Vector Space-1

CE40282-1: Linear Algebra
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What is vector?

- A vector is an ordered finite list of numbers. Written as:

$$a, X, p, \beta, E^{\text{aut}}, \mathbf{g}, \vec{a}$$

$$\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \quad \begin{pmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{pmatrix} \quad (-1.1, 0.0, 3.6, -7.2)$$

- Size (dimension or length): A vector of size n is called an n-vector ($x \in \mathcal{R}^n$)
- Elements (entries, coefficients, components) of a vector
- Two vectors a and b are equal, which we denote a = b, if they have the same size, and each of the corresponding entries is the same. If a and b are n-vectors, then a = b means a1 = b1, . . . , an = bn.
- Numbers are called scalars
- The set of all n-vectors is denoted $\mathbb{R}^n := \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \, a_1, \ldots, a_n \in \mathbb{R} \right\}$

Block vectors

- Suppose b, c, and d are vectors with sizes m, n, p
- *stacked vector* or concatenation of b, c, and d. block vector with entries (blocks) b, c, d is:

$$a = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

- a has size m + n + p:
  - $a = (b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_p)$

Subvector

- $a_{r:s} = (a_r, \ldots, a_s)$ is a subvector of a. It is a vector with size (s-r+1).
- Colon notation is used to denote subvectors.
- The subscript r:s is called the index range
- In a block vector a: $a = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$
  - b, c, and d are subvectors or slices of a, with sizes m, n, and p, respectively.
  - $b = a_{1:m}, \qquad c = a_{(m+1):(m+n)}, \qquad d = a_{(m+n+1):(m+n+p)}$

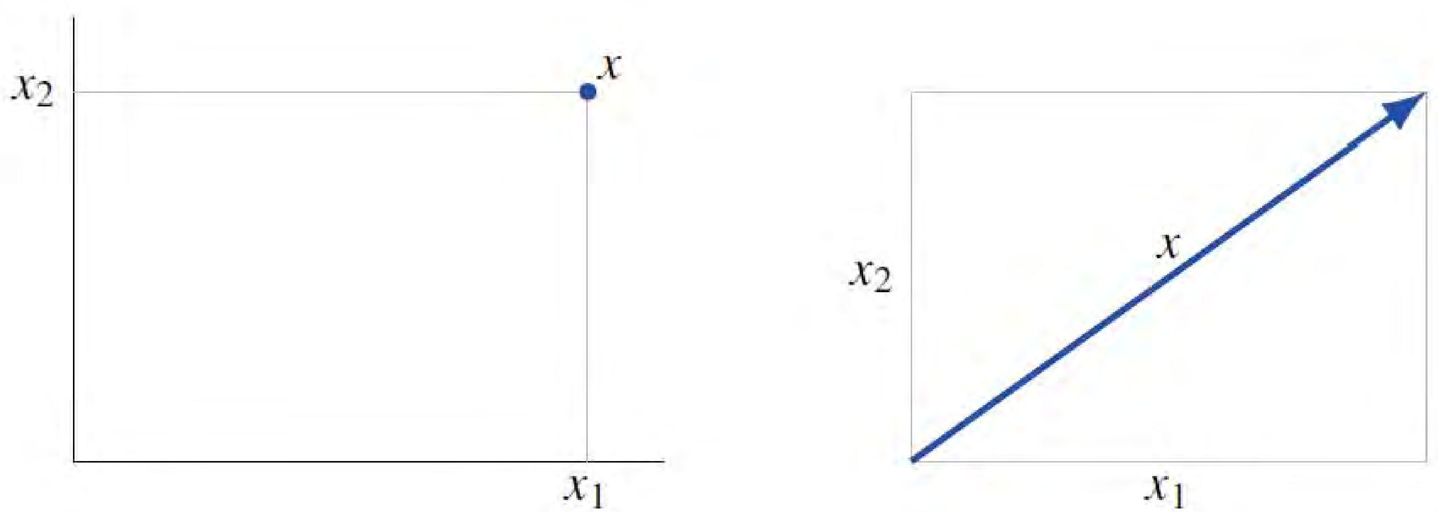
Famous vectors

- Zero vector: O_n
- Ones vector: I_n
- Unit vector: e_i (e_i is the entry with 1 value)
- Question: Write all unit vectors with length of 3?
- Sparse vector: a vector if many of its entries are 0
  - can be stored and manipulated efficiently on a computer
  - **nnz(x)** is number of entries that are nonzero
  - Question: What is the most sparsest vector?

Vectors examples

- Location or displacement in 2-D or 3-D

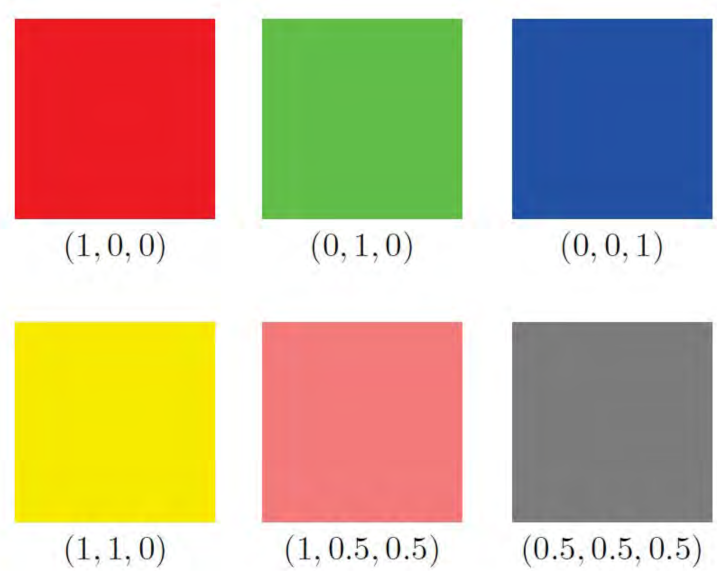
2-vector (x_1, x_2) can represent a location or a displacement in 2-D



- A vector can also be used to represent a displacement in a plane or 3-D space, in which case it is typically drawn as an arrow.
- A vector can also be used to represent the velocity or acceleration, at a given time, of a point that moves in a plane or 3-D space.

Vectors examples

- Color (RGB)
  - A 3-vector can represent a color, with its entries giving the Red, Green, and Blue (RGB) intensity values (often between 0 and 1).



Six colors and their RGB vectors.

Vectors examples

Quantities. An n-vector q can represent the amounts or quantities of n different resources or products held (or produced, or required) by an entity such as a company. Negative entries mean an amount of the resource owed to another party (or consumed, or to be disposed of). For example, a *bill of materials* is a vector that gives the amounts of n resources required to create a product or carry out a task.

Portfolio. An n-vector s can represent a stock portfolio or investment in n different assets, with s_i giving the number of shares of asset i held. The vector $(100, 50, 20)$ represents a portfolio consisting of 100 shares of asset 1, 50 shares of asset 2, and 20 shares of asset 3. Short positions (*i.e.*, shares that you owe another party) are represented by negative entries in a portfolio vector. The entries of the portfolio vector can also be given in dollar values, or fractions of the total dollar amount invested.

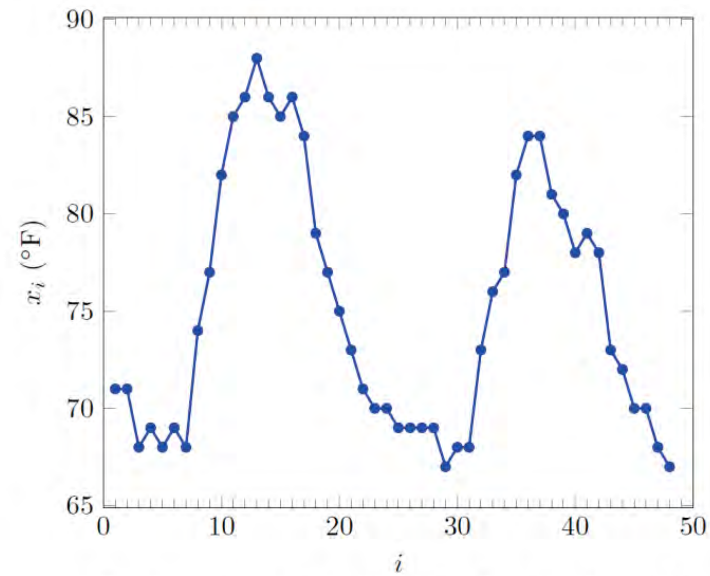
Values across a population. An n-vector can give the values of some quantity across a population of individuals or entities. For example, an n-vector b can give the blood pressure of a collection of n patients, with b_i the blood pressure of patient i, for $i = 1, \ldots, n$.

Proportions. A vector w can be used to give fractions or proportions out of n choices, outcomes, or options, with w_i the fraction with choice or outcome i. In this case the entries are nonnegative and add up to one. Such vectors can also be interpreted as the recipes for a mixture of n items, an allocation across n entities, or as probability values in a probability space with n outcomes. For example, a uniform mixture of 4 outcomes is represented as the 4-vector $(1/4, 1/4, 1/4, 1/4)$.

Vectors examples

Time series

- An n-vector can represent a time series or signal, that is, the value of some quantity at different times.
- The entries in a vector that represents a time series are sometimes called samples, especially when the quantity is something measured.
- An audio (sound) signal can be represented as a vector whose entries
- give the value of acoustic pressure at equally spaced times (typically 48000 or 44100 per second).
- A vector might give the hourly rainfall (or temperature, or barometric pressure) at some location, over some time period.
- These lines carry no information; they are added only to make the plot
- easier to understand visually.



Hourly temperature in downtown Los Angeles on August 5 and 6, 2015 (starting at 12:47AM, ending at 11:47PM).

Vectors examples

Word count vectors

- a short document:

> **Word** count vectors are used **in** computer based **document** analysis. Each entry of the **word** count vector is the **number** of times the associated dictionary **word** appears **in** the **document**.

- a small dictionary (left) and word count vector (right)

$$\begin{array}{l} \text{word} \\ \text{in} \\ \text{number} \\ \text{horse} \\ \text{the} \\ \text{document} \end{array} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$$

- dictionaries used in practice are much larger

Basic Notation

- Column vector $x \in R^n$
- Transport:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 0 \end{bmatrix}^{\mathrm{TT}} = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}$$

$$4^{\mathrm{T}} = 4$$

- Row vector $x^T \in R^{1 \times n}$
- ith element of x is: x_i

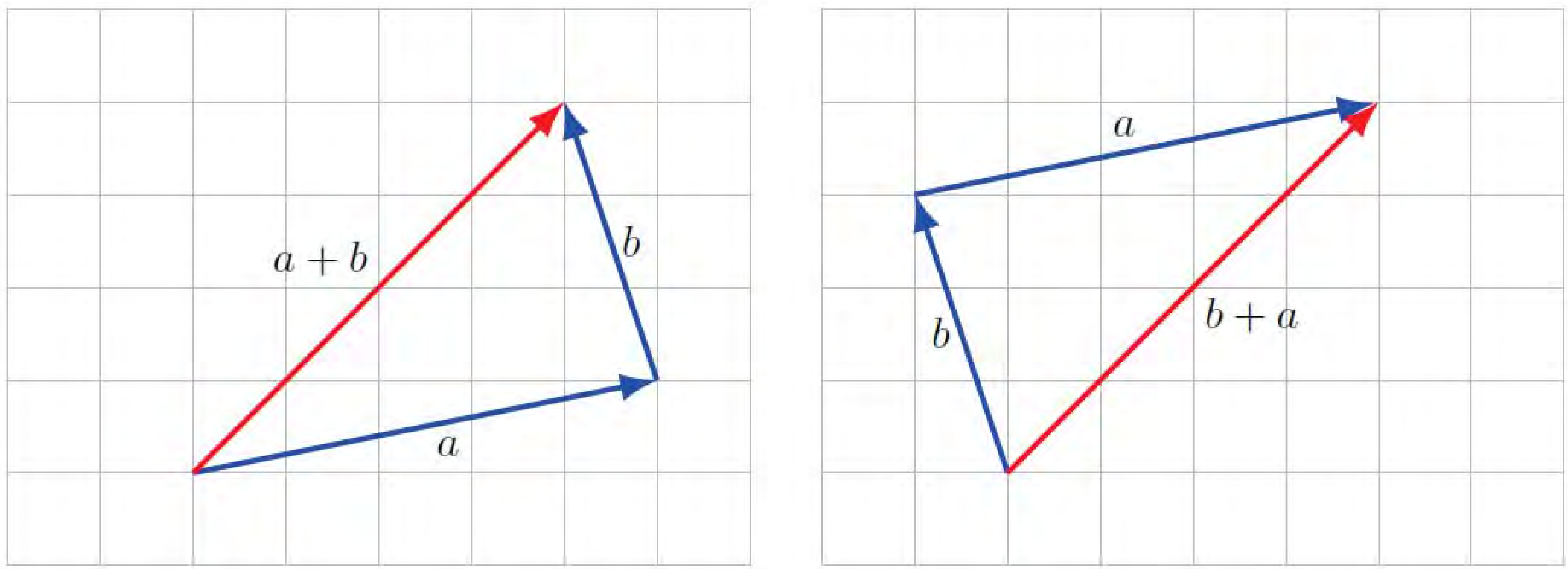
Vector Addition

- n-vectors a and b

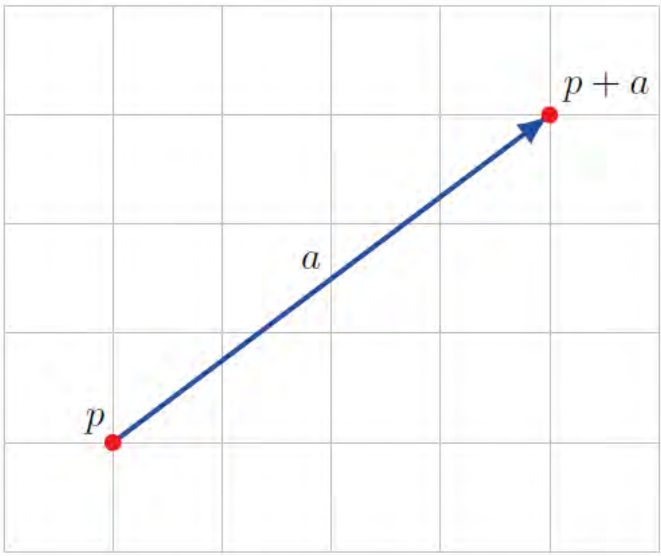
$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad a + b = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

- Can be added, with sum denoted: a + b
- Subtraction is similar: (a-b)
- The result of vector subtraction is called the difference of the two vectors.

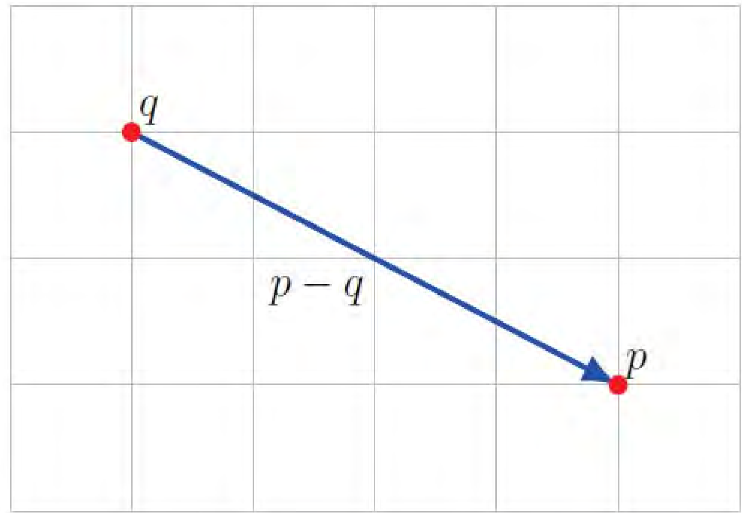
Vector Addition and Subtraction



Vector Addition and Subtraction



The vector $p + a$ is the position of the point represented by p displaced by the displacement represented by a.



The vector $p - q$ represents the displacement from the poin represented by q to the point represented by p.

Vector Addition Properties

- Commutative $a + b = b + a$
- Associative
  - Note: the associative law is that parentheses can be moved around, e.g., (x+y)+z = x+(y+z) and x(yz) = (xy)z

$$(a + b) + c = a + (b + c) = a + b + c$$

- Adding the zero vector to a vector has no effect

$$a + 0 = 0 + a = a$$

  - What constraints should you have?
- Subtracting a vector from itself yields the zero vector

$$a - a = 0$$

  - What is size of 0 here?

Vector Addition Properties

- Transpose: For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$, $(\boldsymbol{u}+\boldsymbol{v})^T = \boldsymbol{u}^T + \boldsymbol{v}^T$
  - Proof?

- Can scalar and vector be added?

$$4 + \begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix} + 4$$

Scalar-Vector Product

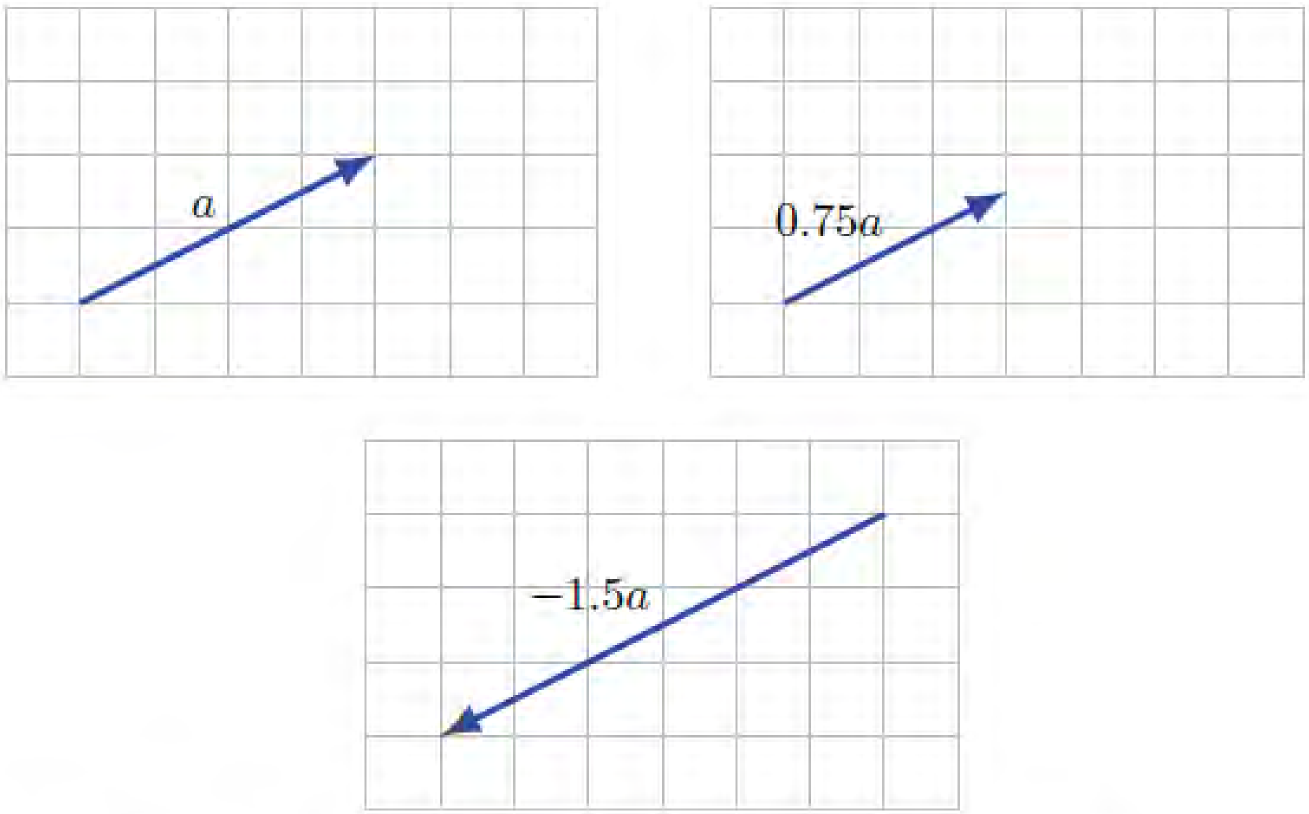
- Scalar multiplication or scalar-vector multiplication: a vector is multiplied by a scalar (i.e., number), which is done by multiplying every element of the vector by the scalar.
  - scalar on the left or scalar on the right

$$(-2)\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix}(1.5) = \begin{bmatrix} 1.5 \\ 13.5 \\ 9 \end{bmatrix}$$

  - Some notations:
    - $\boldsymbol{a}/2$ is a vector means $\left(\frac{1}{2}\right)\boldsymbol{a}$
    - $-\boldsymbol{a}$ is a vector means $(-1)\boldsymbol{a}$
    - $0\boldsymbol{a} = \mathbf{0}$ vector

scalar

Scalar-Vector Product



The vector $0.75a$ represents the displacement in the direction of the displacement a, with magnitude scaled by 0.75; $(-1.5)a$ represents the displacement in the opposite direction, with magnitude scaled by 1.5.

Scalar-Vector Product Properties

- Commutative $\beta \boldsymbol{a} = \boldsymbol{a}\beta$
- Associative

$$(\beta\gamma)\boldsymbol{a} = \beta(\gamma\boldsymbol{a}) = (\beta\boldsymbol{a})\gamma = \beta\boldsymbol{a}\gamma = \beta\gamma\boldsymbol{a}$$

- Left-Distributive

$$(\beta + \gamma)\boldsymbol{a} = \beta\boldsymbol{a} + \gamma\boldsymbol{a}$$

- Right-Distributive

$$\boldsymbol{a}(\beta + \gamma) = \boldsymbol{a}\beta + \boldsymbol{a}\gamma$$

$$\beta(\boldsymbol{a} + \boldsymbol{b}) = \beta\boldsymbol{a} + \beta\boldsymbol{b}$$

Addition of n-vectors

Linear Combinations

- The linear combinations of m vectors $a_1, \dots a_m$, each with size n is:

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

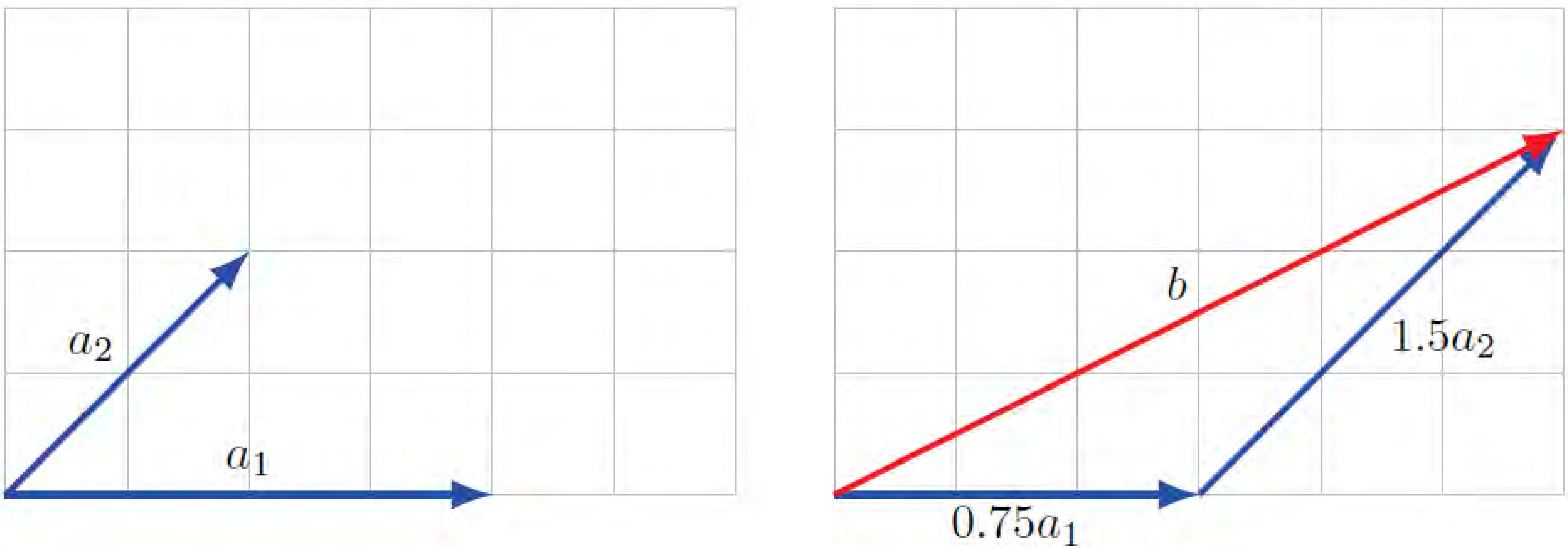
where $\beta_1, \dots, \beta_m$ are scalars and called the coefficients of the linear combination

- We can write any n-vector b as a linear combination of the standard unit vectors, as:

$$b = b_1 e_1 + \cdots + b_n e_n$$

  - Example: What are the coefficients and combination for this vector? $\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$

Linear Combinations



Left. Two 2-vectors a_1 and a_2. *Right.* The linear combination $b = 0.75a_1 + 1.5a_2$

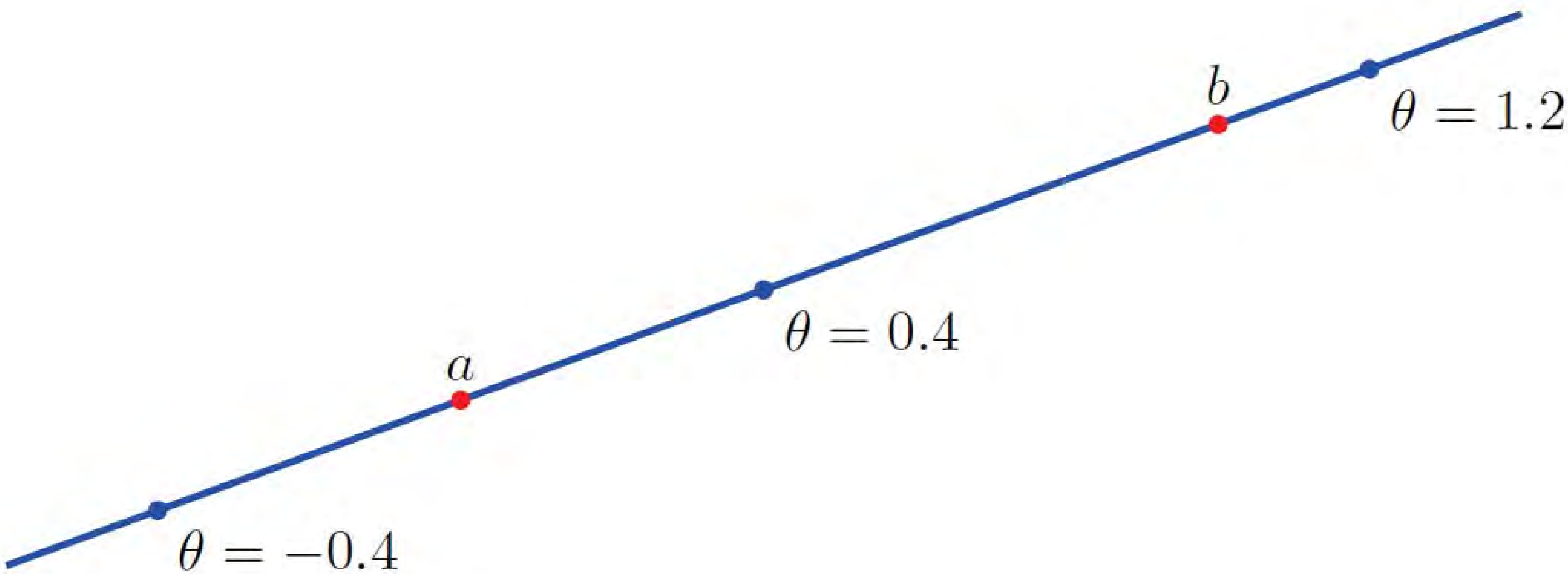
Special Linear Combinations

- Sum of vectors
- Average of vectors
- Affine combination

$$\beta_1 + \cdots + \beta_m = 1$$

- Convex combination, mixture average, weighted average: When the coefficients in an affine combination are nonnegative
  - Note: The coefficients in an affine or convex combination are sometimes given as percentages, which add up to 100%.

Linear Combinations Example



The affine combination $(1-\theta)a + \theta b$ for different values of θ.
These points are on the line passing through a and b; for θ between 0 and 1, the points are on the line segment between a and b.

Vector-Vector Products

- Given two vectors $x, y \in R^n$: (should have same size)
  - $\boldsymbol{x}.\boldsymbol{y}$ is called the inner product or dot product or scalar product of the vectors: $x^T y$ $(y^T x)$

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i .$$

- Dot product is a single number that provides information about the relationship between two vectors
- It is the basic computational building-block from which many operations and algorithms are built, including convolution, correlation, the Fourier transform, matrix multiplication, signal filtering, and so on.
- The term "inner product" is used when the two vectors are continuous functions.
- Why is named scalar product, too?
- Notations: $< a, b >$ $< a|b >$ (a, b) $a.b$

Vector-Vector Products

- Dot product between a vector and itself: magnitude-squared, the length squared, or the squared-norm, of the vector.

$$\mathbf{a}^{\mathrm{T}}\mathbf{a} = \|\mathbf{a}\|^2 = \sum_{i=1}^{n} \mathrm{a}_i \mathrm{a}_i = \sum_{i=1}^{n} \mathrm{a}_i^2$$

- If the vector is mean-centered—the average of all vector elements is subtracted from each element—then the dot product of a vector with itself is call *variance* in statistics lingo.
- When n = 1, the inner product reduces to the usual product of two numbers.

Vector-Vector Products

- The scalar product can be viewed as function taking two vectors as arguments and producing a single scalar as a result. The usual notation in this case is

$$\langle\, ,\rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R},\ \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v} = \sum_{i=1}^{m} u_i v_i$$

with $\mathcal{V} = \mathbb{R}^m$.

- Transpose of dot product:
  - $(a.b)^T = (a^T b)^T = (b^T a) = (b.a) = b^T a$

Dot product properties

- Commutativity
  - The order of the two vector arguments in the inner product does not matter.

$$a^T b = b^T a$$

- Distributivity with vector addition
  - The inner product can be distributed across vector addition.

$$(a + b)^T c = a^T c + b^T c$$
$$a^T (b + c) = a^T b + a^T c$$

Dot product properties

- Bilinear (linear in both a and b)

$$a^T(\lambda b + \beta c) = \lambda a^T b + \beta a^T c$$

- Positive Definite:

$$(a.a) = a^T a \geq 0$$

  - 0 only if a itself is a zero vector $a = \mathbf{0}$

Dot product properties

- Associative
  - Note: the associative law is that parentheses can be moved around, e.g., (x+y)+z = x+(y+z) and x(yz) = (xy)z
- 1) Associative property of the vector dot product with a scalar (scalar-vector multiplication embedded inside the dot product)

scalar

$$\gamma(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = (\gamma\mathbf{u}^{\mathrm{T}})\mathbf{v} = \mathbf{u}^{\mathrm{T}}(\gamma\mathbf{v}) = (\mathbf{u}^{\mathrm{T}}\mathbf{v})\gamma$$

$$= (\gamma \boldsymbol{u})^{T}\boldsymbol{v} = \gamma \boldsymbol{u}^{T}\boldsymbol{v}$$

Dot product properties

- Associative
  - 2) Does vector dot product obey the associative property?

$$\underbrace{\mathbf{u}^{\mathrm{T}}(\mathbf{v}^{\mathrm{T}}\mathbf{w})}_{\substack{\text{vector-scalar product} \\ \text{row vector}}} = \underbrace{(\mathbf{u}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}}\mathbf{w}}_{\substack{\text{scalar-vector product} \\ \text{column vector}}}$$

Dot product properties

- Example
  - For any vectors a, b, c, d with the same size:

$$(a+b)^T(c+d) = a^Tc + a^Td + b^Tc + b^Td$$

  - Specify the vector and scalar additions?
  - Applying the distributive property to the dot product between a vector and itself?

$$(\mathbf{u}+\mathbf{v})^{\mathrm{T}}(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}^{\mathrm{T}}\mathbf{u} + 2\mathbf{u}^{\mathrm{T}}\mathbf{v} + \mathbf{v}^{\mathrm{T}}\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}^{\mathrm{T}}\mathbf{v}$$