Chapter 6

Systems of Linear Equations

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Direct Methods | Indirect Methods | Extra Topics

- Inverse Matrix
- Cramer's Method
- Gaussian Elimination Method
  - Forward Gaussian Elimination
  - Backward Gaussian Elimination
  - Gauss-Jordan
- Triangular Decomposition Method
  - Doolittle's Method
  - Crout's Method
  - Cholesky's Method

Systems of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

This system can be written in the matrix form:

$$AX = B$$

Systems of Linear Equations

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ coefficients matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$ variables matrix $$B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$ constants matrix

Given matrices A and B, we have:

$$\boxed{X = A^{-1}B}$$

Systems of Linear Equations

If B is the zero matrix, the system is called Homogeneous.

In a homogeneous system, in case $|\boldsymbol{A}| \neq \mathbf{0}$ the answer is 0.

This lecture focuses on hetrogeneous, i.e. <u>non-homogeneous</u> systems.

If the matrix A is non-singular ($|A| \neq 0$) and the system is hetrogeneous, a unique solution exists, otherwise the system has either no solution or an infinite number of solutions.

Matrix Determinant

For 2×2 matrix, determinant is calculated as below:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For $\mathbf{3} \times \mathbf{3}$ matrix, rule of Sarrus can be applied:

$$\begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & a_{13} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & a_{23} \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & a_{33} \end{vmatrix}$$

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$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

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$$\left|\begin{matrix} \mathbf{a_{11}} & \mathbf{a_{12}} & a_{13} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & a_{23} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & a_{33} \end{matrix}\right| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

Matrix Determinant

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$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Triangular Matrix

$$L = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix}$$

lower triangular matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

upper triangular matrix

For an upper(or a lower) triangular matrix A , the determinant can be calculated as below:

$$|A| = \prod_{i=1}^{n} a_{ii} = a_{11}a_{22} \dots a_{nn}$$

Identity Matrix

An identity matrix or unit matrix of size n is an $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

If A is an $m \times n$ matrix, it can be easily shown that:

$$\boxed{I_m A = A I_n = A}$$

Transpose Matrix

The transpose of a matrix A is another matrix A^T created by writing the rows of A as the columns of A^T.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Adjoint Matrix

Adjoint of a square matrix is another square matrix such that:

$$A * adj(A) = |A| I_n$$

$$\Rightarrow adj(A) = A^T{}_{cofactor}$$

Cofactor of a matrix of the form $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is:

$$A_{cofactor} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Cofactor Matrix

$$A_{11} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \qquad A_{12} = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$A_{13} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \qquad A_{21} = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$$

$$A_{22} = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \qquad A_{23} = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$A_{31} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \qquad A_{32} = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$A_{33} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Cofactor Matrix

The following notes help you learn how to create the cofactor matrix:

- Sign of each A_{ij} is negative, if $i + j$ is odd and positive, if it is even.
- The magnitude of A_{ij} is determinant of the matrix which come from deleting row i and column j of the primary matrix.

Inverse Matrix

An $n \times n$ matrix A is called invertible if there exists an $n \times n$ matrix B such that:

$$A * B = I_n$$

In this case, B is called inverse of A, denoted by A^{-1}

So we have :

$$\boxed{A^{-1} = \frac{adj(A)}{|A|}}$$

Augmented Matrix

Augmented matrix of matrix A, denoted by $\tilde{A}$, is a matrix obtained by appending matrix B as $n+1$th column of A.

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Some operations are allowed on augmented matrix such as <u>displacement</u> of rows, <u>multiplying</u> a row by a constant, <u>adding or subtracting</u> two rows, etc.

Solving a System of Equations

Two types of methods are proposed for solving a system of equations:

- **Direct methods** like $X = A^{-1}B$ and Cramer's rule.
- **Iterative methods** that begin with an initial guess and improve the answer in each iteration.

Some people believe that Direct methods are not numerical methods, since they give the exact answer in case no number is rounded off in their calculations.

Direct Methods | Indirect Methods | Extra Topics

- Direct Methods
  - **Inverse Matrix**
  - Cramer’s Method
  - Gaussian Elimination Method
    - Forward Gaussian Elimination
    - Backward Gaussian Elimination
    - Gauss-Jordan
  - Triangular Decomposition Method
    - Doolittle’s Method
    - Crout’s Method
    - Cholesky’s Method

Inverse Matrix

$$AX = B \Rightarrow \mathrm{X} = A^{-1}B$$

For calculating determinant of an $n \times n$ matrix, $n!$ multiplication operations are needed.

And $(\mathrm{n} + 1)!$ multiplication operations are needed for calculating inverse of an $n \times n$ matrix.

We can infer that, for example, approximately 40 million multiplication operations are required for a 10×10 matrix!

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Cramer's Rule

For solving an $n \times n$ system of equations, we can generate matrices A_1 to A_n this way:

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(i-1)} & b_1 & a_{1(i+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(i-1)} & b_2 & a_{2(i+1)} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n(i-1)} & b_n & a_{n(i+1)} & \cdots & a_{nn} \end{bmatrix}$$

Then we have:

$$\boxed{x_i = \frac{|A_i|}{|A|}}$$

$(n+1)!$ Multiplications are required when using this method too!

These two methods contain too much error, because the answer is rounded off repeatedly.

An Example

Let's use the following system of equations:

$$
\begin{aligned}
2x + \ \ y + z &= 3 \\
x - \ \ y - z &= 0 \\
x + 2y + z &= 0
\end{aligned}
$$

system of equations	coefficient matrix's determinant	answer column	D_x: coefficient determinant with answer-column values in x-column
$2x + 1y + 1z = 3$ $1x - 1y - 1z = 0$ $1x + 2y + 1z = 0$	$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$	$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix}$

An Example

Similarly, D_y and D_z would then be:

$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} \qquad D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

Evaluating each determinant (using the method explained here), we get:

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (-2)+(-1)+(2) -(-1)-(-4)-(1) = 3$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (-3)+(0)+(0) -(0)-(-6)-(0) = -3+6 = 3$$

$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = (0)+(-3)+(0) -(0)-(0)-(3) = -3-3 = -6$$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = (0)+(0)+(6) -(-3)-(0)-(0) = 6+3 = 9$$

An Example

Cramer's Rule says that $x = D_x \div D$, $y = D_y \div D$, and $z = D_z \div D$. That is:

$$x = {}^{3}/_{3} = 1, \quad y = {}^{-6}/_{3} = -2, \quad \text{and} \quad z = {}^{9}/_{3} = 3$$

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Gaussian Elimination Method

Initialy matrix $\tilde{A}$ is computed. Then its rows are displaced in a way that there are no zeros in the main diagonal.

A is converted to an upper triangular matrix afterwards using

- magnification (multiplying a row by a constant) or
- replacement rules (adding or subtracting rows)

Ultimately, x_n can be easily calculated using the last row of the obtained matrix. Likewise, x_is can be calculated using other rows.

Gaussian Elimination Method

The previous method is called backward Gaussian method.

The other versions of this method are called forward Gaussian and Gauss-Jordan in case the matrix A is finally converted into a lower triangular and the identity matrices, respectively.

Gaussian Elimination Method

In backward method, $\frac{2n^3+3n^2-5n}{6}$ multiplications and divisions with $\frac{n^3-n}{3}$ additions and subtractions are needed for converting A into an upper triangular matrix. Moreover, in replacement step $\frac{n^2+n}{2}$ multiplications and divisions with $\frac{n^2-n}{2}$ additions and subtractions are required.

So, as an example, for a system of 10 equations, 805 operations are needed.

Example

$$\begin{cases} 10x_1 - x_2 + 2x_3 = 4 \\ x_1 + 10x_2 - x_3 = 3 \\ 2x_1 + 3x_2 + 20x_3 = 7 \end{cases} \Rightarrow \tilde{A} = \begin{bmatrix} 10 & -1 & 2 & 4 \\ 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 7 \end{bmatrix} =$$

$$\begin{bmatrix} 10 & -1 & 2 & 4 \\ 0 & 101 & -12 & 26 \\ 0 & 16 & 98 & 31 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 2 & 4 \\ 0 & 101 & -12 & 26 \\ 0 & 0 & 99.901 & 26.881 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_3 = \dfrac{26.881}{99.901} = 0.269 \\ x_2 = \dfrac{26+12x_3}{101} = 0.289 \\ x_1 = \dfrac{4-2x_3+x_2}{10} = 0.375 \end{cases}$$

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Triangular Decomposition Method

First step in this method is called LU decomposition which means decomposing A to 2 matrices L and U such that:

$$A = LU$$

Where:

- L is a lower triangular matrix with all ones in its main diagonal, and
- U is an upper triangular matrix with no zeros in its main diagonal.

Calculating L and U

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

After multiplying, l_{ij}s and u_{ij}s are simply achieved:

$$u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13}$$

$$l_{21} = \frac{a_{21}}{u_{11}}, u_{22} = a_{22} - l_{21}u_{12}, u_{23} = a_{23} - l_{21}u_{13}$$

$$l_{31} = \frac{a_{31}}{u_{11}}, l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}, u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Triangular Decomposition Method

Next step is decomposition of B to L and Y:

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

After multiplying, y_is are easily calculated:

$$y_1 = b_1, y_2 = b_2 - l_{21}y_1, y_3 = b_3 - l_{31}y_1 - l_{32}y_2$$

Triangular Decomposition Method

Knowing $B = LUX$ and $\mathrm{LY} = B$, we obtain:

$$Y = UX$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

After multiplying, x_is are simply achieved:

$$x_3 = \frac{y_3}{u_{33}}, x_2 = \frac{y_2 - u_{23}x_3}{u_{22}}, x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}$$

Triangular Decomposition Method

Triangular decomposition method is called:

- Doolittle's Method in which L is a matrix with all ones on its main diagonal.
- Crout's Method in which U is a matrix with all ones on its main diagonal.
- Cholesky's Method in which $U = L^T$.

Note that Cholesky's Method works only in case A is symmetric and positive definite.

An Example

Example:

Solve the following system of equations using LU Decomposition method:

$$x_1 + x_2 + x_3 = 1$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$3x_1 + 5x_2 + 3x_3 = 4$$

Solution: Here, we have

A = $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ such that A X = C.

An Example

Hence, we get L = $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$

Now, we assume Z $= \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ and solve L Z = C.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

So, we have $z_1 = 1, 4z_1 + z_2 = 6, 3z_1 - 2z_2 + z_3 = 4.$

An Example

Solving, we get $z_1 = 1$, $z_2 = 2$ and $z_3 = 5$.

Now, we solve U X = Z

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Therefore, we get $x_1 + x_2 + x_3 = 1$, $-x_2 - 5x_3 = 2$, $-10x_3 = 5$.

Thus, the solution to the given system of linear equations is $x_1 = 1$, $x_2 = 0.5$, $x_3 = -0.5$ and hence the matrix X = $\begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}$

Direct Methods

Indirect Methods

Extra Topics

- Jacobi Method
- Gauss-Seidel Method

Jacobi Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = 1/a_{11}(b_1 - a_{12}x_2 - a_{13}x_3)$$
$$x_2 = 1/a_{22}(b_2 - a_{21}x_1 - a_{23}x_3)$$
$$x_3 = 1/a_{33}(b_3 - a_{31}x_1 - a_{32}x_2)$$

$$\mathbf{x}^{(0)} = [x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]^t$$

$$x_1^{(1)} = 1/a_{11}(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)})$$
$$x_2^{(1)} = 1/a_{22}(b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)})$$
$$x_3^{(1)} = 1/a_{33}(b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)})$$

Jacobi Method

$$x_{۱}^{(k+۱)} = ۱/a_{۱۱}(b_{۱} - a_{۱۲}x_{۲}^{(k)} - a_{۱۳}x_{۳}^{(k)})$$

$$x_{۲}^{(k+۱)} = ۱/a_{۲۲}(b_{۲} - a_{۲۱}x_{۱}^{(k)} - a_{۲۳}x_{۳}^{(k)})$$

$$x_{۳}^{(k+۱)} = ۱/a_{۳۳}(b_{۳} - a_{۳۱}x_{۱}^{(k)} - a_{۳۲}x_{۲}^{(k)}), \qquad k = ۰, ۱, ۲ \cdots$$

Jacobi Method

An initial guess is required which will be improved as below:

$$x_1^{(k+1)} = \frac{1}{a_{11}}[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}]$$

$$\cdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}[b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots - a_{n(n-1)}x_{n-1}^{(k)}]$$

Thus,

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j=1\ j\neq i}^{n} (a_{ij}x_j^{(k)})]}$$

Example

$$4x_1 - x_2 + x_3 = 4$$
$$x_1 + 6x_2 + 2x_3 = 9$$
$$-x_1 - 2x_2 + 5x_3 = 2$$

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$$

$$x_1 = \frac{1}{4}(4 + x_2 - x_3)$$
$$x_2 = \frac{1}{6}(9 - x_1 - 2x_3)$$
$$x_3 = \frac{1}{5}(2 + x_1 + 2x_2)$$

$k = 0$

$$x_1^{(1)} = \frac{1}{4}(4 + x_2^{(0)} - x_3^{(0)}) = \frac{1}{4}(4 + 0 - 0) = 1$$
$$x_2^{(1)} = \frac{1}{6}(9 - x_1^{(0)} - 2x_3^{(0)}) = \frac{1}{6}(9 - 0 - 0) = 1.5$$
$$x_3^{(1)} = \frac{1}{5}(2 + x_1^{(0)} + 2x_2^{(0)}) = \frac{1}{5}(2 + 0 + 0) = 0.4$$

$k = 1$

$$x_1^{(2)} = \frac{1}{4}(4 + x_2^{(1)} - x_3^{(1)}) = \frac{1}{4}(4 + 1.5 - 0.4) = 1.2750$$
$$x_2^{(2)} = \frac{1}{6}(9 - x_1^{(1)} - 2x_3^{(1)}) = \frac{1}{6}(9 - 1 - 0.8) = 1.2000$$
$$x_3^{(2)} = \frac{1}{5}(2 + x_1^{(1)} + 2x_2^{(1)}) = \frac{1}{5}(2 + 1 + 3) = 1.2000$$

Example

شماره تکرار	$x_۱$	$x_۲$	$x_۳$
۰	۰	۰	۰
۱	۱/۰۰۰۰	۱/۵۰۰۰	۰/۴۰۰۰
۲	۱/۲۷۵۰	۱/۲۰۰۰	۱/۲۰۰۰
۳	۱/۰۰۰۰	۰/۸۸۷۵	۱/۱۳۵۰
۴	۰/۹۳۸۱	۰/۹۵۵۰	۱/۰۵۴۰
۵	۰/۹۷۵۳	۰/۹۹۲۳	۱/۰۰۹۲
۶	۰/۹۹۵۸	۱/۰۰۱۱	۰/۹۹۲۰
۷	۱/۰۰۲۳	۱/۰۰۳۴	۰/۹۹۹۶
۸	۱/۰۰۱۰	۰/۹۹۹۸	۱/۰۰۱۸
۹	۰/۹۹۹۵	۰/۹۹۹۲	۱/۰۰۰۹
۱۰	۰/۹۹۹۶	۰/۹۹۹۸	۰/۹۹۹۶
۱۱	۱/۰۰۰۱	۱/۰۰۰۲	۰/۹۹۹۸
۱۲	۱/۰۰۰۱	۱/۰۰۰۱	۱/۰۰۰۱
۱۳	۱/۰۰۰۰	۱/۰۰۰۰	۱/۰۰۰۱
۱۴	۱/۰۰۰۰	۱/۰۰۰۰	۱/۰۰۰۰

$$x_۱^{(k+۱)} = \frac{۱}{۴}\left(۴ + x_۲^{(k)} - x_۳^{(k)}\right)$$
$$x_۲^{(k+۱)} = \frac{۱}{۶}\left(۹ - x_۱^{(k)} - ۲x_۳^{(k)}\right)$$
$$x_۳^{(k+۱)} = \frac{۱}{۵}\left(۲ + x_۱^{(k)} + ۲x_۲^{(k)}\right)$$

$$x_۱ = x_۲ = x_۳ = ۱$$

Direct Methods

Indirect Methods

Extra Topics

- Jacobi Method
- Gauss-Seidel Method

Gauss-Seidel Method

This Method is similar to the Jacobi method with the same convergence condition, but converges more quickly.

$$x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j=1}^{i-1}\left(a_{ij}x_j^{(k+1)}\right) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k)})]$$

Example

$$۴x_۱ - x_۲ + x_۳ = ۴$$
$$x_۱ + ۶x_۲ + ۲x_۳ = ۹$$
$$-x_۱ - ۲x_۲ + ۵x_۳ = ۲$$

$$x_۱^{(k+۱)} = \frac{۱}{۴}(۴ + x_۲^{(k)} - x_۳^{(k)})$$
$$x_۲^{(k+۱)} = \frac{۱}{۶}(۹ - x_۱^{(k+۱)} - ۲x_۳^{(k)})$$
$$x_۳^{(k+۱)} = \frac{۱}{۵}(۲ + x_۱^{(k+۱)} + ۲x_۲^{(k+۱)})$$

$$k = ۰$$

$$x_۱^{(۱)} = \frac{۱}{۴}(۴ + ۰ + ۰) = ۱$$
$$x_۲^{(۱)} = \frac{۱}{۶}(۹ - ۱ - ۰) = \frac{۴}{۳}$$
$$x_۳^{(۱)} = \frac{۱}{۵}(۲ + ۱ + \frac{۸}{۳}) = \frac{۱۷}{۱۵}$$

Example

شماره تکرار	$x_۱$	$x_۲$	$x_۳$
۰	۰	۰	۰
۱	۱٫۰۰۰۰	۱٫۳۳۳۳	۱٫۱۳۳۳
۲	۱٫۰۵۰۰	۰٫۹۴۷۳	۰٫۹۸۸۹
۳	۰٫۹۸۹۶	۱٫۰۰۵۰	۰٫۹۹۹۹
۴	۱٫۰۰۱۰	۰٫۹۹۹۹	۱٫۰۰۰۰
۵	۱٫۰۰۰۰	۱٫۰۰۰۰	۱٫۰۰۰۰

$$x_۱ = x_۲ = x_۳ = ۱$$

Strictly Diagonally Dominant Matrix

The matrix A is strictly diagonally dominant if:

$$|a_{ii}| > \sum_{j=1\ j\neq i}^{n} |a_{ij}| \qquad i = 1,2,\dots,n$$

Matrices with this property are always invertible.

If coefficient matrix is strictly diagonally dominant, then Jacobi method converges with any initial guess.

Jacobi method converges linearly.

Direct Methods

Indirect Methods

Extra Topics

- Eigenvalues and Eigenvectors
- Power Method
- Gerschgorin's Theorem

Eigenvalues and Eigenvectors

Previous methods work for non-Homogeneous systems.

But what if $B = 0$ and $|A| \neq 0$?

In this case, one trivial answer is zero. However, it might be useful to find non-trivial answers as well.

$$AX = \lambda X \rightarrow AX - \lambda X = 0 \rightarrow (A - \lambda I)X = 0$$

For an $n \times n$ system, we can find n eigenvalues(λ) with n corresponding Xs, called Eigenvectors.

Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$\boxed{|A - \lambda I| = 0}$$

Thus, an n-degree equation is obtained whose roots are λ_is.

Eigenvalues and Eigenvectors

After calculating λs and replacing them in this equation:

$$(A - \lambda I)X = 0$$

We will have a homogeneous system with zero determinant. Hence, the system will have infinitely many answers.

Set x_1 equal to an arbitrary value (usually 1) to achieve all the other x_is.

Eigenvalues and Eigenvectors

- Eigenvalues of A and A^T are equal.
- Eigenvalues of upper and lower triangular matrices are equal to the main diagonal.
- Eigenvalues of symmetric matrices are real numbers.
- Determinant of each matrix is the product of its eigenvalues.
- If eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then eigenvalues of A^{-1} will be $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

$$\boxed{\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i}$$

Eigenvalues and Eigenvectors

- If eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$ and $k \in R$, then
  - eigenvalues of kA will be $\mathrm{k}\lambda_1, \mathrm{k}\lambda_2, \dots, k\lambda_n$
  - eigenvalues of A^k will be ${\lambda_1}^k, {\lambda_2}^k, \dots, {\lambda_n}^k$
  - and eigenvalues of $kI + A$ will be $\mathrm{k} + \lambda_1, \mathrm{k} + \lambda_2, \dots, k + \lambda_n$

- If eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$ and we have $|\lambda_p| > \lambda_i \;\; i = 1, 2, \dots, p-1, p+1, \dots, n$, then λ_p is called

the dominant eigenvalue and its corresponding vector is called the dominant eigenvector.

Example

$$\begin{cases} 2x_1 + x_2 = 0 \\ 3x_1 - 2x_2 = 0 \end{cases} => |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 3 & -2-\lambda \end{vmatrix} =>$$

$$\lambda_1 = \sqrt{7} => \begin{cases} (2-\sqrt{7})x_1 + x_2 = 0 \\ 3x_1 - (2+\sqrt{7})x_2 = 0 \end{cases} =>$$

Eigenvector: $x_1 = 1$, $x_2 = 0.6458 => \begin{pmatrix} 1 \\ 0.6458 \end{pmatrix}$

For $\lambda_2 = -\sqrt{7}$, do likewise and calculate the eigenvector.

Direct Methods

Indirect Methods

Extra Topics

- Eigenvalues and Eigenvectors
- Power Method
- Gerschgorin’s Theorem

Dominant Eigenvalue and Eigenvector

For calculating the dominant eigenvalue and eigenvector, power iteration is applied:

Start with an initial guess X_0:

$$Y_0 = AX_0$$

Divide y_is to c_1 which is maximum of y_is (in respect of the magnitude),

The resulted vector, named X_1, is an approximation of the dominant eigenvector whereas c_1 is an approximation of the dominant eigenvalue.

Iterate previous steps. More iterations will lead to a more precise answer.

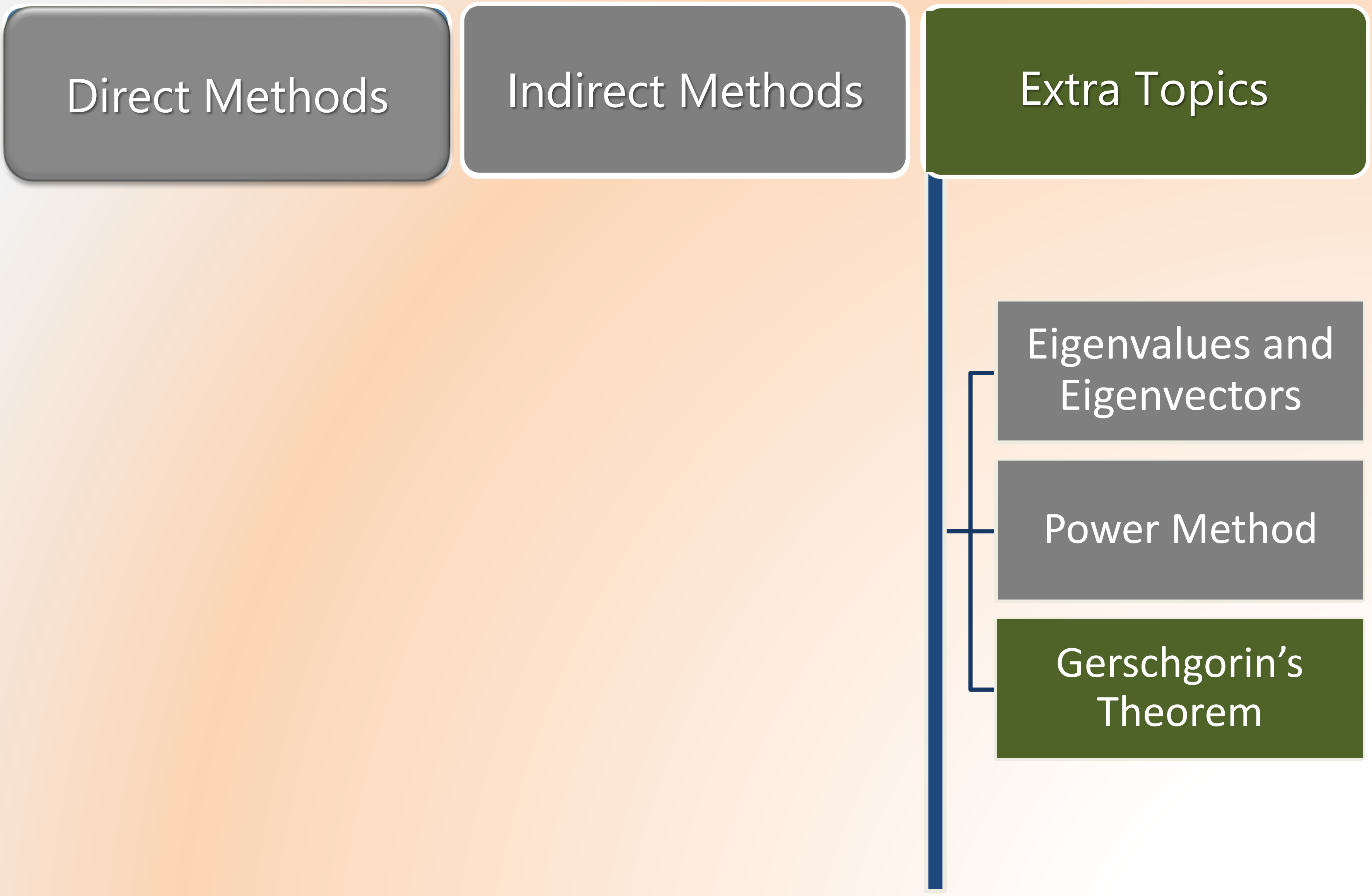
Power iteration converges linearly.

Example

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$

Initial guess: $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} => Y_0 = AX_0 = \begin{bmatrix} -8 \\ 4 \\ 18 \end{bmatrix} => X_1 = 18 * \begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} =>$

Where 18 is an approximation of λ_p and $\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix}$ is an approximation of the dominant eigenvector.


Direct Methods
Indirect Methods
Extra Topics
Eigenvalues and Eigenvectors
Power Method
Gerschgorin's Theorem

Gerschgorin's Theorem

This theorem presents us a series of intervals, any of which contains an eigenvalue.

$$|\lambda_i - a_{ii}| \le \sum_{j=0\ j\neq i}^{n} |a_{ij}|$$

Example :

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \quad \begin{array}{c} |\lambda_1 - 1| \le (|2| + |-1|) \rightarrow -2 \le \lambda_1 \le 4 \\ |\lambda_2 - 1| \le (1 + 1) \rightarrow -1 \le \lambda_2 \le 3 \\ |\lambda_3 + 1| \le (1 + 3) \rightarrow -5 \le \lambda_3 \le 3 \end{array}$$


Any questions?