زبان و ساختار کامپیوتر

فصل پنجم

محاسبات کامپیوتری

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Parts (text & figures) of this lecture are adopted from :

- D. Patterson & J. Hennessey, "Computer Organization & Design, The Hardware/Software Interface", 5th Ed., MK publishing, 2014

Some Concepts

- LSB and MSB
  - Least Significant Bit (LSB)
  - Most Significant Bit (MSB)
- Signed versus Unsigned
  - Unsigned (Assume all non-negative numbers)
    - Used usually for memory addresses
  - Signed
    - Using sign bit
    - Using two's complement notation
- Carry Out
- Overflow

Number Representation

- Weighted number system
  - Can be represented in any base (radix)
    - Value of i^{th} digit "d_i" = $d_i * Base^i$
    - $0 =< d_i < Base$

31	30	29	…	i	…	3	2	1	0
d_{31}	d_{30}	d_{29}	…	d_i	…	d_3	d_2	d_1	d_0

Base Examples

Binary

$$(101101)_2 = 1x2^5 + 0x2^4 + 1x2^3 + 1x2^2 + 0x2^1 + 1x2^0$$

Octal

$$(736.4)_8 = 7 \times 8^2 + 3 \times 8^1 + 6 \times 8^0 + 4 \times 8^{-1}$$
$$= 7 \times 64 + 3 \times 8 + 6 \times 1 + 4/8 = (478.5)_{10}$$

Hexadecimal

$$(F3)_{16} = F \times 16 + 3 = 15 \times 16 + 3 = (243)_{10}$$

Decimal

$$(7245)_{10} = 7x10^3 + 2x10^2 + 4x10^1 + 5x10^0$$

Which Base?

- Number Representation
  - Humans prefer base 10, why?
  - Base 2 best works for computers, why?
  - Base 10 inefficient for computers, why?

Decimal to Base i Conversion

- Convert 65_{10} to base 5
- Convert 19_{10} to base 2

Converting Fractions

- Convert 0.4304_{10} to base 5 = 0.2034

$$\begin{array}{r} .4304 \\ \times \quad 5 \\ \hline 2.1520 \end{array} \qquad \begin{array}{r} 0.7600 \end{array}$$

$$\begin{array}{r} .1520 \\ \times \quad 5 \\ \hline 0.7600 \end{array} \qquad \begin{array}{r} .7600 \\ \times \quad 5 \\ \hline 3.8000 \end{array} \qquad \begin{array}{r} .8000 \\ \times \quad 5 \\ \hline 4.0000 \end{array}$$

- Convert 0.34375_{10} to base 2 = 0.01011

Converting between power of 2 radices

- Convert 1101010100011111_2
  - to base 8
  - to base 16

Binary to Decimal Conversion

- Question:
  - What is decimal value of this 32-bit number?

    $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1000_{two}$
  - Depends on the notation
    - Signed
    - Unsigned

Unsigned Numbers

$$N = (d_{31}*2^{31}) + (d_{30}*2^{30}) + \ldots + (d_1*2^1) + (d_0*2^0)$$

$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_{two} = 0_{ten}$
$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0001_{two} = 1_{ten}$
$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two} = 2_{ten}$
... ...
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two} = 4,294,967,293_{ten}$
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1110_{two} = 4,294,967,294_{ten}$
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111_{two} = 4,294,967,295_{ten}$

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Signed Numbers (2's Complement)

$$N = (d_{31} * -2^{31}) + (d_{30} * 2^{30}) + \ldots + (d_1 * 2^1) + (d_0 * 2^0)$$

```
0000 0000 0000 0000 0000 0000 0000 0000two = 0ten
0000 0000 0000 0000 0000 0000 0000 0001two = 1ten
0000 0000 0000 0000 0000 0000 0000 0010two = 2ten
...                           ...

0111 1111 1111 1111 1111 1111 1111 1101two = 2,147,483,645ten
0111 1111 1111 1111 1111 1111 1111 1110two = 2,147,483,646ten
0111 1111 1111 1111 1111 1111 1111 1111two = 2,147,483,647ten
1000 0000 0000 0000 0000 0000 0000 0000two = -2,147,483,648ten
1000 0000 0000 0000 0000 0000 0000 0001two = -2,147,483,647ten
1000 0000 0000 0000 0000 0000 0000 0010two = -2,147,483,646ten
...                                       ...

1111 1111 1111 1111 1111 1111 1111 1101two = -3ten
1111 1111 1111 1111 1111 1111 1111 1110two = -2ten
1111 1111 1111 1111 1111 1111 1111 1111two = -1ten
```

Other Signed Number Notations

- Signed-Magnitude Notation
- Ones' Complement Notation
- Biased Notation

Signed-Magnitude Notation

- Signed Notation with Sign Flag
- Most positive number
  - 011 … 1
- Most negative number
  - 111 … 1
- There are two zero's
  - 000 … 0
  - 100 … 0
- Used in floating point representation (mantissa)

Ones' Complement Notation

- Positive number same as two's complement
- Negative number:
  - Invert each bit in positive representation
- There are two zero's in ones' complement
  - 000…0
  - 111…1
- Most positive number
  - 0111 … 1
- Most negative number
  - 1000 … 0

Biased Notation (Excess 2^{n-1})

- If n bits used for representation:
  - Add all numbers with 2^{n-1}
- Zero represented by
  - 100 … 0
- Most negative number (-2^{n-1})
  - 000 … 0
- Most positive number ($2^{n-1}-1$)
  - 111 … 1

N	Excess-4	2's Comp
-4	000	100
-3	001	101
-2	010	110
-1	011	111
0	100	000
1	101	001
2	110	010
3	111	011

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Biased Notation (Excess $2^{n-1}-1$)

- If n bits used for representation:
  - Add all numbers with $2^{n-1}-1$
- Zero represented by
  - 011 … 1
- Most negative number ($-2^{n-1}+1$)
  - 000 … 0
- Most positive number (2^{n-1})
  - 111 … 1
- Used in floating point representation (exponent)

N	Excess-3	Excess-4
-4	-	000
-3	000	001
-2	001	010
-1	010	011
0	011	100
1	100	101
2	101	110
3	110	111
4	111	-

Signed Number Notations (Summary)

- Unbiased
  - Positive Binary $N=+14$ 0 0001110
  - Signed-Magnitude $-N=-14$ 1 0001110
  - 1's Complement (2^n-N-1) $-N=-14$ 1 1110001
  - 2's Complement (2^n-N) $-N=-14$ 1 1110010
- Biased $(2^{n-1}+N)$
  - Positive Binary $N=+14$ 1 0001110
  - Negative Binary $M=-14$ 0 1110010

Integer Addition / Subtraction

```
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
  0000000001000000 +          64
  0000000000101010           +42
  ----------------          ----
  0000000001101010           106

  1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0
  0000000001000000 +          64
  1111111111010110           -42
  ----------------          ----
  0000000000010110            22
```

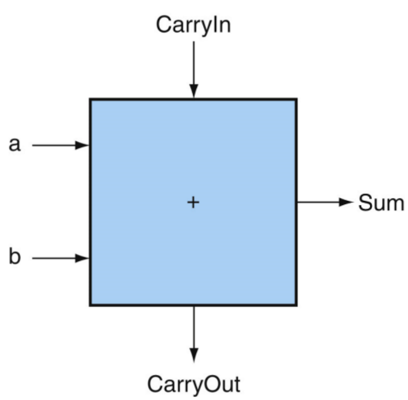
2's compliment

Overflow Conditions for Add/Sub

Operation	Operand A	Operand B	Result indicating overflow
$A + B$	≥ 0	≥ 0	< 0
$A + B$	< 0	< 0	≥ 0
$A - B$	≥ 0	< 0	< 0
$A - B$	< 0	≥ 0	≥ 0

- While adding signed numbers, an overflow occurs when
  - Both operands have the same sign,
  - but the result has the opposite sign
- the carry into and out of the MSB differ

One-bit Full Adder



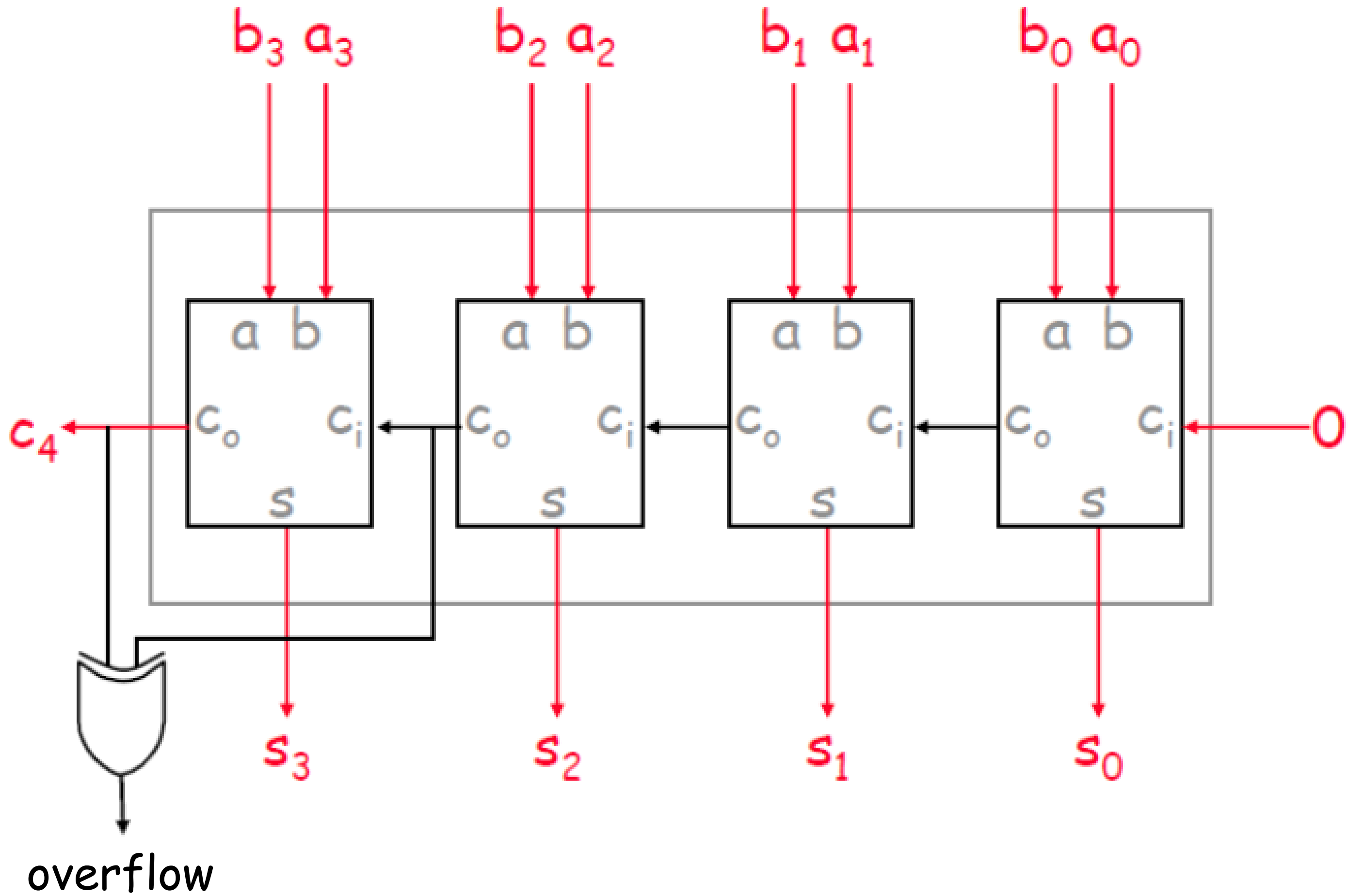
Input and output specification for a 1-bit adder

Inputs			Outputs		Comments
a	**b**	**CarryIn**	**CarryOut**	**Sum**	
0	0	0	0	0	$0 + 0 + 0 = 00_{two}$
0	0	1	0	1	$0 + 0 + 1 = 01_{two}$
0	1	0	0	1	$0 + 1 + 0 = 01_{two}$
0	1	1	1	0	$0 + 1 + 1 = 10_{two}$
1	0	0	0	1	$1 + 0 + 0 = 01_{two}$
1	0	1	1	0	$1 + 0 + 1 = 10_{two}$
1	1	0	1	0	$1 + 1 + 0 = 10_{two}$
1	1	1	1	1	$1 + 1 + 1 = 11_{two}$

$$Sum = a \oplus b \oplus CarryIn$$

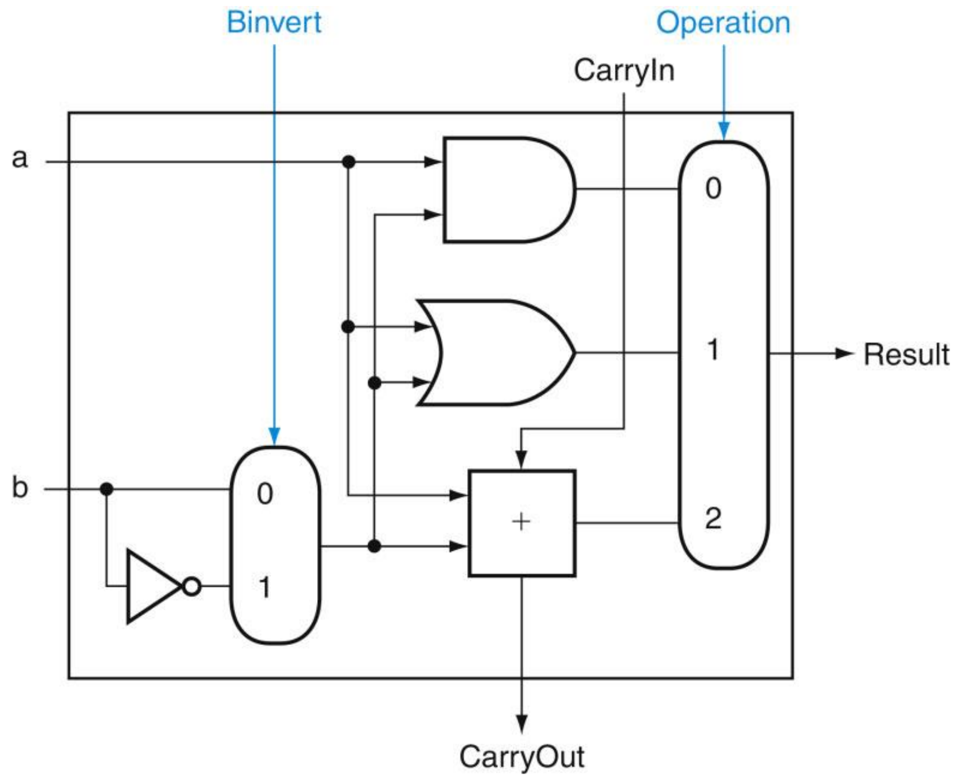
$$CarryOut = a.b + CarryIn.(a \oplus b)$$

Ripple-Carry Adder

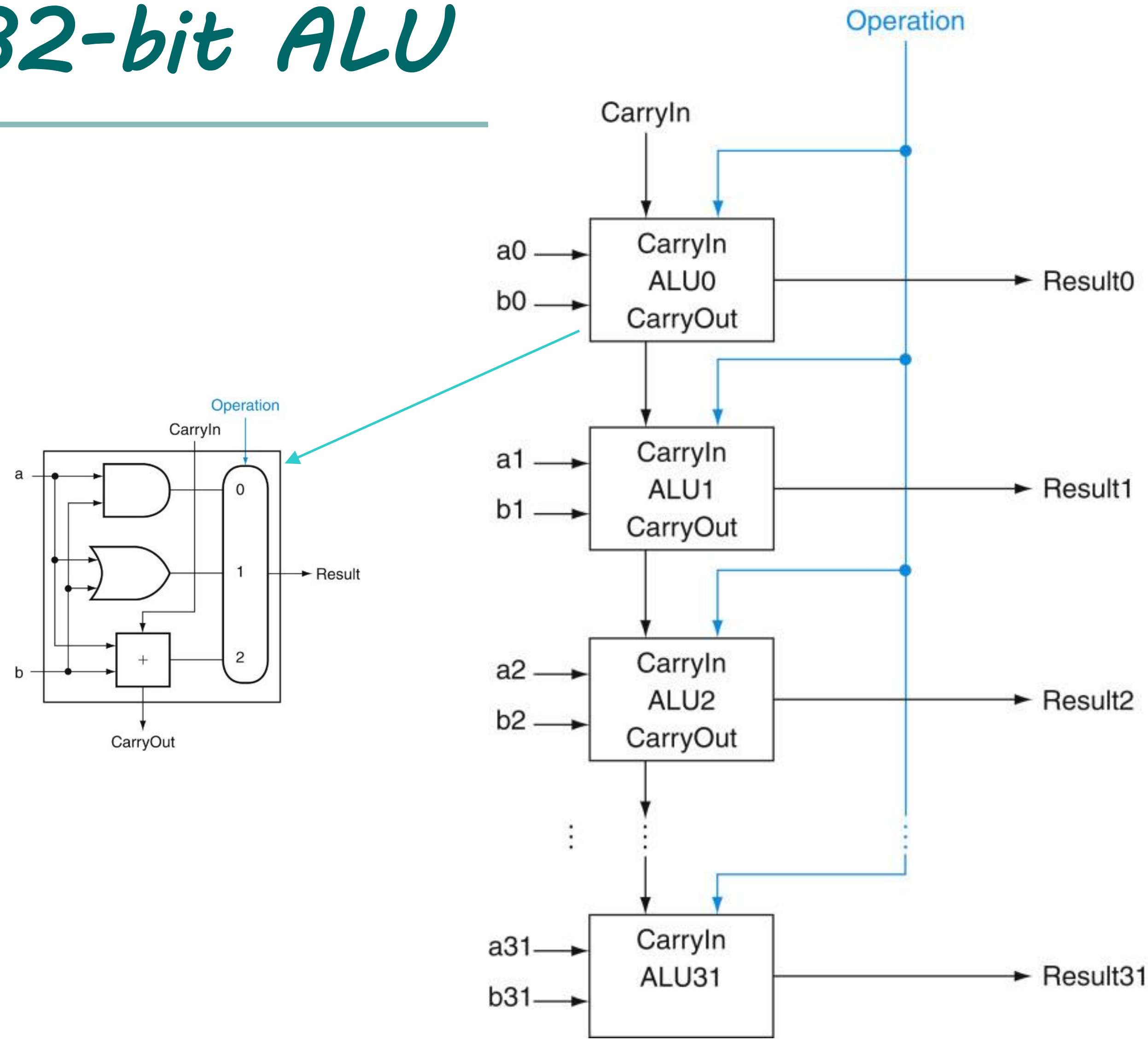


One-bit ALU

Performs AND, OR, and addition on a and b or a and $\underline{b}$



32-bit ALU


Operation
CarryIn
a
b
0
1
2
+
Result
CarryOut
Operation
CarryIn
a0
b0
CarryIn
ALU0
CarryOut
Result0
a1
b1
CarryIn
ALU1
CarryOut
Result1
a2
b2
CarryIn
ALU2
CarryOut
Result2
a31
b31
CarryIn
ALU31
Result31

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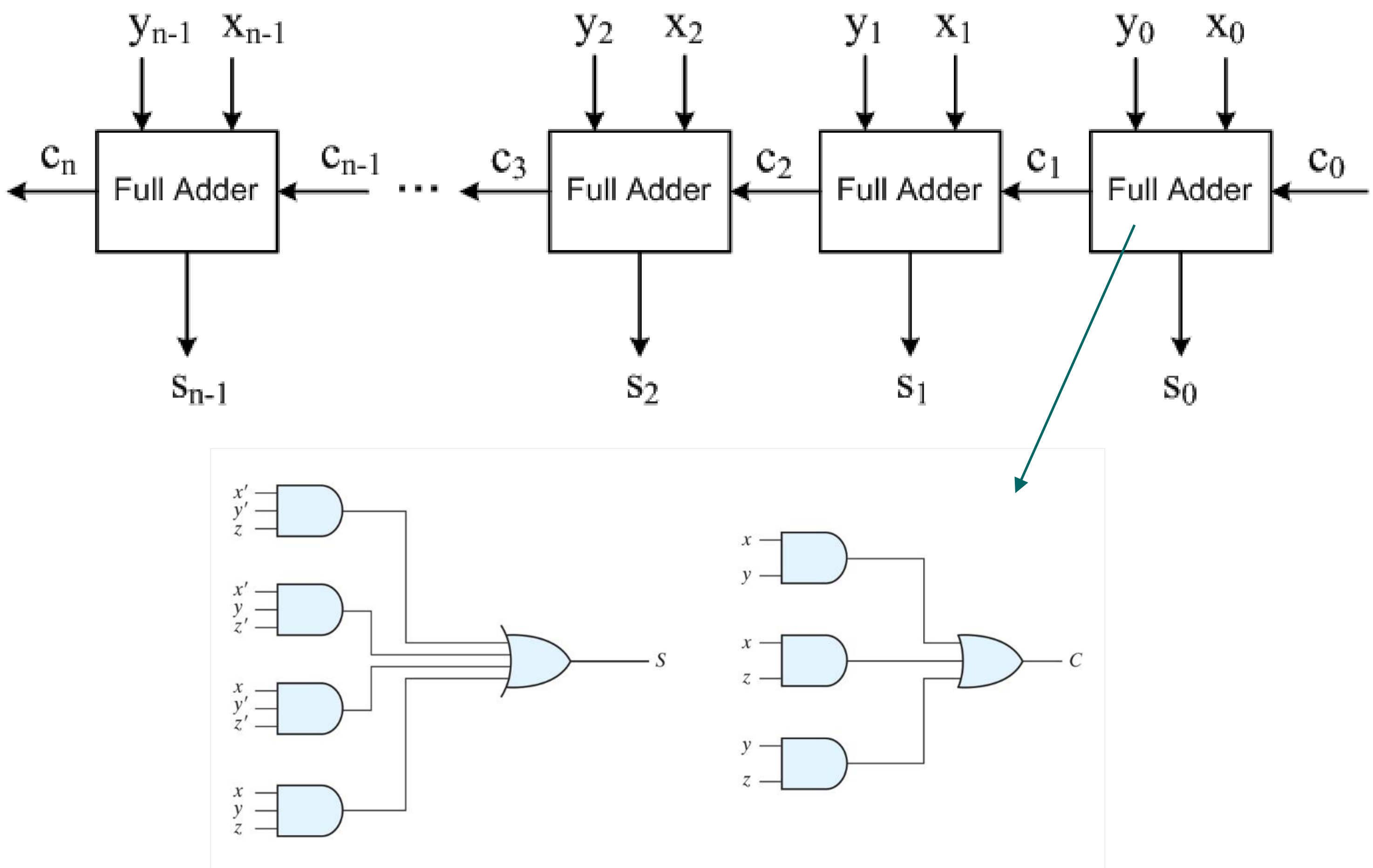
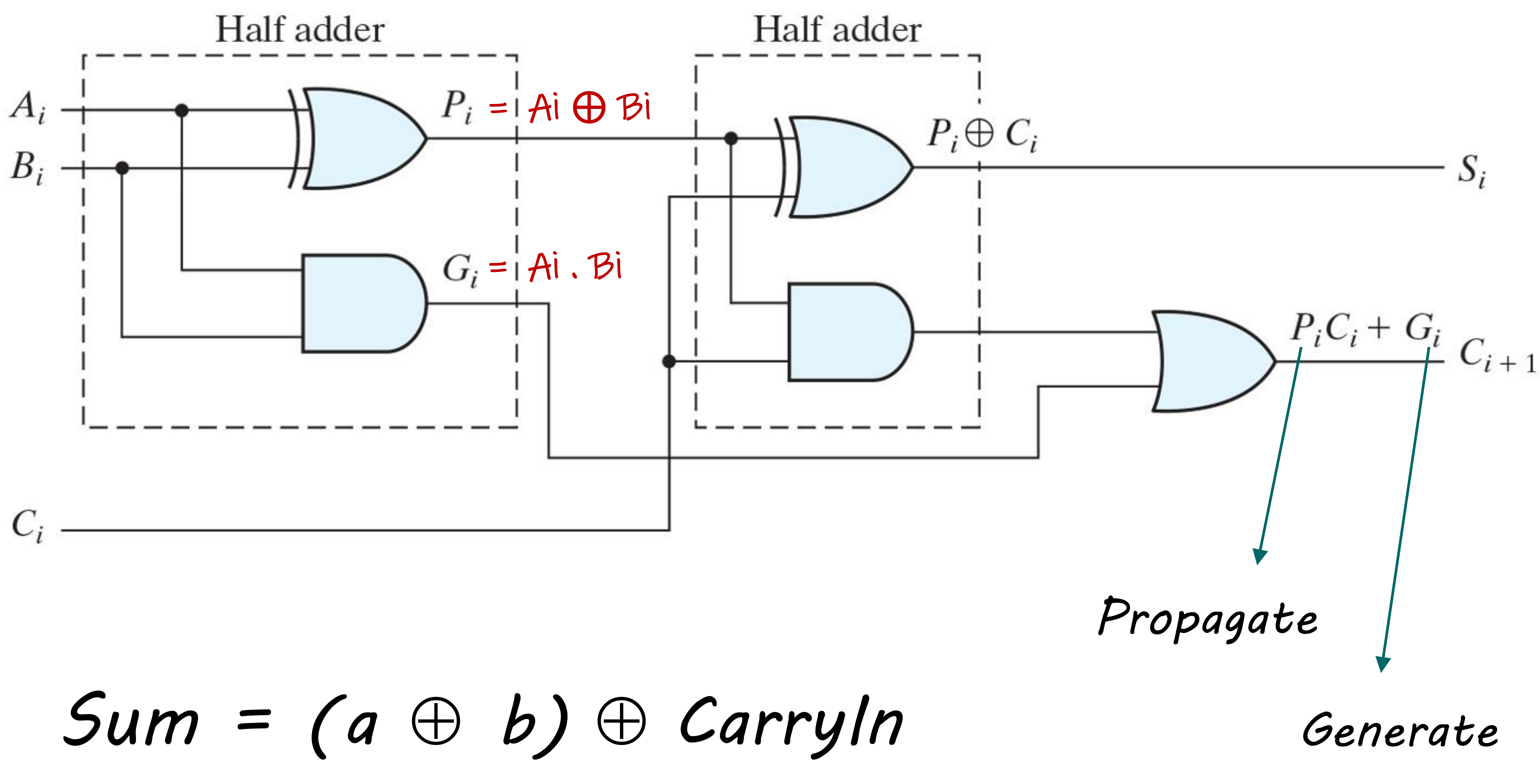
Reminder: Ripple-Carry Adder



Carry Generate / Propagate



$$Sum = (a \oplus b) \oplus CarryIn$$

$$CarryOut = a.b + CarryIn.(a \oplus b)$$

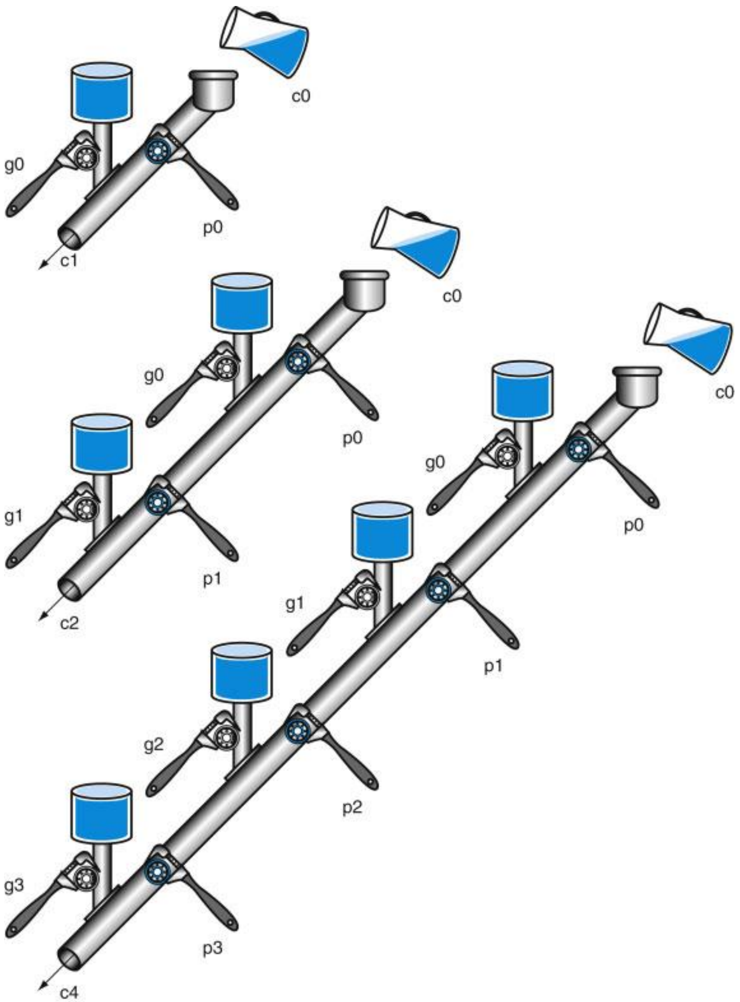
CLA vs. Pipes

$c_1 = g_0 + p_0 \cdot c_0$

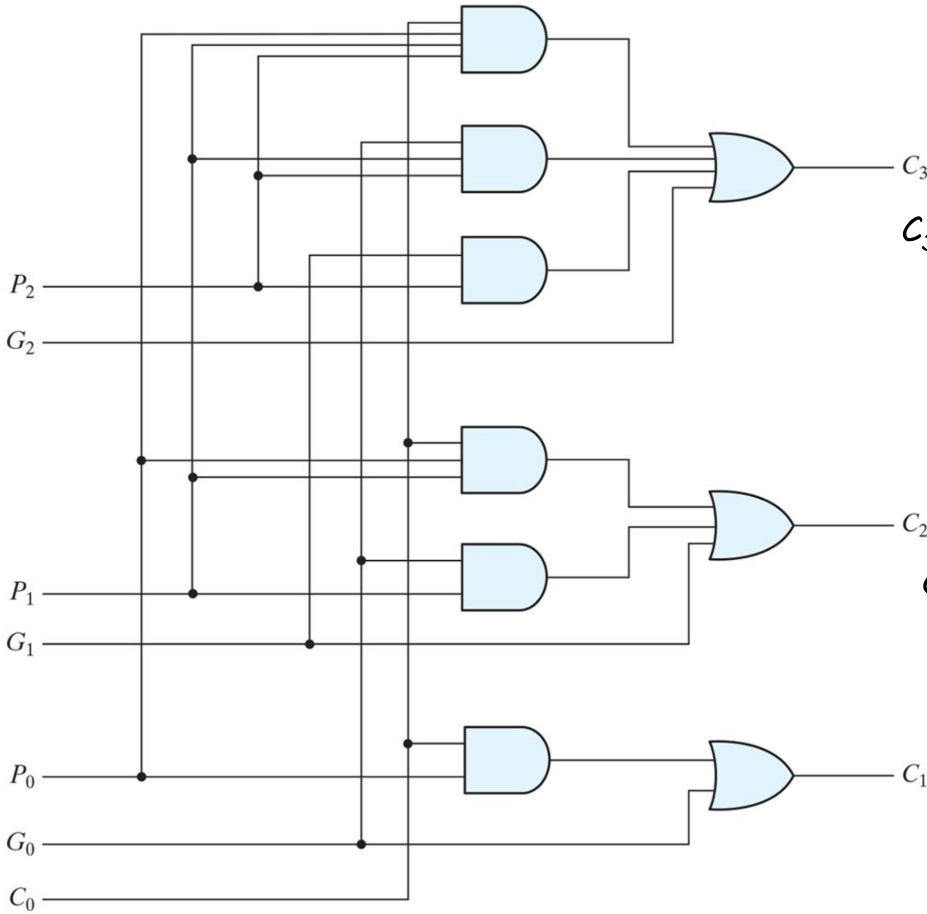
$c_2 = g_1 + p_1 \cdot g_0 + p_1 \cdot p_0 \cdot c_0$

$c_3 = g_2 + p_2 \cdot g_1 + p_2 \cdot p_1 \cdot g_0$
$+ p_2 \cdot p_1 \cdot p_0 \cdot c_0$

$c_4 = g_3 + p_3 \cdot g_2 + p_3 \cdot p_2 \cdot g_1$
$+ p_3 \cdot p_2 \cdot p_1 \cdot g_0 + p_3 \cdot p_2 \cdot p_1 \cdot p_0 \cdot c_0$



CLA Generate/Propagate Circuit

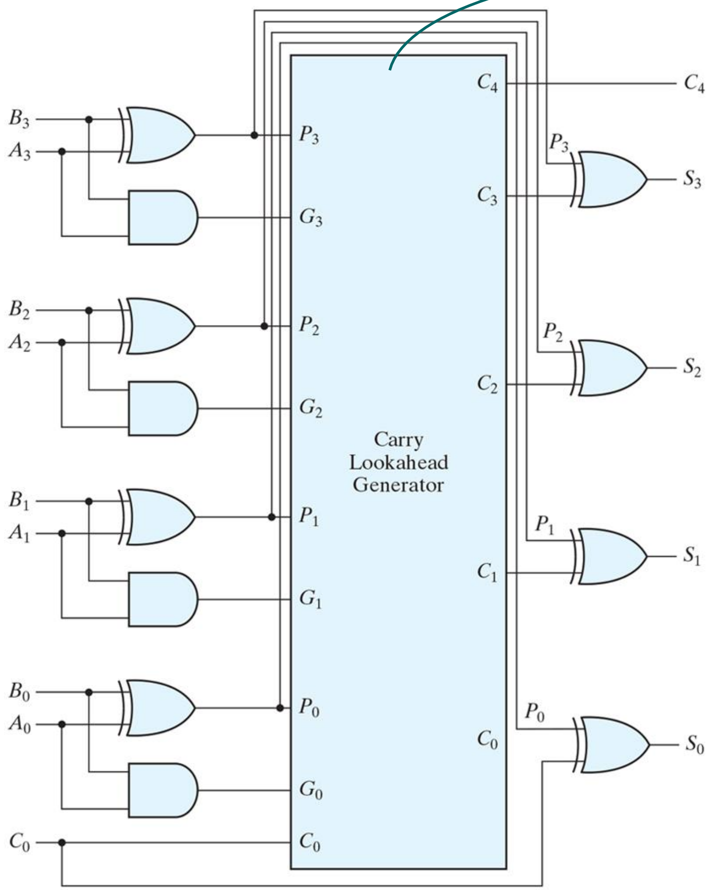


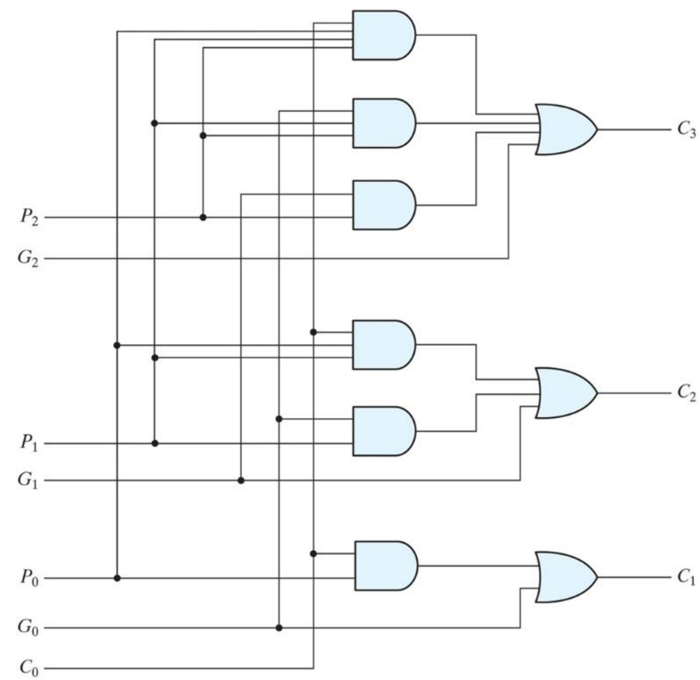
$C_3 = G_2 + P_2 \cdot G_1 + P_2 \cdot P_1 \cdot G_0 + P_2 \cdot P_1 \cdot P_0 \cdot C_0$

$C_2 = G_1 + P_1 \cdot G_0 + P_1 \cdot P_0 \cdot C_0$

$C_1 = G_0 + P_0 \cdot C_0$

CLA 4-bit Adder





Multiplication Approach (1st ver)

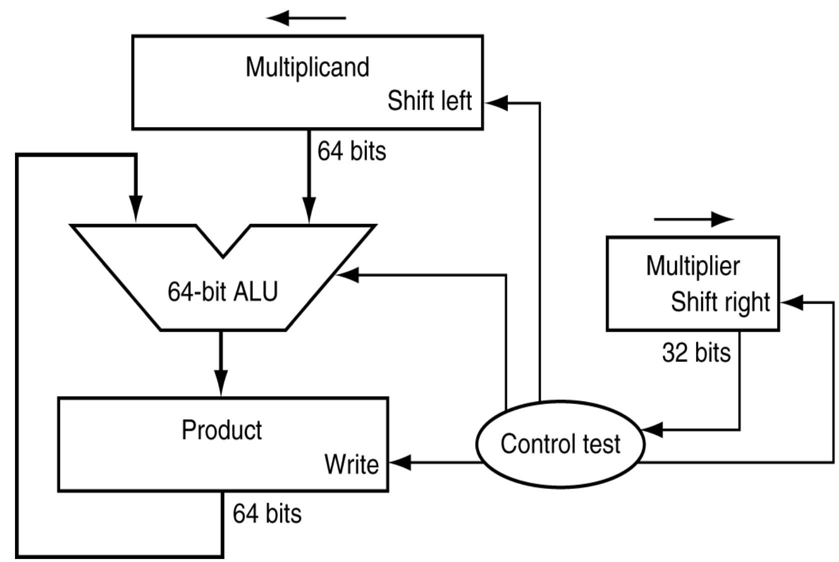
مضروب (multiplicand)

مضروب‌فیه (multiplier)

حاصل‌ضرب (product)

```
      1000
  ×   1001
  --------
      1000
     0000
    0000
   1000
  --------
   1001000
```

Length of product is the sum of operand lengths



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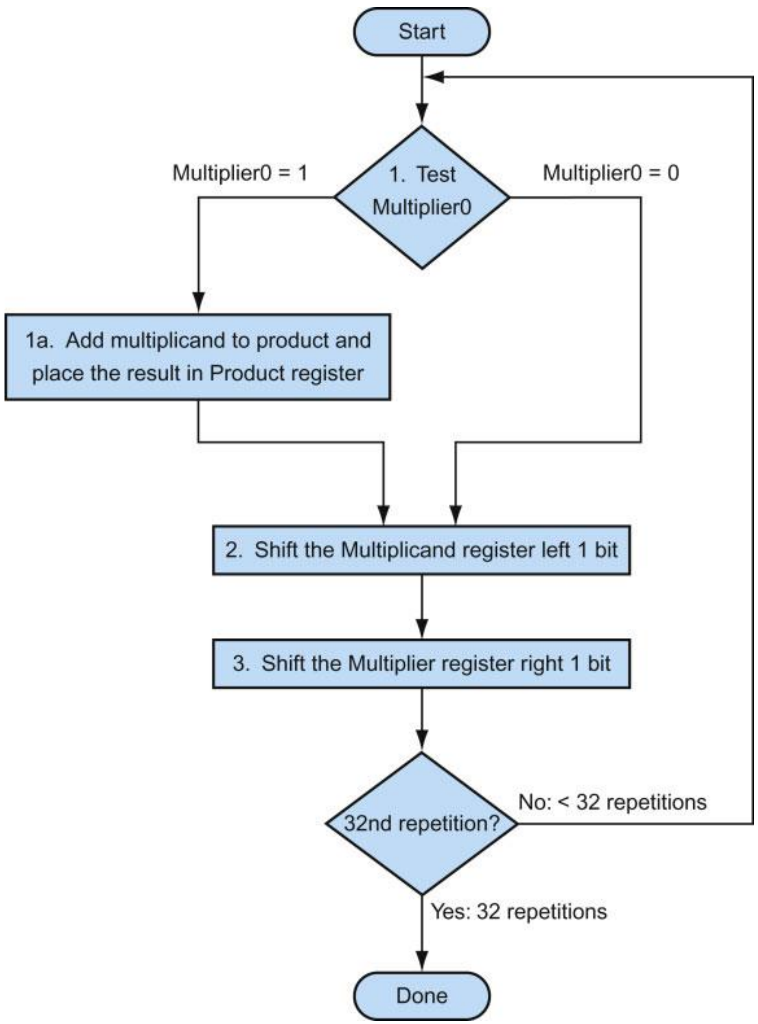
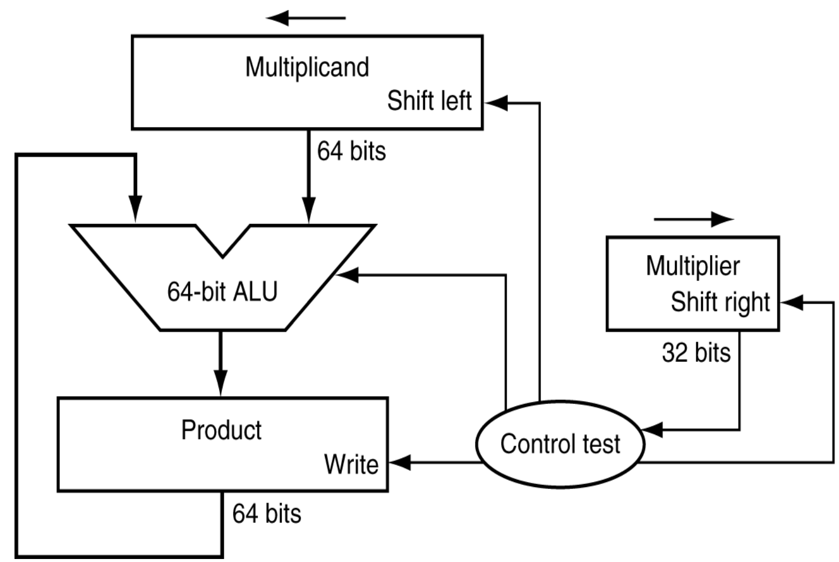
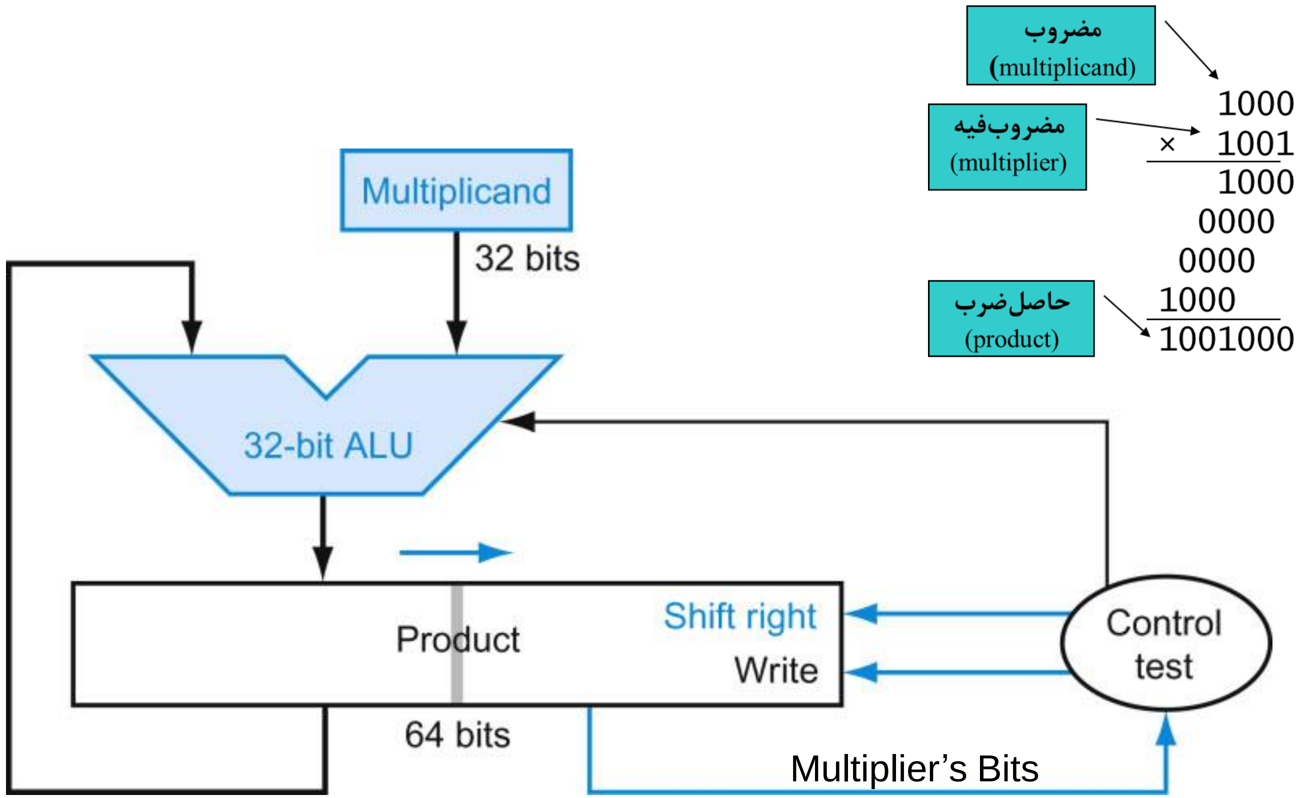
Multiplication Algorithm (1st ver)

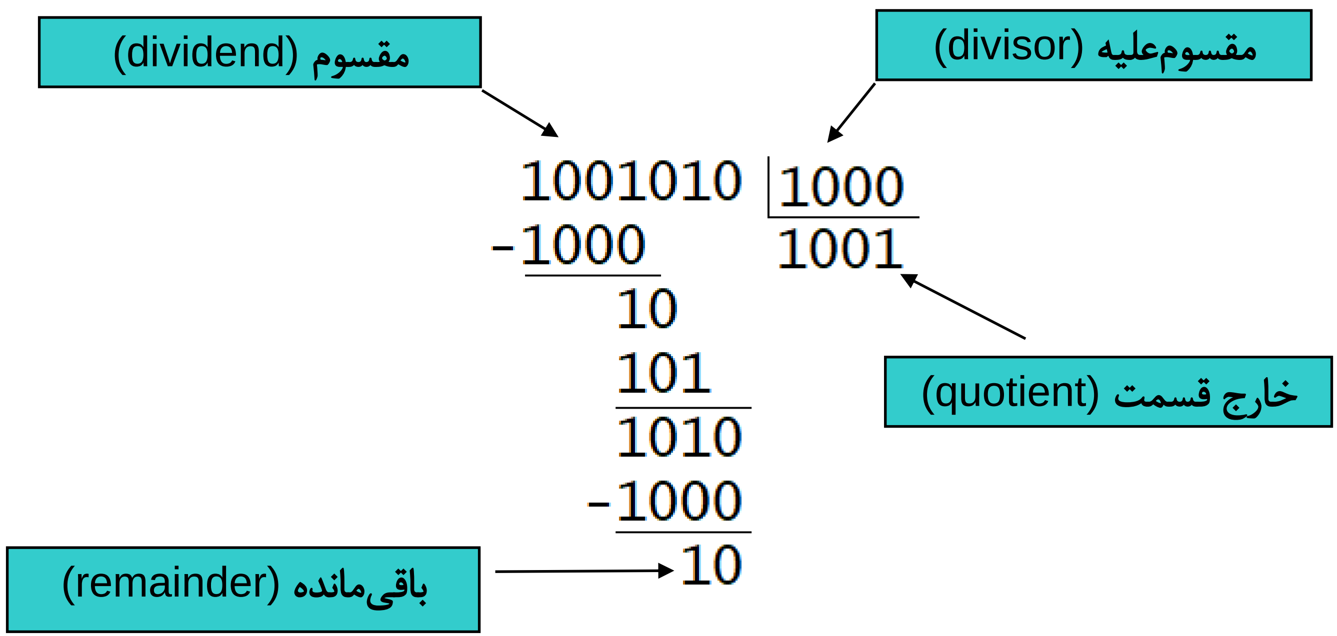




Multiplication (2nd ver)



Division



Dividend = Quotient × Divisor + Remainder, |Remainder| < |Divisor|

Real Numbers

- Numbers with Fractions:
  - 3.14159…
  - 2.17
  - 0.0000001
  - $1.25 * 10^{-12}$
  - $1.43 * 10^{+12}$
- Representation in computers:
  - Fixed point
  - Floating point

Fixed-Point Representation

- A real Example:
  - $d_{23}d_{22}\ldots d_1 d_0 \cdot f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_7$
  - 24-bit: integer bits
  - 8-bit: fraction bits
- Application
  - Used in CPUs with no floating-point unit
    - Embedded microprocessors and microcontrollers
  - Digital Signal Processing (DSP) applications

Fixed-Point Representation (cont.)

- Consider 5-Bit Representation
  - $d_2d_1d_0 \cdot f_1f_0$
  - $(d_2\times -2^2)+(d_1\times 2^1)+(d_0\times 2^0)+(f_1\times 2^{-1})+(f_0\times 2^{-2})$
- Largest positive number?
- Smallest positive number?
- Largest magnitude negative number?
- Smallest magnitude negative number?

Fixed-Point Representation (cont.)

- Arithmetic:
  - $011.11 + 011.11 = 111.10$ out of range (overflow)
  - $010.10 \times 000.10 = 000001.0100$
  - $000.01 \times 000.01 = 000000.0001$ out of range (underflow)
  - $011.01 \times 011.01 = ?$

Fixed-Point Representation (cont.)

- Arithmetic:
  - 011.11 + 011.11 = 111.10
  - 010.10 × 000.10 = 000001.0100
  - 000.01 × 000.01 = 000000.0001
  - 011.01 × 011.01 = 001010.1001

out of range (overflow)

out of range (underflow)

Both overflow & underflow

Fixed-Point Representation (cont.)

- Pros
  - Simple hardware
  - Fast computation
  - Different precisions at different applications
    - 24bits/8bits , 18bits/14bits, 8bits/24bits
- Cons
  - Low precision
  - Small range

Scientific Notation (Decimal)

coefficient

exponent

$$6.02_{10} \times 10^{23}$$

decimal point

radix (base)

- Normalized Form:
  - Exactly one non-zero digit to left of decimal point
- Alternatives to representing 0.0000000012
  - Normalized: 1.2×10^{-9}
  - Not normalized: 0.12×10^{-8}, 12.0×10^{-10}

Normalized Scientific Notation (Binary)

mantissa (fraction)

exponent

$$1.0_{two} \times 2^{-1}$$

binary point

radix (base)

Floating-Point Notation

- Floating Point Notation Consists of:
  - Fraction (F): 23 bits
  - Exponent (E): 8 bits
  - Fraction Sign bit (S)
  - Also called, single precision floating-point
- $N = (-1)^S \times (1+F) \times 2^E$

<table>
<tr><td>31</td><td>30</td><td>…</td><td>24</td><td>23</td><td>22</td><td>21</td><td>…</td><td>1</td><td>0</td></tr>
<tr><td>S</td><td colspan="4">Exponent</td><td colspan="5">Fraction</td></tr>
</table>

Floating-Point Notation (cont.)

- Pros (compared to fixed-point)
  - Very Wide Range
  - More precision bits
- Cons (compared to fixed-point)
  - Arithmetic operation more complicated
  - HW more complicated

<table>
<tr><td>31</td><td>30</td><td>...</td><td>24</td><td>23</td><td>22</td><td>21</td><td>...</td><td>1</td><td>0</td></tr>
<tr><td>S</td><td colspan="4">Exponent</td><td colspan="5">Fraction</td></tr>
</table>

Floating-Point Notation (cont.)

- $N = (-1)^S \times (1+F) \times 2^E$
- Precision versus Range
  - More precision ➔ smaller range?
  - Wider range ➔ less precision?
- True for fixed-point
  - Not necessarily correct for floating point

31	30	...	24	23	22	21	...	1	0
S	Exponent				Fraction				

Floating-Point Notation (cont.)

- Overflow:
  - Exponent too large to fit in “Exponent” field
- Underflow:
  - Non-zero fraction so small to represent
  - Negative exponent too large to fit

31	30	...	24	23	22	21	...	1	0
S	Exponent				Fraction				

IEEE 754 – Single Precision

- Signed-magnitude notation for fraction (mantissa)
- Biased (Excess $2^{n-1}-1$) notation for exponent
- E_{min}=00000001
- E_{max}=11111110
- E=00000000 reserved for zero
- E=11111111 reserved for infinity & NaN
- Smallest positive no: 1.17549435 E-38
- Largest positive no: 3.4028235 E38

31	30	...	24	23	22	21	...	1	0
S	Exponent				Fraction				

$$N = (-1)^S * (1 + F) * 2^E$$

IEEE 754 – Double Precision

- Two words long (64 bits)
- Reduced chances of overflow/underflow
- Format
  - Sign bit (S)
  - Fraction (F): 52 bits
  - Exponent (E): 11 bits
- A bias of 1023 in Exponential part

More on IEEE 754 Standard

- Single precision (32bits)/Double precision (64bits)
- Normalized/ Denormalized forms
- Standard definitions for zero, infinity, NaN
- Check: https://www.h-schmidt.net/FloatConverter/IEEE754.html

Single precision		Double precision		Object represented
Exponent	Fraction	Exponent	Fraction	
0	0	0	0	0
0	Nonzero	0	Nonzero	± denormalized number
1–254	Anything	1–2046	Anything	± floating-point number
255	0	2047	0	± infinity
255	Nonzero	2047	Nonzero	NaN (Not a Number)

Denormalized Forms

- An attempt to squeeze every last bit of precision from a floating-point operation
- Smallest pos. single precision normalized no:
  - $1.00000000000000000000000 \times 2^{-126}$
- Smallest single precision denormalized no:
  - $0.00000000000000000000001 \times 2^{-126} = 1.0 \times 2^{-149}$

31	30	...	24	23	22	21	...	1	0
S	Exponent				Fraction				

Concluding Remarks

- Bits have no inherent meaning
  - Interpretation depends on the operations applied
- Computer representations of numbers
  - Finite range and precision
  - Need to account for this in programs
- Bounded range and precision
  - Operations can overflow and underflow

Outlines

- Weighted Number System
- Signed Number Representation
  - 2's Compliment/ 1's Compliment
  - Signed-Magnitude Notation
  - Biased Notation
- Arithmetic Operations
  - Addition/ Subtraction/ Multiplication/ Division
- Real Numbers
  - Fixed Point / Floating Point Representation
  - IEEE 754 Standard