Stable Marriage

Greedy algorithms

Problem

- W women, each with a rank list

Problem

- W women, each with a rank list
- M men, each with a rank list
- $|M| = |W| = n$

Problem

- W women, ea
- M men
- $|M| =$

How to assign? Or

propose?

Problem

Y
R
B
G

Y
B
R
G

B
G
R
Y

B
Y
G
R

B
G
Y
R

G
R
Y
B

Y
B
G
R

Y
R
B
G

Matching

- A mapping from men to women

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  - Each woman is mapped to exactly one man

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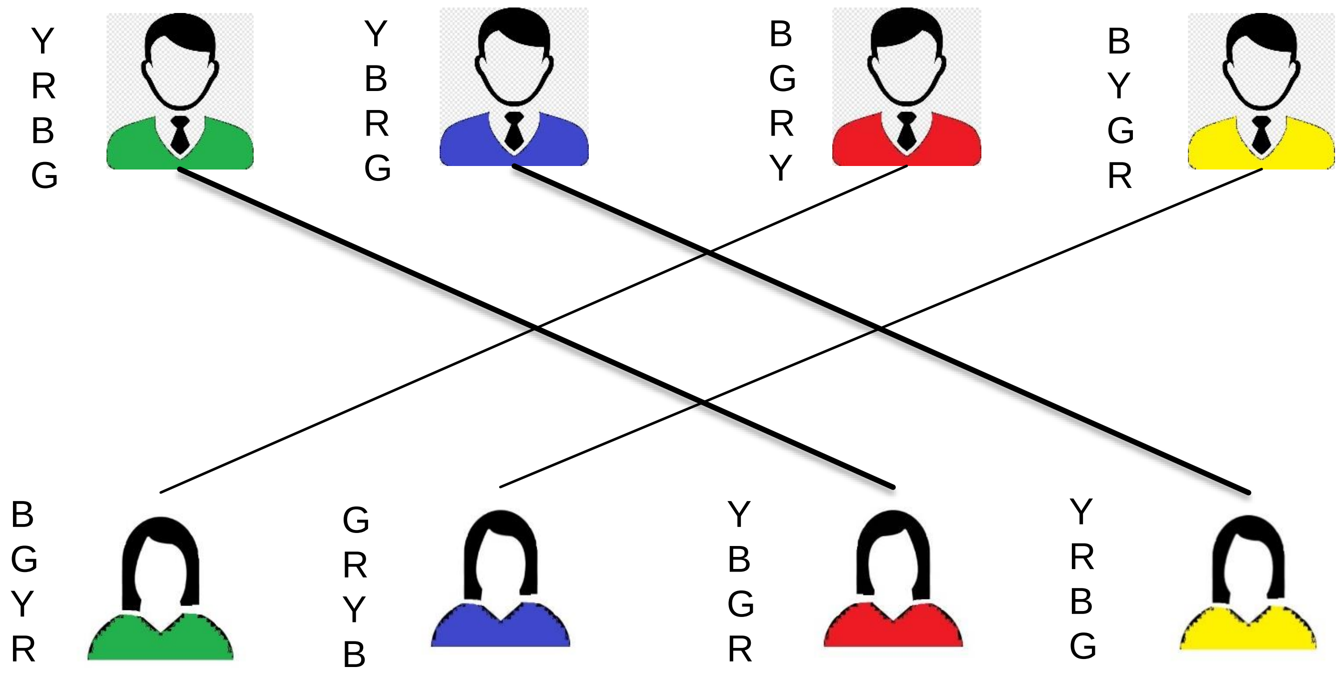
  - Each man is mapped to exactly one woman

Problem



A perfect Matching

Stable Matching

- There is no man M and woman W such that :

Stable Matching

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Stable Matching

- There is no man M and woman W such that :
  - Not mapped to each other
  - M prefers W to his current wife
  - W prefers M to her current husband

No blocking pair

Stable Matching

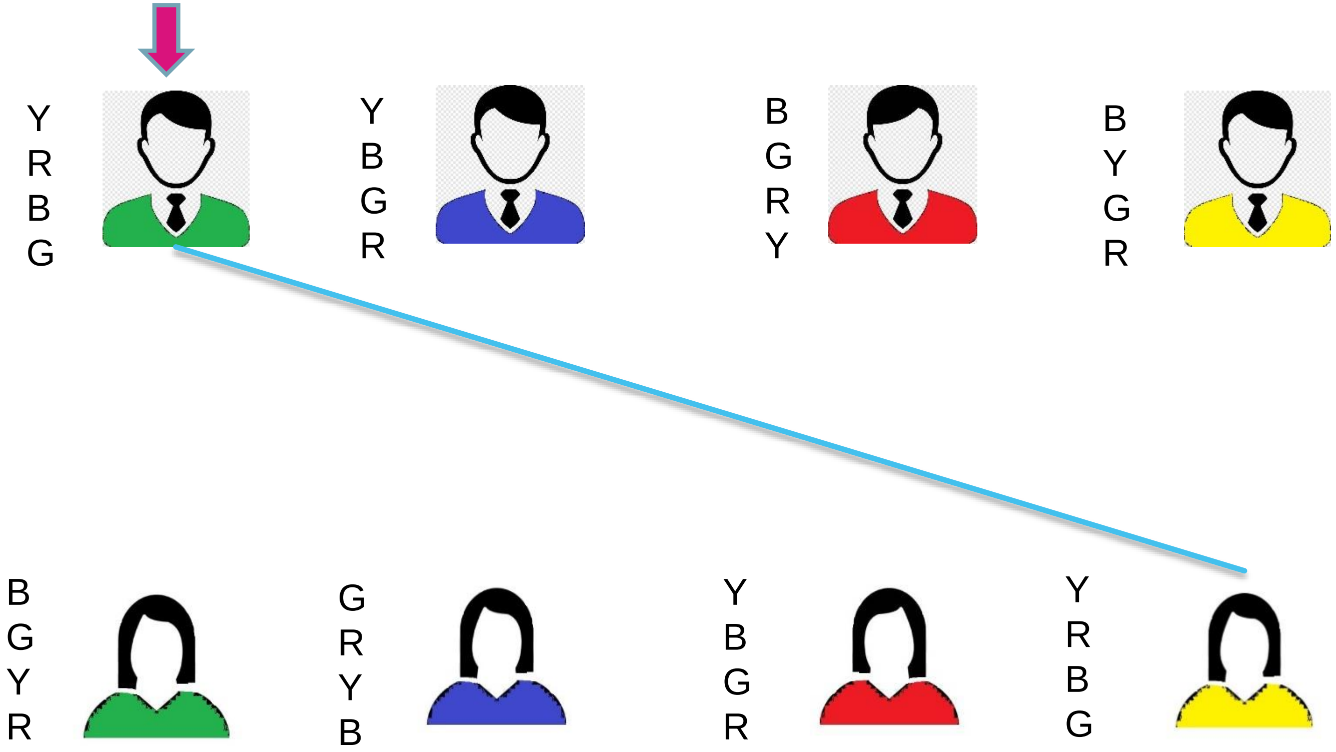
- There is

Gale-Shapely algorithm

- *W* prefers

No blocking pair

Problem



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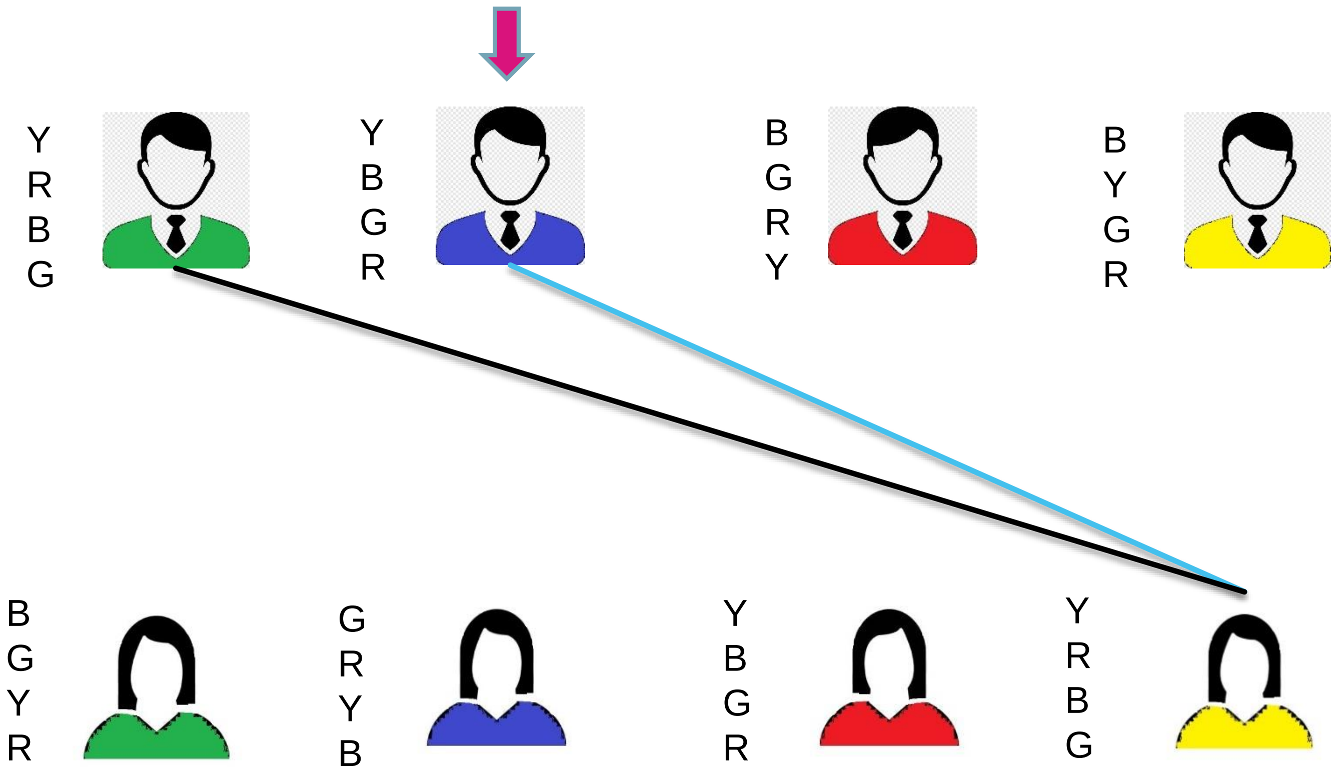

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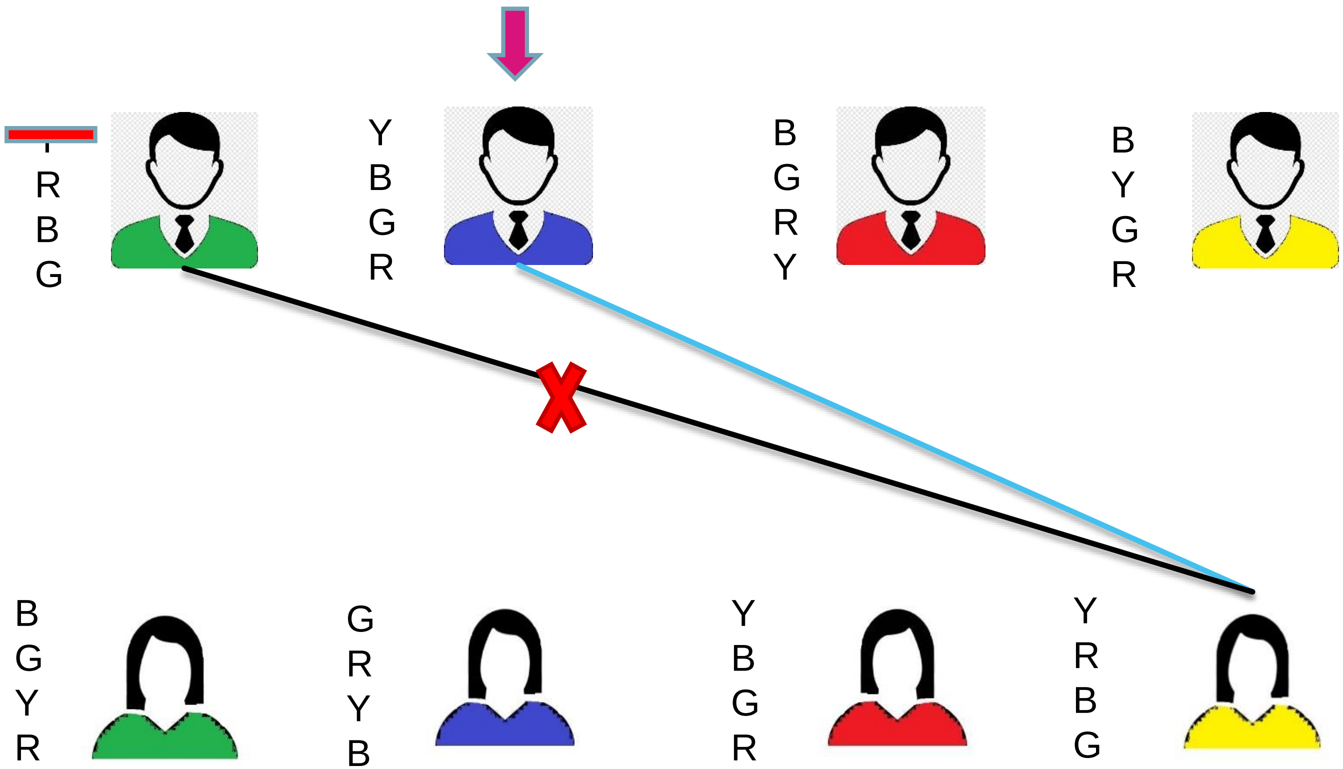

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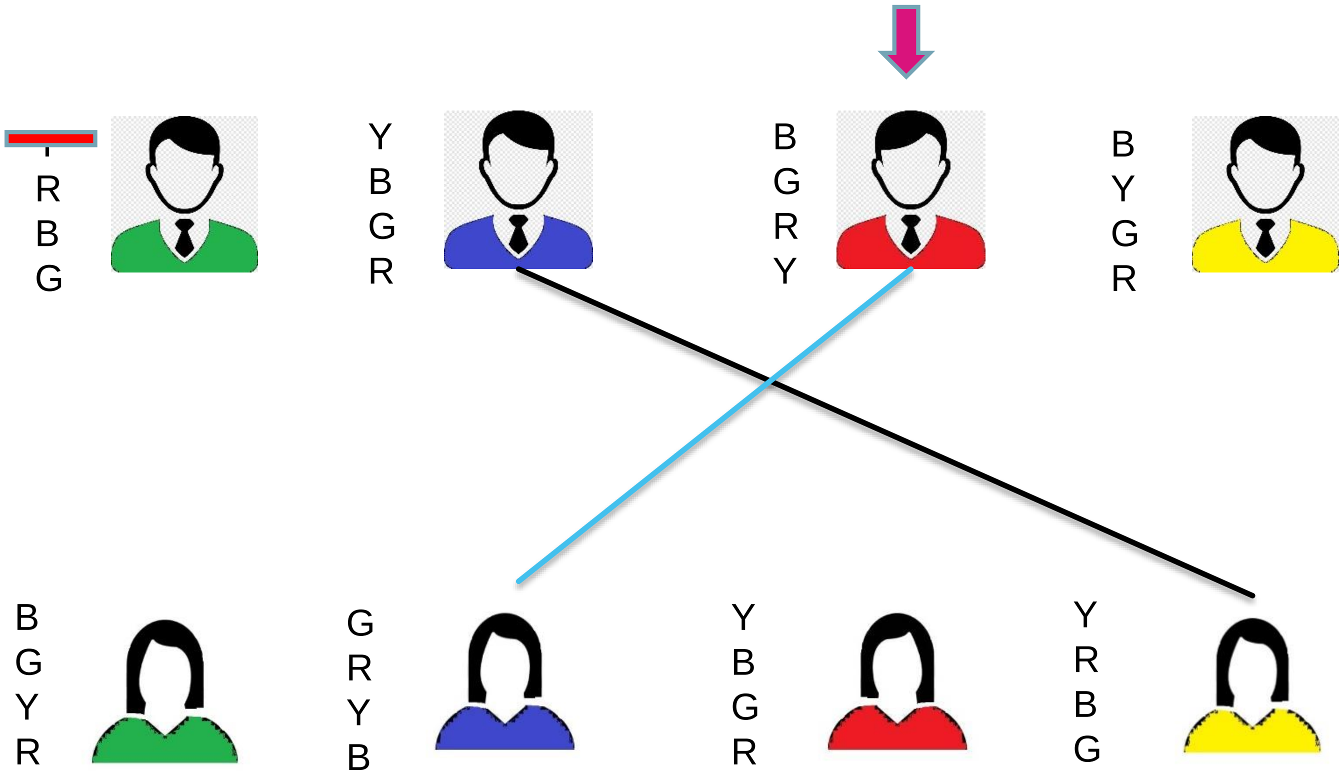

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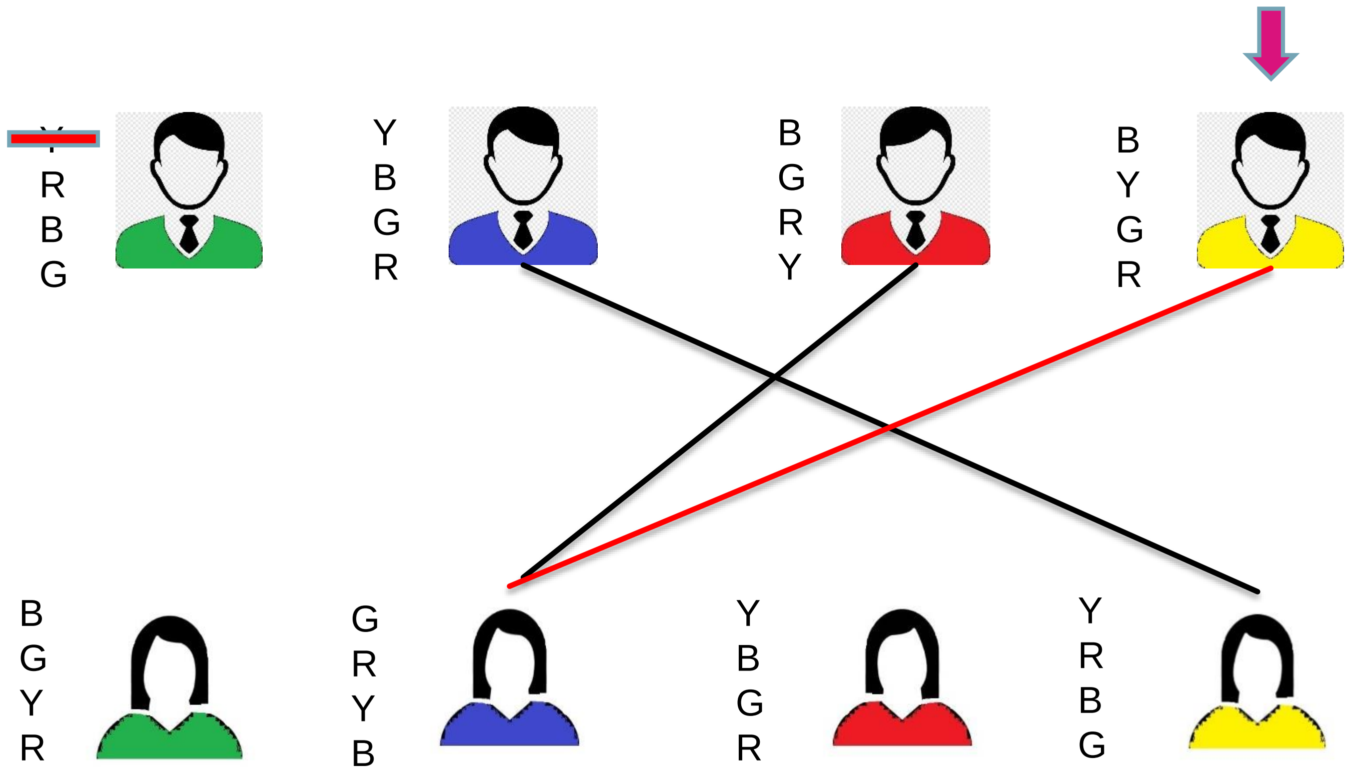

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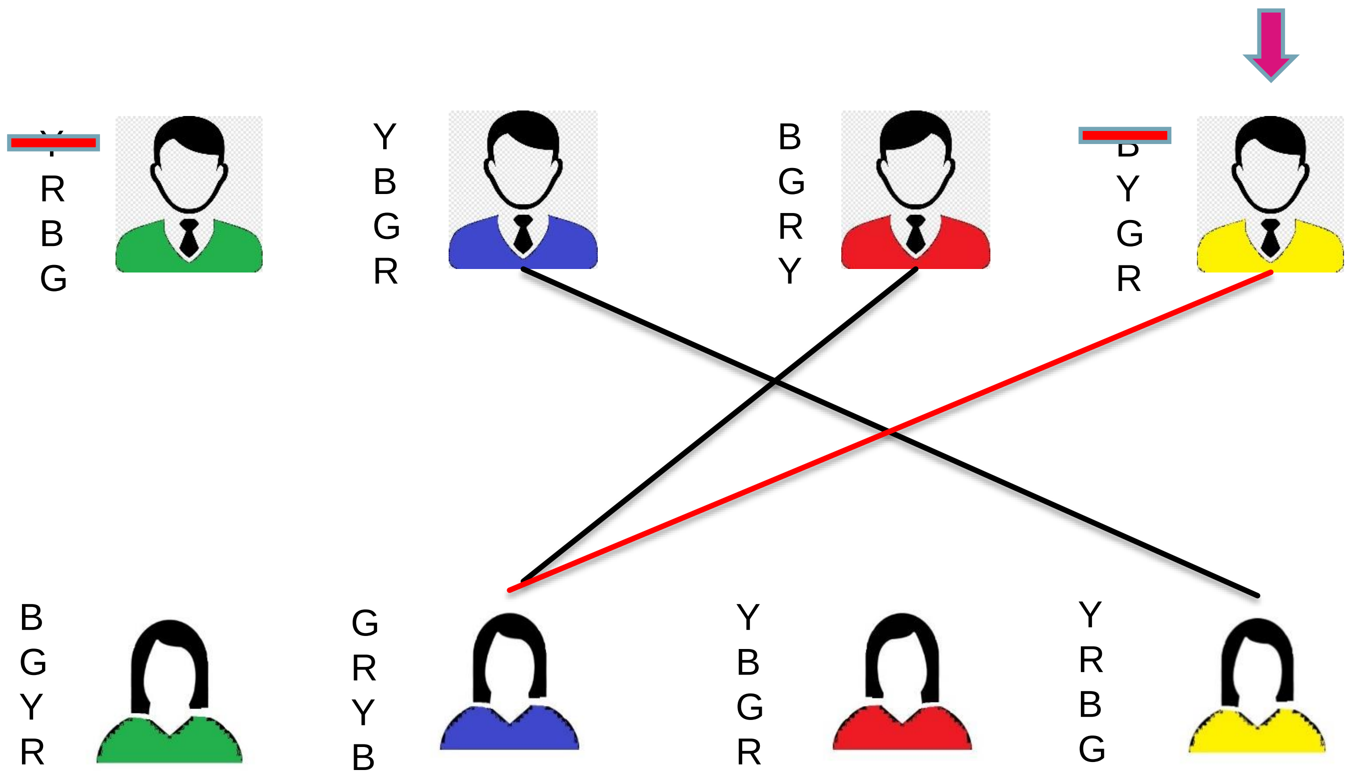

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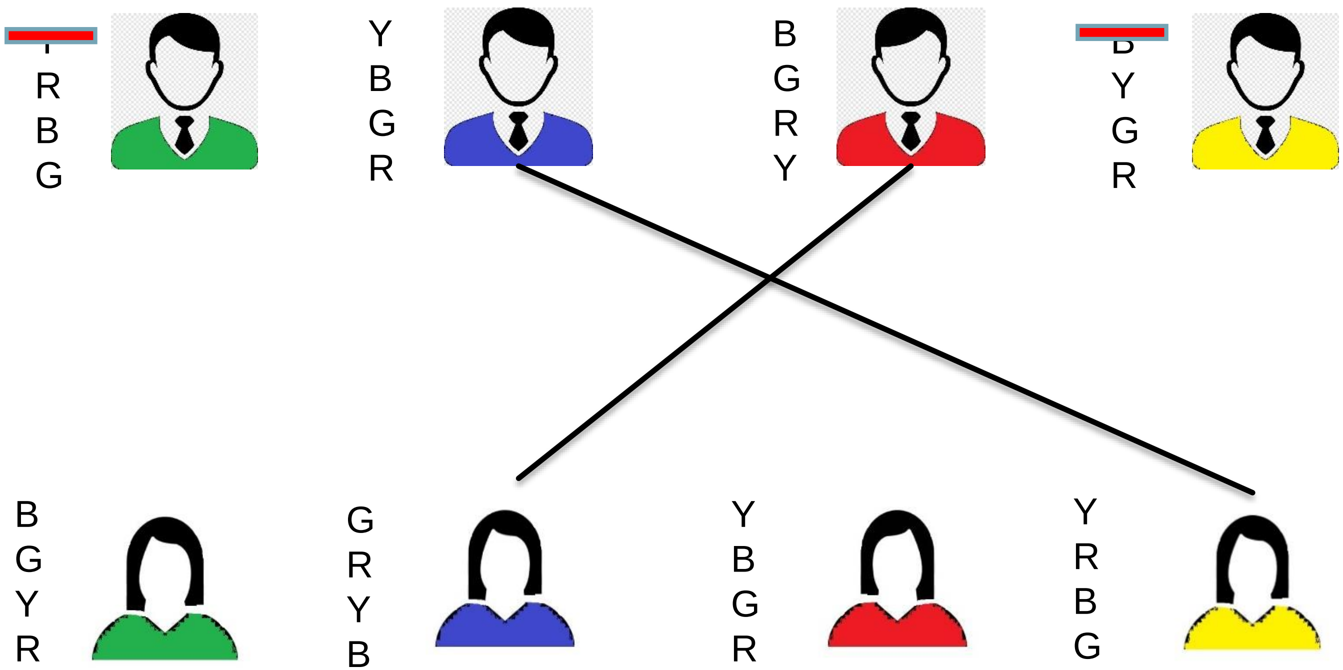

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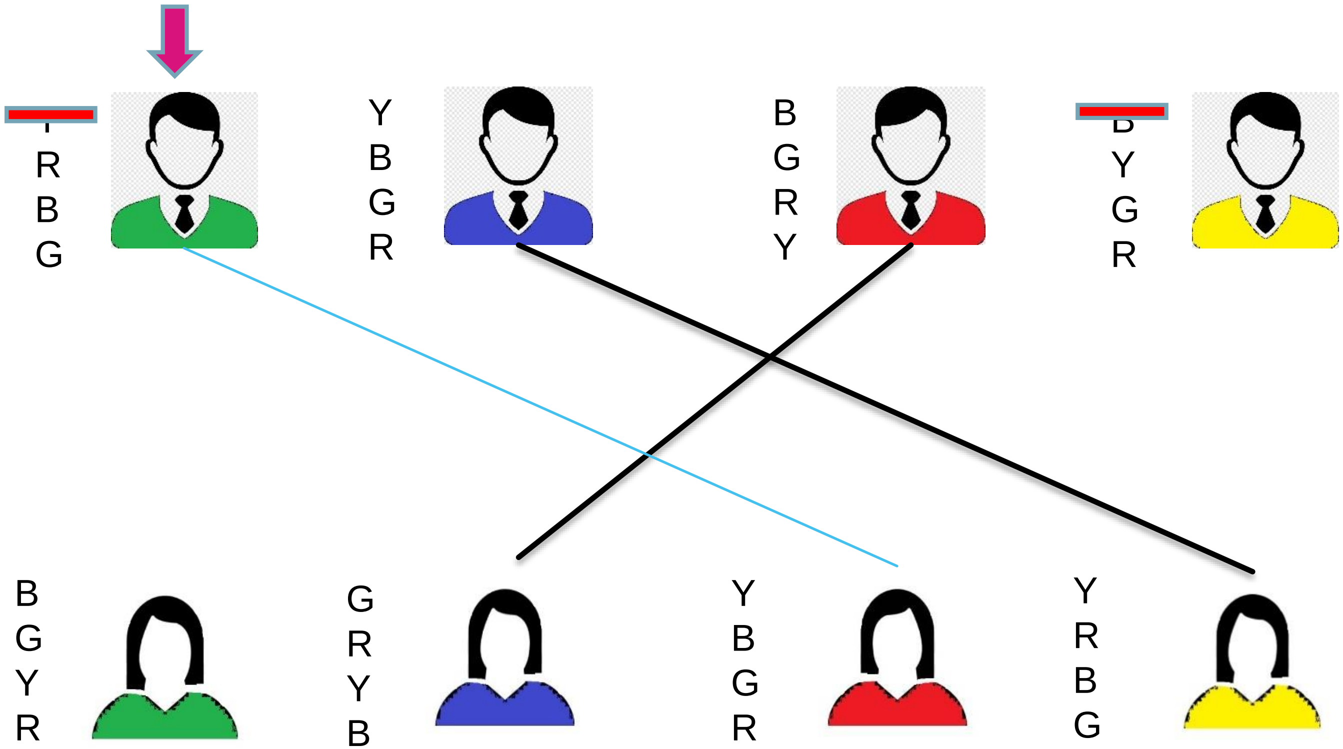

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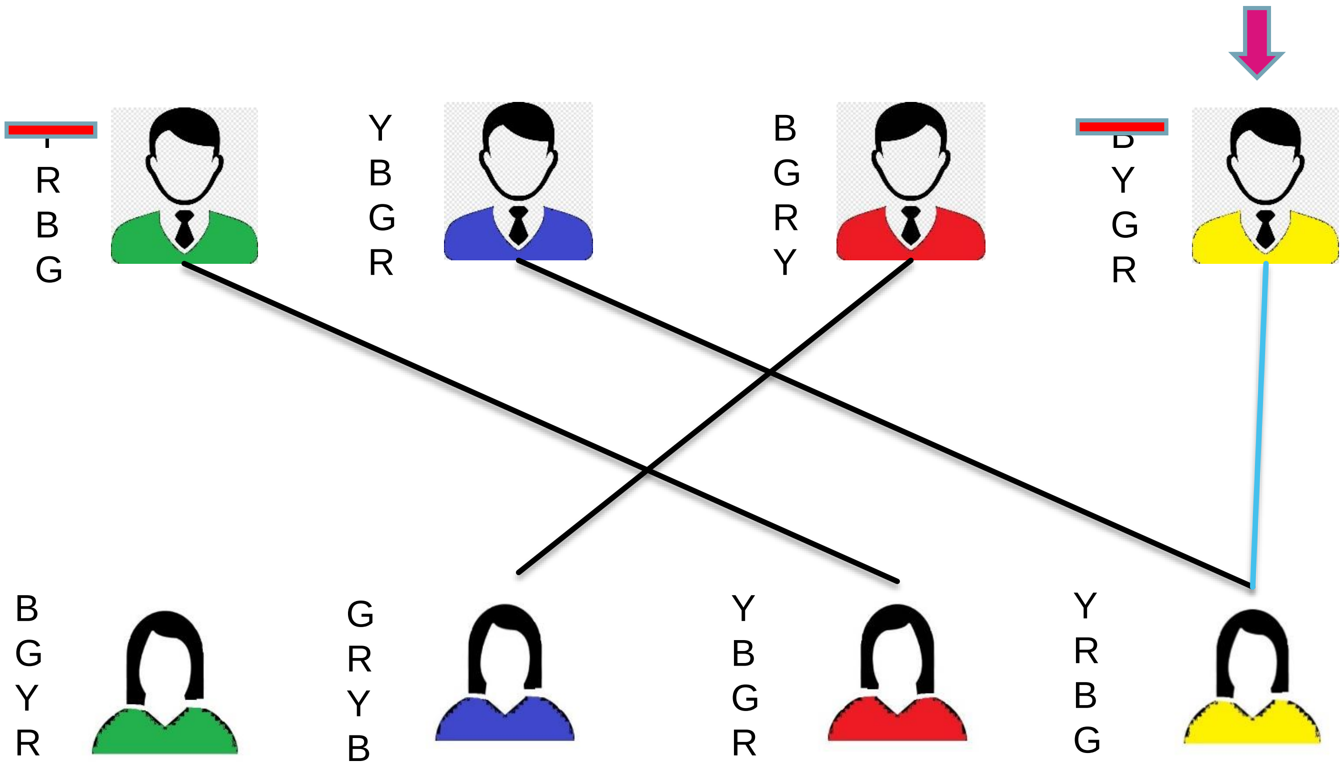

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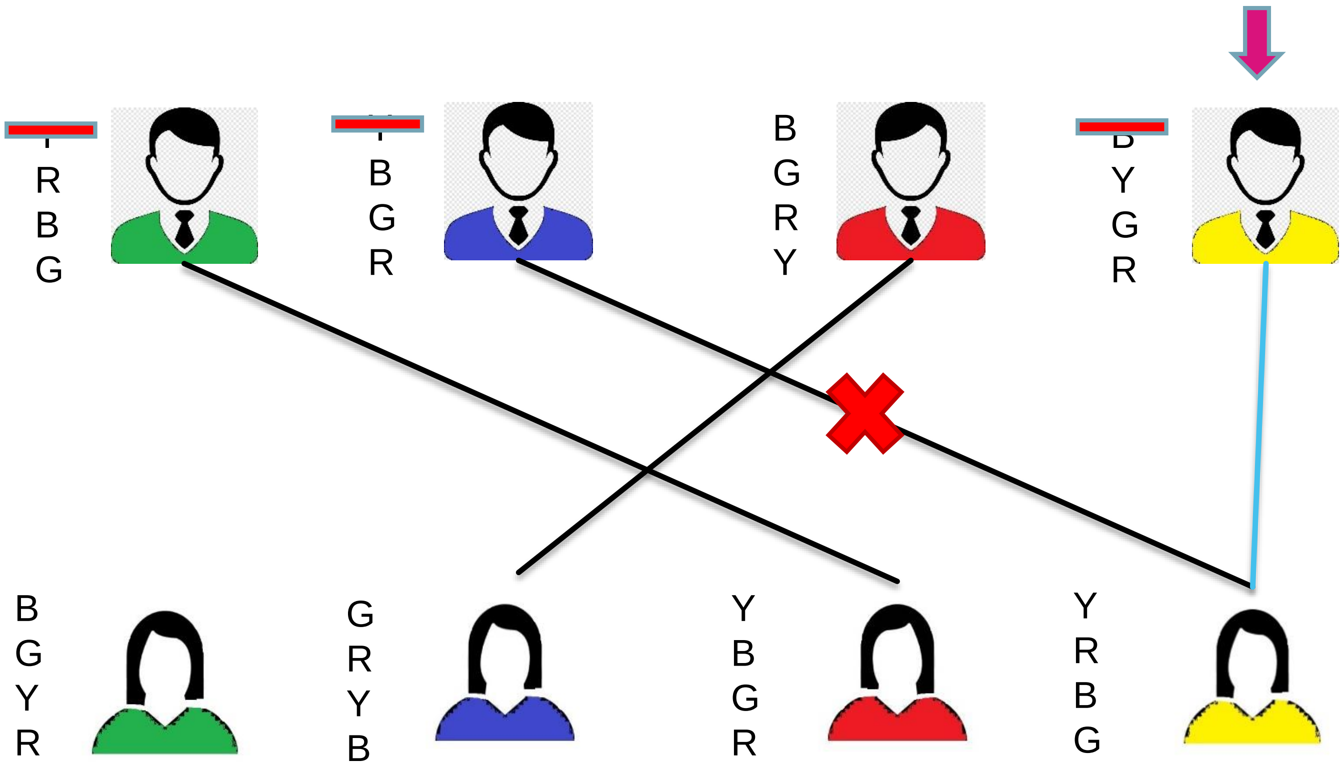

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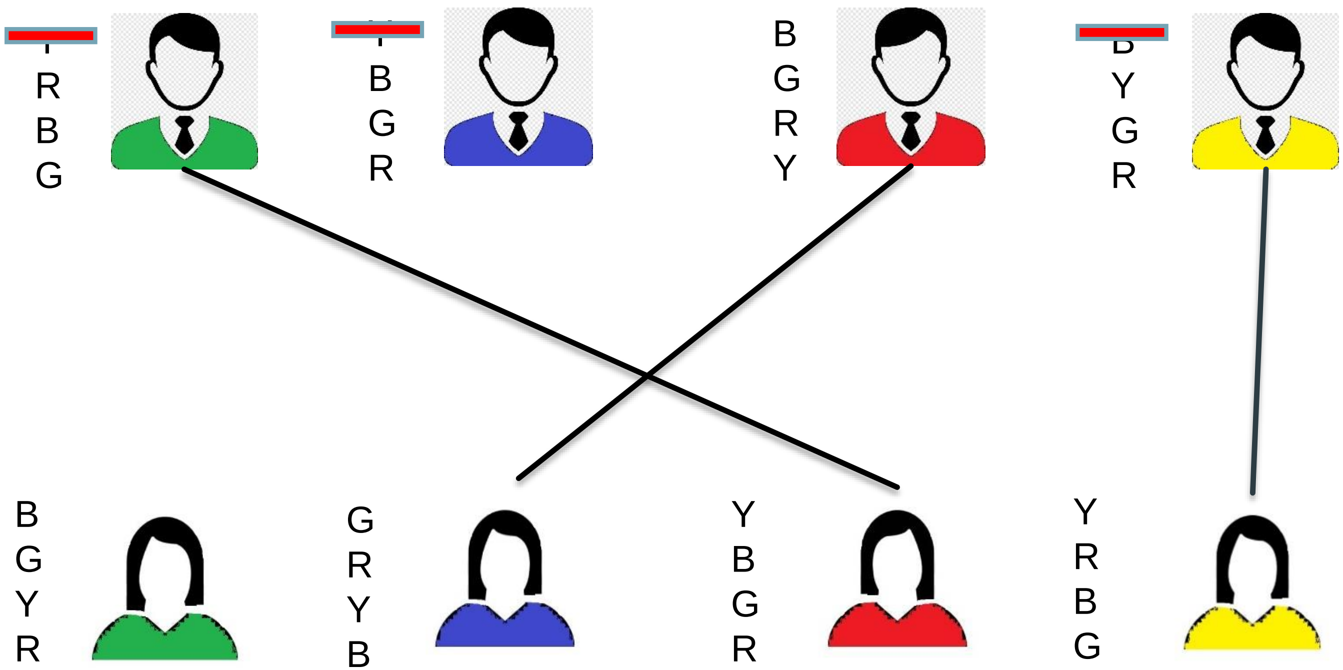

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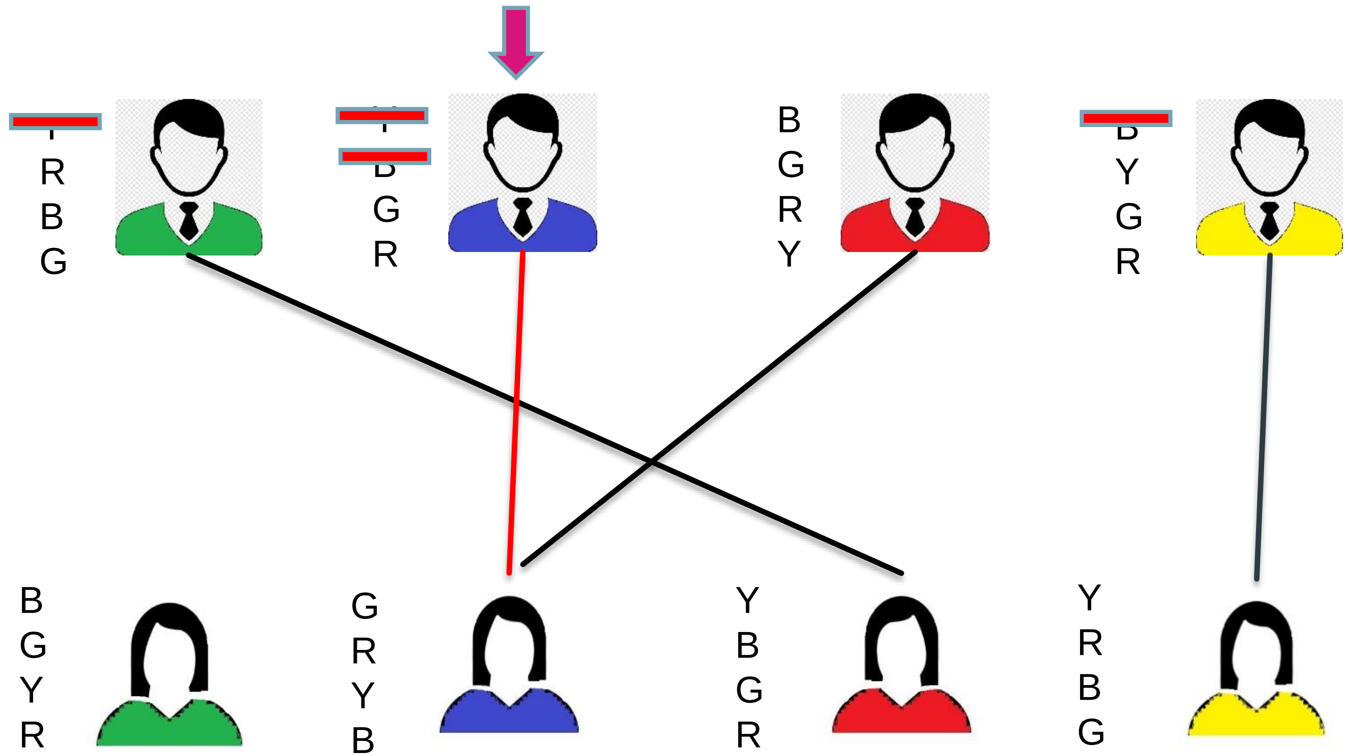

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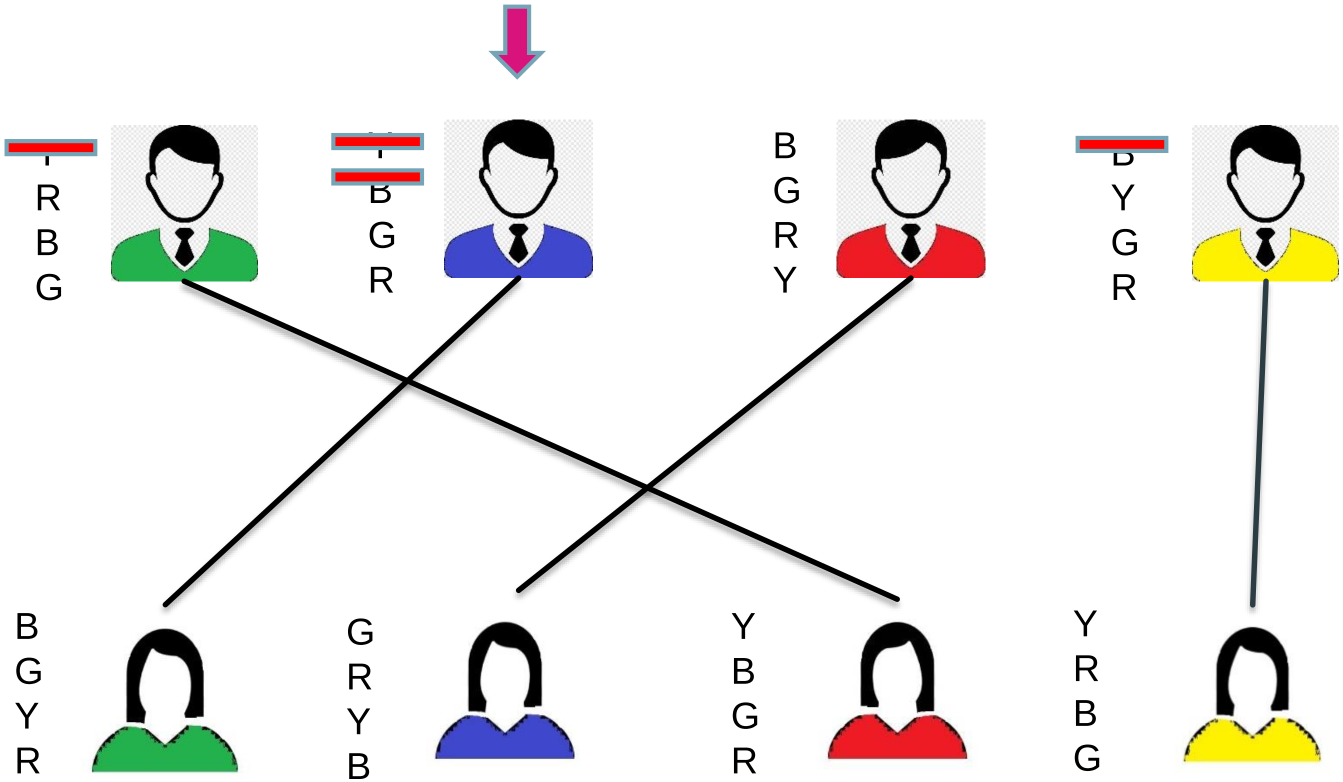

A perfect Matching

Proven Properties

(in class)

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1. The algorithm ends at most after n^2 rounds of proposal
2. The final matching is perfect
3. The final matching is stable

Optimal & Pessimal

Men	Women			
1	A	B	C	D
2	A	D	C	B
3	A	C	B	D
4	A	B	C	D

Women	Men			
A	1	3	2	4
B	4	3	2	1
C	2	3	1	4
D	3	4	2	1

Men	Women			
1	A	B	C	D
2	A	D	C	B
3	A	C	B	D
4	A	B	C	D

W	Men			
A	1	3	2	4
B	4	3	2	1
C	2	3	1	4
D	3	4	2	1

$$S = \{(1, A), (2, D), (3, C), (4, B)\}$$

$$T = \{(1, A), (2, C), (3, D), (4, B)\}$$

So there could be more than one stable pairing!

Best Possible Partner

- Best possible partner for man 2 ?
- (2,A) → Not Stable!
- Best possible realistic outcome for man 2 → (2,D)

$$S = \{(1,A),(2,D),(3,C),(4,B)\}$$
$$T = \{(1,A),(2,C),(3,D),(4,B)\}$$

Men	Women			
1	A	B	C	D
2	A	D	C	B
3	A	C	B	D
4	A	B	C	D

W	Men			
A	1	3	2	4
B	4	3	2	1
C	2	3	1	4
D	3	4	2	1

Optimal Woman for a Man

- The highest woman on his list whom he is paired with in **any** stable pairing.
- The optimal woman is the best that a man can do under the condition of stability.

- Can men achieve optimality **simultaneously** ?

Optimal Pairing

- Male Optimal Pairing:
  the pairing in which each man is paired with his optimal woman.

- Female Optimal Pairing:
  the pairing in which each woman is paired with her optimal man.

Pessimal Pairing

- Pessimal Partner:
  for a person is the lowest ranked partner whom (s)he is ever paired with in some stable pairing.

- Male Pessimal Pairing:
  the pairing in which each man is paired with his pessimal woman.

- Female Pessimal Pairing:
  the pairing in which each woman is paired with her pessimal man.

Traditional propose & reject algorithm

is male optimal !

Proof:

- Suppose for the sake of contradiction that the pairing is not male optimal.
- Consider M to be the first man who got rejected by his optimal woman called W* in favor of M* who proposed to her.
- $T = \{\ldots, (M, W^*), (M^*, W'), \ldots\}$
- $M^*\ likes\ W^*\ at\ least\ as\ much\ as\ his\ optimal\ woman.$
- $(M^*, W^*) \rightarrow rogue\ couple$
- Contradiction!

If a pairing is male optimal, then it is also

female pessimal !

Proof:

- $T = \{\ldots, (M, W), \ldots\} \rightarrow male\ optimal$ pairing
- Suppose for the sake of contradiction that there exists a stable pairing S:
- $S = \{\ldots, (M^*, W), (M, W'), \ldots\}$ that $M^*\ is\ lower\ on\ W\ list\ than\ M$
- $W\ prefers\ M\ to\ M^*$ and M prefers W to his partner W'
- $So\ (M, W)\ is\ a\ rouge\ couple\ in\ S \rightarrow S\ is\ not\ stable$
- Contradiction!

In conclusion,

Make The First Move!!!