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Computer Simulation

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Chapter Seven: Random-Variate Generation



Purpose & Overview



- Understanding the generating process of samples from a **specified distribution** as input to our simulation model
 - Unlike the previous lecture, where the goal was to generate random numbers who follow the uniform distribution
- Explaining some widely-used techniques for generating random variates
 - Inverse-transform technique
 - Acceptance-rejection technique
- Many well-known simulators provide random-variate generators, but some may not
 - In this lecture you learn the essentials for developing routines for generating random variates if the simulator does not support it

Assumptions



- All of the techniques in this lecture consider the two following assumptions:
 - R_i is a previously generated random number readily available
 - R_i is uniformly distributed on $[0,1]$
 - Therefore, the PDF, and CDF of these numbers are defined as follows:
- With these R_i s in one hand, and a specific statistical distribution in another, we generate variates, which support that intended distribution

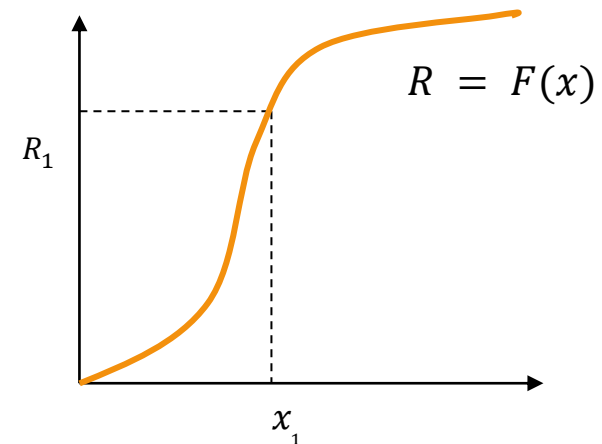
$$f_R(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_R(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Inverse-Transform Technique



- This technique is generally applicable to any distribution, but it gives us the best result if the CDF of the intended distribution is **reversible**
 - You can simply obtain F^{-1} from $F(x)$
- Steps to take:
 - Produce $R_i \sim U[0,1]$
 - Use: $\mathbf{R_i = F(x)} \Rightarrow x = F^{-1}(R_i)$
 - Find x
 - Is it correct to put R , and $F(x)$ equal?
- This technique enables us to sample data with various distributions, and generate similar numbers (x_i) with the same distribution
 - Exponential, Weibull, Uniform, triangular, and even empirical



Exponential Distribution (1)



- Recall: the PDF, and CDF for the exponential distribution are defined as follows:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$F(x) = \int_{-\infty}^x f(t) dt = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- Where λ indicates the rate of events occurred in a time unit, and the average time between two consequent events is λ^{-1}
- Our goal here is to generate a set of random variates $\{x_1, x_2, x_3, \dots\}$, who follow the exponential distribution

Exponential Distribution (2)



■ The step by step guide for exponential variate generation via inverse-transform:

□ **1st step:** Specify the CDF of the ultimate variates (x_i), which you want to generate: $F(x) = 1 - e^{-\lambda x}, x \geq 0$

□ **2nd step:** Place R equal to F(x): $F(x) = 1 - e^{-\lambda x} = R$

■ Since x is a random variable, $1 - e^{-\lambda x}$ is also a random variable, which we have named it as R

□ As we illustrate later, R follows the uniform distribution over [0,1] (as expected)

□ **3rd step:** Solving the equation to obtain x based on R

$$1 - e^{-\lambda x} = R \rightarrow e^{-\lambda x} = 1 - R \rightarrow -\lambda x = \ln(1 - R) \rightarrow x = -\frac{1}{\lambda} \ln(1 - R)$$

■ This is called **exponential variate generator function**

■ Independent from the distribution, the **variate generator function** is denoted with $x = F^{-1}(R)$



Exponential Distribution (3)

- **4th step:** Based on how many variates you want to generate, you must produce R_i s first
 - You can use the techniques you studied in lecture 6
 - Then apply them to the variate generator function to obtain x_i s

$$x_i = F^{-1}(R_i) = -\frac{1}{\lambda} \ln(1 - R_i)$$

- To simplify the exponential variate generation, we can replace $1 - R_i$ with R_i
 - Why?

$$x_i = -\frac{1}{\lambda} \ln(R_i)$$



Exponential Distribution (4)



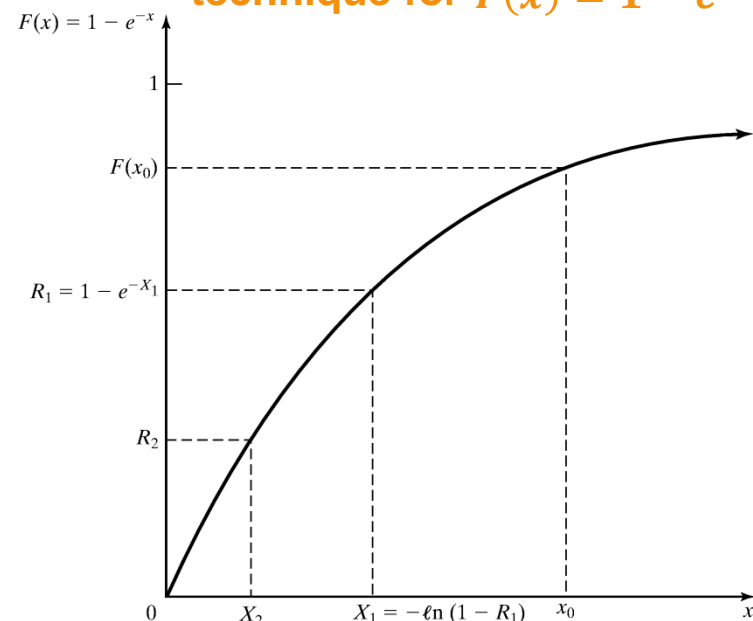
- Does the generated x_i really follows the exponential distribution with using this equation?
 - Consider the CDF of an exponential distribution with $\lambda = 1$
 - Assume x_0 as a prespecified number
- Let's obtain the CDF for x_i ($i = 1$)

$$P(x_1 \leq x_0) = P(R_1 \leq F(x_0)) = F(x_0)$$

A
B

- How A is applicable?
 - $x_1 \leq x_0$ if and only if $R_1 \leq F(x_0)$
- How B is applicable?
 - Because $0 \leq F(x_0) \leq 1$, and R_1 is uniformly distributed on $[0,1]$

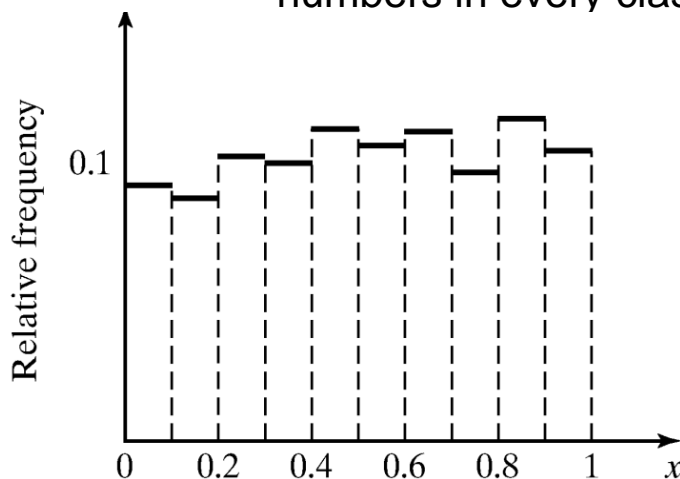
Figure: Inverse-transform technique for $F(x) = 1 - e^{-x}$



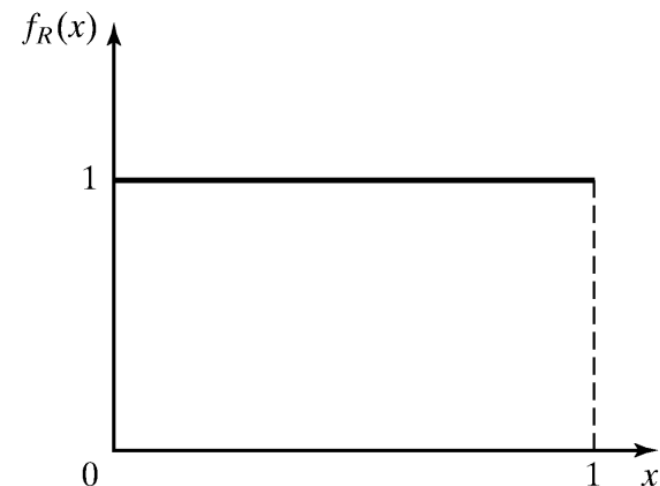
Example (1)



- Generate 200 variates x_i with distribution $\exp(\lambda = 1)$
 - First, we produce 200 $R_i \sim U(0,1)$
 - These random numbers are shown in the following histogram
 - 10 classes have been used for representing the frequency of numbers in every class



Histogram for 200 empirically generated R_i Random numbers $\sim U(0, 1)$



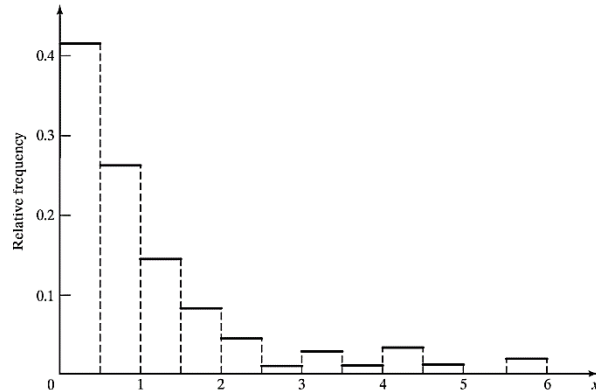
Theoretical representation of R_i Random numbers $\sim U(0, 1)$

Example (2)

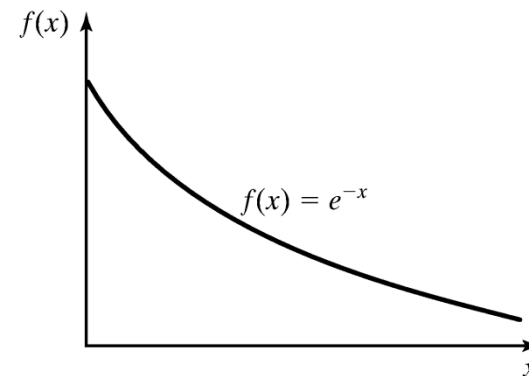
- Then we utilize the following equation:

$$x_i = -\frac{1}{\lambda} \ln(R_i) = -\ln(R_i)$$

- The generated variates are illustrated in the following histogram



Histogram for 200 empirically produced x_i exponential variates $\sim \exp(\lambda = 1)$



Theoretical representation of x_i variates $\sim \exp(\lambda = 1)$

Other Distributions (1)



- Now let's talk about producing variates for other distributions with employing the **inverse-transform**
 - Recall: The CDF of the targeted distribution must be **reversible**
- Uniform distribution:
 - The random variable X is uniformly distributed on $[a,b]$ if:

$$x_i \sim U(a,b) \Rightarrow x_i = a + (b-a)R_i$$

- Where a , and b can get any values ($a < b$)
 - Based on this equation, if R_i is selected from $[0,1]$, x_i resides in $[a,b]$
- Weibull distribution ($v = 0$):

$$f(x) = \begin{cases} \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-(x/\alpha)^\beta}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \longrightarrow F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}, x \geq 0$$

\downarrow

$$x_i = \alpha [-\ln(1 - R_i)]^{\frac{1}{\beta}}$$

Other Distributions (2)



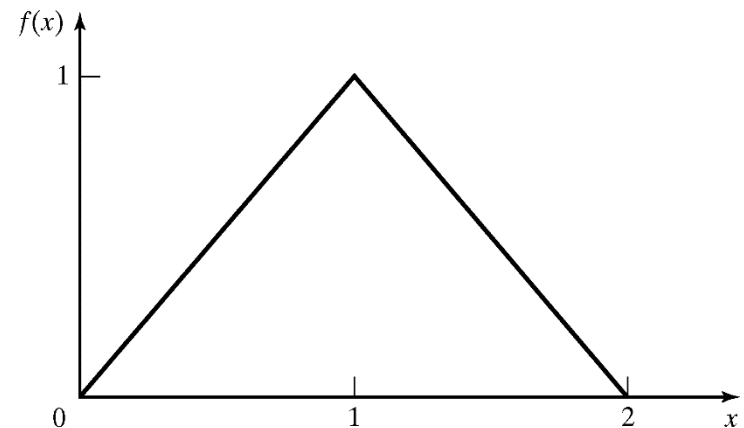
■ Triangular distribution:

- The same process must be applied to its PDF:

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x^2}{2}, & 0 < x \leq 1 \\ 1 - \frac{(2-x)^2}{2}, & 1 < x \leq 2 \\ 1, & x > 2 \end{cases}$$

$$x_i = \begin{cases} \sqrt{2R_i}, & 0 \leq R_i \leq 1/2 \\ 2 - \sqrt{2(1 - R_i)}, & 1/2 < R_i \leq 1 \end{cases}$$



Obtain the x_i equation in triangular distribution with inverse-transform technique



Empirical Continuous Distributions (1)



- When there isn't sufficient data, known theoretical distributions cannot be used for **modeling** and also **generating random variates**
- What approaches could be used here?
 - Resampling more data (if possible), and reaching a better understanding from the collected data
 - Then, try to suggest values for those areas with no data available based on the graphical presentation of the collected data
 - Also known as **discrete** approach
 - Interpolate between observed data points to fill in the gaps in the empirical CDF
 - This is where we want to focus in this section and produce random variates based on empirical **continuous** distribution

Empirical Continuous Distributions (2)



■ Problem statement:

- There is a **small sample set (size n)** composed of values you have observed
- We want to know, **if** supposed to be other observed values in this set, what would they like be?

■ The process:

- Arrange your limited collected data from smallest to largest

$$y_1 \leq y_2 \leq y_3 \leq \dots \leq y_n$$

- Consider the gap between two numbers as an interval, and assign the probability $1/n$ to each interval
- Plot these values and draw a line between every two consequent points
 - This is your empirical CDF composed of segmented lines

Empirical Continuous Distributions (3)



■ The process (Cont.):

- Produce R_i , which is uniformly distributed on $[0,1]$
- Determine the value of j according to $\frac{j-1}{n} < R_i \leq \frac{j}{n}$
 - You can do this with either the CDF or a table composed of your gathered information
- Calculate the value of x_i according to the following equation:

$$x_i = F^{-1}(R_i) = y_{j-1} + a_j \left[R_i - \frac{j-1}{n} \right]$$

- x_i is the value you want to produce, and y_j is the value you have already collected from the system
- a_j is indicating the slope of the j th segmented line in the CDF

$$a_j = \frac{y_j - y_{j-1}}{\frac{j}{n} - \frac{j-1}{n}} = \frac{y_j - y_{j-1}}{1/n}$$



Example (1)

- The speed of preparation for a group of firefighters have been measured and reported in minutes
 - 2.76 1.83 0.80 1.45 1.24
- We sort these data and insert them in a table as follows:

j	Interval (Hours)	Probability 1/n	Cumulative Probability, j/n	Slope, a_j
1	$0.0 \leq x \leq 0.80$	0.2	0.20	4.00
2	$0.8 < x \leq 1.24$	0.2	0.40	2.20
3	$1.24 < x \leq 1.45$	0.2	0.60	1.05
4	$1.45 < x \leq 1.83$	0.2	0.80	1.90
5	$1.83 < x \leq 2.76$	0.2	1.00	4.65

- Since there are only 5 values, the probability of every interval is 1/5
- The slope of every segmented line is also calculated
 - These values are then used to plot the empirical CDF

Example (2)



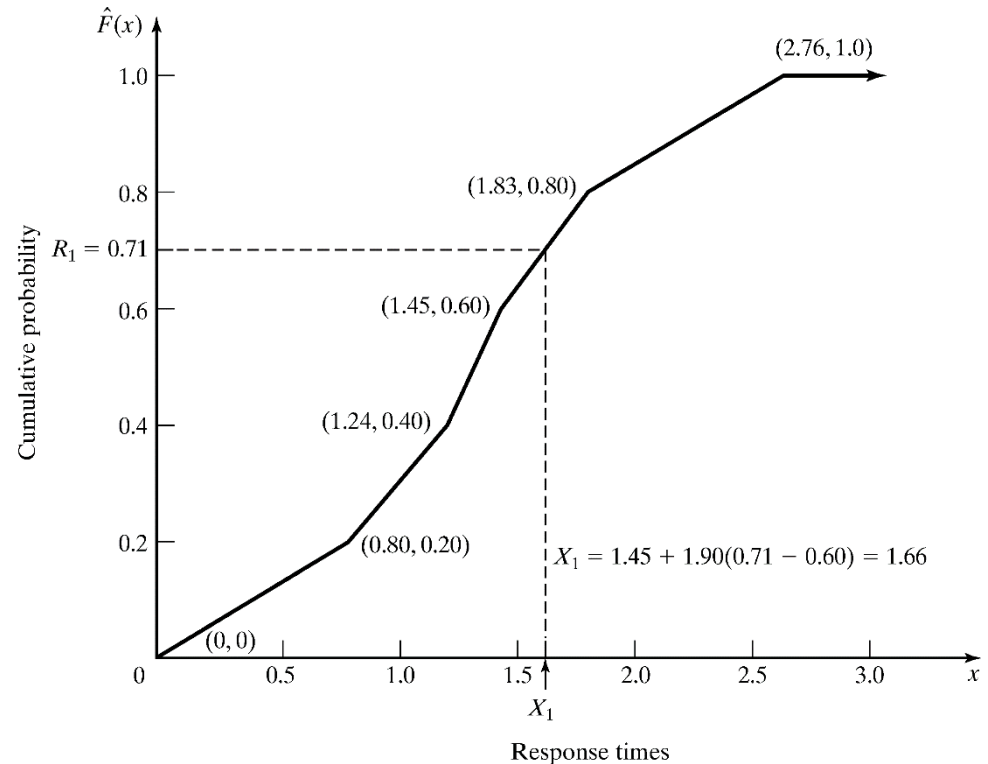
- The CDF is now obtained
- How to generate a random variate based on this empirical CDF?

- Assume $R_1 = 0.71$
- R_1 lies between 0.6, and 0.8

$$0.6 = \frac{j-1}{n} < R_1 \leq 0.8 = \frac{j}{n}$$

- Therefore, $j = 4$ and $a_4 = 1.90$

$$x_1 = y_{j-1} + a_j \left[R_i - \frac{j-1}{n} \right] = 1.45 + a_4 \left[0.71 - \frac{3}{5} \right] = 1.45 + 1.90 \times (0.71 - 0.6) = 1.66$$



Empirical Continuous Distribution with Large Sample Set (1)



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- What if the number of samples is high?
 - We need a classification approach
 - Consider a number of intervals with identical length
 - Every interval embraces a number of samples
 - The **cumulative frequency** of every interval is indicated with c_j
- So, the only difference with the previous approach is in the frequency of observations in every interval
 - Previously, every interval was contained of only one sample (check out the table in slide 16)
 - Therefore, the probability of every interval was identical to others ($1/n$), and the cumulative frequency of the j th interval was j/n
 - In the new approach, intervals contain different number of samples
 - So, we use the actual cumulative frequency of every interval, which may not be equal to other intervals



Empirical Continuous Distribution with Large Sample Set (2)



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■ The process:

- Produce R_i , which is uniformly distributed on $[0,1]$
- Determine the value of j according to $c_{j-1} < R_i \leq c_j$
- Calculate the value of x_i according to the following equation:

$$x_i = F^{-1}(R_i) = y_{j-1} + a_j[R_i - c_{j-1}]$$

- x_i is the value you want to produce, and y_j is the value you have already collected from the system
- a_j is indicating the slope of the j th segmented line in the CDF

$$a_j = \frac{y_j - y_{j-1}}{c_j - c_{j-1}}$$





Example (1)

- Assume that the repairing time of a device has been measured in $n=100$ consecutive observations
 - These values are classified into 4 intervals and added in the following table

j	Interval (Hours)	Frequency	Relative Frequency	Cumulative Frequency, c_j	Slope, a_j
1	$0.25 \leq x \leq 0.5$	31	0.31	0.31	0.81
2	$0.5 < x \leq 1.0$	10	0.10	0.41	5.0
3	$1.0 < x \leq 1.5$	25	0.25	0.66	2.0
4	$1.5 < x \leq 2.0$	34	0.34	1.00	1.47

- One of the differences between this table and the previous version is the **lower bound** of the 1st interval
 - Previously, since we built the intervals from the samples themselves, we put 0, but here we use the sample with minimum value

Example (2)



- Now we could depict the CDF from the table with drawing the segmented lines
- How to generate a random variate based on this empirical CDF?

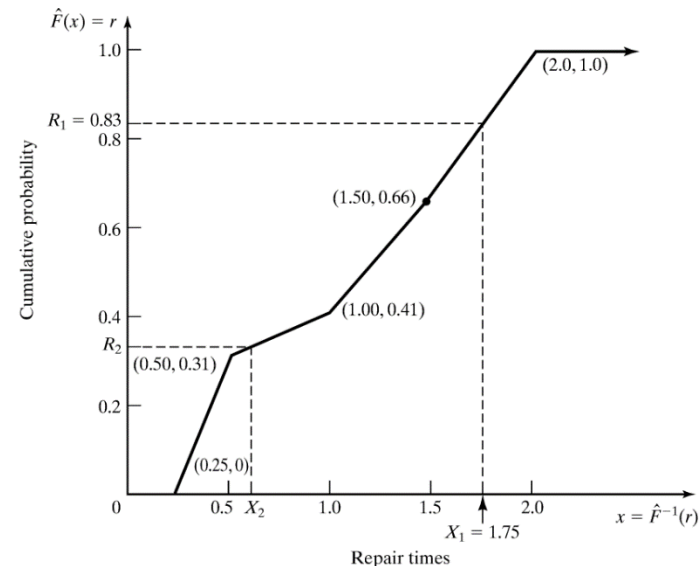
- Assume $R_1 = 0.83$
- R_1 lies between 0.66, and 1.0

$$0.66 = c_3 < R_1 \leq 0.8 = c_4$$

- Therefore, $j = 4$, and $a_4 = 1.47$

$$x_1 = y_{j-1} + a_j[R_i - c_{j-1}] = 1.5 + 1.47[0.83 - 0.66] = 1.75$$

- Calculate x_2 , if $R_2 = 0.33$



Continuous Distributions without a Closed-Form Inverse



- A number of useful continuous distributions do not have a closed-form expression for their CDF or its inverse
 - Example: Normal, Gamma, and Beta distribution
- We are willing to **approximate** the inverse of CDF
 - Example: Simple approximation for the inverse of CDF in the standard normal distribution:

Only 0.5% error

$$x_i = F^{-1}(R_i) \approx \frac{R_i^{0.135} - (1 - R_i)^{0.135}}{0.1975}$$

- In the following table, the approximated inverse has been compared with the exact inverse based on their corresponding R_i values

R	<i>Approximate Inverse</i>	<i>Exact Inverse</i>
0.01	-2.3263	-2.3373
0.10	-1.2816	-1.2813
0.25	-0.6745	-0.6713
0.50	0.0000	0.0000
0.75	0.6745	0.6713
0.90	1.2816	1.2813
0.99	2.3263	2.3373

Discrete Distribution (1)



- Now let's concentrate on **discrete distributions**
- Random variates supporting all kinds of discrete distributions can be generated via inverse-transform technique
 - This could be done either numerically with table-lookup procedures, algebraically, or a formula
- Examples for distributions:
 - Empirical
 - Discrete uniform
 - Gamma

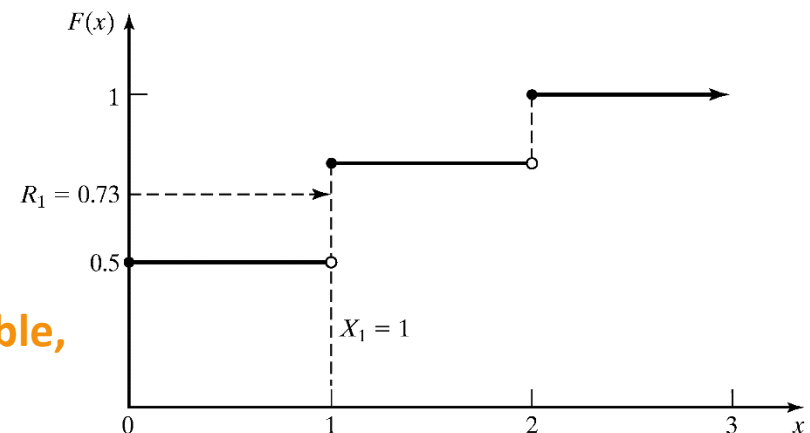
Discrete Distribution (2)



■ Example:

- Suppose that at the end of a working day, the number of shipments of a sailing company (X), on the loading dock of a port is either 0, 1, or 2
- The probability distribution of the gathered data is as follows
 - Accordingly, the cumulative distribution is also calculated, and indicated in the 3rd column

x	$p(x)$	$F(x)$
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00



**The CDF could be plotted based on this table,
and according to $F(X) = P(X \leq x)$**

- In order to take financial decisions, we have been asked to model the number of shipments (X) based on what we have observed



Discrete Distribution (3)

- Assume we want to find the random variate corresponding to $R_1 = 0.73$
 - First, find R_1 on the y-axis of $F(X)$
 - Draw a line to cut the 1st jump step
 - Find the corresponding x_1 to R_1 , which is 1
- You can also use a lookup table to find the corresponding x_i values
 - This table could be simply obtained from the previous CDF

$$x_i = \begin{cases} 0, & \text{if } R_i \leq 0.5 \\ 1, & \text{if } 0.5 < R_i \leq 0.8 \\ 2, & \text{if } 0.8 < R_i \leq 1.0 \end{cases}$$



i	Input r_i	Output x_i
0	0.50	0
1	0.80	1
2	1.00	2

- Since $r_1 = 0.5 < R_1 = 0.73 \leq r_2 = 0.8$, $\rightarrow i = 2 \rightarrow x_i = 1$

Geometric Distribution



- Consider a geometric distribution with a success probability of P
 - The PMF of this distribution is:

$$p(x) = p(1 - p)^x, \quad x = 0, 1, 2, \dots \quad 0 < p < 1$$

- The CDF is given by: $F(x) = 1 - (1 - p)^{x+1}$
- Using the Inverse-Transform Technique:

$$X = \left\lceil \frac{\ln(1 - R)}{\ln(1 - p)} - 1 \right\rceil \quad R \sim U[0,1]$$

- This variate generator gives us geometric variates ≥ 0 , but sometimes we need geometric variates $\geq q$:

$$X = q + \left\lceil \frac{\ln(1 - R)}{\ln(1 - p)} - 1 \right\rceil \quad x_i \in \{q, q + 1, q + 2, \dots\}$$

Acceptance-Rejection Technique (1)

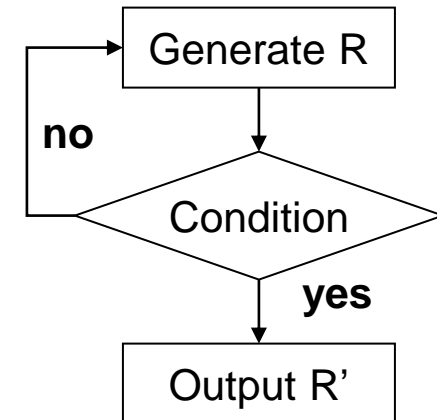


- Useful particularly when inverse CDF **does not exist in closed form**
- Let's discuss this technique using an example
 - Assume we want to generate random variates $X \sim U(1/4, 1)$
 - Follow these steps:
 - 1st : Generate a random number $R \sim U[0,1]$
 - 2nd : If $R \geq 1/4$, it lies in the acceptable range
 - So, we **accept** R and assign it to x
 - Unless, if $R < 1/4$, it will be **rejected**
 - 3rd : Go back to step 1, and produce an other random number R
 - Of course, if you need another variate
- In acceptance-rejection, the number of accepts is always less or equal to the total number of generated R s

Acceptance-Rejection Technique (2)



- For instance, in our example, with producing high enough random numbers R , approximately $1/4$ of the total number of produced R s will be rejected
 - Because they were less than $1/4$
 - Only $3/4$ will be accepted
- The process of variate generation using the AR technique is shown in the flowchart
 - While all of R s do not meet the condition, a fraction of them (R') does
 - So, R' has the desired distribution
- **Efficiency:** heavily depends on the ability to minimize the number of rejections in the algorithm



Poisson Process (1)



- Recall: The PDF of a random variable N supporting the Poisson distribution with $\alpha > 0$:

$$P(N = n) = \frac{e^{-\alpha} \alpha^n}{n!}, n = 0, 1, 2, \dots \quad \alpha = \lambda t$$

- Where n indicates the number of arrivals in the intended observation period
- Consider A_i as the interarrival between customer i , and $i - 1$
 - These interarrivals follow the exponential distribution with rate λ
 - We know how to generate exponential variates with $x_i = -\frac{1}{\lambda} \ln R_i$
- Now, we want to use this relation between Poisson and exponential distributions, to generate Poisson variates

Poisson Process (2)



- Assume $N = n$ arrivals are supposed to happen in our observation period t

- If we assume t to be a time unit ($\lambda = \alpha$), then:

$$A_1 + A_2 + A_3 + \dots + A_n \leq 1 < A_1 + A_2 + A_3 + \dots + A_n + A_{n+1} \quad \text{I}$$

- This indicates that the $n+1$ th customer arrives after our observation period ($t=\text{one time unit}$)

- So, n **MUST** meet this temporal condition

- Hence, let's generate a number of exponential A_i interarrivals with accordance to the above condition:

$$A_i = -\frac{1}{\lambda} \ln R_i \quad \text{II} \quad \longrightarrow \quad \sum_{i=1}^n -\frac{1}{\lambda} \ln R_i \leq 1 < \sum_{i=1}^{n+1} -\frac{1}{\lambda} \ln R_i$$

Poisson Process (3)



- To simplify the unequal relation, multiply the sides by $-\lambda$

$$\boxed{\sum_{i=1}^n \ln R_i} \geq -\lambda > \sum_{i=1}^{n+1} \ln R_i$$

We know that sum of logarithms equals to logarithm of their multiplication

$$\ln \prod_{i=1}^n R_i \geq -\lambda > \ln \prod_{i=1}^{n+1} R_i$$

- Take e^x from both sides $\rightarrow e^{\ln \prod_{i=1}^n R_i} \geq e^{-\lambda} > e^{\ln \prod_{i=1}^{n+1} R_i}$

- We know $e^{\ln x} = x$:
$$\prod_{i=1}^n R_i \geq e^{-\lambda} > \prod_{i=1}^{n+1} R_i$$

Poisson Process (4)



- Finally, according to what we have obtained, based on the question that **is it possible to have n arrivals in a time unit**, we can have an acceptance/rejection criterion
- Therefore, follow these steps to produce Poisson variates:
 - 1st step: Assign $n=0$ (no arrivals yet), and $P=1$ (a probability)
 - 2nd step: Generate a random number R_{n+1} , and replace the old P value with $P \times R_{n+1}$
 - 3rd step: If $P < e^{-\lambda}$, the condition is met (P is actually the multiplication of previous R_i s)
 - n is accepted, and N is assigned with n
 - Otherwise, n is rejected, increase n by one, and return to step 2
- To produce a **single Poisson variate**, the above algorithm must be iteratively executed until an n is accepted

Example



■ Generate 3 Poisson variates with $\lambda = 0.2$

- First calculate $e^{-\lambda} = 0.8187$
- Then generate a sequence of random numbers $R \sim U[0,1]$
 - You can use table A.1 in the textbook
- Then, follow the algorithm:

Step 1. Set $n = 0, P = 1$.

Step 2. $R_1 = 0.4357, P = 1 \cdot R_1 = 0.4357$.

Step 3. Since $P = 0.4357 < e^{-\alpha} = 0.8187$, accept $N = 0$.

Step 1–3. ($R_1 = 0.4146$ leads to $N = 0$.)

Step 1. Set $n = 0, P = 1$.

Step 2. $R_1 = 0.8353, P = 1 \cdot R_1 = 0.8353$.

Step 3. Since $P \geq e^{-\alpha}$, reject $n = 0$ and return to Step 2 with $n = 1$.

Step 2. $R_2 = 0.9952, P = R_1 R_2 = 0.8313$.

Step 3. Since $P \geq e^{-\alpha}$, reject $n = 1$ and return to Step 2 with $n = 2$.

Step 2. $R_3 = 0.8004, P = R_1 R_2 R_3 = 0.6654$.

Step 3. Since $P < e^{-\alpha}$, accept $N = 2$.

n	R_{n+1}	P	Accept/Reject	Result
0	0.4357	0.4357	$P < e^{-\alpha}$ (accept)	$N = 0$
0	0.4146	0.4146	$P < e^{-\alpha}$ (accept)	$N = 0$
0	0.8353	0.8353	$P \geq e^{-\alpha}$ (reject)	
1	0.9952	0.8313	$P \geq e^{-\alpha}$ (reject)	
2	0.8004	0.6654	$P < e^{-\alpha}$ (accept)	$N = 2$



Poisson variates
that we have
generated



What if λ is high? (1)

- As we noticed, higher λ imposes more cost to the acceptance/rejection technique
 - Specially when $\lambda \geq 15$
 - An alternate approach is to use an **approximation** technique based on the standard normal distribution

- Set: $Z = \frac{N-\lambda}{\sqrt{\lambda}}$

- Where Z is approximately a normally distributed variable with mean 0 and variance 1

- Generate standard normal variate Z (pair wise):

$$Z_1 = (-2 \ln R_1)^{1/2} \cos(2\pi R_2)$$

$$Z_2 = (-2 \ln R_1)^{1/2} \sin(2\pi R_2)$$

Generate 2 random numbers, and get 2 standard normal variates



What if λ is high? (2)

- Z_1 , and Z_2 are standard normal variates, which are totally independent
- But your Poisson variates have not been yet generated
 - To do so, set N equal to the following:

$$N = \left\lfloor \lambda + \sqrt{\lambda}Z - 0.5 \right\rfloor$$

- For both Z_1 , and Z_2
 - This equation has been obtained based on the conducted transformation in the previous slide
- Note: If $\lambda + \sqrt{\lambda}Z - 0.5 < 0$, put $N=0$
- Not forget that this approximation technique is not an acceptance/rejection technique!

Non-Stationary Poisson Process (NSPP)



- Recall: a NSPP is a Poisson arrival process with an arrival rate that varies with time
 - Our goal here is to use a special case of acceptance/rejection technique known as **thinning** to produce exponentially distributed interarrivals with rate $\lambda(t)$, $0 \leq t \leq T$
- Idea behind thinning:
 - 1st step: Generate a **stationary** Poisson arrival process at the fastest rate, $\lambda^* = \text{Max } \lambda(t)$, assign $t=0$, and $i=1$
 - i indicates the i th event (or customer)
 - 2nd step: Generate an exponential variate E with rate λ^* , and put $t=t+E$
 - 3rd step: Generate $R \sim U[0,1]$, if $R \leq \lambda(t) / \lambda^* \rightarrow$ assign $\tau_i = t$, $i=i+1$
 - τ_i is the arrival time of the i th NSPP event
 - 4th step: Return to step 2 (whether accepted or not)
- Where does thinning come with low efficiency?



Example



- For the following NSPP, generate a random variate using the thinning technique

	t (min)	Mean Time Between Arrivals (min)	Arrival Rate $\lambda(t)$ (#/min)
1 st 60 min	0	15	1/15
	60	12	1/12
	120	7	1/7
4 th 60 min	180	5	1/5
	240	8	1/8
	300	10	1/10
	360	15	1/15
	420	20	1/20
	480	20	1/20

Step 1: $\lambda^* = \max \lambda(t) = 1/5$, $t = 0$ and $i = 1$

Step 2: For random number $R = 0.2130$

$$E = -5\ln(0.213) = 13.13$$

$$t = t + E = 13.13$$

Within the 1st 60 min

Step 3: Generate $R = 0.8830$

$$\lambda(13.13)/\lambda^* = (1/15)/(1/5) = 1/3$$

Since $R > 1/3$, do not accept the arrival time

Step 2: For random number $R = 0.5530$

$$E = -5\ln(0.553) = 2.96$$

$$t = t + E = 13.13 + 2.96 = 16.09$$

Step 3: Generate $R = 0.0240$

$$\lambda(16.09)/\lambda^* = (1/15)/(1/5) = 1/3$$

Since $R < 1/3$, $T_1 = t = 16.09$

$$\text{and } i = i + 1 = 2$$

Variate Generation for Special Cases



- Among other distributions, there are special techniques used for variate generation, for instance:
 - Normal, and log-normal distributions
 - You can find out more in the textbook
 - The standard normal distribution, which we have already discussed
 - Beta distribution (which could be obtained from gamma distribution)
 - Erlang, and binomial distributions
 - **Convolution technique** is used: In this technique, to produce a random variate following an intended distribution, two or more random variables from another distribution are added up
 - For the Erlang distribution:

$$X = \sum_{i=1}^k X_i \longrightarrow X = \sum_{i=1}^k -\frac{1}{k\theta} \ln R_i = -\frac{1}{k\theta} \ln \left(\prod_{i=1}^k R_i \right)$$

A sequence of R_i s is used for a single X

Summary



- Principles of random-variate generation via:
 - Inverse-transform technique
 - The CDF of the target distribution must be reversible
 - Unless, we need another method
 - Acceptance-rejection technique
 - Discussed for Poisson and NSPP
 - Special cases
 - Convolution technique for Erlang variates
- These techniques are important for generating variates with continuous or discrete distributions to obtain appropriate inputs to be fed into our simulator