Linear Equations

CE40282-1: Linear Algebra
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Matrices

- A matrix is a rectangular array of numbers,

$$\begin{bmatrix} -1.1 & 2 & 3 & 2+3i \\ 5 & -3 & 0.33 & 6.7 \\ 3.6 & 1.2 & 34 & 10.3 \end{bmatrix}$$

- Its size: (row dimension) × (column dimension)
  - e.g. matrix above is 3×4
- More scientific definition, later on course! Do you have any suggestions?!

Matrices

$$A = \begin{bmatrix} -1.1 & 2 & 3 & 2+3i \\ 5 & -3 & 0.33 & 6.7 \\ 3.6 & 1.2 & 34 & 10.3 \end{bmatrix}$$

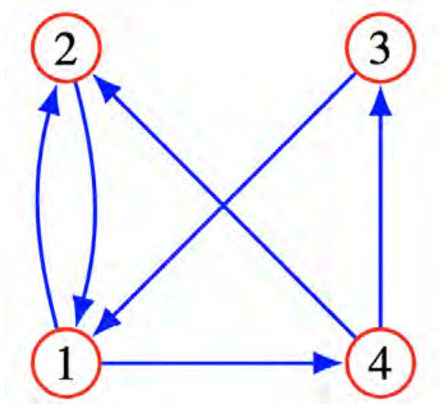
- Elements also called entries of coefficients
- i is the row index, j is column index, indexes start at 1
- $A_{i,j}$ is i, j element of matrix A
  - e.g. above matrix: $A_{2,3} = 0.33$

Examples of Matrices

- Images: $\mathrm{X}_{\mathrm{i,j}}$ is $\mathrm{i, j}$ pixel value in a image
- Rainfall data: $\mathrm{R}_{\mathrm{i,j}}$ is amount of rain at location i on day j
- Feature matrix: $\mathrm{X}_{\mathrm{i,j}}$ is value of feature i for entity j
- In each of these, what do the rows and columns means?

Examples of Matrices

- Consider this directed graph:



- Can be represented as $n{\times}n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Special Matrices

- $m \times n$ ***zero matrix*** has all entries zero, written as $0_{m\times n}$ or just 0

- ***Identity matrix*** is square matrix with $I_{ii} = 1$ and $I_{ij} = 0$ for $i \neq j$, e.g.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Sparse matrix**: most entries are zero

Columns & Rows of Matrix

- Suppose A is a $m \times n$ matrix:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix}$$

Columns & Rows of Matrix

- The j-th column is (the m-vector) $\begin{bmatrix} A_{1,j} \\ \vdots \\ A_{m,j} \end{bmatrix}$

- The i-th row is (the n-row-vector) $\begin{bmatrix} A_{i,1} & \cdots & A_{i,n} \end{bmatrix}$

- Slice of matrix $A_{p:q,r:s}$ is: $\begin{bmatrix} A_{p,r} & \cdots & A_{p,s} \\ \vdots & \vdots & \vdots \\ A_{q,r} & \cdots & A_{q,s} \end{bmatrix}$

Block Matrices

- We can form block matrices, whose entries are matrices,

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where B, C, D and E are matrices (called blocks of A)

- Matrices in each block row must have the same height

- Matrices in each block column must have the same width

Example

- If

$$B = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix} , \; C = \begin{bmatrix} -1 \end{bmatrix} , \; D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \end{bmatrix} , \; E = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

- Then

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & -1 \\ 2 & 2 & 1 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix}$$

Column & Row Representation

- Suppose A is an $m \times n$ matrix
- Can express as block matrix with its columns $\boldsymbol{a_1}, \dots, \boldsymbol{a_n}$

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix}$$

- Or as block matrix with its rows $\boldsymbol{b_1}, \dots, \boldsymbol{b_m}$

$$A = \begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_m \end{bmatrix}$$

Matrix-Vector Multiplication

- Matrix-vector product of $m \times n$ matrix A, n-vector $\boldsymbol{x}$, denoted $\boldsymbol{y} = A\boldsymbol{x}$,

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}_{m\times 1}$$

$$y_i = A_{i1}x_1 + \cdots + A_{in}x_n$$

$$i = 1, \cdots, m$$

Row Interpretation

- $\boldsymbol{y}$=$A\boldsymbol{x}$ can be expressed as

$$\begin{bmatrix} \boldsymbol{b_1} \\ \vdots \\ \boldsymbol{b_m} \end{bmatrix} [\boldsymbol{x}] = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$y_i = \boldsymbol{b_i} \cdot \boldsymbol{x} \qquad i = 1, \cdots, m$$

- $\boldsymbol{y}$ is a ‘batch’ inner product of all rows of A with $\boldsymbol{x}$

Column Interpretation

- $\boldsymbol{y} = A\boldsymbol{x}$ can be expressed as

$$\begin{bmatrix} \boldsymbol{a_1} & \cdots & \boldsymbol{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{y}$$

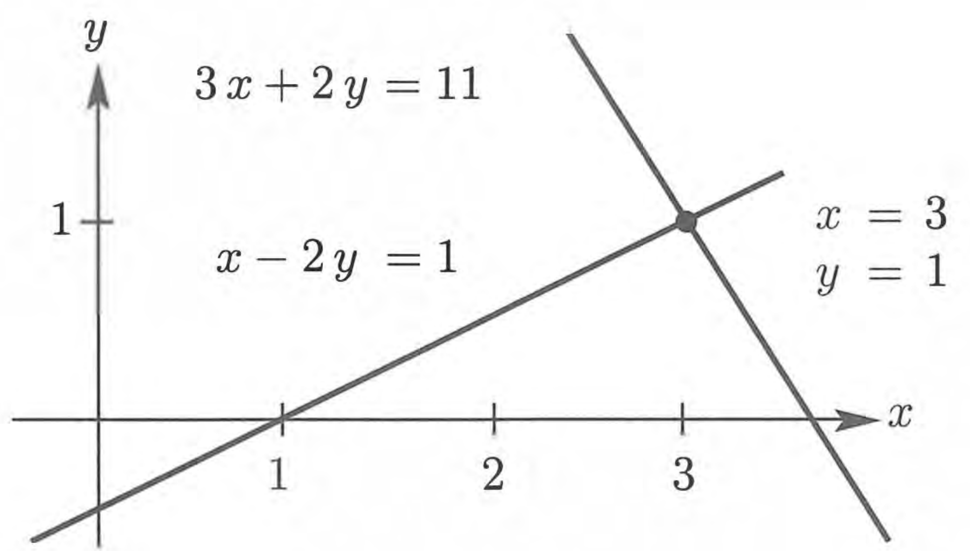
$$\boldsymbol{y} = x_1\boldsymbol{a_1} + \cdots + x_n\boldsymbol{a_n}$$

- $\boldsymbol{y}$ is linear combination of cols of A with coefficients $x_1, \ldots, x_n$

Vectors & Linear Equation

- Consider this simple system of equations,

$$\begin{aligned} x &- 2y &= 1 \\ 3x &+ 2y &= 11 \end{aligned}$$



Vectors & Linear Equation

- Can be expressed as matrix-vector multiplication

$$\begin{aligned} x &- 2y &= 1 \\ 3x &+ 2y &= 11 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{b}$$

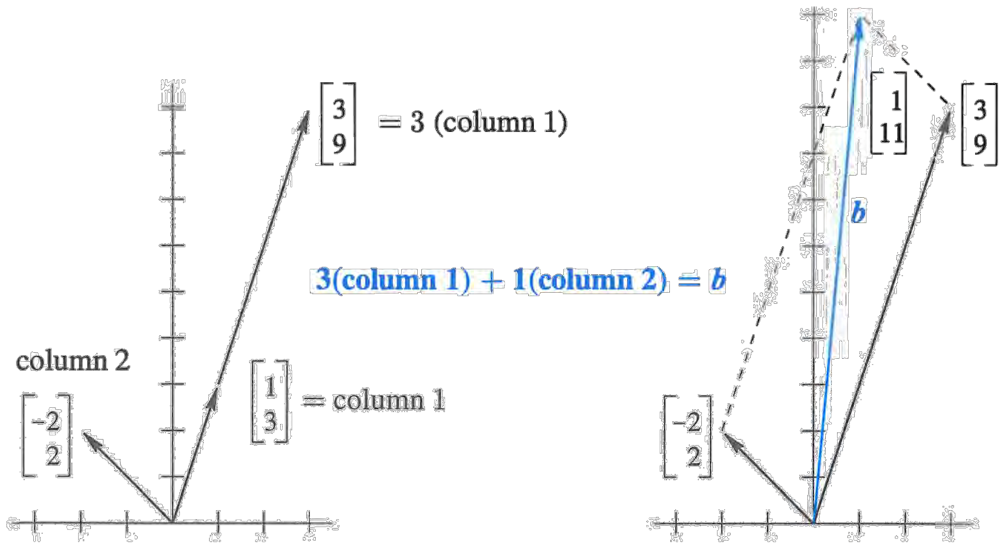
- Matrix Equation, $A\boldsymbol{x} = \boldsymbol{b}$
- A is often called **coefficient matrix**
- **Augmented matrix** is: $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$

Vectors & Linear Equation

- Also, Can be expressed as linear combination of cols:

$$\begin{aligned} x \quad - \quad 2y \quad &= 1 \\ 3x \quad + \quad 2y \quad &= 11 \end{aligned}$$

$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} = \boldsymbol{b}$$



- Same for n equation, n variables

Idea of Elimination

- Subtract a multiple of equation (1) from to (2) to eliminate a variable

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \xrightarrow[\textit{(subtract to eliminate } 3x\textit{)}]{\textit{(multiply equation 1 by 3)}} \quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 8 \end{bmatrix}}_{\boldsymbol{c}}$$

A has become a upper triangle matrix U

Idea of Elimination (Row Reduction Algorithm)

- The **pivots** are on the diagonal of the triangle after elimination (boldface 2 below is the first pivot)

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8. \end{aligned}$$

  - Step 1: by subtracting (1) to (2) eliminate x's in (2)
  - Step 2: subtract (1) from (3) and totally eliminate x
  - Step 3: subtract new(2) from new(3)
- The variables corresponding to pivot columns in the matrix are called basic variables. The other variables are called a free variable.

Idea of Elimination (Row Reduction Algorithm)

- The whole idea is how to make U out of A by eliminate variables column by column

Column 1. *Use the first equation to create zeros below the first pivot.*

Column 2. *Use the new equation 2 to create zeros below the second pivot.*

Columns 3 to n. *Keep going to find all n pivots and the upper triangular U.*

$$\begin{bmatrix} \mathbf{x} & x & x & x \\ 0 & \mathbf{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{x} & x & x & x \\ & \mathbf{x} & x & x \\ & & \mathbf{x} & x \\ & & & \mathbf{x} \end{bmatrix}$$

Example

- Augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \quad\quad - 5x_3 &= 1 \\ x_2 + \quad x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad \begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

- x_1, x_2: basic x_3: free
- This system is consistent, because the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

Definition

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

  1. All nonzero rows are above any rows of all zeros.

  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

  3. All entries in a column below a leading entry are zeros.

  If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

  4. The leading entry in each nonzero row is 1.

  5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Reduced echelon form

Existence and Uniqueness Questions

- Two fundamental question about a linear system:
  - Is the system consistent; that is, does at least one solution exist?
  - 2. If a solution exists, is it the only one; that is, is the solution unique?

Existence and Uniqueness Questions

- Theorem: A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad \cdots \quad 0 \quad b]$ with b nonzero
- If a linear system is consistent, then the solution set contains either:
  - a unique solution, when there are no free variables
  - infinitely many solutions, when there is at least one free variable.

Find all solutions of a linear system

- 1. Write the augmented matrix of the system.
- 2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
- 3. Continue row reduction to obtain the reduced echelon form.
- 4. Write the system of equations corresponding to the matrix obtained in step 3.
- 5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Existence of Solutions

- The equation Ax=b has a solution if and only if b is a linear combination of the columns of A.

EXAMPLE Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if it can be written in the form $Ax=0$, where A is a matrix and 0 is the zero vector.
- Trivial solution: $Ax=0$ always has at least one solution, namely, $x=0$ (the zero vector).
- Nontrivial solution: The non-zero solution for $Ax=0$.
- The homogeneous equation $Ax=0$ has a nontrivial solution if and only if the equation has at least one free variable.

Reference

- Linear Algebra and Its Applications, David C. Lay
- Linear Algebra Done Right, Axler, Chapter 3.D
- Introduction to Linear Algebra, Strange, Chapter 2.1,2.2
- http://vmls-book.stanford.edu/vmls-slides.pdf