Eigenvectors and Eigenvalues

CE40282-1: Linear Algebra
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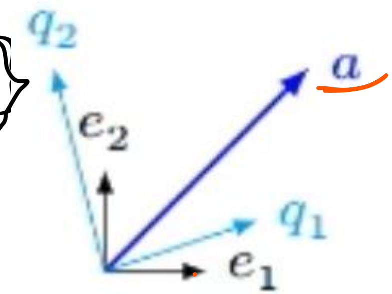
Review

- n-vector a based on basis $\{e_1, \dots, e_n\}$



$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

- n-vector a based on new basis $\{q, \dots, q_n\}$

$$a = \overline{a_1} q_1 + \overline{a_2} q_2 + \cdots + \overline{a_n} q_n = \underbrace{[q_1 \quad \dots \quad q_n]}_{Q} \begin{bmatrix} \overline{a_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{a_n} \end{bmatrix}$$

- Matrix Q is invertible.
- Any invertible matrix is a basic matrix.

$\boxed{a = Q\bar{a}}$

Review

$Aa = AQ\bar{a} = b = Q\bar{b}$

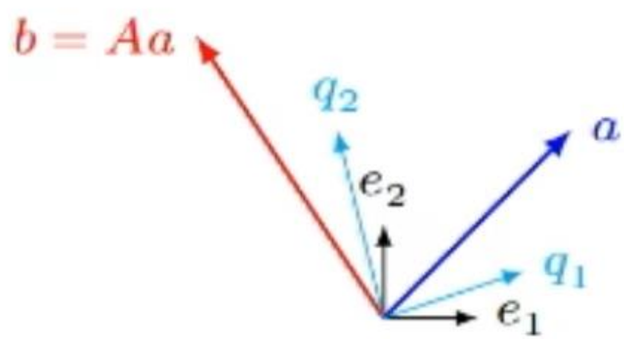
$Aa = b$

- A square matrix for a linear transform

$$A : n \times n \quad A : \mathbb{R}^n \to \mathbb{R}^n \implies \boxed{Aa = b} \quad a, b \in \mathbb{R}^n$$

$$\left.\begin{array}{l} a = Q\bar{a} \\ b = Q\bar{b} \end{array}\right\} \implies AQ\bar{a} = Q\bar{b} \implies \underbrace{Q^{-1}AQ}_{\bar{A}}\,\bar{a} = \bar{b} \implies \bar{A}\bar{a} = \bar{b}$$

$$\underbrace{\boxed{Q^{-1}AQ}}_{\bar{A}}$$



- Linear transform in new basis $\bar{A} = Q^{-1}AQ$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Similar Matrices

- Two n-by-n matrices A and B are called similar if there exists an **invertible n-by-n matrix Q** such that

$$B = QAQ^{-1}$$

$$\boxed{A = Q^{-1}BQ}$$

$$QA = \underbrace{QQ^{-1}}_{I}BQ$$

$$QA = BQ$$

- A and B are similar if $QA = BQ$
- $A = Q^{-1}BQ \ \rightarrow B = QAQ^{-1}$
- Same determinant ✓
- Inverse of A and B are similar (if exists) ✓

$$|A| = \underbrace{|Q^{-1}|}_{\frac{1}{\cancel{|Q|}}} |B| \cancel{|Q|} =$$

Similarity Transformation

- We can use similarity transformation for changing the standard matrix of linear transformation

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

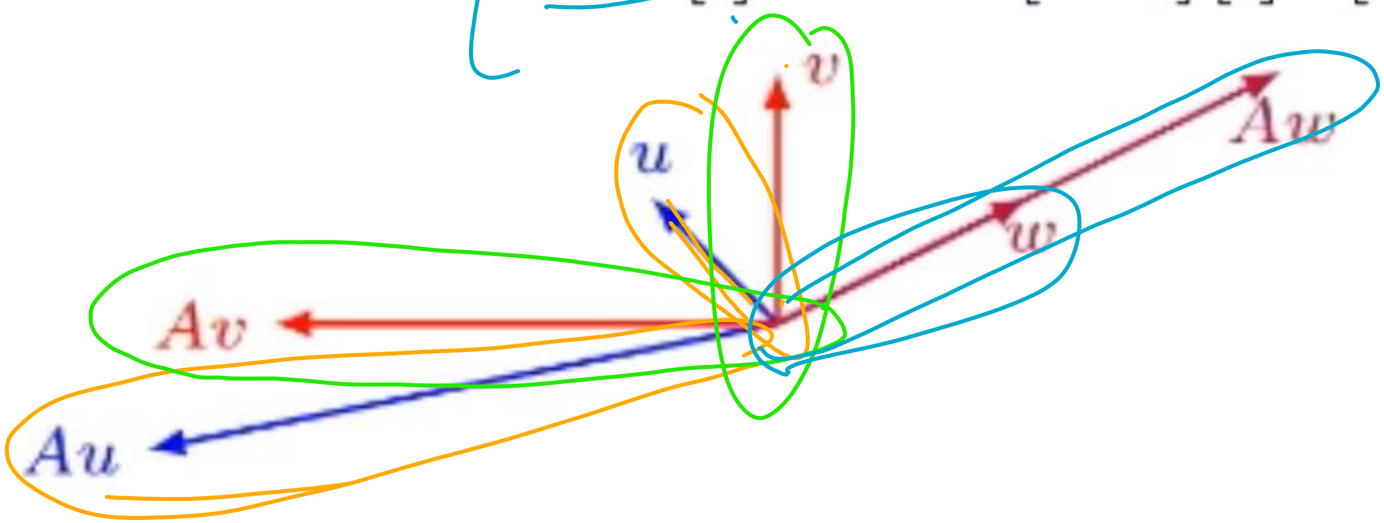
Motivation

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \implies Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2w$$



Definition

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

- An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.
- Example

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad , \lambda = 2.$$

  - Show that 7 is an eigenvalue of matrix A, and find the corresponding eigenvectors.

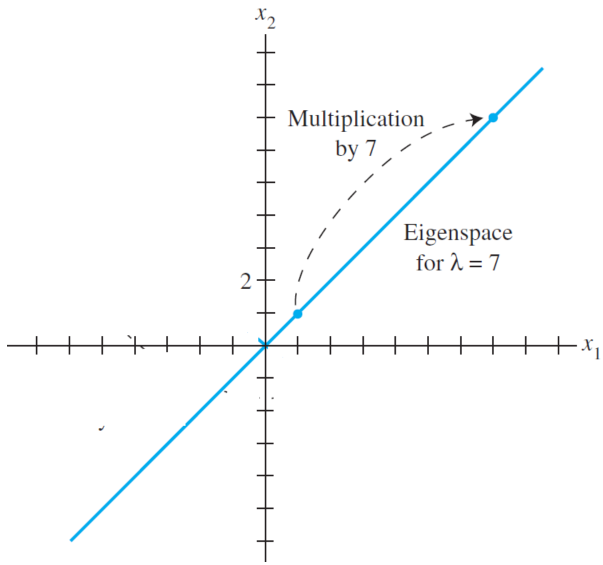
$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.



Characteristic Equation

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $Av = \lambda v \implies Av - \lambda vI = 0 \implies \boxed{(A - \lambda I)v = 0 \quad v \neq 0}$
- Characteristic equation $|A - \lambda I| = 0$
- Characteristic polynomial $|A - \lambda I|$ $\underline{\Delta_A(\lambda)}, \underline{\Delta(\lambda)}$
  - Matrix $n \times n$ has n.... eigenvalue

$\Delta(\lambda) =$ $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = ($

Characteristic Equation

- Example

$$\lambda^4(\lambda^2-4\lambda-12)=\lambda^{4}(\lambda-6)(\lambda+2)$$

$6 \quad -2$

-2

$+6$

0

  - The characteristic polynomial of a 6×6 matrix is $\lambda^6-4\lambda^5-12\lambda^4$. Find the eigenvalues and their multiplicities.

  - $A=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$ $A=\begin{bmatrix}2 & 1\\ -1 & 0\end{bmatrix}$ $A=\begin{bmatrix}-2 & 1\\ -2 & 0\end{bmatrix}$

Matrix spectrum

$\sigma(A)=\{5,1,0,0,6,-2\}$

- Set of all eigenvalues of matrix $\sigma(A)$
- Theorem: The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal
  - Proof?
- . $0 \in \sigma(A) \Leftrightarrow |A| = 0$
- A is invertible if and only if ……… حاصلضرب مقادیر ویژه‌اش صفر نباشد
- 0 is an eigenvalue of A if and only if A is not invertible.

Similar Matrices

- Similar matrices has equal characteristic equation
  - vice versa?

- Example
  - $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}, \bar{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ → $\sigma(\bar{A}) = \{3, 2, 3\}$
  - $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ → $\{2, 2\}$

$\{2, 2\}$

Eigenvectors Linear Independence

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

- One way to prove the statement "If P then Q" is to show that P and the negation of Q leads to a contradiction

- Distinct eigenvalues -> eigenvectors are LI
- Duplicate eigenvalues -> ???
  - Example

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Some notes

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

The number 0 is *not* an eigenvalue of A.

The determinant of A is *not* zero.

WARNINGS:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.