Chapter 2

Numerical Methods for Solving Nonlinear Equations

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Methods

Extra Topics

- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

Methods

Extra Topics

- Horner Method for Evaluating Polynomials
- Generalized Newton–Raphson Method

Solving Non-Linear Equations

$$f(x) = ax + b = 0$$ (a Linear Equation)

Otherwise, the equation is non-linear.

Solving Non-Linear Equations

$f(x)=0$ $\quad\quad\quad x=\ ?$

$$f(x) = ax^2 + bx + c \rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

but $b^2 - 4ac$ may not be an integer root!

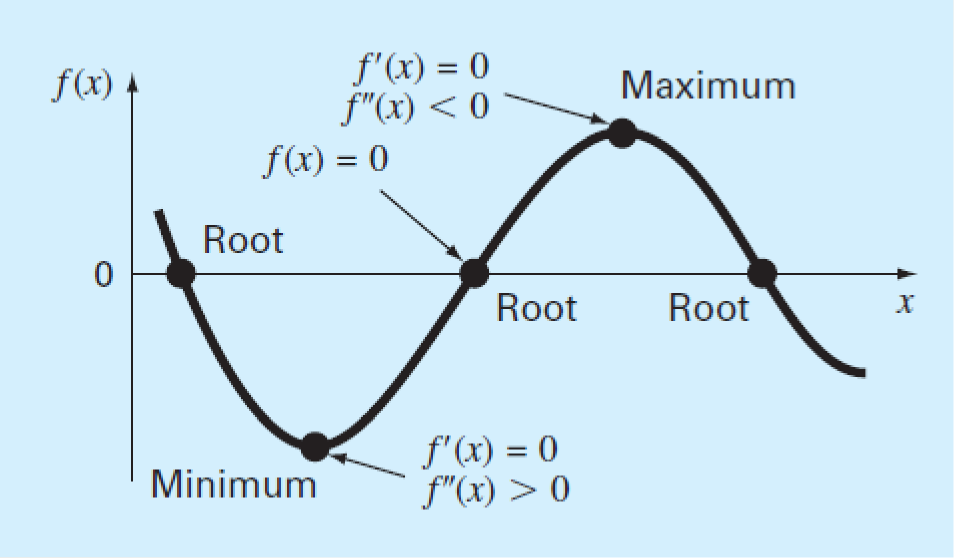
What about other functions?

Solving Non-Linear Equations

$$f(x) = x^{10} + x - 1 = 0$$

$$f(x) = x + \cos(x) = 0$$

$$f(x) = e^{-x} - \cos(x) = 0$$

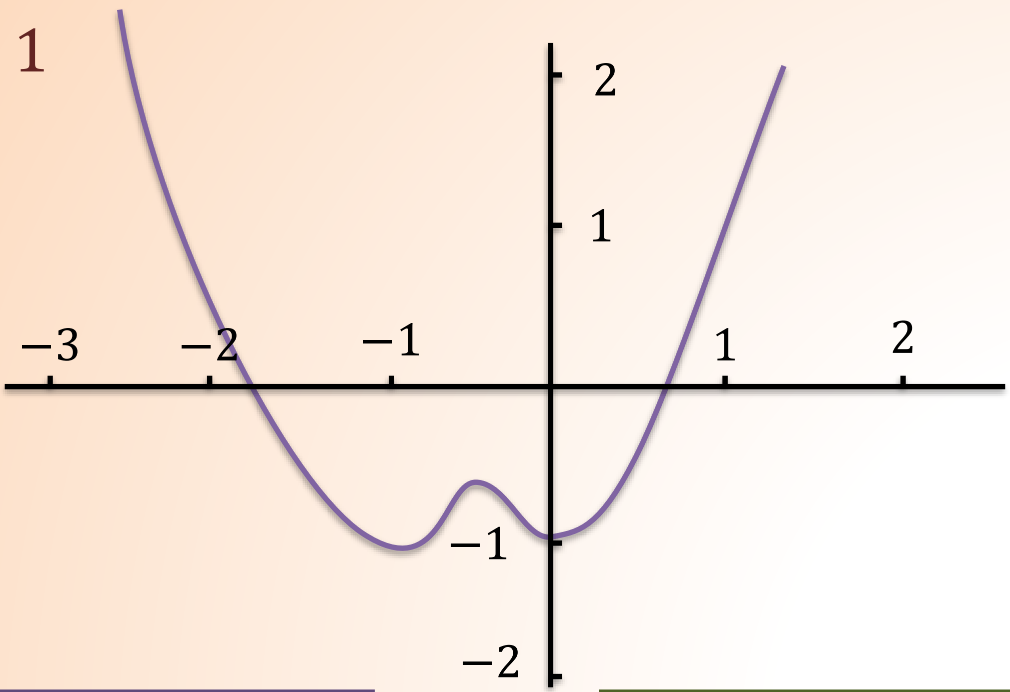


Solution

Plot the graph of $f(x)$ and find its intersection with the x axis.

Example:

$f(x) = x^4 + 2x^3 - x - 1$



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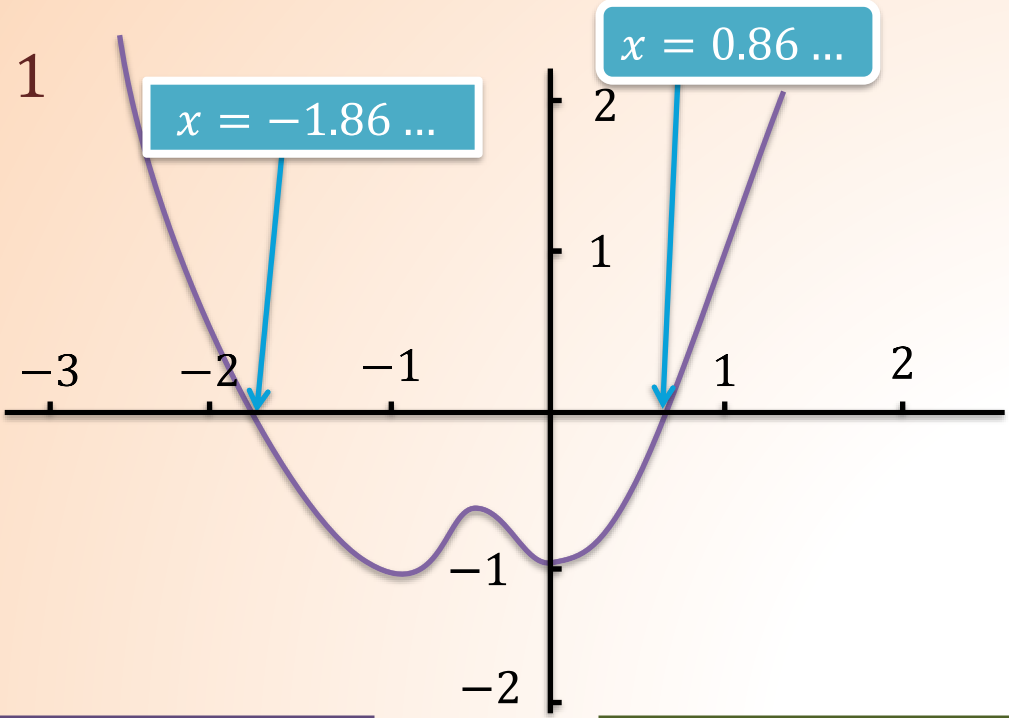
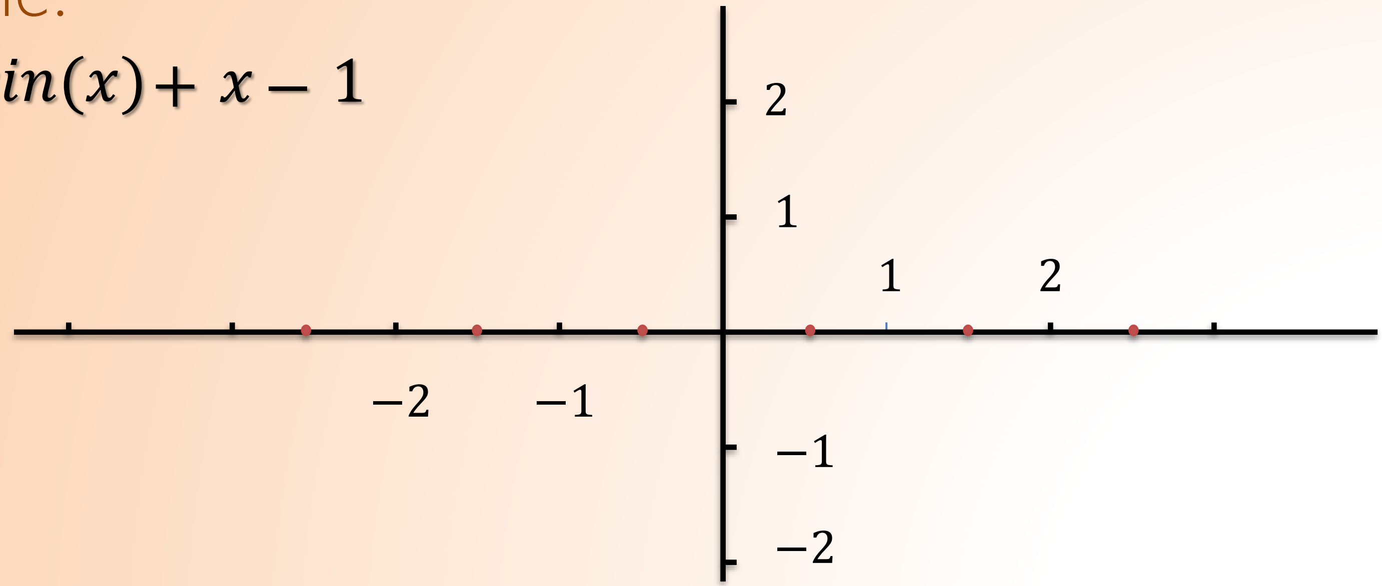


Solution

If $f(x) = f_1(x) - f_2(x)$, we can plot $f_1(x)$ and $f_2(x)$, then find their intersection(s).

- Example:

$f(x) = sin(x) + x - 1$



Solution

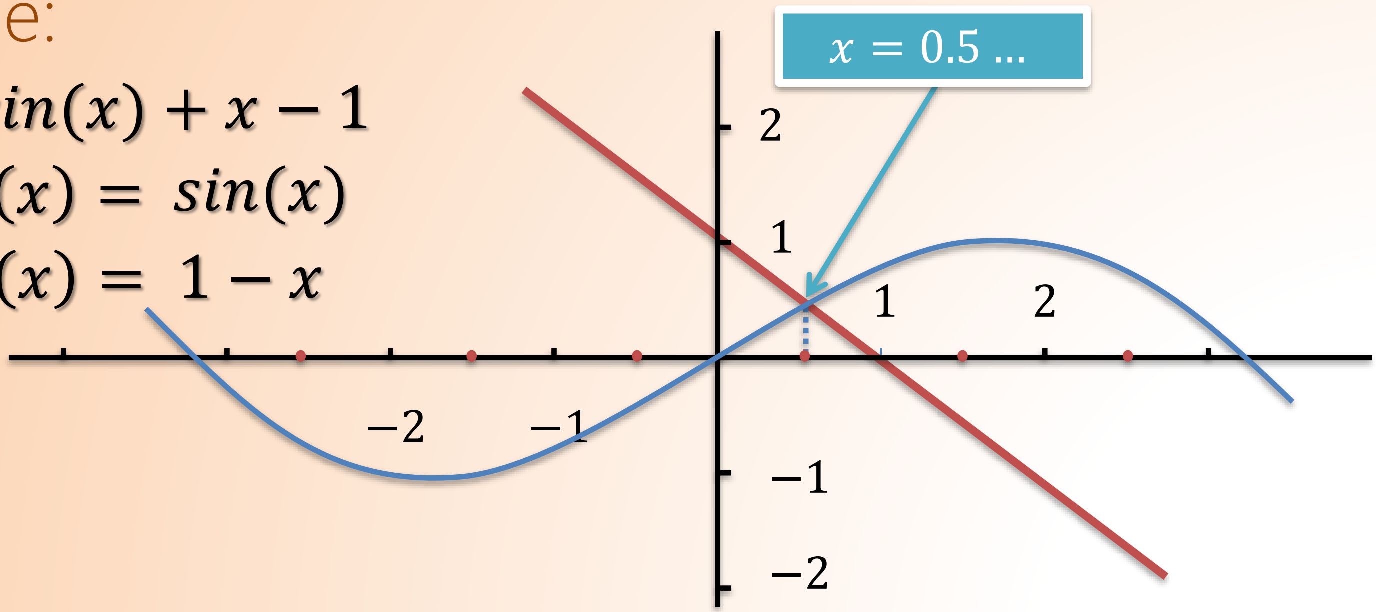
If $f(x) = f_1(x) - f_2(x)$, we can plot $f_1(x)$ and $f_2(x)$, then find their intersection(s).

- Example:

$$f(x) = sin(x) + x - 1$$

$$f_1(x) = sin(x)$$

$$f_2(x) = 1 - x$$



Other Solutions

Problems:

We cannot calculate the exact amount of x by plotting!

Sometimes it is hard to plot $f(x)$.

Solution:

NUMERICAL METHODS

Other Solutions

NUMERICAL METHODS

- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

Numerical Methods

Initially guess an x_0 or an interval (x_0, x_1),

And get the series of $x_i s$.

The sequence x_0, x_1, x_2, … must converge to the exact value of x^*:

- $e(x_0) = |x^* - x_0| = e_0 \,, e(x_1) = |x^* - x_1| = e_1 \,, \ldots, e(x_i) = |x^* - x_i| = e_i$
- $e_0 > e_1 > e_2 > \cdots > e_i \to 0$

Each convergent sequence has a degree of convergence, i.e.

$$e_{i-1}^{m} \times c = e_i \qquad 0 < c < 1$$

m is convergence degree.

Numerical Methods

A sequence (x_n) that converges to x^* is said to have *order of convergence* $\boldsymbol{m \geq 1}$ and *rate of convergence* $\boldsymbol{c}$ if

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^m} = c$$

or

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^m} = c$$

Numerical Methods(cont.)

If the degree of convergence is larger, the sequence will converge faster.

There is usually a maximum expected error ε.
Termination condition is :

$$e_i < \varepsilon \rightarrow |x_i - x^*| < \varepsilon$$

Because x^* is unknown to us, we use x_{i-1} instead:

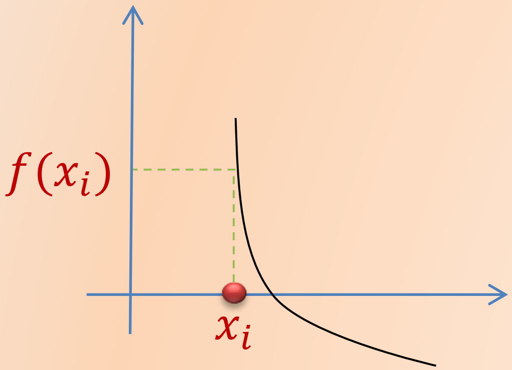
$$|x_{i-1} - x_i| < \varepsilon$$

Numerical Methods(cont.)

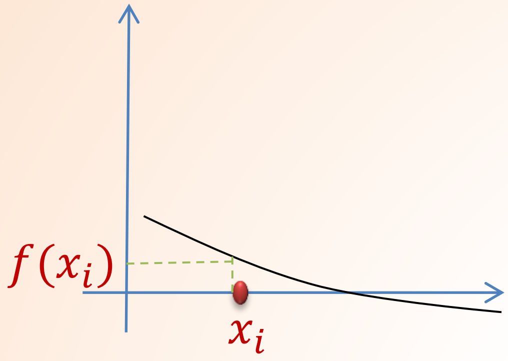
Other termination conditions:

$$f(x_i) < \varepsilon$$

$$i \geq \#iteration$$



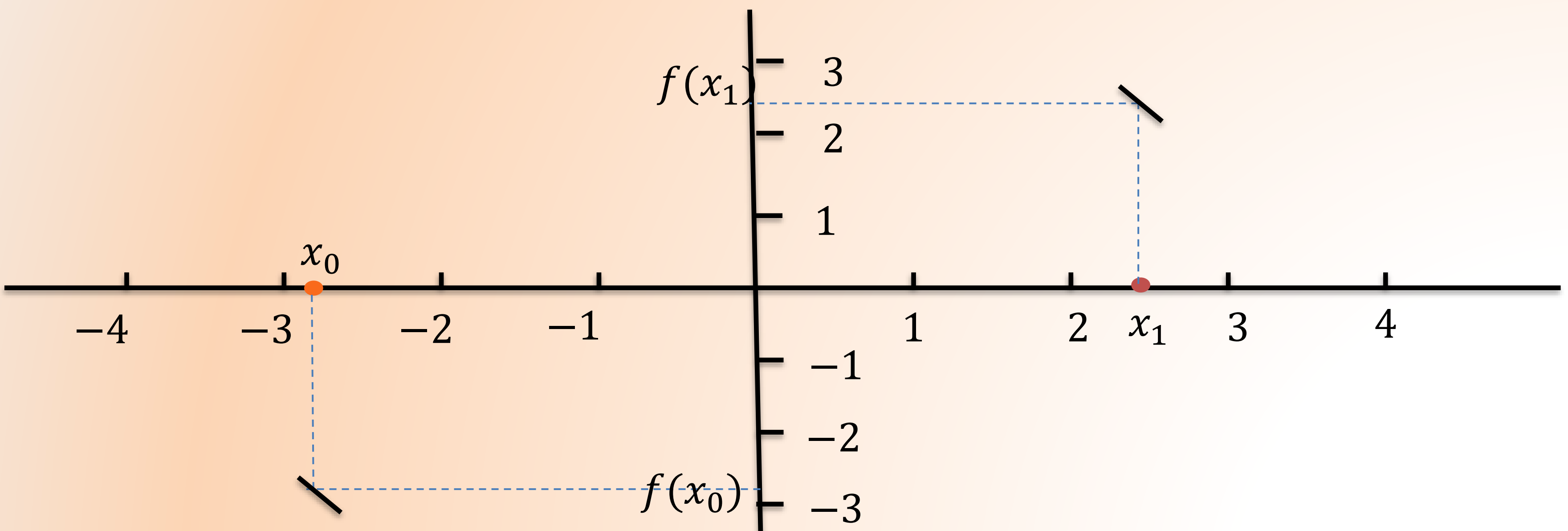
Large $f(x_i)$



Large $|x_i - x^*|$

Bolzano's Theorem

If $f(x)$ is a real-valued continuous function on the interval [a,b], and $f(x_0) \times f(x_1) < 0$, then f has at least one root in this interval.



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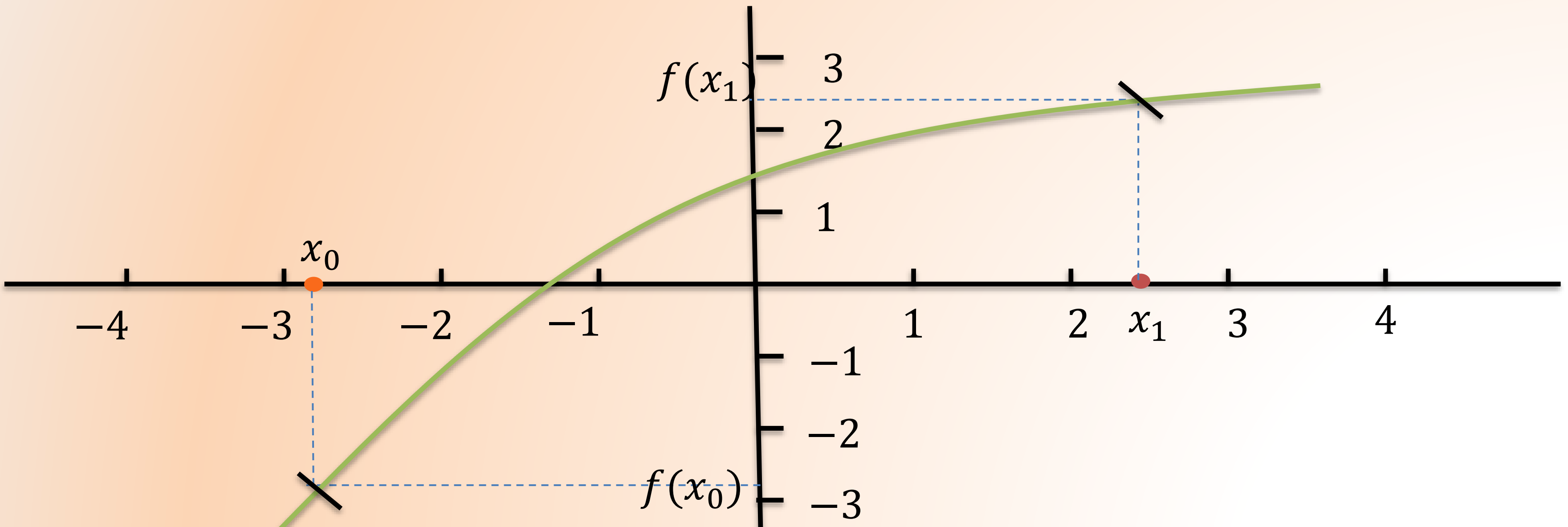

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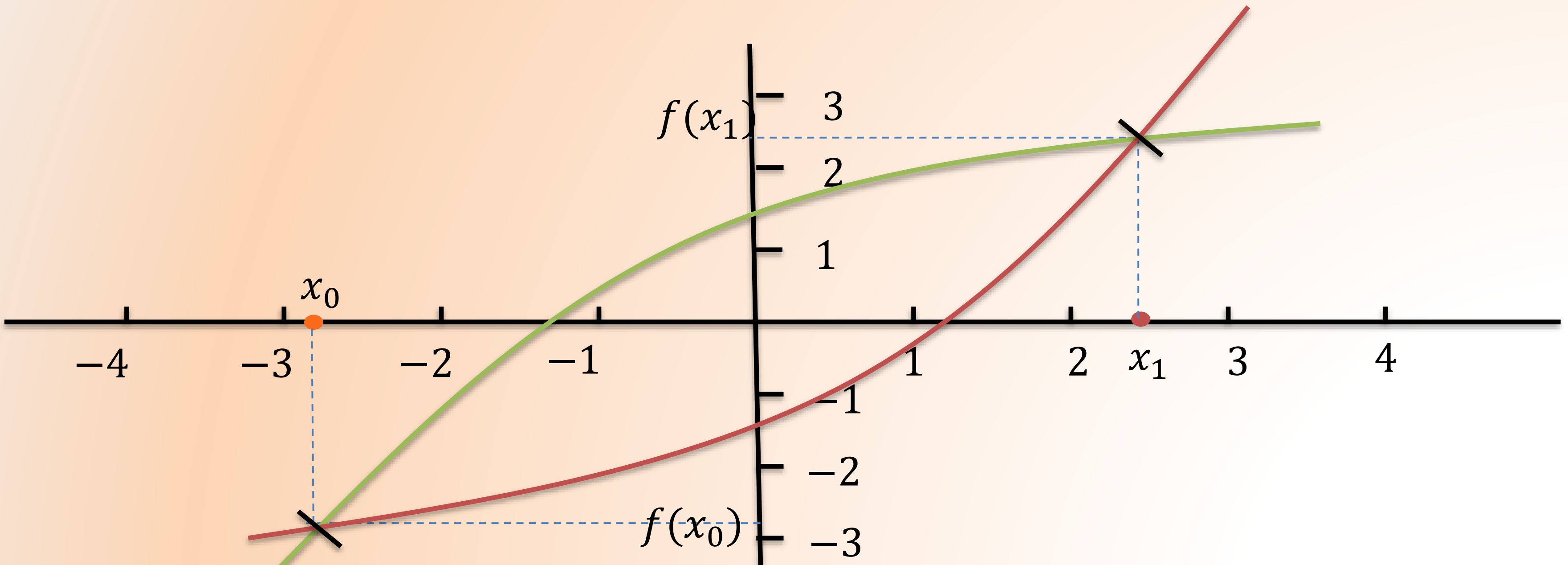

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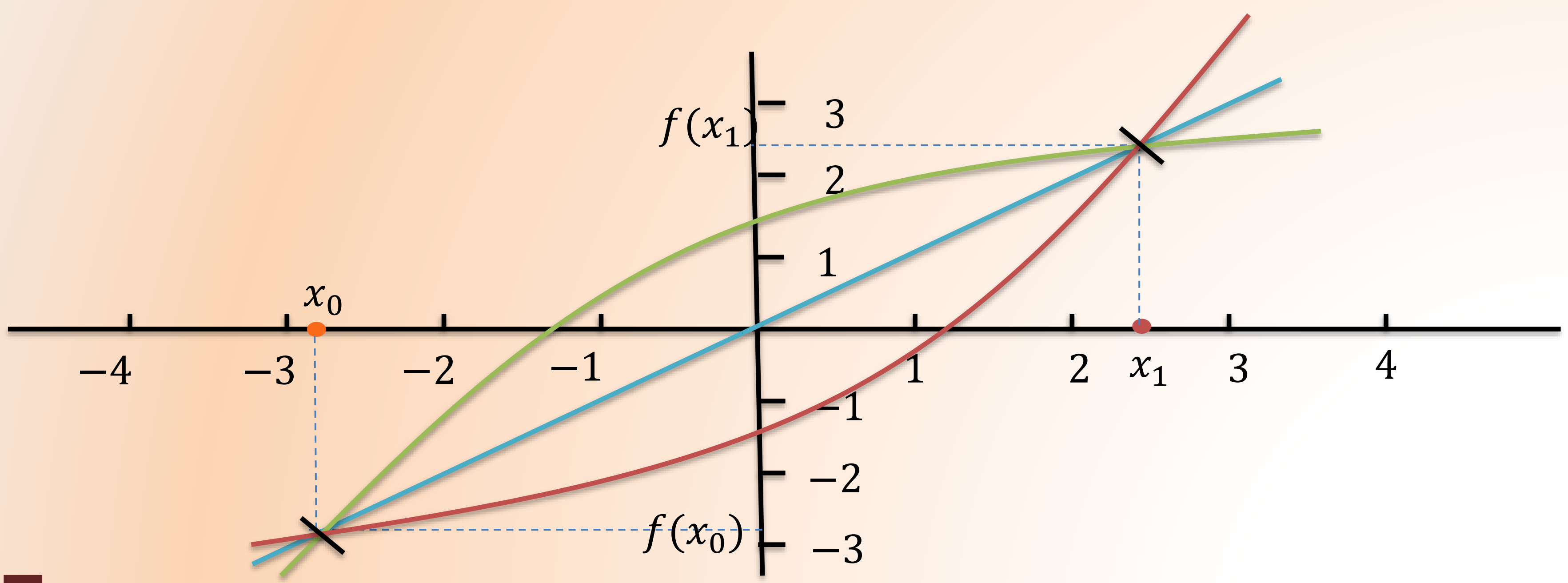

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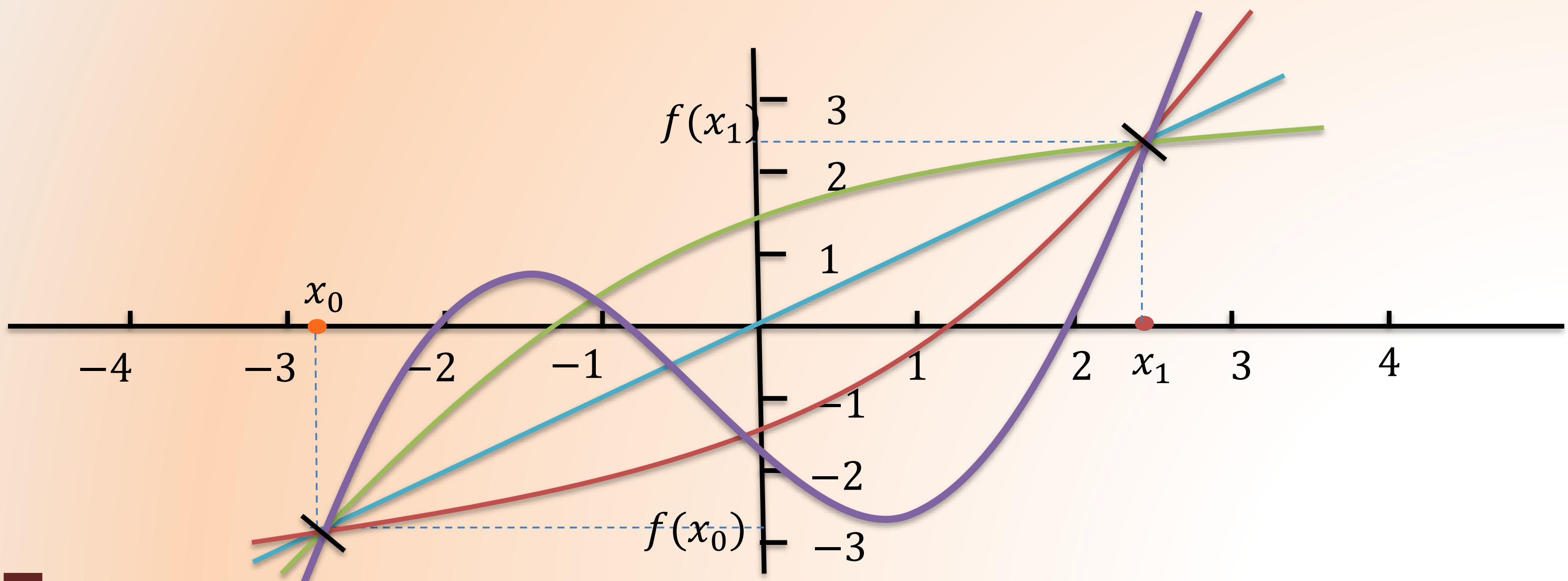
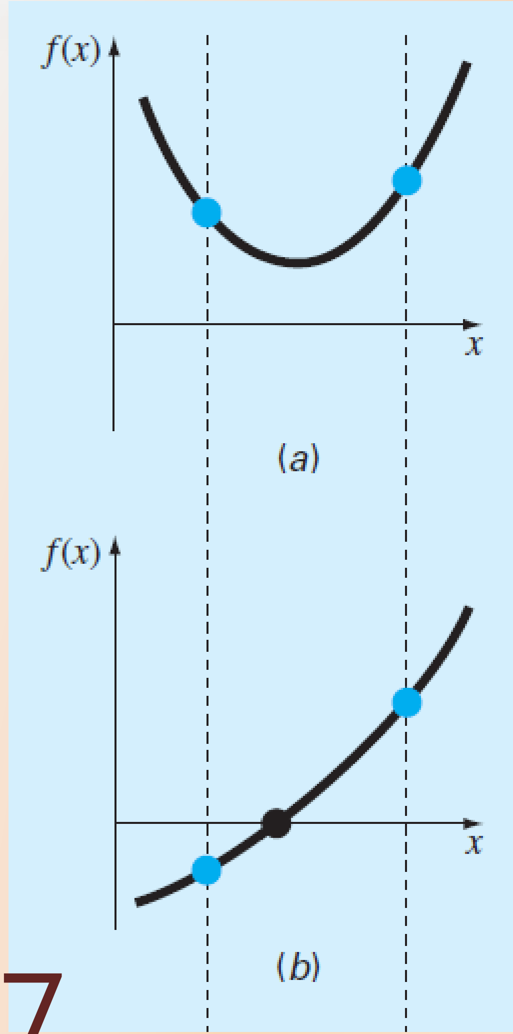


Bolzano's Theorem

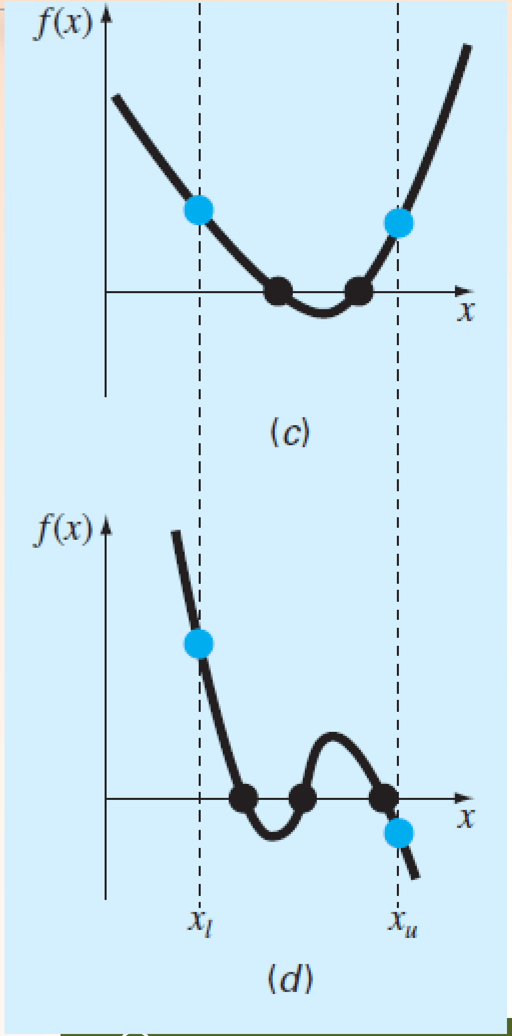
If f is strictly ascending (or descending) in interval $[x_0, x_1]$ then f has exactly one root in this interval.

Roots

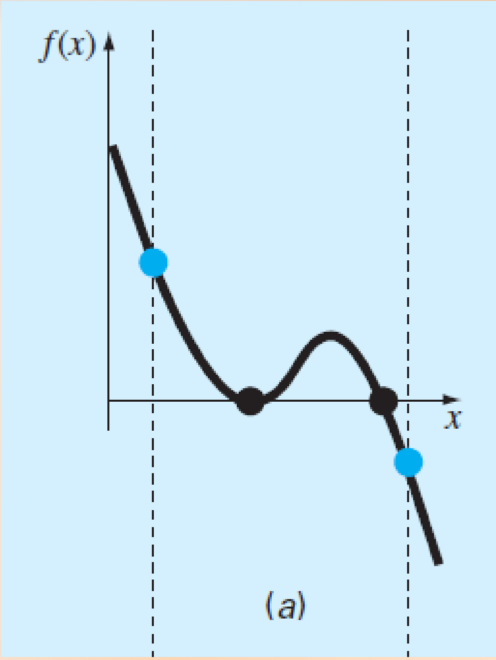


The same signs: #roots=even

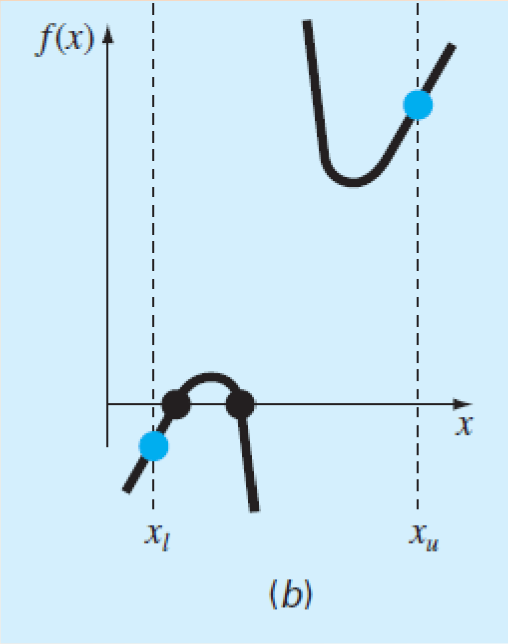
Different signs: #roots=odd



Roots



Different signs



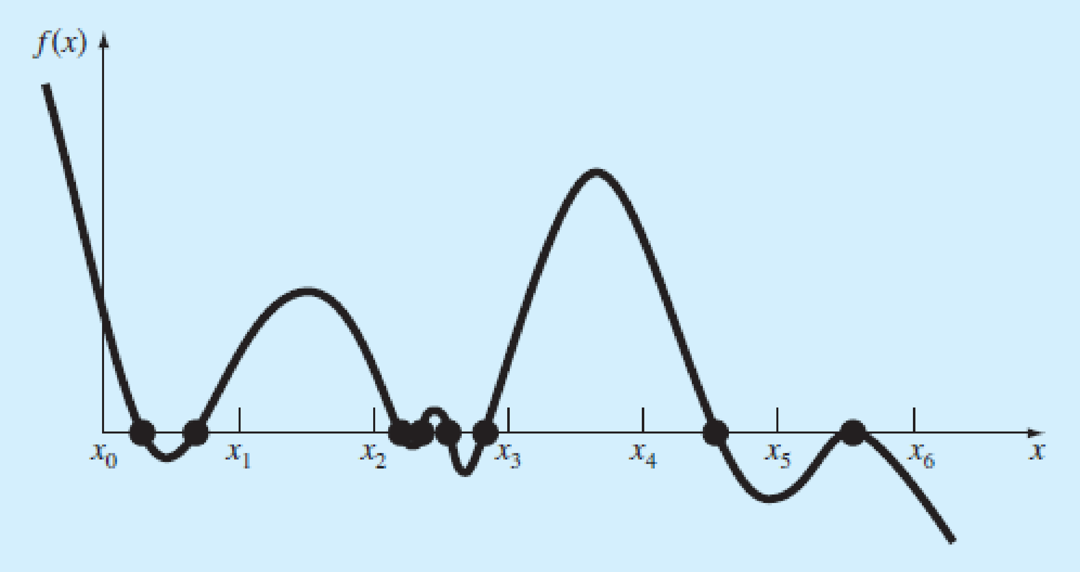
Multiple roots
#roots=even

Discontinuous functions
#roots=even

Incremental Search

Consider Δx and check the sign of $f(x_i) \times f(x_i + \Delta x)$

Cases where roots could be missed because the incremental length of the search procedure is too large. The last root on the right is multiple and would be missed regardless of the increment length.



Methods

Extra Topics

- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

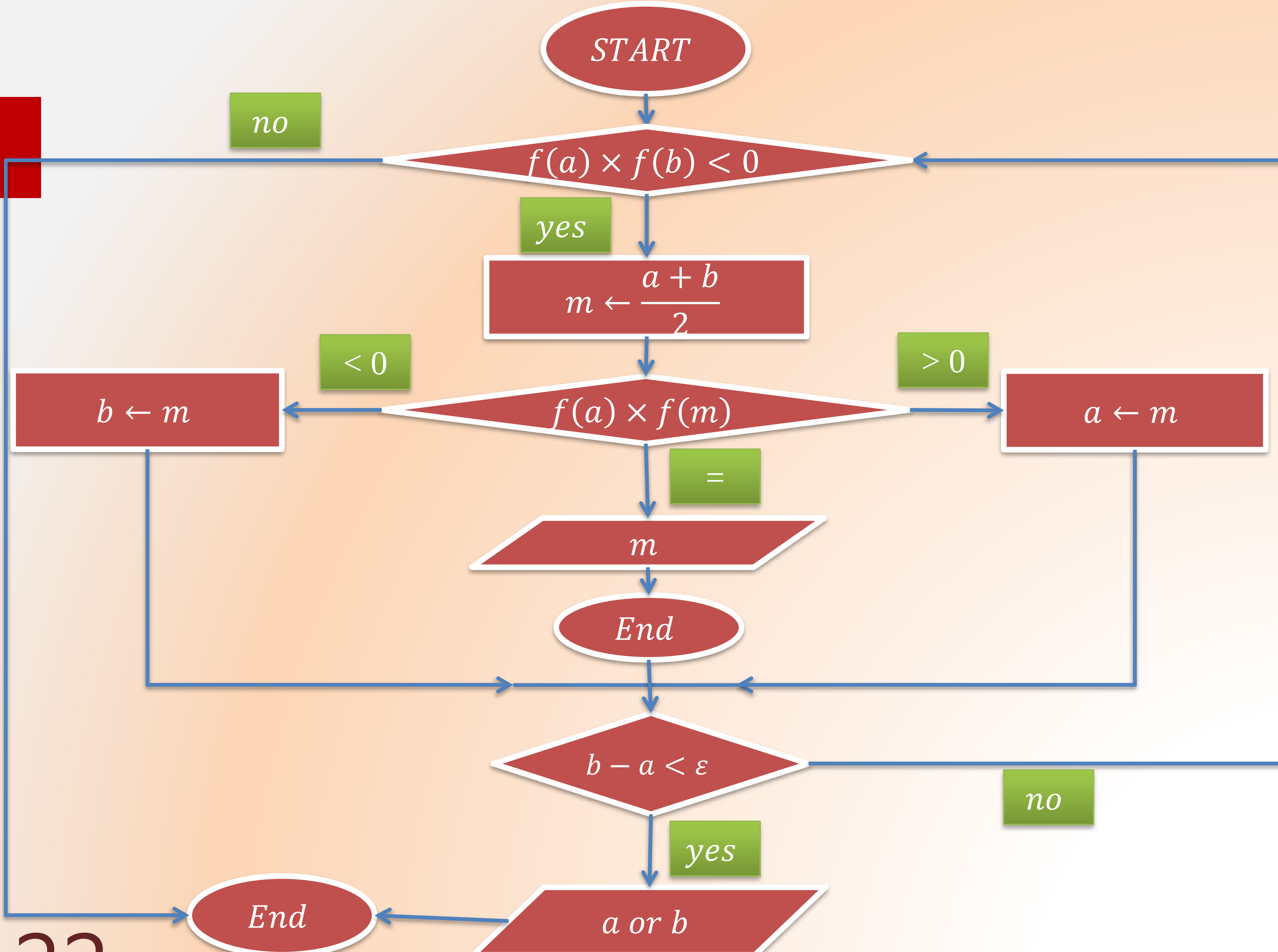
Bisection Method

Assumptions Example Advantages & Disadvantages

$f(x) = 0$

Initial assumption(s):

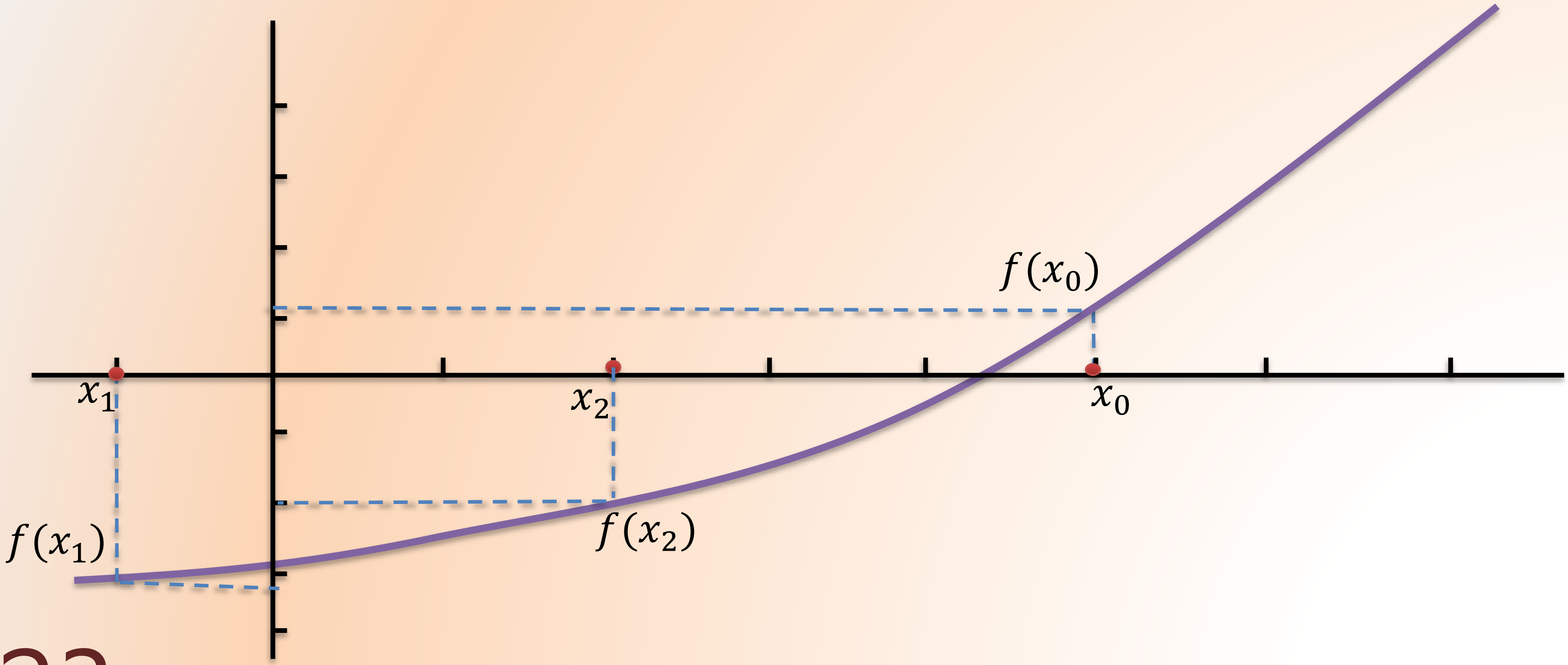
x_0 and x_1 such that $f(x_0) \times f(x_1) < 0$


START
$f(a) \times f(b) < 0$
no
yes
$m \leftarrow \frac{a+b}{2}$
$f(a) \times f(m)$
< 0
> 0
$b \leftarrow m$
$a \leftarrow m$
$=$
m
End
$b - a < \varepsilon$
no
yes
a or b
End

Bisection Method

Assumptions Example Advantages & Disadvantages

$f(x) = 0.25e^{2x} - 1.5 = 0$ $\quad x_0 = 1$ and $x_1 = -0.2$

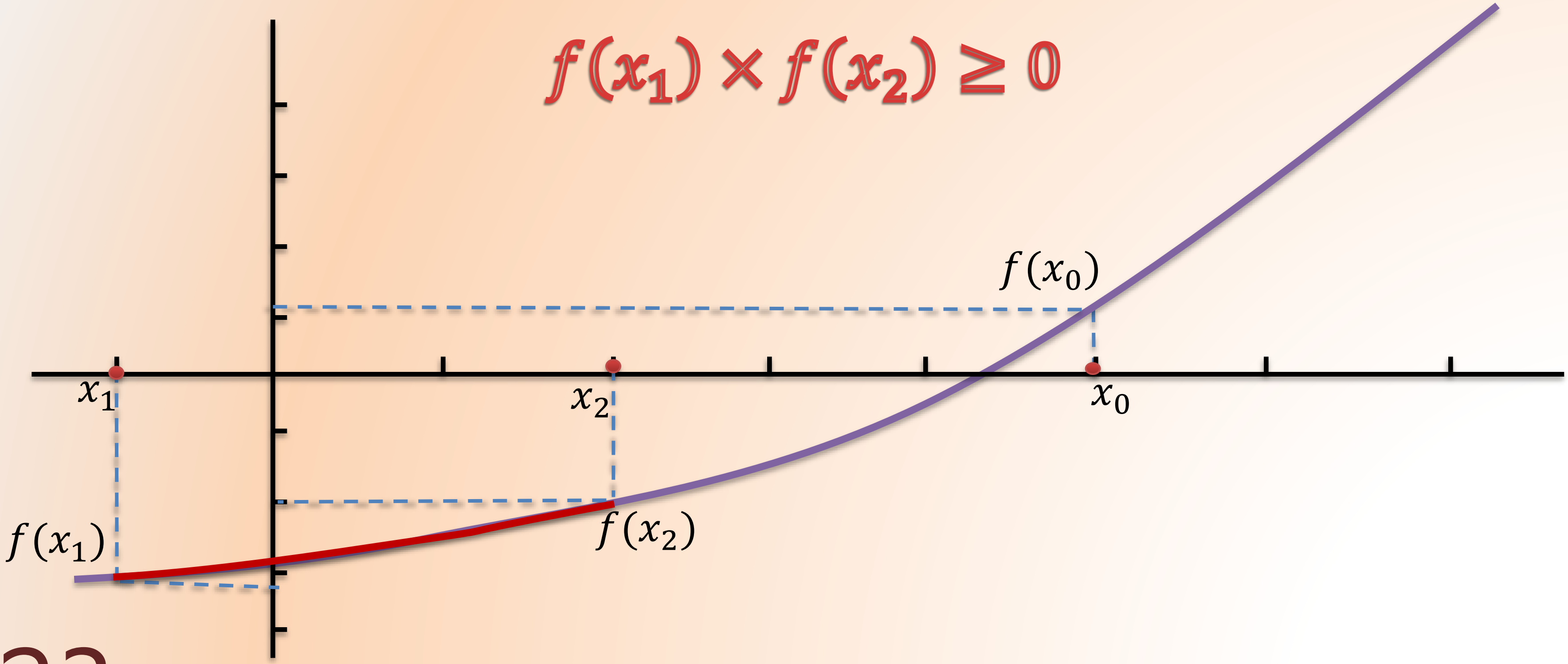


Bisection Method

Assumptions **Example** Advantages & Disadvantages

$$f(x) = 0.25e^{2x} - 1.5 = 0 \qquad x_0 = 1 \text{ and } x_1 = -0.2$$

$$f(x_1) \times f(x_2) \geq 0$$

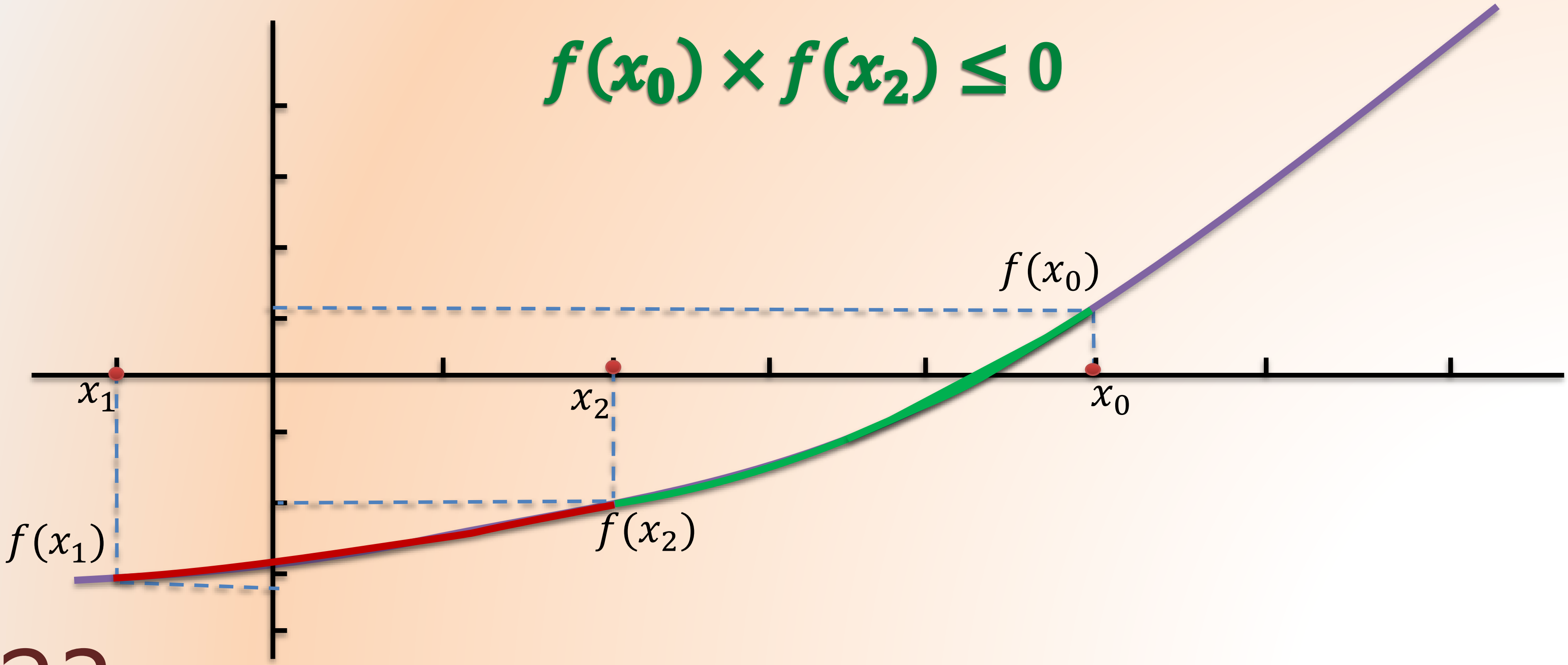


Bisection Method

Assumptions **Example** Advantages & Disadvantages

$$f(x) = 0.25e^{2x} - 1.5 = 0 \qquad x_0 = 1 \text{ and } x_1 = -0.2$$

$$\boldsymbol{f(x_0) \times f(x_2) \leq 0}$$



Bisection Method

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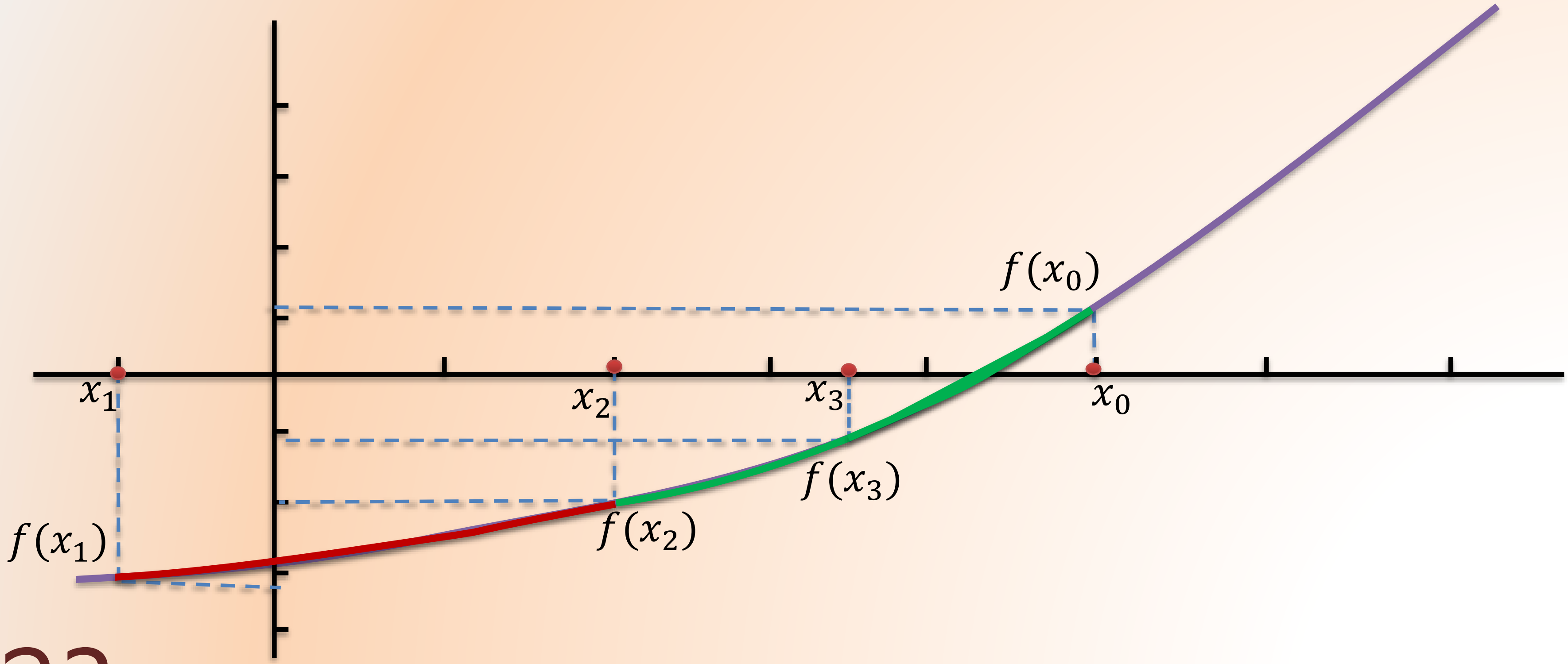
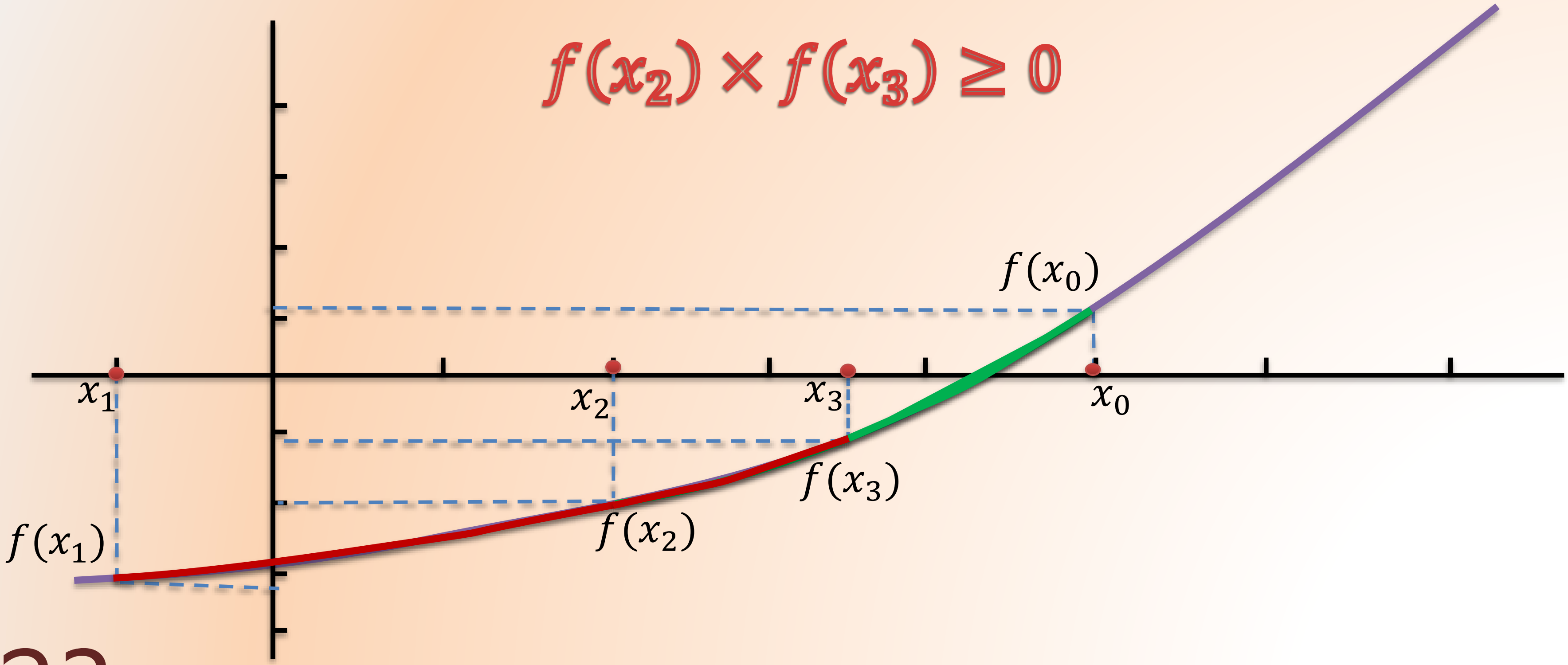


Bisection Method

Assumptions **Example** Advantages & Disadvantages

$f(x) = 0.25e^{2x} - 1.5 = 0$ $x_0 = 1$ and $x_1 = -0.2$

$f(x_2) \times f(x_3) \geq 0$

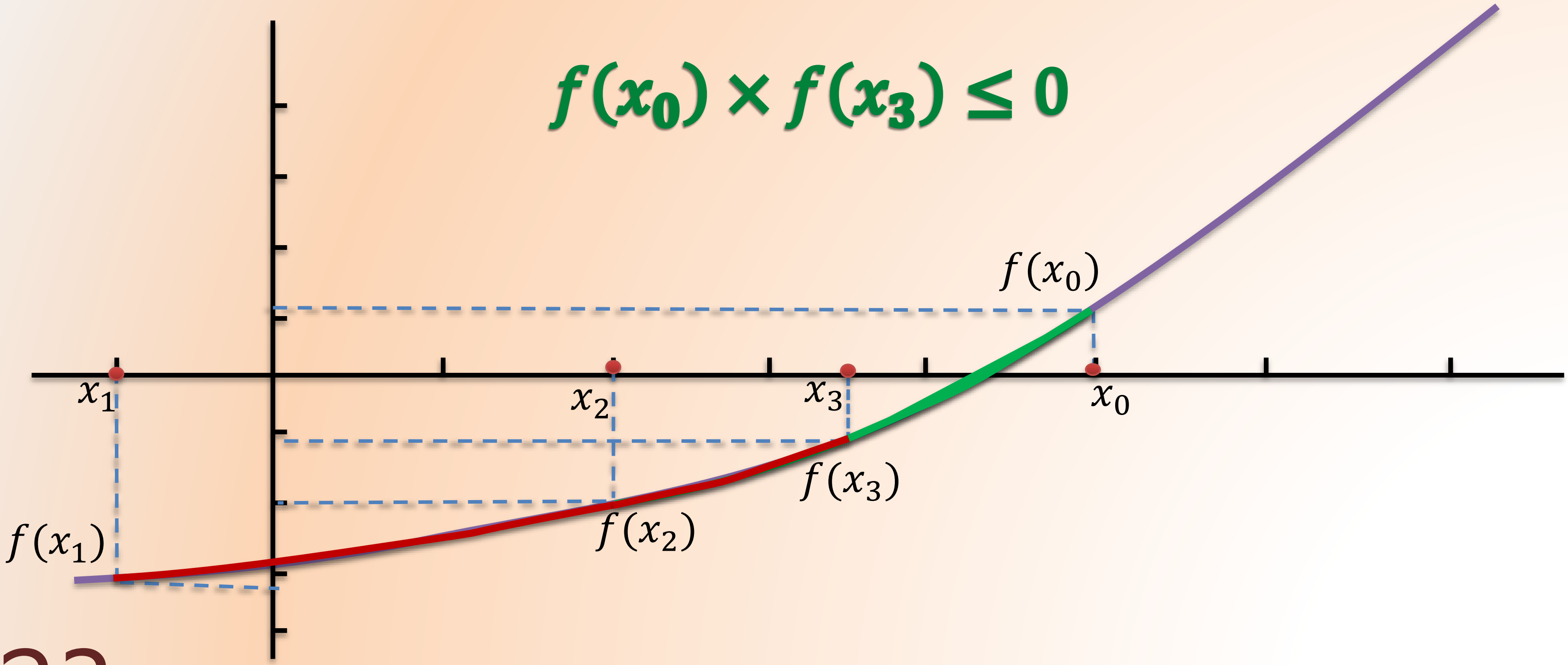


Bisection Method

Assumptions **Example** Advantages & Disadvantages

$f(x) = 0.25e^{2x} - 1.5 = 0$ $\quad x_0 = 1$ and $x_1 = -0.2$

$f(x_0) \times f(x_3) \leq 0$



Bisection Method

Assumptions **Example** Advantages & Disadvantages

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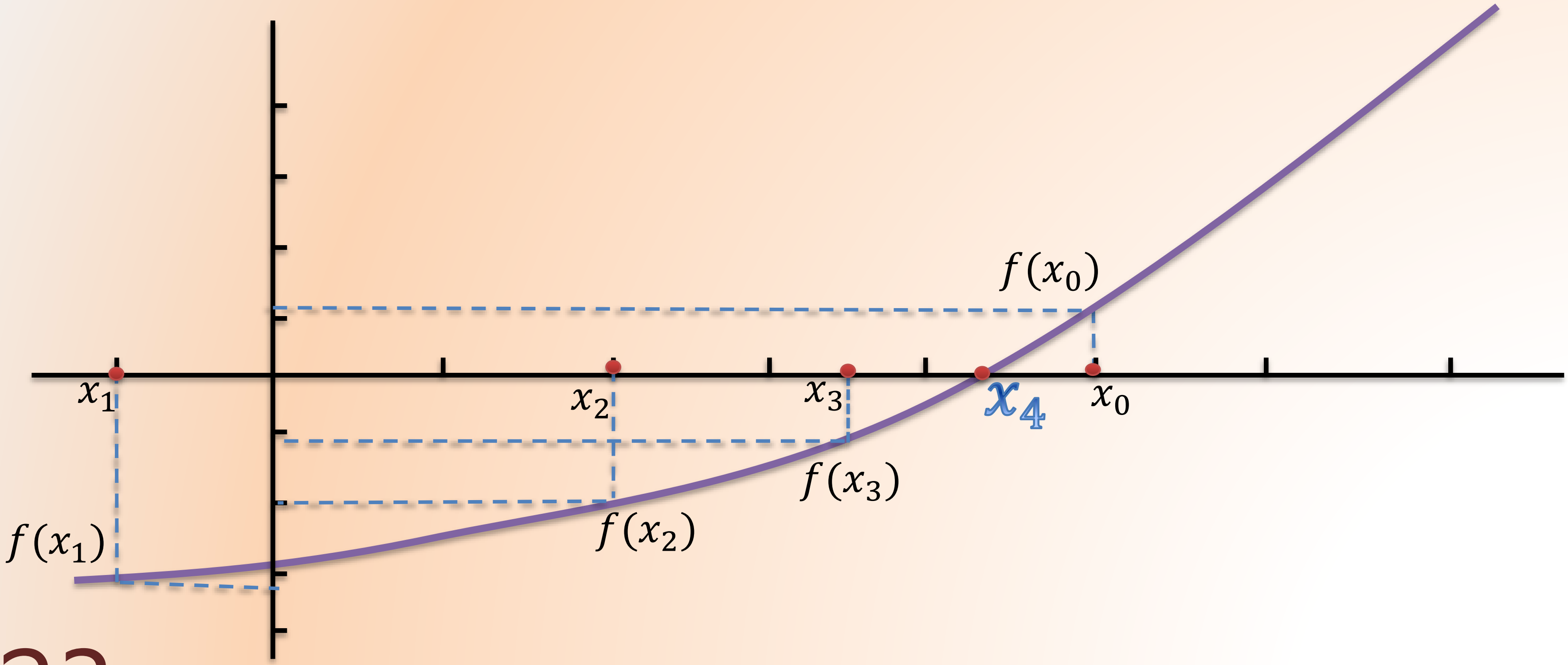


Bisection Method

مثال ۳. تقریبی از ریشه معادله $x^{۲} - (۱ - x)^{۵} = ۰$ را که در فاصله $(۰, ۱)$ قرار دارد با $۴D$ به دست آورید به طوری که داشته باشیم $|f(x_n)| < ۱۰^{-۲}$ که x_n تقریب ریشه در تکرار nام است.

حل:

n	a	b	$x_n = \frac{a+b}{۲}$	علامت $f(a)f(x_n)$	$f(x_n)$
۱	۰	۱	۰٫۵	$-$	۰٫۲۱۸۷۵
۲	۰	۰٫۵	۰٫۲۵	$+$	−۰٫۱۷۴۸۰
۳	۰٫۲۵	۰٫۵	۰٫۳۷۵	$-$	۰٫۰۴۵۲۶
۴	۰٫۲۵	۰٫۳۷۵	۰٫۳۱۲۵۰	$+$	−۰٫۰۵۵۹۳
۵	۰٫۳۱۲۵۰	۰٫۳۷۵	۰٫۳۴۳۷۵	$+$	−۰٫۰۰۳۵۵

چون $|f(x_۵)| = ۰٫۰۰۳۵۵ < ۱۰^{-۲}$ بنابراین تقریب ریشه با $۴D$ عبارتست از:

$$\alpha \simeq ۰٫۳۴۳۸$$

Bisection Method

Number of iterations is n,

if $\frac{b-a}{2^n} \leq \varepsilon$

In the interval $[0, 1]$:

$\boldsymbol{\varepsilon}$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$\boldsymbol{n}$	7	10	14	17	20	24

One step is devoted to the selection of the interval.

The degree of convergence for this method is 1.

(This method converges linearly.)

$$\boldsymbol{e_{k+1} = \frac{1}{2} e_k}$$

Bisection Method

Assumptions Example Advantages & Disadvantages

Advantages :

- Always converges to the answer.
- This method is very simple.
- There is no need to compute the exact value of f. It is only required to know whether f is positive or negative.
- Number of iterations does not relate to the function.

Disadvantages :

- It is very slow (convergence degree is equal to 1).

Methods

- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

Extra Topics

False Position Method

Assumptions Example Advantages & Disadvantages

$$f(x) = 0$$

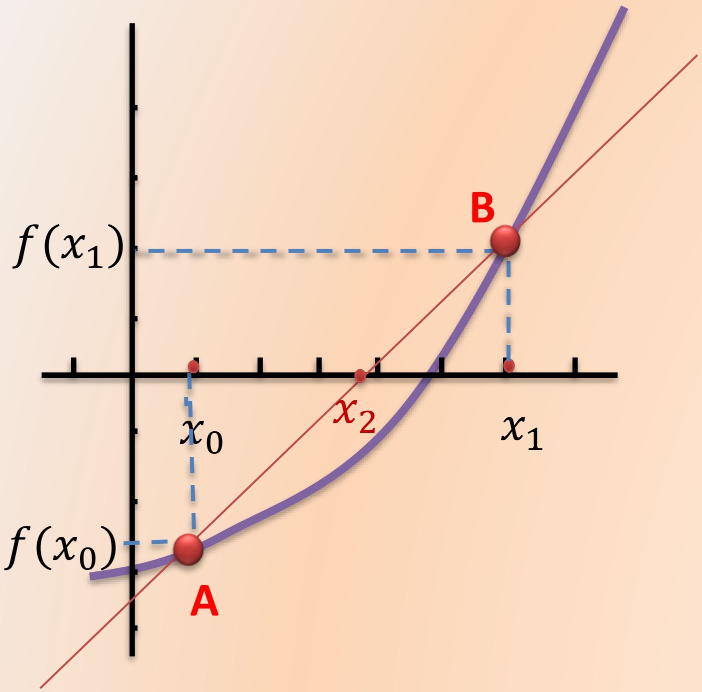
Initial assumptions:

x_0 and x_1 such that $f(x_0) \times f(x_1) < 0$

False Position Method

Assumptions Example Advantages & Disadvantages

$method:$



The equation of line AB:

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The point $(x_2, 0)$ *lies on line AB, so*:

$$0 - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)$$

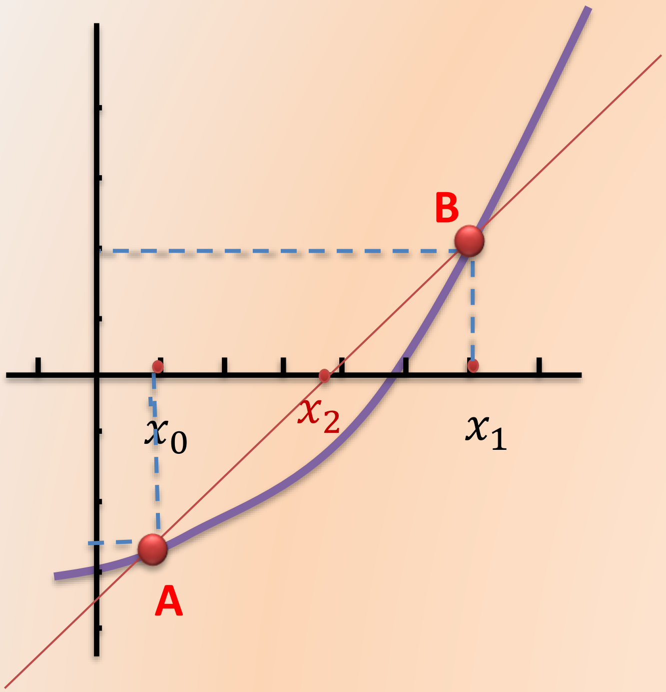
$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

False Position Method

Assumptions Example Advantages & Disadvantages

$method$:

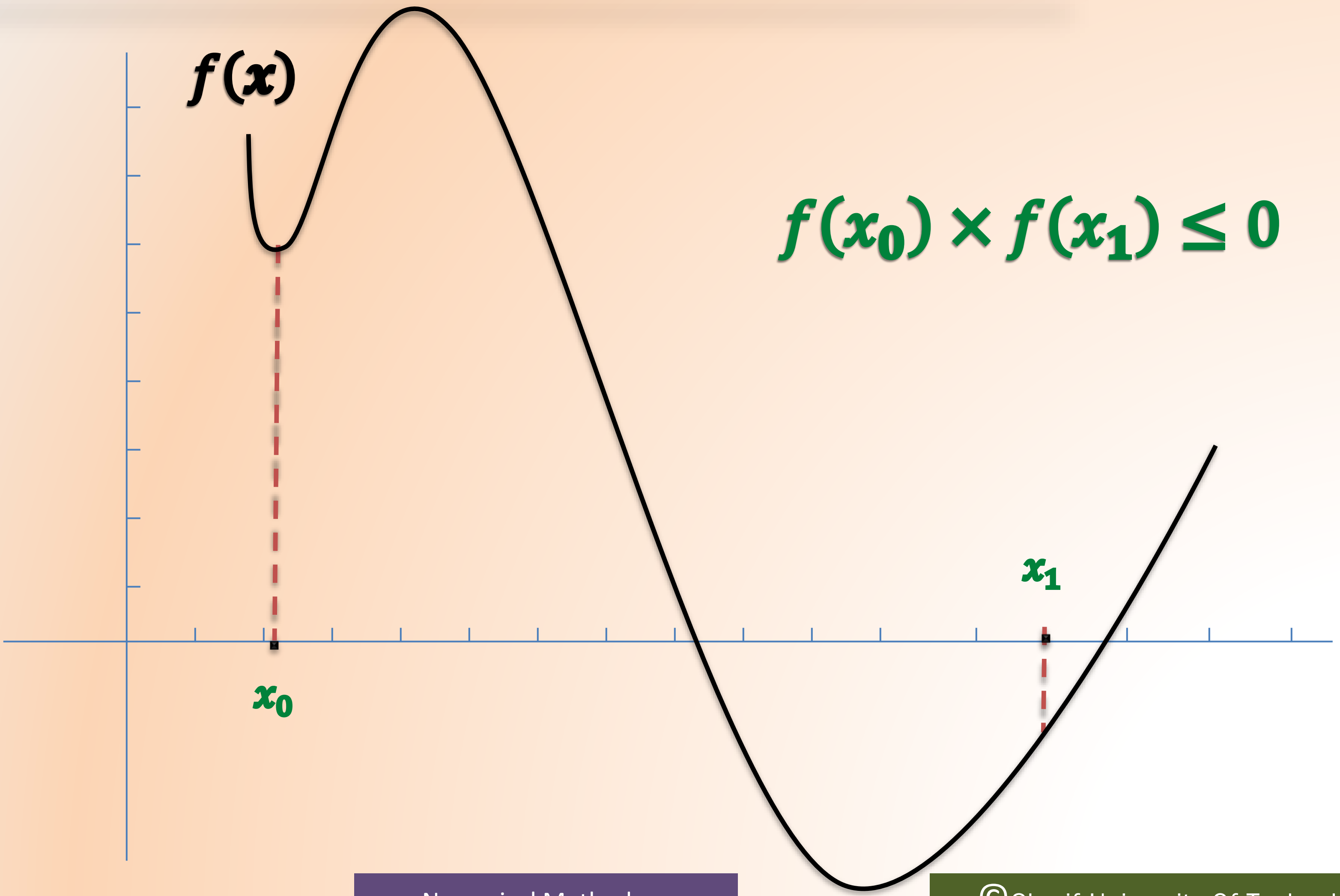
$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$



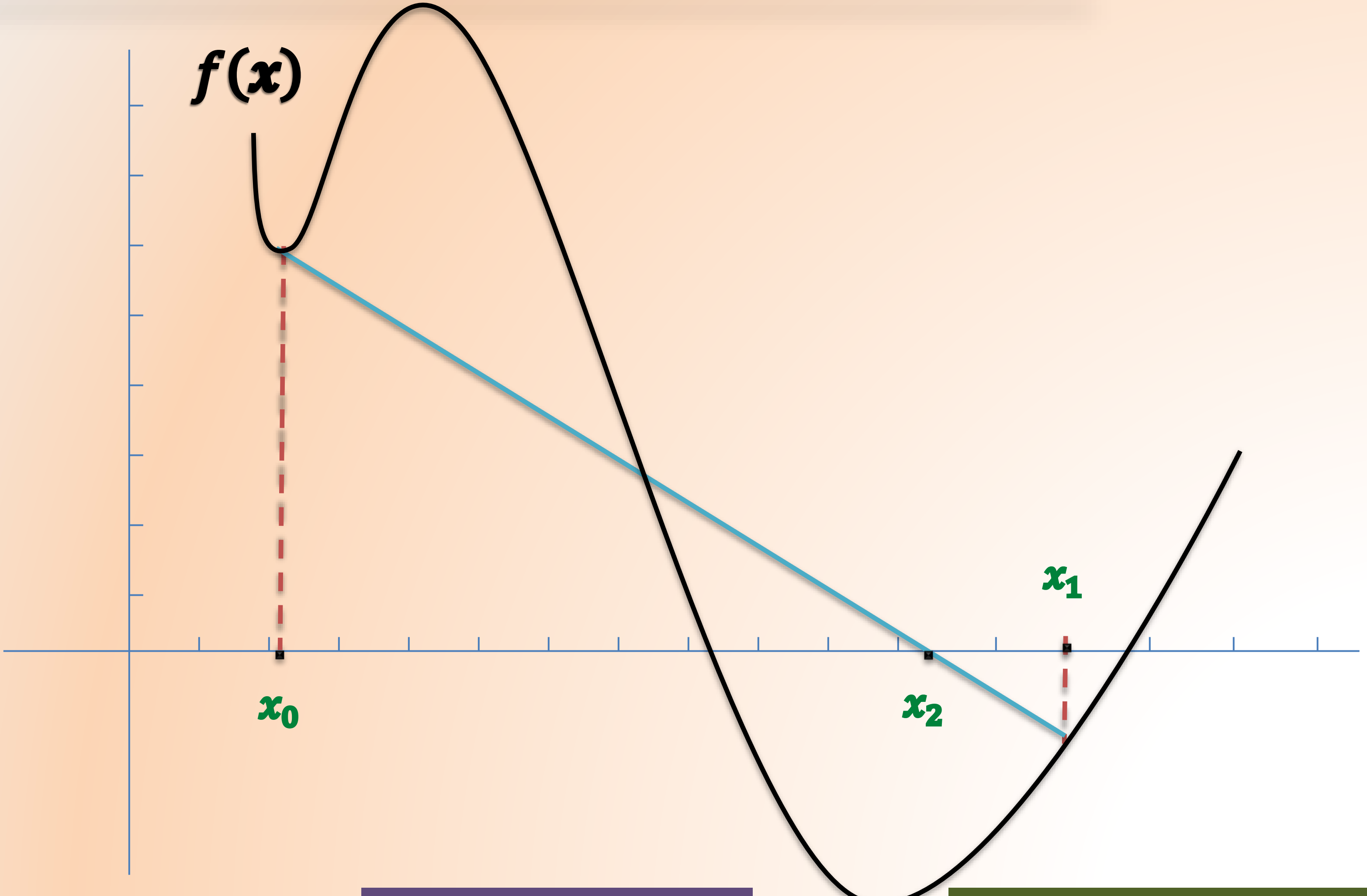
Know, check the Bolzano theorem between
$f(x_2), f(x_0)$
And
$f(x_2), f(x_1)$
To update point A or B

Repeat the above process to get a termination condition

False Position Method



False Position Method



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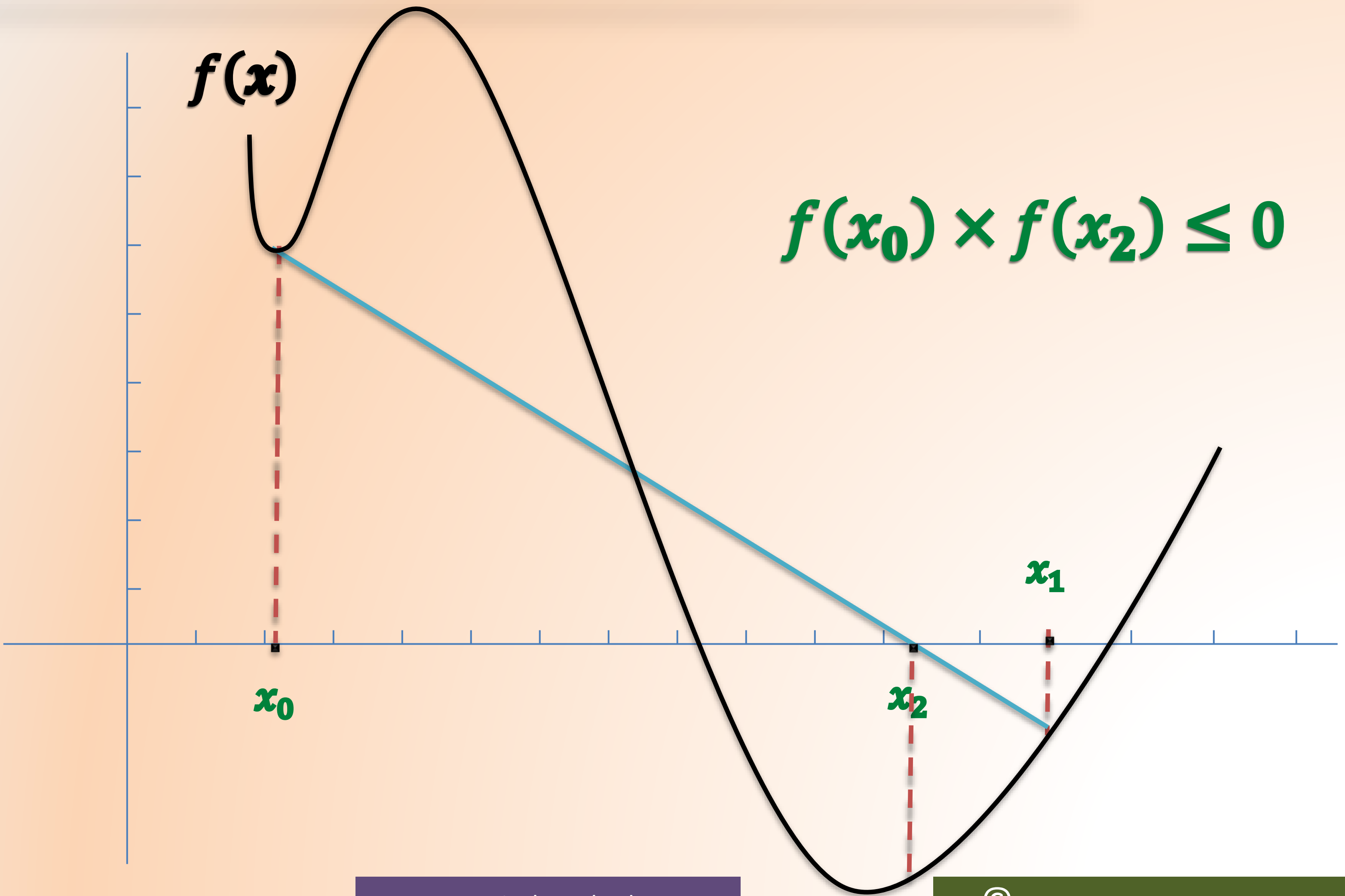

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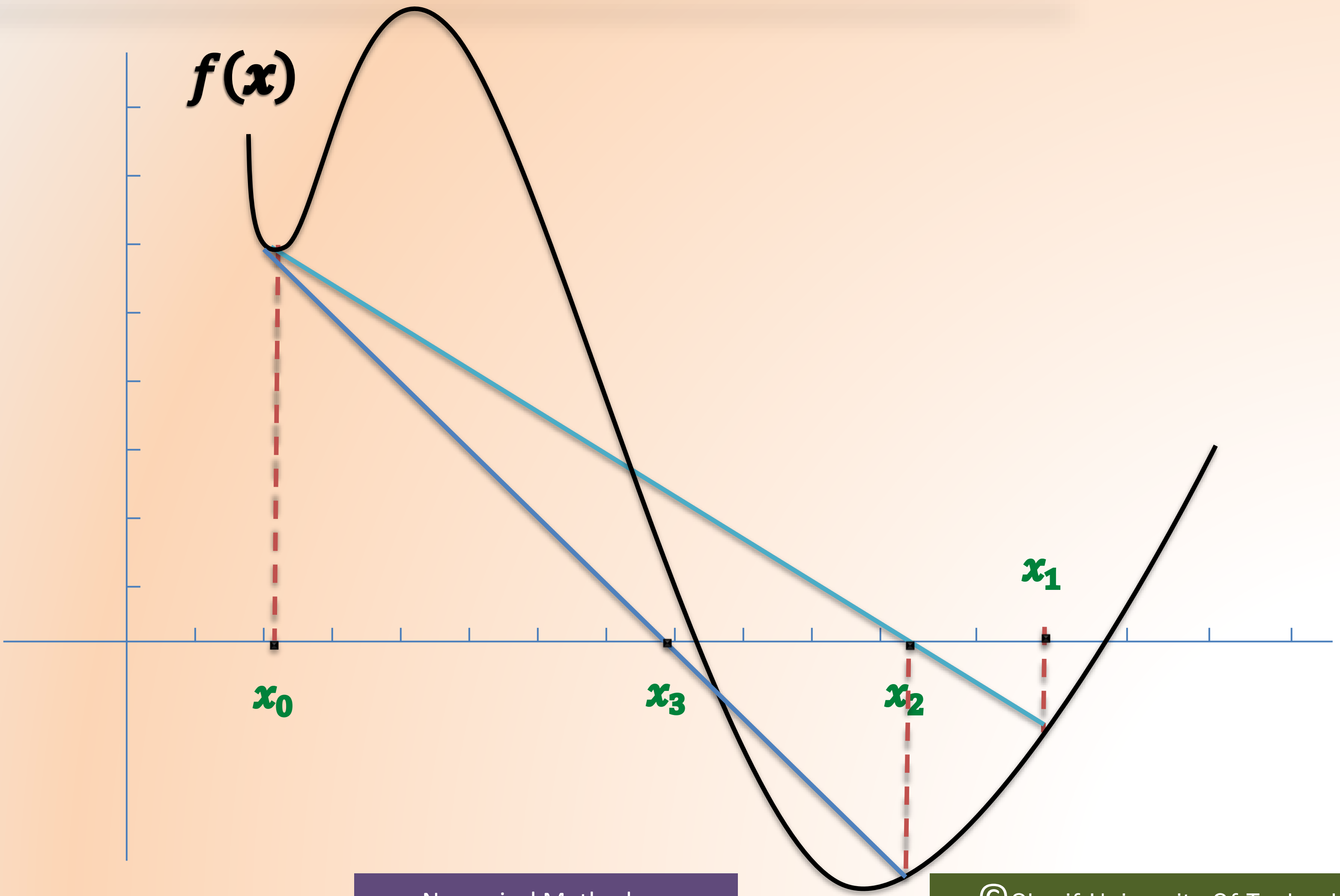

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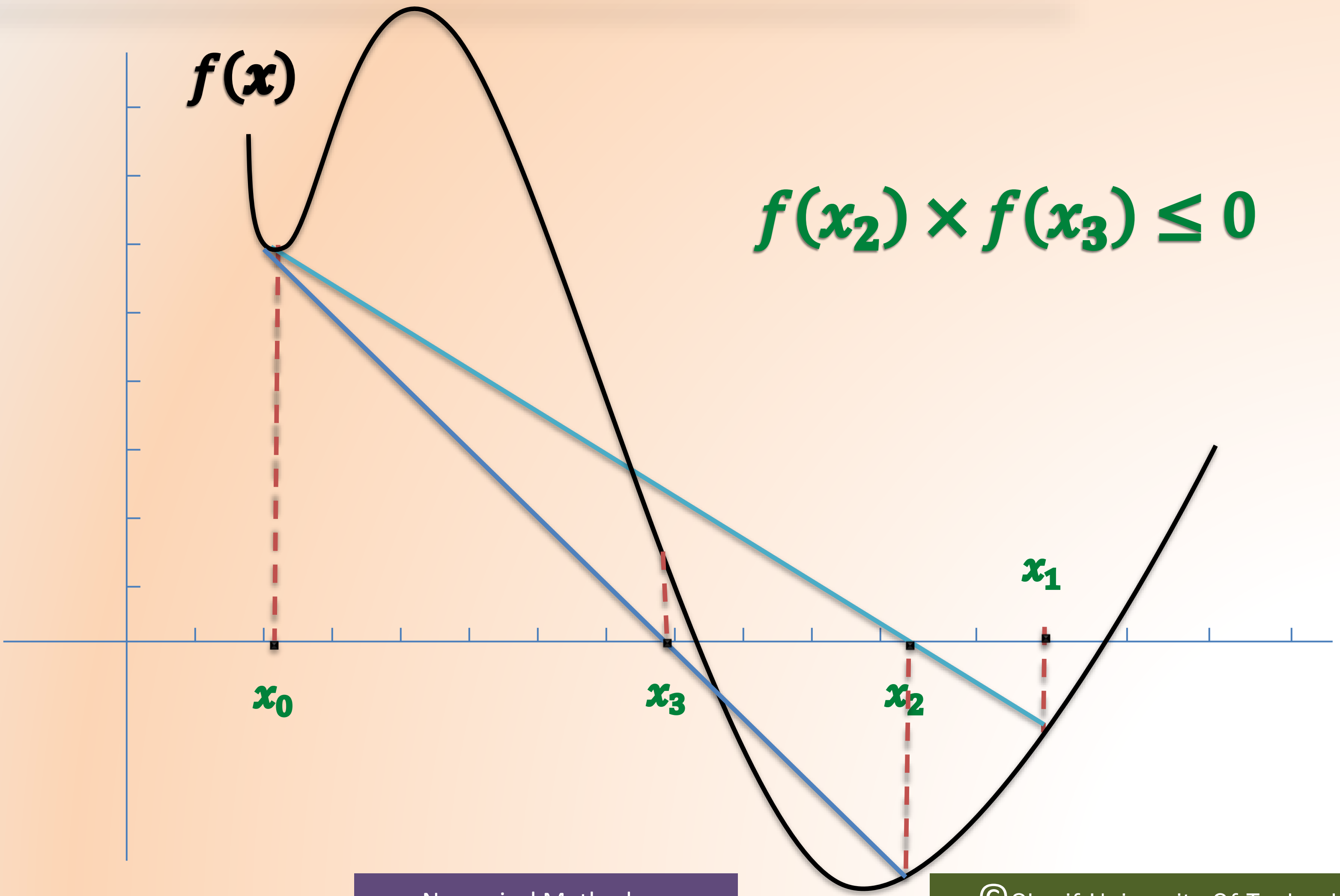

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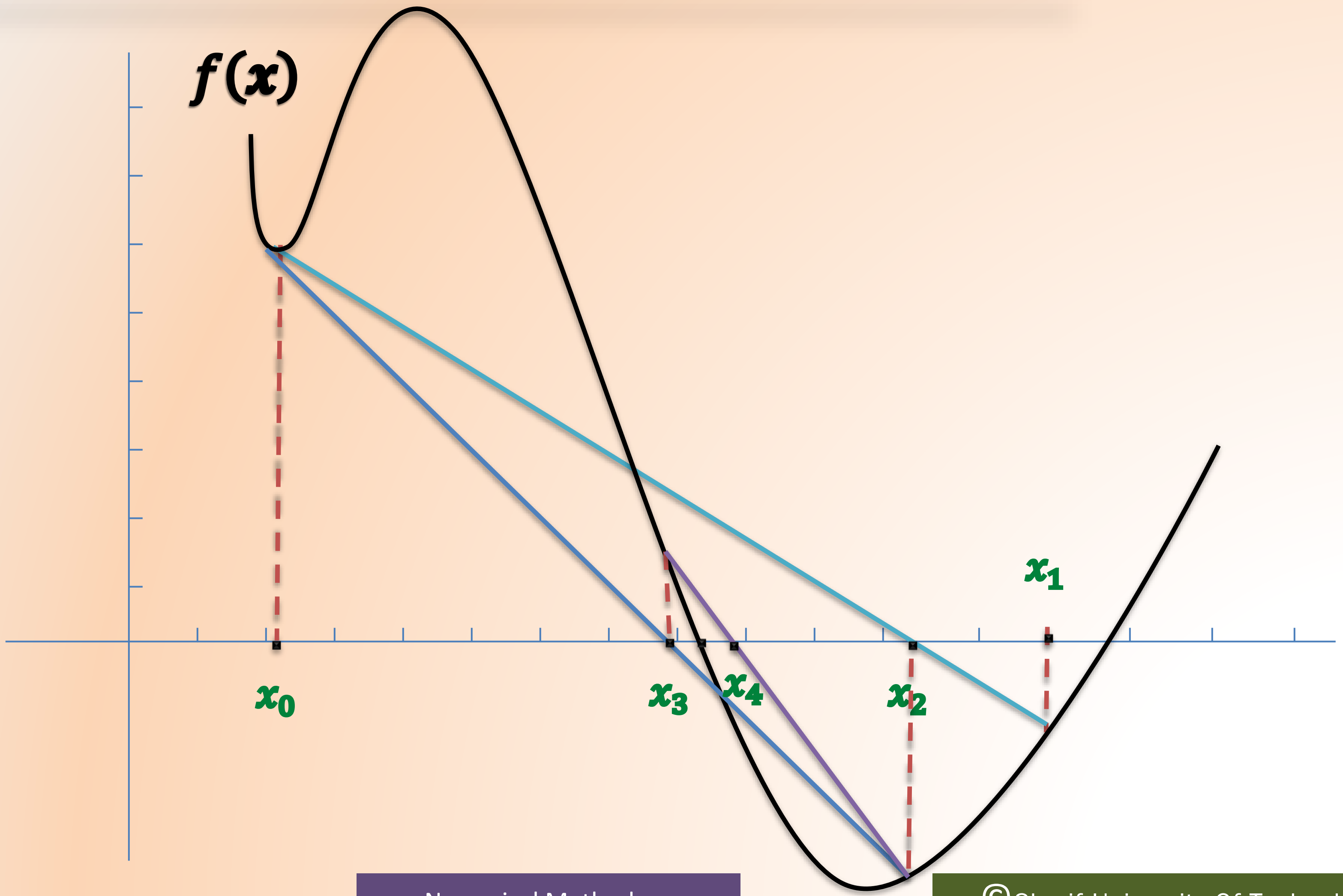

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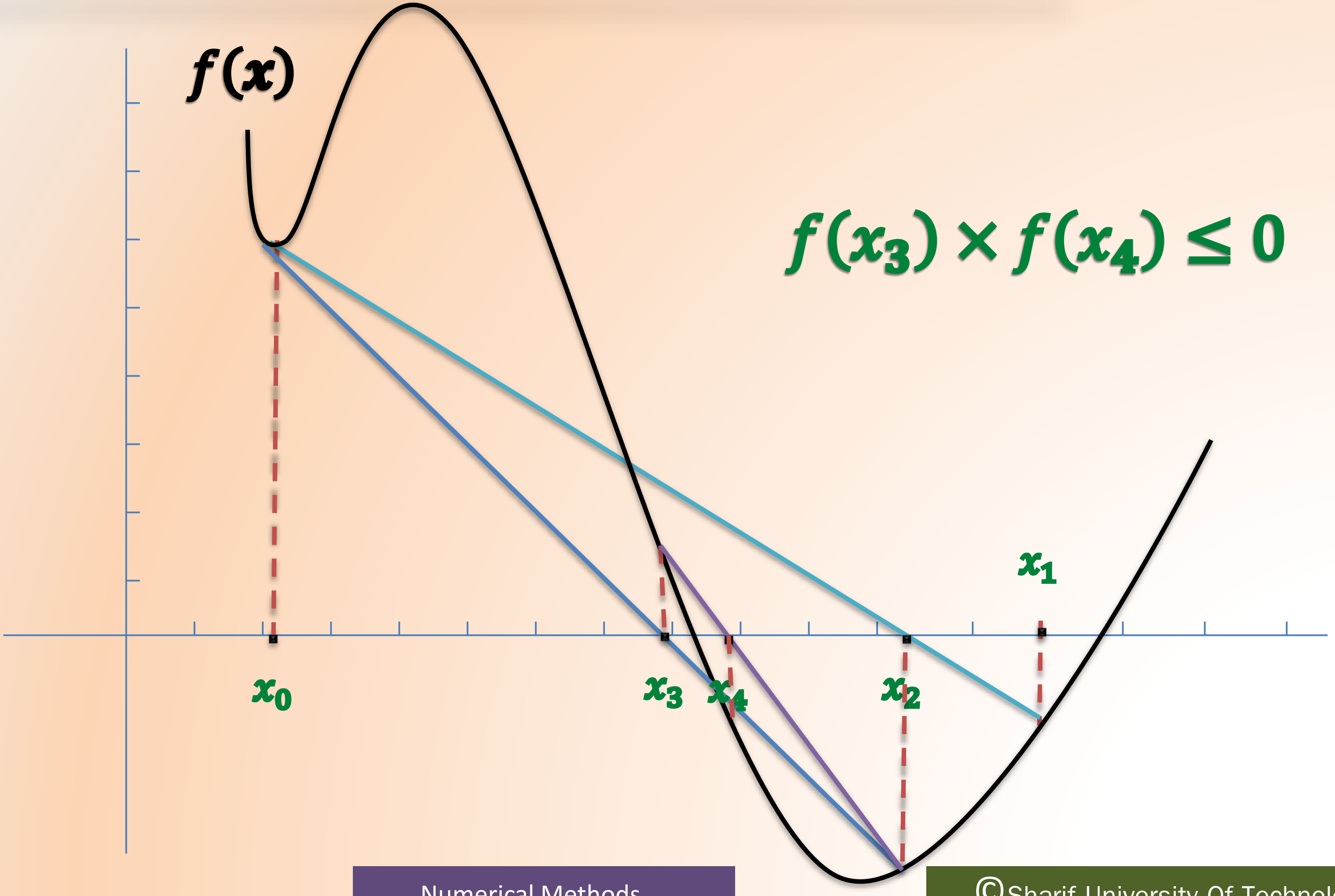
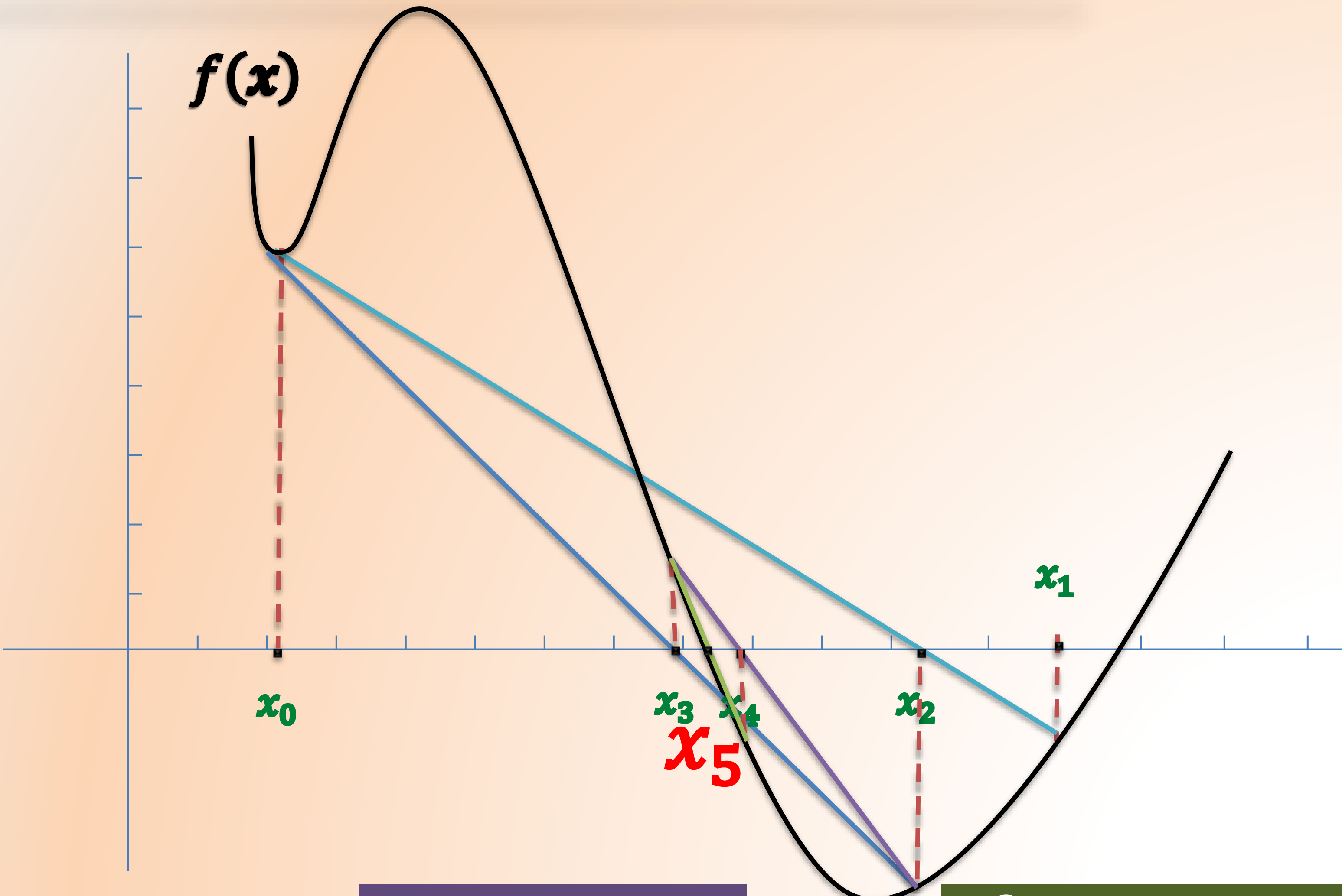


False Position Method



False Position Method

Termination condition:

$$|x_n - x_{n-1}| < \varepsilon$$

Degree of convergence:

$\frac{1+\sqrt{5}}{2}$ **=1.618** **Golden Ratio**

False Position Method

Assumptions **Example** Advantages & Disadvantages

$2x - \log x = 7 \; x \in (3.5, 4)$ (2 steps)

$$f(x) = 2x - \log x - 7 \rightarrow \begin{cases} f(4) = 0.39794 = f_1 \\ f(3.5) = -0.54407 = f_0 \end{cases}$$

Step 1:

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 3.78878$$

$$f_2 = f(x_2) = -0.00104$$

Step 2:

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = 3.78934$$

False Position Method

مثال ۵. تقریبی از ریشه معادله $f(x) = x^2 - 2^x = 0$ را که در فاصله $(-1, 0)$ قرار دارد به روش نابجایی با $4D$ به دست آورید به طوری که $|f(x_n)| < 10^{-2}$.

حل: هرگاه قرار دهیم $x_n = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$، جدول زیر را خواهیم داشت :

n	a	b	x_n	$f(a)$	$f(x_n)$	علامت $f(a)f(x_n)$
۱	$-$۱	۰	$-$۰٫۶۶۶۶۷	۰٫۵	$-$۰٫۱۸۵۵۲	$-$
۲	$-$۱	$-$۰٫۶۶۶۶۷	$-$۰٫۷۵۶۸۸	۰٫۵	$-$۰٫۰۱۸۹۲	$-$
۳	$-$۱	$-$۰٫۷۵۶۸۸	$-$۰٫۷۶۵۷۴	۰٫۵	$-$۰٫۰۰۱۷۹	$-$

چون $|f(x_3)| = 0.00179 < 10^{-2}$ پس x_3 تقریب مورد نظر ریشه است.لذا با $4D$ تقریب ریشه عبارت است از :

$$\alpha \simeq -0.7657$$

False Position Method

مثال ۱. تقریبی از ریشهٔ معادلهٔ $f(x) = ۳x - e^{-x} = ۰$ را که در فاصلهٔ $(۰/۲۵, ۰/۲۷)$ قرار دارد، با سه رقم اعشار به دست آورید، به طوری که داشته باشیم؛ $|f(x_n)| < ۰/۰۰۱$ که x_n تقریب ریشه در تکرارnام است.

Bisection method

n	a	b	$x_n = \frac{a+b}{۲}$	علامت $f(a)f(x_n)$	$f(x_n)$
۱	۰/۲۵	۰/۲۷	۰/۲۶	−	۰/۰۰۸۹
۲	۰/۲۵	۰/۲۶	۰/۲۵۵	+	−۰/۰۰۹۹
۳	۰/۲۵۵	۰/۲۶	۰/۲۵۷۵	+	−۰/۰۰۰۵

چون $|f(x_۳)| = ۰/۰۰۰۵ < ۰/۰۰۱$ بنابراین $x_۳$ را به عنوان تقریب ریشه معادله در نظر می‌گیریم.لذا هرگاه α ریشه مورد نظر باشد، با سه رقم اعشار قرار می‌دهیم :

$$\alpha \simeq ۰/۲۵۸$$

False Position Method

مثال ۱. تقریبی از ریشهٔ معادلهٔ $f(x) = ۳x - e^{-x} = ۰$ را که در فاصلهٔ $(۰{,}۲۵, ۰{,}۲۷)$ قرار دارد، با سه رقم اعشار به دست آورید، به طوری که داشته باشیم؛ $|f(x_n)| < ۰{,}۰۰۱$ که x_n تقریب ریشه در تکرار nام است.

False position method

$$x_۱ = \frac{۰{,}۲۵ \times ۰{,}۰۴۶۶ - ۰{,}۲۷ \times (-۰{,}۰۲۸۸)}{۰{,}۰۴۶۶ - (-۰{,}۰۲۸۸)} = ۰{,}۲۵۷۶$$

$$f(x_۱) = -۰{,}۰۰۰۱$$

لذا $|f(x_۱)| = ۰{,}۰۰۰۱ < ۲ \times ۱۰^{-۴}$.بنابراین $x_۱$ تقریب ریشه بوده و این تقریب با سه رقم اعشار عبارت است از :

$$\alpha \simeq ۰{,}۲۵۸$$

False Position Method

Here bisection method works better.

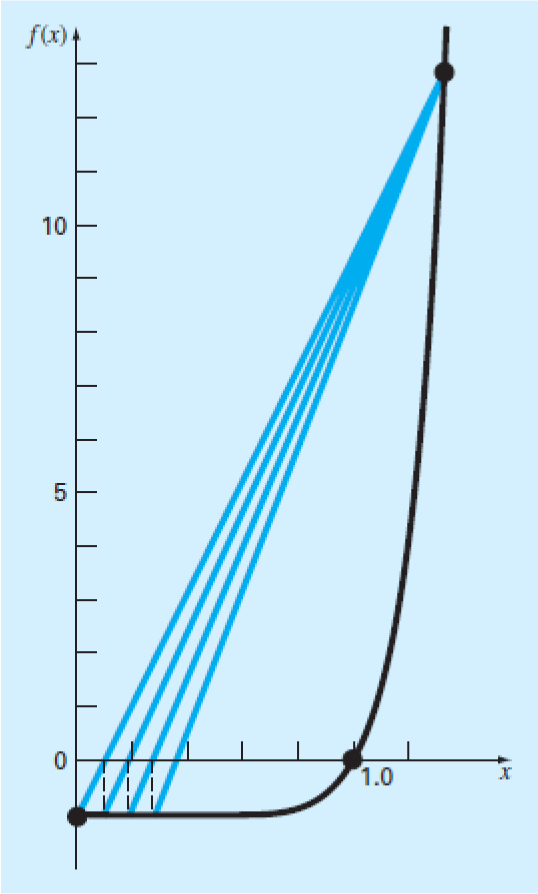


FIGURE 5.9
Plot of $f(x) = x^{10} - 1$, illustrating slow convergence of the false-position method.

False Position Method

Assumptions Example Advantages & Disadvantages

Advantages :

- Always converges to the answer.
- It converges faster than Bisection method.

Disadvantages :

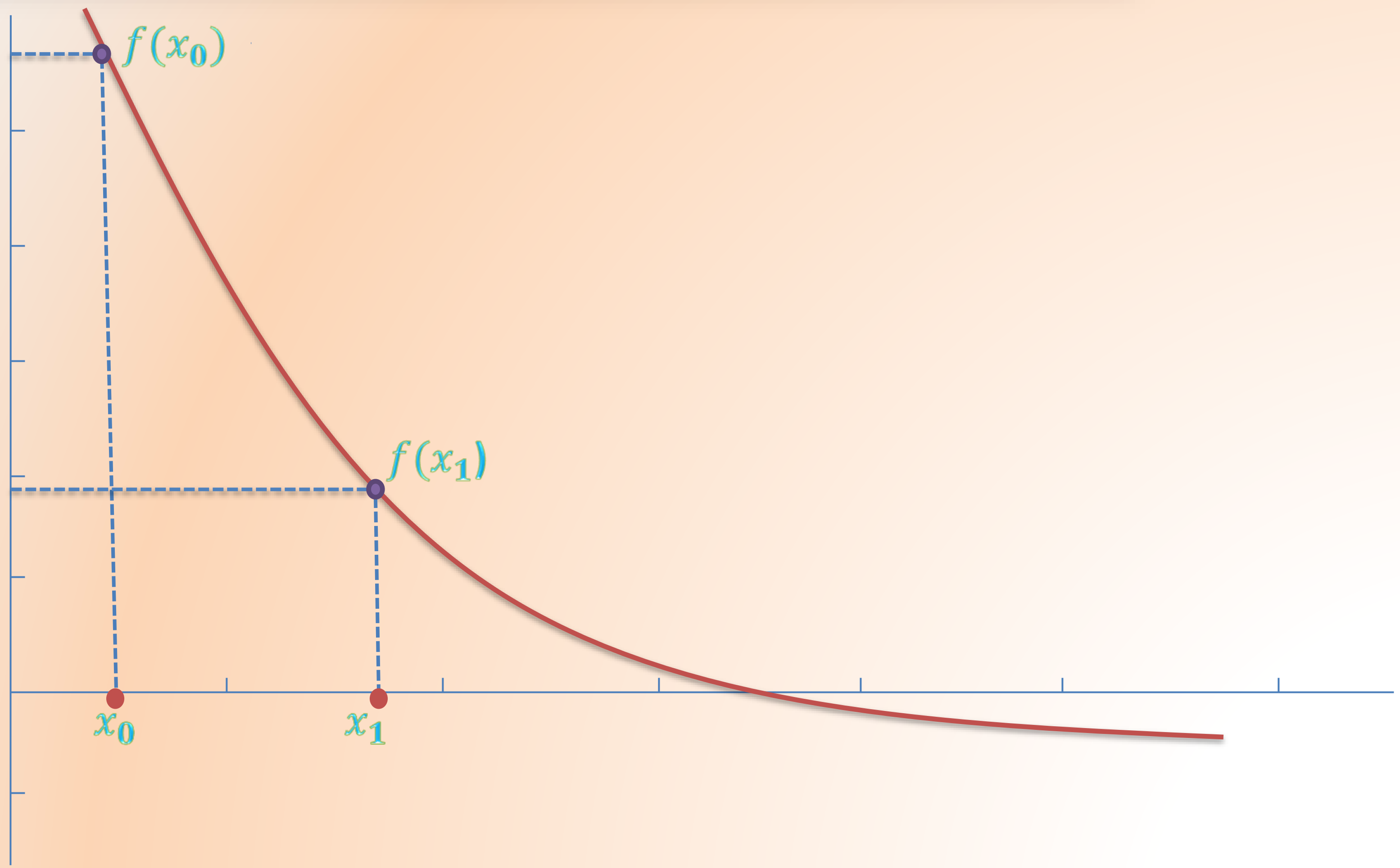
- It is more difficult than bisection method, because the number of operations are more.
- If all of $x_i s$ (or most of them) are in one side of root, this method converges slower.
- All calculations strictly depend on the function $f(x)$.

Methods

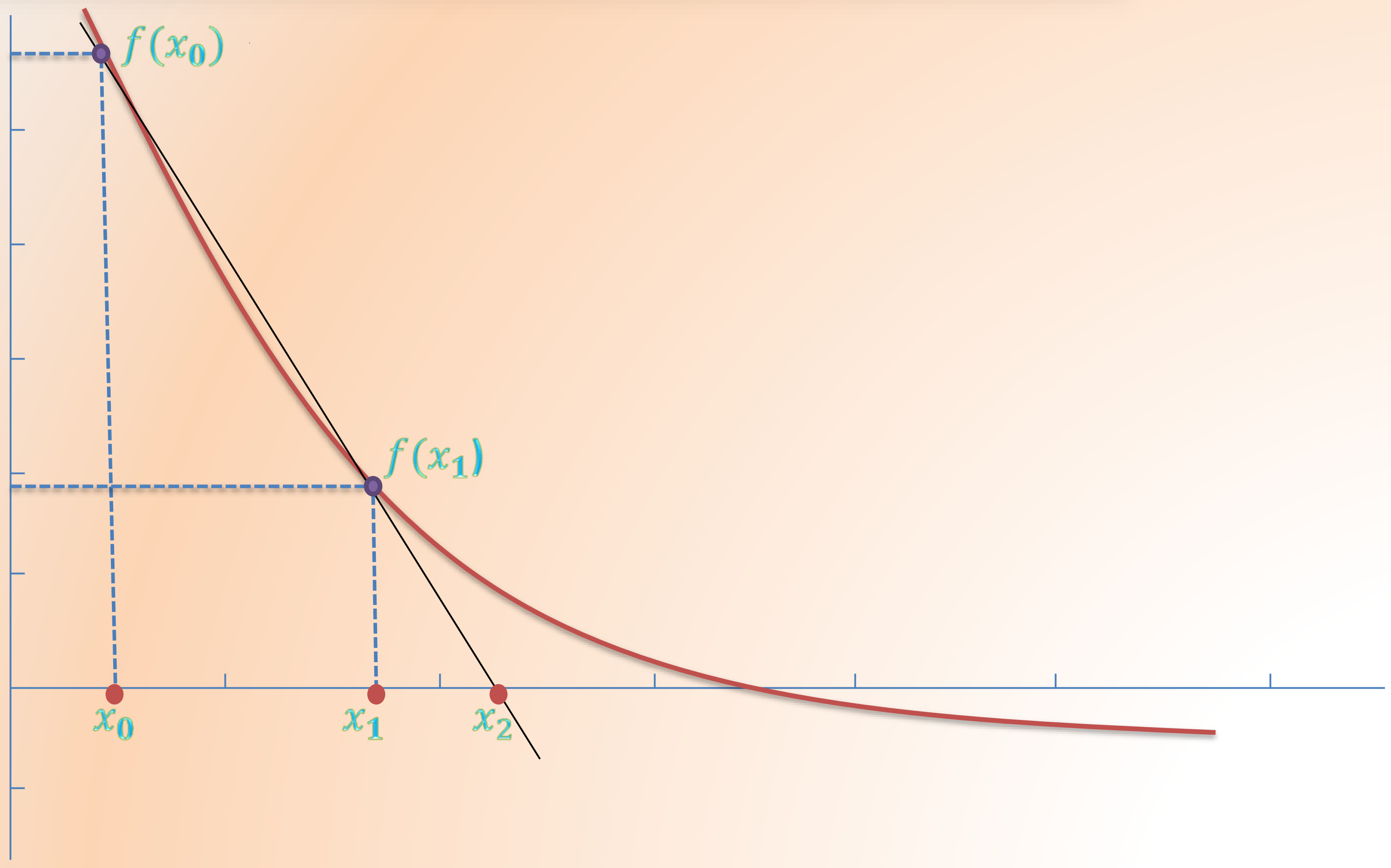
Extra Topics

- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

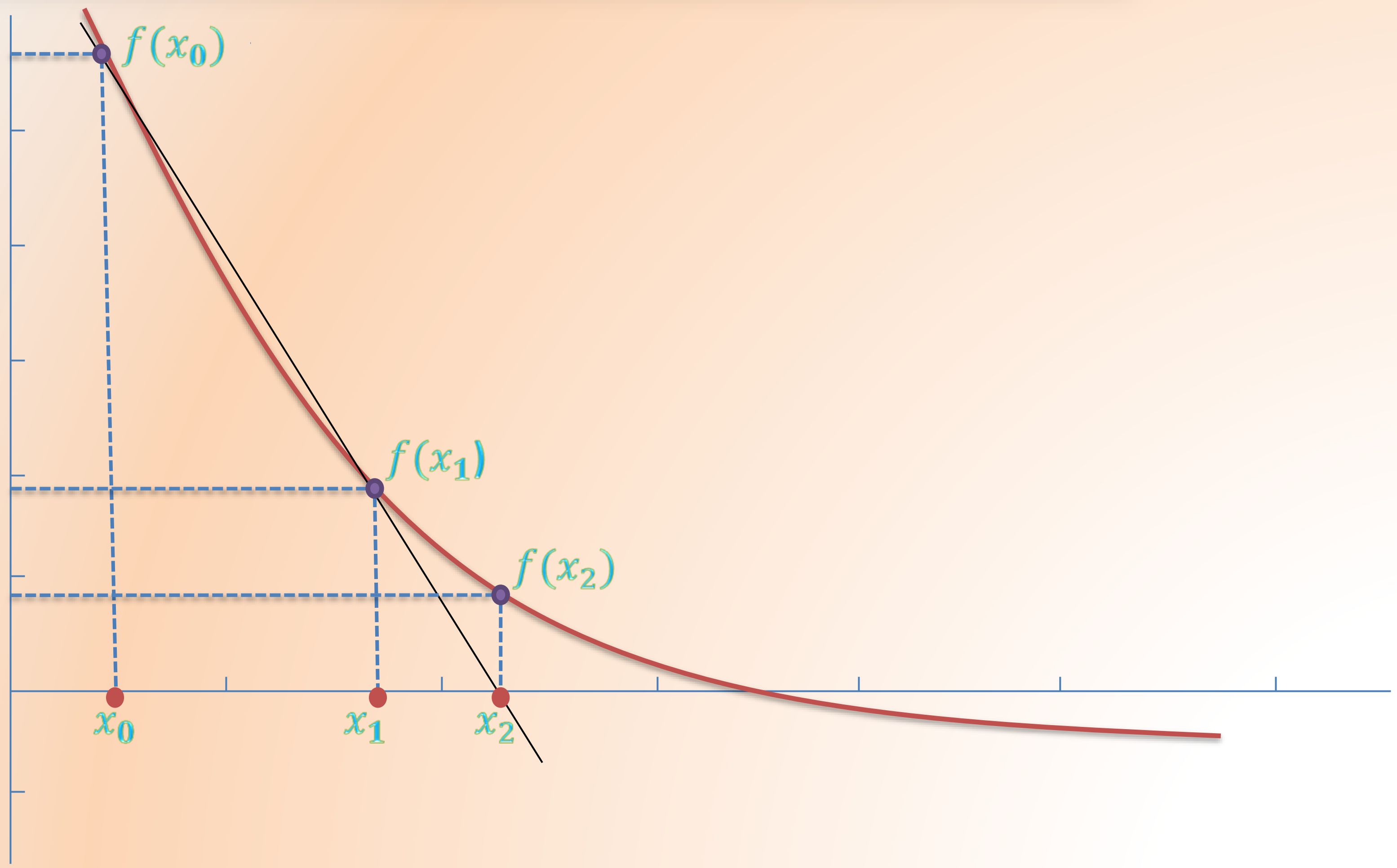
Secant Method



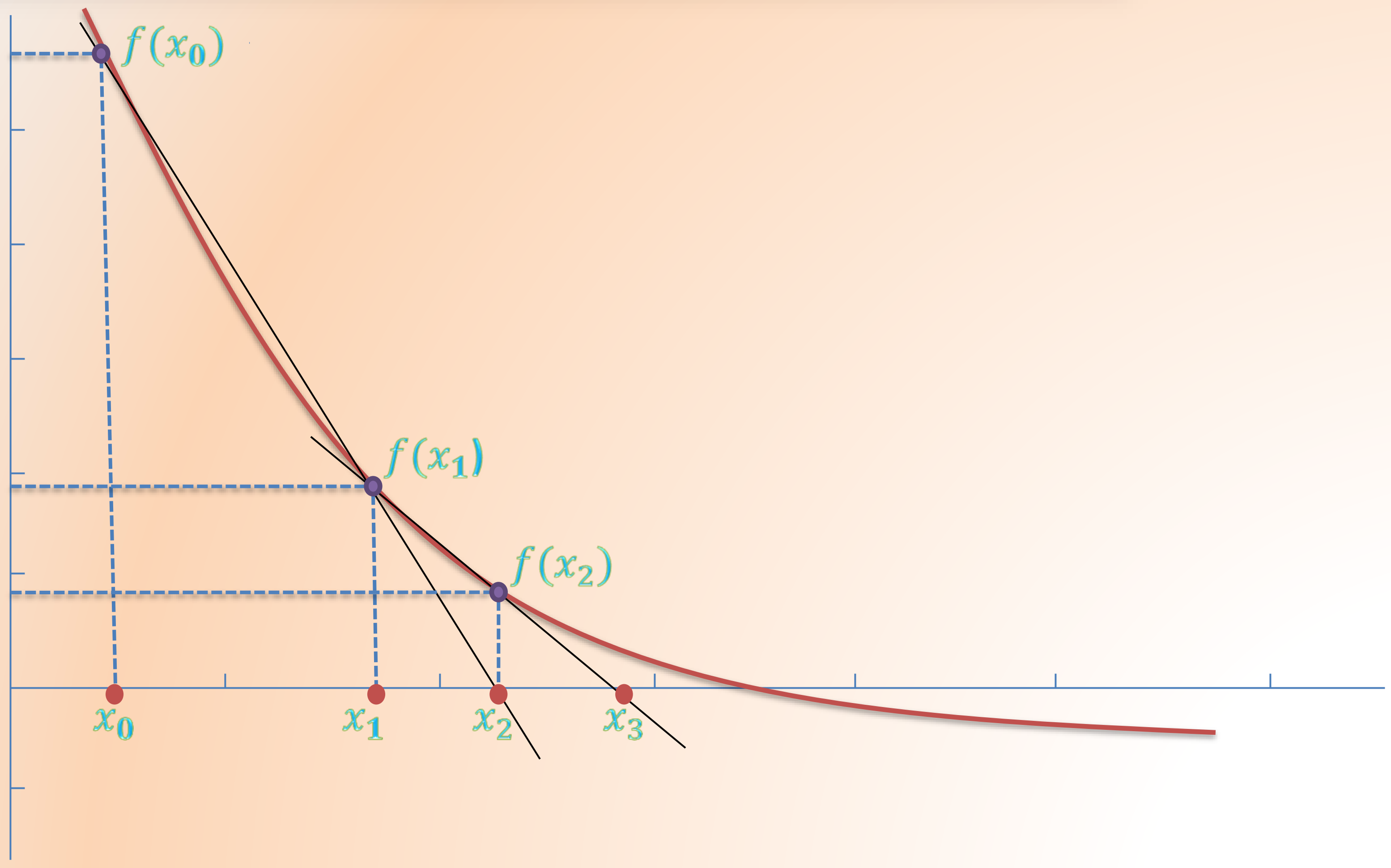
Secant Method



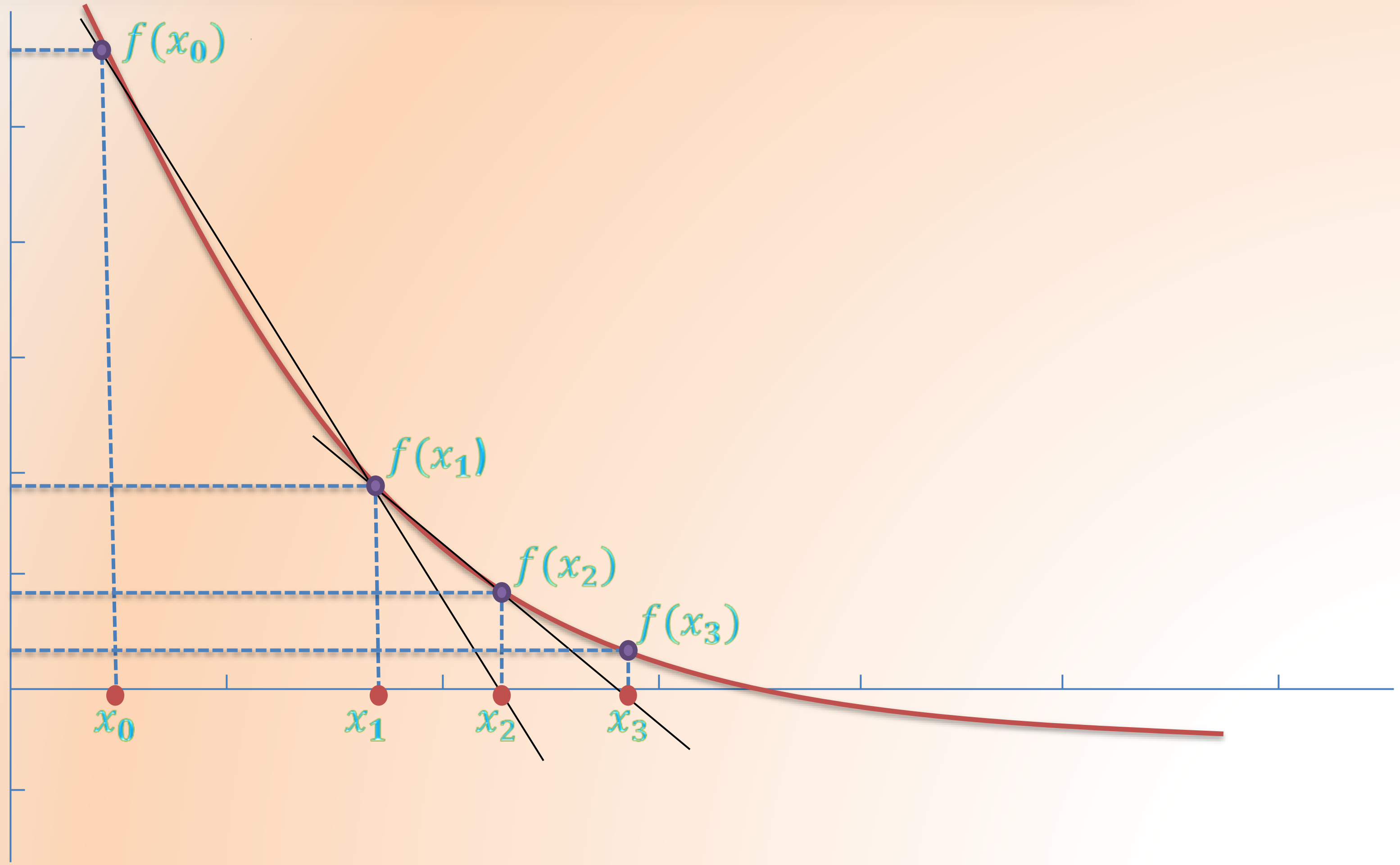
Secant Method



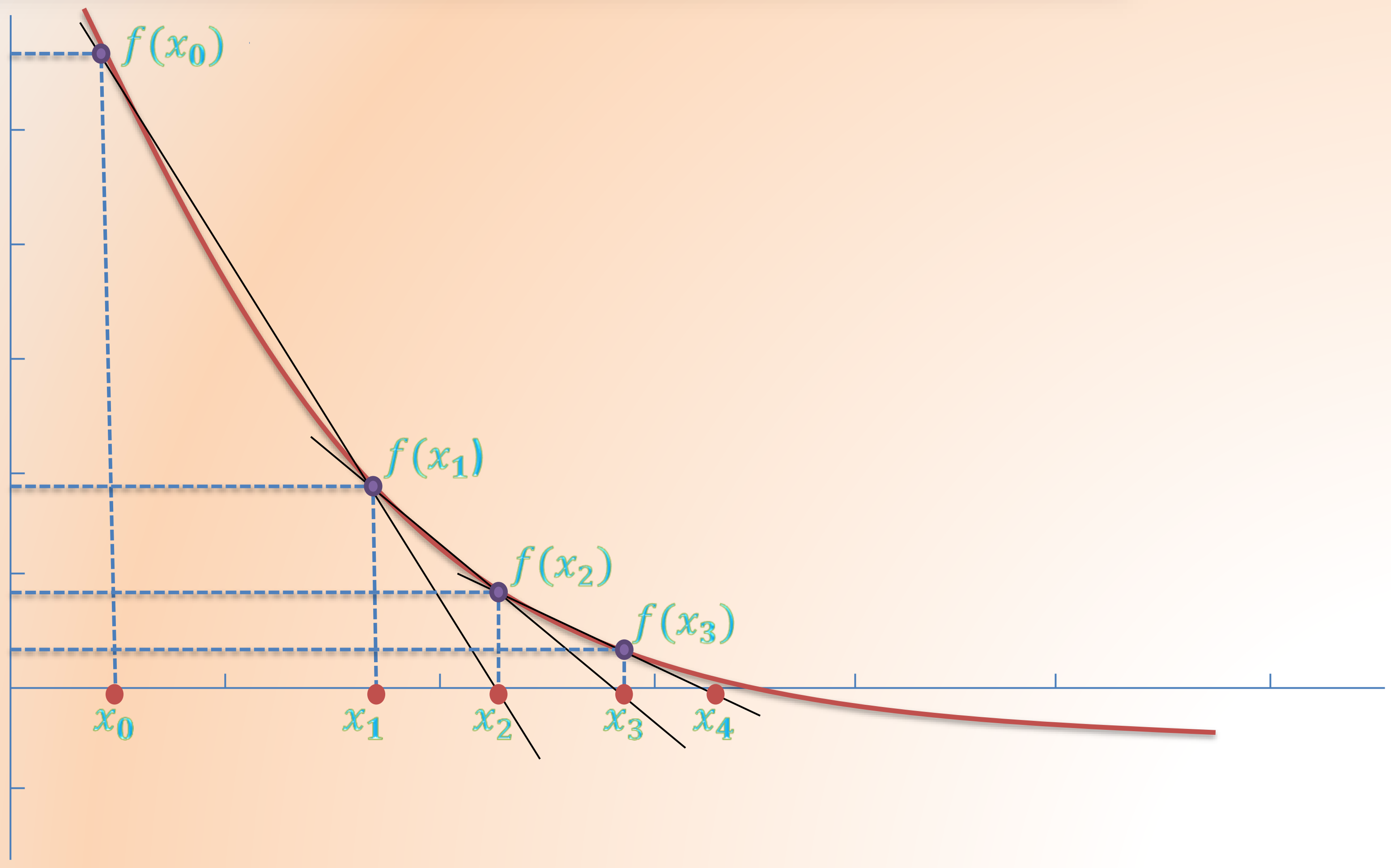
Secant Method



Secant Method



Secant Method



Secant Method

Assumptions Example Advantages & Disadvantages

Secant method is very similar to False-Position method except that Bolzano's theorem is not needed to be checked.

Initial approximations: an interval (x_0, x_1)

$$x_{n+1} = \frac{x_{n-1} \times f(x_n) - x_n \times f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Or

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Secant Method

Assumptions Example Advantages & Disadvantages

- Termination condition: $|x_n - x_{n-1}| < \varepsilon$
- Degree of convergence: $\frac{1+\sqrt{5}}{2}$
- Convergence condition:

The function $f(x)$ should be twice continuously differentiable and the root should be simple.

Secant Method

Assumptions **Example** Advantages & Disadvantages

$f(x) = \cos x - x^2 = 0 \quad (5\ steps) \qquad (x_1 = 1, x_0 = 0)$

$$f(x_0) = 1 - 0 = 1$$

$$f(x_1) = 0.9998 - 1 = -0.0002$$

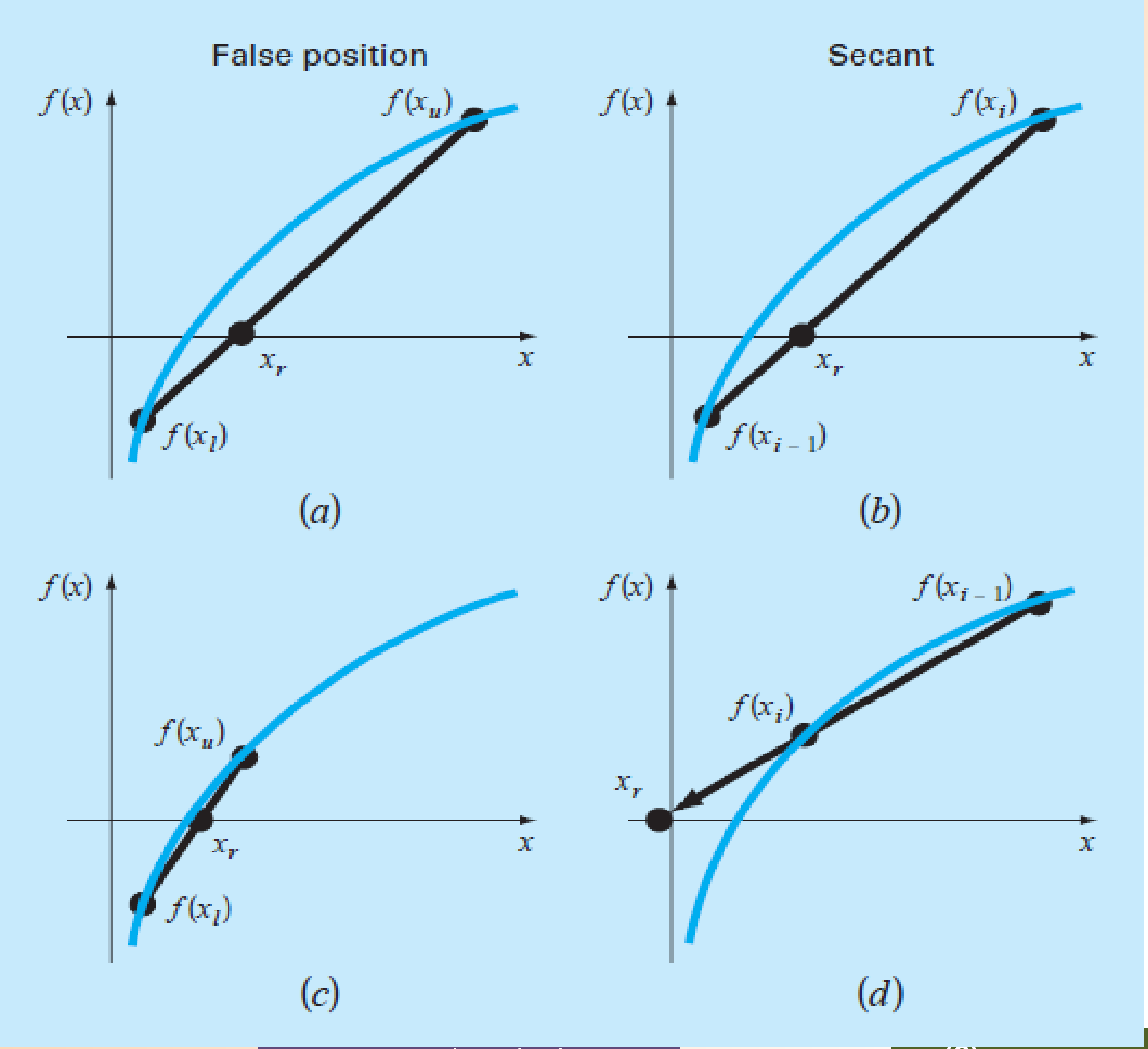
$$x_2 = 1 - \frac{(1-0)(-0.0002)}{0.0006-(-0.0002)} = 0.99995$$

$$f_3 = f(0.99995) = -0.0000023$$

$$x_4 = 0.9995 \qquad f_4 = 0.001347$$

$$x_5 = 0.99995$$

Secant Method



Secant Method

Assumptions Example **Advantages & Disadvantages**

Advantages :

- It converges faster than bisection method.
- Bolzano's theorem is not needed.

Disadvantages :

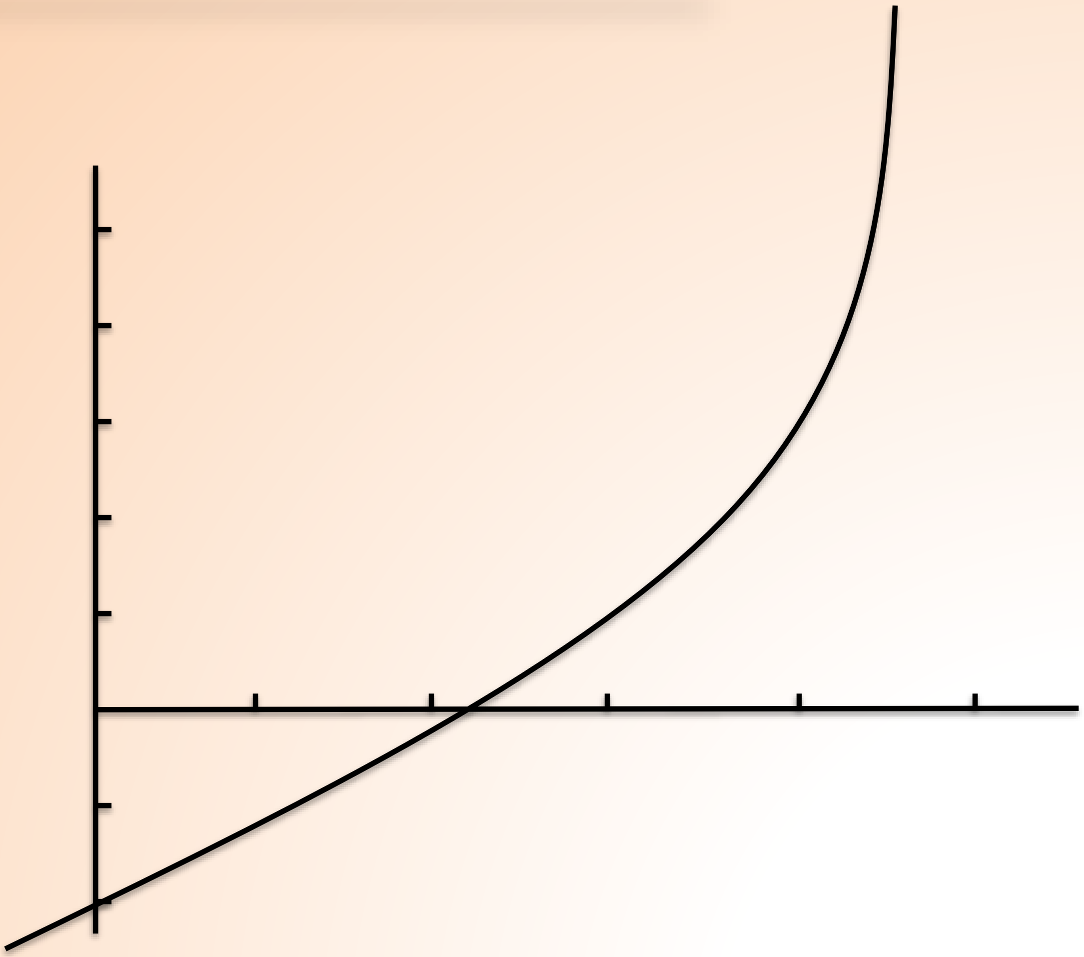
- It is harder than bisection method.
- Sometimes it does not converge.
- If the initial guess is not close to the answer, then there is no guarantee it converges to the answer.

Methods

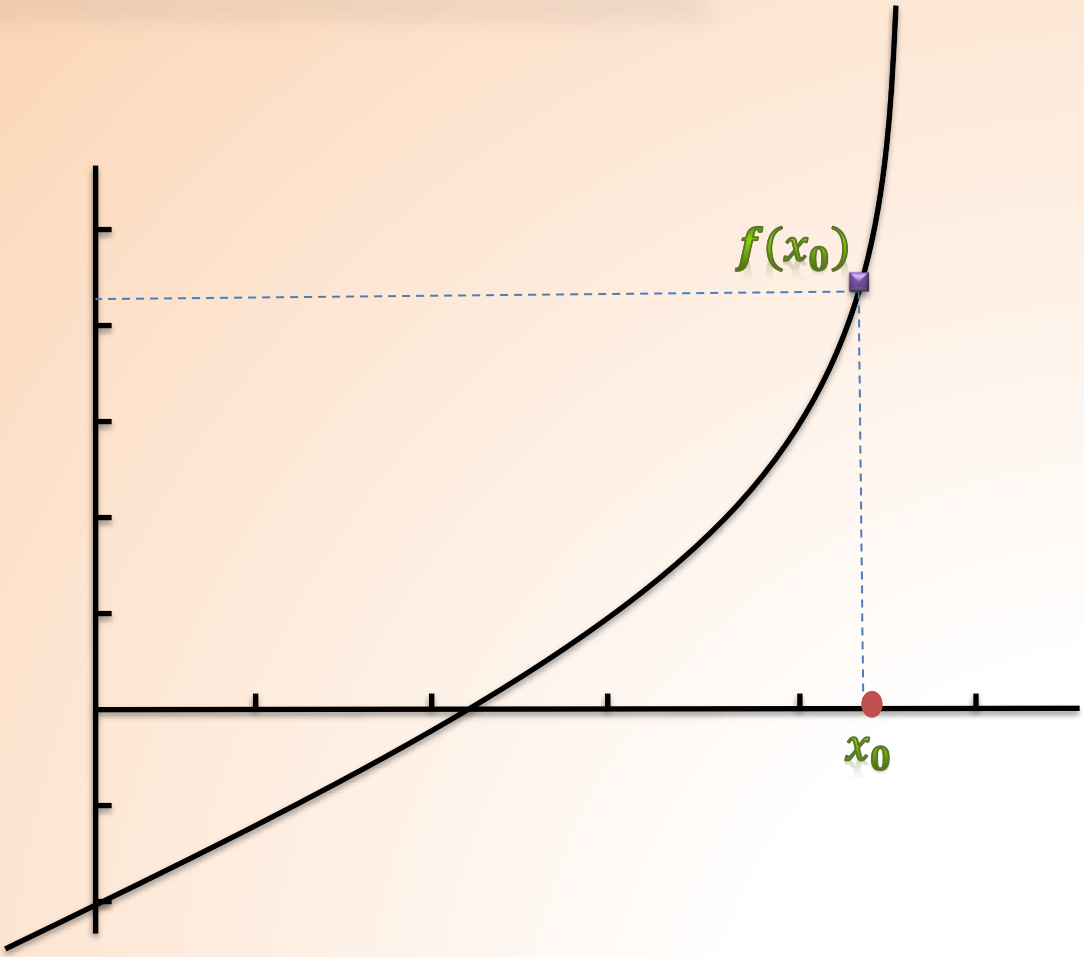
Extra Topics

- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

Newton-Raphson Method



Newton-Raphson Method

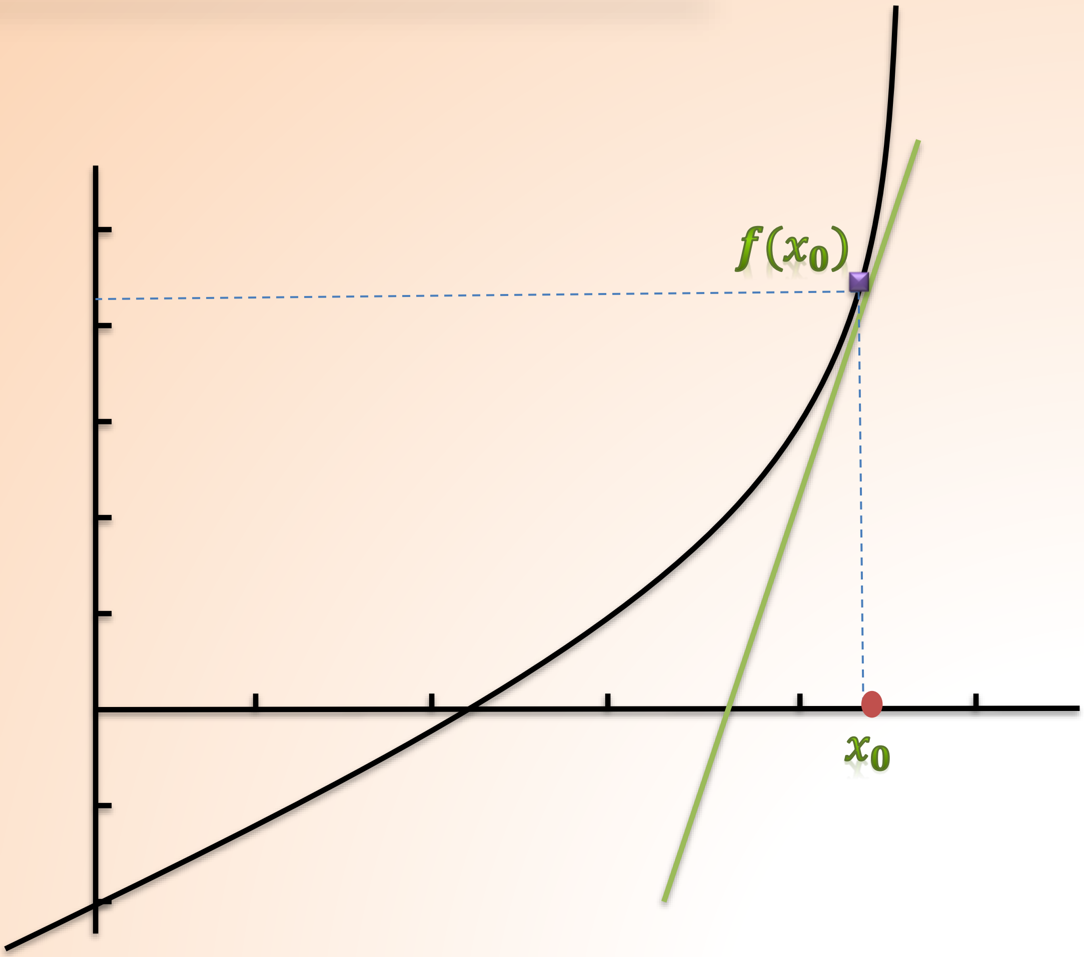


Newton-Raphson Method

Slope of tangent line:

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Newton-Raphson Method

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$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$

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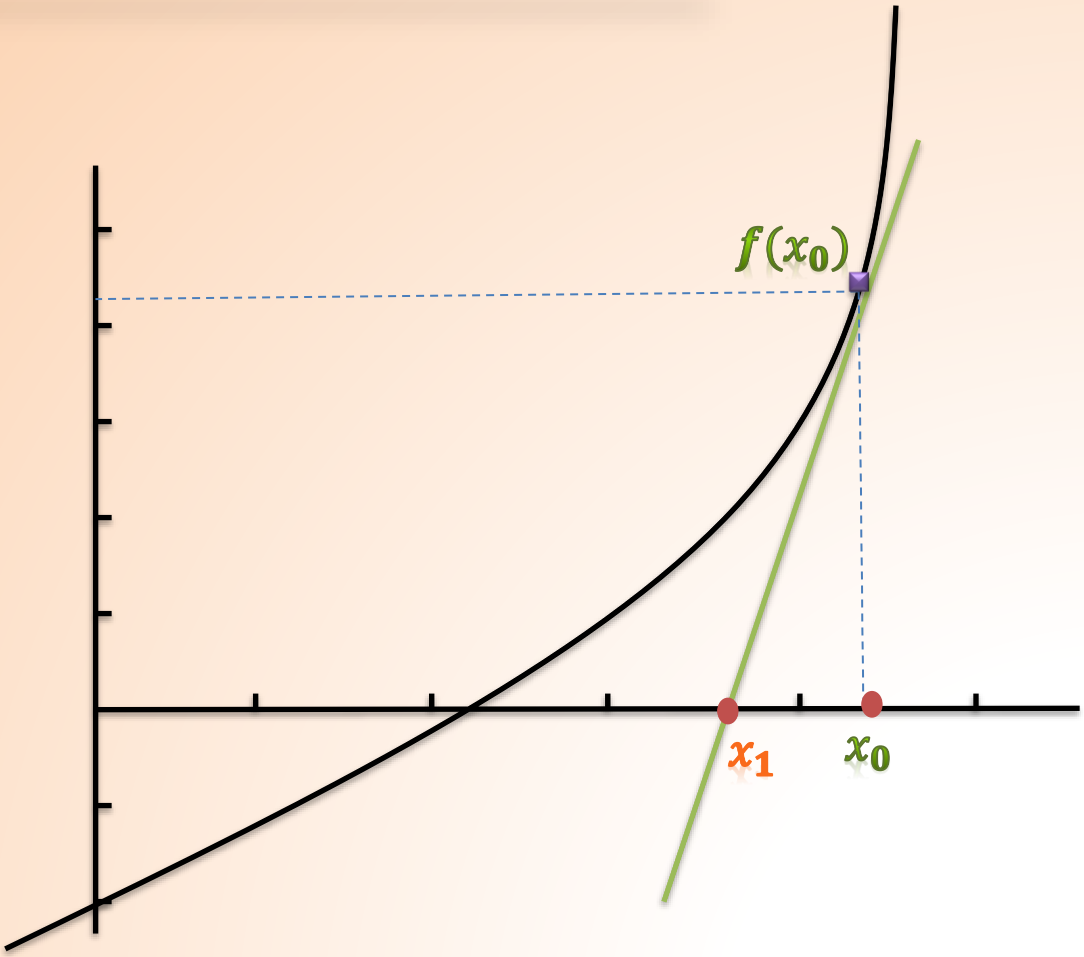

Newton-Raphson Method

Slope of tangent line:

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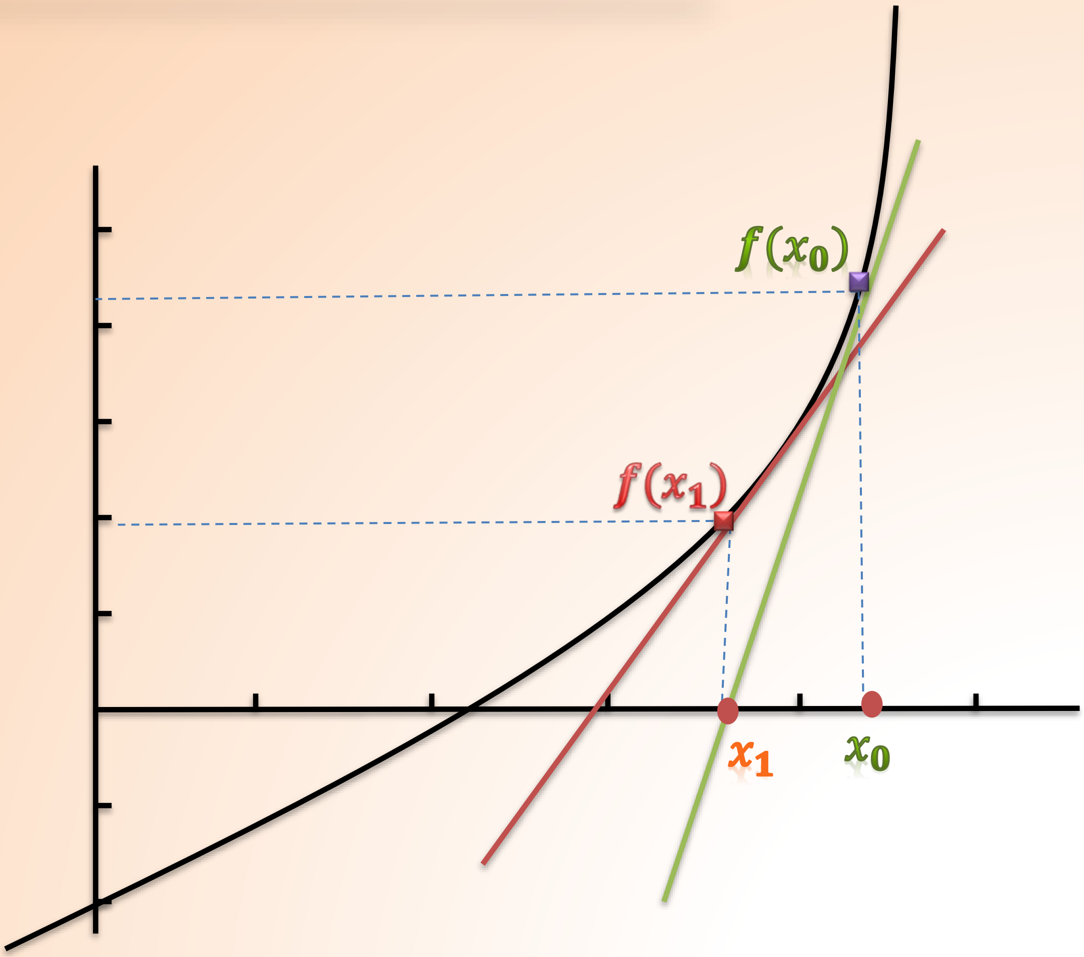
$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Newton-Raphson Method

Slope of tangent line:

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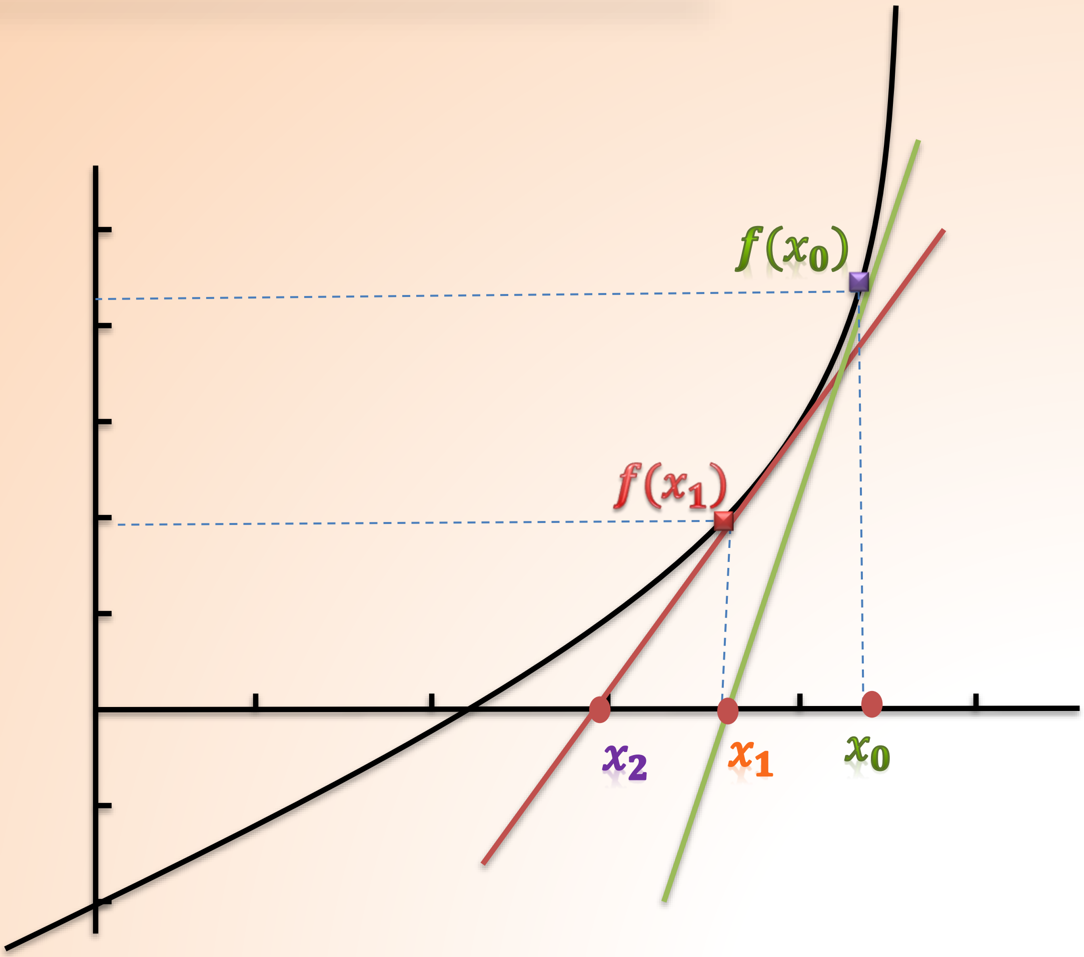

Newton-Raphson Method

Slope of tangent line:

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$

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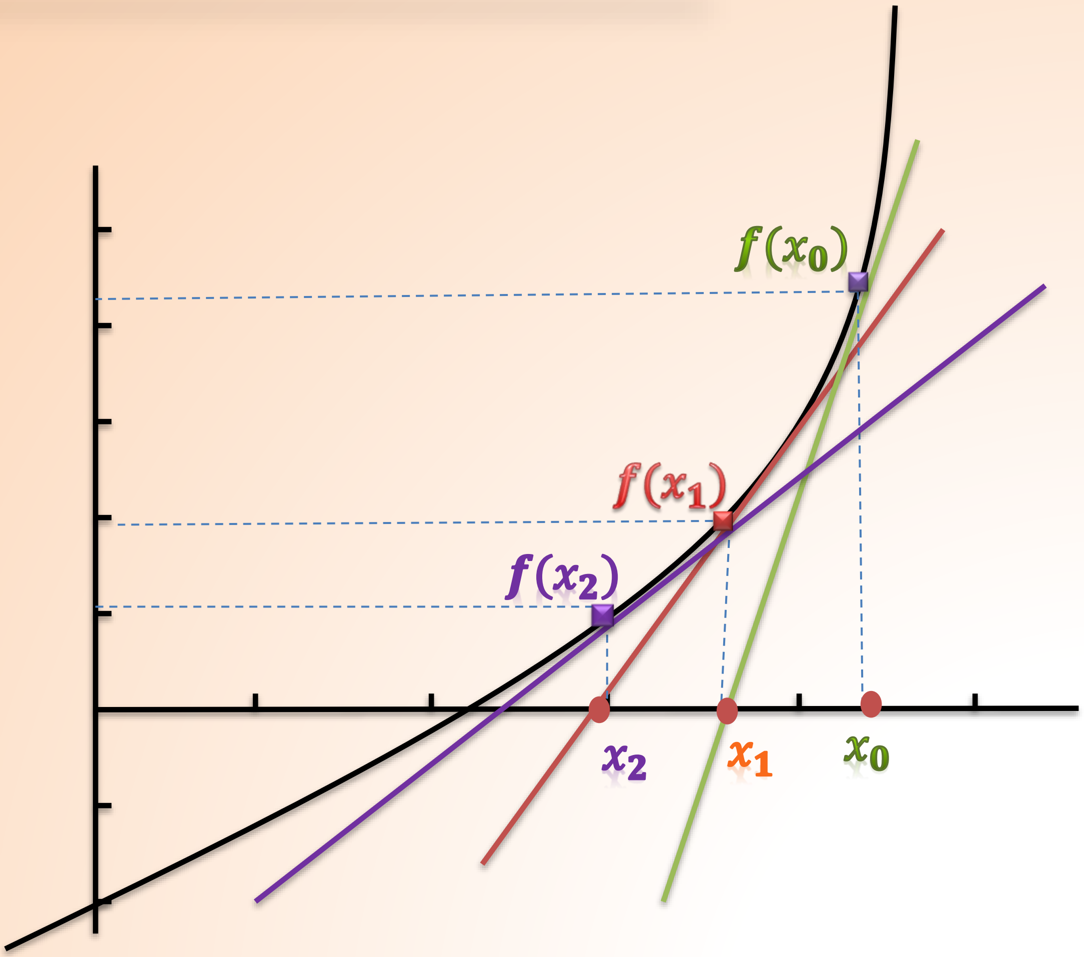
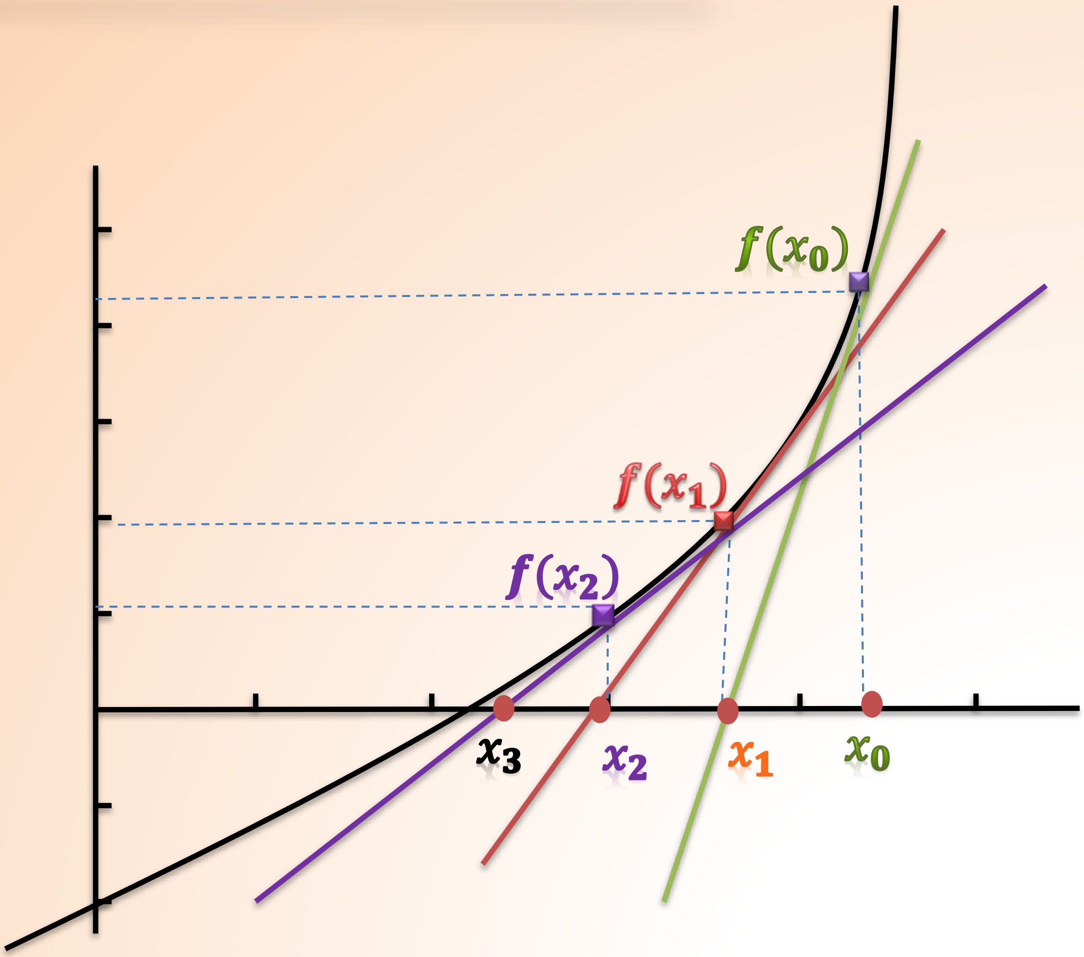


Newton-Raphson Method

Slope of tangent line:

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Newton-Raphson Method

Assumptions Example Advantages & Disadvantages

Initial assumption: x_0

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton-Raphson Method

همگرایی روش نیوتن رافسون :

از بسط تیلور حول نقطه x_n استفاده می‌کنیم:

$$f(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{(x - x_n)^2}{2!} f''(\varepsilon_n) \qquad x_n < \varepsilon_n < x$$

$f(x)$ ساده

$x = \alpha$ که α ریشه است، قرار می‌دهیم:

$$0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2!} f''(\varepsilon_n) \qquad x_n < \varepsilon_n < \alpha$$

عبارت را بر $f'(x_n)$ تقسیم می‌کنیم:

$$\underbrace{\alpha - x_n + \frac{f(x_n)}{f'(x_n)}}_{-x_{n+1}} = -\frac{1}{2} (\alpha - x_n)^2 \frac{f''(\varepsilon_n)}{f'(x_n)}$$

$$\underbrace{\alpha - x_{n+1}}_{e_{n+1}} = -\frac{1}{2} (\underbrace{\alpha - x_n}_{e_n})^2 \frac{f''(\varepsilon_n)}{f'(x_n)} \Longrightarrow e_{n+1} = \left| \frac{f''(\varepsilon_n)}{2 f'(x_n)} \right| e_n^{2}$$

Newton-Raphson Method

Assumptions Example Advantages & Disadvantages

Termination condition: $|x_n - x_{n-1}| < \varepsilon$

Convergence condition: $\left|\frac{f(x)\times f''(x)}{f'^2(x)}\right| < 1$

Degree of convergence: 2

Note that the convergence degree is 2 , only in case of simple roots

Newton-Raphson Method

Assumptions **Example** Advantages & Disadvantages

مثال ۷. ریشه معادلهٔ $f(x) = x - \cos x = ۰$ را که در فاصلهٔ $[۰, ۱]$ قرار دارد به روش نیوتن با چهار رقم اعشار به دست آورید به طوری که $|x_n - x_{n-۱}| < ۱۰^{-۴}$ که x_n تقریب ریشه مورد نظر در تکرار nام است.قرار دهید $x_۰ = ۰/۵$.

Newton-Raphson Method

Assumptions **Example** Advantages & Disadvantages

حل: داریم $f(x) = x - \cos x$ و $f'(x) = ۱ + \sin x$

$$x_{n+۱} = x_n - \frac{x_n - \cos x_n}{۱ + \sin x_n}$$

با قرار دادن $x_۰ = ۰/۵$ داریم:

$$x_۱ = ۰/۷۵۵۲۲$$

$$x_۲ = ۰/۷۳۹۱۴$$

$$x_۳ = ۰/۷۳۹۰۹$$

چون $|x_۳ - x_۲| = ۵ \times ۱۰^{-۵} < ۱۰^{-۴}$ لذا $x_۳$ تقریب ریشه مورد نظر است و با $۴D$ این تقریب عبارتست از:

$$\alpha \simeq ۰/۷۳۹۱$$

Newton-Raphson Method

Assumptions **Example** Advantages & Disadvantages

Calculating $\sqrt{5}$ using Newton-Raphson method.
(Newton-Raphson is a good method for calculating $\sqrt[k]{a}$.)

$$x = \sqrt{5} \rightarrow x^2 = 5$$

$$f(x) = x^2 - 5 = 0 \rightarrow f'(x) = 2x$$

$$x_{i+1} = x_i - \frac{{x_i}^2 - 5}{2x_i}$$

$$\rightarrow x_{i+1} = \frac{{x_i}^2 + 5}{2x_i}$$

$$\rightarrow x_{i+1} = \frac{1}{2}(x_i + \frac{5}{x_i})$$

i	x_i
0	2
1	2.25
2	2.236111111
3	2.236067978
4	2.236067978

Newton-Raphson Method

Assumptions **Example** Advantages & Disadvantages

Calculating $\sqrt{5}$ using Newton-Raphson method.

(Newton-Raphson is a good method for calculating $\sqrt[k]{a}$.)

$$x = \sqrt{5} \rightarrow x^2 = 5$$

$$f(x)$$

$$x_{i+1} = \frac{1}{k}\left[(k-1)x_i + a x_i^{1-k}\right]$$

$$x_{i+1}$$

$$\rightarrow x_{i+1} = \frac{{x_i}^2 + 5}{2x_i}$$

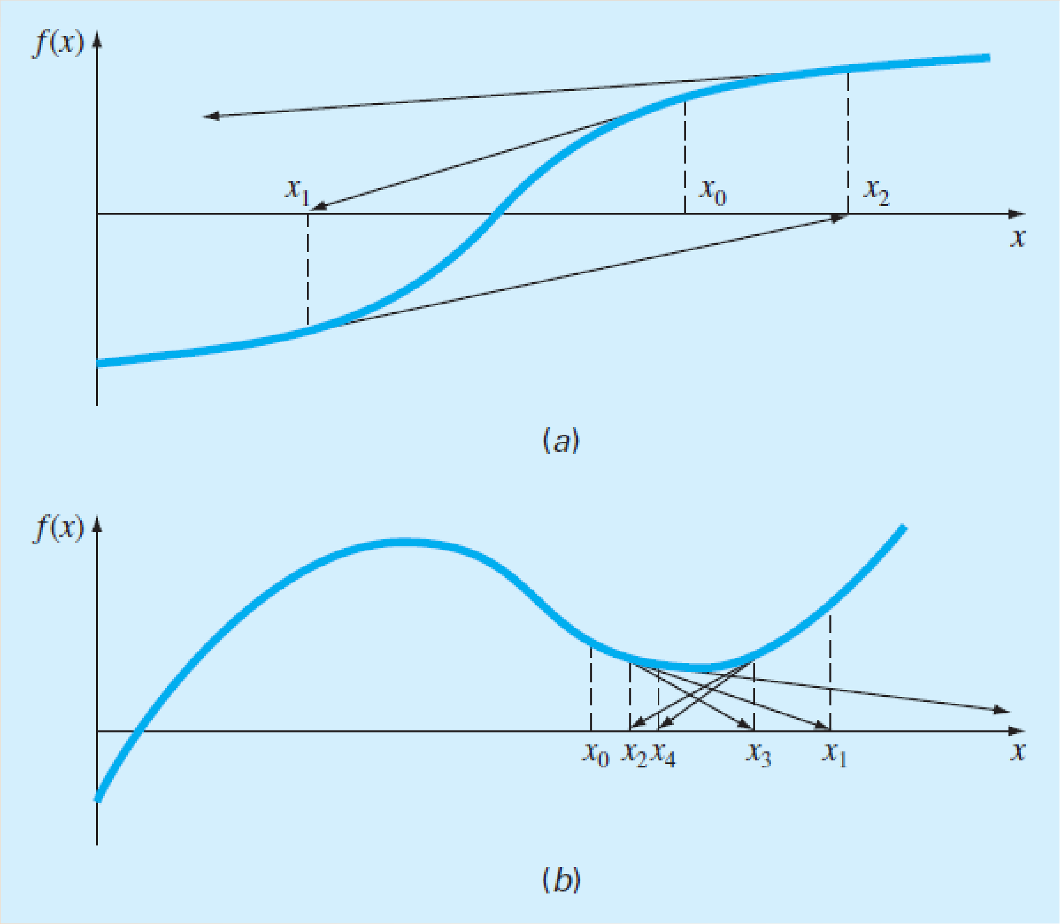
$$\rightarrow x_{i+1} = \frac{1}{2}(x_i + \frac{5}{x_i})$$

2	2.236111111
3	2.236067978
4	2.236067978

Newton-Raphson Method

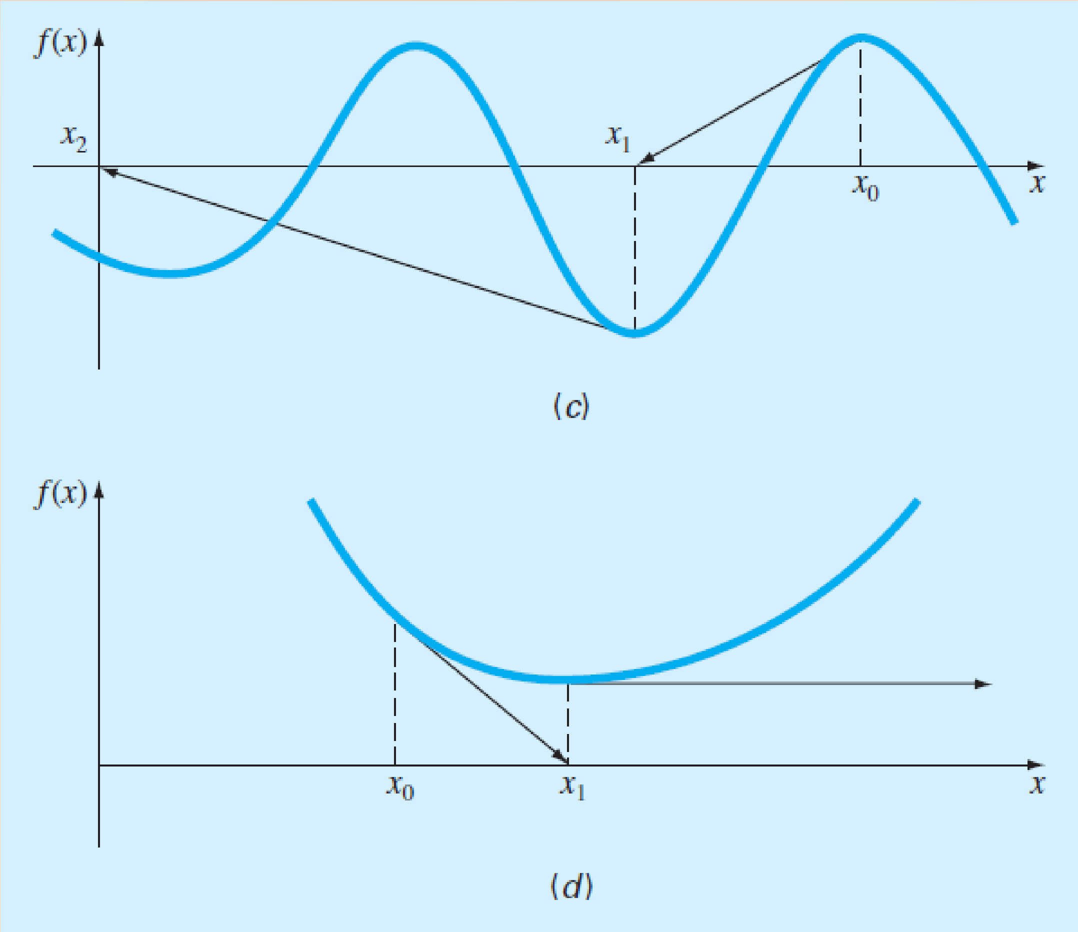
Selecting initial approximation (x_0) is very important so it is better to approximate x_0 by plotting $f(x)$.

Newton-Raphson Method



The cases where the Newton-Raphson method exhibits poor convergence

Newton-Raphson Method



The cases where the Newton-Raphson method exhibits poor convergence

Newton-Raphson Method

Assumptions Example **Advantages & Disadvantages**

Advantages :

- It converges very fast.

Disadvantages :

- No guarantee to converge to answer.
- Calculating derivation may be hard.
- All calculations and convergence of this method strictly depend on the function $f(x)$ and initial value of x_0 .

Methods

Extra Topics

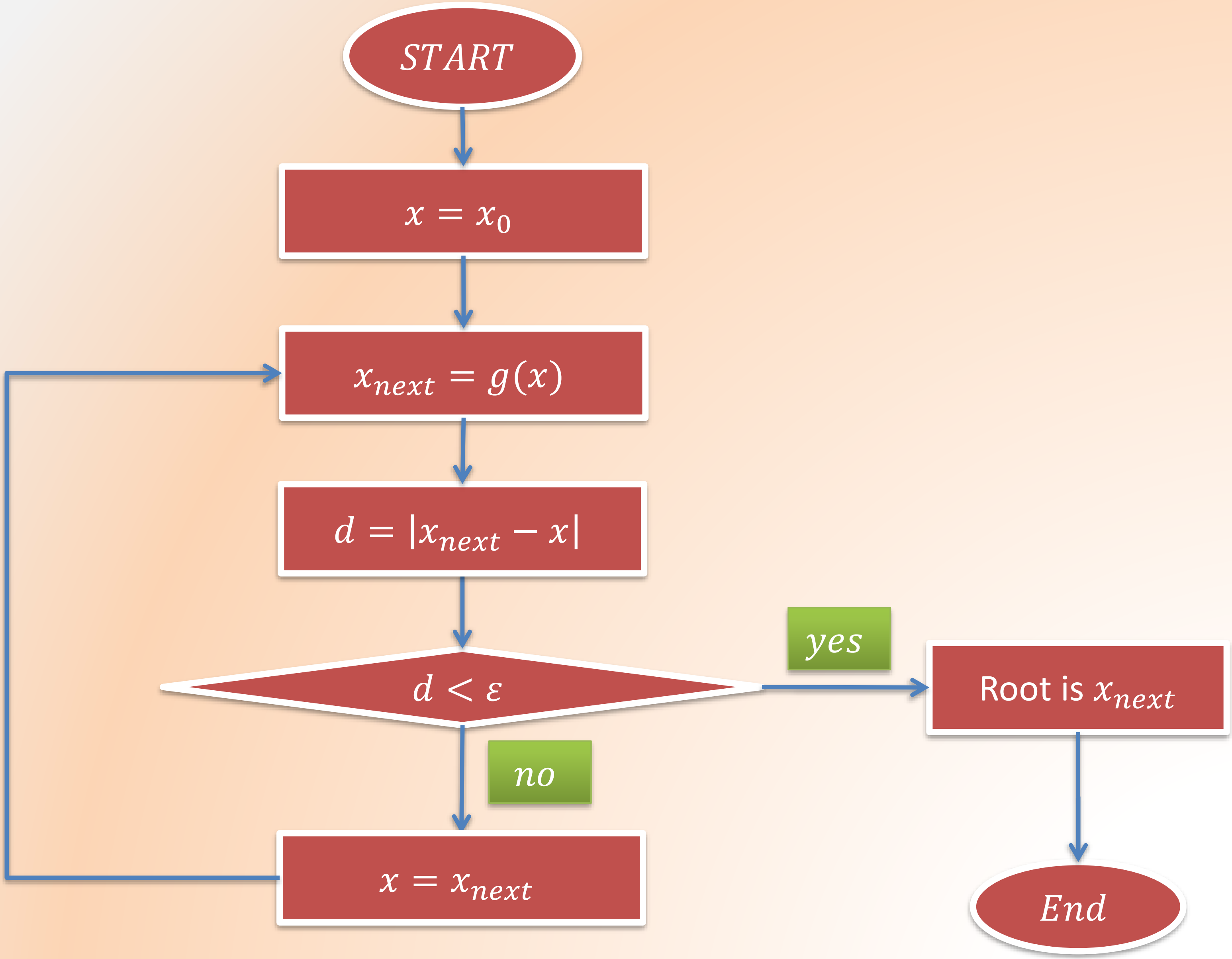
- Bisection Method
- False Position Method
- Secant Method
- Newton–Raphson Method
- Simple Iterative or Fixed Point Method

Fixed Point Method

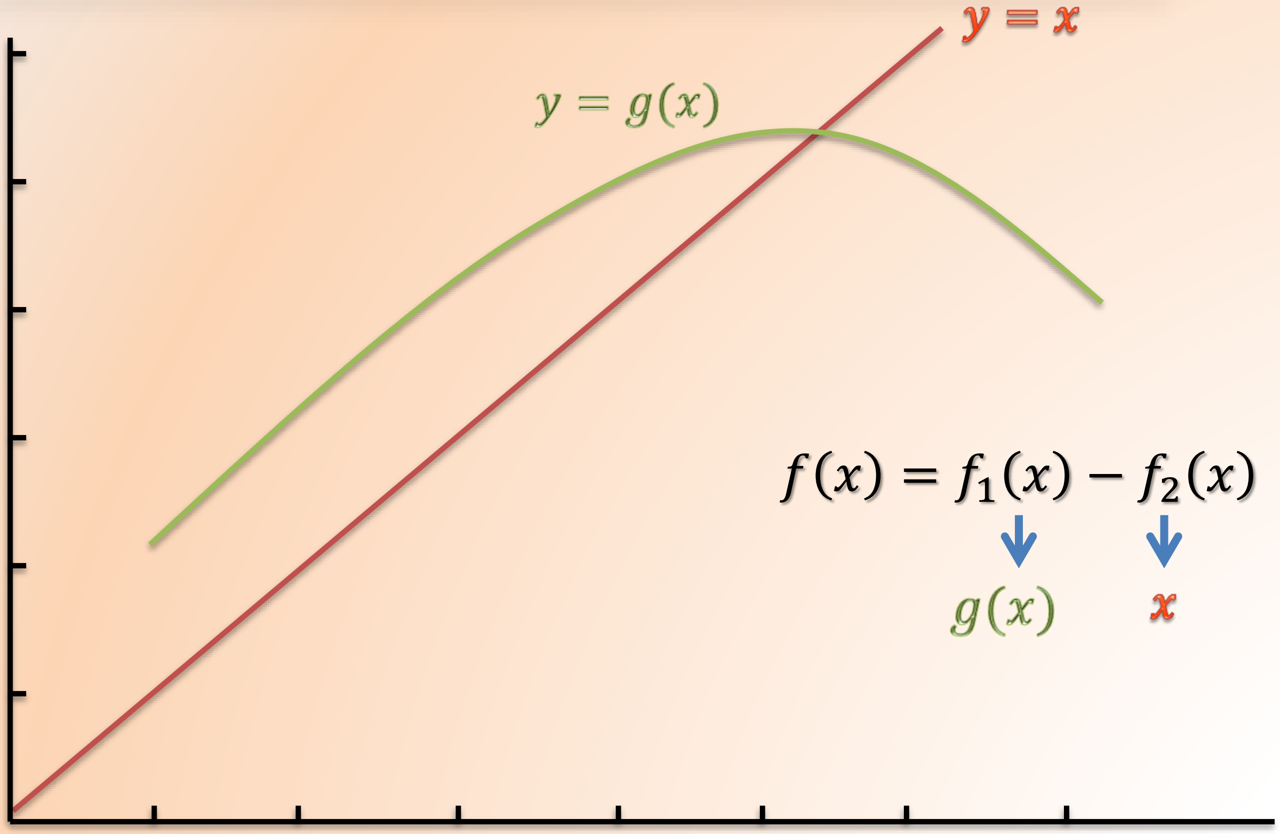
Assumptions Example Advantages & Disadvantages

- $f(x) = 0 \Rightarrow \quad x - g(x) = 0$
  $x = g(x)$
- Initial approximation: x_0

$$\boxed{x_{i+1} = g(x_i)}$$


START
$x = x_0$
$x_{next} = g(x)$
$d = |x_{next} - x|$
$d < \varepsilon$
yes
Root is x_{next}
no
$x = x_{next}$
End

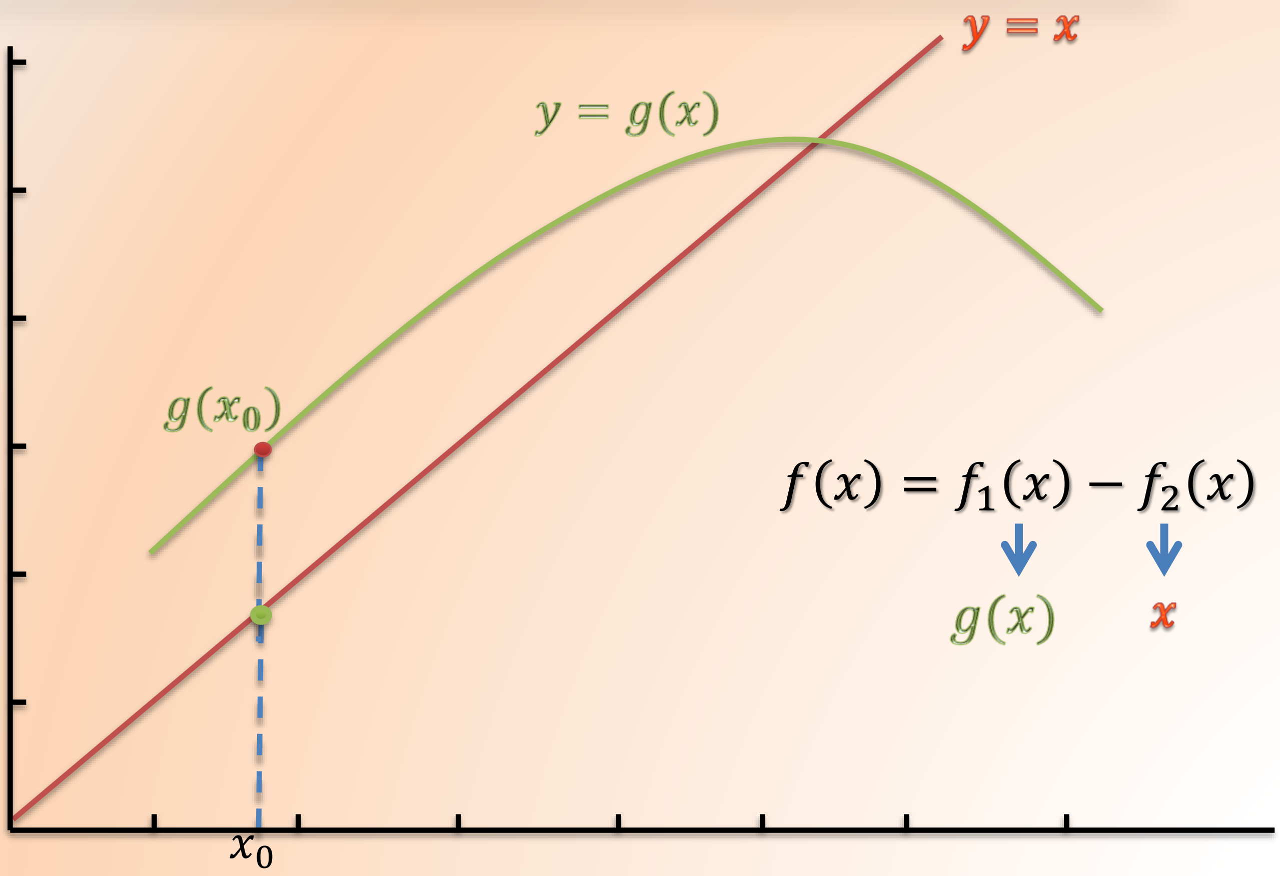
Fixed Point Method



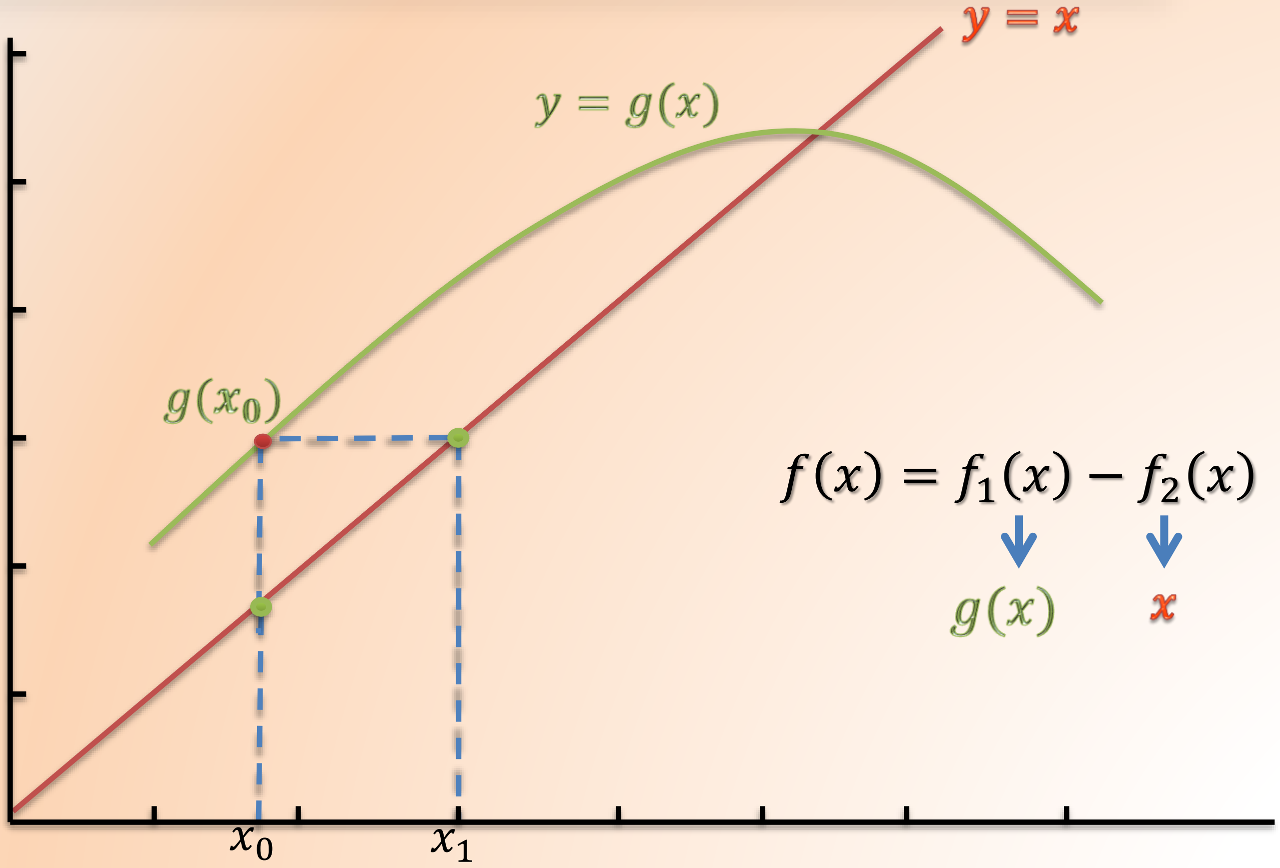
$$f(x) = f_1(x) - f_2(x)$$

$$\underbrace{f_1(x)}_{g(x)} \qquad \underbrace{f_2(x)}_{x}$$

Fixed Point Method



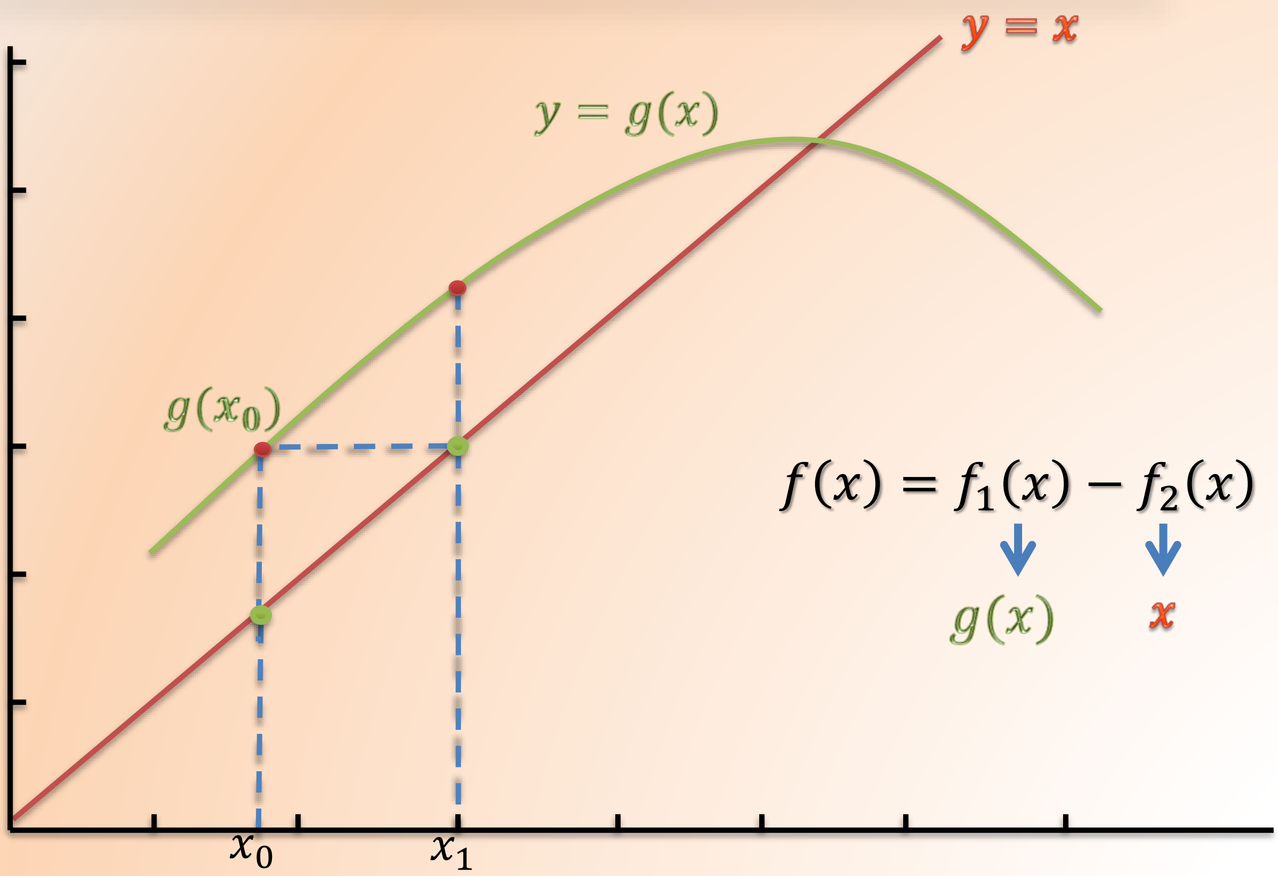
Fixed Point Method



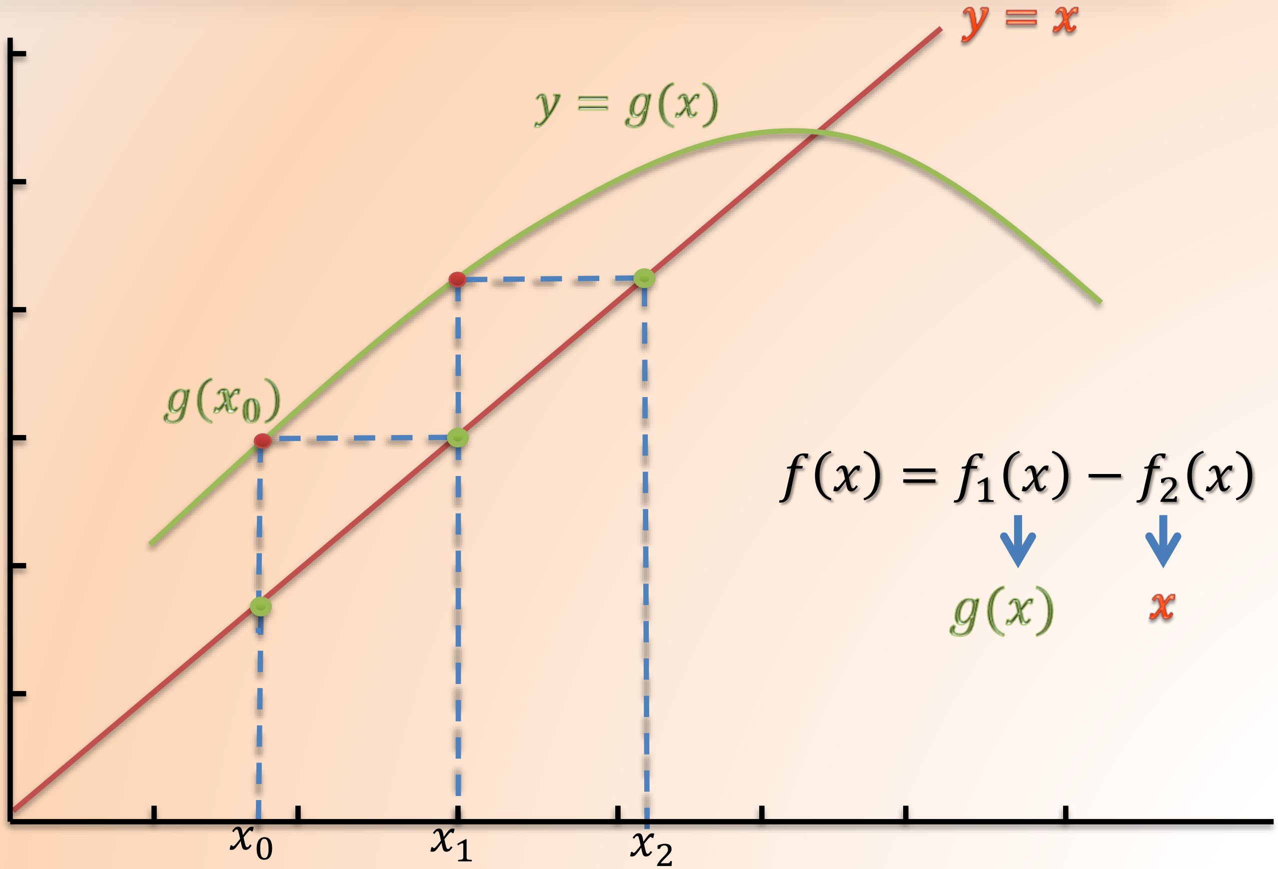
$$f(x) = f_1(x) - f_2(x)$$

$f_1(x) \rightarrow g(x)$, $f_2(x) \rightarrow x$

Fixed Point Method



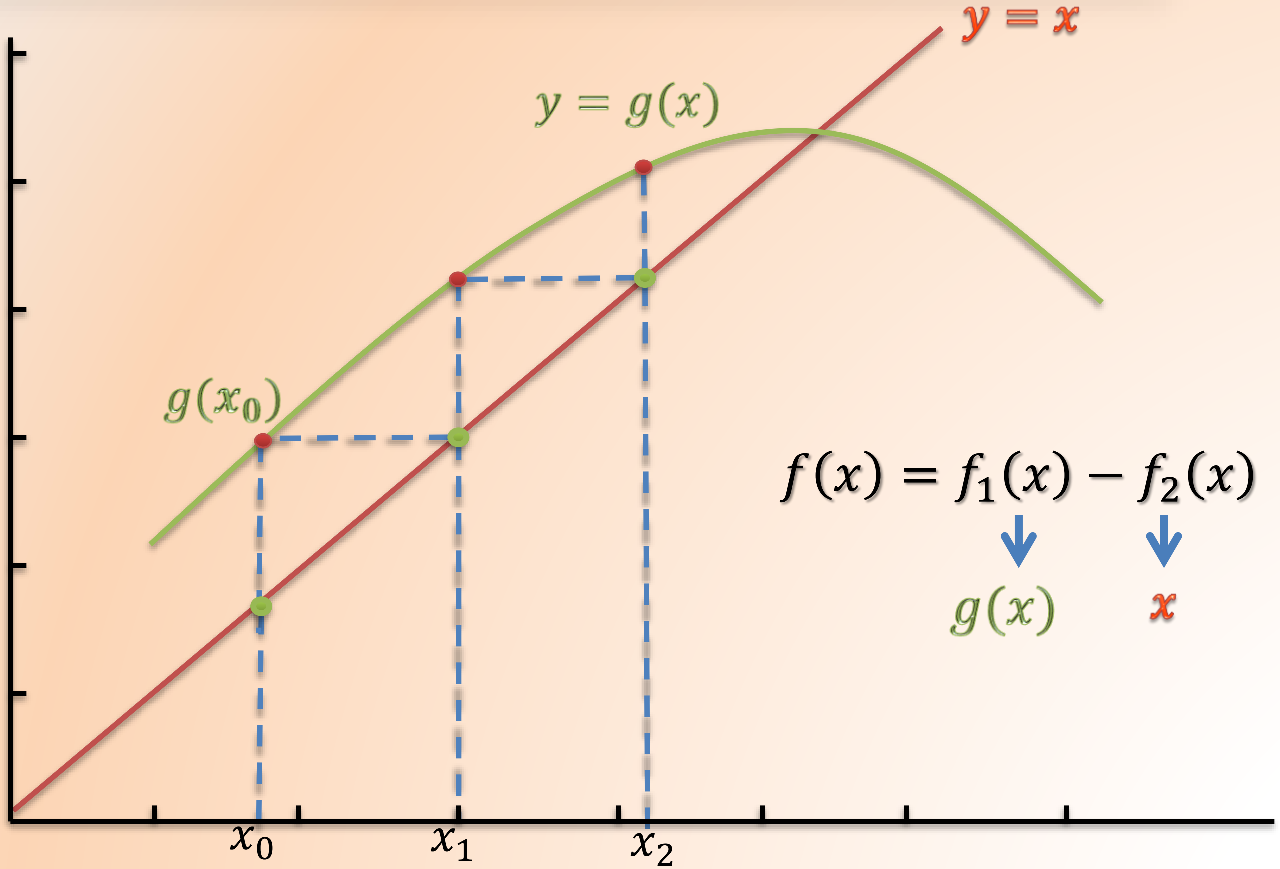
Fixed Point Method



$$f(x) = f_1(x) - f_2(x)$$

$f_1(x) \rightarrow g(x)$, $f_2(x) \rightarrow x$

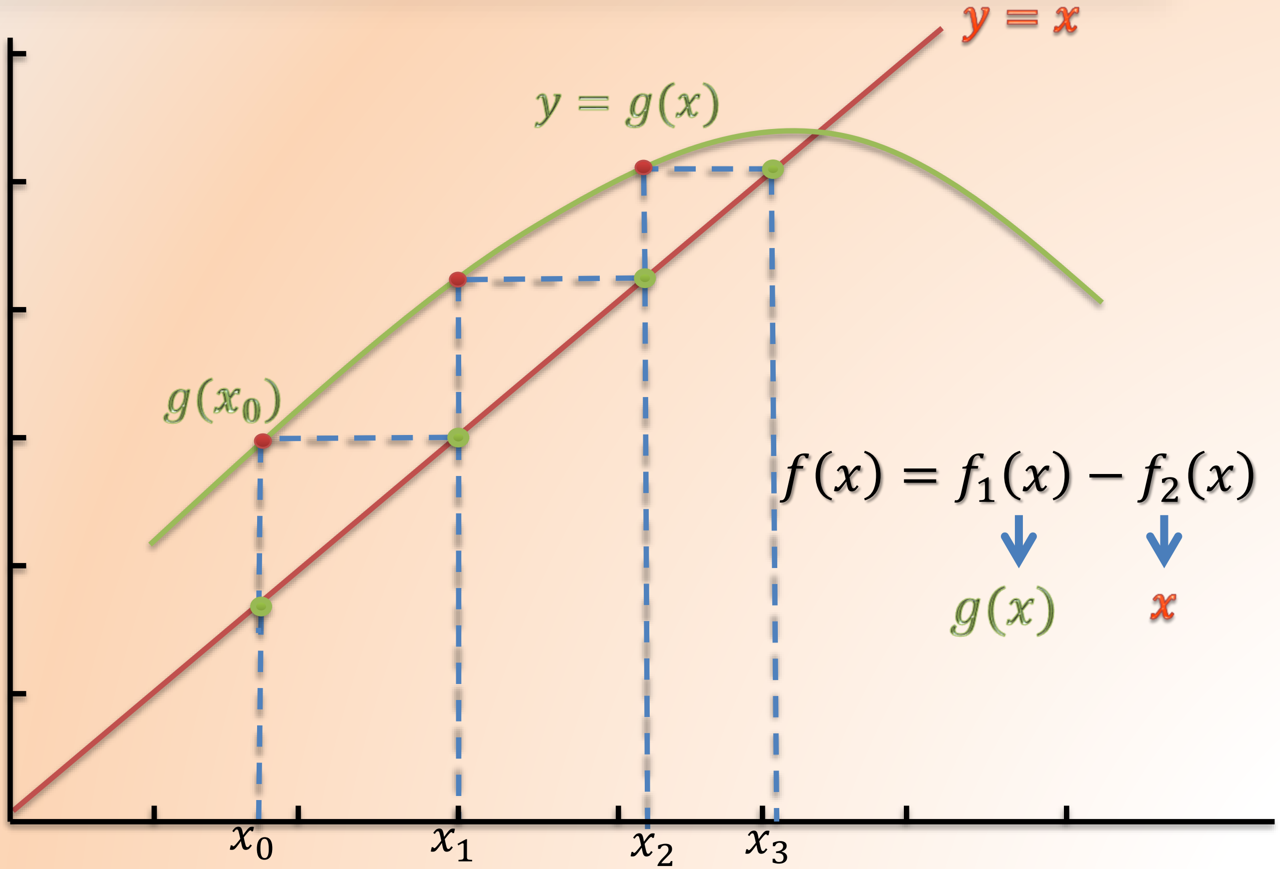
Fixed Point Method



$$f(x) = f_1(x) - f_2(x)$$

$f_1(x) \downarrow g(x)$, $f_2(x) \downarrow x$

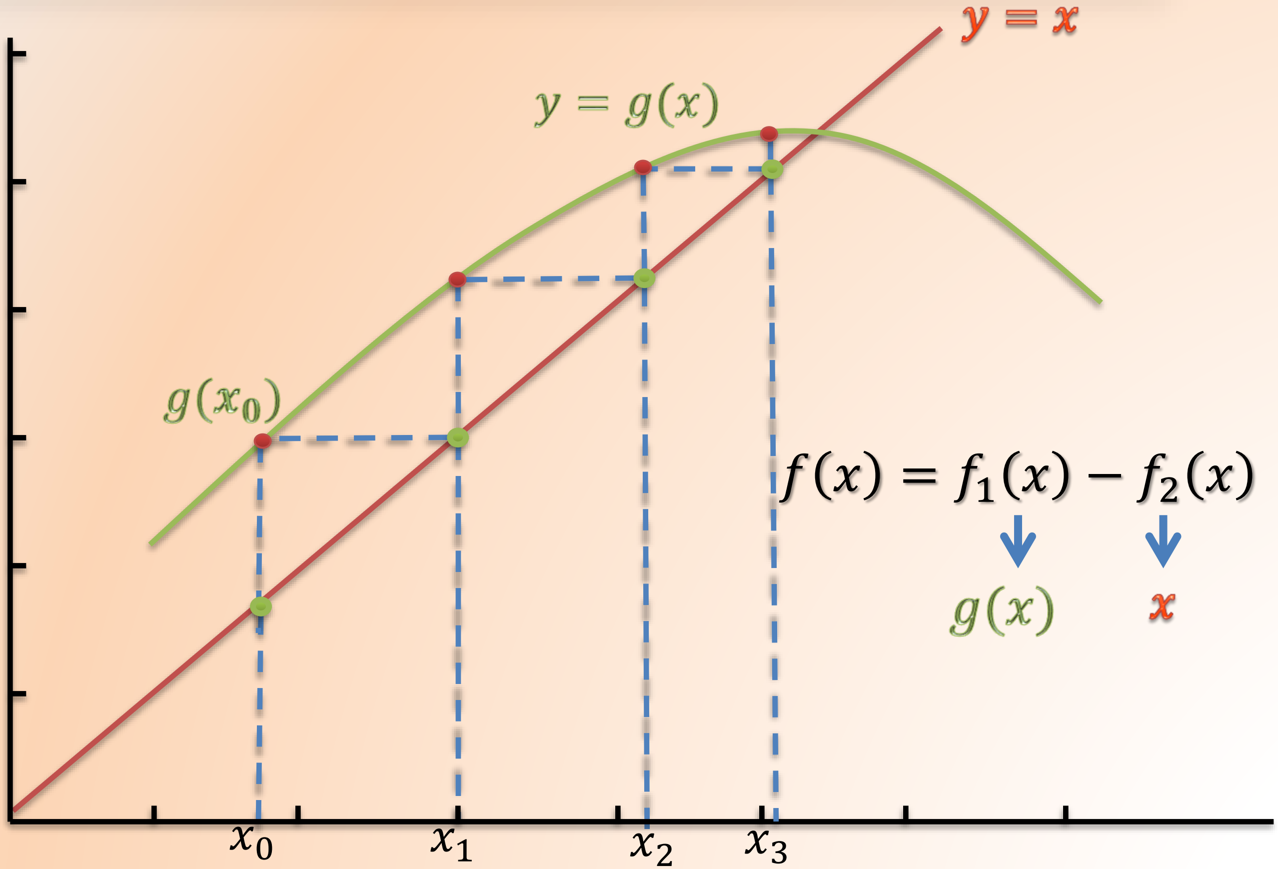
Fixed Point Method



$$f(x) = f_1(x) - f_2(x)$$

$\downarrow$ $g(x)$ $\downarrow$ x

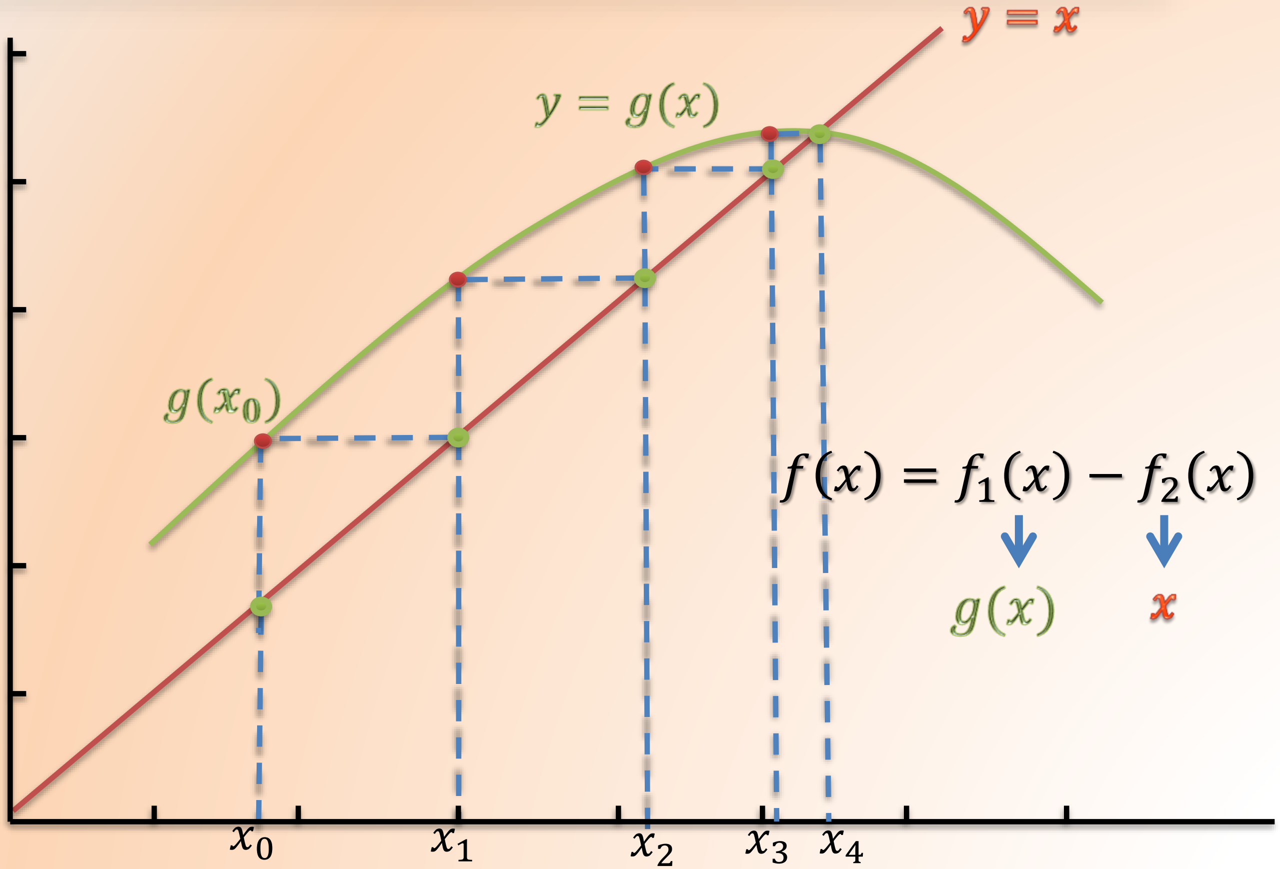
Fixed Point Method



$$f(x) = f_1(x) - f_2(x)$$

$g(x)$ x

Fixed Point Method



$$f(x) = f_1(x) - f_2(x)$$

$$f_1(x) \to g(x), \quad f_2(x) \to x$$

Fixed Point Method

Assumptions Example Advantages & Disadvantages

مثال: تقریبی از ریشه مثبت معادله $x^2+x-1=0$ را به روش گفته شده به دست آورید.

حل: فرض می‌کنیم $x_0=0.5$ و قرار می‌دهیم $g(x)=1-x^2$. پس داریم:

$$x_{n+1}=g(x_n)=1-x_n^2$$

$$x_1=g(x_0)=0.7500\ ,\quad x_2=g(x_1)=0.4375\ ,\quad x_3=g(x_2)=0.8086\ \ldots$$

$$x_7=0.9492\ ,\quad x_8=0.0990.$$

مشاهده می‌شود دنباله حاصل از اندیس‌های فرد به ۱ و دنباله حاصل از اندیس‌های زوج به صفر همگراست. بنابراین در این حالت $\{x_n\}$ همگرا نیست.

پس $g(x)$ دیگری تعریف می‌کنیم:

$$g(x)=\sqrt{1-x}$$

$$x_{n+1}=g(x_n)=\sqrt{1-x_n}$$

در این حالت اگر $x_0=0.5$ آنگاه $\{x_n\}$ بعد از ۲۴ تکرار به جواب معادله همگرا خواهد شد و اگر $x_0=1$، $\{x_n\}$ همگرا نمی‌شود.

Fixed Point Method

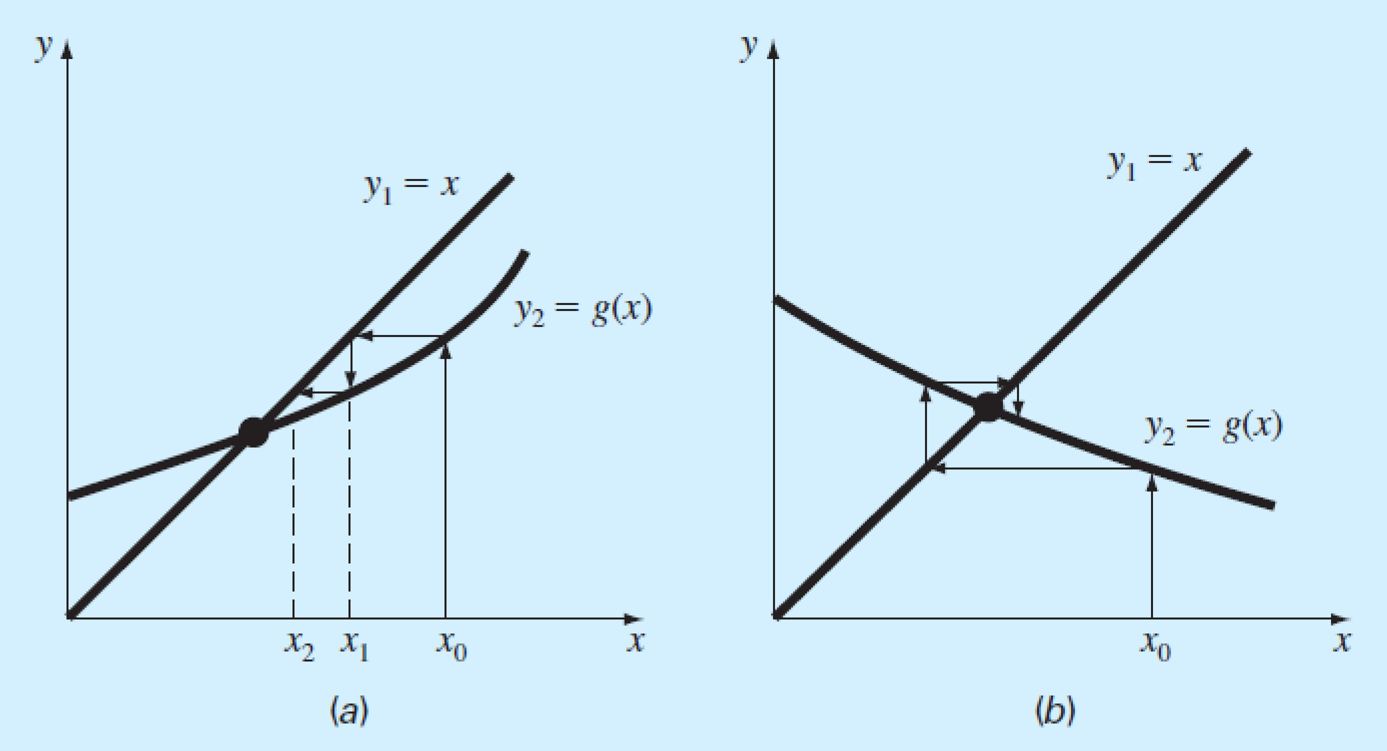
$g(x)$ must be selected such that it can be used for the next step.

Convergence conditions :

- $x_0 \in (\mathrm{a}, \mathrm{b})$
- $g'(x)$ must be continuous in (a, b)
- $\forall x \in (a, b) : |g'(x)| < 1$
- $\forall x \in (a, b) : g(x) \in (a, b)$

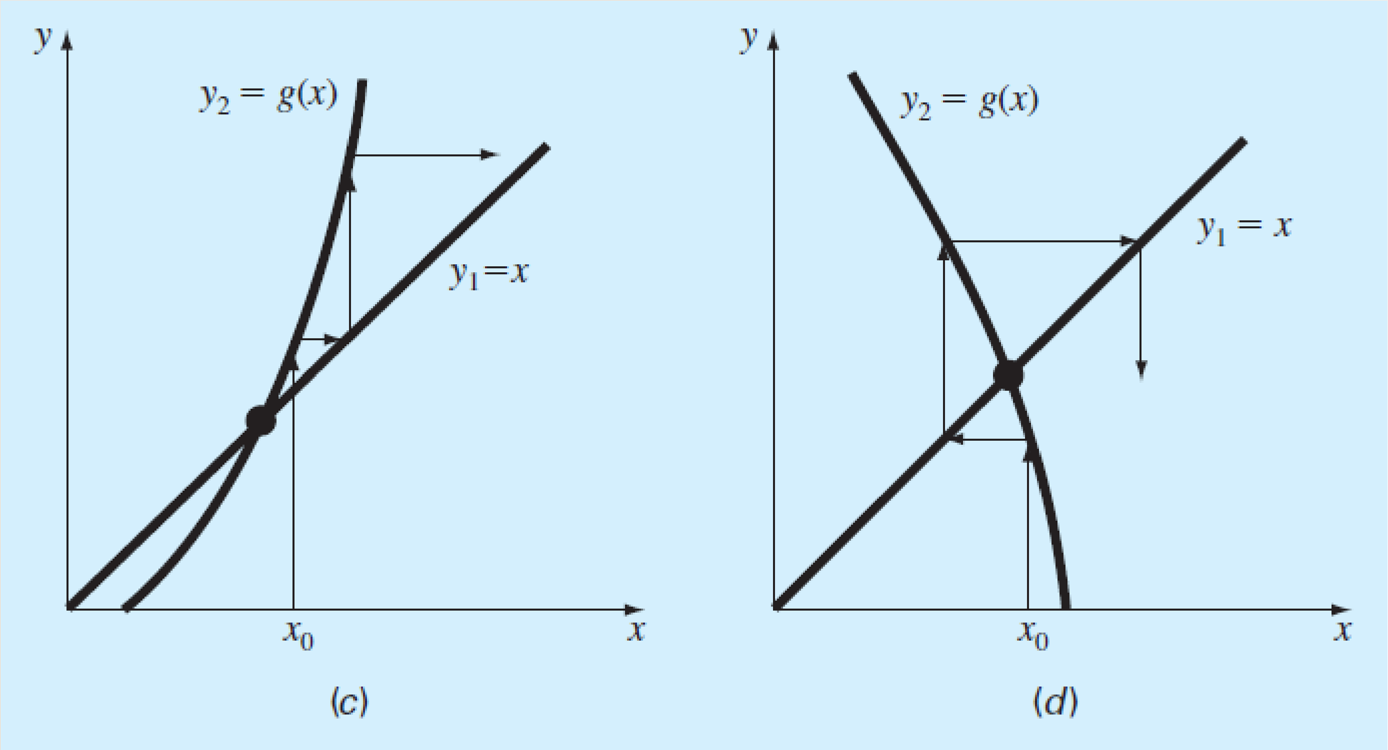
Fixed Point Method

Assumptions **Example** Advantages & Disadvantages



Fixed Point Method

Assumptions **Example** Advantages & Disadvantages



Fixed Point Method

مثال: حل معادله $2x - \ln x - 4 = 0$ با شرط توقف $|x_n - x_{n-1}| < 10^{-4}$

ابتدا سعی می‌کنیم محدوده‌ای که ریشه در آن قرار دارد را به دست آوریم.

$$f(x) = 2x - \ln x - 4$$

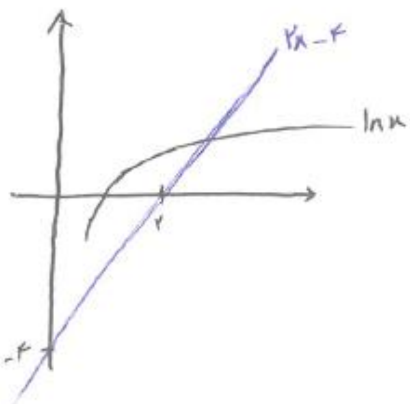
$$\left.\begin{array}{l} f(2) = -\ln 2 < 0 \\ f(3) = 2 - \ln 3 > 0 \end{array}\right\} \rightarrow$$

$f(x)$ در بازه $(2,3)$ پیوسته

پس معادله در این بازه یک ریشه دارد.

همچنین نمودار معادلات $y = 2x - 4$ و $y = \ln x$ نشان می‌دهد که معادله در بازه $(2,3)$ تنها یک ریشه دارد.

(این نمودار برای محدوده x_0 است نه برای روش نقطه ثابت!)



پس بازه $(2,3)$ را در نظر گرفته و $g(x)$ را به صورت زیر می‌سازیم:

$$x = 2 + \frac{1}{2}\ln x = g(x)$$

Fixed Point Method

حال شرایط همگرایی را بررسی می‌کنیم:

$2 < x < 3 \Rightarrow \ln 2 < \ln x < \ln 3 \Rightarrow$

$2 + \frac{1}{2}\ln 2 < 2 + \frac{1}{2}\ln x < 2 + \frac{1}{2}\ln 3 \Rightarrow 2 < 2 + \frac{1}{2}\ln x < 3 \Rightarrow 2 < g(x) < 3$

✓ شرط ۱ برقرار شده است.

$g'(x) = \frac{1}{2x} \Rightarrow 0 < g'(x) \leq \frac{1}{4} \quad \forall x \in (2,3)$ ✓ شرط ۲ برقرار شده است

هر دو شرط برقرار است $\Longleftarrow$

$x_0 = 2.5$

$x_{n+1} = g(x_n) = 2 + \frac{1}{2}\ln x_n \qquad n = 0, 1, \dots$

$x_1 = 2.45815$

$x_2 = 2.44970$

$x_3 = 2.44798$

$x_4 = 2.44763$

$x_5 = 2.44756$

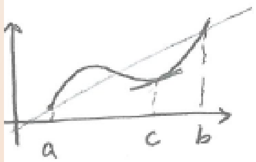
$x_6 = 2.44755 \Rightarrow \underline{\alpha \simeq 2.44755}$

Fixed Point Method

درجه همگرایی روش نقطه ثابت :

قضیه مقدار میانگین : اگر تابع $f(x)$ در بازه $[a,b]$ مشتق پذیر باشد، آنگاه عددی مانند $c\in[a,b]$ وجود دارد که

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$



با توجه به قضیه بالا، برای مقدار خطای روش نقطه ثابت داریم :

$$|e_{k+1}| = |\alpha - x_{k+1}| = |g(\alpha) - g(x_k)| = |g'(c)|\,\underbrace{|\alpha - x_k|}_{e_k} \Longrightarrow$$

قضیه مقدار میانگین

$$|e_{k+1}| = \underbrace{|g'(c)|}_{|C|<1}\,|e_k| \Rightarrow$$ درجه همگرایی ۱ است $m=1$

برای همگرایی در این روش، ضریب $|e_k|$ که $|g'(c)|$ است، باید کمتر از ۱ باشد.

که همان طور که مشاهده شد در شرط همگرایی این مسأله ذکر شده است.

Fixed Point Method

Assumptions Example Advantages & Disadvantages

Advantages :

- Can divide $f(x)$

Disadvantages :

- No guarantee to converge to answer.
- Convergence conditions for this method are hard to be checked.
- The degree of convergence for this method is linear and equal to 1 .
- All calculations and convergence of this method strictly depend on the function $g(x)$ and initial value of x_0 .

Fixed Point Method

در این جا می‌خواهیم شرط همگرایی روش نیوتن - رافسون که حالت خاصی از روش نقطه ثابت است را بررسی کنیم. برای این منظور از قضیه زیر می‌توانیم استفاده کنیم.

قضیه: فرض کنید دنباله $\{x_n\}$ با ضابطه $x_{n+1}=g(x_n)$ به α همگرا باشد و g دارای مشتق دوم پیوسته باشد.

- اگر $g'(\alpha)\neq 0$ ← رتبه همگرایی روش نقطه ثابت برابر ۱ است.
- اگر $g'(\alpha)=0$ ← رتبه همگرایی روش نقطه ثابت حداقل ۲ است.
- اگر $g'(\alpha)=0$ ، $g''(\alpha)\neq 0$ ← رتبه همگرایی روش نقطه ثابت دقیقاً ۲ است.

* به طور کلی می‌توان گفت: اگر دنباله $\{x_n\}$ با ضابطه $x_{n+1}=g(x_n)$ به α همگرا باشد و g دارای مشتق kام پیوسته باشد و

$$g'(\alpha)=g''(\alpha)=\dots=g^{(k-1)}(\alpha)=0 \quad , \quad g^{(k)}(\alpha)\neq 0$$

آنگاه رتبه همگرایی $\{x_n\}$ برابر k است.

Newton-Raphson Method

- حال روش نیوتن رافسون را بررسی می‌کنیم:

$$f(x)=0 \quad , \quad x = x - \frac{f(x)}{f'(x)}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$\Longleftarrow$ $f(x)$ ریشه ساده α ، $f'(\alpha)\neq 0$

$$g(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)} \Longrightarrow g(\alpha) = \alpha$$

$$g'(x) = 1 - \frac{f'(x)\, f'(x) - f''(x)\, f(x)}{[f'(x)]^2} = 1 - 1 + \frac{f(x)\, f''(x)}{[f'(x)]^2} = \frac{f(x)\, f''(x)}{f'^2(x)}$$

* در روش نیوتن رافسون گفتیم شرط همگرایی این است که $\left|\frac{f(x)\, f''(x)}{f'^2(x)}\right| < 1$ که این همان شرط همگرایی $|g'(x)| < 1$ در روش نقطه ثابت است.

Newton-Raphson Method

$$g'(\alpha) = \frac{f(\alpha)\, f''(\alpha)}{f'^{2}(\alpha)} = 0$$

پس تا این جا در روش نیوتن رافسون $g'(\alpha) = 0$ است ← پس ترتیب همگرایی حداقل ۲ است.

ولی با محاسبه $g''(\alpha) \neq 0$ متوجه می‌شویم که ترتیب همگرایی دقیقاً ۲ است.

$$g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} \neq 0$$

چون α ریشه ساده است

Methods

Extra Topics

- Horner Method for Evaluating Polynomials
- Generalized Newton–Raphson Method

Newton-Raphson Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{P_n(x_k)}{P'_n(x_n)}$$

Horner Method for Evaluating Polynomials

اگر n بزرگ باشد، برای محاسبه $p(x_k)$ و $p'(x_k)$ در هر مرحله، تعداد عملیات ضرب زیادی نیاز است:

به عنوان مثال،

(1) $p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \Rightarrow a_3 \cdot x \cdot x \cdot x + a_2 \cdot x \cdot x + a_1 \cdot x + a_0$ $\left.\begin{matrix} \text{۶ ضرب} \\ \text{۳ جمع} \end{matrix}\right\}$

ولی اگر همین مثال را به فرم زیر بنویسیم، تعداد عملیات ضرب کمتر می‌شود:

(2) $p(x) = ((a_3 x + a_2) x + a_1) x + a_0 \Rightarrow ((a_3 \cdot x + a_2) \cdot x + a_1) \cdot x + a_0$ $\left.\begin{matrix} \text{۳ ضرب} \\ \text{۳ جمع} \end{matrix}\right\}$

به صورت کلی، در حالت (1)، به ازای $p_n(x)$، $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ ضرب و
در حالت (2)، به ازای $p_n(x)$، n ضرب نیاز است.

Horner Method for Evaluating Polynomials

i. $p_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$

ii. $p_n(x_0) = ?$

$b_n = a_n$

$b_{n-1} = a_{n-1} + b_n x_0$

$b_{n-2} = a_{n-2} + b_{n-1} x_0$

…

$b_0 = a_0 + b_1 x_0$

Horner Alg:

input a_K $K=0,1,\dots,n$

input x_0

$b_n = a_n$

for $K = n-1$ to 0

$\qquad b_K = b_{K+1} x_0 + a_K$

$p(x_0) = b_0$

Horner Method for Evaluating Polynomials

	x^n	x^{n-1}	x^{n-2}	…	x^0
x_0	a_n	a_{n-1}	a_{n-2}	…	a_0
+					

Horner Method for Evaluating Polynomials

	x^n	x^{n-1}	x^{n-2}	…	x^0
x_0	a_n	a_{n-1}	a_{n-2}	…	a_0
+					
	b_n				

Horner Method for Evaluating Polynomials

	x^n	x^{n-1}	x^{n-2}	…	x^0
x_0	a_n	a_{n-1}	a_{n-2}	…	a_0
$+$		$x_0 b_n$			
	b_n	b_{n-1}			

Horner Method for Evaluating Polynomials

	x^n	x^{n-1}	x^{n-2}	…	x^0
x_0	a_n	a_{n-1}	a_{n-2}	…	a_0
+		$x_0 b_n$	$x_0 b_{n-1}$		
	b_n	b_{n-1}	b_{n-2}		

Horner Method for Evaluating Polynomials

	x^n	x^{n-1}	x^{n-2}	…	x^0
x_0	a_n	a_{n-1}	a_{n-2}	…	a_0
$+$		$x_0 b_n$	$x_0 b_{n-1}$	…	$x_0 b_1$
	b_n	b_{n-1}	b_{n-2}	…	

Horner Method for Evaluating Polynomials

	x^n	x^{n-1}	x^{n-2}	…	x^0
x_0	a_n	a_{n-1}	a_{n-2}	…	a_0
+		$x_0 b_n$	$x_0 b_{n-1}$	…	$x_0 b_1$
	b_n	b_{n-1}	b_{n-2}	…	b_0

Answer

Horner Method for Evaluating Polynomials

Example:

$$f(x) = 2x^3 - 6x^2 + 2x - 1 \quad f(3) = ?$$

$$b_3 = a_3 = 2$$

$$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0$$

$$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2$$

$$b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$$

$$\Rightarrow f(3) = 5$$

Horner Method for Evaluating Polynomials

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0 \qquad f(3) = 5$

$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2, \Rightarrow \; b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$

	x^3	x^2	x^1	x^0
x_0	2	-6	2	-1
+				
	2			

Horner Method for Evaluating Polynomials

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0$ $\qquad f(3) = 5$

$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2,$ ➡ $b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$

	x^3	x^2	x^1	x^0
x_0	2	-6	2	-1
$+$		$2 \times 3 = 6$		
	2			

Horner Method for Evaluating Polynomials

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0 \qquad f(3) = 5$

$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2,$ ➡ $b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$

	x^3	x^2	x^1	x^0
x_0	2	-6	2	-1
$+$		$2 \times 3 = 6$		
	2	$6 - 6 = 0$		

Horner Method for Evaluating Polynomials

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0 \qquad f(3) = 5$

$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2, \Rightarrow b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$

	x^3	x^2	x^1	x^0
x_0	2	-6	2	-1
$+$		$2 \times 3 = 6$	$3 \times 0 = 0$	
	2	$6 - 6 = 0$		

Horner Method for Evaluating Polynomials

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0$ $\qquad f(3) = 5$

$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2, \Rightarrow b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$

	x^3	x^2	x^1	x^0
x_0	2	-6	2	-1
$+$		$2 \times 3 = 6$	$3 \times 0 = 0$	
	2	$6 - 6 = 0$	2	

Horner Method for Evaluating Polynomials

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x_0 = -6 + 2 \times 3 = 0 \qquad f(3) = 5$

$b_1 = a_1 + b_2 x_0 = 2 + 0 = 2, \Rightarrow b_0 = a_0 + b_1 x_0 = -1 + 2 \times 3 = 5$

	x^3	x^2	x^1	x^0
x_0	2	-6	2	-1
$+$		$2 \times 3 = 6$	$3 \times 0 = 0$	$2 \times 3 = 6$
	2	$6 - 6 = 0$	2	5

Answer

Horner Method for Evaluating Polynomials

قضیه: فرض کنید $p(x)$ چندجمله‌ای تعریف شده باشد و b_k ها $(k=0,1,\dots,n)$ توسط الگوریتم هورنر برای نقطه x_0 تعریف شده باشند. اگر تعریف کنیم $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1$

آنگاه،

$$p(x) = (x - x_0)\, Q(x) + b_0$$

پس داریم:

$$\begin{cases} p(x) = (x - x_0)\, Q(x) + b_0 \\ Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1 \end{cases}$$

از $p(x)$ مشتق می‌گیریم:

$$\begin{cases} p'(x) = Q(x) + (x - x_0)\, Q'(x) \\ p'(x_0) = Q(x_0) \end{cases}$$

پس برای محاسبه $p'(x_0)$ کافی است $Q(x_0)$ را محاسبه کنیم. (برای محاسبه $Q(x_0)$ هم از روش هورنر استفاده می‌کنیم)

Horner Method for Evaluating Polynomials

Example: $P(x) = 2x^4 - 3x^2 + 3x - 4 \quad P(-2) = ? \; P'(-2) = ?$

	$x^۴$	$x^۳$	$x^۲$	$x^۱$	$x^۰$	
$x_۰$ (-۲)	۲	۰	-۳	۳	-۴	← a_i ها: ضرایب $P(x)$
+		(-۲)×۲ = -۴	(-۲)(-۴) = ۸	(-۲)×۵ = -۱۰	(-۲)×(-۷) = ۱۴	
	۲	-۴+۰ = -۴	۸+(-۳) = ۵	-۱۰+۳ = -۷	۱۴-۴ = ۱۰	← b_i ها: ضرایب $Q(x)$
+		(-۲)×۲ = -۴	(-۲)(-۸) = ۱۶	(-۲)×۲۱ = -۴۲		۱۰ → جواب $P(-۲)$
	۲	(-۴)+(-۴) = -۸	۱۶+۵ = ۲۱	(-۴۲)+(-۷) = -۴۹		-۴۹ → جواب $p'(-۲)$

Methods

Extra Topics

Horner Method for Evaluating Polynomials

Generalized Newton–Raphson Method

Generalized Newton–Raphson Method

A method for solving a system of n nonlinear functions with n variables.

$$F(X) = 0$$

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{cases}$$

Generalized Newton–Raphson Method

در روش نیوتن رافسون تعمیم یافته، مثل روش نیوتن رافسون، یک ماتریس اولیه به صورت $X_0 = (x_1^0, x_2^0, \dots, x_n^0)$
انتخاب می‌کنیم و روند الگوریتم را با استفاده از فرمول زیر پیش می‌بریم:

$$\boxed{X_{k+1} = X_k - J^{-1}(X_k)\, F(X_k)}$$

در این فرمول، $J(X)$ ماتریس ژاکوبین است که در ادامه آن را تعریف می‌کنیم.

اگر دقت کنید شبیه به این فرمول، فرمول اصلی روش نیوتن رافسون را متوجه می‌شوید.

$$\left(x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\right) \longrightarrow \text{فرمول نیوتن رافسون}$$

(ماتریس ژاکوبین، ماتریس مشتقات است).

Generalized Newton–Raphson Method

در ماتریس $J(X)$، درایه‌ی ij ام مربوط به مشتق معادله f_i نسبت به متغیر j ام است:

$$J(X) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots\cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \dfrac{\partial f_n}{\partial x_1} & \cdots\cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Generalized Newton–Raphson Method

For instance, for 2 variables x and y, we have:

$F(x, y) = 0$

$G(x, y) = 0$

Approximate x_0 and y_0:

$$\boxed{x_{i+1} = x_i + \frac{D_1}{D}}$$

$$\boxed{y_{i+1} = y_i + \frac{D_2}{D}}$$

Generalized Newton–Raphson Method

$$D = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix} = \frac{\partial F}{\partial x} \times \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y} \times \frac{\partial G}{\partial x}$$

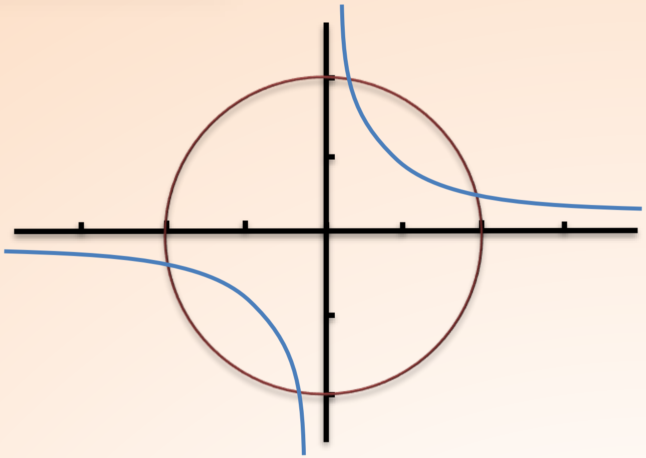
$$D_1 = \begin{vmatrix} -F(x,y) & \frac{\partial F}{\partial y} \\ -G(x,y) & \frac{\partial G}{\partial y} \end{vmatrix} \qquad D_2 = \begin{vmatrix} \frac{\partial F}{\partial x} & -F(x,y) \\ \frac{\partial G}{\partial x} & -G(x,y) \end{vmatrix}$$

In each step, x_i and y_i are obtained from the pervious step to calculate D, D_1 and D_2 .

Generalized Newton–Raphson Method

Example:

$$\begin{cases} F(x,y) = x^2 + y^2 - 4 = 0 \\ G(x,y) = xy - 1 = 0 \end{cases}$$



Generalized Newton–Raphson Method

Example:

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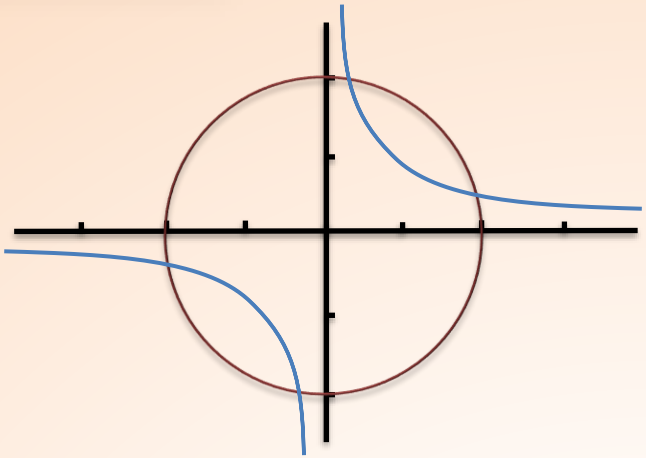


$$\frac{\partial F}{\partial X} = 2x, \qquad \frac{\partial F}{\partial Y} = 2y$$

$$\frac{\partial G}{\partial X} = y, \qquad \frac{\partial G}{\partial y} = x$$

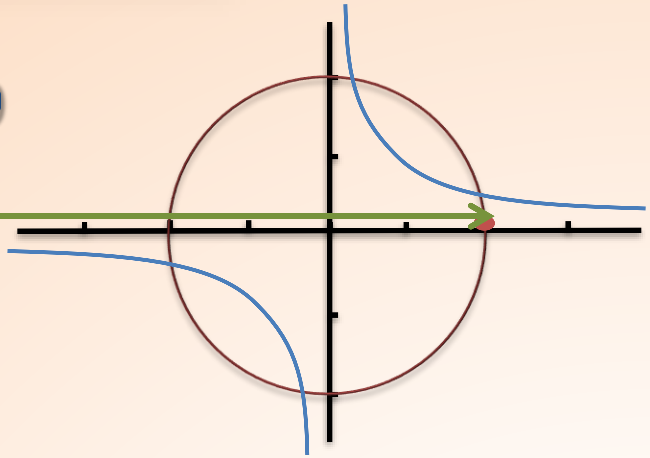
$$D = \begin{vmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} \end{vmatrix} = 2x^2 - 2y^2$$

$$D_1 = \begin{vmatrix} -F(x,y) & \frac{\partial F}{\partial Y} \\ -G(x,y) & \frac{\partial G}{\partial Y} \end{vmatrix} = -(x^2 + y^2 - 4)(x) + (xy - 1)(2y)$$

$$D_2 = \begin{vmatrix} \frac{\partial F}{\partial X} & -F(x,y) \\ \frac{\partial G}{\partial X} & -G(x,y) \end{vmatrix} = (2x)(-xy + 1) - (x^2 + y^2 - 4)(y)$$

Generalized Newton–Raphson Method

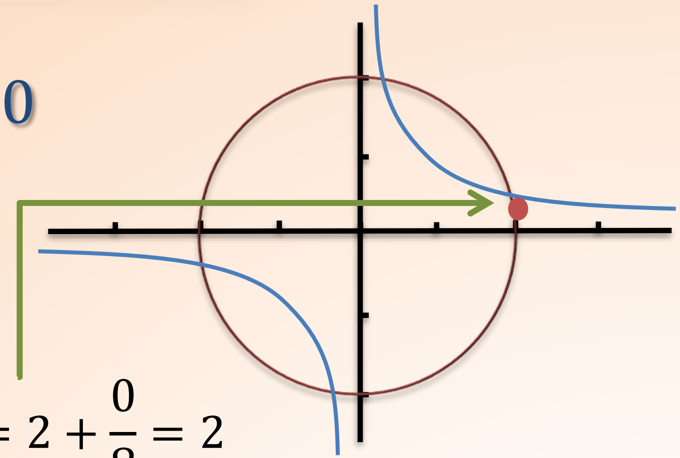
Example: $\begin{cases} F(x,y) = x^2 + y^2 - 4 = 0 \\ G(x,y) = xy - 1 = 0 \end{cases}$



$$\begin{cases} x_{i+1} = x_i + \dfrac{D_1}{D} \\ y_{i+1} = y_i + \dfrac{D_2}{D} \end{cases} \qquad \begin{cases} x_0 = 2 \\ y_0 = 0 \end{cases}$$

Generalized Newton–Raphson Method

Example: $\begin{cases} F(x,y) = x^2 + y^2 - 4 = 0 \\ G(x,y) = xy - 1 = 0 \end{cases}$



$$\begin{cases} x_{i+1} = x_i + \dfrac{D_1}{D} \\ y_{i+1} = y_i + \dfrac{D_2}{D} \end{cases} \qquad \begin{cases} x_0 = 2 \\ y_0 = 0 \end{cases} \rightarrow \begin{cases} D = 8 \\ D_1 = 0 \\ D_2 = 3 \end{cases} \rightarrow \begin{cases} x_1 = 2 + \dfrac{0}{8} = 2 \\ y_1 = 0 + \dfrac{3}{8} = \dfrac{3}{8} \end{cases}$$

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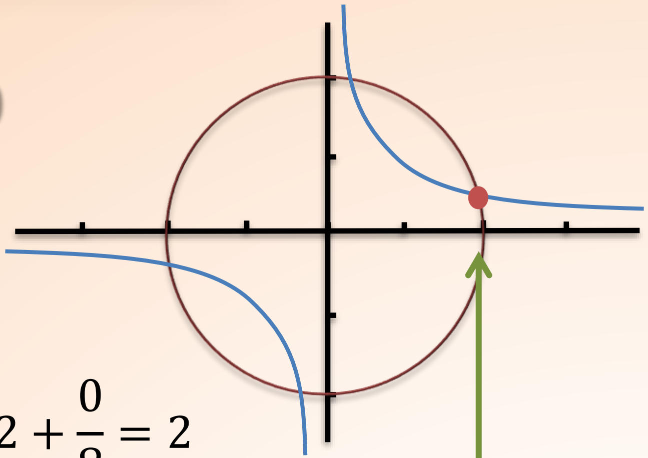

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$$\begin{cases} x_1 = 2 \\ y_1 = \dfrac{3}{8} \end{cases} \rightarrow \begin{cases} D = 7.71875 \\ D_1 = -0.46785 \\ D_2 = -1.0527 \end{cases} \rightarrow \begin{cases} x_2 = 2 + \dfrac{-0.46785}{7.71875} = 1.9393 \\ y_2 = \dfrac{3}{8} + \dfrac{-1.0527}{7.71875} = 0.2386 \end{cases}$$


Any questions?