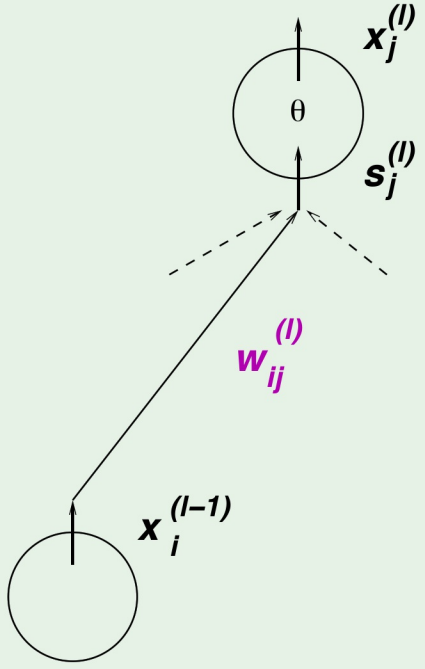
We have $\frac{\partial\ s_j^{(l)}}{\partial\ w_{ij}^{(l)}} = x_i^{(l-1)}$



Computing $\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$

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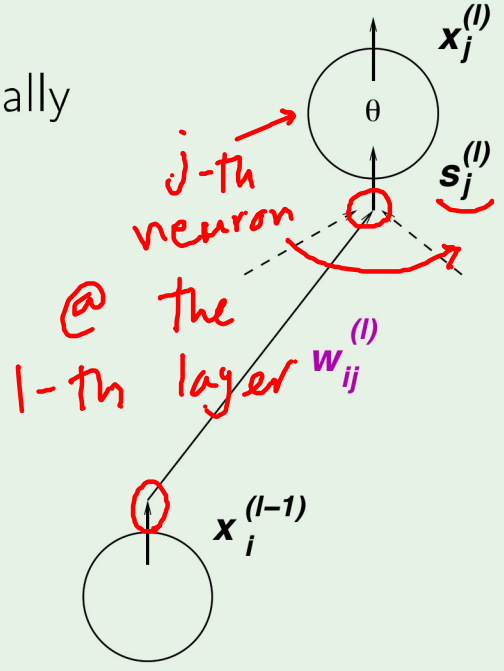
We have $\frac{\partial\ s_j^{(l)}}{\partial\ w_{ij}^{(l)}} = x_i^{(l-1)}$ We only need: $\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ s_j^{(l)}} = \delta_j^{(l)}$



δ for the final layer

$$\delta_j^{(l)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_j^{(l)}}$$

For the final layer $l = L$ and $j = 1$:

δ for the final layer

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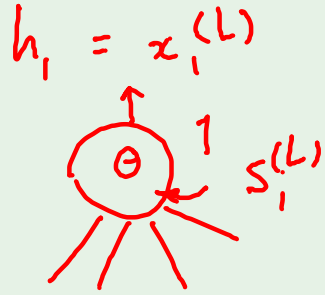
$$\delta_1^{(L)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_1^{(L)}}$$

$$\mathbf{e}(\mathbf{w}) = (x_1^{(L)} - y_n)^2$$

y_n: 1 or -1

$h_1 = \theta(s_1^{(L)})$



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$$\mathbf{e}(\mathbf{w}) = (x_1^{(L)} - y_n)^2$$

$$x_1^{(L)} = \theta(s_1^{(L)})$$

$$\theta'(s) = 1 - \theta^2(s) \quad \text{for the tanh}$$

$$\frac{\partial\, e(w)}{\partial\, s_1^{(L)}} = 2(x_1^{(L)} - y_n) \cdot \theta'(s_1^{(L)})$$

$$\theta'(s_1^{(L)}) = 1 - \theta^2(s_1^{(L)})$$

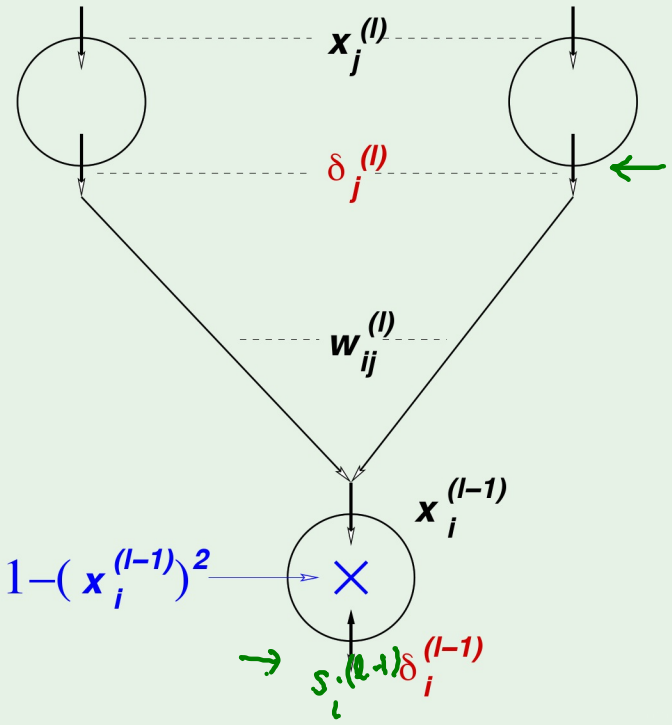
$\forall j \quad \delta_j^{(L)}$

$\forall i \quad ?= \delta_i^{(L-1)}$

base of the induction

Back propagation of δ

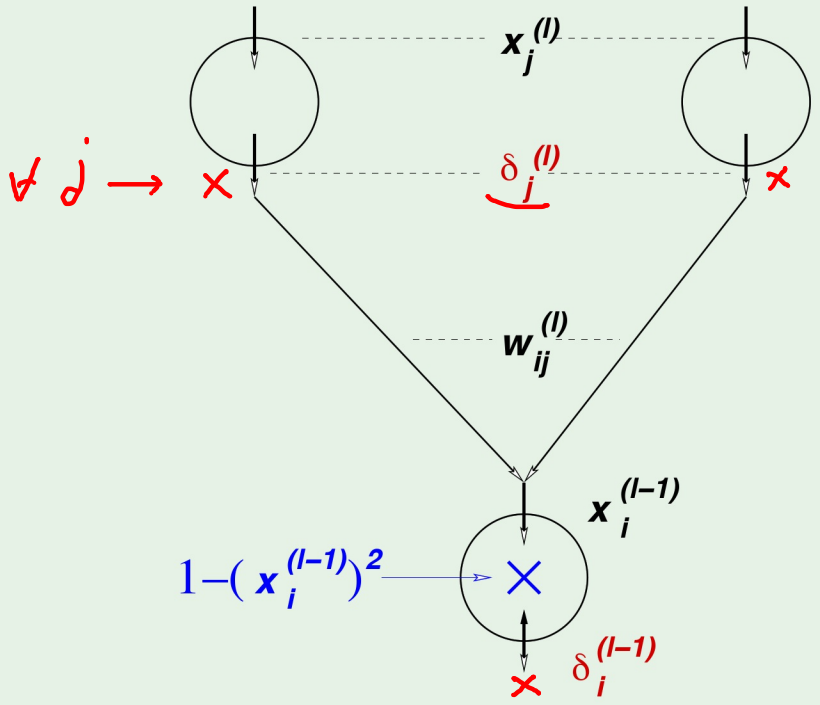
$$\delta_i^{(l-1)} = \frac{\partial\, \mathrm{e}(\mathbf{w})}{\partial\, s_i^{(l-1)}}$$



Back propagation of δ

$$\delta_i^{(l-1)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_i^{(l-1)}}$$

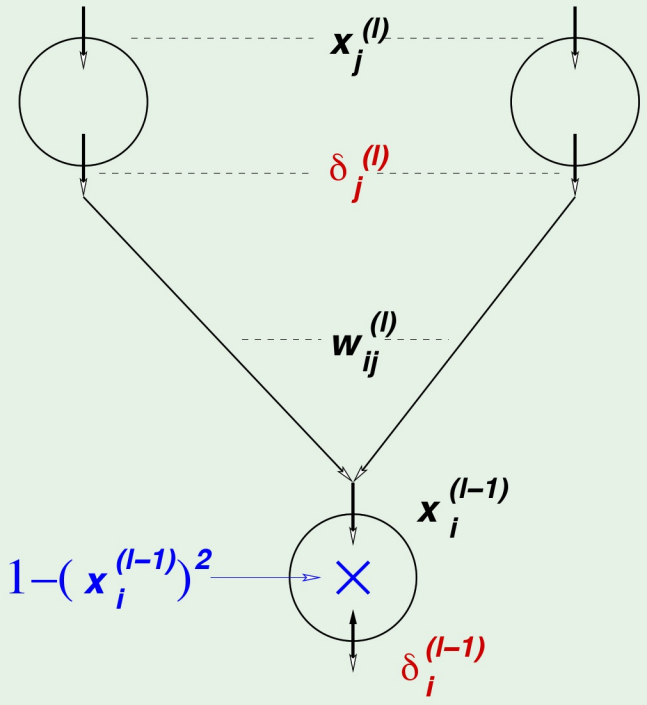
$$= \sum_j \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_j^{(l)}} \times \frac{\partial\, s_j^{(l)}}{\partial\, x_i^{(l-1)}} \times \frac{\partial\, x_i^{(l-1)}}{\partial\, s_i^{(l-1)}}$$



Back propagation of δ

$$\delta_i^{(l-1)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_i^{(l-1)}}$$

$$= \sum_{j=1}^{d^{(l)}} \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_j^{(l)}} \times \frac{\partial\, s_j^{(l)}}{\partial\, x_i^{(l-1)}} \times \frac{\partial\, x_i^{(l-1)}}{\partial\, s_i^{(l-1)}}$$



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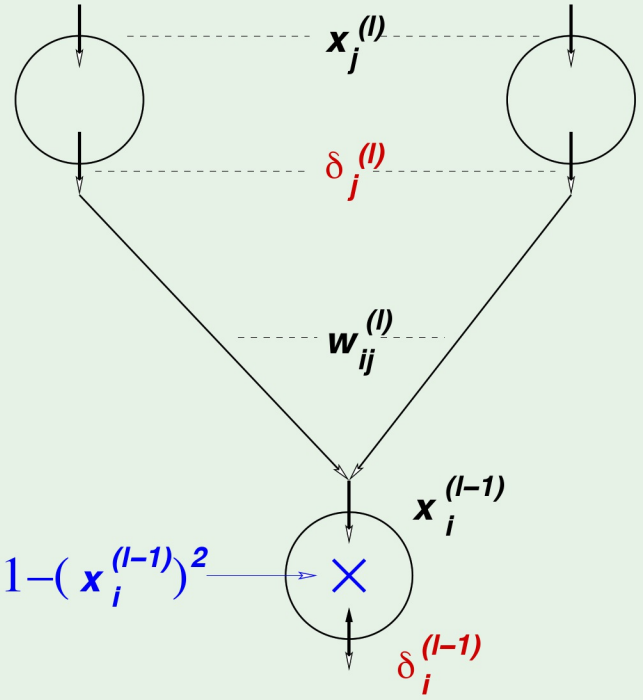
$$= \sum_{j=1}^{d^{(l)}} \delta_j^{(l)} \times w_{ij}^{(l)} \times \theta'(s_i^{(l-1)})$$

$\sum w_{ij}^{(l)}\, x_i^{(l-1)}$

$\theta(s_i^{(l-1)})$



Vanishing Grad.

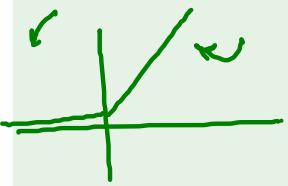
Back propagation of δ

$\delta_1^{(L)} \leftarrow \delta_j^{(L-1)} \leftarrow \delta_i^{(L-2)} \leftarrow \cdots \leftarrow \delta_i^{(1)}$

$$\delta_i^{(l-1)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_i^{(l-1)}}$$

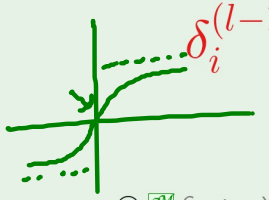
$$= \sum_{j=1}^{d^{(l)}} \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_j^{(l)}} \times \frac{\partial\, s_j^{(l)}}{\partial\, x_i^{(l-1)}} \times \frac{\partial\, x_i^{(l-1)}}{\partial\, s_i^{(l-1)}}$$

$$= \sum_{j=1}^{d^{(l)}} \delta_j^{(l)} \times w_{ij}^{(l)} \times \theta'(s_i^{(l-1)})$$



$\theta'(s_i^{(l-1)}) < 1$

$$\delta_i^{(l-1)} = (1 - (x_i^{(l-1)})^2) \sum_{j=1}^{d^{(l)}} w_{ij}^{(l)}\, \delta_j^{(l)}$$



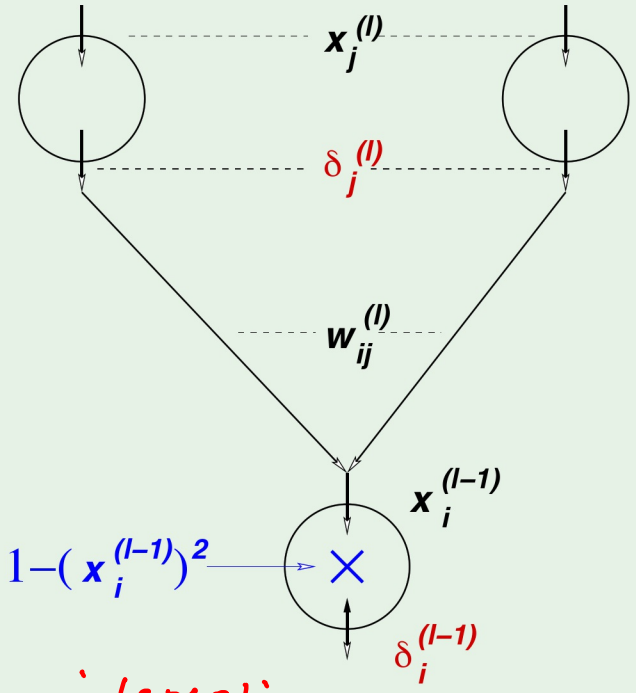
$\theta'(s_i^{(l-1)})$ → Scaling

$\sum w_{ij}^{(l)} \delta_j^{(l)}$ → Weighted Avg.

→ Super interesting

Why? → Similar Forward & Backward

Weighted Avg.



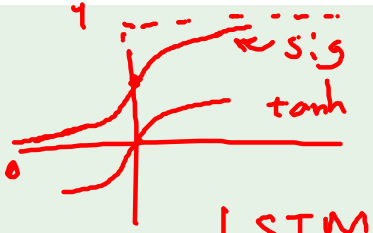
Backpropagation algorithm

$sig(x) = \frac{1}{1+e^{-x}}$ $\quad sig' = \frac{e^{-x}}{(1+e^{-x})^2}$ $\quad sig'(0) = \frac{1}{(1+1)^2} = \frac{1}{4}$

1: Initialize all weights $w_{ij}^{(l)}$ at random
2: **for** $t = 0, 1, 2, \ldots$ **do**
3: Pick $n \in \{1, 2, \cdots, N\}$ ⇝ for m in 1 ... K
4: *Forward:* Compute all $x_j^{(l)}$
5: *Backward:* Compute all $\delta_j^{(l)}$
6: Update the weights: $w_{ij}^{(l)} \leftarrow w_{ij}^{(l)} - \eta\, x_i^{(l-1)} \delta_j^{(l)}$
7: Iterate to the next step until it is time to stop
8: Return the final weights $w_{ij}^{(l)}$

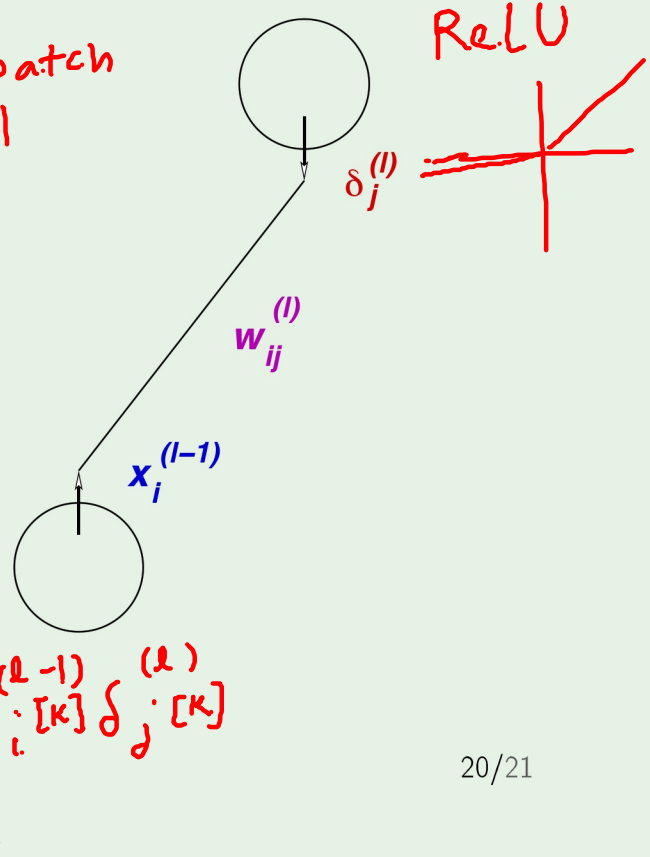
SGD with mini-batch size = 1

$\sum_{k \in M_m} x_i^{(l-1)}[k]\, \delta_j^{(l)}[k]$



LSTM

ReLU



Final remark: hidden layers

learned nonlinear transform

interpretation?