General Vector Space

CE40282-1: Linear Algebra
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Binary Operations

- Any function from $A \times A \longmapsto A$ is a binary operation.

  - Is - operator a binary on natural numbers?

Fields

- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: (F,+,.)

- Field is a set (F) with two binary operations (+ , .) satisfying following properties:

Fields

$\forall$ **a, b, c** $\in \boldsymbol{F}$

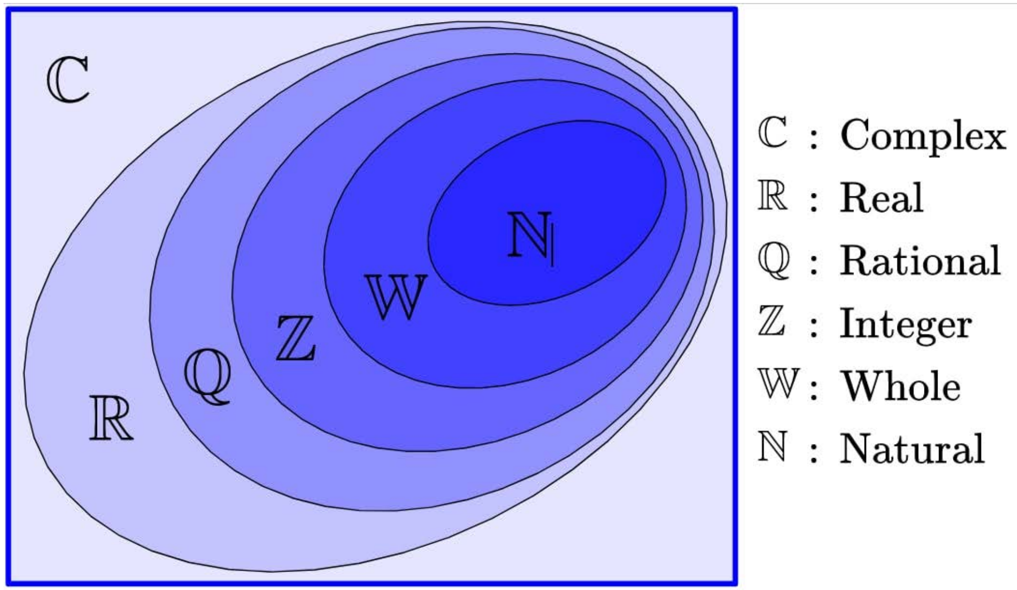
Properties	Binary Operations	
	Addition (+)	Multiplication (.)
Closure	$\exists a + b \in F$	$\exists a.b \in F$
Associative (شرکت‌پذیری)	$a + (b + c)$ $= (a + b) + c$	$a.(b.c) = (a.b).c$
Commutative (جابه‌جایی‌پذیری)	$a + b = b + a$	$a.b = b.a$
Existence of identity e $\in F$	$a + e = a = e + a$	$a.e = a = e.a$
Existence of inverse: For each a in F there must exist b_1 in F	$a + b_1 = e = b_1 + a$	$a.b_1 = e = b_1.a$ $\underline{For\ any\ nonzero\ a}$

Multiplication is distributive over addition

$$a.(b + c) = a.b + a.c$$
$$(a + b).c = a.c + b.c$$

Fields

- Which are fields?
  - $\mathbb{R}$
  - $\mathbb{C}$
  - $\mathbb{Q}$
  - $\mathbb{Z}$
  - $\mathbb{N}$
  - $\mathbb{R}^{2\times 2}$



Vector Space

- Building blocks of linear algebra
- A non empty set V with field F (most of time $\mathbb{R}$ or $\mathbb{C}$) forms a vector space with two operations:
  - " $+$ ": Binary operation on V which is $\mathrm{V} \times V \rightarrow V$
  - ".": $F \times V \rightarrow V$

Vector Space

Definition *A* **vector space** *over a field $\mathcal{F}$ is the set $\mathcal{V}$ equipped with two operations: $(\mathcal{V}, \mathcal{F}, +, \cdot)$*

(i) Vector addition: *denoted by "+" adds two elements $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ to produce another element $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{V}$;*

(ii) Scalar multiplication: *denoted by "$\cdot$" multiplies a vector $\boldsymbol{x} \in \mathcal{V}$ with a scalar $\alpha \in \mathcal{F}$ to produce another vector $\alpha \cdot \boldsymbol{x} \in \mathcal{V}$. We usually omit the "$\cdot$" and simply write this vector as $\alpha\boldsymbol{x}$.*

[A1] Vector addition is commutative: $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

[A2] Vector addition is associative: $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} = \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ *for every* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$.

[A3] Additive identity: *There is an element* $\mathbf{0} \in \mathcal{V}$ *such that* $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in \mathcal{V}$.

[A4] Additive inverse: *For every* $\boldsymbol{x} \in \mathcal{V}$, *there is an element* $(\boldsymbol{x}) \in \mathcal{V}$ *such that* $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.

Addition of vector space

[M1] Scalar multiplication is associative: $a(b\boldsymbol{x}) = (ab)\boldsymbol{x}$ *for every* $a, b \in \mathcal{F}$ *and for every* $\boldsymbol{x} \in \mathcal{V}$.

[M2] First Distributive property: $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ *and for every* $a, b \in \mathcal{F}$ *and for every* $\boldsymbol{x} \in \mathcal{V}$.

[M3] Second Distributive property: $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ *and every* $a \in \Re^1$.

[M4] Unit for scalar multiplication: $1\boldsymbol{x} = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in \mathcal{V}$.

Action of the field of scalars on the vector space

Vector Space

- Which are vector space?
  - Set $\mathbb{R}^n$ over $\mathbb{R}$
  - Set $\mathbb{C}$ over $\mathbb{R}$
  - Set $\mathbb{R}$ over $\mathbb{C}$
  - Set of all polynomials with coefficient from field $\mathbb{R}$
  - Set of all polynomials of degree at most n with coefficient from field $\mathbb{R}$
  - Matrix: $M_{m,n}(\mathbb{R})$
  - Function: $f(x): x \longrightarrow \mathbb{R}$

Vector spaces of functions

The elements of $\mathcal{V}$ can be functions. Define function addition and scalar multiplication as

$$(f+g)(x) = f(x) + g(x) \text{ and } (af)(x) = af(x) \ .$$

Then the following are examples of vector spaces of functions.

(a) The set of all polynomials with real coefficients.

(b) The set of all polynomials with real coefficients and of degree $\leq n-1$. We write this as

$$\mathcal{P}_n = \{p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} : a_i \in \Re\} \ .$$

(c) The set of all real-valued continuous functions defined on $[0,1]$.

(d) The set of real-valued functions that are differentiable on $[0,1]$.

(e) Each of the above examples are special cases for an even more general construction. For *any* non-empty set X and any field $\mathcal{F}$, define $\mathcal{F}^X = \{f : X \to \mathcal{F}\}$ to be a space of functions with addition and scalar multiplication defined for all $x \in X$, $f, g \in \mathcal{F}^X$ and $a \in \mathcal{F}$. Then F^X is a vector space of functions over the field $\mathcal{F}$. This vector space is denoted by $\Re^X$ when the field is chosen to be the real line.

Vector Space of Polynomials

- $P_n(\mathbb{R})$
  - Scalar multiplication
  - Vector addition
  - And other 8 properties!

Conclusion

The operations on a field $\mathbb{F}$ are

- $+ : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$
- $\times : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$

The operations on a vector space $\mathbb{V}$ over a field $\mathbb{F}$ are

- $+ : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$
- $\cdot : \mathbb{F} \times \mathbb{V} \rightarrow \mathbb{V}$

Vector Space of Polynomials

- Linear independence

- Example: Are $(1-x), (1+x), x^2$ linearly independent?

Vector Space of Polynomials

- Span

- Basis
  - Example: Are $(1 - x), (1 + x), x^2$ for $P_2(\mathbb{R})$?

Vector Space of Polynomials

- Standard bases for $P_n(\mathbb{R})$?

- Polynomial is isomorphic to vector space

Polynomial ring

- $p = p_0 + p_1 X + p_2 X^2 + \cdots + p_{m-1} X^{m-1} + p_m X^m$
- Every polynomial is a finite linear combination of the powers of x and if a linear combination of powers of x is 0 then all coefficients are zero (assuming x is an indeterminate, not a number).
- The ring of polynomials with coefficients in a field is a vector space with basis $1, x, x^2, x^3, \dots$.

Functions Linearly Independent

- Let f(t) and g(t) be differentiable functions. Then they are called linearly dependent if there are nonzero constants c1 and c2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise they are called linearly independent.

- Example:

functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - \cos^2(t)$

Review

- What have we learned so far?

Review

Span	vs Lin Indep
Want many vectors in small space	Want few vectors in big space.
Adding vectors to list only helps	Deleting vectors from list only helps
$A = \begin{bmatrix} \vert & & \vert \\ \underline{v}_1 & \cdots & \underline{v}_k \\ \vert & & \vert \end{bmatrix}$ $A\underline{x} = \underline{b}$ has soln $\iff \underline{b} \in \text{Span}\{\underline{v}_1, \dots, \underline{v}_k\}$	$A\underline{x} = \underline{0}$ has only triv soln $\underline{x} = \underline{0}$ $\iff \underline{v}_1, \dots, \underline{v}_k$ lin indep

Example

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Which of the following lists span $\mathbb{R}^3$?

$a)$ $\mathbf{v}_1, \mathbf{v}_2$.

$b)$ $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

$c)$ $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$.

$d)$ $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6$.

$e)$ $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$.

Example

- Let $\mathbb{P}_2$ be the vector space of polynomials of degree less than or equal to 2. Let B be the basis $\mathbf{b}_1 = x^2, \mathbf{b}_2 = -1 + x, \mathbf{b}_3 = x + x^2$.

Find the coordinates of the vector $\mathbf{v} = 1 + 2x - x^2$ with respect to B.

Affine Independence

An indexed set of points $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is **affinely dependent** if there exist real numbers $c_1, \ldots, c_p$, not all zero, such that

$$c_1 + \cdots + c_p = 0 \quad \text{and} \quad c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

Otherwise, the set is **affinely independent**.

- Example:
  - $\{v_1\}$

Affine Independence

Given an indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, with $p \geq 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

a. S is affinely dependent.

b. One of the points in S is an affine combination of the other points in S.

c. The set $\{\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1\}$ in $\mathbb{R}^n$ is linearly dependent.

- Example:
  - Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 14 \\ 6 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_4\}$. Is S affinely dependent?

Barycentric Coordinates

THEOREM 6

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an affinely independent set in $\mathbb{R}^n$. Then each $\mathbf{p}$ in aff S has a unique representation as an affine combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. That is, for each $\mathbf{p}$ there exists a unique set of scalars $c_1, \ldots, c_k$ such that

$$\mathbf{p} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \quad \text{and} \quad c_1 + \cdots + c_k = 1 \tag{7}$$

DEFINITION

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an affinely independent set. Then for each point $\mathbf{p}$ in aff S, the coefficients $c_1, \ldots, c_k$ in the unique representation (7) of $\mathbf{p}$ are called the **barycentric** (or, sometimes, **affine**) **coordinates** of $\mathbf{p}$.

Observe that (7) is equivalent to the single equation

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \cdots + c_k \begin{bmatrix} \mathbf{v}_k \\ 1 \end{bmatrix} \tag{8}$$

involving the homogeneous forms of the points. Row reduction of the augmented matrix $\begin{bmatrix} \tilde{\mathbf{v}}_1 & \cdots & \tilde{\mathbf{v}}_k & \tilde{\mathbf{p}} \end{bmatrix}$ for (8) produces the barycentric coordinates of $\mathbf{p}$.

Barycentric Coordinates

EXAMPLE 4 Let $\mathbf{a} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the barycentric coordinates of $\mathbf{p}$ determined by the affinely independent set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Reference

- LINEAR ALGEBRA: Theory, Intuition, Code
- David Cherney,
- Online Courses!