Vector Space-2

CE40282-1: Linear Algebra

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General Examples

- The inner product of a vector with the ith standard unit vector gives (or `picks out') the ith element of a.

$$e_i^T a = a_i$$

- The inner product of a vector with the vector of ones gives the sum of the elements of the vector.

$$\mathbf{1}^T a = a_1 + \cdots + a_n$$

- The inner product of an n-vector with the vector $\mathbf{1}/n$ gives the average or mean of the elements of the vector.

$$\boldsymbol{aveg}(a) = \boldsymbol{\mu}_a = (\mathbf{1}/n)^T a = (a_1 + \cdots + a_n)/n$$

General Examples

- The inner product of a vector with itself gives the sum of the squares of the elements of the vector.

$$a^T a = a_1^2 + \cdots + a_n^2$$

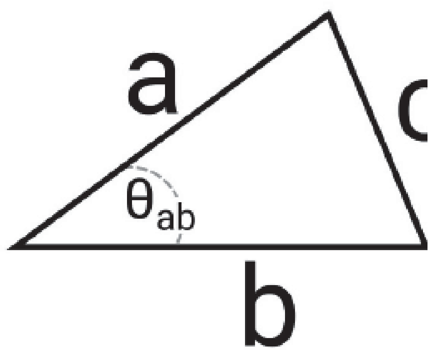
- Selective sum: Let b be a vector all of whose entries are either 0 or 1. Then $b^T a$ is the sum of the elements in a for which $b_i = 1$.

Inner product of block vectors

- If two block vectors conform, then the inner product of them is the sum of inner products of the blocks:
  - Proof?

Vector dot product: Geometry

- Dot Product: the cosine of the angle between the two vectors, times the lengths of the two vectors.

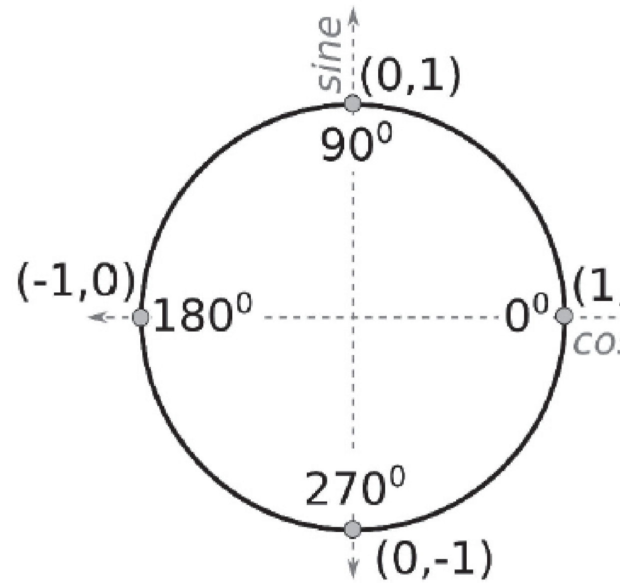


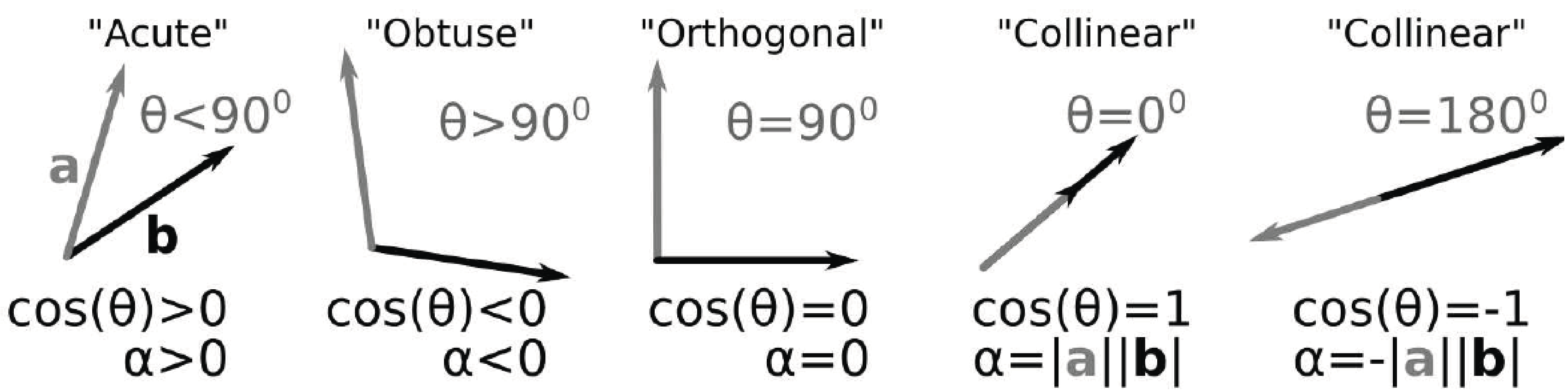
$$\mathbf{a}^T\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta_{ab})$$

  - proof
- In statistics, cos() with suitable normalization is called the Pearson correlation coefficient.

Vector dot product: Geometry

- $\theta < 90^o$
- $\theta > 90^o$
- $\theta = 90^o$: vectors are orthogonal
- $\theta = 0^o$: collinear
- $\theta = 180^o$: collinear





Vector-Vector Products

- Given two vectors $x \in R^m, y \in R^n$:
  - $x \otimes y = xy^T \in R^{m \times n}$ is called the outer product of the vectors: $(xy^T)_{ij} = x_i y_j$

$$xy^T \in \mathbb{R}^{m \times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

  - Is it symmetric?
  - Example: Represent $A \in R^{m \times n}$ with outer product of two vectors:

$$A = \begin{bmatrix} | & | & & | \\ x & x & \cdots & x \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m & \cdots & x_m \end{bmatrix}$$

Outer Products

- Properties:
  - $(u \otimes v)^T = (v \otimes \mathrm{u})$
  - $(v + w) \otimes u = v \otimes u + w \otimes u$
  - $u \otimes (v + w) = u \otimes v + u \otimes w$
  - $c(v \otimes u) = (cv) \otimes u = v \otimes (cu)$
  - $(u.v) = trace(u \otimes v) \;\; (u, v \,\epsilon R^n)$
  - $(u \otimes v)w = (v.w)u$

Hadamard vector product

- Element-wise product

$$\mathbf{c} = \mathbf{a} \odot \mathbf{b} = \begin{bmatrix} \mathrm{a}_1\mathrm{b}_1 & \mathrm{a}_2\mathrm{b}_2 & \dots & \mathrm{a}_n\mathrm{b}_n \end{bmatrix}$$

- Properties:
  - $a \odot b = b \odot a$
  - $a \odot (b \odot c) = (a \odot b) \odot c$
  - $a \odot (b + c) = a \odot b + a \odot c$
  - $(\theta a) \odot b = a \odot (\theta b) = \theta (a \odot b)$
  - $a \odot \mathbf{0} = \mathbf{0} \odot a = \mathbf{0}$

Cross product

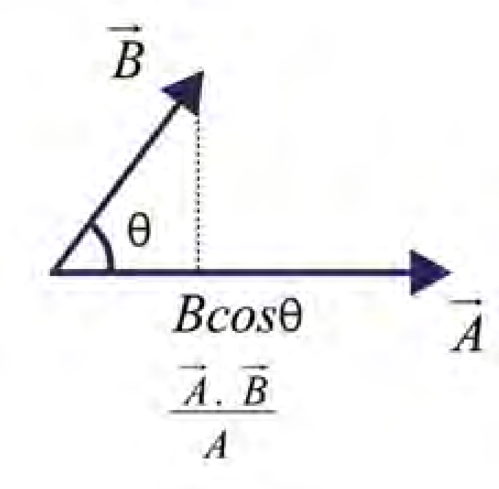
- The cross product is defined only for two 3-element vectors, and the result is another 3-element vector. It is commonly indicated using a multiplication symbol (×).

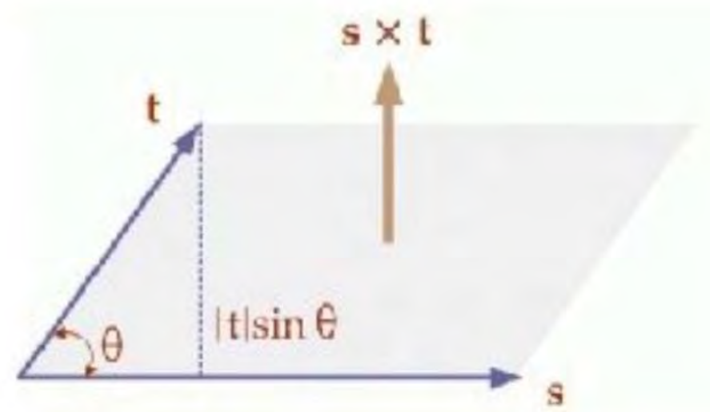
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta_{ab})$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

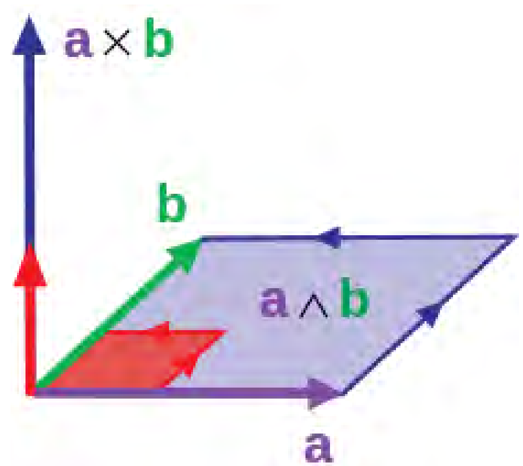
- It used often in geometry, for example to create a vector c that is orthogonal to the plane spanned by vectors a and b. It is also used in vector and multivariate calculus to compute surface integrals.

Products



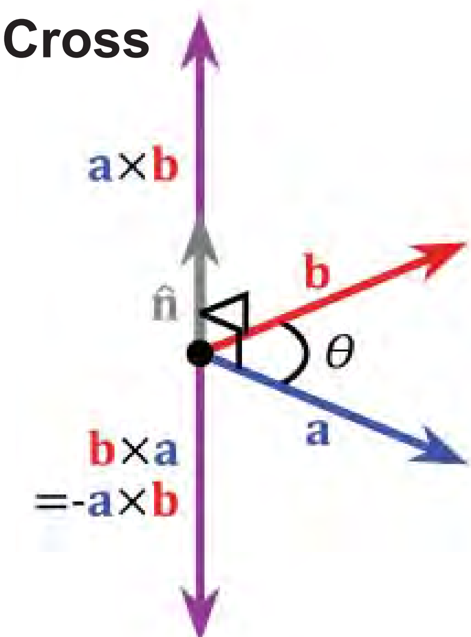


Dot



Wedge and Cross

Cross



Span or linear hull

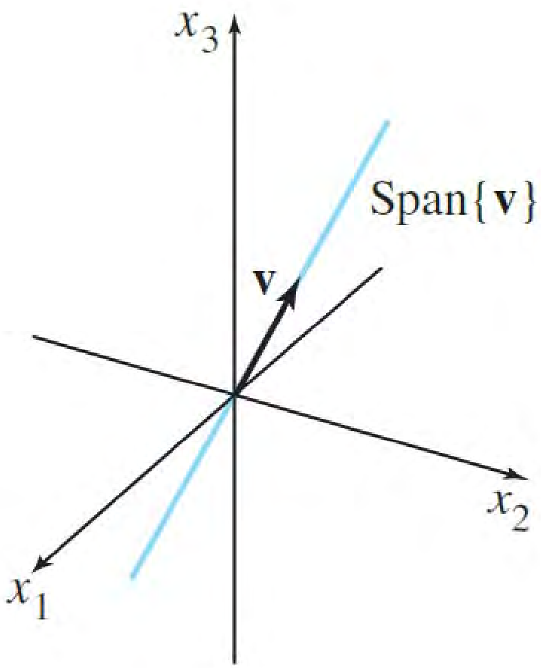
- If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \ldots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

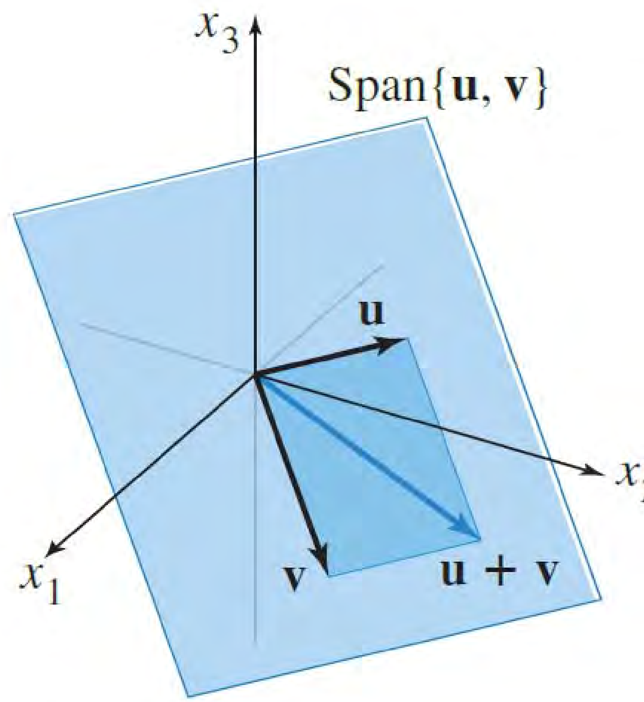
  with $c_1, \ldots, c_p$ scalars.
- Examples:
  - Is vector b in Span$\{v_1, \ldots, v_p\}$?
  - Is vector v_3 in Span$\{v_1, \ldots, v_p\}$?
  - Is vector $\mathbf{0}$ in Span$\{v_1, \ldots, v_p\}$?
  - Is b in Span$\{a_1, a_2\}$? $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

Span

- v and u are non-zero vectors in R^3 where v not a multiple of u



Span {**v**} as a line through the origin.



Span {**u**, **v**} as a plane through the origin.

Linear Combinations

- For vectors $x_1, x_2, \ldots, x_k$:any point y is a linear combination of them iff:

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \in \mathbb{R}$$

- If we restrict α_i's to be positive then we get a conic combination.

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \geq 0 \in \mathbb{R}$$

- Instead of being positive, if we put the restriction that α_i's sum up to 1, it is called an affine combination

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \in \mathbb{R}, \quad \sum \alpha_i = 1$$

- When a combination is affine as well as conic, it is called a convex combination

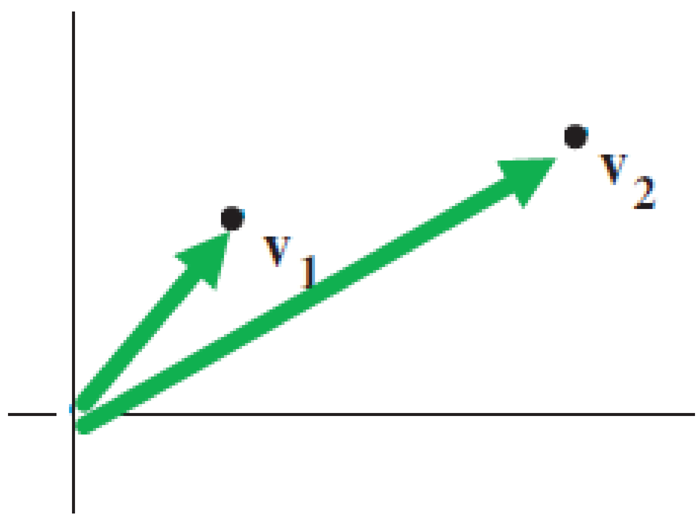
$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \geq 0 \in \mathbb{R}, \quad \sum_i \alpha_i = 1$$

Linear Combinations

	Linear	Affine	Convex	Conic
x in R^2				
x_1, x_2 in R^2				
x_1, x_2 in R^3				

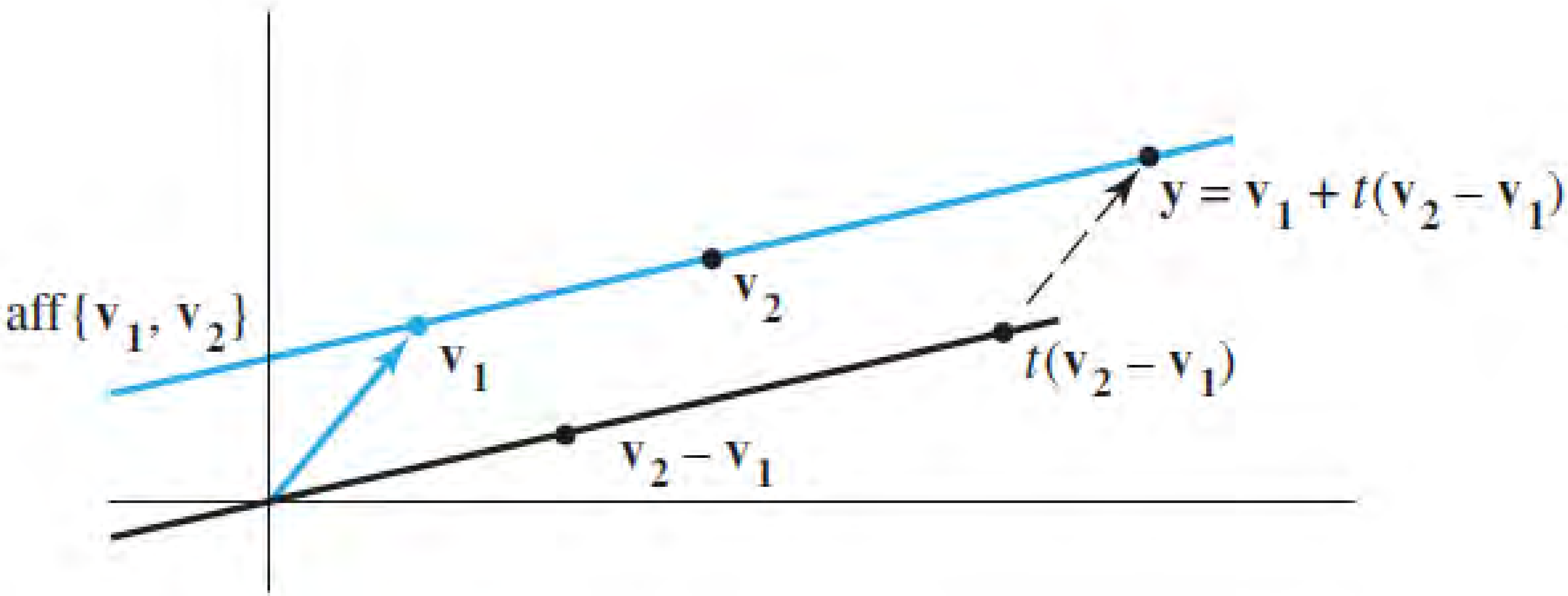
Affine

- The set of all affine combinations of points in a set S is called the affine hull (or affine span) of S, denoted by aff(S).
- What is the affine hull of single point v_1?
- What is the affine hull of two distinct $\{v_1, v_2\}$?



Affine

- What is the affine hull of two distinct $\{v_1, v_2\}$?
  - Answer: the blue line



Affine

- Theorem:

A point y in R^n is an affine combination of $v_1, \ldots, v_p$ in R^n if and only if $y - v_i$ is a linear combination of the translated points $v_2 - v_i, \ldots, v_p - v_i$ (where $1 \leq i \leq p$ and $i \neq 2, p$)

  - Proof?

Affine

- A set is called affine iff for any two points in the set, the line through them is contained in the set. In other words, for any two points in the set, their affine combinations are in the set itself.
- Theorem: A set (S) is affine iff any affine combination of points in the set is in the set itself.
  S is affine if and only if S=aff(S)