Vector Space-3

CE40282-1: Linear Algebra
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Affine

- Which are affine sets?
  - Square in R^2
  - R^2
  - $\{x: Ax = b\}$
  - $\{x\}$

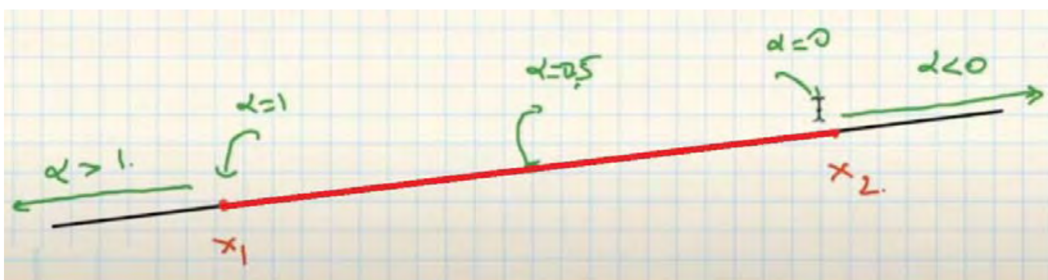
Convex combination

A **convex combination** of points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is a linear combination of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

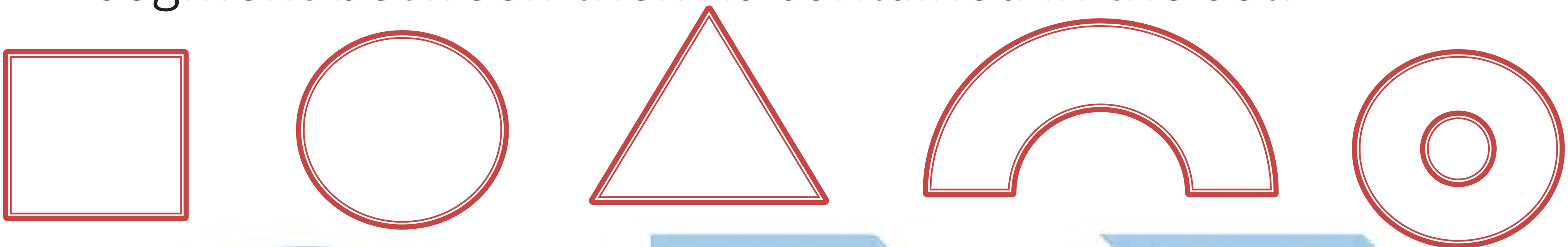
such that $c_1 + c_2 + \cdots + c_k = 1$ and $c_i \geq 0$ for all i. The set of all convex combinations of points in a set S is called the **convex hull** of S, denoted by conv S.

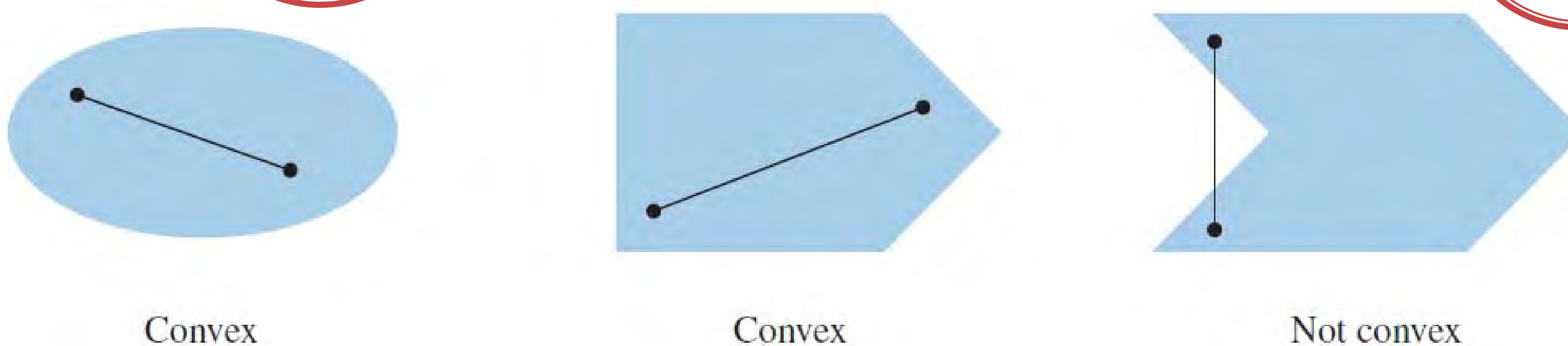
- The convex hull of a single point v_1?
- What are the points in the $conv(v_1, v_2\}$?
  - $y = (1-\alpha)v_1 + \alpha v_2 \quad 0 \leq \alpha \leq 1$
  - The line segment between v_1, v_2 denoted by $\overline{v_1 v_2}$



Convex Set

- A set S is convex any two points in the set, the line segment between them is contained in the set.





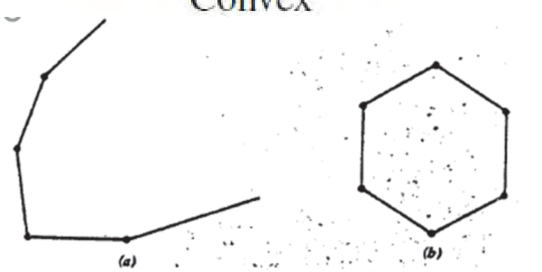


Figure 3.11 (a) A convex polyhedron that is not a polytope. (b) A convex polytope.

Convex Set

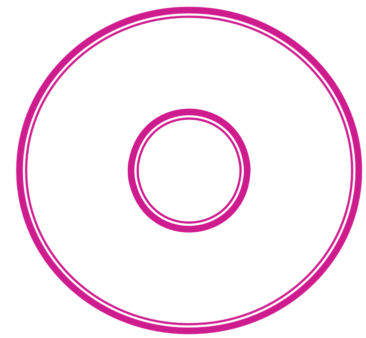
- Theorem: A set (S) is convex iff every convex combination of points of S lines in S.
  S is convex if and only if S=conv(S)

Affine and Convex set

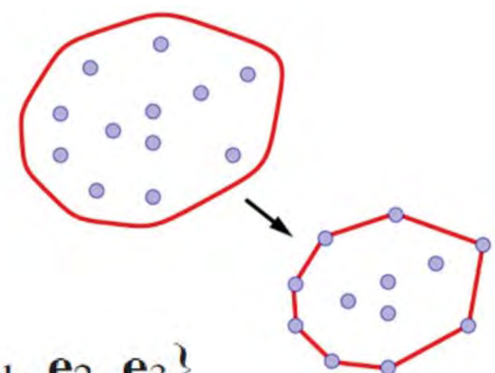
- Theorem: Let $\{S_\alpha : \alpha \in A\}$ be any collection of convex sets. Then $\bigcap_{\alpha \in A} S_\alpha$ is convex.

- Theorem: Let $\{S_\alpha : \alpha \in A\}$ be any collection of affine sets. Then $\bigcap_{\alpha \in A} S_\alpha$ is affine.

Convex Hull

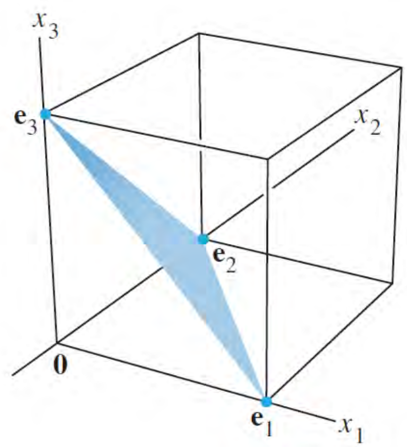


- Convex Hull Conv(S) is set of all convex combinations of points in S
- Convex Hull is the smallest convex polygon containing a given set of points



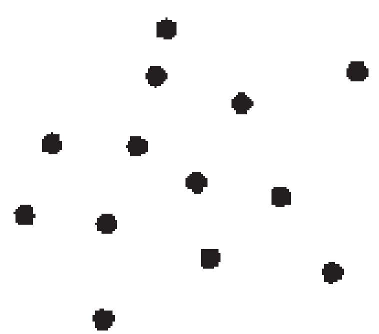
- S be the set consisting of the standard basis for $\mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

  the conv(S) is:

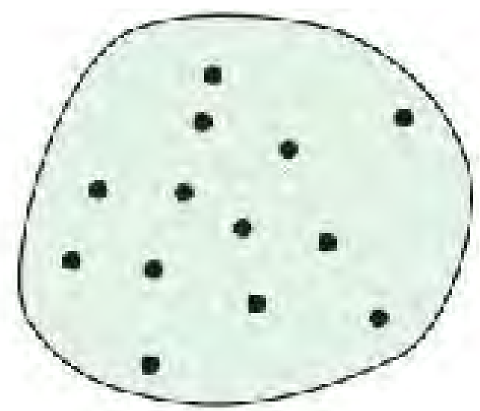


Convex hull

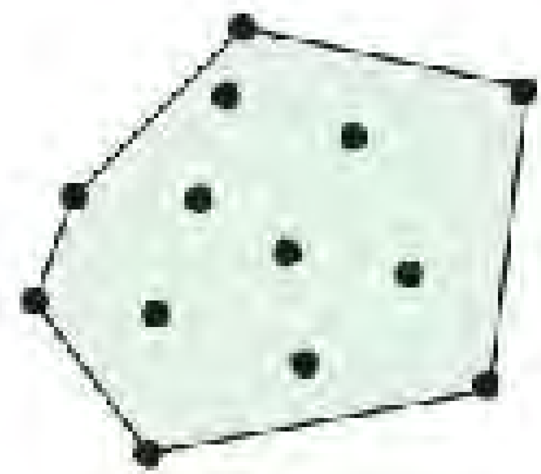
- Theorem: For any set P, the convex hull of P (conv(P)) is the intersection of all the convex sets that contain P.



set P



convex set



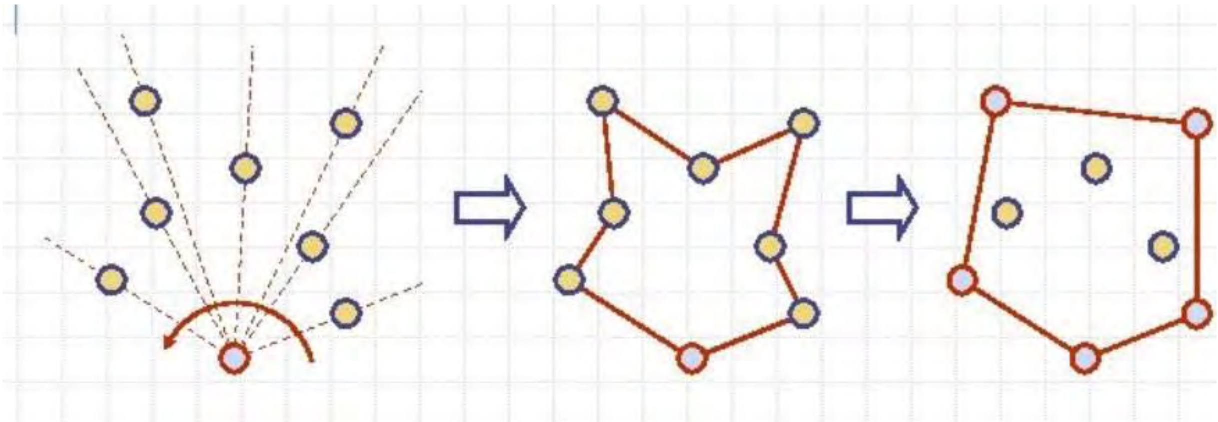
convex hull (conv P)

For any $P \subseteq \mathbb{R}^d$ *we have*

$$\operatorname{conv}(P) = \left\{ \sum_{i=1}^{n} \lambda_i p_i \;\middle|\; n \in \mathbb{N} \wedge \sum_{i=1}^{n} \lambda_i = 1 \wedge \forall i \in \{1, \ldots, n\} : \lambda_i \geqslant 0 \wedge p_i \in P \right\}$$

Convex hull

- One way:
  - Phase 1: Find the lowest point (anchor point)
  - Phase 2: Form a nonintersecting polygon by sorting the points counterclockwise around the anchor point
  - Phase 3: While the polygon has a nonconvex vertex, remove it



- Algorithms for finding a convex hull:
  - Graham Scan
  - Jarvis March
  - Divide & Conquer

Convex hull

- Theorem:

(Caratheodory) If S is a nonempty subset of $\mathbb{R}^n$, then every point in conv S can be expressed as a convex combination of $n+1$ or fewer points of S.

  - Proof later
  - Example

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

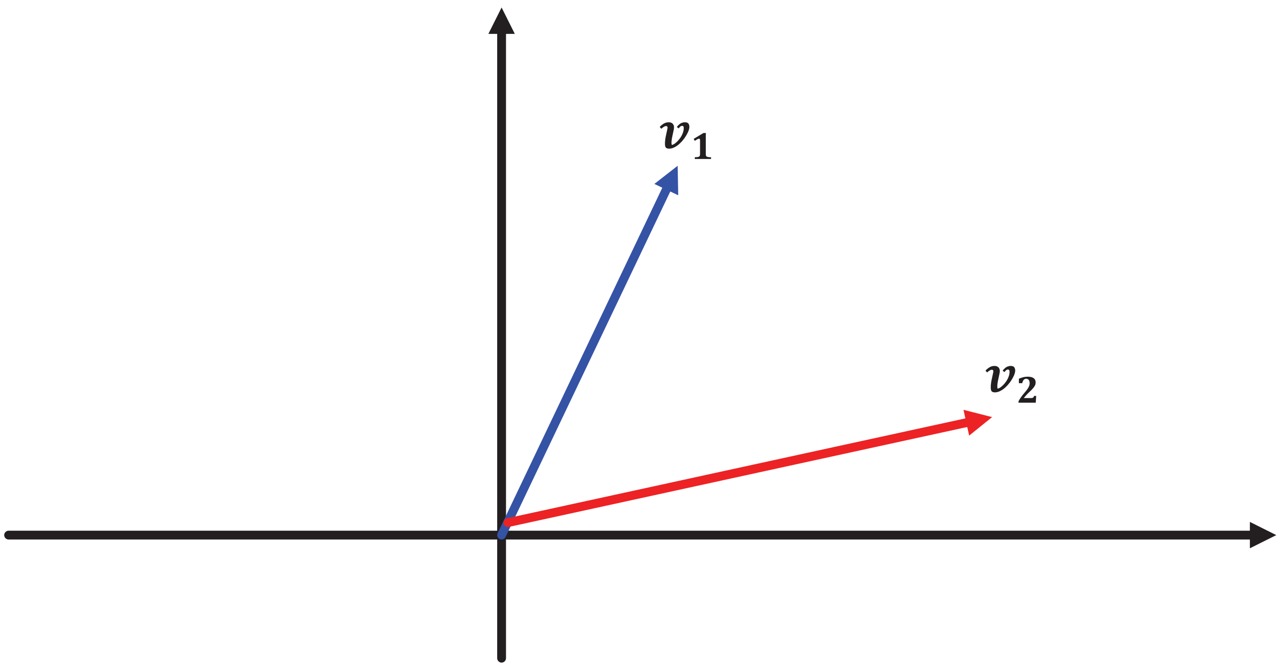
$$\tfrac{1}{4}\mathbf{v}_1 + \tfrac{1}{6}\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3 + \tfrac{1}{12}\mathbf{v}_4 = \mathbf{p} = \begin{bmatrix} \frac{10}{3} \\ \frac{5}{2} \end{bmatrix}$$

    - Then:

$$\tfrac{17}{48}\mathbf{v}_1 + \tfrac{4}{48}\mathbf{v}_2 + \tfrac{27}{48}\mathbf{v}_3 = \mathbf{p}$$

Example

- Find the convex combination and convex hull of following two vectors:



Properties of Convex Hull

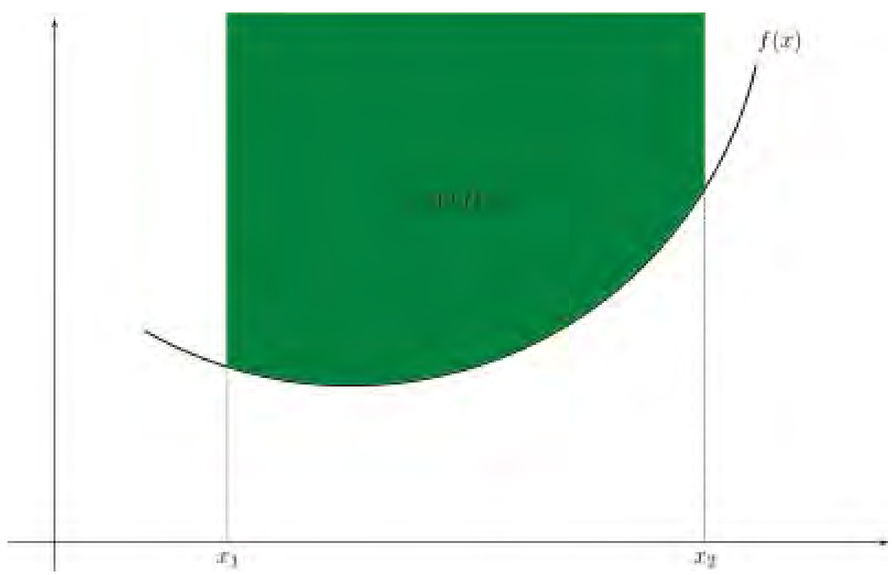
- Convex hull is always a convex set.
- Convex hull is the smallest set that contains the underlying set.

Convex function

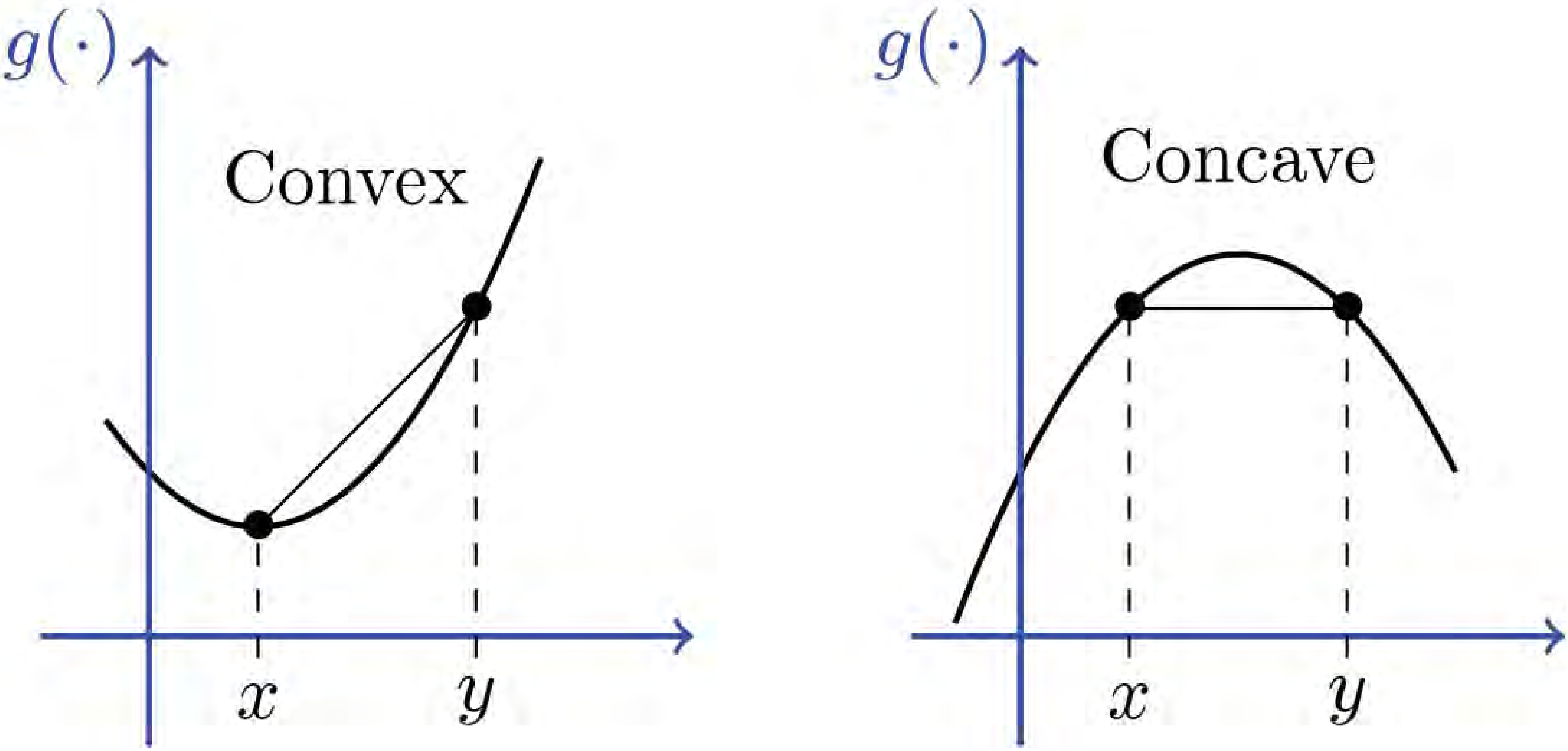
- A function is convex iff its epigraph is a convex set.
- Epigraph or supergraph

$$\operatorname{epi} f = \{(x,\mu)\ :\ x \in \mathbb{R}^n,\ \mu \in \mathbb{R},\ \mu \geq f(x)\} \subseteq \mathbb{R}^{n+1}$$

$$f\left((1-\theta)x^{(0)} + \theta x^{(1)}\right) \leq (1-\theta) f\left(x^{(0)}\right) + \theta f\left(x^{(1)}\right), \quad \forall \theta \in [0,1]$$



Convex and Concave Function



second derivative is nonnegative on its entire domain

Convexity

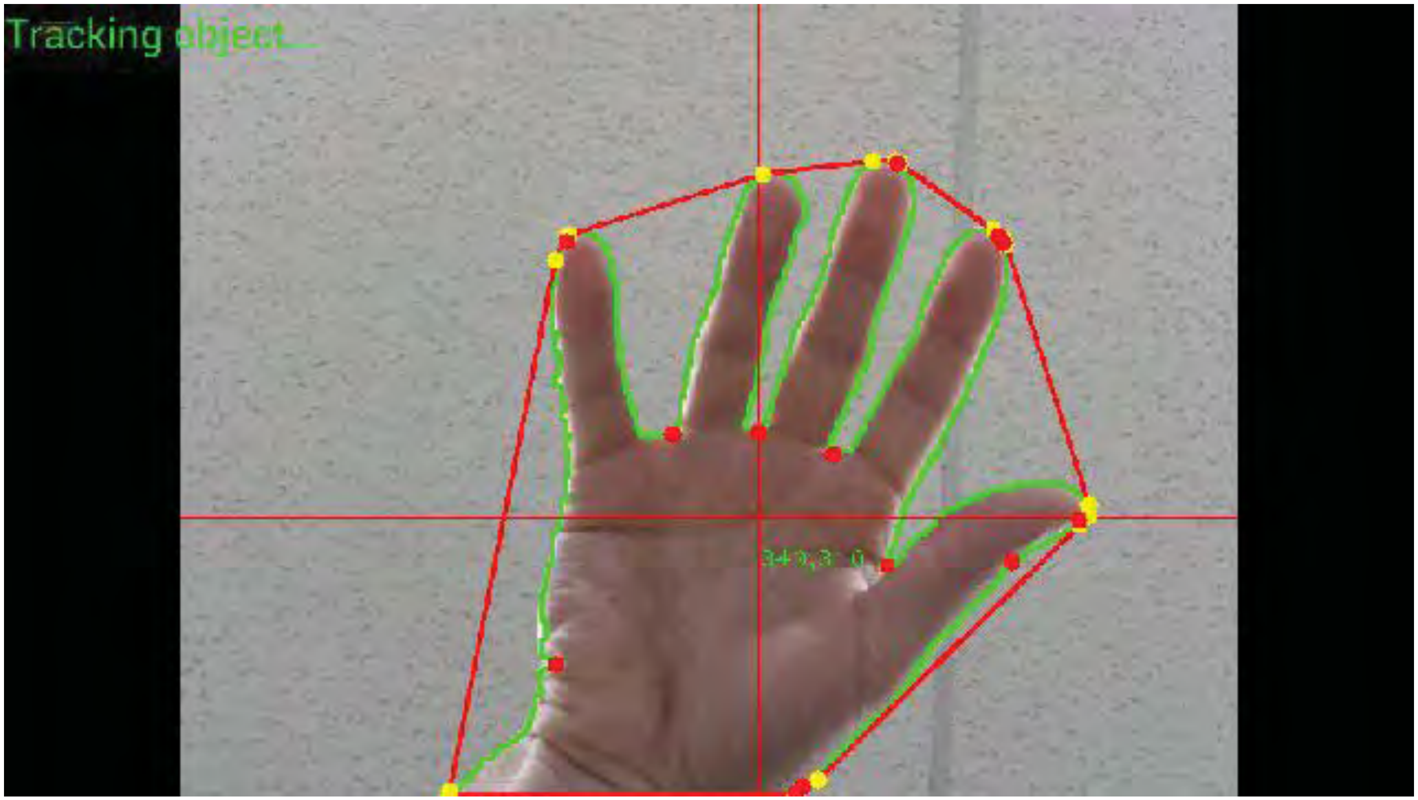
- The following operations preserve convexity:
  - The intersection of (a possibly in finite) set of convex sets
  - The sum two convex sets
  - The product of two convex sets
  - The image of a convex set under an affine function (a linear function plus an set). Similarly, the inverse image of a convex set under an affine function
  - The projection of a convex set onto some of its coordinates.

Hyperplanes

- Hyperplanes play a special role in the geometry of R^n because they divide the space into two disjoint pieces, just as a plane $(ax + by + cz = d)$ separates R^3 into two parts and a line $(ax + by = d)$ cuts through R^2.
- Hyperplane: $\{x | a^T x = b\}$ affine and convex
- Halfspace: $\{x | a^T x \leq b\}$ convex

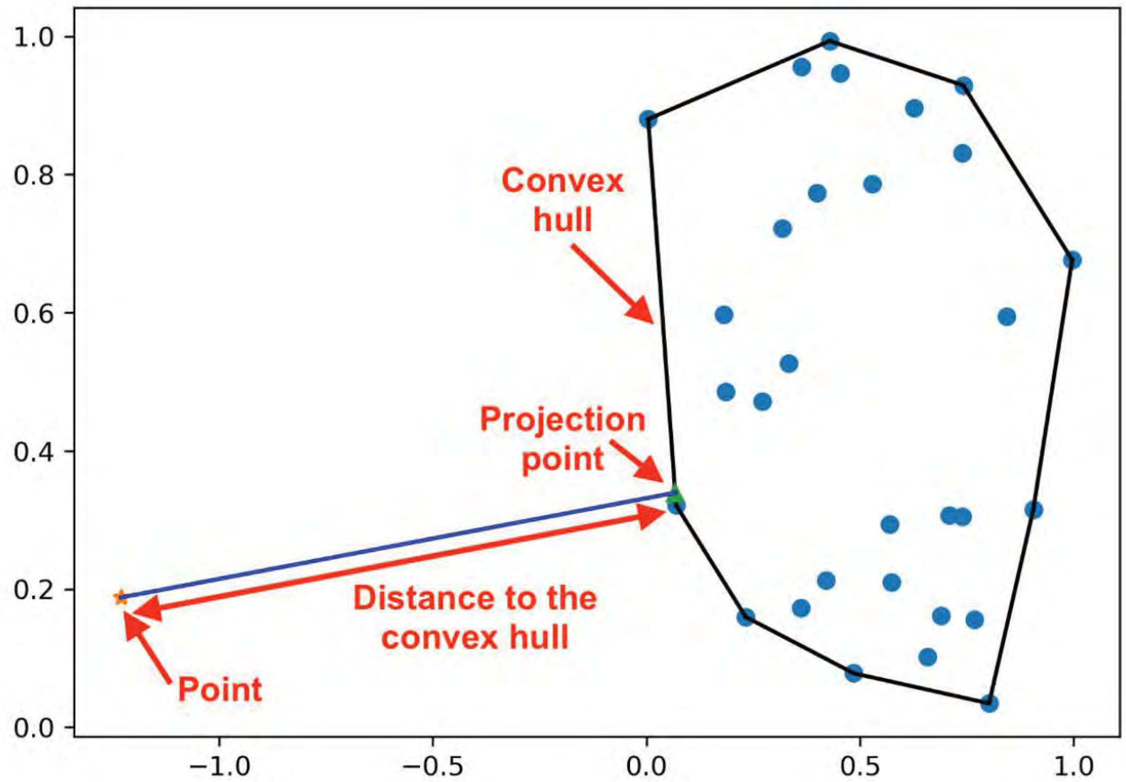
Convex Hull Applications

- Computer Vision



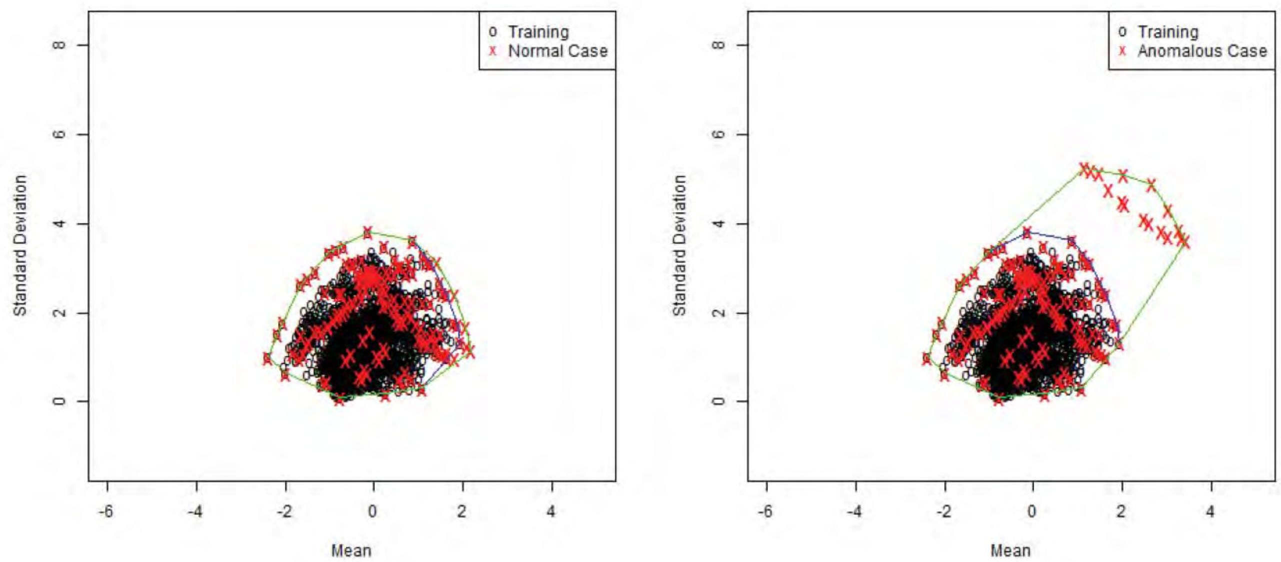
Convex Hull Applications

- Detecting outliers



Convex Hull Applications

- Anomaly Detection
  - The use of a convex hull make it possible to draw the boundary between normal and abnormal data behaviour.

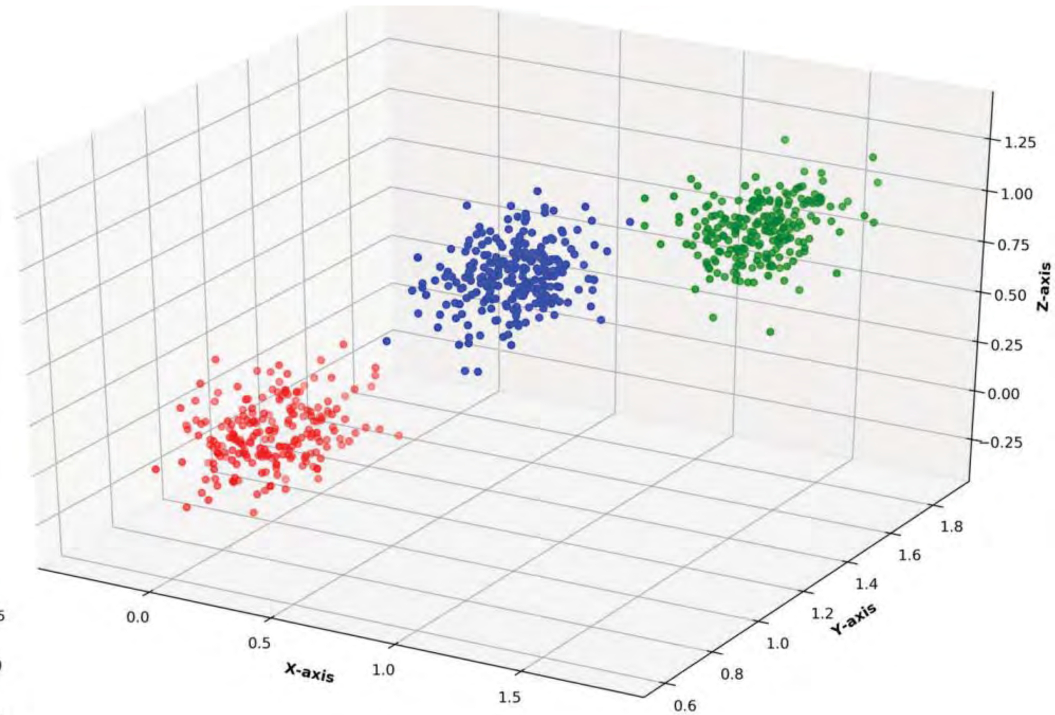


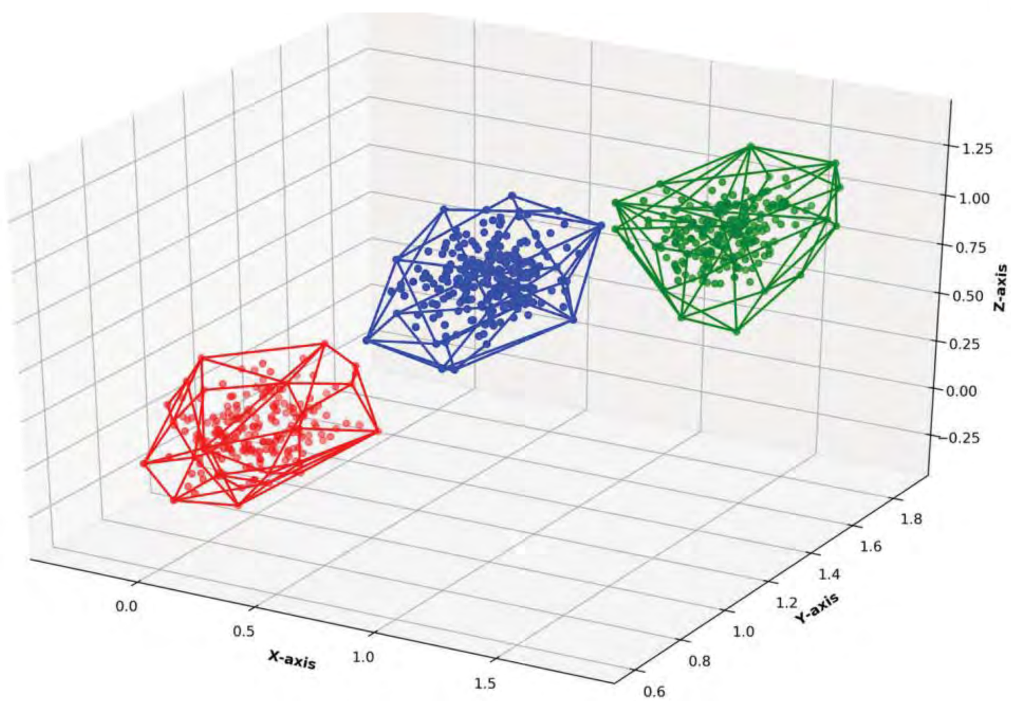
(a) (b)

Fig. 2. Examples of convex hulls on parameter spaces using the dataset Normal-vs-2. The green line shows H_U and the blue line H_N. A normal sample detection is shown in (a) and an anomaly detection shown in (b).

Convex Hull Applications

- Clustering data





Complexity of vector computations

- Computers store (real) numbers in floating-point format
- Floating point= 64 bits or 8 bytes
  - How many possible sequences of bits?
  - How many bytes to store n-vector?
- Current memory and storage devices, with capacities measured in many gigabytes (109 bytes), can easily store vectors with dimensions in the millions or billions.
- Sparse vectors are stored in a more efficient way that keeps track of indices and values of the nonzero entries.
- Note about floating point operations and round-off error.

Complexity of vector computations

- How quickly the vector operations can be carried out by a computer depends very much on the computer hardware and software, and the size of the vector.
- Basic arithmetic operations (addition, multiplication, . . .) are called Floating Point Operations (FLOP)s.
- Estimate the time of computation= counting the total number of Floating Point Operations (FLOP)s.
- The complexity of an operation is the number of flops required to carry it out, as a function of the size or sizes of the input to the operation.
- Crude approximation of time to execute: (flopsneeded)/(computer speed)
- current computers are around 1Gflop/sec (10^9 flops/sec)

Complexity of vector computations

		#FLOPS		Complexity	
Operation		**General**	**Sparse**	**General**	**Sparse**
Scalar product					
Vector sum					
Inner product					
Outer product					
Hadamard product					

Reference

- Chapter 2,3,4: LINEAR ALGEBRA: Theory, Intuition, Code
- Chapter 1: Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares
- Chapter 8: Linear Algebra and its applications
- Chapter 2: Linear Algebra Jim Hefferon
- Chapter 4: Linear Algebra Devid Cherney