Matrix Algebra: Dimension and Rank

CE40282-1: Linear Algebra
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Rank Theorem

- Theorem:
  - $col\ rank(A) = \text{row}\ rank(A)$
  - In general it is called rank of matrix! rank(A)
  - Proof?

Rank Properties

- $col\ rank(A_{m\times n}) \leq \min(m,n)$
- row $rank(A_{m\times n}) \leq \min(m,n)$
- $\dim(\mathrm{range}(A)) = \mathrm{rank}(A)$

$$\mathrm{nullity}(A) + \mathrm{rank}(A) = n$$

$$\mathrm{rank}(A) \leq \min(m,n)$$

Rank Properties

- For $A, B \in \mathbb{R}^{m\times n}$
  1. $\text{rank}(A) \leq \min(m, n)$
  2. $\text{rank}(A) = \text{rank}(A^T)$
  3. $\text{rank}(AB) \leq \min\big(\text{rank}(A), \text{rank}(B)\big)$
  4. $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$
- A has *full rank* if $\text{rank}(A) = \min(m, n)$
- If $m > \text{rank}(A)$ rows not linearly independent
  – Same for columns if $n > \text{rank}(A)$

Rank Properties

- The ***range*** or the columnspace of a matrix $A \in \mathbb{R}^{m\times n}$, denoted $\mathcal{R}(A)$, is the the span of the columns of A. In other words,

$$\mathcal{R}(A) = \{v \in \mathbb{R}^m : v = Ax, x \in \mathbb{R}^n\}.$$

- Assuming A is full rank and $n < m$, the projection of a vector $y \in \mathbb{R}^m$ onto the range of A is given by,

$$\mathrm{Proj}(y; A) = \mathrm{argmin}_{v\in\mathcal{R}(A)} \|v - y\|_2 = A(A^T A)^{-1} A^T y \ .$$

- When A contains only a single column, $a \in \mathbb{R}^m$, this gives the special case for a projection of a vector on to a line:

$$\mathrm{Proj}(y; a) = \frac{aa^T}{a^T a} y \ .$$

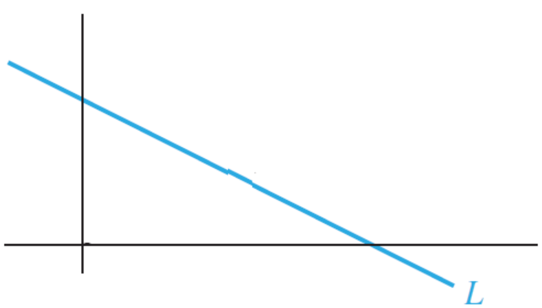
Review: Subspace

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- Examples
  - $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace
  - 

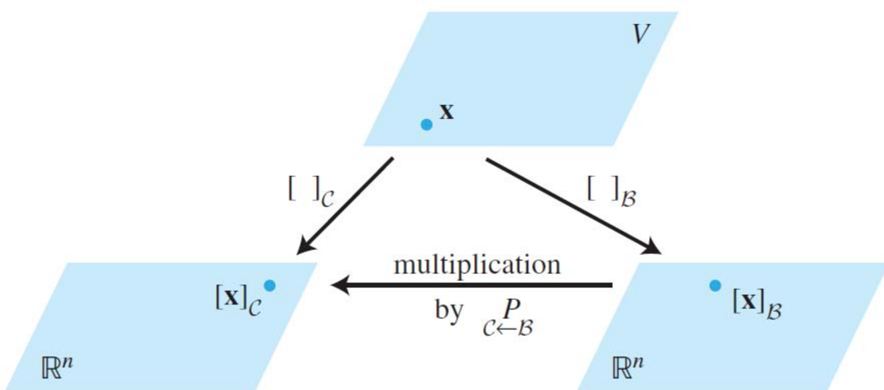
Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$



$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

Change of Basis

- Example

Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

- Final Review!

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

QR Decompose and Matrix Inverse

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Gram Matrix

Consider an $n \times m$ matrix A over $\mathbb{R}$, where

$$A = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$$

The $m \times m$ matrix $A^T A$ is :

$$A^T A = \begin{bmatrix} x_1^T x_1 & x_1^T x_2 & \cdots & x_1^T x_m \\ x_2^T x_1 & x_2^T x_2 & \cdots & x_2^T x_m \\ \vdots & \vdots & \ddots & \vdots \\ x_m^T x_1 & x_m^T x_2 & \cdots & x_m^T x_m \end{bmatrix}$$

Note that $A \colon \mathbb{R}^m \to \mathbb{R}^n$ and $A^T A \colon \mathbb{R}^m \to \mathbb{R}^m$. We've already seen that:

1. $\text{rank}\, A = \text{rank}\, A^T A$ and $\text{nullity}\, A = \text{nullity}\, A^T A$ (in fact, $N_A = N_{A^T A}$)
2. $A^T A \geq 0$
3. If $N_A = 0$, then the projection matrix onto $\text{Span}(x_1, \ldots, x_m)$ is $A(A^T A)^{-1} A^T$.

This is an example of a Gram matrix.

Gram Matrix

- A Gram matrix is Positive Definite and Symmetric
- $G = AA^T$ is left Gram matrix
- Gram Matrix and Left Gram Matrix are symmetric

- Null space: $N\left(A^T A\right) = N(A)$

- Rank: $\text{rank}\left(A^T A\right) = rank(A) = n - nullity(A)$

$$C(A^T A) = R(A^T A) = R(A)$$
$$C(AA^T) = R(AA^T) = C(A)$$

Review: Orthonormal Vectors

- a collection of real m-vectors $a_1, a_2, \ldots, a_n$ is *orthonormal* if
  - the vectors have unit norm: $\|a_i\| = 1$
  - they are mutually orthogonal: $a_i^T a_j = 0$ if $i \neq j$

Example

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Orthogonal Matrix

- Columns of $A_{n\times k} = [a_1 \quad \cdots \quad a_k]$ are orthonormal.

$$n \geq k$$

$$\begin{aligned}
A^T A &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \\
&= \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \cdots & a_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \cdots & a_n^T a_n \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\end{aligned}$$

"matrix with orthonormal columns"

Orthogonal Matrix

- Columns of A are orthonormal $\leftrightarrow$ $\boxed{A^T A = I}$
- Square matrix with orthonormal columns is a orthogonal matrix
  - columns and rows are orthonormal vectors
  - $Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I$
  - is necessarily invertible with inverse $Q^{\mathrm{T}} = Q^{-1}$

Orthogonal Matrix

- Examples
  - Identity matrix $I^T I = I$
  - Rotation matrix

$$R^T R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

  - Reflection matrix

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}^T \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(2\theta) + \sin^2(2\theta) & \cos(2\theta)\sin(2\theta) - \sin(2\theta)\cos(2\theta) \\ \sin(2\theta)\cos(2\theta) - \cos(2\theta)\sin(2\theta) & \sin^2(2\theta) + \cos^2(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Orthogonal Matrix

- All 2x2 orthogonal matrices can be expressed as Rotation or Reflection

Orthonormal Columns Properties

if $A \in \mathbf{R}^{m \times n}$ has orthonormal columns, then the linear function $f(x) = Ax$

This is a mapping with preserving properties of input

- preserves inner products:

$$(Ax)^T(Ay) =$$

- preserves norms:

$$\|Ax\| =$$

- preserves distances: $\|Ax - Ay\| = \|x - y\|$

- preserves angles:

$$\angle(Ax, Ay) = \arccos\left(\frac{(Ax)^T(Ay)}{\|Ax\|\|Ay\|}\right) = \arccos\left(\frac{x^T y}{\|x\|\|y\|}\right) = \angle(x, y)$$