# Linear Independence

CE40282-1: Linear Algebra
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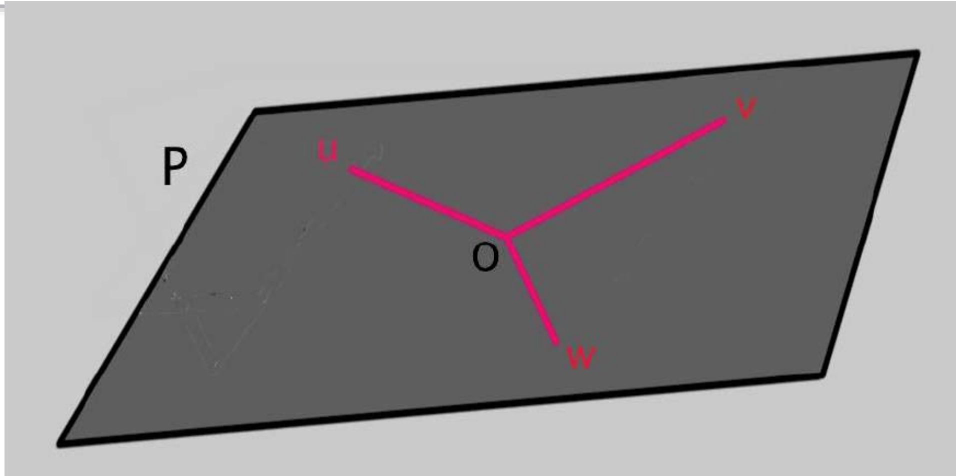
# Linear Independence



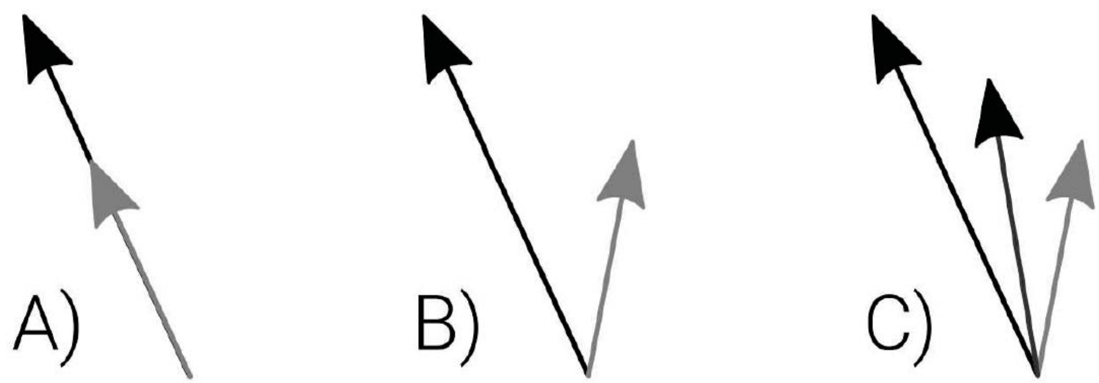

- Plane $P$ includes origin and three non-zero vectors $\{v, u, w\}$ in $P$
- If no two of $\{v, u, w\}$ are parallel, then P=*span{u,v,w}*
- Any two vectors determines a plane and express the other as a linear combination of those two:

$$w = d_1 u + d_2 v \ \ (d_1 \& d_2 \ can't \ both \ be \ zero)$$

- $c_1 u + c_2 v + c_3 w = 0 \longrightarrow u, w, v$ are not all independent.
- Independence is a property of a set of vectors.

# Definition

- Geometry:
  - A set of vectors is independent if the subspace dimensionality (its span) equals the number of vectors.
  - Example: 1,2,3 vectors spans?




Geometric sets of vectors in $\mathbb{R}^2$

# Definition

- Algebra
  - Dependent
    - A set of vectors is dependent if at least one vector in the set can be expressed as a linear weighted combination of the other vectors in that set.
    - For at least one $\lambda \neq 0$ $\quad \mathbf{0} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n, \quad \lambda \in \mathbb{R}$
  - Independence
    - No vector in the set is a linear combination of the others (has only the trivial solution)
    - Only when all $\lambda_i = 0$ $\quad \mathbf{0} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n, \quad \lambda \in \mathbb{R}$

# Example

- Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- A set containing only one vector—say, v—is linearly independent if and only if v is not ...............

- a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

# Properties

- Theorem:

Any set of vectors that contains the zeros vector is guaranteed to be linearly dependent

# Properties

- Theorem:

Any set of $M > N$ vectors in $\mathbb{R}^{\mathrm{N}}$ is necessarily linearly dependent.

Any set of $M \leq N$ vectors in $\mathbb{R}^{\mathrm{N}}$ *could be* linearly independent.

# Properties

- If a collection of vectors is linearly dependent, then any superset of it is linearly dependent.

- Any nonempty subset of a linearly independent collection of vectors is linearly independent.

# Example

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

# Characterization of Linearly Dependent sets

## Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

# Linear Dependent Properties

- Suppose vectors $v_1, \ldots, v_n$ are linearly dependent:

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

with $c_1 \neq 0$. Then:

$$\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{v_2, \ldots, v_n\}$$

- When we write a vector space as the space of a list of vectors, we would like that list to be as short as possible. This can achieved by iterating.

# Linear combinations of linearly independent vectors

- ▸ suppose $x$ is linear combination of linearly independent vectors $a_1, \ldots, a_k$:

$$x = \beta_1 a_1 + \cdots + \beta_k a_k$$

  ▸ the coefficients $\beta_1, \ldots, \beta_k$ are *unique*

- proof

# Conclusion

Step 1: Count the number of vectors (call that number $M$) in the set and compare to $N$ in $\mathbb{R}^{\mathrm{N}}$. As mentioned earlier, if $M > N$, then the set is necessarily dependent. If $M \leq N$ then you have to move on to step 2.

Step 2: Check for a vector of all zeros. Any set that contains the zeros vector is a dependent set.

- The rank of a matrix is the estimate of the number of linearly independent rows or columns in a matrix.