Determinant

CE40282-1: Linear Algebra
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Recall

Definition

The **determinant** of a 2×2 matrix $A = [a_{ij}]$ is the number

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

- A 2×2 matrix is invertible if and only if its determinant is nonzero.

Definition of Submatrix Aij

- **Definition**

  For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A

For instance, if

$$A = \begin{pmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{pmatrix}$$

Recursive Definition of Determinant

The determinant of an $n \times n$ matrix $A = [a_{ij}$ is the sum of n terms of the form $\pm a_{1j}\text{det}A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \cdots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned}\text{det}A &= a_{11}\text{det}A_{11} - a_{12}\text{det}A_{12} + \cdots \\ &\qquad + (-1)^{1+n}a_{1n}\text{det}A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\text{det}A_{1j}\end{aligned}$$

Recursive Definition of Determinant

- 2×2 matrix

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}|$$

$$= a \begin{vmatrix} \square & \square \\ \square & d \end{vmatrix} - b \begin{vmatrix} \square & \square \\ c & \square \end{vmatrix}$$

$$= ad - bc$$

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

Recursive Definition of Determinant

- 3×3 matrix

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}|$$

$$= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= aei + bfg + cdh - afh - bdi - ceg$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -5 + 0 + 6 - (25 + 12 + 0) = -36$$

Cofactor

Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \mathrm{det} A_{ij}$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n},$$

which is a **cofactor expansion across the first row** of A.

Cofactor Expansion

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion **across any row** or **down any column**. The expansion across the ith row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Cofactor Expansion

- Example

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

Properties

- If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

  - Determinant of zero matrix is:

Properties

- If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is:

Properties

- If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

Properties

- If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$$

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = ka_{11}C_{11} + \cdots + ka_{1n}C_{1n} = k\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n\times n}| = k^n |A_{n\times n}|$
- If a row/column is multiple of another row/column then determinant is .....

Properties

- Row and Column Operations
  - If a multiple of one row/column of A is added to another row/column to produce a matrix B, then detB = detA.
  - Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

Properties

- If columns/rows of matrix are linear dependent if and only if its determinant is zero
  - Proof?

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Echelon form

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Determinant of Transpose

Theorem

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Multiplicative Property

- **Theorem**
  If A and B are $n \times n$ matrices, then $\det AB = \det A \det B$.

- **Warning**
  In general, $\det(A + B) \neq \det A + \det B$.

- The determinant of the inverse of an invertible matrix is the inverse of the determinant

$$AA^{-1} = I \implies |AA^{-1}| = |I| = 1 \implies |A||A^{-1}| = 1 \implies |A^{-1}| = |A|^{-1}$$

- The determinant of orthogonal matrix is ……

Cramer's Rule

- Ax=b and A is invertible

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \qquad I = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix}$$

$$AI = A \implies A\begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} Ae_1 & \cdots & Ae_n \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

$$\begin{aligned} A\overbrace{\begin{bmatrix} e_1 & e_2 & \cdots & x & \cdots & e_n \end{bmatrix}}^{I_j(x)} &= \begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ax & \cdots & Ae_n \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & b & \cdots & a_n \end{bmatrix}}_{A_j(b)} \end{aligned}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \implies |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \implies |A||I_j(x)| = |A_j(b)| \implies x_j = \frac{|A_j(b)|}{|A|}$$

Cramer's Rule

- **Cramer's Rule**

  Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

  $$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \dots, n$$

- Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \implies x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

A Formula for A^{-1}

The j-th column of A^{-1} is a vector x that satisfies

$$Ax = e_j$$

By Cramer's rule

$$\{(i,j) - \text{entry of } A^{-1}\} = x_i = \frac{\det A_i(e_j)}{\det A}$$

$$\det A_i(e_j) = (-1)^{i+j} \det A_{ji}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors is called the **adjugate** (or **classical adjoint**) of A, denoted by adj$A$.

A Formula for A^{-1}

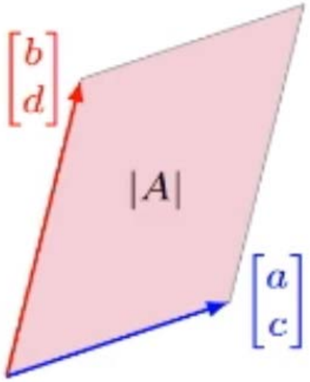
Let A be an invertible $n \times n$ matrix. Then

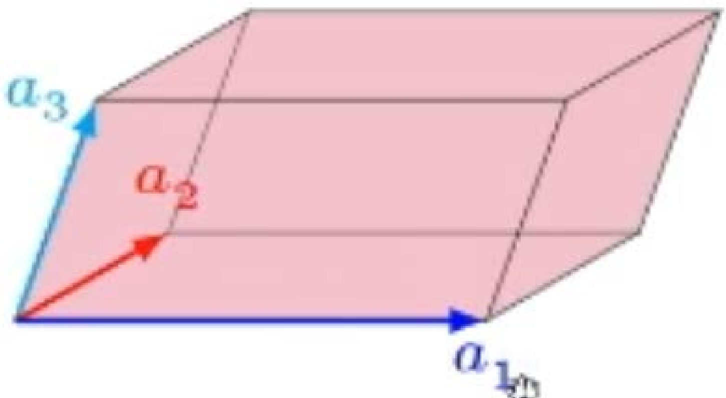
$$A^{-1} = \frac{1}{\det A} \text{adj} A$$

$$\left.\begin{array}{l} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{array}\right\} \Rightarrow A^{-1} = \frac{1}{|A|}[C_{ij}]^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinants as Area or Volume

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $\det A$.
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $\det A$.





Linear Transformations

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$